Algebra & Number Theory

Volume 2

2008

No. 6

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mathematical sciences publishers

ALGEBRA AND NUMBER THEORY 2:6(2008)

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The definition of Rouquier for families of characters of Weyl groups in terms of blocks of the associated Iwahori–Hecke algebra has made possible the generalization of this notion to the complex reflection groups. Here we give an algorithm for the determination of the "Rouquier blocks" of the cyclotomic Ariki–Koike algebras.

Introduction

The work of G. Lusztig [1984] on irreducible characters of reductive groups over finite fields has displayed the important role of the families of characters of the Weyl groups concerned. More recent results of Gyoja [1996] and Rouquier [1999] have made possible the definition of a substitute for families of characters which can be applied to all complex reflection groups. In particular, Rouquier has shown that the families of characters of a Weyl group W are exactly the blocks of the Iwahori–Hecke algebra of W over a suitable coefficient ring; this is now called the Rouquier ring. This definition generalizes without problem to the cyclotomic Hecke algebras of all complex reflection groups. Ever since, we have been interested in the determination of the Rouquier blocks of the cyclotomic Hecke algebras of complex reflection groups.

Broué and Kim [2002] presented an algorithm for the determination of the Rouquier blocks for the cyclotomic Hecke algebras of the groups $G(d, 1, r)$ and $G(d, d, r)$. Later Kim [2005] used the same algorithm to determine the Rouquier blocks for the group $G(de, e, r)$. The Rouquier blocks of the spetsial cyclotomic Hecke algebra of many exceptional complex reflection groups have been determined in [Malle and Rouquier 2003]. Finally, in [Chlouveraki 2007], we determined the Rouquier blocks of the cyclotomic Hecke algebras of all exceptional complex reflection groups.

MSC2000: 20C08.

Keywords: Ariki–Koike algebras, Rouquier blocks, cyclotomic Hecke algebras, families of characters.

However, it was recently realized that the algorithm given in [Broué and Kim 2002] works only in the case where d is the power of a prime number. The aim of this paper is to give a complete description of the Rouquier blocks of the cyclotomic Ariki–Koike algebras of the group $G(d, 1, r)$. In order to achieve that, we use the theory of essential hyperplanes introduced in [Chlouveraki 2007]. According to this theory, the Rouquier blocks of the cyclotomic Hecke algebras of any complex reflection group depend on numerical data determined by the generic Hecke algebra, the essential hyperplanes of the group. Thanks to Theorem 2.15, it suffices to study the blocks of the generic Hecke algebra in a finite number of cases in order to obtain the Rouquier blocks for all cyclotomic Hecke algebras.

An algorithm for the blocks of the Ariki–Koike algebras of $G(d, 1, r)$ over any field has been given in [Lyle and Mathas 2007]. This algorithm can be applied to give us the Rouquier blocks of the cyclotomic Ariki–Koike algebras and we use it to obtain a characterization in the combinatorial terms used in [Broué and Kim 2002]. Our main result is Theorem 3.18, which determines completely the Rouquier blocks of the cyclotomic Ariki–Koike algebras. The most important consequence is that we can obtain the Rouquier blocks of a cyclotomic Ariki–Koike algebra of $G(d, 1, r)$ from the families of characters of the Weyl groups of type B_n, $n \leq r$, already determined by Lusztig. This result can also be deduced from the Morita equivalences established in [Dipper and Mathas 2002]. Moreover, we show that the Rouquier blocks in the important case of the spetsial cyclotomic Hecke algebra are the ones given by the algorithm of [Broué and Kim 2002].

Finally, in the case of the Weyl groups, Lusztig attaches to every irreducible character two integers, denoted by a and A, and shows [1987, 3.3 and 3.4] that they are constant on the families. In an analogue way, we can define integers a and A attached to every irreducible character of a cyclotomic Hecke algebra of a complex reflection group. Proposition 3.21 completes the proof of the result in [Broué and Kim 2002, 3.18] to the effect that the integers a and A are constant on the Rouquier blocks of $G(d, 1, r)$. The same result has been obtained by the author for the exceptional complex reflection groups in [Chlouveraki 2008].

1. Blocks and symmetric algebras

For proofs of results not given in this section, see [Broué and Kim 2002] or [Chlouveraki 2007, Chapter 2].

Generalities. Assume that $\mathcal{O}$ is a commutative integral domain with field of fractions F and A is an $\mathcal{O}$-algebra, free and finitely generated as an $\mathcal{O}$-module. We denote by ZA the center of A.

Definition 1.1. The *block-idempotents*, or simply *blocks*, of A are the primitive idempotents of ZA.

Let K be a field extension of F and suppose the K-algebra $KA := K \otimes_{\mathcal{O}} A$ is semisimple. By assumption, KA is isomorphic to a direct product of simple algebras:

$$KA \simeq \prod_{\chi \in \mathrm{Irr}(KA)} M_\chi,$$

where $\mathrm{Irr}(KA)$ denotes the set of irreducible characters of KA and M_χ is a simple K-algebra.

For all $\chi \in \mathrm{Irr}(KA)$, we denote by $\pi_\chi : KA \twoheadrightarrow M_\chi$ the projection onto the χ-factor and by e_χ the element of KA such that:

$$\pi_{\chi'}(e_\chi) = \begin{cases} 1_{M_\chi} & \text{if } \chi = \chi', \\ 0 & \text{if } \chi \neq \chi'. \end{cases}$$

The blocks of the algebra KA are related to those of A as follows.

Theorem 1.2. (1) *We have* $1 = \sum_{\chi \in \mathrm{Irr}(KA)} e_\chi$ *and the set* $\{e_\chi\}_{\chi \in \mathrm{Irr}(KA)}$ *is the set of all the blocks of the algebra* KA.

(2) *There exists a unique partition* $\mathrm{Bl}(A)$ *of* $\mathrm{Irr}(KA)$ *such that*

(a) *For all* $B \in \mathrm{Bl}(A)$, *the idempotent* $e_B := \sum_{\chi \in B} e_\chi$ *is a block of* A.

(b) *We have* $1 = \sum_{B \in \mathrm{Bl}(A)} e_B$ *and for every central idempotent* e *of* A, *there exists a subset* $\mathrm{Bl}(A, e)$ *of* $\mathrm{Bl}(A)$ *such that*

$$e = \sum_{B \in \mathrm{Bl}(A,e)} e_B.$$

In particular the set $\{e_B\}_{B \in \mathrm{Bl}(A)}$ *is the set of all the blocks of* A.

If $\chi \in B$ for some $B \in \mathrm{Bl}(A)$, we say that χ *belongs to the block* e_B.

Remark. For all $B \in \mathrm{Bl}(A)$, we have $KAe_B \simeq \prod_{\chi \in B} M_\chi$.

Assumptions 1.3. From now on, we make the following assumptions:

(int) The ring $\mathcal{O}$ is a Noetherian and integrally closed domain with field of fractions F and A is an $\mathcal{O}$-algebra which is free and finitely generated as an $\mathcal{O}$-module.

(spl) The field K is a finite Galois extension of F and the algebra KA is split (i.e., for every simple KA-module V, $\mathrm{End}_{KA}(V) \simeq K$) semisimple.

We denote by $\mathcal{O}_K$ the integral closure of $\mathcal{O}$ in K.

Blocks and integral closure. The Galois group $\mathrm{Gal}(K/F)$ acts on $KA = K \otimes_{\mathcal{O}} A$ (viewed as an F-algebra) as follows: if $\sigma \in \mathrm{Gal}(K/F)$ and $\lambda \otimes a \in KA$, then $\sigma(\lambda \otimes a) := \sigma(\lambda) \otimes a$.

If V is a K-vector space and $\sigma \in \mathrm{Gal}(K/F)$, we denote by ${}^\sigma V$ the K-vector space defined on the additive group V with multiplication $\lambda.v := \sigma^{-1}(\lambda)v$ for all

$\lambda \in K$ and $v \in V$. If $\rho : KA \to \mathrm{End}_K(V)$ is a representation of the K-algebra KA, its composition with the action of σ^{-1} is also a representation ${}^\sigma\rho : KA \to \mathrm{End}_K({}^\sigma V)$:

$$KA \xrightarrow{\sigma^{-1}} KA \xrightarrow{\rho} \mathrm{End}_K(V).$$

We denote by ${}^\sigma\chi$ the character of ${}^\sigma\rho$ and we define the action of $\mathrm{Gal}(K/F)$ on $\mathrm{Irr}(KA)$ as follows: if $\sigma \in \mathrm{Gal}(K/F)$ and $\chi \in \mathrm{Irr}(KA)$, then

$$\sigma(\chi) := {}^\sigma\chi = \sigma \circ \chi \circ \sigma^{-1}.$$

This operation induces an action of $\mathrm{Gal}(K/F)$ on the set of blocks of KA:

$$\sigma(e_\chi) = e_{{}^\sigma\chi} \quad \text{for all } \sigma \in \mathrm{Gal}(K/F),\ \chi \in \mathrm{Irr}(KA).$$

Hence, the group $\mathrm{Gal}(K/F)$ acts on the set of idempotents of $Z\mathcal{O}_K A$ and thus on the set of blocks of $\mathcal{O}_K A$. Since $F \cap \mathcal{O}_K = \mathcal{O}$, the idempotents of ZA are the idempotents of $Z\mathcal{O}_K A$ which are fixed by the action of $\mathrm{Gal}(K/F)$. As a consequence, the primitive idempotents of ZA are sums of the elements of the orbits of $\mathrm{Gal}(K/F)$ on the set of primitive idempotents of $Z\mathcal{O}_K A$. Thus, the blocks of A are in bijection with the orbits of $\mathrm{Gal}(K/F)$ on the set of blocks of $\mathcal{O}_K A$. The following proposition is just a reformulation of this result.

Proposition 1.4. (1) *Let B be a block of A and B' a block of $\mathcal{O}_K A$ contained in B. If $\mathrm{Gal}(K/F)_{B'}$ denotes the stabilizer of B' in $\mathrm{Gal}(K/F)$, then*

$$B = \bigcup_{\sigma \in \mathrm{Gal}(K/F)/\mathrm{Gal}(K/F)_{B'}} \sigma(B'), \quad \textit{that is,} \quad e_B = \sum_{\sigma \in \mathrm{Gal}(K/F)/\mathrm{Gal}(K/F)_{B'}} \sigma(e_{B'}).$$

(2) *Two characters $\chi, \psi \in \mathrm{Irr}(KA)$ are in the same block of A if and only if there exists $\sigma \in \mathrm{Gal}(K/F)$ such that $\sigma(\chi)$ and ψ belong to the same block of $\mathcal{O}_K A$.*

Remark. For all $\chi \in B'$, we have $\mathrm{Gal}(K/F)_\chi \subseteq \mathrm{Gal}(K/F)_{B'}$.

Part (2) of the proposition allows us to transfer the problem of the classification of the blocks of A to that of the classification of the blocks of $\mathcal{O}_K A$.

Blocks and prime ideals. We denote by $\mathrm{Spec}(\mathcal{O})$ the set of prime ideals of $\mathcal{O}$. Since $\mathcal{O}$ is Noetherian and integrally closed, we have

$$\mathcal{O} = \bigcap_{\mathfrak{p} \in \mathrm{Spec}(\mathcal{O})} \mathcal{O}_\mathfrak{p},$$

where $\mathcal{O}_\mathfrak{p} := \{x \in F \mid (\exists a \in \mathcal{O} - \mathfrak{p})(ax \in \mathcal{O})\}$ is the localization of $\mathcal{O}$ at $\mathfrak{p}$.

Let $\mathfrak{p}$ be a prime ideal of $\mathcal{O}$ and $\mathcal{O}_\mathfrak{p} A := \mathcal{O}_\mathfrak{p} \otimes_\mathcal{O} A$. The blocks of $\mathcal{O}_\mathfrak{p} A$ are the $\mathfrak{p}$-*blocks of* A. If $\chi, \psi \in \mathrm{Irr}(KA)$ belong to the same block of $\mathcal{O}_\mathfrak{p} A$, we write $\chi \sim_\mathfrak{p} \psi$.

Proposition 1.5. *Two characters $\chi, \psi \in \mathrm{Irr}(KA)$ belong to the same block of A if and only if there exist a finite sequence $\chi_0, \chi_1, \ldots, \chi_n \in \mathrm{Irr}(KA)$ and a finite sequence $\mathfrak{p}_1, \ldots, \mathfrak{p}_n \in \mathrm{Spec}(\mathcal{O})$ such that*

$$\chi_0 = \chi, \quad \chi_n = \psi, \quad \textit{and} \quad \chi_{j-1} \sim_{\mathfrak{p}_j} \chi_j \quad \textit{for } 1 \le j \le n.$$

Blocks and residue blocks. Let $\mathfrak{p}$ be a maximal ideal of $\mathcal{O}$ and set $k_{\mathfrak{p}} := \mathcal{O}/\mathfrak{p}$ its residue field. If $\mathcal{O}_{\mathfrak{p}}$ is the localization of $\mathcal{O}$ at $\mathfrak{p}$, then $k_{\mathfrak{p}}$ is also the residue field of $\mathcal{O}_{\mathfrak{p}}$. The natural surjection $\pi_{\mathfrak{p}} : \mathcal{O}_{\mathfrak{p}} \twoheadrightarrow k_{\mathfrak{p}}$ extends to a morphism $\pi_{\mathfrak{p}} : \mathcal{O}_{\mathfrak{p}}A \twoheadrightarrow k_{\mathfrak{p}}A$, which in turn induces a morphism

$$\pi_{\mathfrak{p}} : Z\mathcal{O}_{\mathfrak{p}}A \to Zk_{\mathfrak{p}}A.$$

The following lemma will serve for the proof of Proposition 1.7.

