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We prove that the moduli stack $\overline{\mathcal{M}}_{g,n}$ of stable curves of genus g with n marked points is rigid, that is, has no infinitesimal deformations. This confirms the first case of a principle proposed by Kapranov. It can also be viewed as a version of Mostow rigidity for the mapping class group.

1. Introduction

Kapranov [1997] has proposed the following informal statement: Given a smooth variety $X = X(0)$, consider the moduli space $X(1)$ of varieties obtained as deformations of $X(0)$, the moduli space $X(2)$ of deformations of $X(1)$, and so on. Then this process should stop after $n = \dim X$ steps, that is, $X(n)$ should be rigid (no infinitesimal deformations). Roughly speaking, one thinks of $X(1)$ as H^1 of a sheaf of nonabelian groups on $X(0)$. Indeed, at least the tangent space to $X(1)$ at $[X]$ is identified with $H^1(T_X)$, where T_X is the tangent sheaf, the sheaf of first order infinitesimal automorphisms of X. Then one regards $X(m)$ as a kind of nonabelian H^m, and the analogy with the usual definition of abelian H^m suggests the statement above.

In particular, the moduli space of curves should be rigid. In this paper, we verify this in the following precise form: the moduli stack of stable curves of genus g with n marked points is rigid for each g and n.

On the other hand, moduli spaces of surfaces should have nontrivial deformations in general. A simple example (for surfaces with boundary) is given in Section 6. It seems plausible that there should be a nontrivial deformation of a moduli space of surfaces whose fibres parametrise "generalised surfaces" in some sense, for example noncommutative surfaces. From this point of view the result of this paper says that the concept of a curve cannot be deformed.

Let us also note that our result can be thought of as a version of Mostow rigidity for the mapping class group. Recall that the moduli space M_g of smooth complex curves of genus g is the quotient of the Teichmüller space T_g by the mapping class

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group Γ_g. The space T_g is a bounded domain in $\mathbb{C}^{3g-3}$, which is homeomorphic to a ball, and Γ_g acts discontinuously on T_g with finite stabilisers. We thus obtain M_g as a complex orbifold with orbifold fundamental group Γ_g. The space T_g admits a natural metric, the Weil–Petersson metric, which has negative holomorphic sectional curvatures. So, roughly speaking, M_g looks like a quotient of a complex ball by a discrete group Γ of isometries, with finite volume. Mostow rigidity predicts that such a quotient is uniquely determined by the group Γ up to complex conjugation. (This is certainly true if Γ acts freely with compact quotient; see [Siu 1980].) In particular, it should have no infinitesimal deformations. Unfortunately I do not know a proof along these lines.

2. Statements

We work over an algebraically closed field k of characteristic zero. Let g and n be nonnegative integers such that $2g-2+n>0$. Let $\overline{\mathcal{M}}_{g,n}$ denote the moduli stack of stable curves of genus g with n marked points. The stack $\overline{\mathcal{M}}_{g,n}$ is a smooth proper Deligne–Mumford stack of dimension $3g-3+n$.

Theorem 2.1. *The stack $\overline{\mathcal{M}}_{g,n}$ is rigid, that is, has no infinitesimal deformations.*

Let $\partial\overline{\mathcal{M}}_{g,n} \subset \overline{\mathcal{M}}_{g,n}$ denote the boundary of the moduli stack, that is, the complement of the locus of smooth curves (with its reduced structure). The locus $\partial\overline{\mathcal{M}}_{g,n}$ is a normal crossing divisor in $\overline{\mathcal{M}}_{g,n}$.

Theorem 2.2. *The pair $(\overline{\mathcal{M}}_{g,n}, \partial\overline{\mathcal{M}}_{g,n})$ has no locally trivial deformations.*

Let $\overline{M}_{g,n}$ denote the coarse moduli space of the stack $\overline{\mathcal{M}}_{g,n}$. The space $\overline{M}_{g,n}$ is a projective variety with quotient singularities.

