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David A. Cox and Evgeny Materov

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We give an explicit description of the terms and differentials of the Tate resolution of sheaves arising from Segre embeddings of $\mathbb{P}^a \times \mathbb{P}^b$. We prove that the maps in this Tate resolution are either coming from Sylvester-type maps, or from Bezout-type maps arising from the so-called toric Jacobian.

1. Introduction

Let V and W be dual vector spaces of dimension $N+1$ over a field K of characteristic 0. It is known that there is a relation between complexes of free graded modules over the exterior algebra $E = \bigwedge V$ and coherent sheaves on projective space $\mathbb{P}(W)$. More precisely, the Bernstein–Gel'fand–Gel'fand (BGG) correspondence [1978] establishes an equivalence between the derived category of bounded complexes of coherent sheaves on $\mathbb{P}(W)$ and the stable category of complexes of finitely generated graded modules over E. The essential part of this correspondence is given via the *Tate resolutions*, namely for any coherent sheaf $\mathcal{F}$ on $\mathbb{P}(W)$ there exists a bi-infinite exact sequence

$$T^\bullet(\mathcal{F}) : \cdots \to T^{-1}(\mathcal{F}) \to T^0(\mathcal{F}) \to T^1(\mathcal{F}) \to \cdots$$

of free graded E-modules. The terms of Tate resolution were described explicitly by Eisenbud, Fløystad and Schreyer [2003a] in the form

$$T^p(\mathcal{F}) = \bigoplus_i \widehat{E}(i-p) \otimes H^i\big(\mathbb{P}(W), \mathcal{F}(p-i)\big),$$

where

$$\widehat{E} = \omega_E = \mathrm{Hom}_K(E, K) = \bigwedge W$$

as an E-module.

While the terms of Tate resolutions are described explicitly, the maps are much more difficult to describe. The knowledge of the maps give us, for example, an

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opportunity to compute generalized resultants (see, for example, [Eisenbud et al. 2003b] or [Khetan 2003; 2005]).

Cox [2007] found an explicit construction of the Tate resolution for the d-fold Veronese embedding

$$\nu_d : \mathbb{P}^n \to \mathbb{P}^{\binom{n+d}{d}-1}$$

of $\mathbb{P}^n$ when $\mathcal{F} = \nu_{d*}\mathcal{O}_{\mathbb{P}^n}(k)$. The construction of differentials in Tate resolution involves the Bezoutian of $n+1$ homogeneous polynomials of degree d in $n+1$ variables. In this paper, we find a similar description of the Tate resolution arising from the Segre embedding

$$\nu : \mathbb{P}^a \times \mathbb{P}^b \to \mathbb{P}^{ab+a+b}$$

of the sheaf $\nu_*\mathcal{O}_{\mathbb{P}^a\times\mathbb{P}^b}(k,l)$. The shape of the Tate resolution depends only on the pair (k,l) and there are three types of possible resolutions:

$$\begin{aligned} &\text{Type 1:} \quad -a \le k-l \le b,\\ &\text{Type 2:} \quad k-l > b,\\ &\text{Type 3:} \quad k-l < -a. \end{aligned}$$

We prove that Type 1 maps involve the toric Jacobian of a sequence bilinear forms $f_0,\dots,f_{a+b}$ in $x_0,\dots,x_a$, $y_0,\dots,y_b$ given by

$$f_j = \sum_{i,k} a_{ijk}\, x_i\, y_k, \quad 0 \le j \le a+b.$$

This result resembles the Bezout-type formulas for hyperdeterminants of a three-dimensional matrix $A = (a_{ijk})$ discussed in [Gel′fand et al. 1994, Chapter 14, Theorem 3.19]. The resolutions of Type 2 and 3 are similar to each other and both arise from the Sylvester forms of $f_0,\dots,f_{a+b}$. Notice that similar formulas appear in the study of Bondal type formulas for hyperdeterminants of A (see [Gel′fand et al. 1994, Chapter 14, Theorem 3.18]).

The situations considered in this paper and [Cox 2007] are special cases when $\mathcal{F}$ is a push-forward of $\mathcal{L} = \mathcal{O}(m_1,\dots,m_r)$ in the projective embedding

$$\nu : \mathbb{P}^{l_1} \times \cdots \times \mathbb{P}^{l_r} \to \mathbb{P}(S^{d_1}K^{l_1+1} \otimes \cdots \otimes S^{d_r}K^{l_r+1})$$

which is a combination of Segre and Veronese embeddings. This case will be studied in a different paper [Cox and Materov $\ge$ 2008]. We conjecture that the maps in the Tate resolutions are essentially the same as in Weyman–Zelevinsky complexes [1994] or the same as in the resultant spectral sequences from [Gel′fand et al. 1994, Chapter 4, Section 3].

Here is the outline of our paper. In Section 2 we give a definition of the Tate resolution and explain its basic properties. We then enter the main part of the paper: in Section 3 we describe the terms of Tate resolution arising from Segre embeddings of sheaves on products of projective spaces, and in Section 4 we find explicit forms for corresponding differentials.

2. Basic definitions and properties of Tate resolutions

2.1. ***Graded exterior algebras.*** Given V and W as above, the algebras $S = \operatorname{Sym} W$ and $E = \bigwedge V$ are graded by the following convention: $\deg(e_i) = 1$ for a basis $e_0, e_1, \dots, e_N$ of W and $\deg(e_i^*) = -1$ for the dual basis $e_0^*, e_1^*, \dots, e_N^*$ of V, so that $E_{-i} = \bigwedge^i V$. Define $E(p)$ as the graded E-module with $E(p)_q = E_{p+q}$. Then any free graded E-module is an E-module of the form

$$M = \bigoplus_i E(-i) \otimes V_i,$$

where V_i is a finite-dimensional K-vector space with $V_i = \{0\}$ for almost all i. Note that V_i gives the degree i generators of M, because

$$(E(-i) \otimes V_i)_i = E(-i)_i \otimes V_i = E_0 \otimes V_i = V_i.$$

The dual to E algebra $\widehat{E} = \omega_E = \operatorname{Hom}_K(E, K)$ is a left E-module with the graded components

$$\widehat{E}_i = \operatorname{Hom}_K(E_{-i}, K) = \operatorname{Hom}_K\big(\bigwedge^i V, K\big).$$

The perfect pairing

$$\bigwedge^i V \times \bigwedge^i W \to K$$

implies $\widehat{E}_i = \bigwedge^i W$ and $\widehat{E} = \bigwedge W$. Moreover, $\widehat{E}$ is Gorenstein, that is, $\widehat{E}$ is isomorphic to E with a shift in grading. Namely, the isomorphism

$$\bigwedge^i V \otimes \bigwedge^{N+1} W \to \bigwedge^{N+1-i} W$$

implies

$$\widehat{E} = E(-N-1) \otimes \bigwedge^{N+1} W,$$

and therefore $\widehat{E} \cong E(-N-1)$ (noncanonically) via a map $\bigwedge^{N+1} W \cong K$. For later purposes, we note the canonical isomorphism

$$\operatorname{Hom}_E\big(\widehat{E}(p) \otimes A, \widehat{E}(q) \otimes B\big)_0 \simeq \operatorname{Hom}_K\big(\bigwedge^{p-q} W \otimes A, B\big), \tag{2-1}$$

where the subscript 0 denotes graded homomorphisms of degree zero.

2.2. *Tate resolutions.* By [Eisenbud et al. 2003a] or [Fløystad 2000] a coherent sheaf $\mathcal{F}$ on $\mathbb{P}(W)$ determines a *Tate resolution* $T^\bullet(\mathcal{F})$, which is an (unbounded) acyclic complex

$$T^\bullet(\mathcal{F}) : \cdots \to T^{-1}(\mathcal{F}) \to T^0(\mathcal{F}) \to T^1(\mathcal{F}) \to \cdots$$

of free graded E-modules with the terms

$$T^p(\mathcal{F}) = \bigoplus_i \widehat{E}(i-p) \otimes H^i\big(\mathbb{P}(W), \mathcal{F}(p-i)\big).$$

For example, in degree k we have

$$T^p(\mathcal{F})_k = \bigoplus_i \bigwedge^{i-p+k} W \otimes H^i\big(\mathbb{P}(W), \mathcal{F}(p-i)\big)$$

since $\widehat{E}(i-p)_k = \widehat{E}_{i-p+k} = \bigwedge^{i-p+k} W$. The Tate resolution is defined by each differential $d^p : T^p(\mathcal{F}) \to T^{p+1}(\mathcal{F})$ since $T^{\geq p}(\mathcal{F})$ is a minimal injective resolution of $\ker(d^p)$ and $T^{<p}(\mathcal{F})$ is a minimal projective resolution of $\ker(d^p)$ [Eisenbud 2005].

When the context is clear, we will write $H^i(\mathcal{F}(j))$ instead of $H^i\big(\mathbb{P}(W), \mathcal{F}(j)\big)$.

Lemma 2.1. *For fixed* k, $T^p(\mathcal{F})_k = 0$ *if either* $p > k+m$ *or* $p < k-N-1$, *where* $m = \dim(\operatorname{supp}(\mathcal{F}))$.