Lemma 1.6. *Let e be an idempotent of $\mathcal{O}_{\mathfrak{p}}A$ whose image $\bar{e}$ in $k_{\mathfrak{p}}A$ is central. Then e is central.*

Proof. Set $R := \mathcal{O}_{\mathfrak{p}}A$. Since $\bar{e}$ is central, we have $\bar{e}k_{\mathfrak{p}}A(1-\bar{e}) = (1-\bar{e})k_{\mathfrak{p}}A\bar{e} = \{0\}$, i.e., $eR(1-e) \subseteq \mathfrak{p}R$ and $(1-e)Re \subseteq \mathfrak{p}R$. Since e and $(1-e)$ are idempotents, we get $eR(1-e) \subseteq \mathfrak{p}eR(1-e)$ and $(1-e)Re \subseteq \mathfrak{p}(1-e)Re$. By Nakayama's lemma, $eR(1-e) = (1-e)Re = \{0\}$. Thus, from

$$R = eRe \oplus eR(1-e) \oplus (1-e)Re \oplus (1-e)R(1-e)$$

we deduce that $R = eRe \oplus (1-e)R(1-e)$. Consequently, e is central. □

Proposition 1.7. *If $\mathcal{O}_{\mathfrak{p}}$ is a discrete valuation ring and $K = F$, then the morphism*

$$\pi_{\mathfrak{p}} : Z\mathcal{O}_{\mathfrak{p}}A \to Zk_{\mathfrak{p}}A$$

induces a bijection between the set of blocks of $\mathcal{O}_{\mathfrak{p}}A$ and the set of blocks of $k_{\mathfrak{p}}A$.

Proof. From now on, the symbol $\widehat{\ }$ will stand for $\mathfrak{p}$-adic completion. Clearly $\pi_{\mathfrak{p}}$ sends a block of $\mathcal{O}_{\mathfrak{p}}A$ to a sum of blocks of $k_{\mathfrak{p}}A$. Now let $\bar{e}$ be a block of $k_{\mathfrak{p}}A$. By the idempotent lifting theorems [Thévenaz 1995, Theorem 3.2] and the preceding lemma, $\bar{e}$ is lifted to a sum of central primitive idempotents in $\hat{\mathcal{O}}_{\mathfrak{p}}A$. However, since KA is split semisimple, the blocks of $\hat{\mathcal{O}}_{\mathfrak{p}}A$ belong to KA. But $K \cap \hat{\mathcal{O}}_{\mathfrak{p}} = \mathcal{O}_{\mathfrak{p}}$ (see [Nagata 1962, 18.4], for example) and $\mathcal{O}_{\mathfrak{p}}A \cap Z\hat{\mathcal{O}}_{\mathfrak{p}}A \subseteq Z\mathcal{O}_{\mathfrak{p}}A$. Therefore, $\bar{e}$ is lifted to a sum of blocks in $\mathcal{O}_{\mathfrak{p}}A$ and this provides the block bijection. □

Symmetric algebras. Let $\mathcal{O}$ be a ring and let A be an $\mathcal{O}$-algebra. We still suppose that the assumptions 1.3 are satisfied.

Definition 1.8. A *trace function* on A is an $\mathcal{O}$-linear map $t : A \to \mathcal{O}$ such that $t(ab) = t(ba)$ for all $a, b \in A$.

Definition 1.9. We say that a trace function $t : A \to \mathbb{O}$ is a symmetrizing form on A or that A is a symmetric algebra if the morphism

$$\hat{t} : A \to \mathrm{Hom}_{\mathbb{O}}(A, \mathbb{O}),\ \ a \mapsto (x \mapsto \hat{t}(a)(x) := t(ax))$$

is an isomorphism of A-modules-A.

Example 1.10. In the case where $\mathbb{O} = \mathbb{Z}$ and $A = \mathbb{Z}[G]$ (G a finite group), we can define the following symmetrizing (or *canonical*) form on A

$$t : \mathbb{Z}[G] \to \mathbb{Z},\ \ \sum_{g \in G} a_g g \mapsto a_1,$$

where $a_g \in \mathbb{Z}$ for all $g \in G$.

If $\tau : A \to \mathbb{O}$ is a linear form, we denote by $\tau^\vee$ its inverse image by the isomorphism $\hat{t}$, i.e., $\tau^\vee$ is the element of A such that

$$t(\tau^\vee a) = \tau(a) \text{ for all } a \in A.$$

Lemma 1.11 (see [Geck and Pfeiffer 2000, §7.1], for example).

(1) *τ is a trace function if and only if $\tau^\vee \in ZA$.*

(2) *Let $(e_i)_{i \in I}$ be a basis of A over $\mathbb{O}$ and $(e'_i)_{i \in I}$ the dual basis with respect to t, so $t(e_i e'_j) = \delta_{ij}$. We have $\tau^\vee = \sum_i \tau(e'_i)e_i = \sum_i \tau(e_i)e'_i$ and more generally, for all $a \in A$, we have $\tau^\vee a = \sum_i \tau(e'_i a)e_i = \sum_i \tau(e_i a)e'_i$.*

Schur elements. If A is a symmetric algebra with a symmetrizing form t, we obtain a symmetrizing form t^K on KA by extension of scalars. Every irreducible character $\chi \in \mathrm{Irr}(KA)$ is a trace function on KA and thus we can define $\chi^\vee \in ZKA$. Since KA is a split semisimple K-algebra, we have that $KA \simeq \prod_{\chi \in \mathrm{Irr}(KA)} M_\chi$, where M_χ is a matrix algebra isomorphic to $\mathrm{Mat}_{\chi(1)}(K)$. The map $\pi_\chi : KA \twoheadrightarrow M_\chi$, restricted to ZKA, defines a map $\omega_\chi : ZKA \twoheadrightarrow K$.

Definition 1.12. For all $\chi \in \mathrm{Irr}(KA)$, the *Schur element* of χ with respect to t, denoted by s_χ, is the element of K defined by

$$s_\chi := \omega_\chi(\chi^\vee).$$

Schur elements lie in the integral closure:

Proposition 1.13 [Geck and Pfeiffer 2000, §7.2]. *For all $\chi \in \mathrm{Irr}(KA)$, $s_\chi \in {\mathbb{O}_K}^*$.*

Example 1.14. Let $\mathbb{O} := \mathbb{Z}$, $A := \mathbb{Z}[G]$ (G a finite group) and t the canonical symmetrizing form. If K is an algebraically closed field of characteristic 0, then KA is a split semisimple algebra and $s_\chi = |G|/\chi(1)$ for all $\chi \in \mathrm{Irr}(KA)$. Because of the integrality of the Schur elements, we must have $|G|/\chi(1) \in \mathbb{Z} = \mathbb{Z}_K \cap \mathbb{Q}$ for all $\chi \in \mathrm{Irr}(KA)$. Thus, we have shown that $\chi(1)$ divides $|G|$.

The following properties of the Schur elements can be derived easily from the above (see also [Broué 1991; Geck 1993; Geck and Pfeiffer 2000; Geck and Rouquier 1997; Broué et al. 1999]).

Proposition 1.15.

(1)
$$t = \sum_{\chi \in \mathrm{Irr}(KA)} \frac{1}{s_\chi} \chi.$$

(2) *For all* $\chi \in \mathrm{Irr}(KA)$, *the central primitive idempotent associated with* χ *is*
$$e_\chi = \frac{1}{s_\chi} \chi^\vee.$$

2. Hecke algebras of complex reflection groups

Generic Hecke algebras. Let μ_∞ be the group of all the roots of unity in $\mathbb{C}$ and K a number field contained in $\mathbb{Q}(\mu_\infty)$. We denote by $\mu(K)$ the group of all the roots of unity of K. For every integer $d > 1$, we set $\zeta_d := \exp(2\pi i/d)$ and denote by μ_d the group of all the d-th roots of unity.

Let V be a K-vector space of finite dimension r. Let W be a finite subgroup of $\mathrm{GL}(V)$ generated by (pseudo-)reflections acting irreducibly on V. Let us denote by $\mathcal{A}$ the set of the reflecting hyperplanes of W. We set $\mathcal{M} := \mathbb{C} \otimes V - \bigcup_{H \in \mathcal{A}} \mathbb{C} \otimes H$. For $x_0 \in \mathcal{M}$, let $P := \Pi_1(\mathcal{M}, x_0)$ and $B := \Pi_1(\mathcal{M}/W, x_0)$. Then there exists a short exact sequence (cf. [Broué et al. 1998], §2B):

$$\{1\} \to P \to B \to W \to \{1\}.$$

We denote by τ the central element of P defined by the loop

$$[0, 1] \to \mathcal{M}, \qquad t \mapsto \exp(2\pi i t) x_0.$$

For every orbit $\mathcal{C}$ of W on $\mathcal{A}$, we denote by $e_{\mathcal{C}}$ the common order of the subgroups W_H, where H is any element of $\mathcal{C}$ and W_H the subgroup formed by id_V and all the reflections fixing the hyperplane H.

We choose a set of indeterminates $\boldsymbol{u} = (u_{\mathcal{C},j})_{(\mathcal{C} \in \mathcal{A}/W)(0 \le j \le e_{\mathcal{C}}-1)}$ and we denote by $\mathbb{Z}[\boldsymbol{u}, \boldsymbol{u}^{-1}]$ the Laurent polynomial ring in all the indeterminates $\boldsymbol{u}$. We define the *generic Hecke algebra* $\mathcal{H}$ of W to be the quotient of the group algebra $\mathbb{Z}[\boldsymbol{u}, \boldsymbol{u}^{-1}]B$ by the ideal generated by the elements of the form

$$(\boldsymbol{s} - u_{\mathcal{C},0})(\boldsymbol{s} - u_{\mathcal{C},1}) \ldots (\boldsymbol{s} - u_{\mathcal{C},e_{\mathcal{C}}-1}),$$

where $\mathcal{C}$ runs over the set $\mathcal{A}/W$ and $\boldsymbol{s}$ runs over the set of monodromy generators around the images in $\mathcal{M}/W$ of the elements of the hyperplane orbit $\mathcal{C}$.

Example 2.1. Let $W := G_4 = \langle s, t \mid sts = tst, s^3 = t^3 = 1 \rangle$. Then s and t are conjugate in W and their reflecting hyperplanes belong to the same orbit in $\mathcal{A}/W$.

The generic Hecke algebra of W has the presentation

$$\mathcal{H}(G_4) = \big\langle S, T \mid STS = TST,\\ (S-u_0)(S-u_1)(S-u_2) = 0,\ (T-u_0)(T-u_1)(T-u_2) = 0\big\rangle.$$

We make some assumptions for the algebra $\mathcal{H}$. They have been verified for all but a finite number of irreducible complex reflection groups; see [Broué et al. 1999, remarks before 1.17 and §2] and [Geck et al. 2000].

Assumptions 2.2. The algebra $\mathcal{H}$ is a free $\mathbb{Z}[\boldsymbol{u}, \boldsymbol{u}^{-1}]$-module of rank $|W|$. Moreover, there exists a linear form $t : \mathcal{H} \to \mathbb{Z}[\boldsymbol{u}, \boldsymbol{u}^{-1}]$ with the following properties:

(1) t is a symmetrizing form for $\mathcal{H}$.

(2) Via the specialization $u_{\mathcal{C},j} \mapsto \zeta_{e_{\mathcal{C}}}^{j}$, the form t becomes the canonical symmetrizing form on the group algebra $\mathbb{Z}W$.

(3) If we denote by $\alpha \mapsto \alpha^*$ the automorphism of $\mathbb{Z}[\boldsymbol{u}, \boldsymbol{u}^{-1}]$ consisting of the simultaneous inversion of the indeterminates, then for all $b \in B$, we have
$$t(b^{-1})^* = \frac{t(b\tau)}{t(\tau)}.$$

We know from [Broué et al. 1999, 2.1] that the form t is unique. From now on we suppose that the assumptions 2.2 are satisfied.

Theorem 2.3 [Malle 1999, 5.2]. *Let*
$$\boldsymbol{v} = (v_{\mathcal{C},j})_{(\mathcal{C}\in\mathcal{A}/W)(0\le j\le e_{\mathcal{C}}-1)}$$
be a set of $\sum_{\mathcal{C}\in\mathcal{A}/W} e_{\mathcal{C}}$ indeterminates such that $v_{\mathcal{C},j}^{|\mu(K)|} = \zeta_{e_{\mathcal{C}}}^{-j} u_{\mathcal{C},j}$ for every $\mathcal{C}$ and j. Then the $K(\boldsymbol{v})$-algebra $K(\boldsymbol{v})\mathcal{H}$ is split semisimple.

By the Tits deformation theorem (see [Broué et al. 1999, 7.2], for example), it follows that the specialization $v_{\mathcal{C},j} \mapsto 1$ induces a bijection $\chi \mapsto \chi_{\boldsymbol{v}}$ from the set $\mathrm{Irr}(K(\boldsymbol{v})\mathcal{H})$ of absolutely irreducible characters of $K(\boldsymbol{v})\mathcal{H}$ to the set $\mathrm{Irr}(W)$ of absolutely irreducible characters of W, such that the following diagram is commutative:

$$\begin{array}{cccc} \chi_{\boldsymbol{v}}: & \mathcal{H} & \to & \mathbb{Z}_K[\boldsymbol{v}, \boldsymbol{v}^{-1}] \\ & \downarrow & & \downarrow \\ \chi: & \mathbb{Z}_K W & \to & \mathbb{Z}_K. \end{array}$$

The following result concerning the form of the Schur elements associated with the irreducible characters of $K(\boldsymbol{v})\mathcal{H}$ is proved using a case by case analysis.

