Algebra & Number Theory

Volume 2

2008

No. 7

Quasimaps, straightening laws, and quantum cohomology for the Lagrangian Grassmannian

James Ruffo

mathematical sciences publishers

Quasimaps, straightening laws, and quantum cohomology for the Lagrangian Grassmannian

James Ruffo

The Drinfel'd Lagrangian Grassmannian compactifies the space of algebraic maps of fixed degree from the projective line into the Lagrangian Grassmannian. It has a natural projective embedding arising from the canonical embedding of the Lagrangian Grassmannian. We show that the defining ideal of any Schubert subvariety of the Drinfel'd Lagrangian Grassmannian is generated by polynomials which give a straightening law on an ordered set. Consequentially, any such subvariety is Cohen–Macaulay and Koszul. The Hilbert function is computed from the straightening law, leading to a new derivation of certain intersection numbers in the quantum cohomology ring of the Lagrangian Grassmannian.

1. Introduction

The space of algebraic maps of degree d from $\mathbb{P}^1$ to a projective variety X has applications to mathematical physics, linear systems theory, quantum cohomology, geometric representation theory, and the geometric Langlands correspondence [Braverman 2006; Sottile 2000; 2001]. This space is (almost) never compact, so various compactifications have been introduced to help understand its geometry. Among these (at least when X is a flag variety) are Kontsevich's space of *stable maps* [Fulton and Pandharipande 1997; Kontsevich 1995], the *quot scheme* (or space of *quasiflags*) [Chen 2001; Laumon 1990; Strømme 1987], and the *Drinfel'd compactification* (or space of *quasimaps*). The latter space is defined concretely as a projective variety, and much information can be gleaned directly from its defining equations.

Inspired by the work of Hodge [1943], standard monomial theory was developed by Lakshmibai [2003], Musili [2003], Seshadri, and others (see also the references therein), to study the *flag varieties* G/P, where G is a semisimple algebraic group and $P \subseteq G$ is a parabolic subgroup. These spaces have a decomposition into

MSC2000: primary 13F50; secondary 13P10, 14N35, 14N15.

Keywords: algebra with straightening law, quasimap, Lagrangian Grassmannian, quantum cohomology.

Schubert cells, whose closures (the *Schubert varieties*) give a basis for cohomology. As consequences of standard monomial theory, Schubert varieties are normal and Cohen–Macaulay, and one has an explicit description of their singularities and defining ideals.

A key part of standard monomial theory is that any flag variety G/P (P a parabolic subgroup) has a projective embedding which presents its coordinate ring as an *algebra with straightening law* (Definition 3.3), a special case of a *Hodge algebra* [De Concini et al. 1982]. This idea originates with the work of Hodge on the Grassmannian [1943], and was extended to the Lagrangian Grassmannian by De Concini and Lakshmibai [1981]. This framework was extended to many other flag varieties by Lakshmibai [2003], Seshadri, and their coauthors [Musili 2003]. Littelmann's path model for representations of algebraic groups [1998] later provided the necessary tools to treat all flag varieties in a unified way, as carried out by Chirivì [2000; 2001]. The general case requires a more expansive notion of an algebra with straightening law (also due to Chirivì), called a *Lakshmibai–Seshadri (LS) algebra*.

Sottile and Sturmfels [2001] have extended standard monomial theory to the *Drinfel'd Grassmannian* parametrizing algebraic maps from $\mathbb{P}^1$ into the Grassmannian. They define Schubert subvarieties of this space and prove that the homogeneous coordinate ring of any Schubert variety (including the Drinfel'd Grassmannian itself) is an algebra with straightening law on a distributive lattice. Using this fact, the authors show that these Schubert varieties are normal, Cohen–Macaulay and Koszul, and have rational singularities.

We extend these results to the *Drinfel'd Lagrangian Grassmannian*, which parametrizes algebraic maps from $\mathbb{P}^1$ into the Lagrangian Grassmannian. In particular, we prove:

Theorem 1.1. *The coordinate ring of any Schubert subvariety of the Drinfel'd Lagrangian Grassmannian is an algebra with straightening law on a doset.*

See Theorems 5.9 and 5.10 in Section 5 for more details.

A doset, as introduced in [De Concini and Lakshmibai 1981], is a certain kind of ordered set (Definition 3.1). As consequences of Theorem 1.1, we show that the coordinate ring is reduced, Cohen–Macaulay, and Koszul, and obtain formulas for its degree and dimension. These formulas have an interpretation in terms of quantum cohomology, as described in Section 4.

In Section 2, we review the basic definitions and facts concerning Drinfel'd compactifications and the Lagrangian Grassmannian. Section 3 provides the necessary background on algebras with straightening law. We discuss an application to the quantum cohomology of the Lagrangian Grassmannian in Section 4. Our main result and its consequences are proved in Section 5.

2. Preliminaries

We first give a precise definition of the Drinfel'd compactification of the space of algebraic maps from $\mathbb{P}^1$ to a homogeneous variety. We then review the basic facts we will need regarding the Lagrangian Grassmannian.

2A. ***Spaces of algebraic maps.*** Let G be a semisimple linear algebraic group. Fix a Borel subgroup $B \subseteq G$ and a maximal torus $T \subseteq B$. Let R be the set of roots (determined by T), and $S := \{\rho_1, \ldots, \rho_r\}$ the simple roots (determined by B). The simple roots form an ordered basis for the Lie algebra $\mathfrak{t}$ of T; let $\{\omega_1, \ldots, \omega_r\}$ be the dual basis (the *fundamental weights*). The *Weyl group* W is the normalizer of T modulo T itself.

Let $P \subseteq G$ be the maximal parabolic subgroup associated to the fundamental weight ω, let $\mathrm{L}(\omega)$ be the irreducible representation of highest weight ω, and let $(\bullet, \bullet)$ denote the Killing form on $\mathfrak{t}$. The flag variety G/P embeds in $\mathbb{P}\mathrm{L}(\omega)$ as the orbit of a point $[v] \in \mathbb{P}\mathrm{L}(\omega)$, where $v \in \mathrm{L}(\omega)$ is a highest weight vector. Define the *degree* of a algebraic map $f : \mathbb{P}^1 \to G/P$ to be its degree as a map into $\mathbb{P}\mathrm{L}(\omega)$. For $\rho \in R$, set $\rho^\vee := 2\rho/(\rho, \rho)$. For simplicity, assume that $(\omega, \rho^\vee) \leq 2$ for all $\rho \in S$ (that is, P is of *classical type* [Lakshmibai 2003]). This condition implies that $\mathrm{L}(\omega)$ has T-fixed lines indexed by certain admissible pairs of elements of W/W_P.

Let $\mathcal{M}_d(G/P)$ be the space of algebraic maps of degree d from $\mathbb{P}^1$ into G/P. If P is of classical type then the set $\mathfrak{D}$ of *admissible pairs* indexes homogeneous coordinates on $\mathbb{P}\mathrm{L}(\omega)$ (see Definition 3.16, and [Lakshmibai 2003; Musili 2003] for a more thorough treatment). Therefore, any map $f \in \mathcal{M}_d(G/P)$ can be expressed as

$$f : [s, t] \mapsto [p_w(s, t) \mid w \in \mathfrak{D}],$$

where the $p_w(s, t)$ are homogeneous forms of degree d. This leads to an embedding of $\mathcal{M}_d(G/P)$ into $\mathbb{P}\big((S^d\mathbb{C}^2)^* \otimes \mathrm{L}(\omega)\big)$, where $(S^d\mathbb{C}^2)^*$ is the space of homogeneous forms of degree d in two variables. The coefficients of the homogeneous forms in $(S^d\mathbb{C}^2)^*$ give coordinate functions on $(S^d\mathbb{C}^2)^* \otimes \mathrm{L}(\omega)$; they are indexed by the set $\{w^{(a)} \mid w \in \mathfrak{D}, a = 0, \ldots, d\}$, a disjoint union of $d+1$ copies of $\mathfrak{D}$.

The closure of

$$\mathcal{M}_d(G/P) \subseteq \mathbb{P}\big((S^d\mathbb{C}^2)^* \otimes \mathrm{L}(\omega)\big)$$

is called the *Drinfel'd compactification* and denoted $\mathfrak{Q}_d(G/P)$. This definition is due to V. Drinfel'd, dating from the mid-1980s. Drinfel'd never published this definition himself; to the author's knowledge its first appearance in print was in [Rosenthal 1994]; see also [Kuznetsov 1997].

Let $G = \mathrm{SL}_n(\mathbb{C})$ and P be the maximal parabolic subgroup stabilizing a fixed k-dimensional subspace of $\mathbb{C}^n$, so that $G/P = \mathrm{Gr}(k, n)$. In this case we denote the *Drinfel'd Grassmannian* $\mathfrak{Q}_d(G/P)$ by $\mathfrak{Q}_d(k, n)$. In [Sottile and Sturmfels 2001] it is shown that the homogeneous coordinate ring of $\mathcal{Q}_d(k,n)$ is an algebra with straightening law on the distributive lattice

$$\binom{[n]}{k}_d := \left\{\alpha^{(a)} \;\middle|\; \alpha \in \binom{[n]}{k}, 0 \le a \le d\right\},$$

with partial order on $\binom{[n]}{k}_d$ defined by $\alpha^{(a)} \le \beta^{(b)}$ if and only if $a \le b$ and $\alpha_i \le \beta_{b-a+i}$ for $i = 1, \dots, k-b+a$. It follows that the homogeneous coordinate ring of $\mathcal{Q}_d(k,n)$ is normal, Cohen–Macaulay, and Koszul, and that the ideal $I^d_{k,n-k} \subseteq \mathbb{C}[\binom{[n]}{k}_d]$ has a quadratic Gröbner basis consisting of the straightening relations.

Taking $d = 0$ above, this partial order is the classical *Bruhat order* on $\binom{[n]}{k}$. In general, given a semisimple algebraic group G with parabolic subgroup P, the Bruhat order is an ordering on the set of maximal coset representatives of the quotient of the Weyl group of G by the Weyl group of P.

Suppose that $d = \ell k + q$ for nonnegative integers ℓ and q with $q < k$, and let $X = (x_{ij})_{1 \le i,j \le n}$ be a matrix with polynomial entries

$$x_{ij} = x_{ij}^{(k_i)} t^{k_i} + \cdots + x_{ij}^{(1)} t + x_{ij}^{(0)},$$

where $k_i = \ell + 1$ if $i \le q$ and $k_i = \ell$ if $i > q$. The ideal $I^d_{k,n-k}$ is the kernel of the map

$$\varphi : \mathbb{C}\left[\binom{[n]}{k}_d\right] \to \mathbb{C}[X]$$

sending the variable $p_\alpha^{(a)}$ indexed by $\alpha^{(a)} \in \binom{[n]}{k}_d$ to the coefficient of t^a in the maximal minor of X whose columns are indexed by α.

The main results of [Sottile and Sturmfels 2001] follow from the next proposition. Given any distributive lattice, we denote by $\wedge$ and $\vee$, respectively, the *meet* and *join*. The symbol $\wedge$ will also be used for exterior products of vectors, but the meaning should be clear from the context.

Proposition 2.1 [Sottile and Sturmfels 2001, Theorem 10]. *Let α, β be a pair of incomparable variables in the poset $\binom{[n]}{k}_d$. There is a quadratic polynomial $S(\alpha, \beta)$ lying in the kernel of $\varphi : \mathbb{C}[\binom{[n]}{k}_d] \to \mathbb{C}[X]$ whose first two monomials are*

$$p_\alpha p_\beta - p_{\alpha\wedge\beta} p_{\alpha\vee\beta}.$$

Moreover, if $\lambda p_\gamma p_\delta$ is any noninitial monomial in $S(\alpha, \beta)$, then α, β lies in the interval $[\gamma, \delta] = \{\theta \in \binom{[n]}{k}_d \mid \gamma \le \theta \le \delta\}$.

The quadratic polynomials $S(\alpha, \beta)$ in fact form a Gröbner basis for the ideal they generate. It is shown in [Sottile and Sturmfels 2001] that there exists a toric (SAGBI) deformation taking $S(\alpha, \beta)$ to its *initial form* $p_\alpha p_\beta - p_{\alpha\wedge\beta} p_{\alpha\vee\beta}$, deforming the Drinfel'd Grassmannian into a toric variety.

Our goal is to extend the main results of standard monomial theory to the *Lagrangian Drinfel'd Grassmannian* $L\mathfrak{Q}_d(n) := \mathfrak{Q}_d(\mathrm{LG}(n))$ of degree-d maps from $\mathbb{P}^1$ into the Lagrangian Grassmannian.

2B. *The Lagrangian Grassmannian.* De Concini and Lakshmibai [1981] showed that, in its natural projective embedding, the Lagrangian Grassmannian $\mathrm{LG}(n)$ is defined by quadratic relations which give a straightening law on a doset. These relations are obtained by expressing $\mathrm{LG}(n)$ as a linear section of $\mathrm{Gr}(n, 2n)$. While this is well known, the author knows of no explicit derivation of these relations which do not require the representation theory of semisimple algebraic groups. We provide a derivation which does not rely upon representation theory (although we adopt the notation and terminology). This will be useful when we consider the Drinfel'd Lagrangian Grassmannian, to which representation theory has yet to be successfully applied.

Set $[n] := \{1, 2, \ldots, n\}$, $\bar{\imath} := -i$, and $\langle n\rangle := \{\bar{n}, \ldots, \bar{1}, 1, \ldots n\}$. If S is any set, let $\binom{S}{k}$ be the collection of subsets $\alpha = \{\alpha_1, \ldots, \alpha_k\}$ of cardinality k.

The projective space $\mathbb{P}(\bigwedge^n \mathbb{C}^{2n})$ has *Plücker coordinates* indexed by the distributive lattice $\binom{\langle n\rangle}{n}$, and the Grassmannian $\mathrm{Gr}(n, 2n)$ is the subvariety, defined by the Plücker relations, of $\mathbb{P}(\bigwedge^n \mathbb{C}^{2n})$.

Proposition 2.2 [Fulton 1997; Hodge 1943]. *For $\alpha, \beta \in \binom{\langle n\rangle}{n}$ there is a Plücker relation*

$$p_\alpha p_\beta - p_{\alpha\wedge\beta} p_{\alpha\vee\beta} + \sum_{\gamma\leq\alpha\wedge\beta<\alpha\vee\beta\leq\delta} c^{\gamma,\delta}_{\alpha,\beta} p_\gamma p_\delta = 0.$$

The defining ideal of $\mathrm{Gr}(n, 2n) \subseteq \mathbb{P}(\bigwedge^n \mathbb{C}^{2n})$ *is generated by the Plücker relations.*

Fix an ordered basis $\{e_{\bar{n}}, \ldots, e_{\bar{1}}, e_1, \ldots, e_n\}$ of the vector space $\mathbb{C}^{2n}$, and let $\Omega := \sum_{i=1}^n e_{\bar{\imath}} \wedge e_i$ be a nondegenerate alternating bilinear form. The *Lagrangian Grassmannian* $\mathrm{LG}(n)$ is the set of maximal isotropic subspaces of $\mathbb{C}^{2n}$ (relative to Ω).

Let $\{h_i := E_{ii} - E_{\bar{\imath}\bar{\imath}} \mid i \in [n]\}$ be the usual basis for the Lie algebra $\mathfrak{t}$ of T [Fulton and Harris 1991], and let $\{h_i^* \mid i \in [n]\} \subseteq \mathfrak{t}^*$ be the dual basis. Observe that $h_{\bar{\imath}}^* = -h_i^*$. The weights of any representation of $\mathrm{Sp}_{2n}(\mathbb{C})$ are $\mathbb{Z}$-linear combinations of the fundamental weights $\omega_i = h_{n-i+1}^* + \cdots + h_n^*$.

The weights of the representation $\bigwedge^n \mathbb{C}^{2n}$, and hence those of the subrepresentation $\mathrm{L}(\omega_n)$, are of the form $\omega = \sum_{i=1}^n h_{\alpha_i}^*$ for some $\alpha \in \binom{\langle n\rangle}{n}$. If $\alpha_j = \bar{\alpha}_{j'}$ for some $j, j' \in [n]$, then $h_{\alpha_j}^* = -h_{\alpha_{j'}}^*$, and thus the support of ω does not contain $h_{\alpha_j}^*$. Hence the set of all such weights ω are indexed by elements $\alpha \in \binom{\langle n\rangle}{k}$ $(k = 1, \ldots, n)$ which do not involve both i and $\bar{\imath}$ for any $i = 1, \ldots, n$.

Let V be a vector space. For simple alternating tensors $v := v_1 \wedge \cdots \wedge v_l \in \bigwedge^l V$ and $\varphi := \varphi_1 \wedge \cdots \wedge \varphi_k \in \bigwedge^k V^*$, there is a *contraction* defined by setting

$$\varphi \lrcorner v := \begin{cases} \sum_{I \in \binom{[l]}{k}} \pm v_1 \wedge \cdots \wedge \varphi_1(v_{i_1}) \wedge \cdots \wedge \varphi_k(v_{i_k}) \wedge \cdots \wedge v_l, & k \leq l, \\ 0, & k > l, \end{cases}$$

and extending bilinearly to a map $\bigwedge^k V^* \otimes \bigwedge^l V \to \bigwedge^{l-k} V$. In particular, for a fixed element $\Phi \in \bigwedge^k V^*$, we obtain a linear map $\Phi \lrcorner \bullet : \bigwedge^l V \to \bigwedge^{l-k} V$.

