Algebra & Number Theory

Volume 3
2009
No. 2

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mathematical sciences publishers

ALGEBRA AND NUMBER THEORY 3:2(2009)

# A jeu de taquin theory for increasing tableaux, with applications to $K$-theoretic Schubert calculus

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We introduce a theory of *jeu de taquin* for *increasing tableaux*, extending fundamental work of Schützenberger (1977) for standard Young tableaux. We apply this to give a new combinatorial rule for the $K$-theory Schubert calculus of Grassmannians via *$K$-theoretic jeu de taquin*, providing an alternative to the rules of Buch and others. This rule naturally generalizes to give a conjectural root-system uniform rule for any minuscule flag variety $G/P$, extending recent work of Thomas and Yong. We also present analogues of results of Fomin, Haiman, Schensted and Schützenberger.



## 1. Introduction

In this paper, we introduce a *jeu de taquin* type theory for *increasing tableaux*, extending Schützenberger's fundamental framework [1977] to the ($K$-theoretic) *Grothendieck polynomial* context introduced a few years later by Lascoux and Schützenberger [1982].

*MSC2000:* primary 05E10; secondary 14M15.

*Keywords:* Schubert calculus, K-theory, jeu de taquin.

Thomas was supported by an NSERC Discovery grant. Yong was supported by NSF grant 0601010.

One motivation and application for this work comes from Schubert calculus. Let $X = \mathrm{Gr}(k, \mathbb{C}^n)$ be the Grassmannian of $k$-planes in $\mathbb{C}^n$ and let $K(X)$ be the *Grothendieck ring* of algebraic vector bundles over $X$; see, for example, the expositions [Brion 2005; Buch 2005b] for definitions and discussion. To each partition, as identified with its *Young shape* $\lambda \subseteq \Lambda := k \times (n-k)$, let $X_\lambda$ be the associated Schubert variety and $\mathbb{O}_{X_\lambda}$ its structure sheaf. The classes $\{[\mathbb{O}_{X_\lambda}]\} \subseteq K(X)$ form an additive $\mathbb{Z}$-basis of $K(X)$. The (*K-theoretic*) *Schubert structure constants* $C_{\lambda,\mu}^{\nu}$ are defined by

$$[\mathbb{O}_{X_\lambda}] \cdot [\mathbb{O}_{X_\mu}] = \sum_{\nu \subseteq \Lambda} C_{\lambda,\mu}^{\nu} [\mathbb{O}_{X_\nu}].$$

Buch's rule [2002b] established alternation of sign, that is,

$$(-1)^{|\nu|-|\lambda|-|\mu|} C_{\lambda,\mu}^{\nu} \in \mathbb{N}.$$

In the *cohomology case* $|\lambda| + |\mu| = |\nu|$ where $|\lambda| = \sum_i \lambda_i$ is the *size* of $\lambda$, the numbers $C_{\lambda,\mu}^{\nu}$ are the classical *Littlewood–Richardson coefficients*. Here, $C_{\lambda,\mu}^{\nu}$ counts points in the intersection of three general Schubert varieties. These numbers determine the ring structure of the cohomology $H^\star(X, \mathbb{Q})$. Combinatorially, they are governed by the tableau theory of Schur polynomials. Schützenberger's *jeu de taquin* theory [1977], by which the first modern statement and proof of a Littlewood–Richardson rule was constructed, has had a central impact here.

While $H^\star(X, \mathbb{Q})$ contains important geometric data about $X$, this is even more true of $K(X)$. The combinatorics of $K(X)$ is encoded by the Grothendieck polynomials of Lascoux and Schützenberger [1982] (for more details, see Appendix). This richer environment parallels the Schur polynomial setting, as demonstrated by, for example, [Lenart 2000; Buch 2002b; Buch et al. 2008]. However, basic gaps in this comparison remain. In particular, one lacks an analogue of the *jeu de taquin* theory. This also raises questions of intrinsic combinatorial interest.

Indeed, there has been significant interest in the Grothendieck ring of $X$ and of related varieties; see work on, for example, quiver loci [Buch 2002a; 2005a; Miller 2005; Buch et al. 2008], Hilbert series of determinantal ideals [Knutson and Miller 2005; Knutson et al. 2008; 2009], applications to invariants of matroids [Speyer 2006], and in relation to representation theory [Griffeth and Ram 2004; Lenart and Postnikov 2007; Willems 2006]. See also work of Lam and Pylyavskyy [2007] concerning combinatorial Hopf algebras.

Consequently, we aim to provide unifying foundational combinatorics in support of further such developments. Evidence of the efficacy of this approach is provided through our study of minuscule Schubert calculus; other uses are also suggested. In particular, as a *non*-algebraic geometric application, in a followup paper [Thomas and Yong 2008b], we relate the ideas in this paper to [Buch et al. 2008] and the study of longest strictly increasing subsequences in random words.

Specifically, we introduce a jeu de taquin construction, thereby allowing for $K$-theoretic generalizations of a number of results from algebraic combinatorics. In particular, we give an analogue of Schützenberger's Littlewood–Richardson rule. In addition, we extend Fomin's *growth diagrams*, allowing for, for example, a generalization of Schützenberger's *evacuation involution*. On the other hand, it is interesting that natural generalizations of some results from the classical theory are *not* true, underlining some basic combinatorial obstructions.

One feature of our rule is that it has a natural conjectural generalization to any minuscule flag variety $G/P$, extending our earlier work [Thomas and Yong 2006; 2007]; this provides the first generalized Littlewood–Richardson formula (even conjectural) for $K$-theory, outside of the Grassmannians. (There are already a number of more specialized $K$-theoretic Schubert calculus formulas proven for any $G/P$, such as the Pieri-type formulas of [Lenart and Postnikov 2007] and others.)

***Main definitions.*** An *increasing tableau* $T$ of shape $\nu/\lambda$ is a filling of the skew shape

$$\mathtt{shape}(T) = \nu/\lambda$$

with $\{1, 2, \ldots, q\}$ where $q \leq |\nu/\lambda|$ such that the entries of $T$ strictly increase along each row and column. We write $\max T$ for the maximum entry in $T$. In particular, when $\max T = |\nu/\lambda|$ and each label appears exactly once, $T$ is a *standard Young tableau*. Let $\mathtt{INC}(\nu/\lambda)$ be the set of these increasing tableaux and $\mathtt{SYT}(\nu/\lambda)$ be the set of standard Young tableaux for $\nu/\lambda$. Below we give an example of an increasing tableau and a standard Young tableau, each of shape $\nu/\lambda = (5, 3, 1)/(2, 1)$:

$$\begin{array}{|c|c|c|c|c|}\hline \ & \ & 1 & 2 & 3 \\ \hline \ & 1 & 3 \\ \cline{1-3} 2 \\ \cline{1-1}\end{array} \in \mathtt{INC}((5,3,1)/(2,1)), \qquad \begin{array}{|c|c|c|c|c|}\hline \ & \ & 1 & 4 & 6 \\ \hline \ & 2 & 5 \\ \cline{1-3} 3 \\ \cline{1-1}\end{array} \in \mathtt{SYT}((5,3,1)/(2,1)).$$

We also need to define the *superstandard Young tableau* $S_\lambda$ of shape $\lambda$ to be the standard Young tableau that fills the first row with $1, 2, \ldots, \lambda_1$, the second row with $\lambda_1{+}1, \lambda_1{+}2, \ldots, \lambda_1{+}\lambda_2$, and so on. For example,

$$S_{(5,3,3,1)} = \begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 3 & 4 & 5 \\ \hline 6 & 7 & 8 \\ \cline{1-3} 9 & 10 & 11 \\ \cline{1-3} 12 \\ \cline{1-1}\end{array}\,.$$

A *short ribbon* $R$ is a connected skew shape that does not contain a $2 \times 2$ subshape and where each row and column contains at most two boxes. A *alternating ribbon* is a filling of a short ribbon $R$ with two symbols where adjacent boxes are filled differently. We define $\mathtt{switch}(R)$ to be the alternating ribbon of the same shape as $R$ but where each box is instead filled with the other symbol. For example, we have

$$R = \begin{array}{cccc} & & \circ & \bullet \\ & \circ & \bullet & \\ \circ & \bullet & & \\ \bullet & & & \end{array} \quad \text{and} \quad \texttt{switch}(R) = \begin{array}{cccc} & & \bullet & \circ \\ & \bullet & \circ & \\ \bullet & \circ & & \\ \circ & & & \end{array}.$$

By definition, if $R$ is a ribbon consisting of a single box, $\texttt{switch}$ does nothing to it. We define $\texttt{switch}$ to act on a skew shape consisting of multiple connected components, each of which is an alternating ribbon, by acting on each separately.

Our starting point is the following new idea. Given $T \in \texttt{INC}(\nu/\lambda)$, an *inner corner* is any maximally southeast box $x \in \lambda$. Now fix a set $\{x_1, \ldots, x_s\}$ of inner corners and let each of these boxes is filled with a $\bullet$. Consider the union of short ribbons $R_1$ which is made of boxes with entries $\bullet$ or 1. Apply $\texttt{switch}$ to $R_1$. Now let $R_2$ be the union of short ribbons consisting of boxes with entries $\bullet$ or 2, and proceed as before. Repeat this process $\max T$ times, in other words, until the $\bullet$'s have been switched past all the entries of $T$. The final placement of the numerical entries gives $K\texttt{jdt}_{\{x_i\}}(T)$.