Proof. Since $H^i(\mathcal{F}(p-i)) = 0$ if $i < 0$ or $i > m$, we may assume $0 \leq i \leq m$. Then the inequalities $k+m < p$, $i \leq m$ easily imply

$$i - p + k \leq m - p + k < -p + p = 0,$$

so that $\bigwedge^{i-p+k} W = 0$. Analogously, if $k - N - 1 > p$, $i \geq 0$, then

$$i - p + k \geq -p + k > -p + p + N + 1 = N + 1,$$

so that we have again $\bigwedge^{i-p+k} W = 0$. □

Lemma 2.2. *If* $i < j$, *then the map*

$$d_{i,j}^p : \widehat{E}(i-p) \otimes H^i(\mathcal{F}(p-i)) \to \widehat{E}(j-p-1) \otimes H^j(\mathcal{F}(p+1-j))$$

from the i*-th summand of* $T^p(\mathcal{F})$ *to the* j*-th summand of* $T^{p+1}(\mathcal{F})$ *is zero.*

Proof. Let

$$A = H^i(\mathcal{F}(p-i)) \quad \text{and} \quad B = H^j(\mathcal{F}(p+1-j)).$$

By (2-1), $d_{i,j}^p$ lies in

$$\operatorname{Hom}_E\big(\widehat{E}(i-p) \otimes A, \widehat{E}(j-p-1) \otimes B\big)_0 \simeq \operatorname{Hom}_K\big(\bigwedge^{i-j+1} W \otimes A, B\big).$$

It follows that $d^p_{i,j} = 0$ when $i+1 < j$ and that $d^p_{i,i+1}$ is constant. Then minimality implies that $d^p_{i,i+1} = 0$. □

Finding an explicit expression for differentials $d^p : T^p(\mathscr{F}) \to T^{p+1}(\mathscr{F})$ seems to be a difficult problem. By Lemma 2.2, the general maps from the i-th summand of $T^p(\mathscr{F})$ in the Tate resolution $T^\bullet(\mathscr{F})$ have the form

$$\widehat{E}(i-p) \otimes H^i(\mathscr{F}(p-i)) \to \bigoplus_{j\geq 0} \widehat{E}(i-j-p-1) \otimes H^{i-j}(\mathscr{F}(p+1-i+j)).$$

The "horizontal" cmponent of this map is explicitly known:

$$\widehat{E}(i-p) \otimes H^i(\mathscr{F}(p-i)) \to \widehat{E}(i-p-1) \otimes H^i(\mathscr{F}(p+1-i)),$$
$$f \otimes m \mapsto \sum_i f e_i^* \otimes e_i m.$$

By (2-1), this corresponds to the multiplication map

$$W \otimes H^i(\mathscr{F}(p-i)) \to H^i(\mathscr{F}(p+1-i)).$$

One of the main results of this paper is an explicit description the entire differential $d^p : T^p(\mathscr{F}) \to T^{p+1}(\mathscr{F})$ in some special situations.

3. Tate resolutions for Segre embeddings of $\mathbb{P}^a \times \mathbb{P}^b$

Let $X = \mathbb{P}^a \times \mathbb{P}^b$, with coordinate ring $S = K[\mathbf{x}, \mathbf{y}]$ for variables $\mathbf{x} = (x_0, \ldots, x_a)$, $\mathbf{y} = (y_0, \ldots, y_b)$. The ring S has a natural bigrading where the $\mathbf{x}$ variables have degree $(1, 0)$ and the $\mathbf{y}$ variables have degree $(0, 1)$. The graded piece of S in degree s, t will be denoted $S_{s,t}$. Set

$$W = H^0(X, \mathscr{O}_X(1, 1)) = S_{1,1}$$

and let

$$\nu : X = \mathbb{P}^a \times \mathbb{P}^b \to \mathbb{P}(W) \simeq \mathbb{P}^{ab+a+b}$$

be the Segre embedding. The sheaf

$$\mathscr{F} = \nu_* \mathscr{O}_X(k, l) \tag{3-1}$$

has Tate resolution $T^\bullet(\mathscr{F})$ with

$$\begin{aligned} T^p(\mathscr{F}) &= \bigoplus_i \widehat{E}(i-p) \otimes H^i(\mathscr{F}(p-i)) \\ &= \bigoplus_i \widehat{E}(i-p) \otimes H^i\big(X, \mathscr{O}_X(k+p-i, l+p-i)\big). \end{aligned}$$

In general, we say that the summand $\widehat{E}(i-p)\otimes H^i(\mathscr{F}(p-i))$ of $T^p(\mathscr{F})$ has *cohomological level* i. Since

$$H^i\big(X,\mathscr{O}_X(k+p-i,l+p-i)\big)=0$$

for $i\notin\{0,a,b,a+b\}$, we see that $T^p(\mathscr{F})$ has at most four nonzero cohomological levels.

In Section 2.2, we observed that the "horizontal" components of the differentials $d^p:T^p(\mathscr{F})\to T^{p+1}(\mathscr{F})$ are explicitly known. The main result of this paper is a description of the "diagonal" components of these maps.

3.1. ***Regularity.*** We recall that a coherent sheaf $\mathscr{F}$ is called *m-regular* if

$$H^i(\mathscr{F}(m-i))=0,\quad \text{for all } i>0.$$

If $\mathscr{F}$ is m-regular, then it is known that it is also $(m+1)$-regular. The *regularity* of $\mathscr{F}$, denoted $\operatorname{reg}(\mathscr{F})$, is the unique integer m such that $\mathscr{F}$ is m-regular, but not $(m-1)$-regular. It follows from the definition of regularity if $m=\operatorname{reg}(\mathscr{F})$, then

$$T^p(\mathscr{F})=\widehat{E}(-p)\otimes H^0(\mathscr{F}(p)),\quad \text{for } p\geq m,$$

and the Tate resolution has the form

$$\cdots\to T^{m-2}(\mathscr{F})\to T^{m-1}(\mathscr{F})\to \widehat{E}(-m)\otimes H^0(\mathscr{F}(m))\to\cdots.$$

We now compute the regularity of the sheaf $\mathscr{F}$ defined in (3-1).

Lemma 3.1. $\operatorname{reg}(\mathscr{F})=\max\big\{-\min\{k,l\},\min\{b-k,a-l\}\big\}.$

Proof. Let m_0 denote the right-hand side of the above equation and let $m\geq m_0$. Then Serre duality implies

$$\begin{aligned}H^{a+b}\big(\mathscr{F}(m-(a+b))\big)&=H^{a+b}\big(X,\mathscr{O}_X\big(k+m-(a+b),l+m-(a+b)\big)\big)\\&\simeq H^0\big(X,\mathscr{O}_X\big(b-k-(m+1),a-l-(m+1)\big)\big)^*.\end{aligned}$$

Since $m\geq m_0$ implies $m\geq b-k$ or $m\geq a-l$, we see that $H^{a+b}\big(\mathscr{F}(m-(a+b))\big)=0$.

Next we use the Künneth formula to write

$$\begin{aligned}H^a(\mathscr{F}(m-a))&=H^a\big(X,\mathscr{O}_X(k+m-a,l+m-a)\big)\\&=H^0(\mathbb{P}^a,\mathscr{O}(k+m-a))\otimes H^a(\mathbb{P}^b,\mathscr{O}(l+m-a))\oplus\\&\qquad H^a(\mathbb{P}^a,\mathscr{O}(k+m-a))\otimes H^0(\mathbb{P}^b,\mathscr{O}(l+m-a)).\end{aligned}$$

Since $m\geq m_0$ implies $m\geq -k$ and $m\geq -l$, we see that $k+m-a\geq -a$, which implies $H^a(\mathbb{P}^a,\mathscr{O}(k+m-a))=0$. Furthermore, $H^a(\mathbb{P}^b,\mathscr{O}(l+m-a))=0$ when $a\neq b$, and when $a=b$, we have $l+m-a=l+m-b\geq -b$, which again implies $H^a(\mathbb{P}^b,\mathscr{O}(l+m-a))=0$. Hence $H^a(\mathscr{F}(m-a))=0$, and $H^b(\mathscr{F}(m-b))=0$ is proved similarly.

It follows that $m_0 \geq \operatorname{reg}(\mathcal{F})$. To prove equality, we will let $m = m_0 - 1$ and show that $H^i(\mathcal{F}(m-i)) \neq 0$ for some $i > 0$. We consider two cases.

Case 1: $m_0 = \min\{b-k, a-l\} \geq -\min\{k,l\}$. This implies the inequalities $b-k-(m+1) \geq 0$ and $a-l-(m+1) \geq 0$. Hence

$$H^{a+b}\big(\mathcal{F}(m-(a+b))\big) \simeq H^0\big(X, \mathcal{O}_X\big(b-k-(m+1), a-l-(m+1)\big)\big)^* \neq 0.$$

Case 2: $m_0 = -\min\{k,l\} > \min\{b-k, a-l\}$. If $m_0 = -k$, then $k+m-a = -a-1$, so that $H^a(\mathbb{P}^a, \mathcal{O}(k+m-a)) \neq 0$. We also have $m_0 > \min\{b-k, a-l\}$, so that $m_0 > b-k$ or $m_0 > a-l$. The former is impossible since $m_0 = -k$, and then the latter implies $l+m-a \geq 0$, so that $H^0(\mathbb{P}^b, \mathcal{O}(l+m-a)) \neq 0$. By Künneth,

$$0 \neq H^a(\mathbb{P}^a, \mathcal{O}(k+m-a)) \otimes H^0(\mathbb{P}^b, \mathcal{O}(l+m-a)) \subseteq H^a(\mathcal{F}(m-a)).$$

The proof when $m_0 = -l$ is similar. □

To see what this says about the Tate resolution of $\mathcal{F}$, we define

$$\begin{aligned} p^+ &= \max\big\{-\min\{k,l\}, \min\{b-k, a-l\}\big\}, \\ p^- &= \min\big\{-\min\{k,l\}, \min\{b-k, a-l\}\big\} - 1. \end{aligned} \tag{3-2}$$

Then we have the following result.

Lemma 3.2.

$$T^p(\mathcal{F}) = \begin{cases} \widehat{E}(-p) \otimes S_{k+p,l+p} & p \geq p^+ \\ \widehat{E}(a+b-p) \otimes S^*_{b-k-1-p,a-l-1-p} & p \leq p^-. \end{cases}$$

Proof. The assertion for $p \geq p^+$ follows immediately from Lemma 3.1 and the discussion preceding the lemma. For $p \leq p^-$, note that

$$\begin{aligned} H^{a+b}\big(\mathcal{F}(p-(a+b))\big) &\simeq H^0\big(X, \mathcal{O}_X(b-k-(p+1), a-l-(p+1))\big)^* \\ &= S^*_{b-k-1-p,a-k-1-p} \end{aligned}$$

and that

$$\begin{aligned} H^{a+b-i}\big(\mathcal{F}(p-(a+b-i))\big) &\simeq H^i\big(X, \mathcal{O}_X(b-k-1-p-i, a-l-1-p-i)\big)^* \\ &= H^i(\mathcal{G}(-p-i)), \end{aligned}$$

where $\mathcal{G} = \nu_*\mathcal{O}_X(b-k-1, a-l-1)$. Applying Lemma 3.1 to $\mathcal{G}$, we see that $H^i(\mathcal{G}(-p-i)) = 0$ whenever $i > 0$ and

$$-p \geq \max\big\{-\min\{b-k-1, a-l-1\}, \min\{b-(b-k-1), a-(a-l-1)\}\big\},$$

which is equivalent to $p \leq p^-$. □

Lemma 3.2 tells us that for p^- and below, the Tate resolution lives at cohomological level $a+b$, and for p^+ and above, it lives at cohomological level 0.