Theorem 2.4 [Chlouveraki 2007, Theorem 3.2.5]. *The Schur element $s_\chi(\boldsymbol{v})$ associated with the character $\chi_{\boldsymbol{v}}$ of $K(\boldsymbol{v})\mathcal{H}$ is an element of $\mathbb{Z}_K[\boldsymbol{v}, \boldsymbol{v}^{-1}]$ of the form*
$$s_\chi(\boldsymbol{v}) = \xi_\chi N_\chi \prod_{i\in I_\chi} \Psi_{\chi,i}(M_{\chi,i})^{n_{\chi,i}}$$

where

- ξ_χ *is an element of* $\mathbb{Z}_K$,
- $N_\chi = \prod_{\mathcal{C},j} v_{\mathcal{C},j}^{b_{\mathcal{C},j}}$ *is a monomial in* $\mathbb{Z}_K[\boldsymbol{v}, \boldsymbol{v}^{-1}]$ *such that* $\sum_{j=0}^{e_{\mathcal{C}}-1} b_{\mathcal{C},j} = 0$ *for all* $\mathcal{C} \in \mathcal{A}/W$,
- I_χ *is an index set*,
- $(\Psi_{\chi,i})_{i \in I_\chi}$ *is a family of K-cyclotomic polynomials in one variable (i.e., minimal polynomials of the roots of unity over K)*,
- $(M_{\chi,i})_{i \in I_\chi}$ *is a family of monomials in* $\mathbb{Z}_K[\boldsymbol{v}, \boldsymbol{v}^{-1}]$ *and if* $M_{\chi,i} = \prod_{\mathcal{C},j} v_{\mathcal{C},j}^{a_{\mathcal{C},j}}$, *then* $\gcd(a_{\mathcal{C},j}) = 1$ *and* $\sum_{j=0}^{e_{\mathcal{C}}-1} a_{\mathcal{C},j} = 0$ *for all* $\mathcal{C} \in \mathcal{A}/W$,
- $(n_{\chi,i})_{i \in I_\chi}$ *is a family of positive integers.*

This factorization is unique in $K[\boldsymbol{v}, \boldsymbol{v}^{-1}]$. *Moreover, the monomials* $(M_{\chi,i})_{i \in I_\chi}$ *are unique up to inversion, whereas the coefficient* ξ_χ *is unique up to multiplication by a root of unity.*

Remark. The bijection $\mathrm{Irr}(K(\boldsymbol{v})\mathcal{H}) \leftrightarrow \mathrm{Irr}(W)$, $\chi_{\boldsymbol{v}} \mapsto \chi$ implies that the specialization $v_{\mathcal{C},j} \mapsto 1$ sends $s_{\chi_{\boldsymbol{v}}}$ to $|W|/\chi(1)$ (which is the Schur element of χ in the group algebra with respect to the canonical symmetrizing form).

Let $A := \mathbb{Z}_K[\boldsymbol{v}, \boldsymbol{v}^{-1}]$ and $\mathfrak{p}$ be a prime ideal of $\mathbb{Z}_K$.

Definition 2.5. Let $M = \prod_{\mathcal{C},j} v_{\mathcal{C},j}^{a_{\mathcal{C},j}}$ be a monomial in A such that $\gcd(a_{\mathcal{C},j}) = 1$. We say that M is $\mathfrak{p}$-essential for a character $\chi \in \mathrm{Irr}(W)$, if there exists a K-cyclotomic polynomial Ψ such that

$$\Psi(M) \text{ divides } s_\chi(\boldsymbol{v}) \quad \text{and} \quad \Psi(1) \in \mathfrak{p}.$$

We say that M is $\mathfrak{p}$-essential for W, if there exists a character $\chi \in \mathrm{Irr}(W)$ such that M is $\mathfrak{p}$-essential for χ.

The next result gives a characterization of $\mathfrak{p}$-essential monomials, which plays an essential role in the proof of Theorem 2.15.

Proposition 2.6 [Chlouveraki 2007, Proposition 3.2.6]. *Let* $M = \prod_{\mathcal{C},j} v_{\mathcal{C},j}^{a_{\mathcal{C},j}}$ *be a monomial in A such that* $\gcd(a_{\mathcal{C},j}) = 1$. *We set* $\mathfrak{q}_M := (M-1)A + \mathfrak{p}A$. *Then*

(1) *The ideal* $\mathfrak{q}_M$ *is a prime ideal of A.*

(2) *M is* $\mathfrak{p}$-*essential for* $\chi \in \mathrm{Irr}(W)$ *if and only if* $s_\chi(\boldsymbol{v})/\xi_\chi \in \mathfrak{q}_M$.

Cyclotomic Hecke algebras. Let y be an indeterminate. We set $x := y^{|\mu(K)|}$.

Definition 2.7. A cyclotomic specialization of $\mathcal{H}$ is a $\mathbb{Z}_K$-algebra morphism $\phi : \mathbb{Z}_K[\boldsymbol{v}, \boldsymbol{v}^{-1}] \to \mathbb{Z}_K[y, y^{-1}]$ with the following properties:

- $\phi : v_{\mathcal{C},j} \mapsto y^{n_{\mathcal{C},j}}$ where $n_{\mathcal{C},j} \in \mathbb{Z}$ for all $\mathcal{C}$ and j.

- For all $\mathcal{C} \in \mathcal{A}/W$, if z is another indeterminate, the element of $\mathbb{Z}_K[y, y^{-1}, z]$ defined by
$$\Gamma_{\mathcal{C}}(y, z) := \prod_{j=0}^{e_{\mathcal{C}}-1} (z - \zeta_{e_{\mathcal{C}}}^j y^{n_{\mathcal{C},j}})$$
is invariant under the action of $\mathrm{Gal}(K(y)/K(x))$.

If ϕ is a cyclotomic specialization of $\mathcal{H}$, the corresponding *cyclotomic Hecke algebra* is the $\mathbb{Z}_K[y, y^{-1}]$-algebra, denoted by $\mathcal{H}_\phi$, which is obtained as the specialization of the $\mathbb{Z}_K[\boldsymbol{v}, \boldsymbol{v}^{-1}]$-algebra $\mathcal{H}$ via the morphism ϕ. It also has a symmetrizing form t_ϕ defined as the specialization of the canonical form t.

Remark. Sometimes we describe the morphism ϕ by the formula
$$u_{\mathcal{C},j} \mapsto \zeta_{e_{\mathcal{C}}}^j x^{n_{\mathcal{C},j}}.$$
If we now set $q := \zeta x$ for some root of unity $\zeta \in \mu(K)$, then the cyclotomic specialization ϕ becomes a ζ-cyclotomic specialization and $\mathcal{H}_\phi$ can be also considered over $\mathbb{Z}_K[q, q^{-1}]$.

Example 2.8. The spetsial Hecke algebra $\mathcal{H}_q^s(W)$ is the 1-cyclotomic algebra obtained by the specialization
$$u_{\mathcal{C},0} \mapsto q, \;\; u_{\mathcal{C},j} \mapsto \zeta_{e_{\mathcal{C}}}^j \text{ for } 1 \le j \le e_{\mathcal{C}} - 1, \;\text{ for all } \mathcal{C} \in \mathcal{A}/W.$$
For example, if $W := G_4$, then
$$\mathcal{H}_q^s(W) = \langle S, T \mid STS = TST, (S-q)(S^2+S+1) = (T-q)(T^2+T+1) = 0 \rangle.$$

Proposition 2.9 [Chlouveraki 2007, remarks following Theorem 3.3.3]. *The algebra $K(y)\mathcal{H}_\phi$ is split semisimple.*

When y specializes to 1, the algebra $K(y)\mathcal{H}_\phi$ specializes to the group algebra KW (the form t_ϕ becoming the canonical form on the group algebra). Thus, by the Tits deformation theorem, the specialization $v_{\mathcal{C},j} \mapsto 1$ defines the bijections
$$\begin{array}{ccccc} \mathrm{Irr}(K(\boldsymbol{v})\mathcal{H}) & \leftrightarrow & \mathrm{Irr}(K(y)\mathcal{H}_\phi) & \leftrightarrow & \mathrm{Irr}(W) \\ \chi_{\boldsymbol{v}} & \mapsto & \chi_\phi & \mapsto & \chi. \end{array}$$

The following result is an immediate consequence of Theorem 2.4.

Proposition 2.10. *The Schur element $s_{\chi_\phi}(y)$ associated with the irreducible character χ_ϕ of $K(y)\mathcal{H}_\phi$ is a Laurent polynomial in y of the form*
$$s_{\chi_\phi}(y) = \psi_{\chi,\phi} y^{a_{\chi,\phi}} \prod_{\Phi \in C_K} \Phi(y)^{n_{\chi,\phi}}$$
where $\psi_{\chi,\phi} \in \mathbb{Z}_K$, $a_{\chi,\phi} \in \mathbb{Z}$, $n_{\chi,\phi} \in \mathbb{N}$ and C_K is a set of K-cyclotomic polynomials.

Rouquier blocks of the cyclotomic Hecke algebras.

Definition 2.11. The *Rouquier ring of* K, denoted by $\mathcal{R}_K(y)$, is the $\mathbb{Z}_K$-subalgebra of $K(y)$ given by

$$\mathcal{R}_K(y) := \mathbb{Z}_K\big[y, y^{-1}, (y^n - 1)^{-1}_{n\geq 1}\big].$$

Let $\phi : v_{\mathcal{C},j} \mapsto y^{n_{\mathcal{C},j}}$ be a cyclotomic specialization and $\mathcal{H}_\phi$ the corresponding cyclotomic Hecke algebra. The *Rouquier blocks* of $\mathcal{H}_\phi$ are the blocks of the algebra $\mathcal{R}_K(y)\mathcal{H}_\phi$.

Remark. Rouquier [1999] showed that if W is a Weyl group and $\mathcal{H}_\phi$ is obtained via the spetsial cyclotomic specialization (see Example 2.8), then its Rouquier blocks coincide with the families of characters defined by Lusztig. Thus, the Rouquier blocks play an essential role in the program Spets [Broué et al. 1999], whose ambition is to give to complex reflection groups the role of Weyl groups of as yet mysterious structures.

The Rouquier ring has the following interesting properties.

Proposition 2.12 [Chlouveraki 2007, Proposition 3.4.2].

(1) *The group of units* $\mathcal{R}_K(y)^\times$ *of the Rouquier ring* $\mathcal{R}_K(y)$ *consists of the elements of the form*

$$uy^n \prod_{\Phi\in \mathrm{Cycl}(K)} \Phi(y)^{n_\phi},$$

where $u \in \mathbb{Z}_K^\times$, $n, n_\phi \in \mathbb{Z}$, $\mathrm{Cycl}(K)$ *is the set of* K*-cyclotomic polynomials and* $n_\phi = 0$ *for all but a finite number of* Φ.

(2) *The prime ideals of* $\mathcal{R}_K(y)$ *are*
- *the zero ideal* $\{0\}$,
- *the ideals of the form* $\mathfrak{p}\mathcal{R}_K(y)$, *where* $\mathfrak{p}$ *is a prime ideal of* $\mathbb{Z}_K$,
- *the ideals of the form* $P(y)\mathcal{R}_K(y)$, *where* $P(y)$ *is an irreducible element of* $\mathbb{Z}_K[y]$ *of degree at least* 1, *prime to* y *and to* $\Phi(y)$ *for all* $\Phi \in \mathrm{Cycl}(K)$.

(3) *The Rouquier ring* $\mathcal{R}_K(y)$ *is a Dedekind ring.*

Now recall the form of the Schur elements of the cyclotomic Hecke algebra $\mathcal{H}_\phi$ given in Proposition 2.10. If χ_ϕ is an irreducible character of $K(y)\mathcal{H}_\phi$, its Schur element $s_{\chi_\phi}(y)$ is of the form

$$s_{\chi_\phi}(y) = \psi_{\chi,\phi}\, y^{a_{\chi,\phi}} \prod_{\Phi\in C_K} \Phi(y)^{n_{\chi,\phi}}$$

where $\psi_{\chi,\phi} \in \mathbb{Z}_K$, $a_{\chi,\phi} \in \mathbb{Z}$, $n_{\chi,\phi} \in \mathbb{N}$ and C_K is a set of K-cyclotomic polynomials.

Definition 2.13. A prime ideal $\mathfrak{p}$ of $\mathbb{Z}_K$ lying over a prime number p is ϕ-bad for W, if there exists $\chi_\phi \in \mathrm{Irr}(K(y)\mathcal{H}_\phi)$ with $\psi_{\chi,\phi} \in \mathfrak{p}$. If $\mathfrak{p}$ is ϕ-bad for W, we say that p is a ϕ-bad prime number for W.

Remark. If W is a Weyl group and ϕ is the spetsial cyclotomic specialization, then the ϕ-bad prime ideals are the ideals generated by the bad prime numbers (in the usual sense) for W (see [Geck and Rouquier 1997, 5.2]).

Note that if $\mathfrak{p}$ is ϕ-bad for W, then p must divide the order of the group (since $s_{\chi_\phi}(1) = |W|/\chi(1)$).

Denote by $\mathcal{O}$ the Rouquier ring. By Proposition 1.5, the Rouquier blocks of $\mathcal{H}_\phi$ are unions of the blocks of $\mathcal{O}_\mathcal{P}\mathcal{H}_\phi$ for all prime ideals $\mathcal{P}$ of $\mathcal{O}$. However, in all of the following cases, due to the form of the Schur elements, the blocks of $\mathcal{O}_\mathcal{P}\mathcal{H}_\phi$ are singletons (i.e., $e_{\chi_\phi} = \chi_\phi^\vee / s_{\chi_\phi} \in \mathcal{O}_\mathcal{P}\mathcal{H}_\phi$ for all $\chi_\phi \in \mathrm{Irr}(K(y)\mathcal{H}_\phi)$):

- $\mathcal{P}$ is the zero ideal $\{0\}$.
- $\mathcal{P}$ is of the form $P(y)\mathcal{O}$, where $P(y)$ is an irreducible element of $\mathbb{Z}_K[y]$ of degree at least 1, prime to y and to $\Phi(y)$ for all $\Phi \in \mathrm{Cycl}(K)$.
- $\mathcal{P}$ is of the form $\mathfrak{p}\mathcal{O}$, where $\mathfrak{p}$ is a prime ideal of $\mathbb{Z}_K$ which is not ϕ-bad for W.