**Example 1.1.** Let $T = \begin{array}{|c|c|c|c|c|} \hline \ & \ & 1 & 2 & 3 \\ \hline \ & 2 & 3 \\ \cline{1-3} 2 \\ \cline{1-1} \end{array}$ be as above and $\{x_i\}$ as indicated below:

$$\begin{array}{ccccc} \ & \bullet & 1 & 2 & 3 \\ \bullet & 2 & 3 & & \\ 2 & & & & \end{array} \mapsto \begin{array}{ccccc} \ & 1 & \bullet & 2 & 3 \\ \bullet & 2 & 3 & & \\ 2 & & & & \end{array} \mapsto \begin{array}{ccccc} \ & 1 & 2 & \bullet & 3 \\ 2 & \bullet & 3 & & \\ \bullet & & & & \end{array} \mapsto \begin{array}{ccccc} \ & 1 & 2 & 3 & \bullet \\ 2 & 3 & \bullet & & \\ \bullet & & & & \end{array}$$

and therefore

$$K\texttt{jdt}_{\{x_i\}} = \begin{array}{cccc} \ & 1 & 2 & 3 \\ 2 & 3 & & \end{array}.$$

It is easy to see that $K\texttt{jdt}_{\{x_i\}}(T)$ is an increasing tableau also. Moreover, if $T$ is a standard Young tableau, and only one corner $x$ is selected, the result is an *ordinary jeu de taquin slide* $\texttt{jdt}_x(T)$. Given $T \in \texttt{INC}(\nu/\lambda)$ we can iterate applying $K\texttt{jdt}$-slides until no such moves are possible. The result $K\texttt{rect}(T)$, which we call *a K-rectification* of $T$, is an increasing tableau of straight shape, that is, one whose shape is given by some partition $\lambda$. We will refer to the choice of intermediate $K\texttt{jdt}$ slides as a *rectification order*.

**Theorem 1.2.** *Let $T \in \texttt{INC}(\nu/\lambda)$. If $K\texttt{rect}(T)$ is a superstandard tableau $S_\mu$ for some rectification order, then $K\texttt{rect}(T) = S_\mu$ for any rectification order.*

It will also be convenient to define *reverse slides*

$$K\texttt{revjdt}_{\{x_i\}}(T)$$

of $T \in \texttt{INC}(\nu/\lambda)$, where now each $x_i$ is an *outer corner*, that is, a maximally north-west box $x \in \Lambda \setminus \nu$. We can similarly define *reverse rectification* $K\texttt{revrect}(T)$.

Clearly, Theorem 1.2 also implies the "reverse version". When we refer to *slides*, we mean either $K\texttt{jdt}$ or $K\texttt{revjdt}$ operations.

Theorem 1.2 may be compared to what is often called the "confluence theorem" or the "First Fundamental Theorem" in the original setting of [Schützenberger 1977]. There, the superstandard assumption is unnecessary and so rectification is always well-defined. However this is not true in our more general context.

**Example 1.3.** Consider the following two $K$-rectifications of the same skew tableau $T$:

$$T = \begin{ytableau} {} & {} & \bullet & 2 \\ {} & {} & 2 \\ 1 & 3 & 4 \end{ytableau} \mapsto \begin{ytableau} {} & {} & 2 \\ {} & \bullet & 4 \\ 1 & 3 \end{ytableau} \mapsto \begin{ytableau} {} & {} & 2 \\ \bullet & 3 & 4 \\ 1 \end{ytableau} \mapsto \begin{ytableau} {} & \bullet & 2 \\ 1 & 3 & 4 \end{ytableau} \mapsto \begin{ytableau} \bullet & 2 & 4 \\ 1 & 3 \end{ytableau} \mapsto \begin{ytableau} 1 & 2 & 4 \\ 3 \end{ytableau}$$

and

$$T = \begin{ytableau} {} & {} & {} & 2 \\ {} & \bullet & 2 \\ 1 & 3 & 4 \end{ytableau} \mapsto \begin{ytableau} {} & {} & \bullet & 2 \\ {} & 2 & 4 \\ 1 & 3 \end{ytableau} \mapsto \begin{ytableau} {} & {} & 2 \\ \bullet & 2 & 4 \\ 1 & 3 \end{ytableau} \mapsto \begin{ytableau} {} & \bullet & 2 \\ 1 & 2 & 4 \\ 3 \end{ytableau} \mapsto \begin{ytableau} \bullet & 2 & 4 \\ 1 & 4 \\ 3 \end{ytableau} \mapsto \begin{ytableau} 1 & 2 & 4 \\ 3 & 4 \end{ytableau}.$$

The two results (rightmost tableaux) are different. However, neither rectification is superstandard.

We need Theorem 1.2 to state our new combinatorial rule for $C_{\lambda,\mu}^{\nu}$:

**Theorem 1.4.** $(-1)^{|\nu|-|\lambda|-|\mu|}C_{\lambda,\mu}^{\nu}$ *counts the number of* $T \in \texttt{INC}(\nu/\lambda)$ *where*

$$K\texttt{rect}(T) = S_{\mu}.$$

**Example 1.5.** The computation $C_{(2,2),(2,1)}^{(3,2,2,1)} = -2$ is witnessed by the increasing tableaux

$$\begin{ytableau} {} & {} & 2 \\ {} & {} \\ 1 & 3 \\ 3 \end{ytableau} \quad \text{and} \quad \begin{ytableau} {} & {} & 2 \\ {} & {} \\ 1 & 2 \\ 3 \end{ytableau},$$

which both rectify to $\begin{ytableau} 1 & 2 \\ 3 \end{ytableau}$.

One can replace the superstandard assumption by some other classes $\{C_{\mu}\}$ of tableau (most obviously the one where we consecutively number columns rather than rows), but we focus on the superstandard choice in this paper.

We will give a self-contained proof of Theorem 1.4, once granted Lenart's Pieri rule [2000].

A short review of past work on $K$-theoretic Littlewood–Richardson rules is in order: The first rule for $C_{\lambda,\mu}^{\nu}$ was given by Buch [2002b], who gave a generalization of the *reverse lattice word* formulation of the classical Littlewood–Richardson rule.

That formula utilized the new idea of *set-valued tableaux* (see the Appendix). Afterwards, another formula was given by Lascoux [2001] in terms of counting paths in a certain tree (generalizing the *Lascoux–Schützenberger tree*; see, for example, [Manivel 1998]). In [Knutson and Yong 2004], Lascoux's rule was reformulated in terms of *diagram marching moves*, and it was also extended to compute a wider class of $K$-theoretic Schubert structure constants. More recently, in [Buch et al. 2008], a rule was given for another class of combinatorial numbers generalizing $C_{\lambda,\mu}^{\nu}$. This rule specializes to a new formula for $C_{\lambda,\mu}^{\nu}$ and in fact gives an independent proof of Buch's rule.

***Organization of this paper.*** In Section 2, we introduce an analogue of Fomin's *growth diagrams*, which compute $K$-rectifications; their symmetries make it possible to give a simple proof of the *infusion involution* of Section 3. In Section 4, we again exploit growth diagrams to give an analogue of Schützenberger's *evacuation involution*. In Section 5, we use the infusion involution to show that if Theorem 1.2 holds, then Theorem 1.4 indeed computes Schubert calculus. Theorem 1.2 itself is actually proved in Section 6, where we also need a connection to *longest strictly increasing subsequences* of reading words of tableaux. In Section 7, we describe a conjectural minuscule Schubert calculus rule, that generalizes our results for the Grassmannian, together with an example. In Section 8, we give counterexamples to natural analogues of various results that are true for classical Young tableau theory. Finally, in Section 9 we give some concluding remarks and further conjectures. In order to be self-contained, we give background about Grothendieck polynomials in the Appendix so that our results can be given a completely elementary and concrete origin.

## 2. Growth diagrams

A construction that is important to this paper is a generalization of Fomin's growth diagram ideas to the $K$-theory context.

Let $\mathbb{Y}$ be the *Young lattice* and $\subseteq$ the partial order on all shapes where $\lambda \subseteq \mu$ when $\lambda$ is contained inside $\mu$. The covering relations on $\mathbb{Y}$ are $\lambda \subseteq \mu$ such that $\mu/\lambda$ is a single box.

Each increasing tableau $T$ can be viewed as a *shape sequence* of increasing shapes in $\mathbb{Y}$ where each successive shape is grown from the previous one by adding some number of boxes, no two in the same row or column.