3.2. ***The shape of the resolution.*** For $k, l \in \mathbb{Z}$, the Tate resolution of

$$\mathcal{F} = \nu_* \mathcal{O}_X(k, l)$$

on $X = \mathbb{P}^a \times \mathbb{P}^b$ has one of the following three types:

$$\begin{aligned} &\text{Type 1:} \quad -a \le k - l \le b, \\ &\text{Type 2:} \quad k - l > b, \\ &\text{Type 3:} \quad k - l < -a. \end{aligned}$$

We will prove three lemmas, one for each type.

Lemma 3.3 (Type 1). *Assume that $\mathcal{F}$ has Type* 1. *Then $p^- = -\min\{k, l\} - 1$ and $p^+ = \min\{b - k, a - l\}$. Furthermore, if $p^- < p < p^+$, then*

$$T^p(\mathcal{F}) = \begin{array}{c} \widehat{E}(a + b - p) \otimes S^*_{b-k-1-p, a-l-1-p} \\ \bigoplus \\ \widehat{E}(-p) \otimes S_{k+p, l+p}. \end{array}$$

Proof. Since a and b are positive, the inequality $-a \le k - l \le b$ implies that $-\min\{k, l\} \le \min\{b - k, a - l\}$. Using (3-2), we get the desired formulas for p^- and p^+.

Now assume that $p^- < p < p^+$. Recall that $H^a(\mathcal{F}(p - a))$ is isomorphic to

$$H^0(\mathbb{P}^a, \mathcal{O}(k + p - a)) \otimes H^a(\mathbb{P}^b, \mathcal{O}(l + p - a)) \\ \oplus H^a(\mathbb{P}^a, \mathcal{O}(k + p - a)) \otimes H^0(\mathbb{P}^b, \mathcal{O}(l + p - a)).$$

If the second summand is nonzero, then $k + p - a < -a$ and $l + p - a \ge 0$, which implies $k - l < -a$, a contradiction. If the first summand is nonzero, then $a = b$, $k + p - a \ge 0$ and $l + p - a < -a$. These imply $k - l > a = b$, again a contradiction. Hence $H^a(\mathcal{F}(p - a)) = 0$. A similar argument shows that $H^b(\mathcal{F}(p - b)) = 0$. $\square$

Thus, when $\mathcal{F}$ has Type 1, the differential $d^p : T^p(\mathcal{F}) \to T^{p+1}(\mathcal{F})$ looks like

$$\begin{tikzcd} \widehat{E}(a + b - p) \otimes S^*_{b-k-1-p, a-l-1-p} \arrow[r] \arrow[dr, "d^p_{a+b,0}"] & \widehat{E}(a + b - p - 1) \otimes S^*_{b-k-p-2, a-l-p-2} \\ \bigoplus & \bigoplus \\ \widehat{E}(-p) \otimes S_{k+p, l+p} \arrow[r] & \widehat{E}(-p - 1) \otimes S_{k+p+1, l+p+1}. \end{tikzcd}$$

Hence a Type 1 Tate resolution has cohomological levels $a + b$ (the top row) and 0 (the bottom row). Section 4.1 will discuss $d^p_{a+b,0}$.

Lemma 3.4 (Type 2). *Assume that $\mathscr{F}$ has Type 2. Then $p^-=b-k-1$ and $p^+=-l$. Furthermore, if $p^- < p < p^+$, then*

$$T^p(\mathscr{F}) = \widehat{E}(b-p)\otimes S_{k+p-b,0}\otimes S^*_{0,-l-p-1}.$$

Proof. Since a and b are positive, the inequality $k-l>b$ implies $\min\{k,l\}=l$, $\min\{b-k,a-l\}=b-k$. Using $k-l>b$ again, (3-2) gives the desired formulas for p^- and p^+.

Now assume that $p^- < p < p^+$. Then

$$H^{a+b}\big(\mathscr{F}(p-(a+b))\big)\simeq H^0\big(X,\mathbb{O}_X(b-k-1-p,a-l-1-p)\big)^*=0$$

since $p>p^-=b-k-1$. Furthermore, $p<p^+=-l$ implies $l+p-b<0$, so that

$$H^b(\mathbb{P}^a,\mathbb{O}(k+p-b))\otimes H^0(\mathbb{P}^b,\mathbb{O}(l+p-b))=0.$$

Hence, by Künneth and Serre duality on $\mathbb{P}^b$,

$$\begin{aligned}H^b(\mathscr{F}(p-b)) &\simeq H^b\big(X,\mathbb{O}_X(k+p-b,l+p-b)\big)\\ &\simeq H^0(\mathbb{P}^a,\mathbb{O}(k+p-b))\otimes H^b(\mathbb{P}^b,\mathbb{O}(l+p-b))\\ &\simeq S_{k+p-b,0}\otimes S^*_{0,-l-p-1}.\end{aligned}$$

Finally, if $a\neq b$, we also have

$$H^a(\mathbb{P}^b,\mathbb{O}(l+p-a))=0,$$

and $H^0(\mathbb{P}^b,\mathbb{O}(l+p-a))=0$ also holds since $l+p-a<0$. Hence $H^a(\mathscr{F}(p-a))=0$ when $a\neq b$. A similar argument shows that $H^0(\mathscr{F}(p))=0$. □

Lemma 3.4 tells us that for Type 2 Tate resolutions, the only nonzero diagonal maps appear in $T^{p^-}(\mathscr{F})\to T^{p^-+1}(\mathscr{F})$:

$$\begin{tikzcd}\widehat{E}(a+1+k)\otimes S^*_{0,a+k-l-b} \arrow[dr, "d^{p^-}_{a+b,b}"] & \\ & \widehat{E}(k)\otimes S_{0,0}\otimes S^*_{0,k-l-b-1}\end{tikzcd}$$

(at cohomological levels $a+b$ and b) and in $T^{p^+-1}(\mathscr{F})\to T^{p^+}(\mathscr{F})$:

$$\begin{tikzcd}\widehat{E}(b+1+l)\otimes S_{k-l-b-1,0}\otimes S^*_{0,0} \arrow[dr, "d^{p^+-1}_{b,0}"] & \\ & \widehat{E}(l)\otimes S_{k-l,0}\end{tikzcd}$$

(at cohomological levels b and 0). The diagonal maps $d^{p^-}_{a+b,b}$ and $d^{p^+-1}_{b,0}$ will be discussed in Section 4.2.

Lemma 3.5 (Type 3). *Assume that $\mathscr{F}$ has Type* 3. *Then $p^- = a-l-1$ and $p^+ = -k$. Furthermore, if $p^- < p < p^+$, then*

$$T^p(\mathscr{F}) = \widehat{E}(a-p) \otimes S^*_{-k-p-1,0} \otimes S_{0,l+p-a}.$$

Proof. The proof is similar to the proof of Lemma 3.4 and hence is omitted. □

Lemma 3.5 tells us that for Type 3 Tate resolutions, the only nonzero diagonal maps appear in $T^{p^-}(\mathscr{F}) \to T^{p^-+1}(\mathscr{F})$:

$$\begin{array}{ccc} \widehat{E}(b+1+l) \otimes S^*_{b-k+l-a,0} & & \\ & \searrow^{d^{p^-}_{a+b,a}} & \\ & & \widehat{E}(l) \otimes S^*_{l-k-a-1,0} \otimes S_{0,0} \end{array}$$

(at cohomological levels $a+b$ and a) and in $T^{p^+-1}(\mathscr{F}) \to T^{p^+}(\mathscr{F})$:

$$\begin{array}{ccc} \widehat{E}(a+1+k) \otimes S^*_{0,0} \otimes S_{0,l-k-a-1} & & \\ & \searrow^{d^{p^+-1}_{a,0}} & \\ & & \widehat{E}(k) \otimes S_{0,l-k} \end{array}$$

(at cohomological levels a and 0). The diagonal maps $d^{p^-}_{a+b,a}$ and $d^{p^+-1}_{a,0}$ will be discussed in Section 4.2.

Remark 3.6. We finish this section by noting that some of the Tate resolutions considered here can be found in [Fløystad 2004]. Specifically, let W_1 and W_2 be finite-dimensional K-vector spaces, and consider the Tate resolution associated to $\mathscr{F} = \nu_*\mathscr{L}$, where

$$\mathscr{L} = \mathbb{O}_{\mathbb{P}(W_1)\times\mathbb{P}(W_2)}(-2, a) \otimes \textstyle\bigwedge^{a+1} W_1,$$

$\dim W_1 = a+1$, and

$$\nu : \mathbb{P}(W_1) \times \mathbb{P}(W_2) \to \mathbb{P}(W_1 \otimes W_2)$$

is the Segre embedding. The results of our paper apply to this Tate resolution.

Now consider a surjective map $W_1^* \otimes W_2^* \to W^*$. This gives a projection

$$\pi : \mathbb{P}(W_1 \otimes W_2) \dashrightarrow \mathbb{P}(W)$$

whose center is disjoint from the image of the Segre map. By [Fløystad 2004, Section 1.2], the Tate resolution for $\mathscr{F}$ gives a Tate resolution for $\mathscr{G} = \pi_*\mathscr{F}$. Fløystad shows that this projected Tate resolution has the form

$$\cdots \to T^{-1}(\mathscr{G}) \to T^0(\mathscr{G}) = \widehat{E}(a) \otimes W_1^* \to T^1(\mathscr{G}) = \widehat{E}(a-1) \otimes W_2^* \to T^2(\mathscr{G}) \to \cdots$$

with the map $d^0 : T^0(\mathscr{G}) \to T^1(\mathscr{G})$ coming from the surjection $W_1^* \otimes W_2^* \to W^*$ [Fløystad 2004, Theorem 2.1].

