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Homology and cohomology of quantum complete intersections

Petter Andreas Bergh and Karin Erdmann

Dedicated to Luchezar Avramov on the occasion of his sixtieth birthday

We construct a minimal projective bimodule resolution for every finite-dimensional quantum complete intersection of codimension two. Then we use this resolution to compute both the Hochschild cohomology and homology for such an algebra. In particular, we show that the cohomology vanishes in high degrees, while the homology is always nonzero.

1. Introduction

The notion of quantum complete intersections originates from the work by Manin [1987], who introduced the concept of *quantum symmetric algebras*. These algebras were used by Avramov, Gasharov and Peeva [1997] to study modules behaving homologically as modules over commutative complete intersections. In particular, they introduced *quantum regular sequences* of endomorphisms of modules, thus generalizing the classical notion of regular sequences.

Benson, Erdmann and Holloway [2007] defined and studied a new rank variety theory for modules over finite-dimensional quantum complete intersections. For this theory to work, it is essential that the commutators defining the quantum complete intersection be roots of unity, so that a linear combination of the generators behave itself as a generator. In this setting, at least for quantum complete intersections of codimension two, the Hochschild cohomology ring is infinite-dimensional, and a priori there might be connections between rank varieties and the support varieties defined by Snashall and Solberg [2004] (see also [Erdmann et al. 2004]).

Whether or not the higher Hochschild cohomology groups of a finite-dimensional algebra of infinite global dimension can vanish, known as "Happel's question", was unknown until the appearance of [Buchweitz et al. 2005]. In that paper, the authors constructed a four-dimensional selfinjective algebra whose total

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Hochschild cohomology is five-dimensional, thus giving a negative answer to Happel's question. The algebra they constructed is the smallest possible noncommutative quantum complete intersection.

In this paper we study finite-dimensional quantum complete intersections of codimension two. For such an algebra, we construct a minimal projective bimodule resolution, and use this to compute the Hochschild homology and cohomology. In particular, we show that the higher Hochschild cohomology groups vanish if and only if the commutator element is not a root of unity, whereas the Hochschild homology groups never vanish. Thus we obtain a large class of algebras having the same homological properties as the algebra used in [Buchweitz et al. 2005].

2. The minimal projective resolution

Throughout this paper, let k be a field and $q \in k$ a nonzero element. In the main results, this element is assumed not to be a root of unity, implying indirectly that k is an infinite field. We fix two integers $a, b \geq 2$, and denote by A the k-algebra

$$A = k\langle X, Y\rangle/(X^a, XY - qYX, Y^b).$$

This is a finite-dimensional algebra of dimension ab, and it is justifiably a quantum complete intersection of codimension 2; it is the quotient of the quantum symmetric algebra

$$k\langle X, Y\rangle/(XY - qYX)$$

by the quantum regular sequence X^a and Y^b (as defined in [Avramov et al. 1997, Section 2]). We denote the generators of A by x and y, and use the set

$$\{y^i x^j\}_{0 \leq i < b,\ 0 \leq j < a}$$

as a k-basis. The opposite algebra of A is denoted by A^{op}, and the enveloping algebra $A \otimes_k A^{\mathrm{op}}$ by A^{e}.

We now construct explicitly a minimal projective bimodule resolution

$$\mathbb{P}\colon \cdots \to P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\mu} A \to 0,$$

in which P_n is free and of rank $n+1$, viewing the bimodules as left A^{e}-modules. The generators $1 \otimes 1$ of P_n are labeled $\epsilon(i, j)$ for $i, j \geq 0$, such that

$$P_n = \bigoplus_{i+j=n} A^{\mathrm{e}} \epsilon(i, j).$$

For each $s \geq 0$, define the four elements of A^{e}:

$$\tau_1(s) = q^s(1 \otimes x) - (x \otimes 1)$$
$$\tau_2(s) = (1 \otimes y) - q^s(y \otimes 1)$$
$$\gamma_1(s) = \sum_{j=0}^{a-1} q^{js}(x^{a-1-j} \otimes x^j)$$
$$\gamma_2(s) = \sum_{j=0}^{b-1} q^{js}(y^j \otimes y^{b-1-j}).$$

Let $P_0 \xrightarrow{\mu} A$ be the multiplication map $w \otimes z \mapsto wz$. The kernel of this map is generated by $\tau_1(0)$ and $\tau_2(0)$. Now let R_1 and R_2 be the commutative subalgebras of A generated by x and y, respectively. The annihilator of $\tau_i(0)$, viewed as an element of R_i^{e}, is $\gamma_i(0)$, and the complex

$$\cdots \to R_i^{\mathrm{e}} \xrightarrow{\tau_i(0)} R_i^{\mathrm{e}} \xrightarrow{\gamma_i(0)} R_i^{\mathrm{e}} \xrightarrow{\tau_i(0)} R_i^{\mathrm{e}} \xrightarrow{\mu} R_i \to 0$$

is a minimal projective bimodule resolution of R_i [Holm 2000].

In general, given any algebra Γ and an automorphism $\Gamma \xrightarrow{\psi} \Gamma$, we may endow every Γ-module X with a new module structure by restricting scalars via ψ. In this way, we obtain a new module ${}_{\psi}X$, whose underlying set is the same as that of X, but where scalar multiplication is given by

$$\gamma \cdot x = \psi(\gamma)x$$

for $\gamma \in \Gamma$ and $x \in X$. This new module is the *twist of* X with respect to ψ. A homomorphism $X \to Y$ of Γ-modules induces a homomorphism

$${}_{\psi}X \to {}_{\psi}Y$$

of twisted modules.

Now for $i = 1, 2$, define an algebra automorphism $R_i^{\mathrm{e}} \xrightarrow{\sigma_i} R_i^{\mathrm{e}}$ by

$$\sigma_1\colon\ x \otimes 1 \mapsto x \otimes 1,\ 1 \otimes x \mapsto q(1 \otimes x),$$
$$\sigma_2\colon\ y \otimes 1 \mapsto q(y \otimes 1),\ 1 \otimes y \mapsto 1 \otimes y.$$

When we twist the above resolution of R_i by the automorphism σ_i^s for some $s \geq 0$, then multiplication by $\tau_i(0)$ and $\gamma_i(0)$ become multiplication by $\tau_i(s)$ and $\gamma_i(s)$, respectively. We denote this twisted resolution by $\mathbf{R}_i(s)$.

We now define a double complex

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\downarrow & & \downarrow & & \downarrow & & \\
A^{\mathrm{e}}\epsilon(0,2) & \leftarrow & A^{\mathrm{e}}\epsilon(1,2) & \leftarrow & A^{\mathrm{e}}\epsilon(2,2) & \leftarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
A^{\mathrm{e}}\epsilon(0,1) & \leftarrow & A^{\mathrm{e}}\epsilon(1,1) & \leftarrow & A^{\mathrm{e}}\epsilon(2,1) & \leftarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
A^{\mathrm{e}}\epsilon(0,0) & \leftarrow & A^{\mathrm{e}}\epsilon(1,0) & \leftarrow & A^{\mathrm{e}}\epsilon(2,0) & \leftarrow & \cdots
\end{array}$$

whose total complex $\mathbb{P}$ turns out to be the projective bimodule resolution we are seeking. Along row $2s$ we use the resolution $\mathbf{R}_1(bs)$, and along row $2s+1$ we use the resolution $\mathbf{R}_1(bs+1)$. Explicitly, the row maps are given by

$$\begin{aligned}
\epsilon(2r,2s) &\mapsto \gamma_1(bs)\epsilon(2r-1,2s),\\
\epsilon(2r+1,2s) &\mapsto \tau_1(bs)\epsilon(2r,2s),\\
\epsilon(2r,2s+1) &\mapsto \gamma_1(bs+1)\epsilon(2r-1,2s+1),\\
\epsilon(2r+1,2s+1) &\mapsto \tau_1(bs+1)\epsilon(2r,2s+1).
\end{aligned}$$