**Example 2.1.**

$$T = \begin{array}{|c|c|c|c|}\hline \ & \ & \ & 2 \\ \hline \ & \ & 1 & 3 \\ \hline \ & 1 & 2 \\ \cline{1-3} 1 & 2 & 4 \\ \cline{1-3}\end{array} \leftrightarrow \begin{array}{|c|c|c|}\hline \ & \ & \ \\ \hline \ & \ \\ \cline{1-2} \ \\ \cline{1-1}\end{array} - \begin{array}{|c|c|c|}\hline \ & \ & \ \\ \hline \ & \ & \ \\ \hline \ & \ \\ \cline{1-2} \ \\ \cline{1-1}\end{array} - \begin{array}{|c|c|c|c|}\hline \ & \ & \ & \ \\ \hline \ & \ & \ \\ \cline{1-3} \ & \ & \ \\ \cline{1-3} \ & \ \\ \cline{1-2}\end{array} - \begin{array}{|c|c|c|c|}\hline \ & \ & \ & \ \\ \hline \ & \ & \ & \ \\ \hline \ & \ & \ \\ \cline{1-3} \ & \ \\ \cline{1-2}\end{array} - \begin{array}{|c|c|c|c|}\hline \ & \ & \ & \ \\ \hline \ & \ & \ & \ \\ \hline \ & \ & \ \\ \cline{1-3} \ & \ & \ \\ \cline{1-3}\end{array}.$$

| | | | | |
|---|---|---|---|---|
| $(3,2,1)$ | $(3,3,2,1)$ | $(4,3,3,2)$ | $(4,4,3,2)$ | $(4,4,3,3)$ |
| $(2,2)$ | $(3,2,1)$ | $(4,3,2,1)$ | $(4,4,2,1)$ | $(4,4,3,2)$ |
| $(2,1)$ | $(3,1,1)$ | $(4,2,1,1)$ | $(4,3,1,1)$ | $(4,3,2,1)$ |
| $(1)$ | $(2,1)$ | $(3,2,1)$ | $(3,3,1)$ | $(3,3,2)$ |
| $\varnothing$ | $(1)$ | $(2,1)$ | $(3,2)$ | $(3,2,1)$ |

**Table 1.** A $K$-theory growth diagram: the leftmost column describes the rectification order of the skew tableau represented by the top row. The bottom row gives the resulting $K$-rectification.

Now, consider the following choice of rectification order:

$$T = \begin{array}{|c|c|c|c|}\hline \phantom{1} & \phantom{1} & \bullet & 2 \\ \hline & & 1 & 3 \\ \hline \bullet & 1 & 2 \\ \cline{1-3} 1 & 2 & 4 \\ \cline{1-3} \end{array} \rightarrow \begin{array}{|c|c|c|c|}\hline \phantom{1} & \phantom{1} & 1 & 2 \\ \hline & \bullet & 2 & 3 \\ \hline 1 & 2 & 4 \\ \cline{1-3} 2 & 4 \\ \cline{1-2} \end{array} \rightarrow \begin{array}{|c|c|c|c|}\hline \phantom{1} & \bullet & 1 & 2 \\ \hline \bullet & 2 & 3 \\ \cline{1-3} 1 & 4 \\ \cline{1-2} 2 \\ \cline{1-1} \end{array} \rightarrow \begin{array}{|c|c|c|}\hline \bullet & 1 & 2 \\ \hline 1 & 2 & 3 \\ \hline 2 & 4 \\ \cline{1-2} \end{array} \rightarrow \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 2 & 3 \\ \cline{1-2} 4 \\ \cline{1-1} \end{array},$$

where the $\bullet$'s indicate the set of boxes to use in each $K\mathtt{jdt}$ step. Each of these increasing tableaux also has a shape sequence, which we put one atop of another so the shapes increase moving up and to the right. The result is a *$K$-theory growth diagram*; in our example, we have Table 1.

Consider the following local conditions on any $2 \times 2$ subsquare

$$\begin{array}{|c|c|}\hline \alpha & \beta \\ \hline \gamma & \delta \\ \hline \end{array}$$

of such a grid of shapes, where by assumption $\gamma \subseteq \alpha \subseteq \beta$ and $\gamma \subseteq \delta \subseteq \beta$, as in the example above:

(G1) $\alpha/\gamma$ is a collection of boxes no two in the same row or column, and similarly for $\beta/\alpha$, $\beta/\delta$, and $\delta/\gamma$.

(G2) $\delta$ is the shape $\alpha \cup \mathtt{shape}(K\mathtt{jdt}_{\alpha/\gamma}(T))$, where $T$ is the skew tableau of shape $\beta/\alpha$ filled with 1's. This uniquely determines $\delta$ from $\gamma$, $\alpha$ and $\beta$. Similarly, $\alpha$ is uniquely determined by $\gamma$, $\delta$ and $\beta$.

**Proposition 2.2.** *If*

$$\begin{array}{|c|c|}\hline \alpha & \beta \\ \hline \gamma & \delta \\ \hline \end{array}$$

*is a* $2 \times 2$ *square in a $K$-theory growth diagram, then* (G1) *and* (G2) *hold. Also, if* $\mathscr{G}$ *is a growth diagram, then so is* $\mathscr{G}$ *reflected about its antidiagonal.*

*Proof.* These are straightforward verifications. The second statement uses the fact that (G1) and (G2) are symmetric in $\alpha$ and $\delta$. $\square$

Let $K\mathtt{GROWTH}(\lambda, \mu; \nu)$ be the set of $K$-theory growth diagrams such that

- the leftmost column encodes the superstandard tableau of shape $\lambda$,
- the bottom-most row encodes the superstandard tableau of shape $\mu$, and
- the top right corner is the shape $\nu$.

The following fact is immediate from Theorem 1.4, and amounts to an alternative formulation for it:

**Corollary 2.3** (of Theorem 1.4). $(-1)^{|\nu|-|\lambda|-|\mu|}C^{\nu}_{\lambda,\mu} = \#K\mathtt{GROWTH}(\lambda, \mu; \nu)$.

By the symmetry of growth diagrams, the roles of the $\lambda$ and $\mu$ can be interchanged, resulting in the same growth diagram (up to reflection). Therefore, the rule of Corollary 2.3 manifests the $\mathbb{Z}_2$ *commutation symmetry*

$$C^{\nu}_{\lambda,\mu} = C^{\nu}_{\mu,\lambda}$$

coming from $[\mathbb{O}_{X_\lambda}][\mathbb{O}_{X_\mu}] = [\mathbb{O}_{X_\mu}][\mathbb{O}_{X_\lambda}]$.

Growth diagrams corresponding to the classical rectifications of a standard tableau (using only `jdt` moves) were first introduced by Fomin; see [Stanley 1999, Appendix 1] and the references therein. In that case, Proposition 2.2 simplifies. Specifically,

(F1) shapes increase by precisely one box in the "up" and "right" directions.

(F2) if $\alpha$ is the unique shape containing $\gamma$ and contained in $\beta$, then $\delta=\alpha$; otherwise there is a unique such shape different than $\alpha$, and this shape is $\delta$.

(Similarly, $\alpha$ is uniquely determined by $\beta$, $\gamma$ and $\delta$.)

Fomin's growth diagrams provide further useful combinatorial ideas that we extend below to the $K$-theory setting. These diagrams also arise (along with other classical tableaux algorithms we generalize) in an elegant geometric context, due to work of van Leeuwen [2000]; there are reasons to hope that one can extend his work to the setting of this paper.

## 3. The infusion involution

Given $T \in \mathtt{INC}(\lambda/\alpha)$ and $U \in \mathtt{INC}(\nu/\lambda)$, define

$$K\mathtt{infusion}(T, U) = \big(K\mathtt{infusion}_1(T, U), K\mathtt{infusion}_2(T, U)\big) \in \mathtt{INC}(\gamma/\alpha) \times \mathtt{INC}(\nu/\gamma)$$

(for some straight shape $\gamma$) as follows: consider the largest label "$m$" that appears in $T$, appearing at $x_1, \ldots, x_k$. Apply the slide $K\mathtt{jdt}_{\{x_i\}}(U)$, leaving some "holes" at the other side of $\nu/\lambda$. Place "$m$" in these holes and repeat, moving the labels originally from $U$ until all labels of $T$ are exhausted. The resulting tableau of shape

$\gamma/\alpha$ and skew tableau of shape $\nu/\gamma$ are the outputted tableaux. To define

$$K\texttt{revinfusion}(T,U) \\ = \big(K\texttt{revinfusion}_1(T,U), K\texttt{revinfusion}_2(T,U)\big) \in \texttt{INC}(\gamma/\alpha)\times\texttt{INC}(\nu/\gamma),$$

we apply $K\texttt{revjdt}$ moves to $T$, moving into boxes of $U$. We begin by removing the labels "1" appearing in $U$ at boxes $\{x_i\}\in\nu/\lambda$, apply $\texttt{revjdt}_{\{x_i\}}(T)$, and place the "1" in the vacated holes of $\lambda$ and continuing with higher labels of $U$.

It is easy to show that

$$K\texttt{infusion} \quad \text{and} \quad K\texttt{revinfusion}$$

are inverses of one another, by inductively applying the observation that if $\{y_i\}$ are the boxes vacated by $K\texttt{jdt}_{\{x_i\}}(T)$ then

$$K\texttt{revjdt}_{\{y_i\}}(K\texttt{jdt}_{\{x_i\}}(T)) = T.$$

We will need the following fact (the "infusion involution"); compare [Haiman 1992; Benkart et al. 1996].

**Theorem 3.1.** *For any increasing tableaux $T$ and $U$ such that* $\texttt{shape}(U)$ *extends* (*the possibly skew shape*) $\texttt{shape}(T)$ *then*

$$K\texttt{infusion}(T,U) = K\texttt{revinfusion}(T,U).$$

*That is*, $K\texttt{infusion}(K\texttt{infusion}(T,U)) = (T,U)$.