The Lagrangian Grassmannian embeds in $\mathbb{P}\mathrm{L}(\omega_n)$, where $\mathrm{L}(\omega_n)$ is the irreducible $\mathrm{Sp}_{2n}(\mathbb{C})$-representation of highest weight $\omega_n = h_1^* + \cdots + h_n^*$. This representation is isomorphic to the kernel of the contraction $\Omega \lrcorner \bullet : \bigwedge^n \mathbb{C}^{2n} \to \bigwedge^{n-2} \mathbb{C}^{2n}$ by Proposition 2.3. We thus have a commutative diagram of injective maps:

$$\begin{array}{ccc} \mathrm{LG}(n) & \longrightarrow & \mathrm{Gr}(n, 2n) \\ \big\downarrow & & \big\downarrow \\ \mathbb{P}\mathrm{L}(\omega_n) & \longrightarrow & \mathbb{P}(\bigwedge^n \mathbb{C}^{2n}). \end{array}$$

The next proposition implies that $\mathrm{LG}(n) = \mathrm{Gr}(n, 2n) \cap \mathbb{P}\mathrm{L}(\omega_n)$.

Proposition 2.3. *The dual of the contraction map*

$$\Omega \lrcorner \bullet : \bigwedge^n \mathbb{C}^{2n} \to \bigwedge^{n-2} \mathbb{C}^{2n}$$

is the multiplication map

$$\Omega \wedge \bullet : \bigwedge^{n-2} \mathbb{C}^{2n*} \to \bigwedge^n \mathbb{C}^{2n*}.$$

Furthermore, the irreducible representation $\mathrm{L}(\omega_n)$ *is defined by the ideal generated by the linear forms*

$$L_n := \operatorname{span}\{\Omega \wedge e^*_{\alpha_1} \wedge \cdots \wedge e^*_{\alpha_{n-2}} \mid \alpha \in \tbinom{\langle n \rangle}{n-2}\}.$$

These linear forms cut out $\mathrm{LG}(n)$ *scheme-theoretically in* $\mathrm{Gr}(n, 2n)$. *Dually*,

$$\mathrm{L}(\omega_n) = \ker(\Omega \lrcorner \bullet).$$

Proof. The proof of first statement is straightforward, and the second can be found in [Weyman 2003, Chapter 3, Exercise 1; Chapter 6, Exercise 24]. $\square$

Since the linear forms spanning L_n are supported on variables indexed by $\alpha \in \binom{\langle n \rangle}{n}$ such that $\{\bar{\imath}, i\} \in \alpha$ for some $i \in [n]$, the set of complementary variables is linearly independent. These are indexed by the set $\mathcal{P}_n$ of *admissible* elements of $\binom{\langle n \rangle}{n}$:

$$\mathcal{P}_n := \left\{\alpha \in \tbinom{\langle n \rangle}{n} \;\middle|\; i \in \alpha \Leftrightarrow \bar{\imath} \notin \alpha\right\},$$

and have a simple description in terms of partitions (see Proposition 2.4).

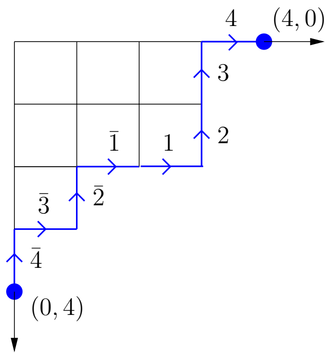


Figure 1. The partition (3, 3, 1) associated to $\bar{4}\bar{2}23$.

Consider the lattice $\mathbb{Z}^2$ with coordinates (a, b) corresponding to the point a units to the right of the origin and b units below the origin. Given an increasing sequence $\alpha \in \binom{\langle n\rangle}{n}$, let $[\alpha]$ be the lattice path beginning at $(0, n)$, ending at $(n, 0)$, and whose i-th step is vertical if $i \in \alpha$ and horizontal if $i \notin \alpha$. We can associate a partition to α by taking the boxes lying in the region bounded by the coordinate axes and $[\alpha]$. For instance, the sequence $\alpha = \bar{4}\bar{2}23 \in \binom{\langle 4\rangle}{4}$ is associated to the partition shown in Figure 1.

Proposition 2.4. *The bijection between increasing sequences and partitions induces a bijection between sequences α which do not contain both i and $\bar{\imath}$ for any $i \in [n]$, and partitions which lie inside the $n \times n$ square (n^n) and are symmetric with respect to reflection about the diagonal $\{(a, a) \mid a \in \mathbb{Z}\} \subseteq \mathbb{Z}^2$.*

Proof. The poset $\mathcal{P}_n$ consists of those $\alpha \in \binom{\langle n\rangle}{n}$ which are fixed upon negating each element of α and taking the complement in $\langle n\rangle$. On the other hand, the composition of these two operations (in either order) corresponds to reflecting the associated diagram about the diagonal. □

Remark 2.5. We will use an element of $\binom{\langle n\rangle}{n}$ and its associated partition interchangeably. We denote by α^t the *transpose* partition obtained by reflecting α about the diagonal in $\mathbb{Z}^2$. As a sequence, α^t is the complement of $\{\bar{\alpha}_1, \ldots, \bar{\alpha}_n\} \subseteq \langle n\rangle$. We denote by α_+ (respectively, α_-) the subsequence of positive (negative) elements of α.

Definition 2.6. The *Lagrangian involution* is the map $\tau : p_\alpha \mapsto \sigma_\alpha p_{\alpha^t}$, where $\sigma_\alpha := \operatorname{sgn}(\alpha_+^c, \alpha_+) \cdot \operatorname{sgn}(\alpha_-, \alpha_-^c) = \pm 1$, and $\operatorname{sgn}(a_1, \ldots, a_s)$ denotes the sign of the permutation sorting the sequence $(a_1, \ldots, a_s)$.

For example, if $\alpha = \bar{4}\bar{1}23$, then $\alpha_+ = 23$, $\alpha_- = \bar{4}\bar{1}$, and $\sigma_\alpha = 1$.

The Grassmannian $\mathrm{Gr}(n, 2n)$ has a natural geometric involution

$$\bullet^\perp : \mathrm{Gr}(n, 2n) \to \mathrm{Gr}(n, 2n)$$

sending an n-plane U to its orthogonal complement

$$U^{\perp} := \{u \in \mathbb{C}^{2n} \mid \Omega(u, u') = 0, \text{ for all } u' \in U\}$$

with respect to Ω. The next proposition relates $\bullet^{\perp}$ to the Lagrangian involution.

Proposition 2.7. *The map* $\bullet^{\perp} : \mathrm{Gr}(n, 2n) \to \mathrm{Gr}(n, 2n)$ *expressed in Plücker coordinates coincides with the Lagrangian involution*:

$$\left[p_\alpha \;\middle|\; \alpha \in \binom{\langle n\rangle}{n}\right] \mapsto \left[\sigma_\alpha p_{\alpha^t} \;\middle|\; \alpha \in \binom{\langle n\rangle}{n}\right].$$

In particular, the relation $p_\alpha - \sigma_\alpha p_{\alpha^t} = 0$ *holds on* $\mathrm{LG}(n)$.

Proof. The set of n-planes in $\mathbb{C}^{2n}$ which do not meet the span of the first n standard basis vectors is open and dense in $\mathrm{Gr}(n, 2n)$. Any such n-plane is the row space of an $n \times 2n$ matrix

$$Y := (I \mid X)$$

where I is the $n \times n$ identity matrix and X is a generic $n \times n$ matrix. We work in the affine coordinates given by the entries in X. For $\alpha \in \binom{\langle n\rangle}{n}$, denote the α-th minor of Y by $p_\alpha(Y)$. For a set of indices $\alpha = \{\alpha_1, \ldots, \alpha_k\} \subseteq [n]$, let $\alpha^c := [n] \setminus \alpha$ be the complement, $\alpha' := \{n-\alpha_k+1, \ldots, n-\alpha_1+1\}$, and $\bar{\alpha} := \{\bar{\alpha}_1, \ldots, \bar{\alpha}_k\}$. Via the correspondence between partitions and sequences (Proposition 2.4), $\alpha^t = \bar{\alpha}^c$.

We claim that $\bullet^{\perp}$ reflects X along the antidiagonal. To see this, we simply observe how the rows of Y pair under Ω. For vectors $u, v \in \mathbb{C}^n$, let $(u, v) \in \mathbb{C}^{2n}$ be the concatenation. Let $r_i := (e_i, v_i) \in \mathbb{C}^{2n}$ be the i-th row of Y. For $k \in \langle n\rangle$, we let $r_{ik} \in \mathbb{C}$ be the k-th entry of r_i. Then, for $i, j \in \langle n\rangle$,

$$\Omega(r_i, r_j) = (e_i, v_i) \cdot (-v_j, e_j)^t = r_{i,n-j+1} - r_{j,n-i+1}.$$

It follows that the effect of $\bullet^{\perp}$ on the minor $X_{\rho,\gamma}$ of X given by row indices ρ and column indices γ is

$$(X^{\perp})_{\rho,\gamma} = X_{\gamma',\rho'}.$$

Let $\alpha = \bar{\epsilon} \cup \phi \in \binom{\langle n\rangle}{n}$, where ϵ and ϕ are subsets of $[n]$ whose cardinalities sum to n. Combining the above description of $\bullet^{\perp}$ with the identity

$$p_\alpha(Y) = \mathrm{sgn}(\epsilon^c, \epsilon) X_{(\epsilon^c)',\phi},$$

we have

$$\begin{aligned} p_\alpha(Y^{\perp}) &= \mathrm{sgn}(\epsilon^c, \epsilon)(X^{\perp})_{(\epsilon^c)',\phi} = \mathrm{sgn}(\epsilon^c, \epsilon) X_{\phi',\epsilon^c} \\ &= \mathrm{sgn}(\epsilon^c, \epsilon)\mathrm{sgn}(\phi, \phi^c) p_{(\bar{\phi}^c,\epsilon^c)}(Y) = \sigma_\alpha p_{\alpha^t}(Y). \end{aligned}$$

It follows that the relation $p_\alpha - \sigma_\alpha p_{\alpha^t} = 0$ holds on a dense Zariski-open subset and hence identically on all of $\mathrm{LG}(n)$. ☐

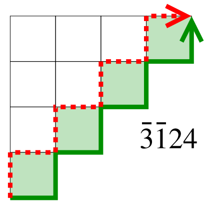
$\bar{4}\bar{2}13$  $\bar{3}\bar{1}24$

Figure 2. The paths associated to $\bar{4}\bar{2}13$ and $\bar{3}\bar{1}24$ in $\mathcal{P}_4$.

By Proposition 2.7, the system of linear forms

$$L_n' := \operatorname{span}\{p_\alpha - \sigma_\alpha p_{\alpha^t} \mid \alpha \in \tbinom{\langle n\rangle}{n}\} \tag{2-1}$$

defines $\mathrm{LG}(n) \subseteq \mathrm{Gr}(n, 2n)$ *set*-theoretically. Since $\mathrm{LG}(n)$ lies in no hyperplane of $\mathbb{P}\mathrm{L}(\omega_n)$, L_n' is a linear subspace of the span L_n of the defining equations of $\mathrm{L}(\omega_n) \subseteq \bigwedge^n \mathbb{C}^{2n}$. The generators of L_n' given in (2-1) suggest that homogeneous coordinates for the Lagrangian Grassmannian should be indexed by some sort of quotient (which we will call $\mathcal{D}_n$) of the poset $\binom{\langle n\rangle}{n}$. The correct notion is that of a doset (Definition 3.1). An important set of representatives for $\mathcal{D}_n$ in $\binom{\langle n\rangle}{n}$ is the set of *Northeast* partitions (Proposition 2.10).

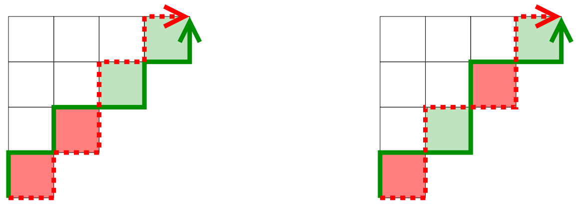
Remark 2.8. The set of *strict* partitions with at most n rows and columns is commonly used to index Plücker coordinates for the Lagrangian Grassmannian. Given a symmetric partition $\alpha \in \mathcal{P}_n$, we can obtain a strict partition by first removing the boxes of α which lie below the diagonal, and then left-justifying the remaining boxes. This gives a bijection between the two sets of partitions.

By Proposition 2.4, we may identify elements of $\binom{\langle n\rangle}{n}$ with partitions lying in the $n \times n$ square (n^n), and $\mathcal{P}_n$ with the set of symmetric partitions. Define a map

$$\pi_n : \tbinom{\langle n\rangle}{n} \to \mathcal{P}_n \times \mathcal{P}_n, \quad \pi_n(\alpha) := (\alpha \wedge \alpha^t, \alpha \vee \alpha^t).$$

Let $\mathcal{D}_n$ be the image of π_n. It is called the set of *admissible pairs*, and is a subset of $\mathbb{O}_{\mathcal{P}_n} := \{(\alpha, \beta) \in \mathcal{P}_n \times \mathcal{P}_n \mid \alpha \le \beta\}$. The image of $\mathcal{P}_n \subseteq \binom{\langle n\rangle}{n}$ under π_n is the diagonal $\Delta_{\mathcal{P}_n} \subseteq \mathcal{P}_n \times \mathcal{P}_n$.

To show that $\mathcal{D}_n$ indexes coordinates on $\mathrm{LG}(n)$, we will work with a convenient set of representatives of the fibers of π_n. The fiber over $(\alpha, \beta) \in \mathcal{D}_n$ can be described as follows. The lattice paths $[\alpha]$ and $[\beta]$ must meet at the diagonal. Since α and β are symmetric, they are determined by the segments of their associated paths to the right and above the diagonal. Let $\Pi(\alpha, \beta)$ be the set of boxes bounded by these segments. Taking $n = 4$ for example, $\Pi(\bar{4}\bar{2}13, \bar{3}\bar{1}24)$ consists of the two shaded boxes above the diagonal in Figure 2. The lattice path $[\bar{4}\bar{2}13]$ is above and to the left of the path $[\bar{3}\bar{1}24]$.



$\bar{4}\bar{2}24$ and $\bar{3}\bar{1}13$ $\bar{4}\bar{1}14$ and $\bar{3}\bar{2}23$

Figure 3. Elements of $\pi_n^{-1}(\bar{4}\bar{2}13, \bar{3}\bar{1}24)$.

For any partition $\alpha \subseteq (n^n)$, the set $\alpha_+ \subseteq \alpha$ consists of the boxes of α on or above the main diagonal, and $\alpha_- \subseteq \alpha$ consists of the boxes of α on or below the main diagonal (compare Remark 2.5). Similarly, let $\Pi_+(\alpha, \beta) \subseteq \Pi(\alpha, \beta)$ be the set of boxes above the diagonal and let $\Pi_-(\alpha, \beta) \subseteq \Pi(\alpha, \beta)$ be the set of boxes below the diagonal.

A subset $S \subseteq (n^n)$ of boxes is *disconnected* if $S = S' \sqcup S''$ and no box of S' shares an edge with a box of S''. A subset S is connected if it is not disconnected. Let $\bigsqcup_{i=1}^k S_i$ be the decomposition of $\Pi_+(\alpha, \beta)$ into its connected components (so that $\bigsqcup_{i=1}^k S_i^t$ is the decomposition of $\Pi_-(\alpha, \beta)$).

Any element γ of the fiber $\pi_n^{-1}(\alpha, \beta)$ is obtained by choosing a subset $I \subseteq [k]$ and setting

$$\gamma = \alpha \cup \left(\bigcup_{i \notin I} S_i^t \right) \cup \left(\bigcup_{i \in I} S_i \right).$$

The elements of $\pi_n^{-1}(\bar{4}\bar{2}13, \bar{3}\bar{1}24)$ are shown in Figure 3.

Definition 2.9. A partition α is *Northeast* if $\alpha_-^t \subseteq \alpha_+$ and is *Southwest* if its transpose is Northeast.

For example, $\bar{4}\bar{2}24 \in \mathcal{P}_4$ is Northeast while $\bar{4}\bar{1}14 \in \mathcal{P}_4$ is neither Northeast nor Southwest. We summarize these ideas as:

Proposition 2.10. *Let*

$$(\alpha, \beta) \in \mathcal{D}_n$$

be an element of the image of π_n. Then $\pi_n^{-1}(\alpha, \beta)$ is in bijection with the set of subsets of connected components of $\Pi_+(\alpha, \beta)$. There exists a unique Northeast element of $\pi_n^{-1}(\alpha, \beta)$, namely, the element corresponding to all the connected components. Similarly, there is a unique Southwest element corresponding to the empty set of components.

Example 2.11. The Lagrangian Grassmannian $\mathrm{LG}(4) \subseteq \mathrm{Gr}(4, 8)$ is defined by the ideal L_4. From the explicit linear generators given in Proposition 2.3, it is evident that $L_4 \subseteq \bigwedge^4 \mathbb{C}^{8*}$ is spanned by weight vectors. For example, the generators of $L_4 \cap (\bigwedge^4 \mathbb{C}^{8^*})_0$ are the vectors of weight zero:

$$\begin{aligned}\Omega \wedge p_{\bar{1}1} &= p_{\bar{4}\bar{1}14} + p_{\bar{3}\bar{1}13} + p_{\bar{2}\bar{1}12},\\ \Omega \wedge p_{\bar{2}2} &= p_{\bar{4}\bar{2}24} + p_{\bar{3}\bar{2}23} + p_{\bar{2}\bar{1}12},\\ \Omega \wedge p_{\bar{3}3} &= p_{\bar{4}\bar{3}34} + p_{\bar{3}\bar{2}23} + p_{\bar{3}\bar{1}13},\\ \Omega \wedge p_{\bar{4}4} &= p_{\bar{4}\bar{3}34} + p_{\bar{4}\bar{2}24} + p_{\bar{4}\bar{1}14}.\end{aligned} \tag{2-2}$$

The following linear forms lie in the span of the right-hand side of (2-2):

$$p_{\bar{2}\bar{1}12} + p_{\bar{4}\bar{3}34},\ \ p_{\bar{3}\bar{1}13} + p_{\bar{4}\bar{2}24},\ \ p_{\bar{3}\bar{2}23} + p_{\bar{4}\bar{1}14},\ \text{and}\ p_{\bar{4}\bar{1}14} + p_{\bar{4}\bar{2}24} + p_{\bar{4}\bar{3}34}. \tag{2-3}$$

Three of the linear forms in (2-3) are supported on a pair $\{p_\alpha, p_{\alpha^t}\}$, and the remaining linear form expresses the Plücker coordinate $p_{\bar{4}\bar{1}14}$ as a linear combination of coordinates indexed by Northeast partitions (this follows from Lemma 5.5 in general).