Similarly, along column $2r$ we use the resolution $\mathbf{R}_2(ar)$, and along column $2r+1$ we use the resolution $\mathbf{R}_2(ar+1)$, introducing a sign in the odd columns. The column maps are therefore given by

$$\begin{aligned}
\epsilon(2r,2s) &\mapsto \gamma_2(ar)\epsilon(2r,2s-1),\\
\epsilon(2r,2s+1) &\mapsto \tau_2(ar)\epsilon(2r,2s),\\
\epsilon(2r+1,2s) &\mapsto -\gamma_2(ar+1)\epsilon(2r+1,2s-1),\\
\epsilon(2r+1,2s+1) &\mapsto -\tau_2(ar+1)\epsilon(2r+1,2s).
\end{aligned}$$

It is straightforward to verify that these maps indeed define a double complex; all the four different types of squares commute. The transpose of the matrices defining the maps in the resulting double complex are given by

$$\begin{pmatrix}
\gamma_1(0) & -\tau_2(as+1) & 0 & 0 & 0 & 0 & \cdots & 0\\
0 & \tau_1(1) & \gamma_2(as) & 0 & 0 & 0 & \cdots & 0\\
0 & 0 & \gamma_1(b) & -\tau_2(a[s-1]+1) & 0 & 0 & \cdots & 0\\
0 & 0 & 0 & \tau_1(b+1) & \gamma_2(a[s-1]) & 0 & \cdots & 0\\
\vdots & \vdots & & \ddots & \ddots & \ddots & \ddots & \vdots\\
0 & 0 & \cdots & & 0 & \gamma_1(bs) & -\tau_2(1) & 0\\
0 & 0 & \cdots & & 0 & 0 & \tau_1(bs+1) & \gamma_2(0)
\end{pmatrix}$$

for the map at stage $2(s+1)$, and

$$\begin{pmatrix} \tau_1(0) & \tau_2(as) & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \gamma_1(1) & -\gamma_2(a[s-1]+1) & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \tau_1(b) & \tau_2(a[s-1]) & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \gamma_1(b+1) & -\gamma_2(a[s-2]+1) & 0 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & & 0 & \gamma_1(b[s-1]+1) & -\gamma_2(1) & 0 \\ 0 & 0 & \cdots & & 0 & 0 & \tau_1(bs) & \tau_2(0) \end{pmatrix}$$

for the map at stage $2s+1$.

Now, for each $n \geq 0$, denote the generator $\epsilon(i, n-i)$ by f_i^n, so that the n-th bimodule in the total complex $\mathbb{P}$ is

$$P_n = \bigoplus_{i=0}^{n} A^{\mathrm{e}} f_i^n,$$

the free A^{e}-module of rank $n+1$ having generators

$$\{f_0^n, f_1^n, \ldots, f_n^n\}.$$

Then the maps $P_n \xrightarrow{d_n} P_{n-1}$ in $\mathbb{P}$ are given by

$$d_{2t}\colon f_i^{2t} \mapsto \begin{cases} \gamma_2(\frac{ai}{2}) f_i^{2t-1} + \gamma_1(\frac{2bt-bi}{2}) f_{i-1}^{2t-1}, & \text{for } i \text{ even}, \\ -\tau_2(\frac{ai-a+2}{2}) f_i^{2t-1} + \tau_1(\frac{2bt-bi-b+2}{2}) f_{i-1}^{2t-1}, & \text{for } i \text{ odd}, \end{cases}$$

$$d_{2t+1}\colon f_i^{2t+1} \mapsto \begin{cases} \tau_2(\frac{ai}{2}) f_i^{2t} + \gamma_1(\frac{2bt-bi+2}{2}) f_{i-1}^{2t}, & \text{for } i \text{ even}, \\ -\gamma_2(\frac{ai-a+2}{2}) f_i^{2t} + \tau_1(\frac{2bt-bi+b}{2}) f_{i-1}^{2t}, & \text{for } i \text{ odd}, \end{cases}$$

where we use the convention $f_{-1}^n = f_{n+1}^n = 0$. The following result shows that the complex is exact.

Proposition 2.1. *The complex $\mathbb{P}$ is exact, and is therefore a minimal projective resolution*

$$\mathbb{P}\colon \cdots \to P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\mu} A \to 0$$

of the left A^{e}-module A.

Proof. We will show that the complex $\mathbb{P} \otimes_A k$ is exact, and a minimal projective resolution of the A-module k. Then the arguments in [Green and Snashall 2004] show that the complex $\mathbb{P}$ is exact.

When applying $-\otimes_A k$ to $A^{\mathrm{e}} = A \otimes_k A^{\mathrm{op}}$, the elements x and y in A^{op} become zero, and so the elements $\tau_i(s) \otimes 1$ and $\gamma_i(s) \otimes 1$ are just given by

$$\begin{aligned}
\tau_1(s) \otimes 1 &= -(x \otimes 1),\\
\tau_2(s) \otimes 1 &= -q^s(y \otimes 1),\\
\gamma_1(s) \otimes 1 &= (x^{a-1} \otimes 1),\\
\gamma_2(s) \otimes 1 &= q^{(b-1)s}(y^{b-1} \otimes 1).
\end{aligned}$$

We shall identify these elements with $-x$, $-q^s y$, x^{a-1} and $q^{(b-1)s} y^{b-1}$, respectively. Moreover, whenever the commutator element q is involved, its precise power does not affect the dimensions of the vector spaces we are considering, so we shall write q^* for simplicity.

Fix a number $n \geq 0$. The free bimodule P_n has generators $\epsilon(i, j)$, with $n = i + j$ and $i, j \geq 0$. When the degree is not ambiguous, we shall denote the element

$$\epsilon(i, j) \otimes 1 \in P_n \otimes_A k$$

by e_j, and we shall denote the map

$$P_n \otimes_A k \xrightarrow{d_n \otimes 1} P_{n-1} \otimes_A k$$

by $\widehat{d}_n$. Moreover, we denote by U_i the left A-submodule of $P_{n-1} \otimes_A k$ generated by $\widehat{d}_n(e_j)$, so that

$$\operatorname{Im} \widehat{d}_n = U_0 + \cdots + U_n \subseteq P_{n-1} \otimes_A k.$$

We now compute the dimensions of these modules U_i. Assume first that n is even. Then

$$\begin{aligned}
U_0 &= Ax^{a-1}e_0,\\
U_i &= A[(q^* y)e_{i-1} + (q^* x)e_i], \text{ for odd } 0 < i < n,\\
U_i &= A\big[(q^* y^{b-1})e_{i-1} + (q^* x^{a-1})e_i\big], \text{ for even } 0 < i < n,\\
U_n &= Ay^{b-1}e_{n-1},
\end{aligned}$$

and so we see that $\dim U_0 = b$, $\dim U_n = a$, and otherwise $\dim U_i = ab - 1$ and $\dim U_j = a + b + 1$ for i odd and j even. When n is odd, then

$$\begin{aligned}
U_0 &= Axe_0,\\
U_i &= A[(-q^* y)e_{i-1} + (q^* x^{a-1})e_i], \text{ for odd } 0 < i < n,\\
U_i &= A\big[(q^* y^{b-1})e_{i-1} + (q^* x)e_i\big], \text{ for even } 0 < i < n,\\
U_n &= Aye_{n-1},
\end{aligned}$$

and so in this case we see that $\dim U_0 = b(a-1)$, $\dim U_n = a(b-1)$, and otherwise $\dim U_i = a(b-1)+1$ and $\dim U_j = b(a-1)+1$ for i odd and j even.