Therefore, applying Proposition 1.5, we obtain:

Proposition 2.14. *Two characters $\chi, \psi \in \mathrm{Irr}(W)$ are in the same Rouquier block of $\mathcal{H}_\phi$ if and only if there exists a finite sequence $\chi_0, \chi_1, \ldots, \chi_n \in \mathrm{Irr}(W)$ and a finite sequence $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ of ϕ-bad prime ideals for W such that*

- *$\chi_0 = \chi$ and $\chi_n = \psi$,*
- *for all j $(1 \leq j \leq n)$, the characters χ_{j-1} and χ_j belong to the same block of $\mathcal{O}_{\mathfrak{p}_j\mathcal{O}}\mathcal{H}_\phi$.*

The above proposition implies that if we know the blocks of the algebra $\mathcal{O}_{\mathfrak{p}\mathcal{O}}\mathcal{H}_\phi$ for every ϕ-bad prime ideal $\mathfrak{p}$ for W, then we know the Rouquier blocks of $\mathcal{H}_\phi$. To determine the former, we can use this result:

Theorem 2.15 [Chlouveraki 2007, Theorem 3.2.17]. *Let $A := \mathbb{Z}_K[\boldsymbol{v}, \boldsymbol{v}^{-1}]$ and $\mathfrak{p}$ be a prime ideal of $\mathbb{Z}_K$. Let $M_1, \ldots, M_k$ be all the $\mathfrak{p}$-essential monomials for W such that $\phi(M_j) = 1$ for all $j = 1, \ldots, k$. Set $\mathfrak{q}_0 := \mathfrak{p}A$, $\mathfrak{q}_j := \mathfrak{p}A + (M_j - 1)A$ for $j = 1, \ldots, k$ and $\mathcal{Q} := \{\mathfrak{q}_0, \mathfrak{q}_1, \ldots, \mathfrak{q}_k\}$. Two irreducible characters $\chi, \psi \in \mathrm{Irr}(W)$ are in the same block of $\mathcal{O}_{\mathfrak{p}\mathcal{O}}\mathcal{H}_\phi$ if and only if there exist a finite sequence $\chi_0, \chi_1, \ldots, \chi_n \in \mathrm{Irr}(W)$ and a finite sequence $\mathfrak{q}_{j_1}, \ldots, \mathfrak{q}_{j_n} \in \mathcal{Q}$ such that*

- *$\chi_0 = \chi$ and $\chi_n = \psi$,*
- *for all i $(1 \leq i \leq n)$, the characters χ_{i-1} and χ_i are in the same block of $A_{\mathfrak{q}_{j_i}}\mathcal{H}$.*

Let $\mathfrak{p}$ be a prime ideal of $\mathbb{Z}_K$ and $\phi : v_{\mathcal{C},j} \mapsto y^{n_{\mathcal{C},j}}$ a cyclotomic specialization. If $M = \prod_{\mathcal{C},j} v_{\mathcal{C},j}^{a_{\mathcal{C},j}}$ is a $\mathfrak{p}$-essential monomial for W, then

$$\phi(M) = 1 \iff \sum_{\mathcal{C},j} a_{\mathcal{C},j} n_{\mathcal{C},j} = 0.$$

Set $m := \sum_{\mathcal{C}\in\mathcal{A}/W} e_{\mathcal{C}}$. The hyperplane defined in $\mathbb{C}^m$ by the relation

$$\sum_{\mathcal{C},j} a_{\mathcal{C},j} t_{\mathcal{C},j} = 0,$$

where $(t_{\mathcal{C},j})_{\mathcal{C},j}$ is a set of m indeterminates, is called a $\mathfrak{p}$-*essential hyperplane* for W. A hyperplane in $\mathbb{C}^m$ is called *essential* for W if it is $\mathfrak{p}$-essential for some prime ideal $\mathfrak{p}$ of $\mathbb{Z}_K$ (Likewise, a monomial is called *essential* for W if it is $\mathfrak{p}$-essential for some prime ideal $\mathfrak{p}$ of $\mathbb{Z}_K$).

Let H be an essential hyperplane corresponding to the monomial M and let $\mathfrak{p}$ be a prime ideal of $\mathbb{Z}_K$. We denote by $\mathcal{B}_{\mathfrak{p}}^H$ the partition of $\mathrm{Irr}(W)$ into blocks of $A_{\mathfrak{q}_M}\mathcal{H}$, where $\mathfrak{q}_M := (M-1)A + \mathfrak{p}A$. Moreover, we denote by $\mathcal{B}_{\mathfrak{p}}^{\varnothing}$ the partition of $\mathrm{Irr}(W)$ into blocks of $A_{\mathfrak{p}A}\mathcal{H}$.

Definition 2.16. Let H be an essential hyperplane for W. By *Rouquier blocks associated with* H we understand the partition $\mathcal{B}^H$ of $\mathrm{Irr}(W)$ generated by the partition $\mathcal{B}_{\mathfrak{p}}^H$, where $\mathfrak{p}$ runs over the set of prime ideals of $\mathbb{Z}_K$. By *Rouquier blocks with no essential hyperplane* we understand the partition $\mathcal{B}^{\varnothing}$ generated by $\mathcal{B}_{\mathfrak{p}}^{\varnothing}$.

With the help of Proposition 2.14 and Theorem 2.15, we obtain the following characterization for the Rouquier blocks of a cyclotomic Hecke algebra:

Proposition 2.17. *Let $\phi : v_{\mathcal{C},j} \mapsto y^{n_{\mathcal{C},j}}$ be a cyclotomic specialization. The Rouquier blocks of the cyclotomic Hecke algebra $\mathcal{H}_\phi$ correspond to the partition of* $\mathrm{Irr}(W)$ *generated by the partitions $\mathcal{B}^H$, where H runs over the set of all essential hyperplanes the integers $n_{\mathcal{C},j}$ belong to. If the $n_{\mathcal{C},j}$ belong to no essential hyperplane, then the Rouquier blocks of $\mathcal{H}_\phi$ coincide with the partition $\mathcal{B}^{\varnothing}$.*

Definition and Corollary 2.18. *Let $\phi : v_{\mathcal{C},j} \mapsto y^{n_{\mathcal{C},j}}$ be a cyclotomic specialization such that the integers $n_{\mathcal{C},j}$ belong to only one essential hyperplane H (resp. to no essential hyperplane). We say that ϕ is a cyclotomic specialization associated with the essential hyperplane H (resp. with no essential hyperplane). The Rouquier blocks of $\mathcal{H}_\phi$ coincide with the partition $\mathcal{B}^H$ (resp. $\mathcal{B}^{\varnothing}$).*

By taking cyclotomic specializations associated to each (or no) essential hyperplane and calculating the Rouquier blocks of the corresponding cyclotomic Hecke algebras, we determined in [Chlouveraki 2007] the Rouquier blocks for all exceptional complex reflection groups. We will do the same for the group $G(d, 1, r)$.

The functions a and A. Following the notations in [Broué et al. 1999, 6B], for every element $P(y) \in \mathbb{C}(y)$, we define

- the *valuation* $\mathrm{val}_y P$ *of* $P(y)$ *at* y as the order of $P(y)$ at 0 (we have $\mathrm{val}_y P < 0$ if 0 is a pole of $P(y)$ and $\mathrm{val}_y P > 0$ if 0 is a zero of $P(y)$), and
- the *degree* $\deg_y P$ *of* $P(y)$ *at* y as the opposite of the valuation of $P(1/y)$.

Moreover, if $x := y^{|\mu(K)|}$, then $\mathrm{val}_x P := \dfrac{\mathrm{val}_y P}{|\mu(K)|}$ and $\deg_x P := \dfrac{\deg_y P}{|\mu(K)|}$. For $\chi \in \mathrm{Irr}(W)$, we define

$$a_{\chi_\phi} := \mathrm{val}_x(s_{\chi_\phi}(y)) \text{ and } A_{\chi_\phi} := \deg_x s_{\chi_\phi}(y).$$

Proposition 2.19 [Broué and Kim 2002, Proposition 2.9]. *Let $\chi, \psi \in \mathrm{Irr}(W)$. If χ_ϕ and ψ_ϕ belong to the same Rouquier block, then*

$$a_{\chi_\phi} + A_{\chi_\phi} = a_{\psi_\phi} + A_{\psi_\phi}.$$

3. Rouquier blocks for the Ariki–Koike algebras

We will start this section by introducing some notations and results in combinatorics [Broué and Kim 2002, §3A] that will be useful for the description of the Rouquier blocks of the cyclotomic Ariki–Koike algebras, i.e., the cyclotomic Hecke algebras associated to the group $G(d, 1, r)$.

Combinatorics. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_h)$ be a partition, i.e., a finite decreasing sequence of positive integers: $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_h \geq 1$. The integer

$$|\lambda| := \lambda_1 + \lambda_2 + \cdots + \lambda_h$$

is called *the size of* λ. We also say that λ *is a partition of* $|\lambda|$. The integer h is called *the height of* λ and we set $h_\lambda := h$. To each partition λ we associate its *β-number*, $\beta_\lambda = (\beta_1, \beta_2, \ldots, \beta_h)$, defined by

$$\beta_1 := h + \lambda_1 - 1,\ \beta_2 := h + \lambda_2 - 2,\ \ldots,\ \beta_h := h + \lambda_h - h.$$

Multipartitions. Fix a positive integer d. Let $\lambda = (\lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(d-1)})$ be a d-partition, i.e., a family of partitions indexed by the set $\{0, 1, \ldots, d-1\}$. Write

$$h^{(a)} := h_{\lambda^{(a)}},\quad \beta^{(a)} := \beta_{\lambda^{(a)}};$$

then

$$\lambda^{(a)} = (\lambda_1^{(a)}, \lambda_2^{(a)}, \ldots, \lambda_{h^{(a)}}^{(a)}).$$

The integer

$$|\lambda| := \sum_{a=0}^{d-1} |\lambda^{(a)}|$$

is called *the size of* λ. We also say that λ *is a d-partition of* $|\lambda|$.

Ordinary symbols. If $\beta = (\beta_1, \beta_2, \ldots, \beta_h)$ is a sequence of positive integers such that $\beta_1 > \beta_2 > \cdots > \beta_h$ and m is a positive integer, then the m-shift of β is the sequence of numbers defined by

$$\beta[m] = (\beta_1 + m,\ \beta_2 + m,\ \ldots,\ \beta_h + m,\ m-1,\ m-2,\ \ldots,\ 1,\ 0).$$

Let $\lambda = (\lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(d-1)})$ be a d-partition. The *d-height of* λ is the family

$(h^{(0)}, h^{(1)}, \dots, h^{(d-1)})$; we define the *height of* λ to be the integer

$$h_\lambda := \max\{h^{(a)} \mid (0 \le a \le d-1)\}.$$

Definition 3.1. The ordinary standard symbol of λ is the family of numbers given by

$$B_\lambda = (B_\lambda^{(0)}, B_\lambda^{(1)}, \dots, B_\lambda^{(d-1)}),$$

where, for all a $(0 \le a \le d-1)$, we have

$$B_\lambda^{(a)} := \beta^{(a)}[h_\lambda - h^{(a)}].$$

An ordinary symbol of λ is a symbol obtained from the ordinary standard symbol by shifting all the rows by the same integer.

The ordinary standard symbol of a d-partition λ is of the form

$$\begin{array}{ccccccc} B_\lambda^{(0)} & = & b_1^{(0)} & b_2^{(0)} & \cdots & b_{h_\lambda}^{(0)} \\ B_\lambda^{(1)} & = & b_1^{(1)} & b_2^{(1)} & \cdots & b_{h_\lambda}^{(1)} \\ \vdots & & \vdots & \vdots & \ddots & \vdots \\ B_\lambda^{(d-1)} & = & b_1^{(d-1)} & b_2^{(d-1)} & \cdots & b_{h_\lambda}^{(d-1)} \end{array}$$

The *ordinary content* of a d-partition of ordinary standard symbol B is the set with repetition

$$\mathrm{Cont}_\lambda = B_\lambda^{(0)} \cup B_\lambda^{(1)} \cup \cdots \cup B_\lambda^{(d-1)}$$

or (with the notations above) the polynomial defined by

$$\mathrm{Cont}_\lambda(x) := \sum_{a,i} x^{b_i^{(a)}}.$$

Example 3.2. Take $d = 2$ and $\lambda = ((2, 1), (3))$. Then

$$B_\lambda = \begin{pmatrix} 3 & 4 \\ 1 & 0 \end{pmatrix}.$$

We have $\mathrm{Cont}_\lambda = \{0, 1, 3, 4\}$ or $\mathrm{Cont}_\lambda(x) = 1 + x + x^3 + x^4$.

Charged symbols. From now on, we fix a weight system, i.e., a family of integers

$$m := (m^{(0)}, m^{(1)}, \dots, m^{(d-1)}).$$

Let $\lambda = (\lambda^{(0)}, \lambda^{(1)}, \dots, \lambda^{(d-1)})$ be a d-partition, and set

$$hc^{(0)} := h^{(0)} - m^{(0)},\ hc^{(1)} := h^{(1)} - m^{(1)},\ \dots,\ hc^{(d-1)} := h^{(d-1)} - m^{(d-1)}.$$

We define the (d, m)-*charged height of* λ as $(hc^{(0)}, hc^{(1)}, \dots, hc^{(d-1)})$, and the m-*charged height of* λ as the integer

$$hc_\lambda := \max\{hc^{(a)} \mid (0 \le a \le d-1)\}.$$

Definition 3.3. The m-charged standard symbol of λ is the family of numbers defined by

$$Bc_\lambda = (Bc_\lambda^{(0)}, Bc_\lambda^{(1)}, \ldots, Bc_\lambda^{(d-1)}),$$

where, for all a $(0 \le a \le d-1)$, we have

$$Bc_\lambda^{(a)} := \beta^{(a)}[hc_\lambda - hc^{(a)}].$$

An m-charged symbol of λ is a symbol obtained from the m-charged standard symbol by shifting all the rows by the same integer.

Remark. The ordinary symbols correspond to the weight system

$$m^{(0)} = m^{(1)} = \cdots = m^{(d-1)} = 0.$$

The m-charged standard symbol of λ is a tableau of numbers arranged into d rows indexed by the set $\{0, 1, \ldots, d-1\}$ such that the a-th row has length equal to $hc_\lambda + m^{(a)}$. For all a $(0 \le a \le d-1)$, we set $l^{(a)} := hc_\lambda + m^{(a)}$ and we denote by

$$Bc_\lambda^{(a)} \;=\; bc_1^{(a)} \;\; bc_2^{(a)} \;\; \ldots \;\; bc_{l^{(a)}}^{(a)}$$

the a-th row of the m-charged standard symbol.

The *m-charged content* of a d-partition of m-charged standard symbol Bc is the set with repetition

$$\mathrm{Contc}_\lambda = Bc_\lambda^{(0)} \cup Bc_\lambda^{(1)} \cup \cdots \cup Bc_\lambda^{(d-1)}$$

or (with the above notations) the polynomial defined by

$$\mathrm{Contc}_\lambda(x) := \sum_{a,i} x^{bc_i^{(a)}}.$$

Example 3.4. Take $d = 2$, $\lambda = ((2, 1), (3))$ and $m = (-1, 2)$. Then

$$Bc_\lambda = \begin{pmatrix} 3 & 1 & & & \\ 7 & 3 & 2 & 1 & 0 \end{pmatrix}$$

We have $\mathrm{Contc}_\lambda = \{0, 1, 1, 2, 3, 3, 7\}$ or $\mathrm{Contc}_\lambda(x) = 1 + 2x + x^2 + 2x^3 + x^7$.