Since each pair $\{p_\alpha, p_{\alpha^t}\}$ is incomparable, there is a Plücker relation which, after reduction by the linear forms (2-2), takes the form

$$\pm p_\alpha^2 - p_\beta p_\gamma + \text{lower order terms}$$

where $\beta := \alpha \wedge \alpha^t$ and $\gamma := \alpha \vee \alpha^t$ are respectively the meet and join of α and α^t. Defining $p_{(\beta,\gamma)} := p_\alpha = \sigma_\alpha p_{\alpha^t}$ we can regard such an equation as giving a rule for rewriting $p^2_{(\beta,\gamma)}$ as a linear combination of monomials supported on a chain. This general case is treated in Section 5.

3. Algebras with straightening law

3A. ***Generalities.*** The following definitions are due to De Concini and Lakshmibai [1981]. Let $\mathcal{P}$ be a poset, $\Delta_{\mathcal{P}}$ the diagonal in $\mathcal{P} \times \mathcal{P}$, and

$$\mathbb{O}_{\mathcal{P}} := \{(\alpha, \beta) \in \mathcal{P} \times \mathcal{P} \mid \alpha \leq \beta\}$$

the subset of $\mathcal{P} \times \mathcal{P}$ defining the order relation on $\mathcal{P}$.

Definition 3.1. [De Concini and Lakshmibai 1981] A *doset* on $\mathcal{P}$ is a set $\mathcal{D}$ such that $\Delta_{\mathcal{P}} \subseteq \mathcal{D} \subseteq \mathbb{O}_{\mathcal{P}}$, and if $\alpha \leq \beta \leq \gamma$, then $(\alpha, \gamma) \in \mathcal{D}$ if and only if $(\alpha, \beta) \in \mathcal{D}$ and $(\beta, \gamma) \in \mathcal{D}$. The ordering on $\mathcal{D}$ is given by $(\alpha, \beta) \leq (\gamma, \delta)$ if and only if $\beta \leq \gamma$ in $\mathcal{P}$. We call $\mathcal{P}$ the *underlying poset*.

Remark 3.2. The doset ordering just defined does not, in general, satisfy the reflexive property. That is, for $(\alpha, \beta) \in \mathcal{D}$, it is not generally true that $(\alpha, \beta) \leq (\alpha, \beta)$. Indeed, this is the case if and only if $(\alpha, \beta) \in \Delta_{\mathcal{P}} \cong \mathcal{P}$, that is, if and only if $\alpha = \beta$.

The *Hasse diagram* of a doset $\mathcal{D}$ on $\mathcal{P}$ is obtained from the Hasse diagram of $\mathcal{P} \subseteq \mathcal{D}$ by drawing a double line for each cover $\alpha \lessdot \beta$ such that (α, β) is in $\mathcal{D}$. The defining property of a doset implies that we can recover all the information in the doset from its Hasse diagram. See Figure 6 in Section 3C for an example.

An algebra with straightening law (Definition 3.3 below) is an algebra generated by indeterminates $\{p_\alpha \mid \alpha \in \mathfrak{D}\}$ indexed by a (finite) doset $\mathfrak{D}$ with a basis consisting of *standard* monomials supported on a chain. That is, a monomial $p_{(\alpha_1,\beta_1)} \cdots p_{(\alpha_k,\beta_k)}$ is standard if $\alpha_1 \leq \beta_1 \leq \alpha_2 \leq \beta_2 \leq \cdots \leq \alpha_k \leq \beta_k$. Furthermore, monomials which are not standard are subject to certain *straightening relations*, as described in the following definition.

Definition 3.3. [De Concini and Lakshmibai 1981] Let $\mathfrak{D}$ be a doset. A graded $\mathbb{C}$-algebra

$$A = \bigoplus_{q \geq 0} A_q$$

is an *algebra with straightening law* on $\mathfrak{D}$ if there is an injection $\mathfrak{D} \ni (\alpha, \beta) \mapsto p_{(\alpha,\beta)} \in A_1$ such that:

(1) The set $\{p_{(\alpha,\beta)} \mid (\alpha, \beta) \in \mathfrak{D}\}$ generates A.

(2) The set of standard monomials are a $\mathbb{C}$-basis of A.

(3) For any monomial $m = p_{(\alpha_1,\beta_1)} \cdots p_{(\alpha_k,\beta_k)}$, $(\alpha_i, \beta_i) \in \mathfrak{D}$ and $i = 1, \ldots, k$, if

$$m = \sum_{j=1}^{N} c_j p_{(\alpha_{j1},\beta_{j1})} \cdots p_{(\alpha_{jk},\beta_{jk})},$$

is the unique expression of m as a linear combination of distinct standard monomials, then the sequence $(\alpha_{j1} \leq \beta_{j1} \leq \cdots \leq \alpha_{jk} \leq \beta_{jk})$ is lexicographically smaller than $(\alpha_1 \leq \beta_1 \leq \cdots \leq \alpha_k \leq \beta_k)$. That is, if $\ell \in [2k]$ is minimal such that $\alpha_{j\ell} \neq \alpha_\ell$, then $\alpha_{j\ell} < \alpha_\ell$.

(4) If $\alpha_1 \leq \alpha_2 \leq \alpha_3 \leq \alpha_4$ are such that for some permutation $\sigma \in S_4$ we have $(\alpha_{\sigma(1)}, \alpha_{\sigma(2)}) \in \mathfrak{D}$ and $(\alpha_{\sigma(3)}, \alpha_{\sigma(4)}) \in \mathfrak{D}$, then

$$p_{(\alpha_{\sigma(1)},\alpha_{\sigma(2)})} p_{(\alpha_{\sigma(3)},\alpha_{\sigma(4)})} = \pm p_{(\alpha_1,\alpha_2)} p_{(\alpha_3,\alpha_4)} + \sum_{i=1}^{N} r_i m_i$$

where the m_i are quadratic standard monomials distinct from $p_{(\alpha_1,\alpha_2)} p_{(\alpha_3,\alpha_4)}$.

The ideal of straightening relations is generated by homogeneous quadratic forms in the p_α $(\alpha \in \mathfrak{D})$, so we may consider the projective variety $X := \operatorname{Proj} A$ they define. For each $\alpha \in \mathcal{P}$, we have the *Schubert variety*

$$X_\alpha := \{x \in X \mid p_{(\beta,\gamma)}(x) = 0 \text{ for } \gamma \not\leq \alpha\}$$

and the *dual Schubert variety*

$$X^\alpha := \{x \in X \mid p_{(\beta,\gamma)}(x) = 0 \text{ for } \beta \not\geq \alpha\}.$$

Remark 3.4. If $\alpha \lessdot \beta$ and $(\alpha, \beta) \in \mathfrak{D}$, then the multiplicity of X_α in X_β is 2, and likewise for the multiplicity of X^α in X^β. This fact will arise in Section 4 when we consider enumerative questions.

We recall the case when X is the Grassmannian of k-planes in $\mathbb{C}^n$, whose coordinate ring is an algebra with straightening law on the poset (with trivial doset structure) $\binom{[n]}{k}$.

For each $i \in [n]$, set $F_i := \langle e_1, \dots, e_i \rangle$ and $F'_i := \langle e_n, \dots, e_{n-i+1} \rangle$, where $\langle \cdots \rangle$ denotes linear span and $\{e_1, \dots, e_n\}$ is the standard basis of $\mathbb{C}^n$. We call $F_\bullet := \{F_1 \subseteq \cdots \subseteq F_n\}$ the *standard coordinate* flag, and $F'_\bullet := \{F'_1 \subseteq \cdots \subseteq F'_n\}$ the *opposite* flag.

We represent any k-plane $E \in \mathrm{Gr}(k, n)$ as the row space of a $k \times n$ matrix. Furthermore, any such k-plane E is the row space of a unique reduced row echelon matrix. The Schubert variety X_α consists of precisely the k-planes E such that the pivot in row i is weakly to the left of column α_i. Since the Plücker coordinate $p_\beta(E)$ is just the β-th maximal minor of this matrix, we see that $E \in X_\alpha$ if and only if $p_\beta(E) = 0$ for all $\beta \not\leq \alpha$; hence the general definition of the Schubert variety X_α (and by a similar argument, the dual Schubert variety X^α) agrees with the well-known geometric definition in the case of the Grassmannian. Namely, for $\alpha \in \binom{[n]}{k}$, the Schubert variety X_α is

$$X_\alpha = \{E \in \mathrm{Gr}(k, n) \mid \dim(E \cap F_{\alpha_i}) \geq i, \text{ for } i = 1, \dots, k\},$$

and the dual Schubert variety X^α is

$$X^\alpha = \{E \in \mathrm{Gr}(k, n) \mid \dim(E \cap F'_{n-\alpha_i+1}) \geq k-i+1, \text{ for } i = 1, \dots, k\}.$$

For a fixed projective variety $X \subseteq \mathbb{P}^n$, there are many homogeneous ideals which cut out X set-theoretically. However, there exists a unique such ideal which is saturated and radical. Under mild hypotheses, any ideal generated by straightening relations on a doset is saturated and radical. The proofs of Theorems 3.8 and 3.9 illustrate the usefulness of Schubert varieties in the study of an algebra with straightening law.

Definition 3.5. An ideal $I \subseteq \mathbb{C}[x_0, \dots, x_n]$ is *saturated* if, given a polynomial $f \in \mathbb{C}[x_0, \dots, x_n]$ and an integer $N \in \mathbb{N}$,

$$x_i^N f = 0 \mod I$$

for all $i = 0, \dots, n$ implies that

$$f = 0 \mod I.$$

Definition 3.6. A ring A is *reduced* if it has no nilpotent elements; that is, if $f \in A$ satisfies $f^N = 0$ for some $N \in \mathbb{N}$, then $f = 0$.

Definition 3.7. An ideal $I \subseteq \mathbb{C}[x_0, \dots, x_n]$ is *radical* if, given a polynomial $f \in \mathbb{C}[x_0, \dots, x_n]$ and an integer $N \in \mathbb{N}$,

$$f^N = 0 \mod I$$

implies that

$$f = 0 \mod I.$$

That is, I is radical if the quotient $\mathbb{C}[x_0, \dots, x_n]/I$ is reduced.

The next two results concern an algebra with straightening law on a doset $\mathcal{D}$ with underlying poset $\mathcal{P}$. We write $A = \mathbb{C}[\mathcal{D}]/J$ for this algebra, where J is the ideal generated by the straightening relations. Proposition 3.9 is a special case of [Chirivì 2000, Proposition 27].

Theorem 3.8. *Let $\mathcal{D}$ be a doset whose underlying poset has a unique minimal element α_0. Then any ideal J of straightening relations on $\mathcal{D}$ is saturated.*

Proof. Let $f \notin J$. Modulo J, we may write $f = \sum_{i=1}^{k} a_i m_i$ where the m_i are (distinct) standard monomials, and $a_i \in \mathbb{C}$. For each $N \in \mathbb{N}$,

$$p_{\alpha_0}^N f = \sum_{i=1}^{k} a_i p_{\alpha_0}^N m_i$$

is a linear combination of standard monomials, since supp $m_i \cup \{\alpha_0\}$ is a chain for each $i \in [k]$. It is nontrivial since $p_{\alpha_0}^N m_i = p_{\alpha_0}^N m_j$ implies $i = j$. Thus $p_{\alpha_0}^N f \notin J$ for any $N \in \mathbb{N}$. □

Proposition 3.9. *An algebra with straightening law on a doset is reduced.*

Proof. Let A be an algebra with straightening law. For $f \in A$ and $\alpha \in \mathcal{P}$, denote by f_α the restriction of f to the dual Schubert variety X^α.

We will show by induction on the poset $\mathcal{P}$ that $f_\alpha^n = 0$ implies $f_\alpha = 0$. Note that by induction on n it suffices to do this for $n = 2$. Indeed, assume that we have shown that $f^2 = 0$ implies $f = 0$ for any f in some ring A, and suppose $f^n = 0$. Then $(f^{\lceil \frac{n}{2} \rceil})^2 = 0$, so that $f^{\lceil \frac{n}{2} \rceil} = 0$ by our assumption, and thus $f = 0$ by induction.

Let $f \in A$ be such that $f_\alpha^2 = 0$. In particular, $f_\beta^2 = 0$ for all $\beta \geq \alpha$ (since $X^\beta \subseteq X^\alpha$), so that $f_\beta = 0$ by induction. It follows that f_α is supported on monomials on X^α which vanish on X^β for all $\beta \geq \alpha$. That is,

$$f_\alpha = \sum_{i=1}^{m} c_i p_\alpha^{e_i} p_{(\alpha,\beta_{1,i})} \cdots p_{(\alpha,\beta_{\ell_i,i})}. \tag{3-1}$$

For the right hand side of (3-1) to be standard, we must have $\ell_i = 1$ for all $i = 1, \dots m$. Also, homogeneity implies that $e := e_1 = \cdots = e_m$ for $i = 1, \dots m$. Thus, if we set $\beta_i := \beta_{1,i}$, then f_α has the form

$$f_\alpha = p_\alpha^e \sum_{i=1}^{m} c_i \, p_{(\alpha,\beta_i)}. \tag{3-2}$$

Choose a linear extension of $\mathcal{D}$ as follows. Begin with a linear extension of $\mathcal{P} \subseteq \mathcal{D}$. For incomparable elements (α, β), (γ, δ) of $\mathcal{D}$, set $(\alpha, \beta) \leq (\gamma, \delta)$ if $\beta < \delta$ or $\beta = \delta$ and $\alpha \leq \gamma$. With respect to the resulting linear ordering of the variables, take the lexicographic term order on monomials in A.

For an element $g \in A$, denote by $\operatorname{lt}(g)$ (respectively, $\operatorname{lm}(g)$) the lead term (respectively, lead monomial) of g. Reordering the terms in (3-2) if necessary, we may assume that

$$\operatorname{lt}(f_\alpha) = c_1 p_\alpha^e p_{(\alpha,\beta_1)}.$$

Writing f_α^2 as a linear combination of standard monomials (by first expanding the square of the right hand side of (3-2) and then applying the straightening relations), we see that

$$\operatorname{lt}(f_\alpha^2) = \pm c_1^2 p_\alpha^{2e+1} p_{\beta_1}.$$

This follows from our choice of term order and the condition in Definition 3.3 (4).

We claim that $\operatorname{lt}(f_\alpha^2)$ cannot be cancelled in the expression for f_α^2 as a sum of standard monomials. Indeed, suppose there are $i, j \in [m]$ such that

$$\operatorname{lm}\big((p_\alpha^e p_{(\alpha,\beta_i)}) \cdot (p_\alpha^e p_{(\alpha,\beta_j)})\big) = p_\alpha^{2e+1} p_{\beta_1}.$$

Then by the straightening relations, $\beta_1 \leq \beta_i, \beta_j$. But $\beta_1 \not< \beta_i$ since

$$\operatorname{lm}(f_\alpha) = p_\alpha^e p_{(\alpha,\beta_1)}.$$

For the same reasons, $\beta_1 \not< \beta_j$. Therefore $\beta_i = \beta_j = \beta_1$, so $c_1 p_\alpha^e p_{(\alpha,\beta_1)}$ is the only term contributing to the monomial $p_\alpha^{2e+1} p_{\beta_1}$ in f_α^2. □

Remark 3.10. Proposition 3.9 was first proved for an algebra with straightening law on a *poset* in [Eisenbud 1980], but the methods used (deformation to the initial ideal) are not well-suited for a doset. The proof given here is essentially an extension of the proof of Bruns and Vetter [1988, Theorem 5.7] to the doset case.

3B. ***Hilbert series of an algebra with straightening law.*** We compute the Hilbert series of an algebra with straightening law A on a doset, and thus obtain formulas for the dimension and degree of $\operatorname{Proj} A$. Let $\mathcal{P}$ be a poset and $\mathcal{D}$ a doset on $\mathcal{P}$. Assume that $\mathcal{P}$ and $\mathcal{D}$ are *ranked*; that is, any two maximal chains in $\mathcal{D}$ (respectively, $\mathcal{P}$) have the same length. Define rank $\mathcal{D}$ (respectively, rank $\mathcal{P}$) to be the length of any maximal chain in $\mathcal{D}$ (respectively, $\mathcal{P}$).

First, we compute the Hilbert series of A with respect to a suitably chosen fine grading of A by the elements of a semigroup, as follows.

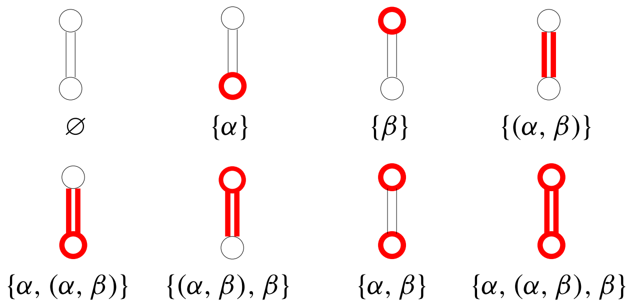


Figure 4. The set $\mathrm{Ch}(\mathcal{D})$ of chains in $\mathcal{D}$.