Our aim is to compute the dimensions of various intersections and sums obtained from the modules U_i. In order to do this, we need the fact that for any elements $z_1, z_2 \in A$, the implication

$$z_1 x^s = z_2 y^t \Longrightarrow z_1 = v_1 y^t + w_1 x^{a-s} \text{ and } z_2 = v_2 x^s + w_2 y^{b-t} \tag{2-1}$$

holds, where v_i and w_i are some elements in A depending on z_1 and z_2. To see this, write

$$z_1 = g_0 + g_1 y + \cdots + g_{b-1} y^{b-1} \quad \text{and} \quad z_2 = h_0 + h_1 y + \cdots + h_{b-1} y^{b-1},$$

where the g_i and h_i are polynomials in x. Then

$$\sum_i h_i y^{t+i} = z_2 y^t = z_1 x^s = \sum_j (q^{-js} g_j x^s) y^j,$$

and comparing the coefficients of y^j, we find that $g_j x^s = 0$ for $j < t$. Therefore, for these values of j, the polynomial g_j must be a multiple of x^{a-s}. Then we can write

$$\sum_{j<t} g_j y^j = w_1 x^{a-s}$$

for some $w_1 \in A$, giving

$$z_1 = \sum_{j<t} g_j y^j + \sum_{j \geq t} g_j y^j = w_1 x^{a-s} + v_1 y^t,$$

where $v_1 = \sum_{j \geq t} g_j y^{j-t}$. This proves the statement for z_1, and the proof for z_2 is similar.

We now compute the intersections of pairs of the modules U_i. Suppose n is even, and fix an even integer $0 \leq j \leq n$. If u belongs to $U_j \cap U_{j+1}$, then there are elements $z_1, z_2 \in A$ such that

$$u = z_1 \big[(q^* y^{b-1}) e_{j-1} + (q^* x^{a-1}) e_j \big] = z_2 [(q^* y) e_j + (q^* x) e_{j+1}].$$

The coefficients of e_{j-1} and e_{j+1} must be zero, whereas those of e_j must be equal, giving

$$(z_1 q^*) x^{a-1} = (z_2 q^*) y.$$

By (2-1), there are elements $v_1, v_2, w_1, w_2 \in A$ such that

$$z_1 q^* = v_1 y + w_1 x, \qquad z_2 q^* = v_2 x^{a-1} + w_2 y^{b-1},$$

hence $u \in A y x^{a-1} e_j$. Conversely, any element in $A y x^{a-1} e_j$ belongs to $U_j \cap U_{j+1}$, showing

$$U_j \cap U_{j+1} = A y x^{a-1} e_j,$$

and that the dimension of this intersection is $b-1$. Similarly, we compute three other types of intersections using the same method, and record everything in the table:

n	j	intersection	dimension
even	even	$U_j \cap U_{j+1} = Ayx^{a-1}e_j$	$b-1$
even	odd	$U_j \cap U_{j+1} = Ay^{b-1}xe_j$	$a-1$
odd	even	$U_j \cap U_{j+1} = Ayxe_j$	$(a-1)(b-1)$
odd	odd	$U_j \cap U_{j+1} = Ay^{b-1}x^{a-1}e_j$	1

Next we show that the equality

$$(U_0 + U_1 + \cdots + U_s) \cap U_{s+1} = U_s \cap U_{s+1} \tag{2-2}$$

holds for any $s \geq 1$. Suppose first that both n and s are even. The inclusion $U_s \cap U_{s+1} \subseteq (U_0 + U_1 + \cdots + U_s) \cap U_{s+1}$ obviously holds, so suppose u is an element belonging to $(U_0 + U_1 + \cdots + U_s) \cap U_{s+1}$. Then u can be written as

$$\begin{aligned} u &= z_0 x^{a-1} e_0 + z_1[(q^* y)e_0 + (q^* x)e_1] + \cdots + z_s\big[(q^* y^{b-1})e_{s-1} + (q^* x^{a-1})e_s\big] \\ &= z_{s+1}[(q^* y)e_s + (q^* x)e_{s+1}], \end{aligned}$$

in which the coefficient of e_{s+1} must be zero. Moreover, the coefficients of e_s must be equal, that is,

$$(z_{s+1} q^*) y = (z_s q^*) x^{a-1},$$

and so from (2-1) we see that there exist elements $v, w \in A$ such that

$$z_{s+1} = v x^{a-1} + w y^{b-1}.$$

This gives

$$u = (v x^{a-1} + w y^{b-1})\, q^* y e_s = v q^* x^{a-1} y e_s,$$

and we see directly that u belongs to $U_s \cap U_{s+1}$. Equation (2-2) therefore holds when n and s are even, and the same arguments show that the equality holds regardless of the parity of n and s.

Using what we just showed, an induction argument gives the equality

$$\dim(U_0 + \cdots + U_s) = \sum_{i=0}^{s} \dim U_i - \sum_{i=0}^{s-1} \dim(U_i \cap U_{i+1}).$$

Then by counting dimensions, we see that the dimension of $\operatorname{Im} \widehat{d}_n$ is given by

$$\dim \operatorname{Im} \widehat{d}_n = \begin{cases} tab + 1, & \text{when } n = 2t, \\ (t+1)ab - 1, & \text{when } n = 2t+1. \end{cases}$$

The exactness of the complex $\mathbb{P} \otimes_A k$ now follows easily; the image of $\widehat{d}_{n+1}$ is contained in the kernel of $\widehat{d}_n$, and the dimension of $P_n \otimes_A k$ is $ab(n+1)$. It follows that $\operatorname{Im} \widehat{d}_{n+1}$ and $\operatorname{Ker} \widehat{d}_n$ are of the same dimension.

As for minimality, it suffices to show that $\operatorname{Im} \widehat{d}_n$ does not have a projective summand. This follows from the description of this module as the sum of U_i. Namely, we see directly that the element $y^{b-1}x^{a-1} \in A$ annihilates each U_i, and therefore also $\operatorname{Im} \widehat{d}_n$. □

3. Hochschild (co)homology

Having obtained the bimodule resolution of $A = k\langle X, Y\rangle/(X^a, XY - qYX, Y^b)$, we turn now to its Hochschild homology and cohomology groups. Let B be a bimodule, and recall that the Hochschild homology of A with coefficients in B, denoted $\mathrm{HH}_*(A, B)$, is the k-vector space

$$\mathrm{HH}_*(A, B) = \mathrm{Tor}_*^{A^{\mathrm{e}}}(B, A),$$

where B is viewed as a right A^{e}-module. Dually, the Hochschild cohomology of A with coefficients in B, denoted $\mathrm{HH}^*(A, B)$, is the k-vector space

$$\mathrm{HH}^*(A, B) = \mathrm{Ext}_{A^{\mathrm{e}}}^*(A, B),$$

where B is viewed as a left A^{e}-module. Of particular interest is the case $B = A$, namely the Hochschild homology and cohomology of A, denoted $\mathrm{HH}_*(A)$ and $\mathrm{HH}^*(A)$, respectively. Now, by viewing A and B as left A^{e}-modules, it follows from [Cartan and Eilenberg 1956, VI.5.3] that $D(\mathrm{HH}^*(A, B))$ is isomorphic, as a vector space, to $\mathrm{Tor}_*^{A^{\mathrm{e}}}(D(B), A)$, where D denotes the usual k-dual $\mathrm{Hom}_k(-, k)$. In particular, by taking $B = A$, we see that

$$\dim_k \mathrm{HH}^n(A) = \dim_k \mathrm{Tor}_n^{A^{\mathrm{e}}}(D(A), A)$$

for all $n \geq 0$.

Our algebra A is Frobenius; it is easy to check that the map $A \xrightarrow{\phi} D(A)$ of left A-modules, defined by

$$\phi(1)\colon \sum_{\substack{0 \leq j \leq b-1 \\ 0 \leq i \leq a-1}} c_{j,i}\, y^j x^i \mapsto c_{b-1,a-1},$$

is an isomorphism. To such a Frobenius isomorphism, one can always associate a k-algebra automorphism $A \xrightarrow{\nu} A$, a *Nakayama automorphism*, with the (defining) property that

$$w \cdot \phi(1) = \phi(1) \cdot \nu(w)$$

for all elements $w \in A$. In our case, the elements x and y generate A, and since

$$x \cdot \phi(1) = \phi(1) \cdot q^{1-b} x \quad \text{and} \quad y \cdot \phi(1) = \phi(1) \cdot q^{a-1} y,$$

we see that the automorphism defined by

$$\nu \colon\ x \mapsto q^{1-b} x,\ \ y \mapsto q^{a-1} y$$

is a Nakayama automorphism. The composite map $\phi \circ \nu^{-1}$ is then a bimodule isomorphism between the right A^{e}-modules ${}_{\nu}A_1$ and $D(A)$, where the scalar action on ${}_{\nu}A_1$ is given by

$$u \cdot (w_1 \otimes w_2) = \nu(w_2) u w_1.$$

Consequently, we see that

$$\dim_k \mathrm{HH}^n(A) = \dim_k \mathrm{Tor}_n^{A^{\mathrm{e}}}({}_{\nu}A_1, A)$$

for all $n \geq 0$.