**Example 3.2.** If

$$T = \begin{array}{ccc} \underline{1} & \underline{2} & \underline{3} \\ \underline{2} & \underline{3} & \\ \underline{4} & & \end{array} \quad \text{and} \quad U = \begin{array}{cccc} & & & 2 \\ & & 1 & 3 \\ & 1 & 3 & \\ 2 & 3 & 4 & \end{array}$$

then we compute $K\texttt{infusion}$ as follows:

$$\begin{array}{cccc} \underline{1} & \underline{2} & \underline{3} & 2 \\ \underline{2} & \underline{3} & 1 & 3 \\ \underline{4} & 1 & 3 & \\ 2 & 3 & 4 & \end{array} \mapsto \begin{array}{cccc} \underline{1} & \underline{2} & \underline{3} & 2 \\ \underline{2} & \underline{3} & 1 & 3 \\ 1 & \underline{4} & 3 & \\ 2 & 3 & 4 & \end{array} \mapsto \begin{array}{cccc} \underline{1} & \underline{2} & \underline{3} & 2 \\ \underline{2} & \underline{3} & 1 & 3 \\ 1 & 3 & \underline{4} & \\ 2 & \underline{4} & 4 & \end{array} \mapsto \begin{array}{cccc} \underline{1} & \underline{2} & \underline{3} & 2 \\ \underline{2} & \underline{3} & 1 & 3 \\ 1 & 3 & 4 & \\ 2 & 4 & \underline{4} & \end{array} \mapsto \begin{array}{cccc} \underline{1} & \underline{2} & 1 & 2 \\ \underline{2} & 1 & \underline{3} & 3 \\ 1 & 3 & 4 & \\ 2 & 4 & \underline{4} & \end{array}$$

$$\mapsto \begin{array}{cccc} \underline{1} & \underline{2} & 1 & 2 \\ \underline{2} & 1 & 3 & \underline{3} \\ 1 & 3 & 4 & \\ 2 & 4 & \underline{4} & \end{array} \mapsto \begin{array}{cccc} \underline{1} & 1 & \underline{2} & 2 \\ 1 & \underline{2} & 3 & \underline{3} \\ \underline{2} & 3 & 4 & \\ 2 & 4 & \underline{4} & \end{array} \mapsto \begin{array}{cccc} \underline{1} & 1 & 2 & \underline{2} \\ 1 & \underline{2} & 3 & \underline{3} \\ 2 & 3 & 4 & \\ \underline{2} & 4 & \underline{4} & \end{array} \mapsto \begin{array}{cccc} \underline{1} & 1 & 2 & \underline{2} \\ 1 & 3 & \underline{2} & \underline{3} \\ 2 & \underline{2} & 4 & \\ \underline{2} & 4 & \underline{4} & \end{array}$$

$$\mapsto \begin{array}{cccc} \underline{1} & 1 & 2 & \underline{2} \\ 1 & 3 & 4 & \underline{3} \\ 2 & 4 & \underline{2} & \\ 4 & \underline{2} & \underline{4} & \end{array} \mapsto \begin{array}{cccc} 1 & \underline{1} & 2 & \underline{2} \\ \underline{1} & 3 & 4 & \underline{3} \\ 2 & 4 & \underline{2} & \\ 4 & \underline{2} & \underline{4} & \end{array} \mapsto \begin{array}{cccc} 1 & 2 & \underline{1} & \underline{2} \\ 2 & 3 & 4 & \underline{3} \\ \underline{1} & 4 & \underline{2} & \\ 4 & \underline{2} & \underline{4} & \end{array} \mapsto \begin{array}{cccc} 1 & 2 & 4 & \underline{2} \\ 2 & 3 & \underline{1} & \underline{3} \\ 4 & \underline{1} & \underline{2} & \\ \underline{1} & \underline{2} & \underline{4} & \end{array}$$

Hence

$$K\texttt{infusion}(T, U) = \left( \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 2 & 3 & \multicolumn{1}{|c}{} \\ \cline{1-2} 4 & \multicolumn{1}{|c}{} & \\ \cline{1-1} \end{array} \;,\; \begin{array}{cccc} & & & \boxed{\underline{2}} \\ & & \boxed{\underline{1}} & \boxed{\underline{3}} \\ & \boxed{\underline{1}} & \boxed{\underline{2}} & \\ \boxed{\underline{1}} & \boxed{\underline{2}} & \boxed{\underline{4}} & \end{array} \right).$$

The reader can check that applying $K\texttt{infusion}$ to this pair returns $(T, U)$, in agreement with Theorem 3.1.

*Proof.* Construct the growth diagram for $K\texttt{rect}(U)$ using the slides suggested by the entries of $T$. It is straightforward to check from the definitions that the bottom row represents $K\texttt{infusion}_1(T, U)$ and the right column $K\texttt{infusion}_2(T, U)$. However, by the antidiagonal symmetry of growth diagrams (see Proposition 2.2), the growth diagram computing $K\texttt{infusion}$ applied to $K\texttt{infusion}(T, U)$ is simply the one for $K\texttt{infusion}(T, U)$ reflected about the antidiagonal. $\square$

Finally, the growth diagram formalism makes it straightforward to observe facts such as the following, which we will need in Section 6:

**Lemma 3.3.** *Let* $T \in \texttt{INC}(\nu/\lambda)$, $R \in \texttt{INC}(\lambda)$ *and fix* $a \in \mathbb{N}$. *If* $A$ *is the increasing tableau consisting of entries from* $1$ *to* $a$ *of* $T$, *and* $B = T \setminus A$ *is the remaining tableau, then*

$$K\texttt{infusion}_1(R, T)$$
$$= K\texttt{infusion}_1(R, A) \cup K\texttt{infusion}_1(K\texttt{infusion}_2(R, A), B).$$

*Proof.* Draw the growth diagram for $K\texttt{infusion}(R, T)$, encoding $R$ on the left and $T$ on the top. The shape $\texttt{shape}(R) \cup \texttt{shape}(A)$ appears on the top row. Draw a vertical line through the growth diagram at that point. The diagram to the left of this line encodes the rectification of $A$ by $R$. The diagram to the right of the line encodes the infusion of $B = T \setminus A$ with the tableau encoded along the dividing line, which is $K\texttt{infusion}_2(R, A)$. $\square$

## 4. A generalization of Schützenberger's evacuation involution

While on the topic of growth diagrams, we take this opportunity to introduce a generalization of another classical result from tableau theory. This section will not be needed in the remainder of the paper.

For $T \in \texttt{INC}(\lambda)$, let ${}^{\circ}T$ be obtained by erasing the (unique) entry 1 in the northwest corner $c$ of $T$ and subtracting 1 from the remaining entries. Let

$$\Delta(T) = K\texttt{jdt}_{\{c\}}({}^{\circ}T).$$

The *K-evacuation* $K\texttt{evac}(T) \in \texttt{INC}(\lambda)$ is defined by the shape sequence

$$\varnothing = \texttt{shape}(\Delta^{\max T}(T)) - \texttt{shape}(\Delta^{\max T - 1}(T)) - \ldots - \texttt{shape}(\Delta^{1}(T)) - T.$$

| | | | | | |
|---|---|---|---|---|---|
| $\varnothing$ | $(1)$ | $(2,1)$ | $(3,2)$ | $(3,3,1)$ | $(4,3,2)$ |
| | $\varnothing$ | $(1)$ | $(2,1)$ | $(3,2,1)$ | $(4,2,2)$ |
| | | $\varnothing$ | $(1)$ | $(2,1)$ | $(3,2,1)$ |
| | | | $\varnothing$ | $(1)$ | $(2,1)$ |
| | | | | $\varnothing$ | $(1)$ |
| | | | | | $\varnothing$ |

**Table 2.** A triangular growth diagram for Example 4.2.

The following result extends Schützenberger's classical theorem for $T \in \mathtt{SYT}(\lambda)$.

**Theorem 4.1.** $K\mathtt{evac} : \mathtt{INC}(\lambda) \to \mathtt{INC}(\lambda)$ *is an involution*, *that is*,

$$K\mathtt{evac}(K\mathtt{evac}(T)) = T.$$

**Example 4.2.** Let

$$T = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 5 \\ \hline 2 & 3 & 4 \\ \cline{1-3} 4 & 5 \\ \cline{1-2} \end{array} \in \mathtt{INC}((4,3,2)).$$

Then the $K$-evacuation is computed by

$$\Delta^1(T) = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline 2 & 3 \\ \cline{1-2} 3 & 4 \\ \cline{1-2} \end{array} \mapsto \Delta^2(T) = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 2 & 3 \\ \cline{1-2} 3 \\ \cline{1-1} \end{array} \mapsto \Delta^3(T) = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1} \end{array} \mapsto \Delta^4(T) = \begin{array}{|c|}\hline 1 \\ \hline \end{array} \mapsto \Delta^5(T) \mapsto \varnothing.$$

Thus

$$K\mathtt{evac}(T) = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline 2 & 3 & 5 \\ \cline{1-3} 3 & 4 \\ \cline{1-2} \end{array}.$$

One checks that applying $K\mathtt{evac}$ to this tableau returns $T$.