Theorem 2.3. *The variety $\overline{M}_{g,n}$ has no locally trivial deformations if*

$$(g,n) \neq (1,2), (2,0), (2,1), (3,0).$$

Remark 2.4. In the exceptional cases, the projection $\overline{\mathcal{M}}_{g,n} \to \overline{M}_{g,n}$ is ramified in codimension one over the interior of $\overline{M}_{g,n}$, and an additional calculation is needed to relate the deformations of the stack and the deformations of the coarse moduli space (see Proposition 5.2). Presumably the result still holds.

3. Proof of Theorem 2.2

Write $\mathcal{B}$ for the boundary of $\overline{\mathcal{M}}_{g,n}$. Let $\Omega_{\overline{\mathcal{M}}_{g,n}}(\log\mathcal{B})$ denote the sheaf of 1-forms on $\overline{\mathcal{M}}_{g,n}$ with logarithmic poles along the boundary, and $T_{\overline{\mathcal{M}}_{g,n}}(-\log\mathcal{B})$ the dual of $\Omega_{\overline{\mathcal{M}}_{g,n}}(\log\mathcal{B})$. The sheaf $T_{\overline{\mathcal{M}}_{g,n}}(-\log\mathcal{B})$ is the subsheaf of the tangent sheaf $T_{\overline{\mathcal{M}}_{g,n}}$ consisting of vector fields on $\overline{\mathcal{M}}_{g,n}$ which are tangent to the boundary. In other words, it is the sheaf of first order infinitesimal automorphisms of the pair

$(\overline{\mathcal{M}}_{g,n}, \mathcal{B})$. Hence the first order locally trivial deformations of the pair $(\overline{\mathcal{M}}_{g,n}, \mathcal{B})$ are identified with the space $H^1\big(T_{\overline{\mathcal{M}}_{g,n}}(-\log \mathcal{B})\big)$. To prove Theorem 2.2, we show

$$H^1\big(T_{\overline{\mathcal{M}}_{g,n}}(-\log \mathcal{B})\big) = 0.$$

Let $\pi : \mathcal{U}_{g,n} \to \overline{\mathcal{M}}_{g,n}$ denote the universal family over $\overline{\mathcal{M}}_{g,n}$. That is, $\mathcal{U}_{g,n}$ is the stack of n-pointed stable curves of genus g together with an extra section (with no smoothness condition). Let Σ denote the union of the n tautological sections of π. We define the boundary $\mathcal{B}_U$ of $\mathcal{U}_{g,n}$ as the union of $\pi^*\mathcal{B}$ and Σ.

Let $\nu \colon \mathcal{B}^\nu \to \mathcal{B}$ be the normalisation of the boundary $\mathcal{B}$ of $\overline{\mathcal{M}}_{g,n}$, and $\mathcal{N}$ the normal bundle of the map $\mathcal{B}^\nu \to \overline{\mathcal{M}}_{g,n}$. Then we have an exact sequence

$$0 \to T_{\overline{\mathcal{M}}_{g,n}}(-\log \mathcal{B}) \to T_{\overline{\mathcal{M}}_{g,n}} \to \nu_*\mathcal{N} \to 0.$$

Let ω_π denote the relative dualising sheaf of the morphism π.

Lemma 3.1. *There is a natural isomorphism*

$$\delta : T_{\overline{\mathcal{M}}_{g,n}}(-\log \mathcal{B}) \xrightarrow{\sim} R^1\pi_*(\omega_\pi(\Sigma)^\vee).$$

Proof. For a pointed stable curve $(C, \Sigma_C = x_1 + \cdots + x_n)$, the space of first order deformations is equal to $\operatorname{Ext}^1(\Omega_C(\Sigma_C), \mathcal{O}_C)$. See [Deligne and Mumford 1969, p. 79–82]. The surjection

$$\operatorname{Ext}^1(\Omega_C(\Sigma_C), \mathcal{O}_C) \to H^0\big(\mathcal{E}xt^1(\Omega_C(\Sigma_C), \mathcal{O}_C)\big) = \bigoplus_{q \in \operatorname{Sing} C} \mathcal{E}xt^1(\Omega_C(\Sigma_C), \mathcal{O}_C)_q$$

sends a global deformation of (C, Σ_C) to the induced deformations of the nodes. Étale locally at the point $[(C, \Sigma_C)] \in \overline{\mathcal{M}}_{g,n}$, the boundary $\mathcal{B}$ is a normal crossing divisor with components B_q indexed by the nodes q of C (the divisor B_q is the locus where the node q is *not* smoothed). The Kodaira–Spencer map identifies the fibre of the normal bundle of B_q at $[(C, \Sigma_C)]$ with the stalk of $\mathcal{E}xt^1(\Omega_C(\Sigma_C), \mathcal{O}_C)$ at q.