4. The maps in the Tate resolution for Segre embeddings of $\mathbb{P}^a \times \mathbb{P}^b$

4.1. ***Type 1 diagonal maps.*** We will use the toric Jacobian from [Cox 1996, §4]. The fan for $\mathbb{P}^a \times \mathbb{P}^b$ has $a+b+2$ 1-dimensional cone generators

$$e_0, \dots, e_a, e'_0, \dots, e'_b,$$

corresponding to $x_0, \dots, x_a, y_0, \dots, y_b$. The generators $e_1, \dots, e_a, e'_0, \dots, e'_{b-1}$ are linearly independent. Given $f_0, \dots, f_{a+b} \in S_{1,1}$, the toric Jacobian is

$$J(f_0, \dots, f_{a+b}) = \frac{1}{x_0 y_b} \det \begin{pmatrix} f_0 & \cdots & f_{a+b} \\ \partial f_0/\partial x_1 & \cdots & \partial f_{a+b}/\partial x_1 \\ \vdots & & \vdots \\ \partial f_0/\partial x_a & \cdots & \partial f_{a+b}/\partial x_a \\ \partial f_0/\partial y_0 & \cdots & \partial f_{a+b}/\partial y_0 \\ \vdots & & \vdots \\ \partial f_0/\partial y_{b-1} & \cdots & \partial f_{a+b}/\partial y_{b-1} \end{pmatrix}. \tag{4-1}$$

Since $f_i \in S_{1,1} = W$, we see that $J(f_0, \dots, f_{a+b}) \in S_{b,a}$, where (b, a) is the "critical degree," often denoted ρ in the literature on toric residues.

This toric Jacobian is closely related to the $(a+1) \times (a+b+1) \times (b+1)$ hyperdeterminant discussed in [Gel′fand et al. 1994, 14.3.D]. The connection becomes especially clear when we use the graph interpretation from [Gel′fand et al. 1994, pp. 473–474]. The idea is as follows.

Fix distinct monomials $f_0, \dots, f_{a+b} \in S_{1,1}$. These give a bipartite graph G with $a+b+2$ vertices $x_0, \dots, x_a, y_0, \dots, y_b$ and $a+b+1$ edges given by the monomials, where $f_\ell = x_i y_j$ is regarded as the edge connecting x_i to y_j. The incidence matrix of G is the $(a+b+2) \times (a+b+1)$ matrix whose rows correspond to vertices and columns correspond to edges, and where an entry is 1 if the vertex lies on the edge and is 0 otherwise.

Let M denote the square matrix obtained from the incidence matrix by removing the bottom row. Then we have the following result.

Lemma 4.1. *Let $f_0, \dots, f_{a+b} \in S_{1,1}$ be distinct monomials and let M be the matrix described above.*

(1) *The toric Jacobian of $f_0, \dots, f_{a+b}$ is given by*

$$J(f_0, \dots, f_{a+b}) = \det M \, \frac{\prod_\ell f_\ell}{\prod_i x_i \prod_j y_j}.$$

(2) $\det M \in \{0, \pm 1\}$, *and* $\det M = \pm 1$ *if and only if G is a tree.*

Proof. Each f_ℓ is homogeneous of degree 1 in $x_0, \dots, x_a$, so $f_\ell = \sum_i x_i \frac{\partial f_\ell}{\partial x_i}$. Hence the toric Jacobian $J(f_0, \dots, f_{a+b})$ can be written as

$$\frac{1}{x_0 y_b} \det \begin{pmatrix} x_0\, \partial f_0/\partial x_0 & \cdots & x_0\, \partial f_{a+b}/\partial x_0 \\ \partial f_0/\partial x_1 & \cdots & \partial f_{a+b}/\partial x_1 \\ \vdots & & \vdots \\ \partial f_0/\partial x_a & \cdots & \partial f_{a+b}/\partial x_a \\ \partial f_0/\partial y_0 & \cdots & \partial f_{a+b}/\partial y_0 \\ \vdots & & \vdots \\ \partial f_0/\partial y_{b-1} & \cdots & \partial f_{a+b}/\partial y_{b-1} \end{pmatrix}$$

$$= \frac{1}{\prod_i x_i \prod_j y_j} \det \begin{pmatrix} x_0\, \partial f_0/\partial x_0 & \cdots & x_0\, \partial f_{a+b}/\partial x_0 \\ x_1\, \partial f_0/\partial x_1 & \cdots & x_1\, \partial f_{a+b}/\partial x_1 \\ \vdots & & \vdots \\ x_a\, \partial f_0/\partial x_a & \cdots & x_a\, \partial f_{a+b}/\partial x_a \\ y_0\, \partial f_0/\partial y_0 & \cdots & y_0\, \partial f_{a+b}/\partial y_0 \\ \vdots & & \vdots \\ y_{b-1}\, \partial f_0/\partial y_{b-1} & \cdots & y_{b-1}\, \partial f_{a+b}/\partial y_{b-1} \end{pmatrix}.$$

For a fixed ℓ, we have $f_\ell = x_i y_j$, which implies

$$f_\ell = x_i \frac{\partial f_\ell}{\partial x_i} = y_j \frac{\partial f_\ell}{\partial y_j},$$

and all other partials vanish. Hence the ℓ-th column is a multiple of f_ℓ, and once we factor out f_ℓ, we are left with the ℓ-th column of the truncated incidence matrix M. Thus

$$J(f_0, \dots, f_{a+b}) = \frac{f_0 \cdots f_{a+b}}{\prod_i x_i \prod_j y_j} \det(M).$$

The second part of the lemma is a standard consequence of the Matrix Tree Theorem [Bondy and Murty 1981, Chapter 12] which counts the number of spanning trees of a graph. □

Now that we have the toric Jacobian, the next step is to introduce duplicate sets of variables

$$\mathbf{X} = (X_0, \dots, X_a),\ \mathbf{Y} = (Y_0, \dots, Y_b),\ \mathbf{x} = (x_0, \dots, x_a),\ \mathbf{y} = (y_0, \dots, y_b).$$

These give the polynomial ring

$$S \otimes S = k[\mathbf{X}, \mathbf{Y}, \mathbf{x}, \mathbf{y}]$$

and the ring homomorphism

$$S = k[\mathbf{x}, \mathbf{y}] \to S \otimes S$$

defined by $x_i \mapsto X_i + x_i$, $y_i \mapsto Y_i + y_i$. The image of $F \in S$ in $S \otimes S$ is denoted $\widetilde{F}$, so that

$$\widetilde{F}(\mathbf{X}, \mathbf{Y}, \mathbf{x}, \mathbf{y}) = F(\mathbf{X}+\mathbf{x}, \mathbf{Y}+\mathbf{y}) \in S \otimes S.$$

From a canonical point of view, the map $F \mapsto \widetilde{F}$ is comultiplication in the natural Hopf algebra structure on S.

The toric Jacobian J gives a linear map

$$J : \bigwedge^{a+b+1} W \to S_{b,a} \subset S$$

and hence a map

$$\widetilde{J} : \bigwedge^{a+b+1} W \to S \otimes S.$$

Looking at homogeneous pieces, we have a decomposition

$$\widetilde{J} = \bigoplus_{\alpha,\beta} J_{\alpha,\beta},$$

where

$$J_{\alpha,\beta} : \bigwedge^{a+b+1} W \to S_{b-\alpha,a-\beta} \otimes S_{\alpha,\beta}$$

lies in

$$\operatorname{Hom}_K\bigl(\bigwedge^{a+b+1} W, S_{b-\alpha,a-\beta} \otimes S_{\alpha,\beta}\bigr) \simeq \operatorname{Hom}_K\bigl(\bigwedge^{a+b+1} W \otimes S^*_{b-\alpha,a-\beta}, S_{\alpha,\beta}\bigr).$$

Using (2-1), $J_{\alpha,\beta}$ gives an element of

$$\operatorname{Hom}_E\bigl(\widehat{E}(a+b-p) \otimes S^*_{b-\alpha,a-\beta}, \widehat{E}(-p-1) \otimes S_{\alpha,\beta}\bigr),$$

which by abuse of notation we write as

$$J_{\alpha,\beta} : \widehat{E}(a+b-p) \otimes S^*_{b-\alpha,a-\beta} \to \widehat{E}(-p-1) \otimes S_{\alpha,\beta}. \tag{4-2}$$

In Section 4.3 we will show that the map $d^p_{a+b,0}$ from a Type 1 Tate resolution (see the discussion of following Lemma 3.3) can be chosen to be $J_{k+p+1,l+p+1}$.

4.2. ***Type 2 and 3 diagonal maps.*** The diagonal maps appearing the Type 2 and 3 Tate resolutions discussed in Section 3.2 are easy to describe. We begin with the map

$$\delta : \bigwedge^{a+1} W \to S_{0,a+1}$$

defined as follows: given $f_0, \dots, f_a \in W$, we get the *Sylvester form*

$$\delta(f_0, \dots, f_a) = \det(\ell_{ij}),$$

where $f_i = \sum_{j=0}^{a} \ell_{ij} x_j$ for $\ell_{ij} \in S_{0,1}$.

Now fix $\alpha \geq 0$. The multiplication map $S_{0,a+1} \otimes S_{0,\alpha} \to S_{0,a+1+\alpha}$ induces

$$S_{0,a+1} \to S^*_{0,\alpha} \otimes S_{0,a+1+\alpha}$$

and gives the composition

$$\bigwedge^{a+1} W \xrightarrow{\delta} S_{0,a+1} \to S^*_{0,\alpha} \otimes S_{0,a+1+\alpha}.$$

This gives maps

$$\begin{aligned}\delta_\alpha &: \bigwedge^{a+1} W \otimes S_{0,\alpha} \to S_{0,a+1+\alpha},\\ \delta^*_\alpha &: \bigwedge^{a+1} W \otimes S^*_{0,a+1+\alpha} \to S^*_{0,\alpha}\end{aligned}$$

and hence (by abuse of notation) maps

$$\begin{aligned}\delta_\alpha &: \widehat{E}(a+1+k) \otimes S_{0,\alpha} \to \widehat{E}(k) \otimes S_{0,a+1+\alpha},\\ \delta^*_\alpha &: \widehat{E}(a+1+k) \otimes S^*_{0,a+1+\alpha} \to \widehat{E}(k) \otimes S^*_{0,\alpha}.\end{aligned} \tag{4-3}$$

In Section 4.3 we will show that the diagonal map $d^{p^-}_{a+b,b}$ from a Type 2 Tate resolution (see the discussion following Lemma 3.4) and the map $d^{p^+-1}_{a,0}$ from a Type 3 Tate resolution (see the discussion following Lemma 3.5) can be chosen to be $\delta^*_{k-l-b-1}$ and $\delta_{l-k-a-1}$ respectively.