Now let $\alpha, \beta \in k$ be nonzero scalars, and let $A \xrightarrow{\psi} A$ be the automorphism defined by $x \mapsto \alpha x$ and $y \mapsto \beta y$. Tensoring the deleted projective bimodule resolution $\mathbb{P}_A$ with the right A^{e}-module ${}_{\psi}A_1$, we obtain an isomorphism

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & {}_{\psi}A_1 \otimes_{A^{\mathrm{e}}} P_{n+1} & \xrightarrow{1 \otimes d_{n+1}} & {}_{\psi}A_1 \otimes_{A^{\mathrm{e}}} P_n & \xrightarrow{1 \otimes d_n} & {}_{\psi}A_1 \otimes_{A^{\mathrm{e}}} P_{n-1} & \longrightarrow \cdots \\
& \downarrow \wr & & \downarrow \wr & & \downarrow \wr & \\
\cdots \longrightarrow & \bigoplus_{i=0}^{n+1} ({}_{\psi}A_1) e_i^{n+1} & \xrightarrow{\delta_{n+1}^{\psi}} & \bigoplus_{i=0}^{n} ({}_{\psi}A_1) e_i^{n} & \xrightarrow{\delta_n^{\psi}} & \bigoplus_{i=0}^{n-1} ({}_{\psi}A_1) e_i^{n-1} & \longrightarrow \cdots
\end{array}$$

of complexes, where $\{e_0^n, e_1^n, \ldots, e_n^n\}$ is the standard generating set of $n+1$ copies of ${}_{\psi}A_1$. The map δ_n^{ψ} is then given by

$$\delta_{2t}^{\psi} \colon y^u x^v e_i^{2t} \mapsto$$
$$\begin{cases} K_1^{\psi}(t,i,u,v) y^{u+b-1} x^v e_i^{2t-1} + K_2^{\psi}(t,i,u,v) y^u x^{v+a-1} e_{i-1}^{2t-1}, & \text{for } i \text{ even}, \\ [q^{(ai-a+2+2v)/2} - \beta] y^{u+1} x^v e_i^{2t-1} & \\ \qquad\qquad + [\alpha q^{(2bt-bi-b+2+2u)/2} - 1] y^u x^{v+1} e_{i-1}^{2t-1}, & \text{for } i \text{ odd}, \end{cases}$$

$$\delta_{2t+1}^{\psi} \colon y^u x^v e_i^{2t+1} \mapsto$$
$$\begin{cases} [\beta - q^{(ai+2v)/2}] y^{u+1} x^v e_i^{2t} + K_3^{\psi}(t,i,u,v) y^u x^{v+a-1} e_{i-1}^{2t}, & \text{for } i \text{ even}, \\ K_4^{\psi}(t,i,u,v) y^{u+b-1} x^v e_i^{2t} + [\alpha q^{(2bt-bi+b+2u)/2} - 1] y^u x^{v+1} e_{i-1}^{2t}, & \text{for } i \text{ odd}, \end{cases}$$

where we use the convention $e_{-1}^n = e_{n+1}^n = 0$. Here the elements $K_j^{\psi}(t,i,u,v)$, which are scalars whose values depend on the parameters ψ, t, i, u and v, are defined as follows:

$$K_1^{\psi}(t,i,u,v)=\sum_{j=0}^{b-1} q^{j(ai+2v)/2}\beta^{b-1-j},$$

$$K_2^{\psi}(t,i,u,v)=\sum_{j=0}^{a-1} q^{j(2bt-bi+2u)/2}\alpha^{j},$$

$$K_3^{\psi}(t,i,u,v)=\sum_{j=0}^{a-1} q^{j(2bt-bi+2+2u)/2}\alpha^{j},$$

$$K_4^{\psi}(t,i,u,v)=\sum_{j=0}^{b-1} q^{j(ai-a+2+2v)/2}\beta^{b-1-j}.$$

When q is not a root of unity, and the characteristic of k does not divide a or b, these scalars are all nonzero when the automorphism ψ is either the identity or the Nakayama automorphism. For, in this case, the elements are of the form

$$q^{s}(1+q^{m}+q^{2m}+\cdots+q^{rm})$$

for some $m,s\in\mathbb{Z}$ and $r=a-1$ or $r=b-1$. When $m=0$, this element is nonzero since the characteristic of k does not divide a or b, and, if it was zero for some $m\neq 0$, then q would be a root of unity because of the equality

$$(1+q^{m}+q^{2m}+\cdots+q^{rm})(1-q^{m})=1-q^{(r+1)m}.$$

In the following result we use this complex to compute the Hochschild homology of our algebra A.

Theorem 3.1. *When q is not a root of unity, the Hochschild homology of A is given by*

$$\dim_k \operatorname{HH}_n(A)=\begin{cases} a+b-1, & \text{when } n=0,\\ a+b, & \text{when } n\geq 1 \text{ and } \operatorname{char} k \text{ divides both } a \text{ and } b,\\ a+b-1, & \text{when } n\geq 1 \text{ and } \operatorname{char} k \text{ divides one of } a \text{ and } b,\\ a+b-2, & \text{when } n\geq 1 \text{ and } \operatorname{char} k \text{ does not divide } a \text{ or } b.\end{cases}$$

Proof. We need to compute the homology groups of the above complex in the case when ψ is the identity automorphism on A, that is, when $\alpha=1=\beta$. We do this by computing $\operatorname{Ker}\delta_{2t}^{1}$ for $t\geq 1$ and $\operatorname{Ker}\delta_{2t+1}^{1}$ for $t\geq 0$, and we treat these two cases separately.

Ker $\boldsymbol{\delta_{2t}^{1}}$. The image under the map δ_{2t}^{1} of a basis vector

$$y^{u}x^{v}e_i^{2t}\in\bigoplus_{i=0}^{2t} Ae_i^{2t}$$

is given by

$$\begin{cases} K_1^1(t,i,u,v)y^{u+b-1}x^v e_i^{2t-1} + K_2^1(t,i,u,v)y^u x^{v+a-1}e_{i-1}^{2t-1}, & \text{for } i \text{ even},\\ [q^{(ai-a+2+2v)/2}-1]y^{u+1}x^v e_i^{2t-1} & \\ \qquad\qquad +[q^{(2bt-bi-b+2+2u)/2}-1]y^u x^{v+1}e_{i-1}^{2t-1}, & \text{for } i \text{ odd}. \end{cases}$$

From the definition of the scalars K_1^1 and K_2^1, we see that

$$K_1^1(t,i,u,v)=0 \iff i=0,\ v=0,\ \operatorname{char} k|b,$$
$$K_2^1(t,i,u,v)=0 \iff i=2t,\ u=0,\ \operatorname{char} k|a,$$

and therefore we first compute the dimension of $\operatorname{Ker}\delta_{2t}^1$ under the assumption that the characteristic of k does not divide a or b.