We now work globally over $\overline{\mathcal{M}}_{g,n}$. We omit the subscripts g, n for clarity. Consider the exact sequence

$$0 \to \pi^*\Omega_{\overline{\mathcal{M}}} \to \Omega_{\mathcal{U}}(\log \Sigma) \to \Omega_{\mathcal{U}/\overline{\mathcal{M}}}(\Sigma) \to 0. \tag{3-1}$$

For a sheaf $\mathcal{F}$ on $\mathcal{U}$, let $\mathcal{E}xt^i_\pi(\mathcal{F}, \cdot)$ denote the i-th right derived functor of

$$\pi_* \circ \mathcal{H}om(\mathcal{F}, \cdot).$$

Applying $\pi_* \circ \mathcal{H}om(\cdot, \mathcal{O}_{\mathcal{U}})$ to the exact sequence (3-1), we obtain a long exact sequence with connecting homomorphism

$$\rho \colon T_{\overline{\mathcal{M}}} \to \mathcal{E}xt^1_\pi(\Omega_{\mathcal{U}/\overline{\mathcal{M}}}(\Sigma), \mathcal{O}_{\mathcal{U}}).$$

The map ρ is the Kodaira–Spencer map for the universal family over $\overline{\mathcal{M}}$ and thus is an isomorphism. (Note that, for a point $p=[(C,\Sigma_C)]\in\overline{\mathcal{M}}$, the base change map

$$\mathcal{E}xt^1_\pi(\Omega_{\mathcal{U}/\overline{\mathcal{M}}}(\Sigma),\mathbb{O}_{\mathcal{U}})\otimes k(p)\to\mathcal{E}xt^1(\Omega_C(\Sigma_C),\mathbb{O}_C)$$

is an isomorphism. Indeed, by relative duality [Kleiman 1980, Theorem 21], it suffices to show that $\pi_*(\Omega_{\mathcal{U}/\overline{\mathcal{M}}}(\Sigma)\otimes\omega_\pi)$ commutes with base change. This follows from cohomology and base change.)

Consider the exact sequences

$$0\to T_{\overline{\mathcal{M}}}(-\log\mathcal{B})\to T_{\overline{\mathcal{M}}}\to\nu_*\mathcal{N}\to 0$$

and

$$0\to R^1\pi_*(\Omega_{\mathcal{U}/\overline{\mathcal{M}}}(\Sigma)^\vee)\to\mathcal{E}xt^1_\pi(\Omega_{\mathcal{U}/\overline{\mathcal{M}}}(\Sigma),\mathbb{O}_{\mathcal{U}})\to\pi_*\,\mathcal{E}xt^1(\Omega_{\mathcal{U}/\overline{\mathcal{M}}}(\Sigma),\mathbb{O}_{\mathcal{U}})\to 0.$$

The Kodaira–Spencer map ρ identifies the middle terms, and induces an identification of the right end terms determined by the deformations of the singularities of the fibres of π. We thus obtain a natural isomorphism δ of the left end terms. Finally, note that $\Omega_{\mathcal{U}/\overline{\mathcal{M}}}(\Sigma)^\vee=\omega_\pi(\Sigma)^\vee$ because $\omega_\pi(\Sigma)$ is invertible and agrees with $\Omega_{\mathcal{U}/\overline{\mathcal{M}}}(\Sigma)$ in codimension 1. This completes the proof. $\square$