We next consider the map

$$\delta' : \bigwedge^{b+1} W \to S_{b+1,0}$$

defined as follows: given $f_0, \ldots, f_b \in W$,

$$\delta'(f_0, \ldots, f_a) = \det(\ell'_{ij}),$$

where $f_i = \sum_{j=0}^{b} \ell'_{ij} y_j$ for $\ell'_{ij} \in S_{1,0}$.

As above, $\beta \geq 0$ gives the multiplication map $S_{b+1,0} \otimes S_{\beta,0} \to S_{b+1+\beta,0}$ and the composition

$$\bigwedge^{b+1} W \xrightarrow{\delta'} S_{b+1,0} \to S^*_{\beta,0} \otimes S_{b+1+\beta,0}.$$

This gives maps

$$\begin{aligned}\delta'_\beta &: \widehat{E}(b+1+l) \otimes S_{\beta,0} \to \widehat{E}(l) \otimes S_{b+1+\beta,0},\\ \delta'^*_\beta &: \widehat{E}(b+1+l) \otimes S^*_{b+1+\beta,0} \to \widehat{E}(l) \otimes S^*_{\beta,0}.\end{aligned} \tag{4-4}$$

In Section 4.3 we will show that the map $d^{p^+-1}_{b,0}$ from a Type 2 Tate resolution (see the discussion following Lemma 3.4) and the map $d^{p^-}_{a+b,a}$ from a Type 3 Tate resolution (see the discussion following Lemma 3.5) can be chosen to be $\delta'_{k-l-b-1}$ and $\delta'^*_{l-k-a-1}$ respectively.

4.3. ***The main theorem.*** Here is the main result of this section.

Theorem 4.2. *For the Tate resolution $T^\bullet(\mathcal{F})$ of the sheaf $\mathcal{F} = \nu_*\mathcal{O}_X(k,l)$, the diagonal maps in $T^p(\mathcal{F}) \to T^{p+1}(\mathcal{F})$ can be chosen as follows*:

(1) (Type 1, $-a \le k-l \le b$): $d^p_{a+b,0} = (-1)^p J_{k+p+1,l+p+1}$.

(2) (Type 2, $k-l > b$): $d^{p^-}_{a+b,b} = \delta^*_{k-l-b-1}$ *and* $d^{p^+-1}_{b,0} = \delta'_{k-l-b-1}$.

(3) (Type 3, $k-l < -a$): $d^{p^-}_{a+b,a} = \delta'^*_{l-k-a-1}$ *and* $d^{p^+-1}_{a,0} = \delta_{l-k-a-1}$.

This uses the maps $J_{\alpha,\beta}$, δ_α, δ^*_α, δ'_β, δ'^*_β defined in (4-2), (4-3) and (4-4).

Proof. We begin with Type 2. Let $\beta = k-l-b-1$ and assume $l=0$ for simplicity, so that $p^+ = 0$. We will show that

$$T^{-2}(\mathcal{F}) \to T^{-1}(\mathcal{F}) \to T^0(\mathcal{F}) \to T^1(\mathcal{F})$$

can be constructed as follows using δ'_β:

$$\begin{tikzcd}[column sep=small]
\widehat{E}(b+2)\otimes S_{\beta-1,0}\otimes S^*_{0,1} \arrow[r, "d^{-2}"] & \widehat{E}(b+1)\otimes S_{\beta,0}\otimes S^*_{0,0} \arrow[dr, "\delta'_\beta"] & & \\
 & & \widehat{E}(0)\otimes S_{\beta+b+1,0} \arrow[r, "d^0"] & \widehat{E}(-1)\otimes S_{\beta+b+2,1}.
\end{tikzcd}$$

The differentials d^{-2} and d^0 are the known horizontal maps. To show that this sequence is exact, the first step is to prove that

$$d^0 \circ \delta'_\beta = \delta'_\beta \circ d^{-2} = 0.$$

Consider the following identity that holds for all $f_0, \dots, f_{b+1} \in W$:

$$\sum_{i=0}^{b+1} (-1)^i f_i\, \delta'\big(f_0 \wedge \cdots \widehat{f_i} \cdots \wedge f_{b+1}\big) = 0. \tag{4-5}$$

If we write $f_i = \sum_{j=0}^{b} \ell'_{ij} y_j$, then (4-5) follows from the obvious identity

$$\det \begin{pmatrix} f_0 & \cdots & f_{b+1} \\ \ell'_{0,0} & \cdots & \ell'_{b+1,0} \\ \vdots & & \vdots \\ \ell'_{0,b} & \cdots & \ell'_{b+1,b} \end{pmatrix} = 0$$

by expanding by minors along the first row and using the definition of δ'.

By (2-1), the composition

$$\widehat{E}(b+1)\otimes S_{\beta,0}\otimes S^*_{0,0} \xrightarrow{\delta'_\beta} \widehat{E}(0)\otimes S_{\beta+b+1,0} \xrightarrow{d^0} \widehat{E}(-1)\otimes S_{\beta+b+2,1}$$

corresponds to a map

$$\bigwedge^{b+2} W \otimes S_{\beta,0} \otimes S_{0,0}^* \to S_{\beta+b+2,1}.$$

We ignore $S_{0,0}^* \simeq k$. Using the definition of δ'_β, this map is given by

$$f_0 \wedge \cdots \wedge f_{b+1} \otimes h \mapsto \sum_{i=0}^{b+1} (-1)^i f_i\, h\, \delta'\big(f_0 \wedge \cdots \widehat{f_i} \cdots \wedge f_{b+1}\big).$$

This reduces to zero (factor out $h \in S_{\beta,0}$ and use (4-5)), so $d^0 \circ \delta'_\beta = 0$.

If $\beta > 0$, we need to consider $\delta'_\beta \circ d^{-2}$. Arguing as above, we determine this map by

$$\bigwedge^{b+2} W \otimes S_{\beta-1,0} \otimes S_{0,1}^* \to S_{\beta+b+1,0},$$

which in turn is determined by the map

$$\bigwedge^{b+2} W \otimes S_{\beta-1,0} \to S_{\beta+b+1,0} \otimes S_{0,1} = S_{\beta+b+1,1}$$

given by

$$f_0 \wedge \cdots \wedge f_{b+1} \otimes h \mapsto \sum_{i=0}^{b+1} (-1)^i f_i\, h\, \delta'\big(f_0 \wedge \cdots \widehat{f_i} \cdots \wedge f_{b+1}\big)$$

for $h \in S_{\beta-1,0}$. As above, this reduces to zero, so that $\delta'_\beta \circ d^{-2} = 0$.

When $\beta = 0$, we have to show that the composition

$$\begin{tikzcd}[column sep=tiny]
\widehat{E}(a+b+2) \otimes S_{0,a+1}^* \arrow[dr, "\delta_0^*"] & & \\
& \widehat{E}(a+1) \otimes S_{0,0} \otimes S_{0,0}^* \arrow[dr, "\delta'_0"] & \\
& & \widehat{E}(0) \otimes S_{b+1,0}
\end{tikzcd}$$

is zero. By (2-1), the composed map corresponds to a map

$$\bigwedge^{a+b+2} W \otimes S_{0,a+1}^* \to S_{b+1,0},$$

which in turn is determined by the a map

$$\bigwedge^{a+b+2} W \to S_{b+1,0} \otimes S_{0,a+1} = S_{b+1,a+1}.$$

Given $f_0, \ldots, f_{a+b+1} \in W$, this map is given by

$$f_0 \wedge \cdots \wedge f_{a+b+1} \mapsto \sum_{|S|=a+1} \varepsilon(S)\, \delta(\mathbf{f}_S)\, \delta'(\mathbf{f}_{S^c}), \tag{4-6}$$

where the sum is over all subsets $S \subset \{0, \dots, a+b+1\}$ of cardinality $a+1$ and $S^c = \{0, \dots, a+b+1\} \setminus S$. Furthermore,

$$\delta(\mathbf{f}_S) = \delta\big(\textstyle\bigwedge_{i \in S} f_i\big),$$
$$\delta'(\mathbf{f}_{S^c}) = \delta'\big(\textstyle\bigwedge_{i \in S^c} f_i\big),$$

and $\varepsilon(S) = \pm 1$ is the sign that appears in the Laplace expansion described below.

To show that the sum in (4-6) is zero, write

$$f_i = \sum_{j=0}^{a} \ell_{ij} x_j = \sum_{j=0}^{b} \ell'_{ij} y_j$$

and consider the matrix

$$\mathcal{M} = \begin{pmatrix} \ell_{0,0} & \cdots & \ell_{a+b+1,0} \\ \vdots & & \vdots \\ \ell_{0,a} & \cdots & \ell_{a+b+1,a} \\ \ell'_{0,0} & \cdots & \ell'_{a+b+1,0} \\ \vdots & & \vdots \\ \ell'_{0,b} & \cdots & \ell'_{a+b+1,b} \end{pmatrix}.$$

If we multiply first $a+1$ rows by suitable $\mathbf{x}$ variables and multiply the last $b+1$ rows by $\mathbf{y}$ variables, we get the same result, namely the row $(f_0, \dots, f_{a+b+1})$. If follows that $\det \mathcal{M} = 0$. If we take the Laplace expansion that involves $(a+1) \times (a+1)$ minors of the first $a+1$ rows multiplied by $(b+1) \times (b+1)$ complementary minors of the last $b+1$ rows, we get the sum in (4-6). Hence this sum is zero, which proves that $\delta'_0 \circ \delta^*_0 = 0$.