*Proof of Theorem 4.1.* Express each of the increasing tableaux

$$T,\ \Delta^1(T), \ldots, \Delta^{\max T - 1}(T),\ \Delta^{\max T}(T) = \varnothing$$

as a shape sequence and place them right justified in a triangular growth diagram. In the example above, we have Table 2. Note that each "minor" of the table whose southwest corner contains a "$\varnothing$" is in fact a growth diagram. It follows that the triangular growth diagram can be reconstructed using (G1) and (G2), by Proposition 2.2. Observe that the right column encodes $K\mathtt{evac}(T)$. By the symmetry of growth diagrams, it follows that applying the above procedure to $K\mathtt{evac}(T)$ would give the same triangular growth diagram, after a reflection across the antidiagonal. Thus the result follows. $\square$

## 5. Proof of the $K$jdt rule

The strategy of our proof is based on the following fact. In the cohomological context, this approach was utilized in [Knutson et al. 2004; Buch et al. 2004].

**Lemma 5.1.** *Let $\{d_{\lambda,\mu}^{\nu}\}$ be integers indexed by shapes $\lambda, \mu, \nu \subseteq \Lambda$ that*

(A) *define a commutative and associative ring $(R, \circ)$ by*

$$a_\lambda \circ a_\mu = \sum_{\nu \subseteq \Lambda} d_{\lambda,\mu}^{\nu} a_\nu$$

*with $\mathbb{Z}$-basis $\{a_\lambda\}$ indexed by shapes $\lambda \subseteq \Lambda$, and such that*

(B) *$d_{\lambda,\rho}^{\nu} = c_{\lambda,\rho}^{\nu}$ whenever $\rho = (t)$ for $0 \leq t \leq n-k$.*

*Then $d_{\lambda,\mu}^{\nu} = c_{\lambda,\mu}^{\nu}$.*

*Proof.* The class $[\mathbb{O}_{X_\lambda}]$ can be expressed as a polynomial in $[\mathbb{O}_{X_{(1)}}], \ldots, [\mathbb{O}_{X_{(n-k)}}]$. This follows by an easy downward induction on $|\lambda|$ using the fact that such an expression exists in cohomology for $[X_\lambda] \in H^\star(X, \mathbb{Q})$ as a polynomial in the classes $[X_{(t)}]$ (the *Jacobi–Trudi identity*) and the lowest order term in $K$-theory agrees with cohomology under the Chern isomorphism. Let this polynomial be $P_\lambda(X_1, \ldots, X_{n-k})$ (where above $X_t = [\mathbb{O}_{X_{(t)}}]$). Now (A) and (B) imply

$$a_\lambda = P_\lambda(a_{(1)}, \ldots, a_{(t)}).$$

Using (B) again, we see that the map from $(R, \circ)$ to $K(X)$ sending $a_\lambda \mapsto [\mathbb{O}_{X_\lambda}]$ is a ring isomorphism, so the desired conclusion follows. □

To apply the lemma, let $d_{\lambda,\mu}^{\nu}$ be the integers computed by the rule given in the statement of the theorem. It remains to check associativity and agreement with Pieri's rule, which we do below. In our proof of associativity we *assume* that Theorem 1.2 is true — this latter result is actually proved in the following section, using some of the elements introduced in the proof of agreement with Pieri's rule, which of course, do not use this assumption. We will also use the commutation symmetry, proved in Section 2 (see after Corollary 2.3), that is, $d_{\lambda,\mu}^{\nu} = d_{\mu,\lambda}^{\nu}$.

***Associativity.*** Let $\alpha, \beta, \gamma, \nu$ be straight shapes and fix superstandard tableaux $S_\alpha$, $S_\beta$, $S_\gamma$ and $S_\nu$.

Associativity is the assertion that

$$\sum_{\sigma} d_{\alpha,\beta}^{\sigma} d_{\sigma,\gamma}^{\nu} = \sum_{\tau} d_{\alpha,\tau}^{\nu} d_{\beta,\gamma}^{\tau}. \tag{5-1}$$

The left-hand side of (5-1) counts pairs of tableaux $(B, C)$ where $B$ is of shape $\sigma/\alpha$ such that $K\mathtt{rect}(B) = S_\beta$, and $C$ is of shape $\nu/\sigma$ such that $K\mathtt{rect}(C) = S_\gamma$.

Let $K\mathtt{infusion}(S_\alpha, B) = (S_\beta, A)$ where $A$ is of shape $\sigma/\beta$, and $K\mathtt{rect}(A) = S_\alpha$. Next compute $K\mathtt{infusion}(A, C) = (D, E)$. We have that $K\mathtt{rect}(E) = S_\alpha$

(since this was the case with $A$) and that $\mathtt{shape}(E) = \nu/\tau$ for some $\tau$, and similarly $K\mathtt{rect}(D) = S_\gamma$ (since this was the case for $C$) and $\mathtt{shape}(D) = \tau/\beta$.

By Theorem 3.1 it follows that the above process establishes a bijection

$$(B, C) \mapsto (E, D)$$

into the set of pairs of tableaux counted by the right-hand side of (5-1). (More precisely, for pairs counted by

$$\sum_\tau d^\nu_{\tau,\alpha}\, d^\tau_{\beta,\gamma} = \sum_\tau d^\nu_{\alpha,\tau}\, d^\tau_{\beta,\gamma}$$

where the equality $d^\nu_{\tau,\alpha} = d^\nu_{\alpha,\tau}$ is the commutation symmetry.) Associativity follows.

***Agreement with Pieri's rule.*** We prove our rule agrees with the following formula, due to Lenart [2000]:

**Theorem 5.2.** *Let $r(\nu/\lambda)$ be the number of rows of $\nu/\lambda$. Then*

$$[\mathbb{O}_{X_\lambda}][\mathbb{O}_{X_{(t)}}] = \sum_\nu (-1)^{|\nu|-|\lambda|-t} \binom{r(\nu/\lambda)-1}{|\nu/\lambda|-t} [\mathbb{O}_{X_\nu}],$$

*where the sum ranges over all $\nu \subseteq \Lambda$ obtained by adding a horizontal strip (no two added boxes are in the same column) to $\lambda$ of size at least $t$.*

Our task is to show that

$$d^\nu_{\lambda,(t)} = \binom{r(\nu/\lambda)-1}{|\nu/\lambda|-t}$$

when $\nu$ is of the form in the statement of Theorem 5.2 and is zero otherwise.

First assume $\nu$ is of the desired form and that $|\nu/\lambda| - t \le r(\nu/\lambda) - 1$. We proceed to construct the required number of increasing tableaux on $\nu/\lambda$, as follows. Select $|\nu/\lambda| - t$ of the non-bottom-most $r(\nu/\lambda) - 1$ rows of $\nu/\lambda$. Now fill the bottom row with consecutive entries $1, 2, \ldots, k$ where $k$ is the number of boxes in that bottom row of $\nu/\lambda$. Proceed to fill the remaining boxes of $\nu/\lambda$ from southwest to northeast. If the current row to be filled was one of the $|\nu/\lambda| - t$ selected rows then begin with the last entry $e$ used in the previously filled row. Otherwise use $e + 1$.

Call these fillings *$t$-Pieri fillings.*

**Example 5.3.** Suppose $\lambda = (5, 3, 2)$, $\nu = (6, 5, 2, 2)$ and $t = 4$. Then $r(\nu/\lambda) = 3$ and $|\nu/\lambda| - t = 1$. Hence the two 4-Pieri fillings we construct are

$$\begin{ytableau} \ & \ & \ & \ & \ & 4 \\ \ & \ & \ & 2 & 3 \\ \ & \ \\ 1 & 2 \end{ytableau} \quad \text{and} \quad \begin{ytableau} \ & \ & \ & \ & \ & 4 \\ \ & \ & \ & 3 & 4 \\ \ & \ \\ 1 & 2 \end{ytableau},$$

which both rectify to $\boxed{1}\boxed{2}\boxed{3}\boxed{4}$ . (In the first tableau we selected the second row and in the second we selected the top row.)

**Lemma 5.4.** *For any rectification order, a $t$-Pieri filling $K$-rectifies to $S_{(t)}$. No other increasing tableau $K$-rectifies to $S_{(t)}$ for any choice of rectification order.*

*Proof.* That the $t$-Pieri fillings all $K$-rectify (under any rectification order) to $S_{(t)}$ follows from a straightforward induction on $|\lambda| \geq 0$ where we show in fact that any $K\mathtt{jdt}$ slide applied to a $t$-Pieri filling results in a $t$-Pieri filling.

A similar induction shows that no other increasing tableau from $\mathtt{INC}(\nu/\lambda)$ $K$-rectifies to $S_{(t)}$ (noting that any such tableau with entries in $\{1,\ldots,t\}$ has a pair of entries $i < j$ where $j$ is southwest of $i$). Separately, but for similar reasons, when $\nu/\lambda$ is not a horizontal strip, one more induction on $|\lambda|$ proves no increasing tableau can $K$-rectify to $S_{(t)}$.

Finally, if $|\nu/\lambda| - t > r(\nu/\lambda) - 1$, then we similarly see that no $t$-Pieri fillings are possible and $d^{\nu}_{\lambda,\mu} = 0$ as desired. □

This completes the proof of Theorem 1.4, assuming Theorem 1.2.

## 6. Proof of the $K\mathtt{rect}$ theorem

We now prove Theorem 1.2. First define the *reading word* of a tableau $T$ to be the word obtained by reading the rows of $T$ from left to right, starting from the bottom and moving up. Let $\mathtt{LIS}(T)$ be the length of the longest strictly increasing subsequence of the reading word of $T$.