The line bundle $\omega_\pi(\Sigma)$ is ample on fibres of π. Hence $\pi_*(\omega_\pi(\Sigma)^\vee)=0$. Also $R^i\pi_*(\omega_\pi(\Sigma)^\vee)=0$ for $i>1$ by dimensions. So

$$H^{i+1}(\omega_\pi(\Sigma)^\vee)=H^i\big(R^1\pi_*(\omega_\pi(\Sigma)^\vee)\big)$$

for all i by the Leray spectral sequence. Hence the isomorphism δ induces an isomorphism

$$H^i\big(T_{\overline{\mathcal{M}}_{g,n}}(-\log\mathcal{B})\big)\xrightarrow{\sim}H^{i+1}(\omega_\pi(\Sigma)^\vee)\tag{3-2}$$

for each i.

Let $U_{g,n}$ denote the coarse moduli space of the stack $\mathcal{U}_{g,n}$ and $p:\mathcal{U}_{g,n}\to U_{g,n}$ the projection. The line bundle $\omega_\pi(\Sigma)$ on the stack $\mathcal{U}_{g,n}$ defines a $\mathbb{Q}$-line bundle $p_*^{\mathbb{Q}}\omega_\pi(\Sigma)$ on the coarse moduli space $U_{g,n}$ (see the Appendix). We use the following important result, which is essentially due to Arakelov [1971, Proposition 3.2, p. 1297]. We refer to [Keel 1999, Section 4] for the proof.

Theorem 3.2. *The $\mathbb{Q}$-line bundle $p_*^{\mathbb{Q}}\omega_\pi(\Sigma)$ is big and nef on $U_{g,n}$.*

It follows by Kodaira vanishing (see Theorem A.1) that $H^i(\omega_\pi(\Sigma)^\vee)=0$ for $i<\dim\mathcal{U}_{g,n}$. Combining with (3-2), we deduce

Proposition 3.3. $H^i\big(T_{\overline{\mathcal{M}}_{g,n}}(-\log\mathcal{B})\big)=0$ *for* $i<\dim\overline{\mathcal{M}}_{g,n}$.

In particular,

$$H^1\big(T_{\overline{\mathcal{M}}_{g,n}}(-\log\mathcal{B})\big)=0$$

if $\dim \overline{\mathcal{M}}_{g,n} > 1$. The remaining cases are easy to check. This completes the proof of Theorem 2.2.

4. Proof of Theorem 2.1

We now prove that $\overline{\mathcal{M}}_{g,n}$ is rigid. Since $\overline{\mathcal{M}}_{g,n}$ is a smooth Deligne–Mumford stack, its first order infinitesimal deformations are identified with the space $H^1(T_{\overline{\mathcal{M}}_{g,n}})$, and we must show that $H^1(T_{\overline{\mathcal{M}}_{g,n}}) = 0$. Consider the exact sequence

$$0 \to T_{\overline{\mathcal{M}}_{g,n}}(-\log \mathcal{B}) \to T_{\overline{\mathcal{M}}_{g,n}} \to \nu_* \mathcal{N} \to 0$$

and the associated long exact sequence of cohomology

$$\cdots \to H^i\big(T_{\overline{\mathcal{M}}_{g,n}}(-\log \mathcal{B})\big) \to H^i(T_{\overline{\mathcal{M}}_{g,n}}) \to H^i(\mathcal{N}) \to \cdots .$$

We prove below that $H^i(\mathcal{N}) = 0$ for $i < \dim \mathcal{B}$. Now

$$H^i\big(T_{\overline{\mathcal{M}}_{g,n}}(-\log \mathcal{B})\big) = 0$$

for $i < \dim \overline{\mathcal{M}}_{g,n}$ by Proposition 3.3, so we deduce

Proposition 4.1. *$H^i(T_{\overline{\mathcal{M}}_{g,n}}) = 0$ for $i < \dim \overline{\mathcal{M}}_{g,n} - 1$.*

In particular, $H^1(T_{\overline{\mathcal{M}}_{g,n}}) = 0$ if $\dim \overline{\mathcal{M}}_{g,n} > 2$. In the remaining cases it is easy to check that $H^1(\mathcal{N}) = 0$, so again $H^1(T_{\overline{\mathcal{M}}_{g,n}}) = 0$.