To complete the proof that δ'_β gives the diagonal map in $T^{-1}(\mathcal{F}) \to T^0(\mathcal{F})$, we follow the strategy used in [Cox 2007, Theorem 1.3]. Let $N' = (a+1)(b+1) = \dim(W)$. Since $\widehat{E} \simeq E(-N')$ and $T^{-1}(\mathcal{F}) \to T^0(\mathcal{F}) \to T^1(\mathcal{F})$ is

$$\widehat{E}(b+1) \otimes S_{\beta,0} \otimes S^*_{0,0} \to \widehat{E}(0) \otimes S_{\beta+b+1,0} \xrightarrow{d^0} \widehat{E}(-1) \otimes S_{\beta+b+2,1},$$

the kernel of d^0 has $\dim(S_{\beta,0} \otimes S^*_{0,0})$ minimal generators of degree $N' - b - 1$. Since we have proved that δ'_β maps into this kernel, it suffices to prove that this map is injective in degree $N' - b - 1$, namely that

$$\delta'_\beta : \textstyle\bigwedge^{N'} W \otimes S_{\beta,0} \to \bigwedge^{N'-b-1} W \otimes S_{\beta+b+1,0}$$

is injective (as above, we ignore $S^*_{0,0}$). A basis of $\bigwedge^{N'} W$ is given by

$$x_0 y_0 \wedge \cdots \wedge x_0 y_b \wedge \omega,$$

where ω is the wedge product of the remaining $N'-b-1$ monomials of W in some order. Since

$$\delta'(x_0y_0\wedge\cdots\wedge x_0y_b)=x_0^{b+1},$$

we see that for $h\in S_{\beta,0}$,

$$\delta'_\beta(x_0y_0\wedge\cdots\wedge x_0y_b\wedge\omega\otimes h)=\omega\otimes x_0^{b+1}h+\cdots\in \textstyle\bigwedge^{N'-b-1}W\otimes S_{\beta+b+1,0},$$

where the omitted terms involve basis elements of $\bigwedge^{N'-b-1}W$ different from ω. The desired injectivity is now obvious.

This completes the proof for $d_{b,0}^{p^+-1}$ in a Type 2 Tate resolution when $l=0$ and $k=\beta+b+1$. The proof for arbitrary l is similar, and the same proof easily adapts to $d_{a,0}^{p^+-1}$ in a Type 3 Tate resolution. As for d^{p^-}, we note that applying $\mathrm{Hom}_E(-,K)\otimes_K\widehat{E}$ to $T^p(\mathcal{F})$ gives $T^{a+b-p}(\mathcal{G})$, where $\mathcal{G}=\nu_*\mathcal{O}_X(-a-1-k,-b-1-l)$. This duality interchanges Type 2 and Type 3 resolutions. Then our results for

$$d_{b,0}^{p^+-1}\quad\text{and}\quad d_{a,0}^{p^+-1}$$

and dualize to give the desired results for

$$d_{a+b,a}^{p^-}\quad\text{and}\quad d_{a+b,b}^{p^-}.$$

It remains to consider Type 1 Tate resolutions. This case will be more complicated since there are two sets of variables to keep track of: the original variables $\mathbf{x}$, $\mathbf{y}$ and the duplicates $\mathbf{X}$, $\mathbf{Y}$ introduced in Section 4.1.

Let $\alpha=k+p+1$ and $\beta=k+p+1$. We will show that the crucial part of $T^p(\mathcal{F})\to T^{p+1}(\mathcal{F})\to T^{p+2}(\mathcal{F})$ can be chosen to be

$$\begin{tikzcd}
\widehat{E}(a+b-p)\otimes S^*_{b-\alpha,a-\beta} \arrow[r, "d^p_{a+b,a+b}"] \arrow[dr, "(-1)^pJ_{\alpha,\beta}"'] & \widehat{E}(a+b-p-1)\otimes S^*_{b-\alpha-1,a-\beta-1} \arrow[dr, "(-1)^{p+1}J_{\alpha+1,\beta+1}"] \arrow[d, phantom, "\bigoplus"] & \\
& \widehat{E}(-p-1)\otimes S_{\alpha,\beta} \arrow[r, "d^{p+1}_{0,0}"] & \widehat{E}(-p-2)\otimes S_{\alpha+1,\beta+1}.
\end{tikzcd}$$

This first step is to show that this is a complex, that is, the composition

$$T^p(\mathcal{F})\to T^{p+1}(\mathcal{F})\to T^{p+2}(\mathcal{F})$$

is zero. Since the horizontal maps behave properly, it suffices to show that

$$d_{0,0}^{p+1}\circ J_{\alpha,\beta}=J_{\alpha+1,\beta+1}\circ d^p_{a+b,a+b}. \tag{4-7}$$

Using (2-1), this is equivalent to showing that the diagram

$$\begin{tikzcd}
\bigwedge^{a+b+2} W \otimes S^*_{b-\alpha,a-\beta} \arrow[r, "d^p_{a+b,a+b}"] \arrow[d, "J_{\alpha,\beta}"] & \bigwedge^{a+b+1} W \otimes S^*_{b-\alpha-1,a-\beta-1} \arrow[d, "J_{\alpha+1,\beta+1}"] \\
W \otimes S_{\alpha,\beta} \arrow[r, "d^{p+1}_{0,0}"] & S_{\alpha+1,\beta+1}
\end{tikzcd}$$

commutes. A key point is that on the top, $d^p_{a+b,a+b}$ uses $\mathbf{X}$, $\mathbf{Y}$, while on the bottom, $d^{p+1}_{0,0}$ uses $\mathbf{x}$, $\mathbf{y}$. We can recast the commutativity of this diagram as saying that

$$d^{p+1}_{0,0} \circ J_{\alpha,\beta} = J_{\alpha+1,\beta+1} \circ d^p_{a+b,a+b}$$

as maps

$$\bigwedge^{a+b+2} W \to \underbrace{S_{b-\alpha,a-\beta}}_{\mathbf{X},\mathbf{Y}} \otimes \underbrace{S_{\alpha+1,\beta+1}}_{\mathbf{x},\mathbf{y}}.$$

Given $a+b+2$ elements of W, we write them as $f_0, \ldots, f_{a+b+1}$ when using $\mathbf{x}$, $\mathbf{y}$ and as $F_0, \ldots, F_{a+b+1}$ when using $\mathbf{X}$, $\mathbf{Y}$. Then (4-7) is equivalent to the identity

$$\sum_{i=0}^{a+b+1} (-1)^i f_i \, J_{\alpha,\beta}\big(f_0 \wedge \cdots \widehat{f_i} \cdots \wedge f_{a+b+1}\big) = \sum_{i=0}^{a+b+1} (-1)^i F_i \, J_{\alpha+1,\beta+1}\big(f_0 \wedge \cdots \widehat{f_i} \cdots \wedge f_{a+b+1}\big)$$

in $S_{b-\alpha,a-\beta} \otimes S_{\alpha+1,\beta+1}$. Summing this over all α and β gives the second identity

$$\sum_{i=0}^{a+b+1} (-1)^i f_i \, \widetilde{J}\big(f_0 \wedge \cdots \widehat{f_i} \cdots \wedge f_{a+b+1}\big) = \sum_{i=0}^{a+b+1} (-1)^i F_i \, \widetilde{J}\big(f_0 \wedge \cdots \widehat{f_i} \cdots \wedge f_{a+b+1}\big),$$

and the first identity follows from the second by taking the appropriate graded piece. However,

- The change of variables $(\mathbf{x}, \mathbf{y}) \leftrightarrow (\mathbf{X}, \mathbf{Y})$ interchanges f_i and F_i;
- $\widetilde{J}\big(f_0 \wedge \cdots \widehat{f_i} \cdots \wedge f_{a+b+1}\big)$ is invariant under $(\mathbf{x}, \mathbf{y}) \leftrightarrow (\mathbf{X}, \mathbf{Y})$.

It follows that the second identity is equivalent to the following

Assertion 4.3.

$$\sum_{i=0}^{a+b+1} (-1)^i f_i \, \widetilde{J}\big(f_0 \wedge \cdots \widehat{f_i} \cdots \wedge f_{a+b+1}\big) \tag{4-8}$$

is invariant under $(\mathbf{x},\mathbf{y}) \leftrightarrow (\mathbf{X},\mathbf{Y})$.

In particular, (4-7) is an immediate consequence of Assertion 4.3.

We will prove Assertion 4.3 by representing (4-8) as a determinant. We begin with the formula

$$J(f_0 \wedge \cdots \wedge f_{a+b}) = \frac{1}{y_b} \det \begin{pmatrix} \partial f_0/\partial x_0 & \cdots & \partial f_{a+b}/\partial x_0 \\ \vdots & & \vdots \\ \partial f_0/\partial x_a & \cdots & \partial f_{a+b}/\partial x_a \\ \partial f_0/\partial y_0 & \cdots & \partial f_{a+b}/\partial y_0 \\ \vdots & & \vdots \\ \partial f_0/\partial y_{b-1} & \cdots & \partial f_{a+b}/\partial y_{b-1} \end{pmatrix},$$

which follows from the proof of Lemma 4.1. This implies

$$\widetilde{J}(f_0 \wedge \cdots \wedge f_{a+b}) = \frac{1}{Y_b + y_b} \det \begin{pmatrix} \widetilde{\partial f_0/\partial x_0} & \cdots & \widetilde{\partial f_{a+b}/\partial x_0} \\ \vdots & & \vdots \\ \widetilde{\partial f_0/\partial x_a} & \cdots & \widetilde{\partial f_{a+b}/\partial x_a} \\ \widetilde{\partial f_0/\partial y_0} & \cdots & \widetilde{\partial f_{a+b}/\partial y_0} \\ \vdots & & \vdots \\ \widetilde{\partial f_0/\partial y_{b-1}} & \cdots & \widetilde{\partial f_{a+b}/\partial y_{b-1}} \end{pmatrix}.$$

It follows easily that

$$\sum_{i=0}^{a+b+1} (-1)^i f_i \, \widetilde{J}\big(f_0 \wedge \cdots \widehat{f_i} \cdots \wedge f_{a+b+1}\big) = \frac{1}{Y_b + y_b} \det \mathbf{M},$$

where $\mathbf{M}$ is the $(a+b+2) \times (a+b+2)$ matrix

$$\mathbf{M} = \begin{pmatrix} f_0 & \cdots & f_{a+b+1} \\ \widetilde{\partial f_0/\partial x_0} & \cdots & \widetilde{\partial f_{a+b+1}/\partial x_0} \\ \vdots & & \vdots \\ \widetilde{\partial f_0/\partial x_a} & \cdots & \widetilde{\partial f_{a+b+1}/\partial x_a} \\ \widetilde{\partial f_0/\partial y_0} & \cdots & \widetilde{\partial f_{a+b+1}/\partial y_0} \\ \vdots & & \vdots \\ \widetilde{\partial f_0/\partial y_{b-1}} & \cdots & \widetilde{\partial f_{a+b+1}/\partial y_{b-1}} \end{pmatrix}.$$