The following result is crucial to our proof of Theorem 1.2.

**Theorem 6.1.** $\mathtt{LIS}(K\mathtt{jdt}_{\{x_i\}}(T)) = \mathtt{LIS}(T)$. *In particular, any rectification order applied to $T$ results in a straight shape whose first row has length equal to* $\mathtt{LIS}(T)$.

**Example 6.2.** Consider the two (different) rectifications of the same tableau $T$ performed in Example 1.3. The reading word of $T$ is $\underline{1}\ \underline{3}\ \underline{4}\ 2\ 2$ (where the unique longest strictly increasing subsequence has been underlined) so $\mathtt{LIS}(T) = 3$. Note that also $\mathtt{LIS}(T_1) = \mathtt{LIS}(T_2) = 3$, that is, the lengths of the first rows of $T_1$ and $T_2$ agree, although $T_1 \neq T_2$.

*Proof of Theorem 6.1.* We will show that if $I$ is a set of boxes of $T$ which forms a strictly increasing subsequence of the reading word of $T$, then there is a string of boxes of equal length in $K\mathtt{jdt}_{\{x_i\}}(T)$ which also forms a strictly increasing subsequence of the reading word. A symmetric argument using reverse slides gives the other desired inequality, thereby implying the theorem.

Fix $I$ as above. We will analyze the slide $K\mathtt{jdt}_{\{x_i\}}(T)$, $\mathtt{switch}$ by $\mathtt{switch}$. Set $T_0 := T$, and let $T_i$ be the result of switching the $\bullet$'s and the $i$'s of $T_{i-1}$. Initially set $I_0 := I$. In a moment, we will describe $I_i$ as a collection of some of the boxes of $T_i$. We emphasize that in what follows $I_i$ does not refer to the actual *contents* of the boxes.

We will show that, at each step, $I_i$ has the following properties:

(P1) The labels of $I_i$ are strictly increasing in the reading word order, except for perhaps one $\bullet$ box.

(P2) If $I_i$ contains a $\bullet$ box, then the labels in $I_i$ preceding the $\bullet$ box in the reading word order are weakly less than $i$, while the labels of boxes following the $\bullet$ box are strictly greater than $i$.

(P3) If there is a $\bullet$ box $y_i$ in $I_i$, then there must be some box $z_i$ in $I_i$, in the same row as $y_i$ and weakly to the right, such that the entry in the box $a_i$ immediately below $z_i$ contains a numerical label. Moreover, if there is a next box $b_i$ in $I_i$ after $z_i$, in the reading order, then it contains a numerical label strictly larger than the one in $a_i$.

**Example 6.3.** (P1) and (P2) are self explanatory. For (P3), a possible configuration that can arise in our discussion below is

$$\begin{array}{|c|c|c|c|c|c|c|}\hline \underline{1} & \underline{\bullet} & \underline{2} & 4 & \underline{5} & 7 & \underline{9} \\ \hline \bullet & 2 & 3 & 6 & 8 & 9 \\ \cline{1-6}\end{array},$$

where the underlined labels indicate members of $I_1$. Here the role of $z_1$ is played by the $\underline{5}$, so $a_1$ is the 8 and $b_i$ is the $\underline{9}$. Note that $b_i$ need not be immediately to the right of the $z_i$. Also, we could have set $z_i$ to be the box containing the $\underline{2}$, but not the $\underline{\bullet}$ nor $\underline{9}$. We emphasize that while it isn't true in the present example, one could have $y_i = z_i$.

**Example 6.4.** Note that in (P3), $b_i$ need not exist. For example, this is the case in

$$\begin{array}{|c|c|}\hline \underline{1} & \underline{\bullet} \\ \hline \bullet & 2 \\ \hline\end{array}$$

which satisfies (P1)–(P3) with $z_i = y_i$.

We now proceed to define $I_i$ inductively for $i \geq 1$. Assume that $I_{i-1}$ satisfies (P1)–(P3). After performing the slide interchanging $\bullet$ boxes with $i$'s we define $I_i$ as follows:

(i) If $I_{i-1}$ has no box containing $i$, then $I_i := I_{i-1}$.

(ii) If $I_{i-1}$ has a box containing $i$ and a $\bullet$ box, then $I_i := I_{i-1}$.

(iii) If $I_{i-1}$ has a box containing $i$, but does not have a $\bullet$ box, and the $i$ in $I_{i-1}$ does not move, then $I_i := I_{i-1}$.

(iv) If $I_{i-1}$ has a box containing $i$, but does not have a $\bullet$ box, and there is a $\bullet$ box (not in $I_{i-1}$) immediately to the left of the $i$ in $I_{i-1}$, then let $I_i$ be $I_{i-1}$ with the box containing $i$ in $I_{i-1}$ replaced by the box to its left (into which $i$ has moved).

(v) If $I_{i-1}$ has a box containing $i$, but does not have a $\bullet$ box, there is a $\bullet$ box (not in $I_{i-1}$) immediately above the $i$, and we are not in case (iv), then let $I_i$ be $I_{i-1}$ with the box containing $i$ in $I_{i-1}$ and all the other boxes in $I_{i-1}$ to the right of it in the same row, replaced by the boxes immediately above them.

Clearly (i)–(v) indeed enumerate all of the intermediate possibilities during a $K$`jdt` slide.

We now prove that $I_i$ satisfies (P1)–(P3).

Case (i): We split this case up into three subcases. First, we consider the case that $I_{i-1}$ has no $\bullet$ box. In this case, (P1) is trivially satisfied (since it held for $I_{i-1}$), and (P2) and (P3) are vacuously true.

Next, we consider the subcase that $I_{i-1}$ has a $\bullet$ box into which an $i$ (not in $I_{i-1}$) moves. Since (P1) and (P2) are satisfied for $I_{i-1}$, (P1) will be satisfied after this, and (P2) and (P3) are vacuous since $I_i$ has no $\bullet$ box.

Finally, we consider the subcase where $I_{i-1}$ has a $\bullet$ box which stays as such in $I_i$. Since the contents of $I_{i-1}$ and $I_i$ are the same, (P1) and (P2) are satisfied. To show (P3) is satisfied, observe that the label in the box below $z_{i-1}$ is strictly greater than $i$ (otherwise $z_{i-1}$ has a label weakly smaller than $i-1$ and is southeast of a $\bullet$, a contradiction), so it does not move, and thus we can take $z_i := z_{i-1}$.

For case (ii), we need the following:

**Lemma 6.5.** *If $I_{i-1}$ satisfies* (P1)–(P3) *and contains a $\bullet$ box and a box labelled $i$ then the $i$ is immediately to the right of the $\bullet$ box.*

*Proof.* By (P2), the next box in $I_{i-1}$ after the $\bullet$ box $y_{i-1}$ must be the box containing $i$. Suppose that that box is not in the same row as $y_{i-1}$. Then $y_{i-1}$ is the last box in $I_{i-1}$ in its row, so we must have $z_{i-1} = y_{i-1}$, and $b_{i-1}$ must be the box from $I_{i-1}$ containing $i$.

Observe that in $T_{i-1}$, there is no label $\ell < i$ which is weakly southeast of a $\bullet$. Thus the entry in $a_{i-1}$ is at least $i$, violating (P3). It follows that the box containing $i$ is in the same row as $y_{i-1}$. Using the same observation again, we see that there are no possible labels for a box between $y_{i-1}$ and the box containing $i$, and therefore, they are adjacent. □

Now, using Lemma 6.5, it is clear that case (ii) preserves (P1) and (P2). To check (P3), as in the previous case, we can take $z_i := z_{i-1}$. This would not work if $z_{i-1} = y_{i-1}$, but this is impossible, because the entry in the box below $z_{i-1}$ should be less than the next entry in $I_{i-1}$ after $z_{i-1}$, which is $i$. So the $\bullet$ box is immediately above a box which is at most $i-1$, and this can't happen in $T_{i-1}$.

Cases (iii) and (iv) are trivial: (P1) holds since the contents of $I_{i-1}$ and $I_i$ are the same, and (P2) and (P3) are vacuously true since $I_i$ contains no $\bullet$ box.

Now we consider case (v). (P1) is trivial, so if $I_i$ has no $\bullet$ box, then we are done. So assume it does. The only way a $\bullet$ box could appear in $I_i$ is in the following situation:

$$\begin{array}{|c|c|}\hline \bullet & i \\ \hline \underline{i} & \underline{k} \\ \hline \end{array} \mapsto \begin{array}{|c|c|}\hline \underline{i} & \underline{\bullet} \\ \hline \bullet & k \\ \hline \end{array},$$

where the box containing $k$ is also in $I_{i-1}$.

In this situation the top two boxes will be in $I_i$, and so we will have introduced a $\bullet$ box into $I_i$. (P2) is clearly satisfied. Set $z_i$ to be the rightmost of the boxes that are in $I_i$ but not in $I_{i-1}$. Now (P3) is satisfied because (P1) was satisfied for $I_{i-1}$.