The irreducible components of the normalisation $\mathcal{B}^\nu$ of the boundary $\mathcal{B}$ of $\overline{\mathcal{M}}_{g,n}$ are finite images of the following stacks [Knudsen 1983a, Definition 3.8, Corollary 3.9]:

(1) $\overline{\mathcal{M}}_{g_1, S_1 \cup \{n+1\}} \times \overline{\mathcal{M}}_{g_2, S_2 \cup \{n+2\}}$, where $g_1 + g_2 = g$ and S_1, S_2 is a partition of $\{1, \ldots, n\}$, and

(2) $\overline{\mathcal{M}}_{g-1,n+2}$.

Here $\overline{\mathcal{M}}_{h,S}$ denotes the moduli stack of stable curves of genus h with marked points labelled by a finite set S. In each case the map to $\mathcal{B}^\nu$ is given by identifying the points labelled by $n+1$ and $n+2$. The map is an isomorphism onto the component of $\mathcal{B}^\nu$ except in case (1) for $g_1 = g_2$ and $n = 0$ and case (2), when it is étale of degree 2.

For $\overline{\mathcal{M}}_{h,S}$ a moduli stack of pointed stable curves as above, let $\pi : \mathcal{U}_{h,S} \to \overline{\mathcal{M}}_{h,S}$ denote the universal family, and $x_i : \overline{\mathcal{M}}_{h,S} \to \mathcal{U}_{h,S}$ for $i \in S$, the tautological sections of π. Define $\psi_i = x_i^* \omega_\pi$, the pullback of the relative dualising sheaf of π along the section x_i. The following result is well known; see, for example, [Harris and Morrison 1998, Proposition 3.32].

Lemma 4.2. *The pullback of $\mathcal{N}^\vee$ to $\overline{\mathcal{M}}_{g_1,S_1\cup\{n+1\}} \times \overline{\mathcal{M}}_{g_2,S_2\cup\{n+2\}}$ is identified with* $\mathrm{pr}_1^* \psi_{n+1} \otimes \mathrm{pr}_2^* \psi_{n+2}$. *Similarly, the pullback of $\mathcal{N}^\vee$ to $\overline{\mathcal{M}}_{g-1,n+2}$ is identified with* $\psi_{n+1} \otimes \psi_{n+2}$.

There is an isomorphism of stacks $c : \overline{\mathcal{M}}_{g,n+1} \to \mathcal{U}_{g,n}$ which identifies the morphism $p_{n+1} : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ given by forgetting the last point with the projection $\pi : \mathcal{U}_{g,n} \to \overline{\mathcal{M}}_{g,n}$; see [Knudsen 1983a, Section 1–2].

Lemma 4.3 [Knudsen 1983b, Theorem 4.1(d), p. 202]. *The line bundle ψ_{n+1} on $\overline{\mathcal{M}}_{g,n+1}$ is identified with the pullback of the line bundle $\omega_\pi(\Sigma)$ under the isomorphism $c\colon \overline{\mathcal{M}}_{g,n+1} \to \mathcal{U}_{g,n}$.*

Corollary 4.4. *The $\mathbb{Q}$-line bundle on the coarse moduli space of $\mathcal{B}^\nu$ defined by $\mathcal{N}^\vee$ is big and nef on each component*

Proof. This follows immediately from Lemmas 4.2, 4.3, and Theorem 3.2. □

We deduce that $H^i(\mathcal{N}) = 0$ for $i < \dim \mathcal{B}$ by Theorem A.1. This completes the proof of Theorem 2.1.

5. Proof of Theorem 2.3

We first prove a basic result which relates the deformations of a smooth Deligne–Mumford stack and its coarse moduli space.

Let $\mathcal{X}$ be a smooth proper Deligne–Mumford stack, X the coarse moduli space of $\mathcal{X}$, and $p : \mathcal{X} \to X$ the projection. Let $T_{\mathcal{X}}$ denote the tangent sheaf of $\mathcal{X}$. Let $D \subset X$ be the union of the codimension one components of the branch locus of $p : \mathcal{X} \to X$ (with its reduced structure). Let $T_X(-\log D)$ denote the subsheaf of the tangent sheaf T_X consisting of derivations which preserve the ideal sheaf of D. It is the sheaf of first order infinitesimal automorphisms of the pair (X, D).