Generic Ariki–Koike algebras. The group $G(d, 1, r)$ is the group of all monomial $r \times r$ matrices with entries in μ_d. It is isomorphic to the wreath product $\mu_d \wr \mathfrak{S}_r$ and its field of definition is $\mathbb{Q}(\zeta_d)$.

The *generic Ariki–Koike algebra* of $G(d, 1, r)$ [Ariki and Koike 1994; Broué and Malle 1993] is the algebra $\mathcal{H}_{d,r}$ generated over the Laurent ring of polynomials in $d+1$ indeterminates

$$\mathbb{O}_d := \mathbb{Z}[u_0, u_0^{-1}, u_1, u_1^{-1}, \ldots, u_{d-1}, u_{d-1}^{-1}, x, x^{-1}]$$

by the elements $\boldsymbol{s}, \boldsymbol{t}_1, \boldsymbol{t}_2, \ldots, \boldsymbol{t}_{r-1}$ satisfying the relations

- $\boldsymbol{s t_1 s t_1} = \boldsymbol{t_1 s t_1 s}$, $\boldsymbol{s t_j} = \boldsymbol{t_j s}$ for $j \neq 1$,
- $\boldsymbol{t_j t_{j+1} t_j} = \boldsymbol{t_{j+1} t_j t_{j+1}}$, $\boldsymbol{t_i t_j} = \boldsymbol{t_j t_i}$ for $|i-j|>1$,
- $(\boldsymbol{s}-u_0)(\boldsymbol{s}-u_1)\dots(\boldsymbol{s}-u_{d-1}) = (\boldsymbol{t_j}-x)(\boldsymbol{t_j}+1) = 0$.

For every d-partition $\lambda = (\lambda^{(0)}, \lambda^{(1)}, \dots, \lambda^{(d-1)})$ of r, we consider the free $\mathbb{O}_d$-module which has as basis the family of standard tableaux of λ. We can give to this module the structure of a $\mathcal{H}_{d,r}$-module [Ariki and Koike 1994; Ariki 1994; Graham and Lehrer 1996] and thus obtain the *Specht module* $\boldsymbol{Sp}^\lambda$ associated to λ.

Set $\mathcal{K}_d := \mathbb{Q}(u_0, u_1, \dots, u_{d-1}, x)$ the field of fractions of $\mathbb{O}_d$. The $\mathcal{K}_d\mathcal{H}_{d,r}$-module $\mathcal{K}_d\boldsymbol{Sp}^\lambda$, obtained by extension of scalars, is absolutely irreducible and every irreducible $\mathcal{K}_d\mathcal{H}_{d,r}$-module is isomorphic to a module of this type. Thus $\mathcal{K}_d$ is a splitting field for $\mathcal{H}_{d,r}$. We denote by χ_λ the (absolutely) irreducible character of the $\mathcal{K}_d\mathcal{H}_{d,r}$-module $\boldsymbol{Sp}^\lambda$.

Since the algebra $\mathcal{K}_d\mathcal{H}_{d,r}$ is split semisimple, the Schur elements of its irreducible characters belong to $\mathbb{O}_d$. The following result by Mathas gives a description of the Schur elements. The same result has been obtained independently by Geck, Iancu and Malle in [Geck et al. 2000].

Proposition 3.5 [Mathas 2004, Corollary 6.5]. *Let λ be a d-partition of r with ordinary standard symbol $B_\lambda = (B_\lambda^{(0)}, B_\lambda^{(1)}, \dots, B_\lambda^{(d-1)})$. Fix $L \geq h_\lambda$, where h_λ is the height of λ. We set $B_{\lambda,L} := (B_\lambda^{(0)}[L-h_\lambda], B_\lambda^{(1)}[L-h_\lambda], \dots, B_\lambda^{(d-1)}[L-h_\lambda]) = (B_{\lambda,L}^{(0)}, B_{\lambda,L}^{(1)}, \dots, B_{\lambda,L}^{(d-1)})$ and $B_{\lambda,L}^{(s)} = (b_1^{(s)}, b_2^{(s)}, \dots, b_L^{(s)})$. Let*

$$a_L := r(d-1) + \binom{d}{2}\binom{L}{2} \quad \text{and} \quad b_L := dL(L-1)(2dL-d-3)/12.$$

Then the Schur element of the irreducible character χ_λ is given by the formulae $s_\lambda = (-1)^{a_L} x^{b_L} (x-1)^{-r} (u_0 u_1 \dots u_{d-1})^{-r} \nu_\lambda / \delta_\lambda$, where

$$\nu_\lambda = \prod_{0 \leq s < t < d} (u_s - u_t)^L \prod_{0 \leq s,t < d} \prod_{b_s \in B_{\lambda,L}^{(s)}} \prod_{1 \leq k \leq b_s} (x^k u_s - u_t)$$

and

$$\delta_\lambda = \prod_{0 \leq s < t < d} \prod_{(b_s, b_t) \in B_{\lambda,L}^{(s)} \times B_{\lambda,L}^{(t)}} (x^{b_s} u_s - x^{b_t} u_t) \prod_{0 \leq s < d} \prod_{1 \leq i < j \leq L} (x^{b_i^{(s)}} u_s - x^{b_j^{(s)}} u_s).$$

We have already mentioned that the field of definition of $G(d,1,r)$ is $K := \mathbb{Q}(\zeta_d)$. If we set

$$v_j^{|\mu(K)|} := \zeta_d^{-j} u_j (0 \leq j \leq d-1) \text{ and } z^{|\mu(K)|} := x,$$

then Theorem 2.3 implies that the algebra $K(v_0, v_1, \dots, v_{d-1}, z)\mathcal{H}_{d,r}$ is split semisimple. Proposition 3.5 implies that the essential monomials for $G(d,1,r)$ are of the form

- $z^k v_s v_t^{-1}$ for $0 \leq s < t < d$ and $-r < k < r$,
- z.

Remark. The monomial z can be seen as a monomial of the form $z_0 z_1^{-1}$, if, in the definition of the Ariki–Koike algebra, we replace the relation

$$(\boldsymbol{t}_j - x)(\boldsymbol{t}_j + 1) = 0 \text{ by } (\boldsymbol{t}_j - x_0)(\boldsymbol{t}_j + x_1) = 0$$

and we set

$$z_0^{|\mu(K)|} := x_0 \text{ and } z_1^{|\mu(K)|} := x_1.$$

Cyclotomic Ariki–Koike algebras. Let y be an indeterminate and let ϕ be a cyclotomic specialization defined by

$$\phi(v_j) = y^{m_j} \quad (0 \leq j < d), \qquad \phi(z) = y^n.$$

If we set $q := y^{|\mu(K)|}$, then ϕ can be described by

$$\phi(u_j) = \zeta_d^j q^{m_j} \quad (0 \leq j < d), \qquad \phi(x) = q^n.$$

The corresponding cyclotomic Hecke algebra $(\mathcal{H}_{d,r})_\phi$ can be considered either over the ring $\mathbb{Z}_K[y, y^{-1}]$ or over the ring $\mathbb{Z}_K[q, q^{-1}]$. We define the Rouquier blocks of $(\mathcal{H}_{d,r})_\phi$ to be the blocks of $(\mathcal{H}_{d,r})_\phi$ defined over the Rouquier ring $\mathcal{R}_K(y)$ in $K(y)$. However, in other texts, as, for example, in [Broué and Kim 2002], the Rouquier blocks are determined over the Rouquier ring $\mathcal{R}_K(q)$ in $K(q)$. Since $\mathcal{R}_K(y)$ is the integral closure of $\mathcal{R}_K(q)$ in the splitting field $K(y)$ for $(\mathcal{H}_{d,r})_\phi$, Proposition 1.4 establishes a relation between the blocks of $\mathcal{R}_K(y)(\mathcal{H}_{d,r})_\phi$ and those of $\mathcal{R}_K(q)(\mathcal{H}_{d,r})_\phi$. Moreover, in our case we can prove:

Proposition 3.6. *The blocks of $\mathcal{R}_K(y)(\mathcal{H}_{d,r})_\phi$ and the blocks of $\mathcal{R}_K(q)(\mathcal{H}_{d,r})_\phi$ coincide.*

Proof. By Proposition 1.4, we know that the blocks of $\mathcal{R}_K(q)(\mathcal{H}_{d,r})_\phi$ are unions of the blocks of $\mathcal{R}_K(y)(\mathcal{H}_{d,r})_\phi$. Now let e be a block-idempotent of $\mathcal{R}_K(y)(\mathcal{H}_{d,r})_\phi$. Since $\mathcal{K}_d$ is a splitting field for $\mathcal{H}_{d,r}$, proposition 1.15 implies that e belongs to $K(q)(\mathcal{H}_{d,r})_\phi$. Thus

$$e \in \mathcal{R}_K(y)(\mathcal{H}_{d,r})_\phi \cap K(q)(\mathcal{H}_{d,r})_\phi = \mathcal{R}_K(q)(\mathcal{H}_{d,r})_\phi,$$

since the ring $\mathcal{R}_K(q)$ is integrally closed and $\mathcal{R}_K(y)$ is integral over it ($y^{|\mu(K)|} - q$ vanishes). Thus, e is a sum of blocks of $\mathcal{R}_K(q)(\mathcal{H}_{d,r})_\phi$. □

Residue equivalence. Let ϕ be a cyclotomic specialization like above and set $\mathcal{O} := \mathcal{R}_K(q)$. Following proposition 2.14, in order to obtain the Rouquier blocks of $(\mathcal{H}_{d,r})_\phi$, we need to calculate the blocks of $\mathcal{O}_{\mathfrak{p}\mathcal{O}}(\mathcal{H}_{d,r})_\phi$ for all ϕ-bad prime ideals $\mathfrak{p}$ of $\mathbb{Z}_K$.

Let $\mathfrak{p}$ be a prime ideal of $\mathbb{Z}_K$ lying over a prime number p. By proposition 2.12 the ring $\mathcal{O}$ is a Dedekind ring and thus $\mathcal{O}_{\mathfrak{p}\mathcal{O}}$ is a discrete valuation ring. If we denote by $k_{\mathfrak{p}}$ its residue field, the blocks of $\mathcal{O}_{\mathfrak{p}\mathcal{O}}(\mathcal{H}_{d,r})_\phi$ coincide with the blocks of $k_{\mathfrak{p}}(\mathcal{H}_{d,r})_\phi$, by Proposition 1.7. We denote the natural surjective map by

$$\pi_{\mathfrak{p}} : \mathcal{O}_{\mathfrak{p}\mathcal{O}} \twoheadrightarrow k_{\mathfrak{p}}.$$

Definition 3.7. The diagram of a d-partition λ is the set

$$[\lambda] := \big\{(i, j, a) \mid (0 \le a \le d-1)(1 \le i \le h^{(a)})(1 \le j \le \lambda_i^{(a)})\big\}.$$

A node is any ordered triple (i, j, a).

The $\mathfrak{p}$-*residue* of the node $x = (i, j, a)$ with respect to ϕ is

$$\mathrm{res}_{\mathfrak{p},\phi}(x) = \begin{cases} \pi_{\mathfrak{p}}(\zeta_d^a q^{m_a} q^{n(j-i)}) & \text{if } n \neq 0, \\ (\pi_{\mathfrak{p}}(j-i), \zeta_d^a q^{m_a}) & \text{if } n = 0 \text{ and } \pi_{\mathfrak{p}}(\zeta_d^a q^{m_a}) \neq \pi_{\mathfrak{p}}(\zeta_d^b q^{m_b}) \text{ for } a \neq b, \\ \pi_{\mathfrak{p}}(\zeta_d^a q^{m_a}) & \text{otherwise.} \end{cases}$$

Let $\mathrm{Res}_{\mathfrak{p},\phi} := \{\mathrm{res}_{\mathfrak{p},\phi}(x) \mid x \in [\lambda] \text{ for some } d\text{-partition } \lambda \text{ of } r\}$ be the set of all possible residues. For any d-partition λ of r and $f \in \mathrm{Res}_{\mathfrak{p},\phi}$, we define

$$C_f(\lambda) = \#\{x \in [\lambda] \mid \mathrm{res}(x) = f\}.$$

Adapting Definition 2.10 of [Lyle and Mathas 2007], we obtain:

Definition 3.8. Let λ and μ be two d-partitions of r. We say that λ and μ are $\mathfrak{p}$-residue equivalent with respect to ϕ if $C_f(\lambda) = C_f(\mu)$ for all $f \in \mathrm{Res}_{\mathfrak{p},\phi}$.

Then Theorem 2.13 of the same reference implies:

Theorem 3.9. *Two irreducible characters $(\chi_\lambda)_\phi$ and $(\chi_\mu)_\phi$ are in the same block of $k_{\mathfrak{p}}(\mathcal{H}_{d,r})_\phi$ if and only if λ and μ are $\mathfrak{p}$-residue equivalent with respect to ϕ.*

The above result, in combination with Proposition 2.14 gives:

Corollary 3.10. *Two irreducible characters $(\chi_\lambda)_\phi$ and $(\chi_\mu)_\phi$ are in the same Rouquier block of $(\mathcal{H}_{d,r})_\phi$ if and only if there exists a finite sequence $\lambda_{(0)}, \lambda_{(1)}, \ldots, \lambda_{(m)}$ of d-partitions of r and a finite sequence $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$ of ϕ-bad prime ideals for W such that*

- *$\lambda_{(0)} = \lambda$ and $\lambda_{(m)} = \mu$,*
- *for all j $(1 \le j \le m)$, the d-partitions $\lambda_{(j-1)}$ and $\lambda_{(j)}$ are $\mathfrak{p}_j$-residue equivalent with respect to ϕ.*

Rouquier blocks and charged content. Theorem 3.13 in [Broué and Kim 2002] gives a description of the Rouquier blocks of the cyclotomic Ariki–Koike algebras when $n \neq 0$. However, in the proof it is supposed that $1-\zeta_d$ always belongs to a prime ideal of $\mathbb{Z}[\zeta_d]$. This is not correct, unless d is the power of a prime number. Therefore, we will state here the part of the theorem that is correct and only for the case $n = 1$.