First, we count the number of single basis vectors in $\bigoplus_{i=0}^{2t} Ae_i^{2t}$ belonging to $\operatorname{Ker}\delta_{2t}^1$. For even i, we have

$$\begin{aligned}\delta_{2t}^1(y^u x^v e_i^{2t})=0 \text{ for all even } i &\iff u+b-1\ge b \text{ and } v+a-1\ge a\\ &\iff 1\le u\le b-1 \text{ and } 1\le v\le a-1,\end{aligned}$$

from which we obtain $(b-1)(a-1)(t+1)$ vectors (there are $t+1$ even numbers in the set $\{0, 1, \dots, 2t\}$). For odd i, we have

$$\begin{aligned}\delta_{2t}^1(y^u x^v e_i^{2t})=0 \text{ for all odd } i &\iff u+1\ge b \text{ and } v+1\ge a\\ &\iff u=b-1 \text{ and } v=a-1,\end{aligned}$$

giving t vectors (there are t odd numbers in the set $\{0, 1, \dots, 2t\}$). Next, we count the other single basis vectors which are mapped to zero, starting with those for which i is even. The element e_{2t}^{2t-1} is zero by definition. Hence when $i=2t$ and $v+a-1\ge a$, that is, when $1\le v\le a-1$, we see that $y^u x^v e_i^{2t}$ maps to zero. But the vectors for which u is nonzero were counted above. Hence the new vectors are $x^v e_{2t}^{2t}$ for $1\le v\le a-1$. Similarly, the element e_{-1}^{2t-1} is zero by definition. Hence when $i=0$ and $u+b-1\ge b$, that is, when $1\le u\le b-1$, we see that $y^u x^v e_i^{2t}$ maps to zero. But here the vectors for which v is nonzero were counted above, and so the new vectors are $y^u e_0^{2t}$ for $1\le u\le b-1$. It is easy to see that except for these $a+b-2$ new vectors, there is no other single basis vector $y^u x^v e_i^{2t}$ in $\operatorname{Ker}\delta_{2t}^1$ for which i is even, since both $K_1^1(t,i,u,v)$ and $K_2^1(t,i,u,v)$ are always nonzero. Moreover, when i is odd, neither e_i^{2t-1} nor e_{i-1}^{2t-1} are zero, and the coefficients

$$[q^{(ai-a+2+2v)/2}-1], \quad \text{and} \quad [q^{(2bt-bi-b+2+2u)/2}-1]$$

are both nonzero. Hence in this case there are no new basis vectors mapped to zero.

Now we count the number of nontrivial linear combinations of two or more basis vectors in $\bigoplus_{i=0}^{2t} Ae_i^{2t}$ belonging to $\operatorname{Ker}\delta_{2t}^1$. Let i be even. If the first term of $\delta_{2t}^1(y^u x^v e_i^{2t})$ is nonzero, then the only way to "kill" it is to involve the second term of $\delta_{2t}^1(y^{u+b-1}x^{v-1}e_{i+1}^{2t})$. Thus to get a nontrivial linear combination, we see that u, v and i must satisfy $u=0$, $1\le v\le a-1$ and $i=0,2,\dots,2t-2$. For these parameter values, the second term of $\delta_{2t}^1(y^u x^v e_i^{2t})$ vanishes, as does the first term of $\delta_{2t}^1(y^{u+b-1}x^{v-1}e_{i+1}^{2t})$. Therefore, for a suitable nonzero scalar $C(a,b,i,u,v)$, the linear combination

$$x^v e_i^{2t} + C(a,b,i,u,v)y^{b-1}x^{v-1}e_{i+1}^{2t}$$

is mapped to zero for $1\le v\le a-1$ and $i=0,2,\dots,2t-2$, and there are $(a-1)t$ such elements. If the second term of $\delta_{2t}^1(y^u x^v e_i^{2t})$ is nonzero, then the only way to "kill" it is to involve the first term of $\delta_{2t}^1(y^{u-1}x^{v+a-1}e_{i-1}^{2t})$. To get a nontrivial linear combination, the parameters u, v and i must satisfy $1\le u\le b-1$, $v=0$ and $i=2,4,\dots,2t$, and for these values the first term of $\delta_{2t}^1(y^u x^v e_i^{2t})$ and the second term of $\delta_{2t}^1(y^{u-1}x^{v+a-1}e_{i-1}^{2t})$ vanish. Thus, for a suitable nonzero scalar $C'(a,b,i,u,v)$, the linear combination

$$y^u e_i^{2t} + C'(a,b,i,u,v)y^{u-1}x^{a-1}e_{i-1}^{2t}$$

is mapped to zero for $1\le u\le b-1$ and $i=2,4,\dots,2t$, and there are $(b-1)t$ such elements.

We have now accounted for all the elements of $\operatorname{Ker}\delta_{2t}^1$, when the characteristic of k does not divide a or b. If the characteristic of k divides a, then we must add to our list the element e_{2t}^{2t}. Similarly, if the characteristic of k divides b, then we must add to our list the element e_0^{2t}. Finally, if the characteristic of k divides both a and b, then we must add both these two elements to our list (and they are different elements since $t\ge 1$). Summing up, we see that the total dimension of $\operatorname{Ker}\delta_{2t}^1$ is given by

$$\dim_k \operatorname{Ker}\delta_{2t}^1 = \begin{cases} abt+ab-1, & \text{when } \operatorname{char} k \text{ does not divide } a \text{ or } b,\\ abt+ab+1, & \text{when } \operatorname{char} k \text{ divides both } a \text{ and } b,\\ abt+ab, & \text{otherwise.}\end{cases}$$

Ker $\boldsymbol{\delta_{2t+1}^1}$. The image under the map δ_{2t+1}^1 of a basis vector

$$y^u x^v e_i^{2t+1} \in \bigoplus_{i=0}^{2t+1} Ae_i^{2t+1}$$

is given by

$$\begin{cases} [1-q^{(ai+2v)/2}]y^{u+1}x^{v}e_i^{2t}+K_3^1(t,i,u,v)y^{u}x^{v+a-1}e_{i-1}^{2t}, & \text{for } i \text{ even} \\ K_4^1(t,i,u,v)y^{u+b-1}x^{v}e_i^{2t}+[q^{(2bt-bi+b+2u)/2}-1]y^{u}x^{v+1}e_{i-1}^{2t}, & \text{for } i \text{ odd.} \end{cases}$$

From the definition of the elements K_3^1 and K_4^1, we see that they are always nonzero, contrary to the case above where there were parameters for which K_1^1 and K_2^1 vanished. Therefore, the characteristic of k does not matter when we compute the dimension of $\operatorname{Ker}\delta_{2t+1}^1$.

We follow the same procedure as we did for $\operatorname{Ker}\delta_{2t}^1$. First we count the number of single basis vectors in $\bigoplus_{i=0}^{2t+1} Ae_i^{2t+1}$ belonging to $\operatorname{Ker}\delta_{2t+1}^1$. For even i, we have

$$\begin{aligned} \delta_{2t+1}^1(y^u x^v e_i^{2t+1})=0 \text{ for all even } i &\iff u+1\geq b \text{ and } v+a-1\geq a \\ &\iff u=b-1 \text{ and } 1\leq v\leq a-1, \end{aligned}$$

resulting in $(a-1)(t+1)$ vectors (there are $(t+1)$ even numbers in the set $\{0,1,\dots,2t+1\}$). When i is odd, we have

$$\begin{aligned} \delta_{2t+1}^1(y^u x^v e_i^{2t+1})=0 \text{ for all odd } i &\iff u+b-1\geq b \text{ and } v+1\geq a \\ &\iff 1\leq u\leq b-1 \text{ and } v=a-1, \end{aligned}$$

giving $(b-1)(t+1)$ vectors (there are $(t+1)$ odd numbers in the set $\{0,1,\dots,2t+1\}$). Next, we count the other single basis vectors in $\bigoplus_{i=0}^{2t+1} Ae_i^{2t+1}$ belonging to $\operatorname{Ker}\delta_{2t+1}^1$, starting with those for which i is even. The element e_{-1}^{2t} is zero; hence for $i=0$ the second term in $\delta_{2t+1}^1(y^u x^v e_i^{2t+1})$ vanishes. If now $v=0$, then the coefficient $[1-q^{(ai+2v)/2}]$ vanishes, and therefore the vector $y^u e_0^{2t+1}$ maps to zero for $0\leq u\leq b-1$. There are b such vectors, and none of them was counted above. Moreover, it is not hard to see that there is no other vector $y^u x^v e_i^{2t+1}$ in $\operatorname{Ker}\delta_{2t+1}^1$ for which i is even. As for the case when i is odd, the element e_{2t+1}^{2t} is zero by definition, and the coefficient $[q^{(2bt-bi+b+2u)/2}-1]$ vanishes for $i=2t+1$ and $u=0$. Therefore, the vector $x^v e_{2t+1}^{2t+1}$ maps to zero for $0\leq v\leq a-1$. These a vectors have not been counted before, and $\operatorname{Ker}\delta_{2t+1}^1$ does not contain more vectors $y^u x^v e_i^{2t+1}$ for which i is odd.