Lemma 5.1. $p_* T_{\mathcal{X}} = T_X(-\log D)$

Proof. The sheaves $p_* T_{\mathcal{X}}$ and $T_X(-\log D)$ satisfy Serre's S_2 condition, and are identified over the locus where p is étale. So it suffices to work in codimension 1. We reduce to the case $\mathcal{X} = [\mathbb{A}^1_x/\mu_e]$, where $\mu_e \ni \zeta : x \mapsto \zeta x$. Then $X = \mathbb{A}^1_x/\mu_e = \mathbb{A}^1_y$, where $y = x^e$, and $D = (y = 0) \subset X$. Let $\pi : \mathbb{A}^1_x \to \mathbb{A}^1_x/\mu_e$ be the quotient map. We compute

$$p_* T_{\mathcal{X}} = \left(\pi_* \mathcal{O}_{\mathbb{A}^1_x} \cdot \frac{\partial}{\partial x}\right)^{\mu_e} = \mathcal{O}_{\mathbb{A}^1_y} \cdot x\frac{\partial}{\partial x} = \mathcal{O}_{\mathbb{A}^1_y} \cdot y\frac{\partial}{\partial y} = T_X(-\log D),$$

as required. □

Proposition 5.2. *The first order deformations of the stack $\mathcal{X}$ are identified with the first order locally trivial deformations of the pair (X, D).*

Proof. By the Lemma, $H^1(T_{\mathcal{X}}) = H^1(p_* T_{\mathcal{X}}) = H^1(T_X(-\log D))$. □

We now apply this result to relate deformations of the stack $\overline{\mathcal{M}}_{g,n}$ and its coarse moduli space $\overline{M}_{g,n}$.

A stable n-pointed curve of genus 0 has no nontrivial automorphisms. Hence the stack $\overline{\mathcal{M}}_{0,n}$ is equal to its coarse moduli space $\overline{M}_{0,n}$, and $\overline{M}_{0,n}$ is rigid by Theorem 2.1. Also, recall that $\overline{M}_{1,1}$ is isomorphic to $\mathbb{P}^1$ and therefore rigid. So, in the following, we assume that $g \neq 0$ and $(g, n) \neq (1, 1)$.

Let $\mathcal{D} \subset \overline{\mathcal{M}}_{g,n}$ be the component of the boundary whose general point is a curve with two components of genus 1 and $g-1$ meeting in a node, with each of the n marked points on the component of genus $g-1$. Note that each point of $\mathcal{D}$ has a nontrivial automorphism given by the involution of the component of genus 1 fixing the node. Let $p : \overline{\mathcal{M}}_{g,n} \to \overline{M}_{g,n}$ be the projection, and $D \subset \overline{M}_{g,n}$ the coarse moduli space of $\mathcal{D}$.

Lemma 5.3 [Harris and Mumford 1982, § 2]. *If* $g+n \geq 4$ *then the automorphism group of a general point of* $\overline{\mathcal{M}}_{g,n}$ *is trivial, and the divisor* $D \subset \overline{M}_{g,n}$ *is the unique codimension* 1 *component of the branch locus of* p.

Assume $g+n \geq 4$. Let $\nu : \mathcal{D}^\nu \to \mathcal{D}$ denote the normalisation of $\mathcal{D}$, so $\mathcal{D}^\nu = \overline{\mathcal{M}}_{1,1} \times \overline{\mathcal{M}}_{g-1,n+1}$. Let $\mathcal{N}_D$ denote the normal bundle of the map $\mathcal{D}^\nu \to \overline{\mathcal{M}}_{g,n}$.