Monomials in $\mathbb{C}[\mathcal{D}]$ are determined by their exponent vectors. We can therefore identify the set of such monomials with the semigroup $\mathbb{N}^{\mathcal{D}}$. Define the *weight map* $\mathbf{w} : \mathbb{N}^{\mathcal{D}} \to \mathbb{Q}^{\mathcal{P}}$ by setting $\mathbf{w}(\alpha, \beta) := \frac{\epsilon_\alpha + \epsilon_\beta}{2}$, where $\epsilon_\alpha \in \mathbb{Q}^{\mathcal{P}}$ ($\alpha \in \mathcal{P}$) is the vector with α-coordinate equal to 1 and all other coordinates equal to 0. This gives a grading of A by the semigroup $\mathrm{im}(\mathbf{w})$. Let $\mathrm{Ch}(\mathcal{D})$ be the set of all chains in $\mathcal{D}$. Since the standard monomials (those supported on a chain) form a $\mathbb{C}$-basis for A, the Hilbert series with respect to this fine grading is

$$H_A(r) = \sum_{c \in \mathrm{Ch}(\mathcal{D})} \sum_{\substack{a \in \mathrm{im}(\mathbf{w}) \\ \mathrm{supp}(a) = c}} r^a,$$

where $r := (r_\alpha \mid \alpha \in \mathcal{P})$, $a = (a_\alpha \mid \alpha \in \mathcal{P})$, and $r^a = \prod_{\alpha \in \mathcal{P}} r_\alpha^{a_\alpha}$. Note that elements of $\mathrm{im}(\mathbf{w})$ correspond to certain monomials with *rational* exponents (supported on $\mathcal{P}$). For example, $(\alpha, \beta) \in \mathcal{D}$ corresponds to $\sqrt{r_\alpha r_\beta}$. Setting all $r_\alpha = r$, we obtain the usual (coarse) Hilbert series, defined with respect to the usual $\mathbb{Z}$-grading on A by degree.

Example 3.11. Consider the doset $\mathcal{D} := \{\alpha, (\alpha, \beta), \beta\}$ on the two element poset $\{\alpha < \beta\}$. The elements of $\mathrm{Ch}(\mathcal{D})$ are shown in Figure 4.

$$\mathrm{Ch}(\mathcal{D}) = \big\{\varnothing, \{\alpha\}, \{\beta\}, \{(\alpha, \beta)\}, \{\alpha, (\alpha, \beta)\}, \{(\alpha, \beta), \beta\}, \{\alpha, \beta\}, \{\alpha, (\alpha, \beta), \beta\}\big\}.$$

We have

$$H_A(r) = 1 + \frac{r_\alpha}{1 - r_\alpha} + \frac{r_\beta}{1 - r_\beta} + \sqrt{r_\alpha r_\beta} + \frac{\sqrt{r_\alpha^3 r_\beta}}{1 - r_\alpha} + \frac{\sqrt{r_\alpha r_\beta^3}}{(1 - r_\beta)} + \frac{r_\alpha r_\beta}{(1 - r_\alpha)(1 - r_\beta)} + \frac{\sqrt{r_\alpha^3 r_\beta^3}}{(1 - r_\alpha)(1 - r_\beta)}.$$

Setting $r = r_\alpha = r_\beta$, we obtain the Hilbert series with respect to the usual $\mathbb{Z}$-grading of $\mathbb{C}[\mathcal{D}]$:

$$h_A(r) = 1 + \frac{2r}{1-r} + r + \frac{2r^2}{1-r} + \frac{r^3+r^2}{(1-r)^2} = \frac{r+1}{(1-r)^2} = 1 + \sum_{i=1}^{\infty}(2i+1)r^i.$$

We see that the Hilbert polynomial is $p(i) = 2i+1$, so $\dim(\operatorname{Proj} A) = 1$, and $\deg(\operatorname{Proj} A) = 2$.

Remark 3.12. The coordinate ring of the Lagrangian Grassmannian LG(2) is an algebra with straightening law on the five-element doset obtained by adding two elements $\hat{0} < \alpha$ and $\hat{1} > \beta$ to the doset of Example 3.11. The addition of these elements does not affect the degree, which is also 2. Theorem 5.9 allows us to carry out such degree computations for the Drinfel'd Lagrangian Grassmannian, giving a new derivation of the intersection numbers computed in quantum cohomology.

Fix a poset $\mathcal{P}$, and a doset $\mathcal{D}$ on $\mathcal{P}$. For the remainder of this section, set $P :=$ rank $\mathcal{P}$ and $D :=$ rank $\mathcal{D}$. Given a chain

$$\{\alpha_1, \ldots, \alpha_u, (\beta_{11}, \beta_{12}), \ldots, (\beta_{v1}, \beta_{v2})\} \subseteq \mathcal{D}$$

(not necessarily written in order), let r_i be the formal variable corresponding to α_i ($i = 1, \cdots, u$), and let s_{jk} correspond to β_{jk} ($j = 1, \ldots, v$, $k = 1, 2$). The variables r and s are not necessarily disjoint; in the example above, the chain $\{\alpha, (\alpha, \beta)\}$ has $r_1 = s_{11}$. We have

$$\sum_{\substack{a\in \operatorname{im}(\mathbf{w})\\ \operatorname{supp}(a)=c}} r^a = \prod_{i=1}^{u} \frac{r_i}{1-r_i} \cdot \prod_{j=1}^{v} \sqrt{s_{j1}s_{j2}}.$$

Recall that we may identify $\mathcal{P}$ with the diagonal $\Delta_{\mathcal{P}} \subseteq \mathcal{D} \subseteq \mathcal{P}\times\mathcal{P}$. Letting c_u^v denote the number of chains consisting of u elements of $\mathcal{P}$ and v elements of $\mathcal{D}\setminus\mathcal{P}$, we have

$$\begin{aligned}\operatorname{HS}_A(r) &= \sum_{u=0}^{P+1}\sum_{v=0}^{D-P} c_u^v \frac{r^{u+v}}{(1-r)^u} = \sum_{u=0}^{P+1}\sum_{v=0}^{D-P} c_u^v r^{u+v}\Bigl(\sum_{k=0}^{\infty} r^k\Bigr)^u \\ &= \sum_{v=0}^{D-P} c_0^v r^v + \sum_{\ell=0}^{\infty}\sum_{u=1}^{P+1}\sum_{v=0}^{D-P} c_u^v \binom{u+\ell-1}{u-1} r^{u+v+\ell}.\end{aligned}$$

When $w > D-P$, the coefficient of r^w agrees with the Hilbert polynomial:

$$\operatorname{HP}_A(w) = \sum_{u=1}^{P+1}\sum_{v=0}^{D-P} c_u^v \binom{w-v-1}{u-1}. \tag{3-3}$$

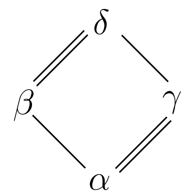


Figure 5. A doset on a four-element poset.

In particular, the dimension of Proj A is P, since this is the largest value of

$$u-1=\deg_w\binom{w-v-1}{u-1}.$$

The leading monomial of $HP_A(w)$ is

$$\sum_{v=0}^{D-P} c_{P+1}^{v}\binom{w-v-1}{P-1}.$$

By our assumption that the maximal chains in $\mathcal{P}$ (respectively, $\mathcal{D}$) have the same length, we have $c_{P+1}^{v}=\binom{D-P}{v}c_{P+1}^{0}$, so that the leading coefficient of $HP_A(w)$ is

$$\frac{c_{P+1}^{0}}{(P-1)!}\sum_{v=0}^{D-P}\binom{D-P}{v}=\frac{2^{D-P}c_{P+1}^{0}}{(P-1)!},$$

from which we deduce the degree and dimension of Proj A.

Theorem 3.13. *The degree of* Proj A *is* $2^{D-P}c_{P+1}^{0}$. *The dimension of* Proj A *is* P.

Example 3.14. Let A be an algebra with straightening law on the doset $\mathcal{D}$ shown in Figure 5.

We have rank $\mathcal{P}=2$, rank $\mathcal{D}=3$, and

$$\begin{aligned}\mathrm{Ch}(\mathcal{D})=\big\{&\varnothing, \{\alpha\}, \{\beta\}, \{\gamma\}, \{\delta\}, \{(\alpha,\gamma)\}, \{(\beta,\delta)\},\\ &\{\alpha,\gamma\}, \{\alpha,\beta\}, \{\alpha,\delta\}, \{\beta,\delta\}, \{\gamma,\delta\}, \{\alpha,(\alpha,\gamma)\},\\ &\{\alpha,(\beta,\delta)\}, \{(\alpha,\gamma),\gamma\}, \{(\alpha,\gamma),\delta\}, \{\beta,(\beta,\delta)\},\\ &\{(\beta,\delta),\delta\}, \{\alpha,\beta,\delta\}, \{\alpha,\gamma,\delta\}, \{\alpha,(\alpha,\gamma),\gamma\},\\ &\{\alpha,(\alpha,\gamma),\delta\}, \{\alpha,\beta,(\beta,\delta)\}, \{\alpha,(\beta,\delta),\delta\},\\ &\{\beta,(\beta,\delta),\delta\}, \{\alpha,(\alpha,\gamma),\gamma,\delta\}, \{\alpha,\beta,(\beta,\delta),\delta\}\big\},\end{aligned}$$

and the values of c_u^v are given by the matrix

$$\begin{pmatrix}1&4&5&2\\2&6&5&2\end{pmatrix},$$

whose entry in row i and column j is c^{i-1}_{j-1}.

In view of (3-3), the Hilbert polynomial is therefore

$$\begin{aligned}\mathrm{HP}_A(w)&\\ &=4\binom{w-1}{0}+5\binom{w-1}{1}+2\binom{w-1}{2}+6\binom{w-2}{0}+5\binom{w-2}{1}+2\binom{w-2}{2}\\ &=2w^2+2w+3=4\frac{w^2}{2!}+2w+3.\end{aligned}$$

In particular, $\dim(\operatorname{Proj} A)=2$ and $\deg(\operatorname{Proj} A)=4$.

Theorems 5.9 and 3.13 will allow us to compute intersection numbers in quantum cohomology in the same manner as Example 3.14. The essential step is to show that the Drinfel'd Lagrangian Grassmannian is a algebra with straightening law on the doset of admissible pairs $\mathscr{D}_{d,n}$.

3C. ***The doset of admissible pairs.*** We define the doset of admissible pairs on the poset $\mathscr{P}_{d,n}$. Let us first consider an example.

Example 3.15. Consider the poset

$$\mathscr{P}_{2,4}:=\{\alpha^{(a)}\in\tbinom{\langle 4\rangle}{4}_2 \mid i\in\alpha \Longleftrightarrow \bar{\imath}\notin\alpha\}$$

of admissible elements of $\binom{\langle 4\rangle}{4}_2$. Let $\mathscr{D}_{2,4}$ be the set of elements $(\alpha,\beta)^{(a)}\in\mathscr{O}_{\mathscr{P}_{2,4}}$ such that α and β have the same number of negative elements. It is a doset on $\mathscr{P}_{2,4}$. The Hasse diagram (drawn so that going up in the doset corresponds to moving to the right) for $\mathscr{D}_{2,4}$ is shown in Figure 6.

To each $(\alpha,\beta)^{(a)}\in\mathscr{P}_{2,4}$, we have the Plücker coordinate

$$p^{(a)}_{(\alpha,\beta)}:=u^a v^{d-a}\otimes p_{(\alpha,\beta)}\in S^d\mathbb{C}^2\otimes \mathrm{L}(\omega_n)^*,$$

where $\{u,v\}\subseteq\mathbb{C}^2$ is a basis dual to $\{s,t\}\subseteq(\mathbb{C}^2)^*$.

Let $\binom{\langle n\rangle}{n}_d\cong\binom{[2n]}{n}_d$ be the poset associated to the (ordinary) Drinfel'd Grassmannian $\mathfrak{Q}_d(n,2n)$, and recall that $\mathscr{P}_{d,n}\subseteq\binom{\langle n\rangle}{n}_d$ is the subposet consisting of the elements $\alpha^{(a)}$ such that $\alpha^t=\alpha$. There are three types of covers in $\mathscr{P}_{d,n}$.

(1) $\alpha^{(a)}\lessdot\beta^{(a)}$, where α and β have the same number of negative elements. For example, $\bar{4}\bar{2}13^{(a)}\lessdot\bar{4}\bar{1}23^{(a)}\in\mathscr{P}_{d,4}$ for any nonnegative integers $a\le d$.

(2) $\alpha^{(a)}\lessdot\beta^{(a)}$, where the number of negative elements in β is one less than the number of negative elements of α. For example, $\bar{4}\bar{1}23^{(a)}\lessdot\bar{4}123^{(a)}\in\mathscr{P}_{d,4}$ for any nonnegative integers $a\le d$.

(3) $\alpha^{(a)}\lessdot\beta^{(a+1)}$, where the number of negative elements of β is one more than the number of negative elements of α, $\bar{n}\in\beta$, and $n\in\alpha$. For example, $\bar{3}\bar{2}14^{(a)}\lessdot\bar{4}\bar{3}\bar{2}1^{(a+1)}$ for any nonnegative integers $a\le d$.

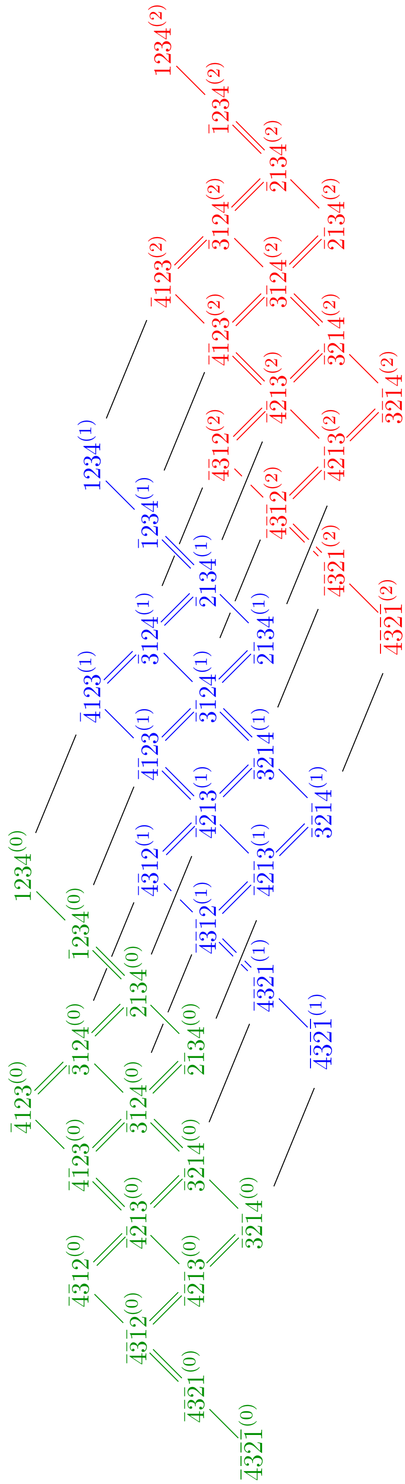


Figure 6. The doset $\mathscr{D}_{2,4}$. Elements increase as one moves to the right.

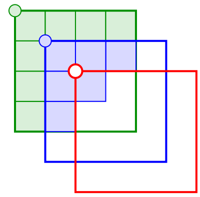


Figure 8. The subset of $S_{2,4}$ associated to $\bar{4}\bar{2}13^{(1)} \in \mathcal{P}_{2,4}$.

The first two types are those appearing in the classical Bruhat order on $\mathcal{P}_{0,n}$. It follows that $\mathcal{P}_{d,n}$ is a union of levels $\mathcal{P}_{d,n}^{(a)}$, each isomorphic to the Bruhat order, with order relations between levels imposed by covers of the type (3) above. We define the doset $\mathcal{D}_{d,n}$ of admissible pairs in $\mathcal{P}_{d,n}$.

Definition 3.16. A pair $(\alpha^{(a)} < \beta^{(a)})$ is *admissible* if there exists a saturated chain $\alpha = \alpha_0 \lessdot \alpha_1 \lessdot \cdots \lessdot \alpha_s = \beta$, where each $\alpha_i \lessdot \alpha_{i+1}$ is a cover of type (1).

We denote the set of admissible pairs by $\mathcal{D}_{d,n}$. Observe that the pair $(\alpha^{(a)} < \beta^{(b)})$ is never admissible if $a < b$.

Proposition 3.17. *The set $\mathcal{D}_{d,n} \subseteq \mathcal{P}_{d,n} \times \mathcal{P}_{d,n}$ is a doset on $\mathcal{P}_{d,n}$. The poset $\mathcal{P}_{d,n}$ is a distributive lattice.*

Proof. In view of our description of the covers in $\mathcal{D}_{d,n}$, it is clear that, for all $d \geq 0$, $\mathcal{D}_{d,n}$ is a doset if and only if $\mathcal{D}_n = \mathcal{D}_{0,n}$ is a doset. The latter is proved in [De Concini and Lakshmibai 1981]. To prove that $\mathcal{P}_{d,n}$ is a distributive lattice, we give an isomorphism with a certain lattice of subsets of the union of $d+1$ shifted $n \times n$ squares in $\mathbb{Z}^2$ which generalize the usual notion of a partition.

Let

$$S_{d,n} := \bigcup_{a=0}^{d} \{(i+a,\, j+a) \mid 0 \leq i,\, j \leq n\}.$$

To $\alpha^{(a)} \in \mathcal{P}_{d,n}$, we associate the subset of $S_{d,n}$ obtained by shifting the (open) squares in α by (a, a), and adding the boxes obtained by translating a box of α by a vector (v_1, v_2) with $v_1, v_2 \leq 0$ and the points (i, i) for $i = 0, \ldots, a$. See Figure 8 for an example. It is straightforward to check that the (symmetric) subsets obtained in this way form a distributive lattice (ordered by inclusion) isomorphic to $\mathcal{P}_{d,n}$. $\square$

4. Schubert varieties and Gromov–Witten invariants

Gromov–Witten invariants are solutions to enumerative questions involving algebraic maps from $\mathbb{P}^1$ to a projective variety X. When X is the Lagrangian Grassmannian (or the ordinary Grassmannian [Sottile and Sturmfels 2001]), these questions can be studied geometrically via the Drinfel'd compactification, as advocated by A. Braverman [2006]. We do this in Section 4A, and relate our findings to the quantum cohomology of the Lagrangian Grassmannian in Section 4B. See [Braverman 2006; Sottile 2000; 2001] for further reading on applications of Drinfel'd compactifications to quantum cohomology. The study of Gromov–Witten invariants (in various special cases) has also been approached via the quot scheme [Bertram 1997; Chen 2003; Ciocan-Fontanine 1999; Fulton and Pandharipande 1997] and the space of stable maps [Bertram et al. 2005; Givental 1996; Oprea 2006].