At last we count the number of nontrivial linear combinations of two or more basis vectors in $\bigoplus_{i=0}^{2t+1} Ae_i^{2t+1}$ belonging to $\operatorname{Ker}\delta_{2t+1}^1$. Let i be even, and suppose the first term of $\delta_{2t+1}^1(y^u x^v e_i^{2t+1})$ is nonzero. The only way to cancel this term is to involve the second term of $\delta_{2t+1}^1(y^{u+1}x^{v-1}e_{i+1}^{2t+1})$. Now, the first term in the latter vanishes, as does the second term of $\delta_{2t+1}^1(y^u x^v e_i^{2t+1})$, since v must be nonzero. Thus, for a suitable nonzero scalar $C''(a,b,i,u,v)$, the element

$$y^u x^v e_i^{2t+1}+C''(a,b,i,u,v)y^{u+1}x^{v-1}e_{i+1}^{2t+1}$$

belongs to $\operatorname{Ker}\delta^1_{2t+1}$, when the parameters satisfy $0\le u\le b-2$, $1\le v\le a-1$ and $i=0,2,\dots,2t$. There are $(a-1)(b-1)(t+1)$ such elements. Finally, suppose the second term of $\delta^1_{2t+1}(y^u x^v e_i^{2t+1})$ is nonzero. To cancel it, we must involve the first term in $\delta^1_{2t+1}(y^{u-b+1}x^{v+a-1}e_{i-1}^{2t+1})$, and so we see that the only possibility for u and v is $u=b-1$ and $v=0$. Therefore, for a suitable nonzero scalar $C'''(a,b,i,u,v)$, the element

$$y^{b-1}e_i^{2t+1}+C'''(a,b,i,u,v)x^{a-1}e_{i-1}^{2t+1}$$

is mapped to zero for $i=2,4,\dots,2t$. There are t such linear combinations.

All the elements of $\operatorname{Ker}\delta^1_{2t+1}$ are now accounted for, and so when summing up we obtain the dimension of this vector space:

$$\dim_k \operatorname{Ker}\delta^1_{2t+1}=abt+ab+a+b-1.$$

Using the identities

$$\dim_k \operatorname{Ker}\delta^1_n+\dim_k \operatorname{Im}\delta^1_n=\dim_k A^{n+1}=(n+1)ab,$$

we can now calculate the Hochschild homology of A. The dimension formula gives $\dim_k \operatorname{Im}\delta^1_{2t+1}=abt+ab-a-b+1$, in particular $\dim_k \operatorname{Im}\delta^1_1=2ab-a-b+1$, giving

$$\dim_k \mathrm{HH}_0(A)=\dim_k A-\dim_k \operatorname{Im}\delta^1_1=a+b-1.$$

Applying the formula to the results we obtained, when computing $\operatorname{Ker}\delta^1_{2t}$,

$$\dim_k \operatorname{Im}\delta^1_{2t+2}=\begin{cases} abt+ab+1, & \text{when } \operatorname{char}k \text{ does not divide } a \text{ or } b,\\ abt+ab-1, & \text{when } \operatorname{char}k \text{ divides both } a \text{ and } b,\\ abt+ab, & \text{otherwise,}\end{cases}$$

and so by calculating $\dim_k \mathrm{HH}_n(A)=\dim_k \operatorname{Ker}\delta^1_n-\dim_k \operatorname{Im}\delta^1_{n+1}$ for $n\ge 1$, we get

$$\dim_k \mathrm{HH}_n(A)=\begin{cases} a+b-2, & \text{when } \operatorname{char}k \text{ does not divide } a \text{ or } b,\\ a+b, & \text{when } \operatorname{char}k \text{ divides both } a \text{ and } b,\\ a+b-1, & \text{otherwise.}\end{cases}$$

This completes the proof. □

In particular, since a and b are both at least 2, the Hochschild homology of A does not vanish in high degrees (or in any degree). This was conjectured by Han [2006] to hold for all finite-dimensional algebras of infinite global dimension, and in the same paper it was proved that this conjecture holds for monomial algebras.

The converse of this conjecture always holds when the algebra modulo its radical is separable over the ground field. Namely, in this situation, if the global dimension of the algebra is finite, then the algebra has finite projective dimension as a bimodule, and hence its Hochschild homology vanishes in high degrees. The same holds of course for Hochschild cohomology, and following this easy observation, Happel [1989] remarked that “the converse seems to be not known”. Thus the cohomology version of Han’s conjecture came to be known as “Happel’s question”. However, this cohomology version is false in general; it was proved in [Buchweitz et al. 2005] that there do exist finite-dimensional algebras of infinite global dimension for which Hochschild cohomology vanishes in high degrees. The counterexample used in the paper was precisely our algebra A with $a = 2 = b$, and the following result shows that the same holds for arbitrary a and b. Contrary to the homology case, the dimensions of the cohomology groups do not depend on the characteristic of k.

Theorem 3.2. *When q is not a root of unity, the Hochschild cohomology of A is given by*

$$\dim_k \mathrm{HH}^n(A) = \begin{cases} 2, & \textit{for } n = 0, \\ 2, & \textit{for } n = 1, \\ 1, & \textit{for } n = 2, \\ 0, & \textit{for } n \geq 3. \end{cases}$$

In particular, the Hochschild cohomology of A vanishes in high degrees.

Proof. It is well known and easy to see that, in general, $\mathrm{HH}^0(A)$ is isomorphic to the center of A, that is, the subalgebra

$$\{w \in A \mid wz = zw \text{ for all } z \in A\}.$$

The center of our algebra A is the vector space spanned by the “first” and the “last” elements in its basis, namely the elements 1 and $y^{b-1}x^{a-1}$. Hence $\mathrm{HH}^0(A)$ is 2-dimensional.

To compute the Hochschild cohomology groups of positive degree, we compute the homology of the complex obtained prior to Theorem 3.1, in the case when ψ is the Nakayama automorphism ν. In this case, the scalars α and β are given by

$$\alpha = q^{1-b}, \quad \text{and} \quad \beta = q^{a-1}.$$

We apply the same method as we did when computing homology; we compute $\operatorname{Ker} \delta^{\nu}_{2t}$ for $t \geq 1$ and $\operatorname{Ker} \delta^{\nu}_{2t+1}$ for $t \geq 0$, treating the two cases separately.

Ker $\boldsymbol{\delta^{\nu}_{2t}}$. The result when applying the map δ^{ν}_{2t} to a basis vector

$$y^u x^v e_i^{2t} \in \bigoplus_{i=0}^{2t} ({}_{\nu}A_1) e_i^{2t}$$

is given by

$$\begin{cases} K_1^{\nu}(t,i,u,v)y^{u+b-1}x^{v}e_i^{2t-1}+K_2^{\nu}(t,i,u,v)y^{u}x^{v+a-1}e_{i-1}^{2t-1}, & \text{for } i \text{ even} \\ [q^{(ai-a+2+2v)/2}-q^{a-1}]y^{u+1}x^{v}e_i^{2t-1} & \\ \qquad\qquad +[q^{(2bt-bi-3b+4+2u)/2}-1]y^{u}x^{v+1}e_{i-1}^{2t-1}, & \text{for } i \text{ odd.} \end{cases}$$

From the definition of the elements K_1^{ν} and K_2^{ν}, we see that

$$K_1^{\nu}(t,i,u,v)=0 \iff i=0,\ v=a-1,\ \operatorname{char} k|b,$$
$$K_2^{\nu}(t,i,u,v)=0 \iff i=2t,\ u=b-1,\ \operatorname{char} k|a,$$

and so we first compute the dimension of $\operatorname{Ker}\delta_{2t}^{\nu}$ in the case when the characteristic of k does not divide a or b.