Lemma 5.4. *There is an exact sequence*

$$0 \to T_{\overline{M}_{g,n}}(-\log D) \to T_{\overline{M}_{g,n}} \to p_* \nu_* \mathcal{N}_D^{\otimes 2} \to 0.$$

Proof. This is a straightforward calculation similar to [Harris and Mumford 1982, Lemma, p. 52]. □

We have

$$H^1\big(T_{\overline{M}_{g,n}}(-\log D)\big) = H^1(T_{\overline{\mathcal{M}}_{g,n}}) = 0$$

by Proposition 5.2 and Theorem 2.1. Also $H^1(\mathcal{N}_D^{\otimes 2}) = 0$ by Theorem A.1 because the $\mathbb{Q}$-line bundle defined by $\mathcal{N}_D^\vee$ on the coarse moduli space of $\mathcal{D}^\nu$ is big and nef by Corollary 4.4. So $H^1(T_{\overline{M}_{g,n}}) = 0$ by Lemma 5.4, that is, $\overline{M}_{g,n}$ has no locally trivial deformations. This concludes the proof of Theorem 2.3.

6. Nonrigidity of moduli of surfaces

We exhibit a moduli space of surfaces with boundary that is not rigid.

Let $P_1, \dots, P_4$ be 4 points in linear general position in $\mathbb{P}^2$. Let l_{ij} be the line through P_i and P_j. Let l be a line through the point $Q = l_{12} \cap l_{34}$ such that l does not pass through $l_{13} \cap l_{24}$ or $l_{14} \cap l_{23}$ and is not equal to l_{12} or l_{34}. Let $S \to \mathbb{P}^2$ be the blowup of the points $P_1, \dots, P_4, Q$, and B the sum of the strict transforms of l and the l_{ij} and the exceptional curves. Then (S, B) is a smooth surface with normal crossing boundary such that $K_S + B$ is very ample. We fix an ordering $B_1, \dots, B_{12}$ of the components of B. The moduli stack $\mathcal{M}$ of deformations of

(S, B) is isomorphic to $\mathbb{P}^1 \setminus \{q_1, \dots, q_4\}$ where the q_i are distinct points. Indeed, it suffices to observe that all deformations of (S, B) are obtained by the construction above. The moduli space $\mathcal{M}$ has a modular compactification $(\overline{M}, \partial\overline{M})$, the Kollár–Shepherd-Barron–Alexeev moduli stack of stable surfaces with boundary, which is isomorphic to $(\mathbb{P}^1, \sum q_i)$. In particular, the pair $(\overline{M}, \partial\overline{M})$ has nontrivial deformations.

Remark 6.1. The compact moduli space $\overline{M}$ is an instance of the compactifications of moduli spaces of hyperplane arrangements described in [Lafforgue 2003] (see also [Hacking et al. 2006]).

Appendix: Kodaira vanishing for stacks

Let $\mathscr{X}$ be a smooth proper Deligne–Mumford stack, X the coarse moduli space of $\mathscr{X}$, and $p : \mathscr{X} \to X$ the projection. Étale locally on X, $p : \mathscr{X} \to X$ is of the form $p : [U/G] \to U/G$, where U is a smooth affine variety and G is a finite group acting on U [Abramovich and Vistoli 2002, Lemma 2.2.3, p. 32]. A sheaf $\mathscr{F}$ on $[U/G]$ corresponds to a G-equivariant sheaf $\mathscr{F}_U$ on U, and $p_*\mathscr{F} = (\pi_*\mathscr{F}_U)^G$, where $\pi : U \to U/G$ is the quotient map.

Let $\mathscr{L}$ be a line bundle on $\mathscr{X}$. Let $n \in \mathbb{N}$ be sufficiently divisible so that for each open patch $[U/G]$ of $\mathscr{X}$ as above and point $q \in U$ the stabilizer G_q of q acts trivially on the fibre of $\mathscr{L}_U^{\otimes n}$ over q. Then the pushforward $p_*(\mathscr{L}^{\otimes n})$ is a line bundle on X. We define $p_*^{\mathbb{Q}}\mathscr{L} = \frac{1}{n} p_*(\mathscr{L}^{\otimes n}) \in \operatorname{Pic}(X) \otimes \mathbb{Q}$, and call $p_*^{\mathbb{Q}}\mathscr{L}$ the $\mathbb{Q}$-line bundle on X defined by $\mathscr{L}$.