4A. ***Intersection problems on the Drinfel'd compactification.*** Given an isotropic flag $F_\bullet$ and a symmetric partition $\alpha \in \mathcal{P}_n$, we have the Schubert variety

$$X^\alpha(F_\bullet) := \{E \in \mathrm{LG}(n) \mid \dim(E \cap F_{n-\alpha_i+i}) \geq i\}.$$

The enumerative problems we consider involve conditions that the image of a map $M \in L\mathcal{M}_d(X)$ pass through Schubert varieties at prescribed points of $\mathbb{P}^1$.

Question 4.1. Let $F^1_\bullet, \dots, F^N_\bullet$ be general Lagrangian flags, $\alpha_1, \dots, \alpha_N \in \mathcal{P}_n$, and let $s_1, \dots, s_N \in \mathbb{P}^1$ be distinct points. Assume

$$\sum_{i=1}^{N} |\alpha_i| = \dim \mathrm{LG}(n) + d(n+1).$$

How many degree-d algebraic maps $M : \mathbb{P}^1 \to \mathrm{LG}(n)$ satisfy

$$M(s_i) \in X^{\alpha_i}(F^i_\bullet)$$

for all $i = 1, \dots, N$?

Our answer to Question 4.1 is given in Theorem 4.3. In order to prove this result, we must first establish some results on the geometry of certain subvarieties of $L\mathcal{Q}_d(n)$ defined in terms of the *universal evaluation map*

$$\mathrm{ev} : \mathbb{P}^1 \times L\mathcal{M}_d(n) \to \mathrm{LG}(n), \quad \mathrm{ev}(s, M) := M(s)$$

for $s \in \mathbb{P}^1$ and $M \in L\mathcal{M}_d(n)$.

Fix a point $s \in \mathbb{P}^1$ and define

$$\mathrm{ev}_s := \mathrm{ev}(s, \bullet) : L\mathcal{M}_d(n) \to \mathrm{LG}(n).$$

Given a Schubert variety $X^\alpha(F_\bullet) \subseteq \mathrm{LG}(n)$, the set of maps $M \in L\mathcal{M}_d(n)$ such that $M(s)$ lies in $X^\alpha(F_\bullet)$ is the preimage $\mathrm{ev}_s^{-1}(X^\alpha(F_\bullet))$. This is a general translate of the locally closed subset $X^{\alpha^{(0)}} \cap L\mathcal{M}_d(n)$ under the action of the group $\mathrm{SL}_2\mathbb{C} \times \mathrm{Sp}_{2n}\mathbb{C}$. By a Schubert variety, we will mean the closure of $\mathrm{ev}_s^{-1}(X^\alpha(F_\bullet))$ in $L\mathcal{Q}_d(n)$, and denote it by $X^{\alpha^{(0)}}(s; F_\bullet)$. In order to understand these subvarieties, we extend the evaluation map to a globally defined map $\mathbb{P}^1 \times L\mathcal{Q}_d(n) \to \mathrm{LG}(n)$. To do this, we must first study the boundary $L\mathcal{Q}_d(n) \setminus L\mathcal{M}_d(n)$.

The embedding

$$L\mathcal{M}_d(n) \hookrightarrow \mathbb{P}\big((S^d\mathbb{C}^2)^* \otimes \mathrm{L}(\omega_n)\big)$$

is defined by regarding a map $M \in L\mathcal{M}_d(n)$ as a $\binom{2n}{n}$-tuple of degree-d homogeneous forms. We identify the space $L\mathcal{M}_d(n)$ of maps with its image, which is a locally closed subset of $\mathbb{P}\big((S^d\mathbb{C}^2)^* \otimes \mathrm{L}(\omega_n)\big)$. The Drinfel'd compactification $L\mathcal{Q}_d(n)$ is by definition the closure of the image.

On the other hand, $L\mathcal{M}_d(n) \subseteq L\mathcal{Q}_d(n)$ is the set of points corresponding to a $\binom{2n}{n}$-tuple of homogeneous forms satisfying the Zariski open condition that they have no common factor. Therefore, the boundary $L\mathcal{Q}_d(n) \setminus L\mathcal{M}_d(n)$ consists of $\binom{2n}{n}$-tuples of homogeneous forms which do have a common factor. Such a list of forms gives a regular map of degree $a < d$ together with an effective Weil divisor of degree $d - a$ on $\mathbb{P}^1$ defined by the base points of the map. We thus have a stratification

$$L\mathcal{Q}_d(n) = \bigsqcup_{a=0}^{d} \mathbb{P}(S^a\mathbb{C}^{2^*}) \times L\mathcal{M}_{d-a}(n)$$

where $\mathbb{P}(S^a\mathbb{C}^{2^*})$ is the space of degree-a forms in two variables, or alternatively, the space of effective Weil divisors on $\mathbb{P}^1$ of degree a. In particular, the boundary of $L\mathcal{Q}_d(n)$ is simply $\bigsqcup_{a=1}^{d} \mathbb{P}(S^a\mathbb{C}^{2^*}) \times L\mathcal{M}_{d-a}(n)$. We may regard any point of $L\mathcal{Q}_d(n)$ as a pair (D, M), where $M \in L\mathcal{M}_{d-a}(n)$ and D is a divisor on $\mathbb{P}^1$.

Fixing a point $s \in \mathbb{P}^1$, the evaluation map $\mathrm{ev}_s := \mathrm{ev}(s, \bullet)$ is undefined at each point $(D, M) \in \mathbb{P}(S^a\mathbb{C}^{2^*}) \times L\mathcal{M}_{d-a}(n)$ such that $s \in D$. Thus, restricting to the stratum $\mathbb{P}(S^a\mathbb{C}^{2^*}) \times L\mathcal{M}_{d-a}(n)$, the map ev_s is defined on $U_s^a \times L\mathcal{M}_{d-a}(n)$, where $U_s^a \subseteq \mathbb{P}(S_a\mathbb{C}^2)$ is the set of forms which do not vanish at $s \in \mathbb{P}^1$.

For each $a = 0, \ldots, d$, define a map

$$\epsilon_s^a : \mathbb{P}(S^a\mathbb{C}^{2^*}) \times L\mathcal{M}_{d-a}(n) \to \mathrm{LG}(n)$$

by the formula $\epsilon_s^a(D, M) := M(s)$, and let

$$\epsilon_s : L\mathcal{Q}_d(n) = \bigsqcup_{a=0}^{d} \mathbb{P}(S^a\mathbb{C}^{2^*}) \times L\mathcal{M}_{d-a}(n) \to \mathrm{LG}(n)$$

be the (globally-defined) map which restricts to ϵ_s^a on $\mathbb{P}(S^a\mathbb{C}^{2^*}) \times L\mathcal{M}_{d-a}(n)$. The evaluation map ev_s agrees with ϵ_s wherever it is defined. Hence ϵ_s extends ev_s to a globally defined map, which is a morphism on each stratum $\mathbb{P}(S^a\mathbb{C}^{2^*}) \times L\mathcal{M}_{d-a}(n)$. The Schubert variety $X^{\alpha^{(0)}}(s; F_\bullet)$ is the preimage of $X^\alpha(F_\bullet)$ under this globally defined map; hence we have the following fact.

Lemma 4.2. *Given a point $s \in \mathbb{P}^1$ and a isotropic flag $F_\bullet$, the Schubert variety $X^{\alpha^{(0)}}(s, F_\bullet)$ is the disjoint union of the strata*

$$\mathbb{P}(S^a\mathbb{C}^{2*}) \times \big(X^{\alpha^{(a)}}(s; F_\bullet) \cap L\mathcal{M}_{d-a}(n)\big).$$

Proof. For each $a \in \{0, \ldots d\}$, we have

$$\big(\mathbb{P}(S^a\mathbb{C}^{2*}) \times L\mathcal{M}_{d-a}(n)\big) \cap \epsilon_s^{-1}(X^\alpha(F_\bullet)) = \mathbb{P}(S^a\mathbb{C}^{2*}) \times X^{\alpha^{(a)}}(s; F_\bullet). \qquad \square$$

We now state and prove the main theorem of this section.

Theorem 4.3. *Given partitions $\alpha_1, \ldots \alpha_N \in \mathcal{P}_n$ such that*

$$\sum_{i=1}^{N} |\alpha_i| = \binom{n+1}{2} + d(n+1),$$

general isotropic flags $F_\bullet^1, \ldots, F_\bullet^N$, and distinct points $s_1, \ldots, s_N \in \mathbb{P}^1$, the intersection

$$X^{\alpha_1^{(0)}}(s_1; F_\bullet^1) \cap \cdots \cap X^{\alpha_N^{(0)}}(s_N; F_\bullet^N) \tag{4-1}$$

is transverse, and hence consists only of reduced points. Each point of the intersection (4-1) *lies in $L\mathcal{M}_d(n)$, that is, corresponds to a degree-d map whose image $M(s_i)$ lies in $X^{\alpha_i}(F_\bullet^i)$ for $i = 1, \ldots, N$.*

Proof. For each $a = 0, \ldots, d$, the Schubert variety $X^{\alpha^{(a)}}(s, F_\bullet)$ is the preimage of the Schubert variety $X^\alpha(F_\bullet) \subseteq \mathrm{LG}(n)$ under the evaluation map ϵ_s, which is regular on the stratum $\mathbb{P}(S^a\mathbb{C}^{2*}) \times L\mathcal{M}_{d-a}(n)$. By Lemma 4.2, it suffices to consider the intersection (4-1) on each of these strata. Fix $a \in \{0, \ldots, d\}$, and consider the product of evaluation maps

$$\prod_{i=1}^{N} \epsilon_{s_i} : \big(\mathbb{P}(S^a\mathbb{C}^{2*}) \times L\mathcal{M}_{d-a}(n)\big)^N \to \mathrm{LG}(n)^N$$

and the injection

$$X^{\alpha_1}(F_\bullet^1) \times \cdots \times X^{\alpha_n}(F_\bullet^N) \hookrightarrow \mathrm{LG}(n)^N.$$

The intersection $\big(X^{\alpha_1^{(0)}} \cap \cdots \cap X^{\alpha_N^{(0)}}\big) \cap \mathbb{P}(S^a\mathbb{C}^{2*}) \times L\mathcal{M}_{d-a}(n)$ is isomorphic to the fiber product

$$\big(\mathbb{P}(S^a\mathbb{C}^{2*}) \times L\mathcal{M}_{d-a}(n)\big)^N \times_{\mathrm{LG}(n)^N} \big(X^{\alpha_1}(F_\bullet^1) \times \cdots \times X^{\alpha_N}(F_\bullet^N)\big).$$

For each $a = 0, \ldots, d$, Kleiman's theorem [1974, Corollary 2] implies that this intersection is proper and transverse. Considering the dimensions of these subvarieties, we see that this intersection is therefore zero-dimensional when $a = 0$ and empty when $a > 0$. $\square$

4B. ***Gromov–Witten invariants and quantum cohomology.*** A common approach to Question 4.1 is through the quantum cohomology ring of the Lagrangian Grassmannian $QH^*(\mathrm{LG}(n))$, defined as follows. The cohomology ring $H^*(\mathrm{LG}(n); \mathbb{Z})$ has a $\mathbb{Z}$-basis consisting of the classes of Schubert varieties (the *Schubert classes*) $\sigma_\alpha := [X^\alpha]$, where $\alpha \in \mathcal{P}_n$. We will denote by α^* the dual partition, defined so that $\sigma_\alpha \cdot \sigma_{\alpha^*} = [pt] \in H^*(\mathrm{LG}(n); \mathbb{Z})$ [Hiller and Boe 1986]. The correspondence $\alpha \leftrightarrow \alpha^*$ is bijective and order reversing.

The (small) *quantum cohomology ring* is the $\mathbb{Z}[q]$-algebra isomorphic to

$$H^*(\mathrm{LG}(n); \mathbb{Z}) \otimes \mathbb{Z}[q]$$

as a $\mathbb{Z}[q]$-module, and with multiplication defined by the formula

$$\sigma_\alpha \cdot \sigma_\beta = \sum \langle \alpha, \beta, \gamma^* \rangle_d \, \sigma_\gamma q^d,$$

where the sum is over all $d \geq 0$ and γ such that $|\gamma| = |\alpha| + |\beta| - d\binom{n+1}{2}$. For partitions α, β, and γ in $\mathcal{P}_n$, the coefficients $\langle \alpha, \beta, \gamma^* \rangle_d$ are the *Gromov–Witten invariants*, defined as the number of algebraic maps $M : \mathbb{P}^1 \to \mathrm{LG}(n)$ of degree d such that

$$M(0) \in X^\alpha(F_\bullet), \quad M(1) \in X^\beta(G_\bullet), \quad \text{and} \quad M(\infty) \in X^{\gamma^*}(H_\bullet),$$

where $F_\bullet$, $G_\bullet$, and $H_\bullet$ are general isotropic flags (that is, general translates of the standard flag under the action of the group $\mathrm{Sp}_{2n}\mathbb{C}$).

A special case of Pieri's rule gives a formula for the product of a Schubert class $\sigma_\alpha \in H^*(\mathrm{LG}(n); \mathbb{Z})$ with the *simple* Schubert class $\sigma_\square$ [Hiller and Boe 1986]:

$$\sigma_\alpha \cdot \sigma_\square = \sum_{\alpha \lessdot \beta} 2^{N(\alpha,\beta)} \sigma_\beta,$$

the sum over all partitions β obtained from α by adding a box above the diagonal, along with its image under reflection about the diagonal. The exponent $N(\alpha, \beta) = 1$ if $(\alpha, \beta) \in \mathcal{D}_n$ and $N(\alpha, \beta) = 0$ otherwise (compare Proposition 2.10). Kresch and Tamvakis [2003] give a quantum analogue of Pieri's rule. We state the relevant special case of this rule:

Proposition 4.4 [Kresch and Tamvakis 2003]. *For any $\alpha \in \mathcal{P}_n$, we have*

$$\sigma_\alpha \cdot \sigma_\square = \sum_{\alpha \lessdot \beta} 2^{N(\alpha,\beta)} \sigma_\beta + \sigma_\gamma q$$

in $QH^*(\mathrm{LG}(n))$, where the first sum is from the classical Pieri rule, and $\sigma_\gamma = 0$ unless α contains the hook-shaped partition $(n, 1^{n-1})$, in which case γ is the partition obtained from α by removing this hook.

For $\alpha^{(a)} \lessdot \beta^{(b)}$ in $\mathcal{D}_{d,n}$, let $N'(\alpha^{(a)}, \beta^{(b)}) = N(\alpha, \beta)$ if $b = a$, and let $N'(\alpha^{(a)}, \beta^{(b)}) = 0$ if $b = a+1$. Let $\alpha^{(0)} \in \mathcal{P}_{d,n}$, and let $\pi \in \mathbb{N}$ be its *corank* in $\mathcal{P}_{d,n}$; that is, π is the length of any saturated chain of elements $\alpha^{(d)} = x_0 \lessdot \cdots \lessdot x_\pi = (n^n)^{(d)}$, where $x_i \in \mathcal{P}_{d,n}$ for all $i = 0, \ldots, \pi$ and $(n^n)^{(d)}$ is the maximal element of $\mathcal{P}_{d,n}$. By Theorem 3.13, π is the dimension of $X^{\alpha^{(0)}}$. The quantum Pieri rule of Proposition 4.4 has a simple formulation in terms of the distributive lattice $\mathcal{P}_{d,n}$:

Theorem 4.5. *The quantum Pieri rule in Proposition 4.4 has the formulation in terms of the poset* $\mathcal{P}_{d,n}$:

$$(\sigma_\alpha q^a) \cdot \sigma_\square = \sum_{\alpha^{(a)} \lessdot \beta^{(b)}} 2^{N'(\alpha^{(a)}, \beta^{(b)})} \sigma_\beta q^b.$$

As a consequence, *we have*

$$\sigma_\alpha \cdot (\sigma_\square)^\pi = \deg(X^{\alpha^{(0)}}) \cdot \sigma_{(n^n)} q^d \pmod{d+1}. \tag{4-2}$$

Proof. The element $\gamma^{(a+1)} \in \mathcal{P}_{d,n}$, where γ is as the partition obtained by removing a maximal hook from α in Proposition 4.4, is the unique cover of $\alpha^{(a)} \in \mathcal{P}_{d,n}$ with superscript $a+1$. The remaining covers (with superscript a) index the sum in Proposition 4.4.

The second formula follows by induction from the first. □

The appearance of the number $\deg(X^{\alpha^{(0)}})$ in (4-2) is for purely combinatorial reasons: it is the number of saturated chains $\alpha^{(0)} \lessdot \cdots \lessdot (n^n)^{(d)}$ in $\mathcal{D}_{d,n}$, counted with multiplicity. Since $X^{\square^{(0)}}$ is a hyperplane section of $L\mathcal{Q}_d(n)$, this is also the number of points in the intersection

$$X^{\alpha^{(0)}}(s; F_\bullet) \cap \left(\bigcap_{i=1}^{\pi} X^{\square^{(0)}}(s_i; F_\bullet^i) \right), \tag{4-3}$$

the intersection of $X^{\alpha^{(0)}}(s; F_\bullet)$ with $\pi = \operatorname{codim}(X^{\alpha^{(0)}}(s; F_\bullet))$ general translates of the hyperplane section $X^{\square^{(0)}}$. On the other hand, multiplication in $QH^*(\mathrm{LG}(n))$ represents the conjunction of conditions that a map takes values in Schubert varieties at generic points of $\mathbb{P}^1$. In this way, the quantum cohomology identity of Theorem 4.5 has an interpretation as the number of points in the intersection (4-3) of Schubert varieties in $L\mathcal{Q}_d(n)$.