Theorem 3.11. *Let ϕ be a cyclotomic specialization such that $\phi(x) = q$. If two irreducible characters $(\chi_\lambda)_\phi$ and $(\chi_\mu)_\phi$ are in the same Rouquier block of $(\mathcal{H}_{d,r})_\phi$, then* $\mathrm{Contc}_\lambda = \mathrm{Contc}_\mu$ *with respect to the weight system $m = (m_0, m_1, \dots, m_{d-1})$. The converse holds when d is the power of a prime number.*

Determination of the Rouquier blocks. In this section, we are going to determine the Rouquier blocks for all cyclotomic Ariki–Koike algebras by determining the Rouquier blocks associated with no and each essential hyperplane for $G(d, 1, r)$. Due to Corollary 2.18, it suffices to consider a cyclotomic specialization associated with no and each essential hyperplane and calculate the Rouquier blocks of the corresponding cyclotomic Hecke algebra. Following the description of the essential monomials in the section on generic Ariki–Koike algebras (page 706), we obtain that the essential hyperplanes for $G(d, 1, r)$ are of the form

- $kN + M_s - M_t = 0$ for $0 \le s < t < d$ and $-r < k < r$.
- $N = 0$.

Case 1: No essential hyperplane. If ϕ is a cyclotomic specialization associated with no essential hyperplane, then the description of the Schur elements by Proposition 3.5 implies that there are no ϕ-bad prime ideals for $G(d, 1, r)$. Therefore, every irreducible character is a block by itself.

Proposition 3.12. *The Rouquier blocks associated with no essential hyperplane are trivial.*

Case 2: Essential hyperplane of the form $kN + M_s - M_t = 0$. The following result is an immediate consequence of the description of the Schur elements by Proposition 3.5.

Proposition 3.13. *Let s, t, k be three integers such that $0 \le s < t < d$ and $-r < k < r$. The hyperplane*

$$H : kN + M_s - M_t = 0$$

is essential for $G(d, 1, r)$ if and only if there exists a prime ideal $\mathfrak{p}$ of $\mathbb{Z}[\zeta_d]$ such that $\zeta_d^s - \zeta_d^t \in \mathfrak{p}$. Moreover, in this case, H is $\mathfrak{p}$-essential for $G(d, 1, r)$.

Example 3.14. The hyperplane $M_0 = M_1$ is 2-essential for $G(2, 1, r)$, whereas it isn't essential for $G(6, 1, r)$, for all $r > 0$.

From now on, we assume that $kN+M_s-M_t=0$ is an essential hyperplane for $G(d,1,r)$, i.e., that there exists a prime ideal $\mathfrak{p}$ of $\mathbb{Z}[\zeta_d]$ such that $\zeta_d^s-\zeta_d^t\in\mathfrak{p}$. Let ϕ be a cyclotomic specialization associated with this essential hyperplane, defined by

$$\phi(u_j)=\zeta_d^j q^{m_j}\ (0\le j<d) \quad \text{and} \quad \phi(x)=q^n.$$

Our aim is the determination of the Rouquier blocks of $(\mathcal{H}_{d,r})_\phi$.

For the notations used in the following theorem, see pages 702–704.

Proposition 3.15. *Let λ, μ be two d-partitions of r. The irreducible characters $(\chi_\lambda)_\phi$ and $(\chi_\mu)_\phi$ are in the same Rouquier block of $(\mathcal{H}_{d,r})_\phi$ if and only if the following conditions are satisfied*:

(1) *We have $\lambda^{(a)}=\mu^{(a)}$ for all $a\notin\{s,t\}$.*

(2) *If $\lambda^{st}:=(\lambda^{(s)},\lambda^{(t)})$ and $\mu^{st}:=(\mu^{(s)},\mu^{(t)})$, then* $\mathrm{Contc}_{\lambda^{st}}=\mathrm{Contc}_{\mu^{st}}$ *with respect to the weight system (m_s,m_t).*

Proof. We can assume, without loss of generality, that $n=1$. We can also assume that $m_s=0$ and $m_t=k$.

Suppose that $(\chi_\lambda)_\phi$ and $(\chi_\mu)_\phi$ belong to the same Rouquier block of $(\mathcal{H}_{d,r})_\phi$. Due to Theorem 3.11, we have $\mathrm{Contc}_\lambda=\mathrm{Contc}_\mu$ with respect to the weight system $m=(m_0,m_1,\dots,m_{d-1})$. Since the m_a, $a\notin\{s,t\}$ could take any value (as long as they don't belong to another essential hyperplane), we must have that $\lambda^{(a)}=\mu^{(a)}$ for all $a\notin\{s,t\}$. Moreover, the equality $\mathrm{Contc}_\lambda=\mathrm{Contc}_\mu$ implies that the corresponding m-charged standard symbols Bc_λ and Bc_μ have the same cardinality and thus $hc_\lambda=hc_\mu$. Therefore, we obtain

$$Bc_\lambda^{(a)}=\beta_\lambda^{(a)}[hc_\lambda-hc_\lambda^{(a)}]=\beta_\mu^{(a)}[hc_\mu-hc_\mu^{(a)}]=Bc_\mu^{(a)} \text{ for all } a\notin\{s,t\}.$$

Consequently, we have the following equality between sets with repetition:

$$Bc_\lambda^{(s)}\cup Bc_\lambda^{(t)}=Bc_\mu^{(s)}\cup Bc_\mu^{(t)}.$$

We can assume that the m_a, $a\notin\{s,t\}$ are sufficiently large so that

$$hc_\lambda\in\{hc_\lambda^{(s)},hc_\lambda^{(t)}\} \text{ and } hc_\mu\in\{hc_\mu^{(s)},hc_\mu^{(t)}\}.$$

In this case, if $\lambda^{st}:=(\lambda^{(s)},\lambda^{(t)})$ and $\mu^{st}:=(\mu^{(s)},\mu^{(t)})$, then

$$Bc_{\lambda^{st}}^{(0)}=Bc_\lambda^{(s)},\ Bc_{\lambda^{st}}^{(1)}=Bc_\lambda^{(t)},\ Bc_{\mu^{st}}^{(0)}=Bc_\mu^{(s)},\ Bc_{\mu^{st}}^{(1)}=Bc_\mu^{(t)}$$

and we obtain $\mathrm{Contc}_{\lambda^{st}}=\mathrm{Contc}_{\mu^{st}}$ with respect to the weight system (m_s,m_t).

Now suppose that the conditions 1 and 2 are satisfied. Set $l:=|\lambda^{st}|$. Due to the first condition, we have $|\mu^{st}|=l$. Let $\mathcal{H}_{2,l}$ be the generic Ariki–Koike algebra of the group $G(2,1,l)$ defined over the ring

$$\mathbb{Z}[U_s,U_s^{-1},U_t,U_t^{-1},X,X^{-1}].$$

The group $G(2,1,l)$ is isomorphic to the cyclic group of order 2 for $l=1$ and to the Coxeter group B_l for $l \geq 2$. Let us consider the cyclotomic specialization

$$\vartheta : U_s \mapsto q^{m_s}, U_t \mapsto -q^{m_t}, X \mapsto q.$$

Due to Theorem 3.11, condition 2 implies that the characters $(\chi_{\lambda^{st}})_\vartheta$ and $(\chi_{\mu^{st}})_\vartheta$ belong to the same Rouquier block of $(\mathcal{H}_{2,l})_\vartheta$. We conclude that $kN+M_s-M_t=0$ is a 2-essential hyperplane for $G(2,1,l)$ and that, due to Corollary 3.10, λ^{st} and μ^{st} are 2-residue equivalent with respect to ϑ. In order to check whether λ and μ are $\mathfrak{p}$-residue equivalent with respect to ϕ, we only need to consider the nodes with third entry s or t (thanks to condition 1). The nodes of λ (resp. of μ) with third entry s or t are the nodes of λ^{st} (resp. μ^{st}). The $\mathfrak{p}$-residues of these nodes with respect to ϕ can be obtained by replacing q^{m_s} by $\zeta_d^s q^{m_s}$ and $-q^{m_t}$ by $\zeta_d^t q^{m_t}$ into the 2-residues with respect to ϑ of the nodes belonging to $[\lambda^{st}]$ and $[\mu^{st}]$. Since λ^{st} and μ^{st} are 2-residue equivalent and $\zeta_d^s - \zeta_d^t \in \mathfrak{p}$ (when before we had $1-(-1) \in (2)$), we obtain that λ and μ are $\mathfrak{p}$-residue equivalent with respect to ϕ. Thus, by Corollary 3.10, $(\chi_\lambda)_\phi$ and $(\chi_\mu)_\phi$ belong to the same Rouquier block of $(\mathcal{H}_{d,r})_\phi$. □

The following result is a corollary of the above proposition. However, it can also be obtained independently using the Morita equivalences established by [Dipper and Mathas 2002]:

Proposition 3.16. *Let λ, μ be two d-partitions of r. The irreducible characters $(\chi_\lambda)_\phi$ and $(\chi_\mu)_\phi$ are in the same Rouquier block of $(\mathcal{H}_{d,r})_\phi$ if and only if the following conditions are satisfied*:

(1) *We have $\lambda^{(a)} = \mu^{(a)}$ for all $a \notin \{s,t\}$.*

(2) *If $\lambda^{st} := (\lambda^{(s)}, \lambda^{(t)})$, $\mu^{st} := (\mu^{(s)}, \mu^{(t)})$ and $l := |\lambda^{st}| = |\mu^{st}|$, then the characters $(\chi_{\lambda^{st}})_\vartheta$ and $(\chi_{\mu^{st}})_\vartheta$ belong to the same Rouquier block of the cyclotomic Ariki–Koike algebra of $G(2,1,l)$ obtained via the specialization*

$$\vartheta : U_s \mapsto q^{m_s}, U_t \mapsto -q^{m_t}, X \mapsto q^n.$$

Proof. Following [Dipper and Mathas 2002, Theorem 1.1], we obtain that the algebra $(\mathcal{H}_{d,r})_\phi$ is Morita equivalent to the algebra

$$A := \bigoplus_{\substack{n_1,\ldots,n_{d-1}\geq 0 \\ n_1+\cdots+n_{d-1}=r}} (\mathcal{H}_{2,n_1})_{\phi'} \otimes \mathcal{H}(\mathfrak{S}_{n_2})_{\phi''} \otimes \cdots \otimes \mathcal{H}(\mathfrak{S}_{n_{d-1}})_{\phi''},$$

where ϕ' is the restriction of ϕ to $\mathbb{Z}[u_s, u_s^{-1}, u_t, u_t^{-1}, x, x^{-1}]$ and ϕ'' is the restriction of ϕ to $\mathbb{Z}[x, x^{-1}]$. Therefore, $(\mathcal{H}_{d,r})_\phi$ and A have the same blocks.

Since $n \neq 0$, the Rouquier blocks of $\mathcal{H}(\mathfrak{S}_{n_2})_{\phi''}, \ldots, \mathcal{H}(\mathfrak{S}_{n_2})_{\phi''}$ are trivial. Thus we obtain that two irreducible characters $(\chi_\lambda)_\phi$ and $(\chi_\mu)_\phi$ are in the same Rouquier block of $(\mathcal{H}_{d,r})_\phi$ if and only if the following conditions are satisfied:

(1) We have $\lambda^{(a)} = \mu^{(a)}$ for all $a \notin \{s, t\}$.

(2) If $\lambda^{st} := (\lambda^{(s)}, \lambda^{(t)})$, $\mu^{st} := (\mu^{(s)}, \mu^{(t)})$ and $l := |\lambda^{st}| = |\mu^{st}|$, then the characters $(\chi_{\lambda^{st}})_{\phi'}$ and $(\chi_{\mu^{st}})_{\phi'}$ belong to the same block of $(\mathcal{H}_{2,l})_{\phi'}$ over the Rouquier ring of $\mathbb{Q}(\zeta_d)$.

Since the hyperplane $kN + M_s - M_t = 0$ is a $\mathfrak{p}$-essential hyperplane for $G(d, 1, r)$, Corollary 3.10 implies that the second condition is equivalent to saying that the 2-partitions λ^{st} and μ^{st} are $\mathfrak{p}$-residue equivalent with respect to ϕ'. By replacing $\zeta_d^s q^{m_s}$ by q^{m_s} and $\zeta_d^t q^{m_t}$ by $-q^{m_t}$ into the $\mathfrak{p}$-residues with respect to ϕ' of the nodes of $[\lambda^{st}]$ and $[\mu^{st}]$, we obtain their 2-residues with respect to ϑ. Therefore, the 2-partitions λ^{st} and μ^{st} are $\mathfrak{p}$-residue equivalent with respect to ϕ' if and only if they are 2-residue equivalent with respect to ϑ, i.e., the characters $(\chi_{\lambda^{st}})_\vartheta$ and $(\chi_{\mu^{st}})_\vartheta$ belong to the same Rouquier block of $(\mathcal{H}_{2,l})_\vartheta$. □

Case 3: Essential hyperplane $N = 0$. Let ϕ be a cyclotomic specialization associated with the essential hyperplane $N = 0$, defined by

$$\phi(u_j) = \zeta_d^j q^{m_j} \ (0 \le j < d) \quad \text{and} \quad \phi(x) = 1.$$

Proposition 3.17. *Let λ, μ be two d-partitions of r. The following assertions are equivalent*:

(i) *The characters $(\chi_\lambda)_\phi$ and $(\chi_\mu)_\phi$ are in the same Rouquier block of $(\mathcal{H}_{d,r})_\phi$.*

(ii) $|\lambda^{(a)}| = |\mu^{(a)}|$ *for all $a = 0, 1, \ldots, d-1$.*

Proof. (i) ⇒ (ii) Thanks to Proposition 2.14, we can assume that there exists a prime ideal $\mathfrak{p}$ of $\mathbb{Z}[\zeta_d]$ such that $(\chi_\lambda)_\phi$ and $(\chi_\mu)_\phi$ belong to the same block of $k_\mathfrak{p}\mathcal{H}_\phi$ (where $k_\mathfrak{p}$ is the $\mathfrak{p}$-residue field of the Rouquier ring). Therefore, by Theorem 3.9, they must be $\mathfrak{p}$-residue equivalent with respect to ϕ. Due to the form of the $\mathfrak{p}$-residue with respect to ϕ and the fact that the m_a $(0 \le a < d)$ can take any value, we must have

$$\begin{aligned} |\lambda^{(a)}| &= \#\{(i, j, a) \mid (1 \le i \le h_\lambda^{(a)})(1 \le j \le \lambda_i^{(a)})\} \\ &= \#\{(i, j, a) \mid (1 \le i \le h_\mu^{(a)})(1 \le j \le \mu_i^{(a)})\} = |\mu^{(a)}| \end{aligned}$$

for all $a = 0, 1, \ldots, d-1$.