To prove Assertion 4.3, it suffices to show that the determinant of the matrix $\mathbf{M}$ is unchanged when we replace its top row with $(F_0, \ldots, F_{a+b+1})$. For this purpose, consider the $(a+b+3)\times(a+b+3)$ matrix

$$\overline{\mathbf{M}} = \begin{pmatrix} & & & 0 \\ & \mathbf{M} & & \vdots \\ & & & 0 \\ \widetilde{\frac{\partial f_0}{\partial y_b}} & \cdots & \widetilde{\frac{\partial f_{a+b+1}}{\partial y_b}} & 1 \end{pmatrix}$$

and observe that $\det \mathbf{M} = \det \overline{\mathbf{M}}$. Write $\overline{\mathbf{M}}$ as

$$\overline{\mathbf{M}} = \begin{pmatrix} f_0 & \cdots & f_{a+b+1} & 0 \\ & & & 0 \\ & \widetilde{Q} & & \vdots \\ & & & 1 \end{pmatrix}.$$

Since $f_\ell \in W = S_{1,1}$, we have the easily proved identity

$$F_\ell - f_\ell = -\sum_{i=0}^{a} x_i \widetilde{\frac{\partial f_\ell}{\partial x_i}} + \sum_{j=0}^{b} Y_j \widetilde{\frac{\partial f_\ell}{\partial y_j}}.$$

Multiplying the last $a+b+2$ rows of $\overline{\mathbf{M}}$ by $-x_i$ or Y_j as appropriate and adding to the first row gives the matrix

$$\overline{\mathbf{M}}' = \begin{pmatrix} F_0 & \cdots & F_{a+b+1} & Y_b \\ & & & 0 \\ & \widetilde{Q} & & \vdots \\ & & & 1 \end{pmatrix}.$$

Note that $\det \overline{\mathbf{M}}' = \det \overline{\mathbf{M}}$. This is *almost* what we need, except for the Y_b in the first row of $\overline{\mathbf{M}}'$.

We claim that $\det \widetilde{Q} = 0$. Assuming this for the moment, it follows that we can replace Y_b with 0 in $\overline{\mathbf{M}}'$ without changing its determinant. This easily implies $\det \mathbf{M}$ is unchanged when we replace its top row with $(F_0, \ldots, F_{a+b+1})$ and will complete the proof of (4-7).

It remains to study $\det \widetilde{Q}$. The matrix $\widetilde{Q}$ is obtained from

$$Q = \begin{pmatrix} \partial f_0/\partial x_0 & \cdots & \partial f_{a+b}/\partial x_0 \\ \vdots & & \vdots \\ \partial f_0/\partial x_a & \cdots & \partial f_{a+b}/\partial x_a \\ \partial f_0/\partial y_0 & \cdots & \partial f_{a+b}/\partial y_0 \\ \vdots & & \vdots \\ \partial f_0/\partial y_b & \cdots & \partial f_{a+b}/\partial y_b \end{pmatrix}$$

by the $F \mapsto \widetilde{F}$ operation described in Section 4.1. But $\det Q = 0$ since $f_\ell = \sum_{i=0}^{a} x_i \,\partial f_\ell/\partial x_i = \sum_{j=0}^{b} y_j \,\partial f_\ell/\partial y_j$, and then

$$\det \widetilde{Q} = \widetilde{\det Q} = 0.$$

Hence we have proved that the maps

$$T^p(\mathscr{F}) \to T^{p+1}(\mathscr{F})$$

defined using $(-1)^p J_{\alpha,\beta}$ give a complex. To show that the complex is exact, we again use the strategy of [Cox 2007, Theorem 1.3]. Lemma 3.3 tells us that $p^+ = \min\{b-k, a-l\}$. For simplicity, we assume $b-k \le a-l$, so that $p^+ = b-k$. Let $\beta = b-k+l$ and $p = p^+ - 1$. Type 1 and $b-k \le a-l$ imply $0 \le \beta \le a$. Then $T^p(\mathscr{F}) \to T^{p+1}(\mathscr{F})$ becomes

$$\begin{tikzcd} \widehat{E}(a+k+1) \otimes S^*_{0,a-\beta} \arrow[drr, "(-1)^p J_{b,\beta}"] & & \\ \bigoplus & & \\ \widehat{E}(k-b+1) \otimes S_{b-1,\beta-1} \arrow[rr, "d^p_{0,0}"] & & \widehat{E}(k-b) \otimes S_{b,\beta}. \end{tikzcd} \tag{4-9}$$

Let $N' = (a+1)(b+1) = \dim(W)$. Then the shape of the Tate resolution tells us that there are $\dim(S_{b-1,\beta-1})$ minimal generators of degree $N' - (k-b+1)$ and $\dim(S^*_{0,a-\beta})$ minimal generators of degree $N' - (a+k+1)$. The former are taken care of by the known map $d^p_{0,0}$, and for the latter, we see that in degree $N' - (a+k+1)$, the above diagram becomes

$$\begin{tikzcd} \bigwedge^{N'} W \otimes S^*_{0,a-\beta} \arrow[drr, "(-1)^p J_{b,\beta}"] & & \\ \bigoplus & & \\ \bigwedge^{N'-a-b} W \otimes S_{b-1,\beta-1} \arrow[rr, "d^p_{0,0}"] & & \bigwedge^{N'-a-b-1} W \otimes S_{b,\beta}. \end{tikzcd} \tag{4-10}$$

As in [Cox 2007, Lemma 2.2], we need to show that $(-1)^p J_{b,\beta}$ is injective and that its image has trivial intersection with the image of $d^p_{0,0}$.

For the former, let $\theta \in \bigwedge^{N'} W$ be the wedge product of the monomials in W in some order, and let $\varphi \in S^*_{0,a-\beta}$ satisfy $J_{b,\beta}(\theta \otimes \varphi) = 0$. Suppose that $\mathbf{Y}^u$ is a monomial in the $\mathbf{Y}$ variables of degree $|u| = a-\beta$. We prove $\varphi(\mathbf{Y}^u) = 0$ as follows.

Pick $\mathbf{Y}^v$ such that $\mathbf{Y}^u \,|\, \mathbf{Y}^v$ and $|v| = a$, and write

$$\mathbf{Y}^v = Y_{j_1} \cdots Y_{j_a}.$$

Then consider the following collection $f_0, \dots, f_{a+b}$ of monomials in $W = S_{1,1}$:

$$x_0 y_j,\ \ j = 0, \dots, b \ \text{ and } \ x_i y_{j_i},\ \ i = 1, \dots, a.$$

The graph of these monomials (in the sense of Section 4.1) is easily seen to be a tree. Then Lemma 4.1 implies that

$$J(f_0 \wedge \cdots \wedge f_{a+b}) = \pm \frac{\prod_{j=0}^{b} x_0 y_j \prod_{i=1}^{a} x_i y_{j_i}}{\prod_{i=0}^{a} x_i \prod_{j=0}^{b} y_j} = \pm x_0^b \prod_{i=1}^{a} y_{j_i} = \pm x_0^b \, \mathbf{y}^v.$$

Thus $\widetilde{J}(f_0 \wedge \cdots \wedge f_{a+b}) = \pm (X_0 + x_0)^b (\mathbf{Y} + \mathbf{y})^v$. Taking those terms of degree (b, β) in $(\mathbf{x}, \mathbf{y})$, we obtain

$$J_{b,\beta}(f_0 \wedge \cdots \wedge f_{a+b}) = \pm \sum_{w} \binom{v}{w} x_0^b \, \mathbf{Y}^{v-w} \, \mathbf{y}^w,$$

where $\binom{v}{w} = \prod_{j=0}^{b} \binom{v_j}{w_j}$ and $\sum_w$ denotes the sum over all exponent vectors w satisfying $|w| = \beta$ and $0 \le w_j \le v_j$ for all j. Writing $\theta = f_0 \wedge \cdots \wedge f_{a+b} \wedge \omega$, we obtain

$$\begin{aligned} 0 &= J_{b,\beta}(f_0 \wedge \cdots \wedge f_{a+b} \wedge \omega \otimes \varphi) \\ &= \omega \otimes \varphi\big(J_{b,\beta}(f_0 \wedge \cdots \wedge f_{a+b})\big) + \cdots \\ &= \omega \otimes \left(\pm \sum_{w} \binom{v}{w} \varphi(\mathbf{Y}^{v-w})\, x_0^b \, \mathbf{y}^w \right) + \cdots, \end{aligned}$$

where the omitted terms involve basis elements of $\bigwedge^{N'-a-b-1} W$ different from ω. Since we are in characteristic 0, it follows that $\varphi(\mathbf{Y}^{v-w}) = 0$ for all w under consideration. Our choice of v guarantees that our original monomial $\mathbf{Y}^u$ is one of these $\mathbf{Y}^{v-w}$'s. Hence $\varphi(\mathbf{Y}^u) = 0$, which implies $\varphi = 0$ since $\mathbf{Y}^u$ was an arbitrary monomial of degree $a - \beta$. This completes the proof $(-1)^p J_{b,\beta}$ is injective.

It remains to show that the image of this map has trivial intersection with the image of $d_{0,0}^p$. Following a suggestion of Jenia Tevelev, we use representation theory to finish the proof.