5. The straightening law

5A. ***A basis for*** $\boldsymbol{S^d\mathbb{C}^2 \otimes \mathrm{L}(\omega_n)^*}$**.** The Drinfel'd Lagrangian Grassmannian embeds in the projective space $\mathbb{P}\big((S^d\mathbb{C}^2)^* \otimes \mathrm{L}(\omega_n)\big)$. We begin by describing convenient bases for the representation $\mathrm{L}(\omega_n)$ and its dual $\mathrm{L}(\omega_n)^*$.

For $\alpha \in \binom{\langle n\rangle}{n}$ and positive integers, set

$$v_\alpha^{(a)} := s^a t^{d-a} \otimes e_{\alpha_1} \wedge \cdots \wedge e_{\alpha_n} \in (S^d\mathbb{C}^2)^* \otimes \bigwedge^n \mathbb{C}^{2n},$$

and let

$$p_\alpha^{(a)} := u^a v^{d-a} \otimes e^*_{\alpha_1} \wedge \cdots \wedge e^*_{\alpha_n} \in S^d\mathbb{C}^2 \otimes \bigwedge^n \mathbb{C}^{2n*}$$

be the *Plücker coordinate* indexed by $\alpha^{(a)} \in \mathcal{D}_{d,n}$, where $\{u, v\} \in \mathbb{C}^2$ and $\{s, t\} \in (\mathbb{C}^2)^*$ are dual bases.

The representation $\mathrm{L}(\omega_n)^*$ is the quotient of $\bigwedge^n \mathbb{C}^{2n*}$ by the linear subspace

$$L_n = \Omega \wedge \bigwedge^{n-2} \mathbb{C}^{2n*}$$

described in Proposition 2.3. Thus $S^d\mathbb{C}^2 \otimes \mathrm{L}(\omega_n)^*$ is the quotient of $S^d\mathbb{C}^2 \otimes \bigwedge^n \mathbb{C}^{2n*}$ by the linear subspace

$$L_{d,n} := S^d\mathbb{C}^2 \otimes L_n.$$

Note that $L_{d,n}$ is spanned by the linear forms

$$\ell_\alpha^{(a)} := u^a v^{d-a} \otimes \sum_{i \mid \{\bar\imath, i\}\cap\alpha=\varnothing} e^*_{\bar\imath} \wedge e^*_i \wedge e^*_{\alpha_1} \wedge \cdots \wedge e^*_{\alpha_{n-2}} \tag{5-1}$$

for $\alpha \in \binom{\langle n\rangle}{n-2}$ and $a = 0, \dots, d$. The linear form (5-1) is simply $u^a v^{d-a}$ tensored with a linear form generating L_n. Each term in the linear form (5-1) is a Plücker coordinate indexed by a sequence involving both i and $\bar\imath$, for some $i \in [n]$.

Let $S \subseteq \mathrm{SL}_2(\mathbb{C})$ and $T \subseteq \mathrm{Sp}_{2n}(\mathbb{C})$ be maximal tori. The torus S is one-dimensional, so that its Lie algebra $\mathfrak{s}$ has basis consisting of a single element $H \in \mathfrak{s}$. For $i \in \langle n\rangle$, let $h_i := E_{ii} - E_{\bar\imath\bar\imath}$. The set $\{h_i \mid i \in [n]\}$ is a basis for the Lie algebra $\mathfrak{t}$ of $T \subseteq \mathrm{Sp}_{2n}(\mathbb{C})$. The weights of the maximal torus $S \times T \subseteq \mathrm{SL}_2(\mathbb{C}) \times \mathrm{Sp}_{2n}(\mathbb{C})$ are elements of $\mathfrak{s}^* \oplus \mathfrak{t}^*$. The Plücker coordinate $p_\alpha^{(a)} \in S^d\mathbb{C}^2 \otimes (\bigwedge^n \mathbb{C}^{2n})^*$ is a weight vector of weight

$$(d-2a)H^* + \sum_{i \mid \bar\alpha_i \notin \alpha} h^*_{\alpha_i}. \tag{5-2}$$

Each linear form (5-1) lies in a unique weight space. Thus, to find a basis for $S^d\mathbb{C}^2 \otimes \mathrm{L}(\omega_n)^*$, it suffices to find a basis for each weight space. We therefore fix the weight (5-2) and its corresponding weight space in the following discussion. We reduce to the case that the weight (5-2) is in fact 0, as follows.

For each $\alpha \in \binom{\langle n\rangle}{n-2}$, we have an element $\ell_\alpha = \Omega \wedge p_\alpha \in L_n$. This is a weight vector of weight

$$\omega_\alpha := h^*_{\alpha_1} + \cdots + h^*_{\alpha_k} \in \mathfrak{t}^*.$$

Set $\widetilde{\alpha} := \{i \in \alpha \mid \bar{\imath} \notin \alpha\}$ and observe that $\omega_{\widetilde{\alpha}} = \omega_\alpha$. The elements $\alpha \in \binom{\langle n\rangle}{n-2}$ such that $\ell_\alpha \in (L_n)_\omega$ are those satisfying $\omega_\alpha = \omega$. That is,

$$(L_n)_\omega = \langle \Omega \wedge p_\alpha \mid \omega_\alpha = \omega \rangle.$$

The shape of the linear form ℓ_α is determined by the number of pairs $\{\bar{\imath}, i\} \subseteq \alpha$; it is the same, up to multiplication of some variables by -1, as the linear form $\ell_{\alpha\setminus\widetilde{\alpha}} = \Omega \wedge p_{\alpha\setminus\widetilde{\alpha}} \in L_{n-|\widetilde{\alpha}|}$, of weight $\omega_{\alpha\setminus\widetilde{\alpha}} = 0$. It follows that the generators of $(L_n)_{\omega_\alpha}$ have the same form as those of $(L_{n-|\widetilde{\alpha}|})_{\omega_{\alpha\setminus\widetilde{\alpha}}}$, up to some signs arising from sorting the indices. Since these signs do not affect linear independence, it suffices to find a basis for $(L_n)_0$, from which it is then straightforward to deduce a basis for $(L_n)_{\omega_\alpha}$. We thus assume that the weight space in question is $(L_n)_0$. This implies that n is even; set $m := \frac{n}{2}$.

Example 5.1. We consider linear forms which span $(L_6)_{h_1^*+h_3^*}$. Let $m = 3$ (so $n = 6$) and $\omega = h_1^* + h_3^*$. If $\alpha = \bar{6}136$, then $\widetilde{\alpha} = 13$ and $\omega_\alpha = \omega$. We have

$$\ell_\alpha = p_{\bar{6}\bar{5}1356} + p_{\bar{6}\bar{4}1346} - p_{\bar{6}\bar{2}1236}.$$

The equations for the weight space $(L_n)_\omega$ are

$$\begin{aligned}
\ell_{\bar{6}136} &= p_{\bar{6}\bar{5}1356} + p_{\bar{6}\bar{4}1346} - p_{\bar{6}\bar{2}1236},\\
\ell_{\bar{5}135} &= p_{\bar{6}\bar{5}1356} + p_{\bar{5}\bar{4}1345} - p_{\bar{5}\bar{2}1235},\\
\ell_{\bar{4}134} &= p_{\bar{6}\bar{4}1346} + p_{\bar{5}\bar{4}1345} - p_{\bar{4}\bar{2}1234},\\
\ell_{\bar{2}123} &= p_{\bar{6}\bar{2}1236} + p_{\bar{5}\bar{2}1235} + p_{\bar{4}\bar{2}1234}.
\end{aligned}$$

We can obtain the linear forms which span $(L_4)_0$ (see Example 2.11) by first removing every occurrence of 1 and 3 in the subscripts above and then flattening the remaining indices. That is, we apply the following replacement (and similarly for the negative indices): $6 \mapsto 4$, $5 \mapsto 3$, $4 \mapsto 2$, and $2 \mapsto 1$. We then replace a variable by its negative if 2 appears in its index; this is to keep track of the sign of the permutation sorting the sequence $(\bar{\imath}, i, \alpha_1, \ldots, \alpha_{n-2})$ in each term of ℓ_α (see (5-1)).

By Proposition 2.3, the map

$$\Bigl(\bigwedge^{2m} \mathbb{C}^{4m}\Bigr)_0 \to \Bigl(\bigwedge^{2m-2} \mathbb{C}^{4m}\Bigr)_0$$

given by contraction with the form

$$\Omega \in \bigwedge^2 (\mathbb{C}^{4m})^*$$

is surjective, with kernel $(\mathrm{L}(\omega_{2m}))_0$. Since the set $\left\{(\bar{\alpha},\alpha) \;\middle|\; \alpha\in\binom{[2m]}{k}\right\}$ is a basis of $(\bigwedge^{2k}\mathbb{C}^{4m})_0$ (for any $k\le m$), we have

$$\begin{aligned}\dim(\mathrm{L}(\omega_{2m}))_0 &= \dim\Bigl(\bigwedge^{2m}\mathbb{C}^{4m}\Bigr)_0 - \dim\Bigl(\bigwedge^{2m-2}\mathbb{C}^{4m}\Bigr)_0\\ &= \binom{2m}{m}-\binom{2m}{m-2} = \frac{1}{m+1}\binom{2m}{m}.\end{aligned}$$

This (Catalan) number is equal to the number of admissible pairs of weight 0.

Lemma 5.2. $\dim(\mathrm{L}(\omega_n))_0$ *is equal to the number of admissible pairs* $(\alpha,\beta)\in\mathfrak{D}_n$ *of weight* $\frac{\omega_\alpha+\omega_\beta}{2}=0$.

Proof. Recall that each trivial admissible pair (α,α), where

$$\alpha=(\bar{a}_1,\ldots,\bar{a}_s,b_1,\ldots,b_{n-s})\in\mathfrak{D}_n,$$

indexes a weight vector of weight $\sum_{i=1}^{n-s}h^*_{b_i}-\sum_{i=1}^{s}h^*_{a_i}$. Also, the nontrivial admissible pairs are those (α,β) for which $\alpha<\beta$ have the same number of negative elements. Therefore, the admissible pairs of weight zero are the $(\alpha,\beta)\in\mathfrak{D}_n$ such that $\beta=(\bar{a}_m,\ldots,\bar{a}_1,b_1,\ldots,b_m)$, $\alpha=(\bar{b}_m,\ldots,\bar{b}_1,a_1,\ldots,a_m)$, and the sets $\{a_1,\ldots,a_m\}$ and $\{b_1,\ldots,b_m\}$ are disjoint. This last condition is equivalent to $a_i>b_i$ for all $i\in[m]$. The number of such pairs is equal to the number of standard tableaux of shape (m^2) (that is, a rectangular box with 2 rows and m columns) with entries in $[2m]$. By the hook length formula [Fulton 1997] this number is $\frac{1}{m+1}\binom{2m}{m}$. ☐

The weight vectors $p_\alpha\in(\bigwedge^n\mathbb{C}^{4m*})_0$ are indexed by sequences of the form

$$\alpha=(\bar{\alpha}_m,\ldots,\bar{\alpha}_1,\alpha_1,\ldots,\alpha_m)$$

which can be abbreviated by the positive subsequence $\alpha_+:=(\alpha_1,\ldots,\alpha_m)\in\binom{[2m]}{m}$ without ambiguity. We take these as an indexing set for the variables appearing in the linear forms (5-1).

With this notation, the positive parts of Northeast sequences are characterized in Proposition 5.4. The proof requires the following definition.

Definition 5.3. A *tableau* is a partition whose boxes are filled with integers from the set $[n]$, for some $n\in\mathbb{N}$. A tableau is *standard* if the entries strictly increase from left to right and top to bottom.

Proposition 5.4. *Let* $\alpha\in\binom{\langle 2m\rangle}{2m}$ *be a Northeast sequence. Then the positive part of* α *satisfies* $\alpha_+\ge 24\cdots(2m)\in\binom{[2m]}{m}$. *In particular, no Northeast sequence contains* $1\in[2m]$ *and every Northeast sequence contains* $2m\in[2m]$.

Proof. $\alpha_+ \geq 24\cdots(2m)$ if and only if the tableau of shape (m^2) whose first row is filled with the sequence $(\alpha^t)_+ = [n] \setminus \alpha_+$ and whose second row is filled with the α_+ is standard. This is equivalent to α being Northeast. $\square$

It follows from Proposition 2.10 that the set $\mathcal{NE}$ of Northeast sequences indexing vectors of weight zero has cardinality equal to the dimension of the zero-weight space of the representation $\mathrm{L}(\omega_{2m})^*$. This weight space is the cokernel of the map

$$\Omega \wedge \bullet : \Big(\bigwedge^{2m-2} \mathbb{C}^{4m}\Big)^*_0 \to \Big(\bigwedge^{2m} \mathbb{C}^{4m}\Big)^*_0.$$

Similarly, the weight space $\mathrm{L}(\omega_{2m})_0$ is the kernel of the dual map

$$\Omega \lrcorner \bullet : \Big(\bigwedge^{2m} \mathbb{C}^{4m}\Big)_0 \to \Big(\bigwedge^{2m-2} \mathbb{C}^{4m}\Big)_0.$$

We fix the positive integer m, and consider only the positive subsequence α_+ of the sequence $\alpha \in \binom{\langle 2m\rangle}{2m}$. When the weight of p_α is 0, α_+ is an element of $\binom{[2m]}{m}$. For $\alpha \in \binom{[2m]}{m}$, we call a bijection $M : \alpha \to \alpha^c$ a *matching* of α. Fixing a matching $M : \alpha \to \alpha^c$, we have an element of the kernel $\mathrm{L}(\omega_{2m})$, as follows. Let H_α be the set of all sequences in $\binom{[2m]}{m}$ obtained by interchanging $M(\alpha_i)$ and α_i, for $i \in I$, $I \subseteq [m]$.

Elements of the set H_α are the vertices of an m-dimensional hypercube, whose edges connect pairs of sequences which are related by the interchange of a single element. Equivalently, a pair of sequences are connected by an edge if they share a subsequence of size $m-1$. For any such subsequence $\beta \subseteq \alpha$ there exists a unique edge of H_α connecting the two vertices which share the subsequence β. Let $I \cdot \alpha$ denote the element of H_α obtained from α by the interchange of $M(\alpha_i)$ and α_i for $i \in I$. The element

$$K_\alpha := \sum_{I \subseteq [m]} (-1)^{|I|} v_{I\cdot\alpha}$$

lies in the kernel $\mathrm{L}(\omega_{2m})$. Indeed, for each $I \subseteq [m]$, we have

$$\Omega \lrcorner v_{I\cdot\alpha} = \sum_{i=1}^{m} v_{(I\cdot\alpha)\setminus\{(I\cdot\alpha)_i\}}.$$

For each term $v_{(I\cdot\alpha)\setminus\{(I\cdot\alpha)_i\}}$ on the right hand side, let $j \in [m]$ be such that either $(I\cdot\alpha)_i = \alpha_j$ or $(I\cdot\alpha)_i = \alpha_j^c$. Set

$$J = \begin{cases} I \cup \{j\}, & \text{if } (I\cdot\alpha)_i = \alpha_j, \\ I \setminus \{j\}, & \text{if } (I\cdot\alpha)_i = \alpha_j^c. \end{cases}$$

The set J is the unique subset of $[m]$ such that $(I\cdot\alpha) \setminus \{(I\cdot\alpha)_i\}$ is in the support of $\Omega \lrcorner v_{J\cdot\alpha}$, with coefficient $(-1)^{|J|} = (-1)^{|I|+1}$. Hence these terms cancel in K_α, and we see that the coefficient of each v_β for $\beta \in \binom{[2m]}{m-1}$ in the support of $\Omega \lrcorner K_\alpha$ is zero. Therefore $\Omega \lrcorner K_\alpha = 0$. See Example 5.8 for the case $m = 2$.

If $\alpha \in \mathcal{NE}$ then there exists a *descending matching*, that is, $M(\alpha_i) < \alpha_i$ for all $i \in [m]$. For example, the condition that the matching $M(\alpha_i) := \alpha_i^c$ be descending is equivalent to the condition that α be Northeast. If we choose a descending matching for each $\alpha \in \mathcal{NE}$, the element $K_\alpha \in \mathrm{L}(\omega_{2m})$ is supported on sequences which precede α in the poset $\binom{[2m]}{m}$. It follows that the set $\mathcal{B} := \{K_\alpha \in \mathrm{L}(\omega_{2m}) \mid \alpha \in \mathcal{NE}\}$ is a basis for $\mathrm{L}(\omega_{2m})$.

Lemma 5.5. *The Plücker coordinates p_α with $\alpha \in \mathcal{NE}$ are a basis for* $\mathrm{L}(\omega_{2m})^*$.

Proof. Fix a basis $\mathcal{B}$ of $\mathrm{L}(\omega_{2m})$ obtained from descending matchings of each Northeast sequence. We use this basis to show that the set of Plücker coordinates p_α such that α is Northeast is a basis for the dual $\mathrm{L}(\omega_{2m})^*$.

Suppose not. Then there exists a linear form

$$\ell = \sum_{\alpha \in \mathcal{NE}} c_\alpha p_\alpha$$

vanishing on each element of the basis $\mathcal{B}$. We show by induction on the poset $\mathcal{NE}$ that all of the coefficients c_α appearing in this form vanish.