(ii) ⇒ (i) Let $a \in \{0, 1, \ldots, d-1\}$. It is enough to show that $(\chi_\lambda)_\phi$ and $(\chi_\mu)_\phi$ are in the same Rouquier block, whenever λ and μ are two d-partitions of r such that $|\lambda^{(a)}| = |\mu^{(a)}|$ and $\lambda^{(b)} = \mu^{(b)}$ for all $b \neq a$, Set $l := |\lambda^{(a)}| = |\mu^{(a)}|$. The generic Ariki–Koike algebra of the symmetric group $\mathfrak{S}_l$ specializes to the group algebra $\mathbb{Z}[\mathfrak{S}_l]$ when x specializes to 1 . For any finite group, it is well known that 1 is the only block-idempotent of the group algebra over $\mathbb{Z}$ (see also [Rouquier 1999], §3, Remark 1). Thus, all irreducible characters of $\mathfrak{S}_l$ belong to the same Rouquier block of $\mathbb{Z}[\mathfrak{S}_l]$. Corollary 3.10 implies that there exist a finite sequence of partitions of l, $\lambda_{(0)}, \lambda_{(1)}, \ldots, \lambda_{(m)}$ and a finite sequence of prime numbers of $\mathbb{Z}$, $p_1, p_2, \ldots, p_m$ such that

- $\lambda_{(0)} = \lambda^{(a)}$ and $\lambda_{(m)} = \mu^{(a)}$,
- $\lambda_{(i-1)}$ and $\lambda_{(i)}$ are (p_i)-residue equivalent with respect to the specialization sending x to 1, for all $i = 1, \ldots, m$.

We define $\lambda_{d,i}$ to be the d-partition of r with

$$\lambda_{d,i}^{(a)} = \lambda_{(i)} \quad \text{and} \quad \lambda_{d,i}^{(b)} = \lambda^{(b)} \text{ for all } b \neq a.$$

Let $\mathfrak{p}_i$ be a prime ideal of $\mathbb{Z}[\zeta_d]$ lying over the prime number p_i. Then we have

- $\lambda_{d,0} = \lambda$ and $\lambda_{d,m} = \mu$,
- $\lambda_{d,i-1}$ and $\lambda_{d,i}$ are $\mathfrak{p}_i$-residue equivalent with respect to ϕ, for all $i = 1, \ldots, m$.

Corollary 3.10 implies that $(\chi_\lambda)_\phi$ and $(\chi_\mu)_\phi$ are in the same Rouquier block of $(\mathcal{H}_{d,r})_\phi$. □

Conclusion. Let ϕ be a cyclotomic specialization for $\mathcal{H}_{d,r}$, defined by

$$\phi(u_j) = \zeta_d^j q^{m_j} \ (0 \leq j < d) \quad \text{and} \quad \phi(x) = q^n.$$

Let λ and μ be d-partitions of r. We write $\lambda \sim_{R,\phi} \mu$ if there exist two integers s and t with $0 \leq s < t < d$ such that the following conditions are satisfied:

(1) We have $\lambda^{(a)} = \mu^{(a)}$ for all $a \notin \{s, t\}$.

(2) If $\lambda^{st} := (\lambda^{(s)}, \lambda^{(t)})$ and $\mu^{st} := (\mu^{(s)}, \mu^{(t)})$, then $\mathrm{Contc}_{\lambda^{st}} = \mathrm{Contc}_{\mu^{st}}$ with respect to the weight system (m_s, m_t) (or, equivalently, the irreducible characters $(\chi_{\lambda^{st}})_\vartheta$ and $(\chi_{\mu^{st}})_\vartheta$ belong to the same Rouquier block of the cyclotomic Ariki–Koike algebra of $G(2, 1, l)$ obtained via the specialization $\vartheta : U_s \mapsto q^{m_s}, U_t \mapsto -q^{m_t}, X \mapsto q^n$).

(3) There exists a prime ideal $\mathfrak{p}$ of $\mathbb{Z}[\zeta_d]$ such that $\zeta_d^s - \zeta_d^t \in \mathfrak{p}$.

Thanks to Propositions 3.12, 3.15 and 3.17, we have this consequence of Proposition 2.17:

Theorem 3.18. *If $n \neq 0$, then two irreducible characters $(\chi_\lambda)_\phi$ and $(\chi_\mu)_\phi$ are in the same Rouquier block of $(\mathcal{H}_{d,r})_\phi$ if and only if there exists a finite sequence $\lambda_{(0)}, \lambda_{(1)}, \ldots, \lambda_{(m)}$ of d-partitions of r such that*

- *$\lambda_{(0)} = \lambda$ and $\lambda_{(m)} = \mu$,*

- *for all* i $(1 \le i \le m)$, *we have* $\lambda_{(i-1)} \sim_{R,\phi} \lambda_{(i)}$.

If $n = 0$, *then two irreducible characters* $(\chi_\lambda)_\phi$ *and* $(\chi_\mu)_\phi$ *are in the same Rouquier block of* $(\mathcal{H}_{d,r})_\phi$ *if and only if there exists a finite sequence* $\lambda_{(0)}, \lambda_{(1)}, \dots, \lambda_{(m)}$ *of* d*-partitions of* r *such that*

- $\lambda_{(0)} = \lambda$ *and* $\lambda_{(m)} = \mu$,
- *for all* i $(1 \le i \le m)$, *we have* $\lambda_{(i-1)} \sim_{R,\phi} \lambda_{(i)}$ *or* $|\lambda^{(a)}_{(i-1)}| = |\lambda^{(a)}_{(i)}|$ *for all* $a = 0, 1, \dots, d-1$.

The spetsial case. In this section, we will show that the Rouquier blocks calculated by the algorithm of [Broué and Kim 2002] are correct, when ϕ is the spetsial cyclotomic specialization (see Example 2.8). We are mostly interested in this case, because, as we have already mentioned, the Rouquier blocks of the spetsial cyclotomic Hecke algebra of a Weyl group coincide with its families of characters.

Let ϕ be a cyclotomic specialization for $\mathcal{H}_{d,r}$, defined by

$$\phi(u_j) = \zeta_d^j q^{m_j} \ (0 \le j < d) \quad \text{and} \quad \phi(x) = q.$$

Let λ and μ be two d-partitions of r. We write $\lambda \sim_{C,\phi} \mu$ if there exist two integers s and t with $0 \le s < t < d$ such that the following conditions are satisfied:

(1) We have $\lambda^{(a)} = \mu^{(a)}$ for all $a \notin \{s, t\}$.

(2) If $\lambda^{st} := (\lambda^{(s)}, \lambda^{(t)})$ and $\mu^{st} := (\mu^{(s)}, \mu^{(t)})$, then $\mathrm{Contc}_{\lambda^{st}} = \mathrm{Contc}_{\mu^{st}}$ with respect to the weight system (m_s, m_t).

Proposition 3.19. *Let* λ *and* μ *be two* d*-partitions of* r. *We have that* $\mathrm{Contc}_\lambda = \mathrm{Contc}_\mu$ *with respect to the weight system* $(m_0, m_1, \dots, m_{d-1})$ *if and only if there exists a finite sequence* $\lambda_{(0)}, \lambda_{(1)}, \dots, \lambda_{(m)}$ *of* d*-partitions of* r *such that*

- $\lambda_{(0)} = \lambda$ *and* $\lambda_{(m)} = \mu$,
- *for all* i $(1 \le i \le m)$, *we have* $\lambda_{(i-1)} \sim_{C,\phi} \lambda_{(i)}$.

Proof. We first show that if $\lambda \sim_{C,\phi} \mu$, then $\mathrm{Contc}_\lambda = \mathrm{Contc}_\mu$. Let s, t be as in the definition of the relation $\sim_{C,\phi}$. Since $\mathrm{Contc}_{\lambda^{st}} = \mathrm{Contc}_{\mu^{st}}$ with respect to the weight system (m_s, m_t), we have that $hc_{\lambda^{st}} = hc_{\mu^{st}}$. Moreover, $hc_\lambda^{(a)} = hc_\mu^{(a)}$ for all $a \neq s, t$. Therefore,

$$hc_\lambda = \max\{hc_{\lambda^{st}}, (hc_\lambda^{(a)})_{a \neq s,t}\} = \max\{hc_{\mu^{st}}, (hc_\mu^{(a)})_{a \neq s,t}\} = hc_\mu.$$

Set $h := hc_\lambda - hc_{\lambda^{st}} = hc_\mu - hc_{\mu^{st}}$. We have

$$Bc_\lambda^{(s)} = \beta_\lambda^{(s)}[hc_\lambda - h_\lambda^{(s)} + m_s] = \beta_\lambda^{(s)}[hc_{\lambda^{st}} - h_\lambda^{(s)} + m_s + h] = Bc_{\lambda^{st}}^{(0)}[h].$$

Similarly, we obtain that

$$Bc_\lambda^{(t)} = Bc_{\lambda^{st}}^{(1)}[h], \ Bc_\mu^{(s)} = Bc_{\mu^{st}}^{(0)}[h] \text{ and } Bc_\mu^{(t)} = Bc_{\mu^{st}}^{(1)}[h].$$

Since

$$Bc^{(0)}_{\lambda^{st}} \cup Bc^{(1)}_{\lambda^{st}} = Bc^{(0)}_{\mu^{st}} \cup Bc^{(1)}_{\mu^{st}},$$

we have

$$Bc^{(0)}_{\lambda^{st}}[h] \cup Bc^{(1)}_{\lambda^{st}}[h] = Bc^{(0)}_{\mu^{st}}[h] \cup Bc^{(1)}_{\mu^{st}}[h]$$

and thus,

$$Bc^{(s)}_{\lambda} \cup Bc^{(t)}_{\lambda} = Bc^{(s)}_{\mu} \cup Bc^{(t)}_{\mu}.$$

Since $Bc^{(a)}_{\lambda} = Bc^{(a)}_{\mu}$ for all $a \neq s, t$, we deduce that $\mathrm{Contc}_\lambda = \mathrm{Contc}_\mu$.

Now let λ and μ be two d-partitions of r such that $\mathrm{Contc}_\lambda = \mathrm{Contc}_\mu$. Let p be a prime number such that $p \geq d$. We consider the cyclotomic specialization $\bar{\phi}$ for $\mathcal{H}_{p,r}$, defined by

$$\bar{\phi}(u_j) = \zeta_p^j q^{m_j}\ (0 \leq j < d), \quad \bar{\phi}(u_i) = \zeta_p^i q^M\ (d \leq i < p), \quad \bar{\phi}(x) = q,$$

where $M > m_j + r$ for all j $(0 \leq j < d)$. We define the p-partition $\bar{\lambda}$ of r by

$$\bar{\lambda}^{(j)} := \lambda^{(j)} \text{ for all } j\ (0 \leq j < d) \text{ and } \bar{\lambda}^{(i)} := \varnothing \text{ for all } i\ (d \leq i < p).$$

Similarly, we define $\bar{\mu}$ by

$$\bar{\mu}^{(j)} := \mu^{(j)} \text{ for all } j\ (0 \leq j < d) \text{ and } \bar{\mu}^{(i)} := \varnothing \text{ for all } i\ (d \leq i < p).$$

We have $hc^{(i)}_{\bar{\lambda}} = hc^{(i)}_{\bar{\mu}} = -M$ for all i $(d \leq i < p)$. Moreover, $hc^{(j)}_{\bar{\lambda}} > -M$ and $hc^{(j)}_{\bar{\mu}} > -M$ for all j $(0 \leq j < d)$. Thus $hc_{\bar{\lambda}} = hc_\lambda = hc_\mu = hc_{\bar{\mu}}$. It is immediate, that $\mathrm{Contc}_{\bar{\lambda}} = \mathrm{Contc}_{\bar{\mu}}$ with respect to the weight system

$$(m_0, m_1, \ldots, m_{d-1}, M, M, \ldots, M).$$

Since p is a prime number, Theorem 3.11 implies that the irreducible characters $\chi_{\bar{\lambda}}$ and $\chi_{\bar{\mu}}$ belong to the same Rouquier block of $(\mathcal{H}_{p,r})_{\bar{\phi}}$. Due to Theorem 3.18, there exists a finite sequence $\bar{\lambda}_{(0)}, \bar{\lambda}_{(1)}, \ldots, \bar{\lambda}_{(m)}$ of p-partitions of r such that

- $\bar{\lambda}_{(0)} = \bar{\lambda}$ and $\bar{\lambda}_{(m)} = \bar{\mu}$,
- for all l $(1 \leq l \leq m)$, we have $\bar{\lambda}_{(l-1)} \sim_{R,\bar{\phi}} \bar{\lambda}_{(l)}$ (and thus $\bar{\lambda}_{(l-1)} \sim_{C,\bar{\phi}} \bar{\lambda}_{(l)}$).

Since $\bar{\lambda} \sim_{R,\bar{\phi}} \bar{\lambda}_{(1)}$ and $\bar{\phi}(x) = q \neq 1$, there exist two integers s and t with $0 \leq s < t < p$ such that $\bar{\lambda}$ and $\bar{\lambda}_{(1)}$ belong to the same Rouquier block associated with an essential hyperplane of the form

$$kN + M_s - M_t = 0, \text{ where } -r < k < r$$

and we have $k + m_s - m_t = 0$. Since $M - m_j > r$ for all j with $0 \leq j < d$, we can't have $s < d \leq t$. If $s \geq d$, then $\bar{\lambda}_{(1)}$ is a p-partition of r if and only if $\bar{\lambda}_{(1)} = \bar{\lambda}$. Thus, we must have $t < d$ and since $\bar{\lambda}^{(i)} = \varnothing$ for all i with $d \leq i < p$, we also have $\bar{\lambda}^{(i)}_{(1)} = \varnothing$ for all such i. Inductively, we obtain $\bar{\lambda}^{(i)}_{(l)} = \varnothing$ for all i such that $d \leq i < p$

and all l such that $1 \leq l \leq m$. (The same result can be obtained from the fact that the charged content of two p-partitions linked by $\sim_{R,\bar{\phi}}$ is the same.)