Recall that there is a natural isomorphism

$$W = S_{1,1} \cong W_1 \otimes W_2,$$

where $W_1 = S_{1,0} = \mathbb{C}^{a+1}$ and $W_2 = S_{0,1} = \mathbb{C}^{b+1}$. First, we show that an action of the group $G = \mathrm{SL}(W_1) \times \mathrm{SL}(W_2)$ on the diagram (4-10) is G-invariant on the maps $d_{0,0}^p$ and $(-1)^p J_{b,\beta}$. Indeed, since the map $d_{0,0}^p$ is induced by the multiplication map

$$W \otimes S_{b-1,\beta-1} \to S_{b,\beta},$$

we conclude that $d^p_{0,0}$ is G-invariant. Now observe that the toric Jacobian can be written as a linear combination of monomials

$$J(f_0,\ldots,f_{a+b})=\sum_{\mu,\nu}c_{\mu,\nu}x^{\mu}y^{\nu},$$

where $c_{\mu,\nu}$ are the entries of the square matrix whose determinant is a hyperdeterminant [Gel′fand et al. 1994, p. 473]. By [Gel′fand et al. 1992, Proposition 1.4], the hyperdeterminant is G-invariant, so the toric Jacobian (4-1) (and respectively the map $(-1)^p J_{b,\beta}$) is G-invariant.

It follows from Schur's Lemma that the images of $d^p_{0,0}$ and $J_{b,\beta}$ have trivial intersection if the representation of G corresponding to

$$\bigwedge\nolimits^{N'-a-b}W\otimes S_{b-1,\beta-1}=\bigwedge\nolimits^{ab+1}(W_1\otimes W_2)\otimes\mathrm{Sym}^{b-1}(W_1)\otimes\mathrm{Sym}^{\beta-1}(W_2)\quad(4\text{-}11)$$

does not contain the representation corresponding to

$$\bigwedge\nolimits^{N'}W\otimes S^*_{0,a-\beta}=\bigwedge\nolimits^{ab+a+b+1}(W_1\otimes W_2)\otimes\mathrm{Sym}^{a-\beta}(W_2^*).$$

To prove this, we use some basic facts from the representation theory of the special linear group (see, for example, [Fulton and Harris 1991, §6.1 and §15.3]). Given a partition $\lambda=(\lambda_1,\ldots,\lambda_s)$ with $\lambda_1\geq\cdots\geq\lambda_s\geq 0$, we get a *Young diagram* D_λ, which consists of s rows of boxes, all starting at the same column, of lengths $\lambda_1\geq\cdots\geq\lambda_s$.

For a vector space V over K, $\mathbb{S}_\lambda(V)$ denotes the irreducible $\mathrm{SL}(V)$-representation corresponding to the partition λ. We use notation

$$\lambda=(d_1^{a_1},\ldots,d_\ell^{a_\ell})$$

to denote the partition having a_i copies of the integer d_i for $1\leq i\leq\ell$. The corresponding Young diagram D_λ has a_i rows of boxes of length d_i. Thus $\lambda=(d)$ gives the symmetric product $\mathbb{S}_\lambda(V)=\mathrm{Sym}^d(V)$ and $\lambda=(1^d)$ gives the exterior product $\mathbb{S}_\lambda(V)=\bigwedge^d V$.

Recall that $\mathbb{S}_\lambda(V)=0$ when the Young diagram of λ has more than $\dim V$ nonzero rows, and that two Young diagrams give the same $\mathrm{SL}(V)$-representation if and only if one can be obtained from the other by adding or deleting columns of height $\dim V$ at the beginning of the Young diagram.

By the Cauchy formula [Fulton and Harris 1991, §6.1], we have the following decomposition for the exterior powers of $W=W_1\otimes W_2$:

$$\bigwedge\nolimits^{ab+1}W=\bigwedge\nolimits^{ab+1}(W_1\otimes W_2)=\bigoplus_{|\lambda|=ab+1}\mathbb{S}_\lambda(W_1)\otimes\mathbb{S}_{\lambda'}(W_2),$$

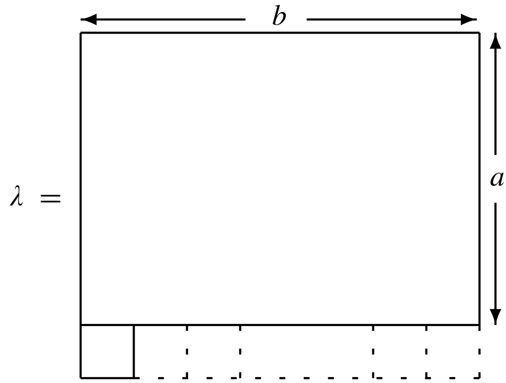
where the direct sum runs over all partitions λ of $ab+1$ with at most $\dim W_1=a+1$ rows, at most $\dim W_2=b+1$ columns, and λ' is the conjugate partition to λ. Note that the representation corresponding to the highest power of determinant $\bigwedge^{N'} W$ is one-dimensional, that is, a trivial representation.

When we combine this with (4-11), we see that it is enough to show that it cannot happen simultaneously that $\mathbb{S}_\lambda(W_1) \otimes \mathrm{Sym}^{b-1}(W_1)$ contains the trivial representation and $\mathbb{S}_{\lambda'}(W_2) \otimes \mathrm{Sym}^{\beta-1}(W_2)$ contains $\mathrm{Sym}^{a-\beta}(W_2^*)$. Since $\dim W_1 = a+1$ and $\dim W_2 = b+1$, we can assume that the Young diagram of λ has at most $a+1$ rows (otherwise $\mathbb{S}_\lambda(W_1) = 0$) and at most $b+1$ columns (otherwise $\mathbb{S}_{\lambda'}(W_2) = 0$).

By the Pieri formula [Fulton and Harris 1991, (6.8)], for any partition λ, we have

$$\mathbb{S}_\lambda(W_1) \otimes \mathrm{Sym}^{b-1}(W_1) \cong \bigoplus_\nu \mathbb{S}_\nu(W_1),$$

where the sum is over all ν whose Young diagram is obtained by adding $b-1$ boxes to the Young diagram of λ, with no two boxes in the same column. Note also that each ν is a partition of $(ab+1)+(b-1) = (a+1)b$. Since D_λ has $|\lambda| = ab+1$ boxes and fits inside a $(a+1)\times(b+1)$ rectangle, the only way for ν to give the trivial representation is for D_λ to be the Young diagram.



You can see how adding $b-1$ boxes to the bottom row (the dashed boxes in the drawing) gives the trivial representation, since D_ν is trivial if and only if it consists entirely of columns of height $a+1$.

This shows that the only case when

$$\mathbb{S}_\lambda(W_1) \otimes \mathrm{Sym}^{b-1}(W_1)$$

contains the trivial representation is when $\lambda = (b^a, 1)$. Hence, λ' must be

$$(a+1, a^{b-1}).$$

On the other hand, $\mathrm{Sym}^{a-\beta}(W_2^*)$ corresponds to the partition $(b^{a-\beta})$ [Fulton and Harris 1991, §15.5, Exercise 15.50], so from the Pieri formula we see that it is impossible to get $(b^{a-\beta})$ from the tensor product $\mathbb{S}_{\lambda'}(W_2) \otimes \mathrm{Sym}^{\beta-1}(W_2)$ by adding

$\beta - 1$ boxes to $(a+1, a^{b-1})$, no two in the same column, and then deleting columns of height $b+1$.

The final step is to prove exactness when $T^p \to T^{p+1}$ is given by

$$\begin{tikzcd}[column sep=large]
\widehat{E}(a+k+1) \otimes S^*_{b-\alpha,a-\beta} \arrow[r, "d^p_{a+b,a+b}"] \arrow[dr, "(-1)^p J_{\alpha,\beta}"] & \widehat{E}(a+k) \otimes S^*_{b-\alpha-1,a-\beta-1} \\
\bigoplus & \bigoplus \\
\widehat{E}(k-b+1) \otimes S_{\alpha-1,\beta-1} \arrow[r, "d^p_{0,0}"] & \widehat{E}(k-b) \otimes S_{\alpha,\beta}.
\end{tikzcd}$$

Here, we use the same conventions as in (4-9), except that we now assume that $b-\alpha$ and $a-\beta$ are positive. As before, the shape of the Tate resolution tells us that there are $\dim(S_{\alpha-1,\beta-1})$ minimal generators of degree $N'-(k-b+1)$ and $\dim(S^*_{b-\alpha,a-\beta})$ minimal generators of degree $N'-(a+k+1)$. The former are taken care of by the known map $d^p_{0,0}$, and for the latter, we see that in degree $N'-(a+k+1)$, the above diagram becomes

$$\begin{tikzcd}[column sep=large]
\bigwedge^{N'} W \otimes S^*_{b-\alpha,a-\beta} \arrow[r, "d^p_{a+b,a+b}"] \arrow[dr, "(-1)^p J_{\alpha,\beta}"] & \bigwedge^{N'-1} W \otimes S^*_{b-\alpha-1,a-\beta-1} \\
\bigoplus & \bigoplus \\
\bigwedge^{N'-a-b} W \otimes S_{\alpha-1,\beta-1} \arrow[r, "d^p_{0,0}"] & \bigwedge^{N'-a-b-1} W \otimes S_{\alpha,\beta}.
\end{tikzcd} \tag{4-12}$$

The map $d^p_{a+b,a+b}$ is injective since it is dual to the surjective multiplication map $W \otimes S_{b-\alpha-1,a-\beta-1} \to S_{b-\alpha,a-\beta}$. As in the proof of [Cox 2007, Theorem 1.3], it follows immediately that the map (4-12) is injective on $\bigwedge^{N'} W \otimes S^*_{b-\alpha,a-\beta}$ and that the images of

$$\bigwedge^{N'} W \otimes S^*_{b-\alpha,a-\beta} \quad \text{and} \quad \bigwedge^{N'-a-b} W \otimes S_{\alpha-1,\beta-1}$$

have trivial intersection. This completes the proof of the theorem. □

Remark 4.4. In the proof of Section 4.3, we used the relation between the toric Jacobian of $f_0, \dots, f_{a+b} \in S_{1,1}$ and the hyperdeterminants studied in [Gel′fand et al. 1994; 1992] to prove the equivariance we needed. The theorem implies that certain hyperdeterminants are explicitly encoded into the Tate resolutions considered here. This is another example of the amazing amount of information contained in these resolutions.

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dac@cs.amherst.edu *Department of Mathematics and Computer Science, Amherst College, Amherst, MA 01002-5000, United States*
www.amherst.edu/~dacox

materov@math.umass.edu *Department of Mathematics and Statistics, University of Massachusetts, Amherst, MA 01003-9305, United States*