This completes the proof that $I_i$ satisfies (P1)–(P3). Thus after iteration, we eventually terminate with a set of boxes $I_m$ in $T_m := K\texttt{jdt}_{\{x_i\}}(T)$ which satisfies (P1)–(P3). We wish to show that $I_m$ contains no $\bullet$ box. Suppose that it did. This $\bullet$ box of $I_m$ must be an outer corner of $T$ (by the way $K\texttt{jdt}$ is defined). This contradicts (P3), since the square below $z_i$ is southeast of the $\bullet$ box, and thus contains no label. Thus $I_m$ contains no $\bullet$ box, so (P1) implies that there is a strictly increasing subsequence of the reading word of $K\texttt{jdt}_{\{x_i\}}(T)$ whose length equals the length of $I$, as desired. □

**Remark 6.6.** Theorem 6.1 may be regarded as a generalization of the classical result of Schensted which asserts that the longest increasing subsequence of a permutation $w = w_1 w_2 \dots w_n$ in the symmetric group $S_n$ (written in one-line notation) is equal to the first row of the common shape of the corresponding insertion and recording tableaux under the Robinson–Schensted algorithm; see, for example, [Stanley 1999]. To see this, one needs to use the well-known fact that the insertion tableau of $w$ is equal to the (classical) rectification of the "permutation tableau" $T_w$ of skew shape

$$(n, n-1, n-2, \dots, 3, 2, 1)/(n-1, n-2, \dots, 3, 2, 1),$$

where $w_1$ occupies the southwest-most box, followed by $w_2$ in the box to its immediate northeast, and so on. In [Thomas and Yong 2008b] we further explore this observation, and connect $K\texttt{rect}$ to the Hecke algorithm of [Buch et al. 2008].

Recall the definition of $t$-Pieri filling given in Section 5.

**Lemma 6.7.** *If an increasing tableau $T$ rectifies* (*with respect to any rectification order*) *to a tableau $V$ which has precisely* $1, 2, \dots, t$ *in the first row and no labels weakly smaller than $t$ elsewhere*, *then*

(1) *the labels* $1, 2, \dots, t$ *form a subtableau of $T$ that is a $t$-Pieri filling*, *and*

(2) $\texttt{LIS}(T) = t$.

*Proof.* By Lemma 3.3, $V$ contains the rectification of the subtableau of $T$ consisting of the entries between 1 and $t$; by results of the previous section, it follows that these entries must form a $t$-Pieri filling; this proves that (1) holds.

By Theorem 6.1, $\mathtt{LIS}(T) = \mathtt{LIS}(V) = t$, proving (2). $\square$

*Proof of Theorem 1.2.* Let $R \in \mathtt{INC}(\lambda)$ encode a rectification where

$$K\mathtt{infusion}_1(R, T) = S_\mu.$$

Let us suppose that the first row of $S_\mu$ is $S_{(t)}$. By Theorem 6.1, $\mathtt{LIS}(T) = t$. By Lemma 6.7, the subtableau $P$ of $T$, consisting of the boxes containing one of the labels $1, 2, \ldots, t$, is a $t$-Pieri filling.

Suppose $Q \in \mathtt{INC}(\lambda)$ is another rectification order. Since the labels of $P$ are weakly smaller than $t$ and those of $T \setminus P$ are strictly larger than $t$, by Lemma 3.3, we can compute $V := K\mathtt{infusion}_1(Q, T)$ in two stages. First, by Lemma 5.4, $K\mathtt{infusion}_1(Q, P)$ is simply $S_{(t)}$, because $P$ is a $t$-Pieri filling. Secondly, we use $K\mathtt{infusion}_2(Q, P)$ to (partially) rectify $T \setminus P$. *A priori*, this could contribute extra boxes to first row of $V$ but since, by Theorem 6.1, $\mathtt{LIS}(V) = \mathtt{LIS}(T) = t$, it does not. Thus the rectification of $T$ by $Q$ consists of the row $S_{(t)}$ with a rectification of $T \setminus P$ to a straight shape underneath it.

Now, by assumption $T \setminus P$ has a (partial) rectification to a superstandard tableaux (using labels starting from $t + 1$), namely $S_\mu \setminus S_{(t)}$. So by induction on the number of boxes of the starting shape, we can conclude that $T \setminus P$ will (partially) rectify to $S_\mu \setminus S_{(t)}$ under any rectification order. Therefore $V = S_\mu$, as desired. $\square$

## 7. Minuscule Schubert calculus conjectures: example and discussion

In earlier work [Thomas and Yong 2006; 2007], we introduced root-system uniform combinatorial rules for minuscule Schubert calculus. Theorem 1.4 has the advantage that it admits a straightforward conjectural generalization to the minuscule setting. We state one form of our conjecture below; more details will appear in forthcoming work.

Let $G$ be a complex, connected reductive Lie group with root system $\Phi$, positive roots $\Phi^+$ and base of simple roots $\Delta$. To each subset of $\Delta$ is associated a parabolic subgroup $P$. The *generalized flag variety* $G/P$ has *Schubert varieties*

$$X_w := \overline{B_- w P/P}$$

for $wW_P \in W/W_P$, where $W$ is the Weyl group of $G$ and $W_P$ is the parabolic subgroup of $W$ corresponding to $P$. Let $K(G/P)$ be the Grothendieck ring of $G/P$, with a basis of Schubert structure sheaves $\{[\mathbb{O}_{X_w}]\}$. Define Schubert structure constants $C_{u,v}^{w}(G/P)$ as before, by

$$[\mathbb{O}_{X_u}] \cdot [\mathbb{O}_{X_v}] = \sum_{wW_P \in W/W_P} C_{u,v}^{w}(G/P)[\mathbb{O}_{X_w}].$$

Brion [2005] has established that

$$(-1)^{\ell(w)-\ell(u)-\ell(v)}C_{u,v}^{w}(G/P)\in\mathbb{N},$$

where $\ell(w)$ is the *Coxeter length* of the minimal length coset representative of $wW_P$.

A maximal parabolic subgroup $P$ is said to be *minuscule* if the associated fundamental weight $\omega_P$ satisfies $\langle\omega_P,\alpha^\vee\rangle\leq 1$ for all $\alpha\in\Phi^+$ under the usual pairing between weights and coroots. The *minuscule flag varieties* $G/P$ are classified into five infinite families and two exceptional cases (the type $A_{n-1}$ cases are the Grassmannians $\mathrm{Gr}(k,\mathbb{C}^n)$).

Associated to each minuscule $G/P$ is a planar poset $(\Lambda_{G/P},\prec)$, obtained as a subposet of the poset of positive roots $\Omega_{G^\vee}$ for the dual root system of $G$; this fact has been known for some time, and recently has been exploited by various authors; see, for example, [Perrin 2007; Purbhoo and Sottile 2008] among others. In this context, *shapes* $\lambda$ are lower order ideals in this poset. These shapes are in bijection with the cosets $wW_P$ indexing the Schubert varieties; in particular, if $wW_P\leftrightarrow\nu$ under this bijection, $\ell(w)=|\nu|$. Define a *skew shape* $\nu/\lambda:=\nu\setminus\lambda$ to be a set theoretic difference of two shapes. Define an *increasing tableau* of shape $\nu/\lambda$ to be an assignment

$$\texttt{label}:\nu/\lambda\to\{1,2,\ldots,q\}$$

such that $\texttt{label}(x)<\texttt{label}(y)$ whenever $x\prec y$, and where each label appears at least once. An *inner corner* of $\nu/\lambda$ is a maximal element $x\in\Lambda_{G/P}$ that is below some element in $\nu/\lambda$. With these definitions, we define notions of $\texttt{INC}_{G/P}(\nu/\lambda)$, $K\texttt{jdt}_{G/P;\{x_i\}}$, $K\texttt{rect}_{G/P}$, superstandard $S_\mu$, and so on, in a manner analogous to those we have given for the Grassmannian. The following rule is new for all minuscule $G/P$:

**Conjecture 7.1.** *For any minuscule $G/P$, $(-1)^{|\nu|-|\lambda|-|\mu|}C_{\lambda,\mu}^{\nu}(G/P)$ equals the number of $T\in\texttt{INC}_{G/P}(\nu/\lambda)$ such that $K\texttt{rect}_{G/P}(T)=S_\mu$.*

Implicit in this conjecture is the conjecture that an analogue of Theorem 1.2 holds. A weaker form of these conjectures is that there is a tableau $C_\mu$ for each shape $\mu$ such that the aforementioned conjectures hold after replacing $S_\mu$ by $C_\mu$.

Briefly, using the ideas contained in this paper, together with those in [Thomas and Yong 2006; 2007] it is not hard to show that $K\texttt{jdt}_{G/P;\{x_i\}}$ is well-defined. The next aim is to establish the analogue of Theorem 1.2. Once this is achieved we can prove that our conjectural rule defines an associative, commutative ring with an additive $\mathbb{Z}$-basis indexed by shapes. It would then remain to show that such rules compute the correct geometric numbers.

The interested reader may find details compatible with the notation used here in [Thomas and Yong 2006]; in particular, there we concretely describe $\Lambda_{G/P}$ in each of these cases. Thus, for brevity, we content ourselves with an example to illustrate our conjecture.