Fix a Northeast sequence $\alpha \in \mathcal{NE}$, and assume that $c_\beta = 0$ for all Northeast $\beta < \alpha$. Since K_α involves only the basis vectors v_β with $\beta \leq \alpha$, we have $\ell(K_\alpha) = c_\alpha$, hence $c_\alpha = 0$. This completes the inductive step of the proof.

The initial step of the induction is simply the inductive step applied to the unique minimal Northeast sequence $\alpha = 24 \cdots (2m)$. □

It follows that every Plücker coordinate p_α indexed by a non-Northeast sequence α can be written uniquely as a linear combination of Plücker coordinates indexed by Northeast sequences. We can be more precise about the form of these linear combinations. Recall that each fiber of the map π_{2m} contains a unique Northeast sequence. For a sequence α_0, let α be the Northeast sequence in the same fiber as α_0.

Lemma 5.6. *For each non-Northeast sequence α_0, let ℓ'_{α_0} be the linear relation among the Plücker coordinates expressing p_{α_0} as a linear combination of the p_β with β Northeast. Then p_α appears in ℓ'_{α_0} with coefficient $(-1)^{|I|}$, where $\alpha = I \cdot \alpha_0$, and every other Northeast β with p_β in the support of ℓ'_{α_0} satisfies $\beta > \alpha$.*

Proof. Let M be the descending matching of α with α^c defined by $M(\alpha_i) := \alpha_i^c$. Let K_α be the kernel element obtained by the process described above. Any linear form

$$\ell = p_{\alpha_0} + (-1)^{|I|+1} p_\alpha + \sum_{\alpha < \beta \in \mathcal{NE}} c_\beta p_\beta$$

vanishes on K_α.

We extend this relation to one which vanishes on all of $\mathrm{L}(\omega_n)_0$, proceeding inductively on the poset of Northeast sequences greater than or equal to α. Suppose that $\beta > \alpha$ is Northeast. By induction, suppose that for each Northeast sequence γ in the interval $[\alpha, \beta]$ the coefficient c_γ of ℓ has been determined in such a way that $\ell(K_\gamma) = 0$.

Let S be the set of Northeast sequences γ in the open interval (α, β) such that v_γ appears in K_β. Then

$$\ell(K_\beta) = \left(\sum_{\gamma \in S} c_\gamma\right) + c_\beta,$$

so setting $c_\beta := -\sum_{\gamma \in S} c_\gamma$ implies that $\ell(K_\beta) = 0$.

This completes the inductive part of the proof. We now have a linear form ℓ vanishing on $\mathrm{L}(\omega_n)_0$ which expresses p_{α_0} as a linear combination of Plücker coordinates indexed by Northeast sequences. Since such a linear form is unique, $\ell = \ell'_{\alpha_0}$. □

By Lemmas 5.5 and 5.6 and the argument preceding them, we deduce:

Theorem 5.7. *The system of linear relations*

$$\{\ell_\alpha^{(a)} = u^a v^{d-a} \otimes \Omega \wedge p_\alpha \mid a = 0, \dots, d,\ \alpha \in \tbinom{\langle n \rangle}{n-2}\}$$

has a reduced normal form consisting of linear forms expressing each Plücker coordinate $p_\beta^{(b)}$ with $\beta \notin \mathcal{NE} \subseteq \binom{\langle n \rangle}{n}$ as a linear combination of Plücker coordinates indexed by Northeast elements of $\binom{\langle n \rangle}{n}$.

Proof. We have seen that the linear relations preserve weight spaces, and Lemmas 5.5 and 5.6 provide the required normal form on each of these. The union of the relations constitute a normal form for the linear relations generating the entire linear subspace $L_{d,n}$. □

Example 5.8. Consider the zero weight space $(\bigwedge^4 \mathbb{C}^8)_0$ (so that $m = 2$). This is spanned by the vectors

$$v_\alpha := e_{\alpha_1} \wedge e_{\alpha_2} \wedge e_{\alpha_3} \wedge e_{\alpha_4}$$

(with dual basis the Plücker coordinates $p_\alpha = v_\alpha^*$), where

$$\alpha \in \{\bar{4}\bar{3}34, \bar{4}\bar{2}24, \bar{4}\bar{1}14, \bar{3}\bar{2}23, \bar{3}\bar{1}13, \bar{2}\bar{1}12\}.$$

The Northeast sequences are $\bar{4}\bar{3}34$ and $\bar{4}\bar{2}24$. The kernel of

$$\Omega \lrcorner \bullet : \left(\bigwedge^4 \mathbb{C}^8\right)_0 \to \left(\bigwedge^2 \mathbb{C}^8\right)_0$$

is spanned by the vectors

$$K_{\bar{4}\bar{2}24} = v_{\bar{4}\bar{2}24} - v_{\bar{4}\bar{1}14} - v_{\bar{3}\bar{2}23} + v_{\bar{3}\bar{1}13}, \text{ and}$$
$$K_{\bar{4}\bar{3}34} = v_{\bar{4}\bar{3}34} - v_{\bar{4}\bar{1}14} - v_{\bar{3}\bar{2}23} + v_{\bar{2}\bar{1}12}.$$

To see this concretely, we compute:

$$\Omega \lrcorner K_{\bar{4}\bar{2}24} = v_{\bar{4}4} + v_{\bar{2}2} - v_{\bar{4}4} - v_{\bar{1}1} - v_{\bar{3}3} - v_{\bar{2}2} + v_{\bar{3}3} + v_{\bar{1}1} = 0,$$

and similarly $\Omega \lrcorner K_{\bar{4}\bar{3}34} = 0$. The fibers of the map $\pi_4 : \binom{\langle 4\rangle}{4} \to \mathscr{D}_4$ are

$$\pi_4^{-1}(\bar{4}\bar{3}12, \bar{2}\bar{1}34) = \{\bar{4}\bar{3}34, \bar{2}\bar{1}12\}, \text{ and}$$
$$\pi_4^{-1}(\bar{4}\bar{2}13, \bar{3}\bar{1}24) = \{\bar{4}\bar{2}24, \bar{4}\bar{1}14, \bar{3}\bar{2}23, \bar{3}\bar{1}13\}.$$

The expression for $p_{\bar{4}\bar{1}14}$ as a linear combination of Plücker coordinates indexed by Northeast sequences is

$$\ell_{\bar{4}\bar{1}14} = p_{\bar{4}\bar{1}14} + c_{\bar{4}\bar{2}24} p_{\bar{4}\bar{2}24} + c_{\bar{4}\bar{3}34} p_{\bar{4}\bar{3}34},$$

for some $c_{\bar{4}\bar{2}24}, c_{\bar{4}\bar{3}34} \in \mathbb{C}$, which we can compute as follows. Since

$$0 = \ell_{\bar{4}\bar{1}14}(K_{\bar{4}\bar{2}24}) = c_{\bar{4}\bar{2}24} - 1,$$

we have $c_{\bar{4}\bar{2}24} = 1$. Similarly,

$$0 = \ell_{\bar{4}\bar{1}14}(K_{\bar{4}\bar{3}34}) = c_{\bar{4}\bar{3}34} - 1,$$

so $c_{\bar{4}\bar{3}34} = 1$. Hence $\ell_{\bar{4}\bar{1}14} = p_{\bar{4}\bar{1}14} + p_{\bar{4}\bar{2}24} + p_{\bar{4}\bar{3}34}$, which agrees with (2-3).

5B. *Proof of the straightening law.* We find generators of $(I_{d,n}+L_{d,n}) \cap \mathbb{C}[\mathscr{D}_{d,n}]$ which express the quotient as an algebra with straightening law on $\mathscr{D}_{d,n}$. Such a generating set is automatically a Gröbner basis with respect to the degree reverse lexicographic term order where variables are ordered by a refinement of the doset order. We begin with a Gröbner basis $G_{I_{d,n}+L_{d,n}}$ for $I_{d,n}+L_{d,n}$ with respect to a similar term order. For $\alpha^{(a)} \in \binom{\langle n\rangle}{n}_d$, write

$$\check{\alpha}^{(a)} := \alpha^{(a)} \vee (\alpha^t)^{(a)} \quad \text{and} \quad \hat{\alpha}^{(a)} := \alpha^{(a)} \wedge (\alpha^t)^{(a)},$$

so that

$$\pi_n(\alpha^{(a)}) = (\hat{\alpha}^{(a)}, \check{\alpha}^{(a)}).$$

We call an element $\alpha^{(a)} \in \binom{\langle n\rangle}{n}_d$ *Northeast* if $\alpha \in \binom{\langle n\rangle}{n}$ is Northeast.

Let $<$ be a linear refinement of the partial order on $\mathscr{P}_{d,n}$ satisfying the following conditions. First, the Northeast sequence is minimal among those in a given fiber of π_n. This is possible since every weight space is an antichain (that is, no two elements are comparable). Second, $\alpha^{(a)} < \beta^{(b)}$ if $(\hat{\alpha}^{(a)}, \check{\alpha}^{(a)})$ is lexicographically smaller than $(\hat{\beta}^{(b)}, \check{\beta}^{(b)})$.

With respect to any such refinement, consider the degree reverse lexicographic term order. A reduced Gröbner basis $G_{d,n}$ for $I_{d,n}+L_{d,n}$ with respect to this term order will have standard monomials indexed by chains (in $\mathcal{P}_{d,n}$) of Northeast partitions. While every monomial supported on a chain of Northeast partitions is standard modulo $I_{d,n}$, this is not always the case modulo $I_{d,n}+L_{d,n}$. In other words, upon identifying each Northeast partition appearing in a given monomial with an element of $\mathcal{D}_{d,n}$, we do not necessarily obtain a monomial supported on a chain in $\mathcal{D}_{d,n}$. It is thus necessary to identify precisely which Northeast chains in $\binom{\langle n\rangle}{n}$ correspond to chains in $\mathcal{D}_{d,n}$ via the map π_n.

A monomial $p_\alpha^{(a)} p_\beta^{(b)}$ such that $\alpha^{(a)} < \beta^{(b)}, (\beta^t)^{(b)}$, and $\alpha^{(a)}, \beta^{(b)} \in \mathcal{NE}$ cannot be reduced modulo $G_{I_{d,n}}$ or $G_{L_{d,n}}$. On the other hand, if $\alpha^{(a)} < \beta^{(b)}$ (say), but $\alpha^{(a)}$ and $(\beta^t)^{(b)}$ are incomparable (written $\alpha^{(a)} \not\sim (\beta^t)^{(b)}$) then there is a relation in $G_{I_{d,n}}$ with leading term $p_\alpha^{(a)} p_{\beta^t}^{(b)}$. It follows that the degree-two standard monomials are indexed by Northeast partitions $p_\alpha^{(a)} p_\beta^{(b)}$ with $\alpha^{(a)} < \beta^{(b)}, (\beta^t)^{(b)}$.

Conversely, any monomial $p_\alpha^{(a)} p_\beta^{(b)}$ with $\alpha^{(a)} < \beta^{(b)}, (\beta^t)^{(b)}$, and $\alpha^{(a)}, \beta^{(b)} \in \mathcal{NE}$ cannot be the leading term of any element of $G_{I_{d,n}+L_{d,n}}$. To see this, observe that $G_{I_{d,n}+L_{d,n}}$ is obtained by Buchberger's algorithm [1965] applied to $G_{I_{d,n}} \cup G_{L_{d,n}}$, and we may consider only the S-polynomials $S(f, g)$ with $f \in G_{I_{d,n}}$ and $g \in G_{L_{d,n}}$. In this case we may assume $\mathrm{in}_< g$ divides $\mathrm{in}_< f$.

Let α_0 be the partition such that $\mathrm{in}_< g = p_{\alpha_0}^{(a)}$ (that is, g is the unique expression of $p_{\alpha_0}^{(a)}$ as a linear combination of Plücker coordinates indexed by Northeast partitions), and let α be the unique Northeast partition such that $\pi_n(\alpha_0) = \pi_n(\alpha)$. By the reduced normal form given in Theorem 5.7, $S(f, g)$ is the obtained by replacing $p_{\alpha_0}^{(a)}$ with $\pm p_\alpha^{(a)} + \ell$, where ℓ is a linear combination of Plücker coordinates $p_\gamma^{(a)}$ with γ Northeast and $\alpha_+ < \gamma_+$. This latter condition implies that $\hat{\alpha} < \hat{\gamma}$ (also, $\check{\alpha} > \check{\gamma}$), and therefore $(\hat{\alpha}, \check{\alpha})$ is lexicographically smaller than $(\hat{\gamma}, \check{\gamma})$.

Hence, with respect to the reduced Gröbner basis $G_{I_{d,n}+L_{d,n}}$, the standard monomials are precisely the monomials $p_\alpha^{(a)} p_\beta^{(b)}$ with $\alpha^{(a)} \le \beta^{(b)}, (\beta^t)^{(b)}$, and $\alpha^{(a)}, \beta^{(b)} \in \mathcal{NE}$.

Recall that elements of the doset $\mathcal{D}_{d,n}$ are pairs (α, β) of admissible elements (Definition 2.6) of $\binom{\langle n\rangle}{n}_d$ such that (regarded as sequences):

- $\alpha \le \beta$;
- α and β have the same number of negative (or positive) elements.

Equivalently, regarding α and β as partitions, the elements of $\mathcal{D}_{d,n}$ are pairs (α, β) of symmetric partitions such that:

- $\alpha \subseteq \beta$;
- α and β have the same Durfee square, where the *Durfee square* of a partition α is the largest square subpartition $(p^p) \subseteq \alpha$ (for some $p \le n$).

Theorem 5.9. *$\mathbb{C}[\binom{\langle n\rangle}{n}_d]/\langle I_{d,n}+L_{d,n}\rangle$ is an algebra with straightening law on $\mathcal{D}_{d,n}$.*

Proof. Since standard monomials with respect to a Gröbner basis are linearly independent, the arguments above establish the conditions in Definition 3.3 (1) and (2).

To establish the condition (3), note that it suffices to consider the expression for a degree-2 monomial as a sum of standard monomials. For simplicity, we absorb the superscripts into our notation and write $\alpha \in \binom{\langle n\rangle}{n}_d$ and similarly for the corresponding Plücker coordinate. Let

$$p_{(\hat{\alpha},\check{\alpha})}p_{(\hat{\beta},\check{\beta})} = \sum_{j=1}^{k} c_j\, p_{(\hat{\alpha}_j,\check{\alpha}_j)}p_{(\hat{\beta}_j,\check{\beta}_j)} \tag{5-3}$$

be a reduced expression in $G_{I_{d,n}+L_{d,n}}$ for $p_{(\hat{\alpha},\check{\alpha})}p_{(\hat{\beta},\check{\beta})}$ as a sum of standard monomials. That is, $p_{(\hat{\alpha},\check{\alpha})}p_{(\hat{\beta},\check{\beta})}$ is nonstandard and $p_{(\hat{\alpha}_j,\check{\alpha}_j)}p_{(\hat{\beta}_j,\check{\beta}_j)}$ is standard for $j = 1,\dots,k$. We assume that α (respectively, β) be the unique Northeast partition such that $\pi_n(\alpha) = (\hat{\alpha},\check{\alpha})$ (respectively, $\pi_n(\beta) = (\hat{\beta},\check{\beta})$), and similarly for each α_j and β_j appearing in (5-3).

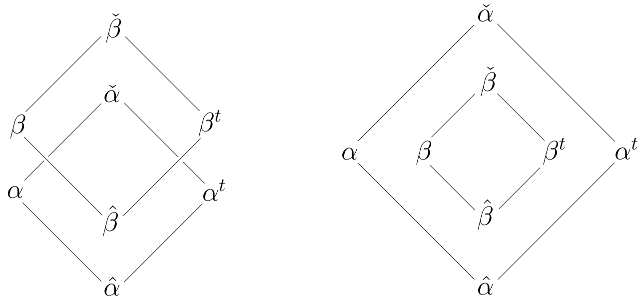
Fix $j = 1,\dots,k$. The standard monomial $p_{(\hat{\alpha}_j,\check{\alpha}_j)}p_{(\hat{\beta}_j,\check{\beta}_j)}$ is obtained by the reduction modulo $G_{L_{d,n}}$ of a standard monomial $p_\gamma p_\delta$ appearing in the straightening relation for $p_\alpha p_\beta$, which is an element of the Gröbner basis $G_{I_{d,n}}$. If γ and δ are both Northeast, then nothing happens, that is, $\gamma = \alpha_j$ and $\delta = \beta_j$. If γ is not Northeast, then we rewrite p_γ as a linear combination of Plücker coordinates indexed by Northeast sequences. Lemma 5.6 ensures that the leading term of the new expression is $p_{(\hat{\gamma},\check{\gamma})}$, and the lower order terms $p_{(\hat{\epsilon},\check{\epsilon})}$ satisfy $\hat{\epsilon} < \hat{\gamma}$.

It follows that the lexicographic comparison in the condition of Definition 3.3 (3) terminates with the first Plücker coordinate. That is, if $(\hat{\alpha}_j \le \check{\alpha}_j \le \hat{\beta}_j \le \check{\beta}_j)$ is lexicographically smaller than $(\hat{\alpha} \le \check{\alpha} \le \hat{\beta} \le \check{\beta})$, then either $\hat{\alpha}_j < \hat{\alpha}$ or $\hat{\alpha}_j = \hat{\alpha}$ and $\hat{\alpha}_j < \hat{\alpha}$. Therefore the reduction process applied to p_δ does not affect the result, and the condition (3) is proven.