Let $l \in \{0, 1, \ldots, m\}$. Define $\lambda_{(l)}$ to be the d-partition of r such that $\lambda_{(l)}^{(j)} := \bar{\lambda}_{(l)}^{(j)}$ for all j $(0 \leq j < d)$. Then

- $\lambda_{(0)} = \lambda$ and $\lambda_{(m)} = \mu$, and
- for all l $(1 \leq l \leq m)$, we have $\lambda_{(l-1)} \sim_{C,\phi} \lambda_{(l)}$. □

Now assume that ϕ is the spetsial cyclotomic specialization, i.e.,

$$m_0 = 1 \text{ and } m_1 = \cdots = m_{d-1} = 0.$$

Proposition 3.20. *Let ϕ be the spetsial cyclotomic specialization. Two irreducible characters $(\chi_\lambda)_\phi$ and $(\chi_\lambda)_\phi$ belong to the same Rouquier block of $(\mathcal{H}_{d,r})_\phi$ if and only if* $\mathrm{Contc}_\lambda = \mathrm{Contc}_\mu$.

Proof. If $(\chi_\lambda)_\phi$ and $(\chi_\lambda)_\phi$ belong to the same Rouquier block of $(\mathcal{H}_{d,r})_\phi$, then, by Theorem 3.11, we have $\mathrm{Contc}_\lambda = \mathrm{Contc}_\mu$.

Now let λ and μ be two d-partitions of r such that $\mathrm{Contc}_\lambda = \mathrm{Contc}_\mu$. Thanks to Proposition 3.19, we can assume that $\lambda \sim_{C,\phi} \mu$. Then there exist two integers s and t with $0 \leq s < t < d$ such that

$$\mathrm{Contc}_{\lambda^{st}} = \mathrm{Contc}_{\mu^{st}} \text{ and } \lambda^{(a)} = \mu^{(a)} \text{ for all } a \neq s, t.$$

Let us suppose that $d = p_1^{a_1} p_2^{a_2} \ldots p_n^{a_n}$, where p_i are prime numbers such that $p_i \neq p_j$ for $i \neq j$. For $i = 1, \ldots, n$, we set $c_i := d/p_i^{a_i}$. Then $\gcd(c_i) = 1$ and, by Bézout's theorem, there exist integers $(b_i)_{1 \leq i \leq n}$ such that $\sum_{i=1}^{n} b_i c_i = 1$. We have $s - t = \sum_{i=1}^{n} (s-t) b_i c_i$. We set $k_i := (s-t) b_i c_i$ and we obtain that $s - t = \sum_{i=1}^{n} k_i$.

For all $i = 1, \ldots, n$, the element $1 - \zeta_d^{c_i}$ belongs to the prime ideal of $\mathbb{Z}[\zeta_d]$ lying over the prime number p_i. So is $1 - \zeta_d^{k_i}$.

Let I be a subset of $\{1, \ldots, n\}$ minimal (with respect to inclusion) for the property

$$s - t \equiv \sum_{i \in I} k_i \pmod{d},$$

i.e., , if $J \subseteq I$ and

$$s - t \equiv \sum_{j \in J} k_j \pmod{d},$$

then $J = I$. Without loss of generality, we can assume that $I = \{1, \ldots, n\}$. Now, for all $1 \leq m \leq n$, set

$$l_m := \sum_{i=1}^{m} k_i \pmod{d} \text{ and } l_0 := 0.$$

Due to the minimality of I, we have $t + l_i \not\equiv s \pmod{d}$ for all $i < n$.

The group $\mathfrak{S}_d$ acts naturally on the set of d-partitions of r: Let

$$\nu = (\nu^{(0)}, \nu^{(1)}, \ldots, \nu^{(d-1)})$$

be a d-partition of r. If $\tau \in \mathfrak{S}_d$, then

$$\tau(\nu) = (\nu^{(\tau(0))}, \nu^{(\tau(1))}, \ldots, \nu^{(\tau(d-1))}).$$

For $a, b \in \{0, \ldots, d-1\}$, we denote by (a, b) the corresponding transposition. If $a, b \neq 0$, then $\nu \sim_{C,\phi} (a, b)\nu$ (since the ordinary content is stable under the action of (a, b)).

For $i \in I$, set $\sigma_i := (t + l_{i-1} \pmod d, t + l_i \pmod d)$. We have that the element

$$\zeta_d^{t+l_{i-1}} - \zeta_d^{t+l_i} = \zeta_d^{t+l_{i-1}}(1 - \zeta_d^{k_i})$$

belongs to the prime ideal of $\mathbb{Z}[\zeta_d]$ lying over the prime number p_i. Therefore, if $t + l_{i-1}, t + l_i \not\equiv 0 \pmod d$, then $\nu \sim_{R,\phi} \sigma_i(\nu)$ for any d-partition ν of r.

Assume that $t + l_i \not\equiv 0 \pmod d$ for all $i < n$. If $\sigma := (t, t + l_{n-1} \pmod d)$, then

$$\sigma = \sigma_1 \circ \sigma_2 \circ \cdots \circ \sigma_{n-2} \circ \sigma_{n-1} \circ \sigma_{n-2} \circ \cdots \circ \sigma_2 \circ \sigma_1.$$

Theorem 3.18 implies that $(\chi_\lambda)_\phi$ and $(\chi_{\sigma(\lambda)})_\phi$ belong to the same Rouquier block of $(\mathcal{H}_{d,r})_\phi$. The same holds for $(\chi_\mu)_\phi$ and $(\chi_{\sigma(\mu)})_\phi$. Since $\lambda \sim_{C,\phi} \mu$ (with respect to s, t), we have that $\sigma(\lambda) \sim_{C,\phi} \sigma(\mu)$ (with respect to $s, t + l_{n-1} \pmod d$). Moreover, the element $\zeta_d^s - \zeta_d^{t+l_{n-1}} = \zeta_d^s(1 - \zeta_d^{-k_n})$ belongs to the prime ideal of $\mathbb{Z}[\zeta_d]$ lying over the prime number p_n and thus, $\sigma(\lambda) \sim_{R,\phi} \sigma(\mu)$. Consequently, $(\chi_\lambda)_\phi$ and $(\chi_\mu)_\phi$ belong to the same Rouquier block of $(\mathcal{H}_{d,r})_\phi$.

Now assume that there exists $1 \leq m < n$ such that

$$t + l_i \not\equiv 0 \pmod d \text{ for all } i < m \text{ and } t + l_m \equiv 0 \pmod d.$$

We will prove that $(\chi_\lambda)_\phi$ and $(\chi_\mu)_\phi$ belong to the same Rouquier block of $(\mathcal{H}_{d,r})_\phi$ by induction on $n - m$.

Let $m = n - 1$. We have to distinguish two cases: If $k_{n-1} \not\equiv k_n \pmod d$, then we have that $t + l_{n-2} + k_n \not\equiv 0 \pmod d$ and we can rearrange the k_i (exchanging k_{n-1} and k_n) so that $t + l_i \not\equiv 0 \pmod d$ for all $i < n$. This case has been covered above.

If $k_{n-1} \equiv k_n \pmod d$, we set

$$\sigma := (t, t + l_{n-2} \pmod d) = \sigma_1 \circ \sigma_2 \circ \cdots \circ \sigma_{n-3} \circ \sigma_{n-2} \circ \sigma_{n-3} \circ \cdots \circ \sigma_2 \circ \sigma_1.$$

As above, $(\chi_\lambda)_\phi$ and $(\chi_{\sigma(\lambda)})_\phi$ belong to the same Rouquier block of $(\mathcal{H}_{d,r})_\phi$. So do $(\chi_\mu)_\phi$ and $(\chi_{\sigma(\mu)})_\phi$. Since the element

$$\zeta_d^s - \zeta_d^{t+l_{n-2}} = \zeta_d^s - \zeta_d^{s-k_{n-1}-k_n} = \zeta_d^s(1 - \zeta_d^{-2k_n})$$

belongs to the prime ideal of $\mathbb{Z}[\zeta_d]$ lying over the prime number p_n, we obtain that $\sigma(\lambda) \sim_{R,\phi} \sigma(\mu)$ and thus $(\chi_\lambda)_\phi$ and $(\chi_\mu)_\phi$ belong to the same Rouquier block of $(\mathcal{H}_{d,r})_\phi$.

Now assume that the result holds for integers greater than m. We will show that it holds for m. Suppose that

$$t + l_i \not\equiv 0 \pmod{d} \text{ for all } i < m \quad \text{and} \quad t + l_m \equiv 0 \pmod{d}.$$

We again distinguish two cases: If there exists $i_0 > m$ such that $k_{i_0} \not\equiv k_m \pmod{d}$, then we have that $t+l_{m-1}+k_{i_0} \not\equiv 0 \pmod{d}$ and we can rearrange the k_i (exchanging k_m and k_{i_0}) so that $t+l_i \not\equiv 0 \pmod{d}$ for all $i < m+1$. Now, the induction hypothesis and the case $t + l_i \not\equiv 0 \pmod{d}$ for all $i < n$ cover all possibilities. Thus, the result is true.

If $k_i \equiv k_m \pmod{d}$, for all $i > m$, we set

$$\sigma := (t, t + l_{m-1} \pmod{d}) = \sigma_1 \circ \sigma_2 \circ \cdots \circ \sigma_{m-2} \circ \sigma_{m-1} \circ \sigma_{m-2} \circ \cdots \circ \sigma_2 \circ \sigma_1.$$

Again we have $(\chi_\lambda)_\phi$ and $(\chi_{\sigma(\lambda)})_\phi$ in the same Rouquier block of $(\mathcal{H}_{d,r})_\phi$, and likewise $(\chi_\mu)_\phi$ and $(\chi_{\sigma(\mu)})_\phi$. Since the element

$$\zeta_d^s - \zeta_d^{t+l_{m-1}} = \zeta_d^{t+l_n} - \zeta_d^{t+l_{m-1}} = \zeta_d^{t+l_{m-1}}(\zeta_d^{(n-m+1)k_m} - 1)$$

belongs to the prime ideal of $\mathbb{Z}[\zeta_d]$ lying over the prime number p_m, we obtain that $\sigma(\lambda) \sim_{R,\phi} \sigma(\mu)$ and thus $(\chi_\lambda)_\phi$ and $(\chi_\mu)_\phi$ belong to the same Rouquier block of $(\mathcal{H}_{d,r})_\phi$. □

The functions a and A. Let ϕ be a cyclotomic specialization for $\mathcal{H}_{d,r}$, given by

$$\phi(u_j) = \zeta_d^j q^{m_j} \ (0 \le j < d), \quad \phi(x) = q^n.$$

If $n \neq 0$, it follows from [Broué and Kim 2002, Proposition 3.18] that the functions a and A (page 702) are constant on the Rouquier blocks of $(\mathcal{H}_{d,r})_\phi$. We will show that this is also true for $n = 0$. The results in Theorem 3.18 reduce this to proving the following:

Proposition 3.21. *Let λ and μ be two d-partitions of r. Let ϕ be a cyclotomic specialization associated with the essential hyperplane $N = 0$. If $(\chi_\lambda)_\phi$ and $(\chi_\mu)_\phi$ belong to the same Rouquier block of $(\mathcal{H}_{d,r})_\phi$, then*

$$a((\chi_\lambda)_\phi) = a((\chi_\mu)_\phi) \text{ and } A((\chi_\lambda)_\phi) = A((\chi_\mu)_\phi).$$

Proof. Thanks to Proposition 2.19, we have that

$$a((\chi_\lambda)_\phi) + A((\chi_\lambda)_\phi) = a((\chi_\mu)_\phi) + A((\chi_\mu)_\phi).$$

Thus, it is enough to show that $A((\chi_\lambda)_\phi) = A((\chi_\mu)_\phi)$.

Set $L := \max\{h_\lambda, h_\mu\}$. Using the notations of Proposition 3.5, it is straightforward to check that, for $x = 1$, the term δ_λ doesn't depend on the d-partition λ. Consequently, we obtain that $A((\chi_\lambda)_\phi) = A((\chi_\mu)_\phi)$ if and only if

$$\deg_q\Big(\prod_{0\le s,t<d}\ \prod_{b_s\in B^{(s)}_{\lambda,L}}\ \prod_{1\le k\le b_s}\big(\zeta_d^s q^{m_s}-\zeta_d^t q^{m_t}\big)\Big) = \deg_q\Big(\prod_{0\le s,t<d}\ \prod_{b_s\in B^{(s)}_{\mu,L}}\ \prod_{1\le k\le b_s}\big(\zeta_d^s q^{m_s}-\zeta_d^t q^{m_t}\big)\Big).$$

Set

$$f_\lambda(q) := \prod_{0\le s,t<d}\ \prod_{b_s\in B^{(s)}_{\lambda,L}}\ \prod_{1\le k\le b_s}\big(\zeta_d^s q^{m_s}-\zeta_d^t q^{m_t}\big).$$

We have

$$\begin{aligned} f_\lambda(q) &= \prod_{0\le s,t<d}\ \prod_{b_s\in B^{(s)}_{\lambda,L}}\big(\zeta_d^s q^{m_s}-\zeta_d^t q^{m_t}\big)^{b_s} \\ &= \prod_{0\le s,t<d}\big(\zeta_d^s q^{m_s}-\zeta_d^t q^{m_t}\big)^{\sum b_s} = \prod_{0\le s,t<d}\big(\zeta_d^s q^{m_s}-\zeta_d^t q^{m_t}\big)^{|\lambda^{(s)}|+\binom{L}{2}}. \end{aligned}$$

Since $(\chi_\lambda)_\phi$ and $(\chi_\mu)_\phi$ belong to the same Rouquier block of $(\mathcal{H}_{d,r})_\phi$, by Proposition 3.17, we have $|\lambda^{(s)}| = |\mu^{(s)}|$ for all $s = 0, 1, \ldots, d-1$. Thus, $f_\lambda(q) = f_\mu(q)$, which implies that $\deg_q f_\lambda(q) = \deg_q f_\mu(q)$. Therefore $A((\chi_\lambda)_\phi) = A((\chi_\mu)_\phi)$. □

Acknowledgements

I thank Cédric Bonnafé and Nicolas Jacon for our fruitful conversations and their useful remarks. I would also thank the Mathematical Sciences Research Institute in Berkeley and the École Polytechnique Fédérale de Lausanne for their financial support.

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Communicated by Dave Benson
Received 2008-04-18 Revised 2008-07-22 Accepted 2008-07-24

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