Theorem A.1. *Assume that the coarse moduli space X is an algebraic variety. If the $\mathbb{Q}$-line bundle $p_*^{\mathbb{Q}}\mathscr{L}$ on X is big and nef then $H^i(\mathscr{L}^\vee) = 0$ for $i < \dim \mathscr{X}$.*

Remark A.2. If the coarse moduli space X is smooth then Theorem A.1 follows from [Matsuki and Olsson 2005, Theorem 2.1].

Theorem A.1 is proved by reducing to the following generalisation of the Kodaira vanishing theorem.

Theorem A.3 [Kollár and Mori 1998, Theorem 2.70, p. 73]. *Let X be a proper normal variety and Δ a $\mathbb{Q}$-divisor on X such that the pair (X, Δ) is Kawamata log terminal (klt). Let N be a $\mathbb{Q}$-Cartier Weil divisor on X such that $N \equiv M + \Delta$, where M is a big and nef $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor. Then $H^i(X, \mathscr{O}_X(-N)) = 0$ for $i < \dim X$.*

Proof of Theorem A.1. Observe first that X is a normal variety with quotient singularities. Consider the sheaf $p_*(\mathscr{L}^\vee)$ on X. If the automorphism group of a general point of $\mathscr{X}$ acts nontrivially on $\mathscr{L}$, then $p_*\mathscr{L}^\vee = 0$, and so $H^i(\mathscr{L}^\vee) = H^i(p_*\mathscr{L}^\vee) = 0$ for each i. Suppose now that the automorphism group of a general point acts trivially on $\mathscr{L}$. Then $p_*\mathscr{L}^\vee$ is a rank 1 reflexive sheaf on X. Write $p_*\mathscr{L}^\vee = \mathscr{O}_X(-N)$, where N is a Weil divisor on X. Let $n \in \mathbb{N}$ be sufficiently divisible so that

$$p_*^{\mathbb{Q}}(\mathcal{L}) = \frac{1}{n} p_*(\mathcal{L}^{\otimes n})$$

as above. Let M be a $\mathbb{Q}$-divisor corresponding to the $\mathbb{Q}$-line bundle $p_*^{\mathbb{Q}}\mathcal{L}$. There is a natural map $(p_*\mathcal{L}^\vee)^{\otimes n} \to p_*(\mathcal{L}^{\vee\otimes n})$, that is, a map $\mathbb{O}_X(-nN) \to \mathbb{O}_X(-nM)$, which is an isomorphism over the locus where p is étale. So $N \equiv M + \Delta$, where Δ is an effective $\mathbb{Q}$-divisor supported on the branch locus of p. Let $D_1, \dots, D_r$ be the codimension 1 components of the branch locus. Let e_i be the ramification index at D_i, and a_i the age of the line bundle $\mathcal{L}^\vee$ along D_i. That is, after removing the automorphism group of a general point of $\mathcal{X}$, a transverse slice of $\mathcal{X}$ at a general point of D_i is of the form $[\mathbb{A}^1_x/\mu_{e_i}]$, where $\mu_{e_i} \ni \zeta : x \mapsto \zeta \cdot x$, and μ_{e_i} acts on the fibre of $\mathcal{L}^\vee$ by the character $\zeta \mapsto \zeta^{-a_i}$, where $0 \le a_i \le e_i - 1$. We compute that $\Delta = \sum \frac{a_i}{e_i} D_i$.

We claim that (X, Δ) is klt. Let $\Delta' = \sum \frac{e_i - 1}{e_i} D_i$, then $K_{\mathcal{X}} = p^*(K_X + \Delta')$, and $\mathcal{X}$ is smooth, so (X, Δ') is klt by [Kollár and Mori 1998, Proposition 5.20(4), p. 160]. Now $\Delta \le \Delta'$ and X is $\mathbb{Q}$-factorial, so (X, Δ) is also klt. We deduce that $H^i(\mathcal{L}^\vee) = H^i(p_*\mathcal{L}^\vee) = H^i(\mathbb{O}_X(-N)) = 0$ for $i < \dim \mathcal{X}$ by Theorem A.3. □

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