It remains to prove the condition (4). Suppose that $(\hat{\alpha},\check{\alpha})$ and $(\hat{\beta},\check{\beta})$ are incomparable elements of $\mathcal{D}_{d,n}$ (α and β Northeast). This means that α is incomparable to either β or β^t (possibly both). Without loss of generality, we will deal only with the more complicated case that α and β^t are incomparable. The hypothesis of the condition (4) is that the set $\{\hat{\alpha},\check{\alpha},\hat{\beta},\check{\beta}\}$ forms a chain in $\binom{\langle n\rangle}{n}_d$. Up to interchanging the roles of α and β, there are two possible cases (see Figure 9):

$$\hat{\alpha} < \hat{\beta} < \check{\alpha} < \check{\beta}, \quad \text{or} \quad \hat{\alpha} < \hat{\beta} < \check{\beta} < \check{\alpha}.$$

First, suppose $\hat{\alpha} < \hat{\beta} < \check{\alpha} < \check{\beta}$. Recall that for any $\gamma_0 \in \binom{\langle n\rangle}{n}_d$, with Northeast sequence γ in the same fiber of π_n, the expression for the Plücker coordinate p_{γ_0} as a linear combination of Plücker coordinates indexed by Northeast sequences is



$\hat{\alpha} < \hat{\beta} < \check{\alpha} < \check{\beta}$ $\qquad$ $\check{\alpha} < \hat{\beta} < \check{\beta} < \check{\alpha}$

Figure 9. The two cases in the proof of the condition of Definition 3.3 (4).

supported on Plücker coordinates p_δ such that $\delta_+ \geq \gamma_+$ with equality if and only if $\delta = \gamma$, and the Plücker coordinate p_γ appears with coefficient ± 1 (Lemma 5.6).

Upon replacing each Northeast (or Southwest) partition with its associated doset element using the map π_n from Section 2B, the first two terms of straightening relation for $p_\alpha p_{\beta^t}$ are

$$
\begin{aligned}
&p_\alpha p_{\beta^t} - p_{\alpha\wedge\beta^t} p_{\alpha\vee\beta^t}\\
&\qquad = p_\alpha p_{\beta^t} - \sigma p_{((\alpha\wedge\beta^t)^\wedge,(\alpha\wedge\beta^t)^\vee)} p_{((\alpha\vee\beta^t)^\wedge,(\alpha\vee\beta^t)^\vee)} + \text{lower order terms}\\
&\qquad = \sigma_\beta p_{(\hat{\alpha},\check{\alpha})} p_{(\hat{\beta},\check{\beta})} - \sigma p_{(\hat{\alpha},\hat{\beta})} p_{(\check{\alpha},\check{\beta})} + \text{lower order terms},
\end{aligned}
$$

where $\sigma = \pm 1$. The second equation is justified as follows. For any element $\alpha \in \binom{\langle n\rangle}{n}_d$, recall that α_+ (respectively, α_-) denotes the subsequence of positive (negative) elements of α. This was previous defined for elements of $\binom{\langle n\rangle}{n}$, but extends to elements of $\binom{\langle n\rangle}{n}_d$ in the obvious way, that is, by ignoring the superscript. The condition

$$\hat{\alpha} < \hat{\beta} < \check{\alpha} < \check{\beta}$$

is equivalent to

$$\bar{\alpha}_-^c < \bar{\beta}_-^c < \alpha_+ < \beta_+.$$

Note that this implies that

$$\alpha \wedge \beta^t = \alpha_- \cup \bar{\beta}_-^c, \quad \text{and} \quad \alpha \vee \beta^t = \bar{\alpha}_+^c \cup \beta_+.$$

We compute in the distributive lattice $\binom{\langle n\rangle}{n}_d$.

$$
\begin{aligned}
(\alpha\wedge\beta^t)\wedge(\alpha^t\wedge\beta) &= (\alpha_-\cup\bar{\beta}_-^c)\wedge(\beta_-\cup\bar{\alpha}_-^c) = \hat{\alpha}\\
(\alpha\wedge\beta^t)\vee(\alpha^t\wedge\beta) &= (\alpha_-\cup\bar{\beta}_-^c)\vee(\beta_-\cup\bar{\alpha}_-^c) = \hat{\beta}
\end{aligned}
$$

Similarly,

$$(\alpha \vee \beta^t) \wedge (\alpha^t \vee \beta) = \check{\alpha}, \quad \text{and} \quad (\alpha \vee \beta^t) \vee (\alpha^t \vee \beta) = \check{\beta}.$$

In the remaining case, we have $\check{\alpha} < \hat{\beta} < \check{\beta} < \check{\alpha}$, and it follows that $\alpha \not\sim \beta$ and $\alpha \not\sim \beta^t$ both hold. We use the relation for the incomparable pair $\alpha \not\sim \beta^t$:

$$\begin{aligned} p_\alpha p_{\beta^t} - p_{\alpha\wedge\beta^t} p_{\alpha\vee\beta^t} \\ &= p_\alpha p_{\beta^t} - \sigma p_{((\alpha\wedge\beta^t)^\wedge,(\alpha\wedge\beta^t)^\vee)} p_{((\alpha\vee\beta^t)^\wedge,(\alpha\vee\beta^t)^\vee)} + \text{lower order terms} \\ &= \sigma_\beta p_{(\hat{\alpha},\check{\alpha})} p_{(\hat{\beta},\check{\beta})} - \sigma p_{(\hat{\alpha},\hat{\beta})} p_{(\check{\beta},\check{\alpha})} + \text{lower order terms,} \end{aligned}$$

where the second equality holds by a similar computation in $\binom{\langle n\rangle}{n}_d$. □

The next result shows that the algebra with straightening law just constructed is indeed the coordinate ring of $L\mathfrak{Q}_d(n)$.

Theorem 5.10. $\mathbb{C}[\binom{\langle n\rangle}{n}_d]/\langle I_{d,n}+L_{d,n}\rangle \cong \mathbb{C}[L\mathfrak{Q}_d(n)]$.

Proof. Let $I' := I(L\mathfrak{Q}_d(n))$. By definition, we have $I_{d,n}+L_{d,n} \subseteq I'$. Since the degree and codimension of these ideals are equal, I' is nilpotent modulo $I_{d,n}+L_{d,n}$. On the other hand $I_{d,n}+L_{d,n}$ is radical, so $I_{d,n}+L_{d,n} = I'$. □

The arguments of De Concini and Lakshmibai [1981, Theorem 4.5] extend to the case of Schubert subvarieties of $L\mathfrak{Q}_d(n)$.

Corollary 5.11. *The coordinate ring of any Schubert subvariety of $L\mathfrak{Q}_d(n)$ is an algebra with straightening law on a doset, hence Cohen–Macaulay and Koszul.*

Proof. For $\alpha^{(a)} \in \mathfrak{D}_{d,n}$, the Schubert variety $X_{\alpha^{(a)}}$ is defined by the vanishing of the Plücker coordinates $p^{(b)}_{(\beta,\gamma)}$ for $\gamma^{(b)} \not\leq \alpha^{(a)}$. The conditions in Definition 3.3 (4) are stable upon setting these variables to zero, so we obtain an algebra with straightening law on the doset

$$\{(\beta,\gamma)^{(b)} \in \mathfrak{D}_{d,n} \mid \gamma^{(b)} \leq \alpha^{(a)}\}.$$

Let $\mathfrak{D} \subseteq \mathcal{P}\times\mathcal{P}$ be a doset on the poset $\mathcal{P}$, A any algebra with straightening law on $\mathfrak{D}$, and $\mathbb{C}\{\mathcal{P}\}$ the unique *discrete* algebra with straightening law on $\mathcal{P}$. That is, $\mathbb{C}\{\mathcal{P}\}$ has algebra generators corresponding to the elements of $\mathcal{P}$, and the straightening relations are $\alpha\beta = 0$ if α and β are incomparable elements of $\mathcal{P}$. Then A is Cohen–Macaulay if and only if $\mathbb{C}\{\mathcal{P}\}$ is Cohen–Macaulay [De Concini and Lakshmibai 1981].

On the other hand, $\mathbb{C}\{\mathcal{P}\}$ is the face ring of the order complex of $\mathcal{P}$. The order complex of a locally upper semimodular poset is shellable. The face ring of a shellable simplicial complex is Cohen–Macaulay [Bruns and Herzog 1993]. By Proposition 3.17, any interval in the poset $\mathcal{P}_{d,n}$ is a distributive lattice, hence locally upper semimodular. This proves that $\mathbb{C}[L\mathfrak{Q}_d(n)]$ is Cohen–Macaulay. The

Koszul property is a consequence of the quadratic Gröbner basis consisting of the straightening relations. □

The main results of this paper suggest that the space of quasimaps is an adequate setting for the study of the enumerative geometry of curves into a general flag variety. They also give a new and interesting example of a family of varieties whose coordinate rings are Hodge algebras.

After the ordinary Grassmannian, the Lagrangian Grassmannian was the first space to be well understood in terms of (classical) standard monomial theory. Our results thus lend credence to the expectation that further study of the space of quasimaps into a flag variety of general type, possibly incorporating the ideas of Chirivì [2000; 2001], will yield new results in parallel (to some extent) with the classical theory.

Acknowledgments

We thank Frank Sottile for suggesting this problem, and for valuable feedback during the preparation of this paper.

References

[Bertram 1997] A. Bertram, "Quantum Schubert calculus", *Adv. Math.* **128**:2 (1997), 289–305. MR 98j:14067 Zbl 0945.14031

[Bertram et al. 2005] A. Bertram, I. Ciocan-Fontanine, and B. Kim, "Two proofs of a conjecture of Hori and Vafa", *Duke Math. J.* **126**:1 (2005), 101–136. MR 2006e:14077 Zbl 1082.14055

[Braverman 2006] A. Braverman, "Spaces of quasi-maps into the flag varieties and their applications", pp. 1145–1170 in *International Congress of Mathematicians* (Madrid, 2006), vol. II, edited by M. Sanz-Solé et al., Eur. Math. Soc., Zürich, 2006. MR 2008i:14019 Zbl 1118.22007

[Bruns and Herzog 1993] W. Bruns and J. Herzog, *Cohen–Macaulay rings*, Cambridge Studies in Advanced Mathematics **39**, Cambridge University Press, Cambridge, 1993. MR 95h:13020 Zbl 0788.13005

[Bruns and Vetter 1988] W. Bruns and U. Vetter, *Determinantal rings*, Lecture Notes in Mathematics **1327**, Springer, Berlin, 1988. MR 89i:13001 Zbl 0673.13006

[Buchberger 1965] B. Buchberger, *Ein Algorithmus zum Auffinden der Basiselemente des Restklassenringes nach einem nulldimensionalen Polynomideal*, PhD thesis, University of Innsbruck, Austria, 1965.

[Chen 2001] L. Chen, "Poincaré polynomials of hyperquot schemes", *Math. Ann.* **321**:2 (2001), 235–251. MR 2002g:14075 Zbl 1076.14536

[Chen 2003] L. Chen, "Quantum cohomology of flag manifolds", *Adv. Math.* **174**:1 (2003), 1–34. MR 2004a:14055 Zbl 1061.14061

[Chirivì 2000] R. Chirivì, "LS algebras and application to Schubert varieties", *Transform. Groups* **5**:3 (2000), 245–264. MR 2001h:14060 Zbl 1019.14019

[Chirivì 2001] R. Chirivì, "Deformation and Cohen–Macaulayness of the multicone over the flag variety", *Comment. Math. Helv.* **76**:3 (2001), 436–466. MR 2002f:14064 Zbl 1074.14529

[Ciocan-Fontanine 1999] I. Ciocan-Fontanine, "On quantum cohomology rings of partial flag varieties", *Duke Math. J.* **98**:3 (1999), 485–524. MR 2000d:14058 Zbl 0969.14039

[De Concini and Lakshmibai 1981] C. De Concini and V. Lakshmibai, "Arithmetic Cohen–Macaulayness and arithmetic normality for Schubert varieties", *Amer. J. Math.* **103**:5 (1981), 835–850. MR 83e:14035 Zbl 0475.14045

[De Concini et al. 1982] C. De Concini, D. Eisenbud, and C. Procesi, *Hodge algebras*, Astérisque **91**, Société Mathématique de France, Paris, 1982. MR 85d:13009 Zbl 0509.13026

[Eisenbud 1980] D. Eisenbud, "Introduction to algebras with straightening laws", pp. 243–268 in *Ring theory and algebra, III* (Norman, OK, 1979), edited by B. R. McDonald, Lecture Notes in Pure and Appl. Math. **55**, Marcel Dekker, New York, 1980. MR 82i:14030 Zbl 0448.13010

[Fulton 1997] W. Fulton, *Young tableaux: With applications to representation theory and geometry*, London Mathematical Society Student Texts **35**, Cambridge University Press, Cambridge, 1997. MR 99f:05119 Zbl 0878.14034

[Fulton and Harris 1991] W. Fulton and J. Harris, *Representation theory: A first course*, Graduate Texts in Mathematics **129**, Springer, New York, 1991. MR 93a:20069 Zbl 0744.22001

[Fulton and Pandharipande 1997] W. Fulton and R. Pandharipande, "Notes on stable maps and quantum cohomology", pp. 45–96 in *Algebraic geometry* (Santa Cruz, 1995), edited by J. Kollár et al., Proc. Sympos. Pure Math. **62**, Amer. Math. Soc., Providence, RI, 1997. MR 98m:14025 Zbl 0898.14018

[Givental 1996] A. B. Givental, "Equivariant Gromov–Witten invariants", *Internat. Math. Res. Notices* 13 (1996), 613–663. MR 97e:14015 Zbl 0881.55006

[Hiller and Boe 1986] H. Hiller and B. Boe, "Pieri formula for $\mathrm{SO}_{2n+1}/\mathrm{U}_n$ and $\mathrm{Sp}_n/\mathrm{U}_n$", *Adv. in Math.* **62**:1 (1986), 49–67. MR 87k:14058 Zbl 0611.14036

[Hodge 1943] W. V. D. Hodge, "Some enumerative results in the theory of forms", *Proc. Cambridge Philos. Soc.* **39** (1943), 22–30. MR 4,184e Zbl 0060.04107

[Kleiman 1974] S. L. Kleiman, "The transversality of a general translate", *Compositio Math.* **28** (1974), 287–297. MR 50 #13063 Zbl 0288.14014

[Kontsevich 1995] M. Kontsevich, "Enumeration of rational curves via torus actions", pp. 335–368 in *The moduli space of curves* (Texel, Netherland, 1994), edited by R. Dijkgraaf et al., Progr. Math. **129**, Birkhäuser, Boston, 1995. MR 97d:14077 Zbl 0885.14028

[Kresch and Tamvakis 2003] A. Kresch and H. Tamvakis, "Quantum cohomology of the Lagrangian Grassmannian", *J. Algebraic Geom.* **12**:4 (2003), 777–810. MR 2004h:14063 Zbl 1051.53070

[Kuznetsov 1997] A. Kuznetsov, "Laumon's resolution of Drinfel′d's compactification is small", *Math. Res. Lett.* **4**:2-3 (1997), 349–364. MR 98d:14022 Zbl 0910.14026

[Lakshmibai 2003] V. Lakshmibai, "The development of standard monomial theory, II", pp. 283–309 in *A tribute to C. S. Seshadri* (Chennai, 2002), edited by V. Lakshmibai et al., Birkhäuser, Basel, 2003. MR 2004m:14107b Zbl 1056.14066

[Laumon 1990] G. Laumon, "Faisceaux automorphes liés aux séries d'Eisenstein", pp. 227–281 in *Automorphic forms, Shimura varieties, and L-functions* (Ann Arbor, MI, 1988), vol. I, edited by L. Clozel and J. S. Milne, Perspect. Math. **10**, Academic Press, Boston, 1990. MR 91k:11106 Zbl 0773.11032

[Littelmann 1998] P. Littelmann, "Contracting modules and standard monomial theory for symmetrizable Kac–Moody algebras", *J. Amer. Math. Soc.* **11**:3 (1998), 551–567. MR 99d:17027 Zbl 0915.20022

[Musili 2003] C. Musili, "The development of standard monomial theory, I", pp. 385–420 in *A tribute to C. S. Seshadri* (Chennai, 2002), edited by V. Lakshmibai et al., Birkhäuser, Basel, 2003. MR 2004m:14107a Zbl 1056.14067

[Oprea 2006] D. Oprea, "The tautological rings of the moduli spaces of stable maps to flag varieties", *J. Algebraic Geom.* **15**:4 (2006), 623–655. MR 2007i:14015 Zbl 1117.14056

[Rosenthal 1994] J. Rosenthal, "On dynamic feedback compensation and compactification of systems", *SIAM J. Control Optim.* **32**:1 (1994), 279–296. MR 95b:93084 Zbl 0797.93018

[Sottile 2000] F. Sottile, "Real rational curves in Grassmannians", *J. Amer. Math. Soc.* **13**:2 (2000), 333–341. MR 2000j:14080 Zbl 0946.14035

[Sottile 2001] F. Sottile, "Rational curves on Grassmannians: systems theory, reality, and transversality", pp. 9–42 in *Advances in algebraic geometry motivated by physics* (Lowell, MA, 2000), edited by E. Previato, Contemp. Math. **276**, Amer. Math. Soc., Providence, RI, 2001. MR 2002d:14080 Zbl 1095.14503

[Sottile and Sturmfels 2001] F. Sottile and B. Sturmfels, "A sagbi basis for the quantum Grassmannian", *J. Pure Appl. Algebra* **158**:2-3 (2001), 347–366. MR 2002a:13027 Zbl 0992.14023

[Strømme 1987] S. A. Strømme, "On parametrized rational curves in Grassmann varieties", pp. 251–272 in *Space curves* (Rocca di Papa, 1985), edited by F. Ghione et al., Lecture Notes in Math. **1266**, Springer, Berlin, 1987. MR 88i:14020 Zbl 0625.14027

[Weyman 2003] J. Weyman, *Cohomology of vector bundles and syzygies*, Cambridge Tracts in Mathematics **149**, Cambridge University Press, Cambridge, 2003. MR 2004d:13020 Zbl 1075.13007

Communicated by Bernd Sturmfels
Received 2008-06-04 Revised 2008-07-31 Accepted 2008-09-12

ruffojv@oneonta.edu *Department of Mathematics, Science, and Statistics, State University of New York - College at Oneonta, Oneonta, NY 13820, United States*
http://employees.oneonta.edu/ruffojv