First, we count the number of single basis vectors in $\bigoplus_{i=0}^{2t}({}_{\nu}A_1)e_i^{2t}$ belonging to $\operatorname{Ker}\delta_{2t}^{\nu}$. As in the homology case, we have

$$\begin{aligned} \delta_{2t}^{\nu}(y^{u}x^{v}e_i^{2t})=0 \text{ for all even } i &\iff u+b-1\geq b \text{ and } v+a-1\geq a \\ &\iff 1\leq u\leq b-1 \text{ and } 1\leq v\leq a-1, \\ \delta_{2t}^{\nu}(y^{u}x^{v}e_i^{2t})=0 \text{ for all odd } i &\iff u+1\geq b \text{ and } v+1\geq a \\ &\iff u=b-1 \text{ and } v=a-1, \end{aligned}$$

from which we obtain $(b-1)(a-1)(t+1)+t$ vectors. Next, we count the other single basis vectors in $\bigoplus_{i=0}^{2t}({}_{\nu}A_1)e_i^{2t}$ belonging to $\operatorname{Ker}\delta_{2t}^{\nu}$. Since K_1^{ν} and K_2^{ν} are always nonzero, the number of such vectors for which i is even is the same as in the homology case, namely $a+b-2$. As for the vectors for which i is odd, it is no longer true that the coefficients are always nonzero. The coefficient $[q^{(ai-a+2+2v)/2}-q^{a-1}]$ vanishes when $i=1$ and $v=a-2$, whereas $[q^{(2bt-bi-3b+4+2u)/2}-1]$ vanishes when $i=2t-1$ and $u=b-2$. Both these cases will occur, since t is at least 1 when we compute $\operatorname{Ker}\delta_{2t}^{\nu}$. However, these coefficients need to vanish *simultaneously* for the basis vector to belong to $\operatorname{Ker}\delta_{2t}^{\nu}$, and this only happens when $t=1$, since then $2t-1=1$. Thus, when $t=1$ the vector $y^{b-2}x^{a-2}e_1^2$ maps to zero, whereas when $t\geq 2$ there are no new basis vectors in $\operatorname{Ker}\delta_{2t}^{\nu}$ for which i is odd.

Now we count the number of nontrivial linear combinations of two or more basis vectors in $\bigoplus_{i=0}^{2t}({}_{\nu}A_1)e_i^{2t}$ belonging to $\operatorname{Ker}\delta_{2t}^{\nu}$. These elements are precisely the same as in the homology case, and we do not encounter problems because of the "new" basis vector in $\operatorname{Ker}\delta_2^{\nu}$ we obtained above. Therefore, the number of such linear combinations is $(a-1)t+(b-1)t$.

We now look at what happens when the characteristic of k divides a or b. If $\operatorname{char} k$ divides a, then we must add the vector $x^{a-1}e_0^{2t}$ to the list of single basis vectors mapped to zero. However, this vector already appears in one of the nontrivial linear combinations; hence it does not contribute to the total dimension. Similarly, when $\operatorname{char} k$ divides b, then the new vector $y^{b-1}e_{2t}^{2t}$ belongs to the list of single basis vectors mapped to zero. But again this vector already appears in one of the nontrivial linear combinations, and it will therefore not contribute to the total dimension. This argument is still valid if $\operatorname{char} k$ divides both a and b. This shows that the dimension of $\operatorname{Ker} \delta_{2t}^{\nu}$ is independent of the characteristic of k.

In total, we see that the dimension of $\operatorname{Ker} \delta_{2t}^{\nu}$ is almost the same as it was in the homology case when the characteristic of k did not divide a or b; we need one additional vector when $t = 1$. Therefore, the dimension is given by

$$\dim_k \operatorname{Ker} \delta_{2t}^{\nu} = \begin{cases} 2ab, & \text{when } t = 1, \\ abt + ab - 1, & \text{when } t \geq 2. \end{cases}$$

Ker $\boldsymbol{\delta_{2t+1}^{\nu}}$. The image under the map δ_{2t+1}^{ν} of a basis vector

$$y^u x^v e_i^{2t+1} \in \bigoplus_{i=0}^{2t+1} ({}_{\nu}A_1) e_i^{2t+1}$$

is given by

$$\begin{cases} [q^{a-1} - q^{(ai+2v)/2}] y^{u+1} x^v e_i^{2t} + K_3^{\nu}(t,i,u,v) y^u x^{v+a-1} e_{i-1}^{2t}, & \text{for } i \text{ even}, \\ K_4^{\nu}(t,i,u,v) y^{u+b-1} x^v e_i^{2t} + [q^{(2bt-bi-b+2+2u)/2} - 1] y^u x^{v+1} e_{i-1}^{2t}, & \text{for } i \text{ odd}. \end{cases}$$

Now, from the definition of the scalars K_3^{ν} and K_4^{ν}, we see that K_3^{ν} is always nonzero, while we have

$$K_4^{\nu}(t,i,u,v) = 0 \iff i = 1,\ v = a-2,\ \operatorname{char} k \mid b.$$

Therefore, we first compute the dimension of $\operatorname{Ker} \delta_{2t+1}^{\nu}$ under the assumption that the characteristic of k does not divide b.

First, we count the number of single basis vectors in $\bigoplus_{i=0}^{2t+1} ({}_{\nu}A_1) e_i^{2t+1}$ belonging to $\operatorname{Ker} \delta_{2t+1}^{\nu}$. As in the homology case, we have

$$\begin{aligned} \delta_{2t+1}^{\nu}(y^u x^v e_i^{2t+1}) = 0 \text{ for all even } i &\iff u+1 \geq b \text{ and } v+a-1 \geq a \\ &\iff u = b-1 \text{ and } 1 \leq v \leq a-1, \\ \delta_{2t+1}^{\nu}(y^u x^v e_i^{2t+1}) = 0 \text{ for all odd } i &\iff u+b-1 \geq b \text{ and } v+1 \geq a \\ &\iff 1 \leq u \leq b-1 \text{ and } v = a-1, \end{aligned}$$

from which we obtain $(a-1)(t+1) + (b-1)(t+1)$ vectors. Next, we count the other single basis vectors in $\bigoplus_{i=0}^{2t+1} ({}_{\nu}A_1) e_i^{2t+1}$ belonging to $\operatorname{Ker} \delta_{2t+1}^{\nu}$, treating first the ones for which i is even. When $i = 0$, the second term of $\delta_{2t+1}^{\nu}(y^u x^v e_i^{2t+1})$ vanishes, and the first term then vanishes if $u = b-1$ or $v = a-1$. Some of these vectors are among the ones counted above, the new ones are $y^{b-1}e_0^{2t+1}$ and $y^u x^{a-1} e_0^{2t+1}$ for $0 \le u \le b-2$. Except for these b elements, there are no other single basis elements in $\operatorname{Ker}\delta_{2t+1}^{\nu}$ for which i is even. As for those for which i is odd, we see that the first term of $\delta_{2t+1}^{\nu}(y^u x^v e_i^{2t+1})$ vanishes when $i = 2t+1$. In this case, the second term vanishes if $u = b-1$ or $v = a-1$, and of these vectors the ones which have not been counted before are the a elements $x^{a-1}e_{2t+1}^{2t+1}$ and $y^{b-1}x^v e_{2t+1}^{2t+1}$ for $0 \le v \le a-2$. It is not hard to see that $\operatorname{Ker}\delta_{2t+1}^{\nu}$ does not contain any other element $y^u x^v e_i^{2t+1}$ for which i is odd.

Finally, we count the number of nontrivial linear combinations of two or more basis elements in $\bigoplus_{i=0}^{2t+1}({}_{\nu}A_1)e_i^{2t+1}$ belonging to $\operatorname{Ker}\delta_{2t+1}^{\nu}$. In the homology case, these were

$$y^u x^v e_i^{2t+1} + C''(a,b,i,u,v) y^{u+1} x^{v-1} e_{i+1}^{2t+1}$$

for $0 \le u \le b-2$, $1 \le v \le a-1$ and $i = 0, 2, \ldots, 2t$, and

$$y^{b-1} e_i^{2t+1} + C'''(a,b,i,u,v) x^{a-1} e_{i-1}^{2t+1}$$

for $i = 2, 4, \ldots, 2t$, where C'' and C''' are suitable scalars. The t latter elements also belong to $\operatorname{Ker}\delta_{2t+1}^{\nu}$, but among the $(a-1)(b-1)(t+1)$ first elements there are some combinations that are not mapped to zero. Namely, we must discard the $b-1$ elements for which $i = 0$ and $v = a-1$, since we showed above that $y^u x^{a-1} e_0^{2t+1}$ maps to zero for $0 \le u \le b-2$. Similarly, we must discard the $a-1$ combinations for which $i = 2t$ and $u = b-2$, since $y^{b-1} x^v e_{2t+1}^{2t+1}$ maps to zero for $0 \le v \le a-2$. However, when $t = 0$, then the situations $i = 0$ and $i = 2t$ are the same, and the element

$$y^{b-2} x^{a-1} e_0^1 + C''(a,b,i,u,v) y^{b-1} x^{a-2} e_1^1$$

has been discarded twice. Thus the total number of nontrivial linear combinations is $(a-1)(b-1)(t+1) + t - (a-1) - (b-1)$ when $t \ge 1$, and one more when $t = 0$.

What happens when $\operatorname{char} k$ divides b? The element $y^u x^{a-2} e_1^{2t+1}$ is not mapped to zero for any u, and it does not "interfere" with one of the nontrivial linear combinations. Hence the dimension of $\operatorname{Ker}\delta_{2t+1}^{\nu}$ is also independent of the characteristic of k.

In total, we see that the dimension of $\operatorname{Ker}\delta_{2t+1}^{\nu}$ differs from that in the homology case, since we need to subtract $(a-1)+(b-1)$ when $t \ge 1$ and $(a-1)+(b-1)-1$ when $t = 0$. Thus, the dimension is given by

$$\dim_k \operatorname{Ker}\delta_{2t+1}^{\nu} = \begin{cases} ab+2, & \text{when } t = 0, \\ abt+ab+1, & \text{when } t \ge 1. \end{cases}$$

We can now calculate the positive degree cohomology groups. We have

$$\dim_k \operatorname{Ker}\delta_1^{\nu} = ab+2,$$

and, since $\dim_k \operatorname{Ker} \delta_2^\nu = 2ab$, we must have $\dim_k \operatorname{Im} \delta_2^\nu = ab$, giving

$$\dim_k \mathrm{HH}^1(A) = \dim_k \operatorname{Ker} \delta_1^\nu - \dim_k \operatorname{Im} \delta_2^\nu = 2.$$

Furthermore, since $\dim_k \operatorname{Ker} \delta_3^\nu = 2ab + 1$, we must have $\dim_k \operatorname{Im} \delta_3^\nu = 2ab - 1$, giving

$$\dim_k \mathrm{HH}^2(A) = \dim_k \operatorname{Ker} \delta_2^\nu - \dim_k \operatorname{Im} \delta_3^\nu = 1.$$

Similarly, direct computations show that the cohomology groups $\mathrm{HH}^n(A)$ vanish when $n \geq 3$, thereby completing the proof. □

When the commutator element q is a root of unity, it is not hard to see that the dimensions of infinitely many of the kernels in the complex we used to compute (co)homology will increase. Therefore, the Hochschild homology of A is still nonzero in all degrees, while it is no longer true that all the higher Hochschild cohomology groups vanish. We record this fact in the final result, which also gives the multiplicative structure of the Hochschild cohomology ring when q is not a root of unity.

Theorem 3.3. *The Hochschild cohomology ring* $\mathrm{HH}^*(A)$ *is finite-dimensional if and only if q is not a root of unity. When this is the case, the algebra is isomorphic to the (five-dimensional graded) fibre product*

$$k[U]/(U^2) \times_k k\langle V, W\rangle/(V^2, VW + WV, W^2),$$

where U is in degree zero and V and W are in degree one.

Proof. Suppose q is not a root of unity. Recall first the initial part

$$P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\mu} A \to 0$$

of the projective bimodule resolution of A, where μ is the multiplication map. The maps d_1 and d_2 are defined on generators as follows:

$$\begin{aligned}
d_1\colon\ & f_0^1 \mapsto [(1 \otimes y) - (y \otimes 1)]\, f_0^0, \\
& f_1^1 \mapsto [(1 \otimes x) - (x \otimes 1)]\, f_0^0, \\
d_2\colon\ & f_0^2 \mapsto \big[(1 \otimes y^{b-1}) + (y \otimes y^{b-2}) + \cdots + (y^{b-1} \otimes 1)\big]\, f_0^1, \\
& f_1^2 \mapsto [q(1 \otimes x) - (x \otimes 1)]\, f_0^1 + [(1 \otimes y) - q(y \otimes 1)]\, f_1^1, \\
& f_2^2 \mapsto \big[(1 \otimes x^{a-1}) + (x \otimes x^{a-2}) + \cdots + (x^{a-1} \otimes 1)\big]\, f_1^1.
\end{aligned}$$

Define two bimodule maps

$$g\colon P_1 \to A, \begin{cases} f_0^1 \mapsto y, \\ f_1^1 \mapsto 0, \end{cases}$$

$$h\colon P_1 \to A, \begin{cases} f_0^1 \mapsto 0, \\ f_1^1 \mapsto x. \end{cases}$$

One checks directly that

$$g \circ d_2 = 0 = h \circ d_2,$$

and that neither of the two maps is liftable through d_1. Consequently they represent the two basis elements of $\mathrm{HH}^1(A) = \mathrm{Ext}^1_{A^{\mathrm{e}}}(A, A)$.

We may identify the degree zero part of $\mathrm{HH}^*(A)$ with the center of A, the two-dimensional vector space spanned by the elements 1 and $y^{b-1}x^{a-1}$. The latter element annihilates both g and h; hence $\mathrm{HH}^*(A)$ is isomorphic to the k-fibre product of the algebra generated by $y^{b-1}x^{a-1}$ with the algebra generated by g and h. Since the Hochschild cohomology ring of a finite-dimensional algebra is always graded commutative (see [Snashall and Solberg 2004, Corollary 1.2]), both g and h square to zero. Therefore, as $\mathrm{HH}^2(A)$ is one-dimensional, we are done if we can show that the product $hg \in \mathrm{HH}^2(A)$ is nonzero.

Define a bimodule map $g_0\colon P_1 \to P_0$ by

$$g_0\colon\ f_0^1 \mapsto (y \otimes 1) f_0^0,\ \ f_1^1 \mapsto 0.$$

It is not hard to see that there exists an element $w \in A^{\mathrm{e}}$ such that the map $g_1\colon P_2 \to P_1$, defined by

$$g_1\colon\ f_0^2 \mapsto w f_0^1,\ \ f_1^2 \mapsto q(y \otimes 1) f_1^1,\ \ f_2^2 \mapsto 0$$

gives a commutative diagram

$$\begin{tikzcd} P_2 \arrow[r, "d_2"] \arrow[d, "g_1"] & P_1 \arrow[d, "g_0"] \arrow[dr, "g"] & \\ P_1 \arrow[r, "d_1"] & P_0 \arrow[r, "\mu"] & A. \end{tikzcd}$$

The product $hg \in \mathrm{HH}^2(A)$ is then represented by the composite map $h \circ g_1$, under which the images of the generators in P_2 are given by

$$h \circ g_1\colon\ f_0^2 \mapsto 0,\ \ f_1^2 \mapsto qyx,\ \ f_2^2 \mapsto 0.$$

This map is not liftable through d_2, and therefore it represents a nonzero element of $\mathrm{HH}^2(A)$. Consequently, the product hg is nonzero. □

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