**Example 7.2.** Let $G/P = \mathbb{OP}^2$ be the Cayley plane. Here we have

$$\Lambda_{\mathbb{OP}^2} : \quad \begin{ytableau} \none & \none & \none & {} & {} & {} & {} & {} \\ \none & \none & \none & {} & {} & {} \\ \none & \none & {} & {} & {} \\ {} & {} & {} & {} & {} \end{ytableau} .$$

We conjecturally compute $C_{\lambda,\mu}^{\nu}(\mathbb{OP}^2)$ where

$$\lambda = \mu = \begin{ytableau} \none & \none & \none & {} & {} & {} & {} & {} \\ \none & \none & \none & {} & {} & {} \\ \none & \none & \star & {} & {} \\ \star & \star & \star & \star & {} \end{ytableau} \quad \text{and} \quad \nu = \begin{ytableau} \none & \none & \none & \star & {} & {} & {} & {} \\ \none & \none & \none & \star & \star & \star \\ \none & \none & \star & \star & \star \\ \star & \star & \star & \star & \star \end{ytableau},$$

where the southwest-most box is the unique minimum of $\Lambda_{\mathbb{OP}^2}$ and the poset increases as one moves "right" or "up".

The relevant shapes/lower order ideals of $\Lambda_{\mathbb{OP}^2}$ are indicated by the boxes filled with $\star$'s. We can encode the shapes by the size of columns as read from left to right, so $\lambda = \mu = (1, 1, 2, 1)$ and $\nu = (1, 1, 2, 4, 3, 1)$. Here "superstandard" means that we consecutively fill the first row, followed by the second row, and so on.

Below, we observe there are only two tableaux $T, U$ on $\nu/\lambda$ that $K$-rectify to $S_\mu$:

$$S_\mu = \begin{ytableau} \none & \none & \none & {} & {} & {} & {} & {} \\ \none & \none & \none & {} & {} & {} \\ \none & \none & 5 & {} & {} \\ 1 & 2 & 3 & 4 & {} \end{ytableau}, \quad T = \begin{ytableau} \none & \none & \none & 5 & {} & {} & {} & {} \\ \none & \none & \none & 3 & 4 & 5 \\ \none & \none & {} & 1 & 2 \\ {} & {} & {} & {} & 1 \end{ytableau}, \quad U = \begin{ytableau} \none & \none & \none & 3 & {} & {} & {} & {} \\ \none & \none & \none & 2 & 4 & 5 \\ \none & \none & {} & 1 & 2 \\ {} & {} & {} & {} & 1 \end{ytableau}.$$

Therefore, our conjecture states that

$$C_{(1,1,2,1),(1,1,2,1)}^{(1,1,2,4,3,1)}(\mathbb{OP}^2) = (-1)^{12-5-5} 2 = 2.$$

The reader can check that the rectification order does not affect the result. For either $T$ or $U$, there are three initial ways to begin the $K$-rectification, after which, all further $K$`jdt` slides are forced.

Note that once one establishes an analogue of Theorem 1.2, one can give an easy modification of the proof of associativity in Section 6 to establish that Conjecture 7.1 defines an associative product. One can check that the analogue of Theorem 1.2 holds in specific instances, say, with the help of a computer. Indeed, we have made exhaustive checks when $G/P$ is the odd orthogonal Grassmannian $OG(5, 11)$ and when it is the Cayley plane $\mathbb{OP}^2$, corresponding to the types $B_5$ and $E_6$. We also made numerous checks in the case of the Freudenthal variety $G_\omega(\mathbb{O}^3, \mathbb{O}^6)$

associated to $E_7$, which while not exhaustive, left us convinced. In particular, our choice of definition of superstandard passes these checks (although we also expect that other choices of $S_\mu$ would as well, such as the ones obtained by rastering by columns, rather than rows).

We emphasize that this rule agrees in type $A$ with the correct product, and as well as in cohomology for all minuscule cases. We also have some computational evidence that our numbers agree with small known cases of Schubert structure constants in type $B$ (as supplied to us by M. Shimozono in private correspondence), although admittedly this is not a convincing amount of evidence on its own. Part of the difficulty in checking Conjecture 7.1 is that it seems to be a challenging task to construct efficient software to compute the $K$-theory Schubert structure constants for the main cases of the minuscule $G/P$'s outside of type $A$. In principle, such an algorithm is linear algebra using torus-equivariant fixed-point localization methods such as [Willems 2006].

Granted associativity, the conjectures would follow if they agree with multiplication in $K(G/P)$ whenever $\mu$ is drawn from some set of multiplicative generators $\mathcal{P}$ for $K(G/P)$. (That is, they agree with a "Pieri rule".)

We also mention that the results of Sections 2–4 also have straightforward minuscule generalizations in cohomology; see [Thomas and Yong 2007].

## 8. Counterexamples

It is interesting that natural analogues of a number of results valid in the standard Young tableau theory are actually false in our setting. We have already seen in the introduction that in general $K\texttt{rect}$ is not well-defined. This aspect can also be blamed for the following two other situations where counterexamples exist:

***Haiman's dual equivalence.*** One can define *K-theoretic dual equivalence*, extending ideas in [Haiman 1992]. Two increasing tableaux are *K-dual equivalent* if any sequence of slides $(\{x_{i_1}^{(1)}\}, \ldots, \{x_{i_k}^{(k)}\})$ for $T$ and $U$ results in increasing tableaux of the same shape. In this case we write

$$T \equiv_D U.$$

By definition, $T \equiv_D U$ implies

$$\texttt{shape}(T) = \texttt{shape}(U).$$

One application of this theory (in the classical setting) is that it leads to a proof of the fundamental theorem of jeu de taquin. For a minuscule (but not $K$-theoretic) generalization, see [Thomas and Yong 2007]. However, it is important for this application that all standard Young tableaux of the same shape are dual equivalent. In view of Theorem 1.2, it is not surprising that this is not true in our setting.

Consider the computations

$$K\texttt{infusion}_2\left(\begin{ytableau} 1 & 3 \\ 2 \end{ytableau}\ ,\ \begin{ytableau} \none & \none & 2 \\ \none & 1 & 4 \\ 1 & 3 \end{ytableau}\right) = \begin{ytableau} \none & \none & 1 \\ 2 & 3 \end{ytableau}\ ,$$

$$K\texttt{infusion}_2\left(\begin{ytableau} 1 & 2 \\ 3 \end{ytableau}\ ,\ \begin{ytableau} \none & \none & 2 \\ \none & 1 & 4 \\ 1 & 3 \end{ytableau}\right) = \begin{ytableau} \none & 1 & 2 \\ 1 & 3 \end{ytableau}\ .$$

These calculations represent two sequences of $K\texttt{jdt}$ slides applied to different tableaux of the same shape (2, 1), but whose results are tableaux of different (skew) shapes.

***Cartons.*** In an earlier paper [Thomas and Yong 2008a], we gave an $S_3$-symmetric Littlewood–Richardson rule in terms of *cartons*. This idea also has a minuscule extension (which we will report on elsewhere). However, the naïve $K$-theoretic generalization does not work.

Briefly, the *carton* of [Thomas and Yong 2008a] is a three-dimensional box with a grid drawn rectilinearly on the six faces of its surface, each of whose sides are growth diagrams. We fix at the outset standard Young tableaux of shape $\lambda$, $\mu$ and $\nu$ along three edges. Shapes are associated to each vertex so that the Fomin growth conditions (F1) and (F2) reproduced in Section 2 hold. The number of such cartons (with fixed initial data) is equal to the classical Littlewood–Richardson number.

The temptation is to attempt to generalize this to $K$-theory by replacing the initial standard Young tableau with superstandard tableau of shapes $\lambda$, $\mu$ and $\nu$, and to instead utilize the growth conditions (G1) and (G2) we introduced in Section 2. This does not work: one computes using Theorem 1.4 that if $k = n - k = 3$, $\lambda = \mu = (2, 1)$ and $\nu = (2)$ then the constant $C_{(2,1),(2,1),(2)} := C_{(2,1),(2,1)}^{(3,3,1)} = -2$. However one cannot consistently complete a legal filling of this $K$-carton.

**Remark 8.1.** These obstructions are closely related to failure of associativity of a certain tableau product defined in [Buch et al. 2008, Section 3.7].

An $\mathbb{Z}_3$-symmetric rule preserving the *triality symmetry*

$$C_{\lambda,\mu,\nu^\vee} = C_{\mu,\nu^\vee,\lambda} = C_{\nu^\vee,\lambda,\mu}$$

where $C_{\lambda,\mu,\nu^\vee} := C_{\lambda,\mu}^{\nu}$ and so on exists in the form of *puzzles*; see [Vakil 2006]). (Unlike in cohomology, in $K$-theory, this latter symmetry is not immediate from the geometric definitions; for a proof see [Buch 2002b; Vakil 2006]. In fact, this symmetry is not expected to hold for general $G/P$, although A. Knutson has informed us, in private communication, that it holds in the minuscule setting.)

## 9. Concluding remarks

***Proctor's d-complete posets.*** Proctor [2004] has studied the class of *$d$-complete posets*. These posets generalize those required in our discussion of minuscule $G/P$ Schubert calculus; see also [Thomas and Yong 2006; 2007]. In particular, $d$-complete posets were shown by Proctor to have a well-defined *jeu de taquin* procedure.

It would be interesting to generalize our arguments to show that for any $d$-complete poset $D$, there is an associative ring $K(D)$ with an additive $\mathbb{Z}$-basis indexed by lower order ideals of $D$ and structure constants defined by a rule generalizing Theorem 1.4. Observing that our notions of $K\mathtt{jdt}$, $K\mathtt{rect}$ *a priori* make sense in this more general context, we ask:

**Problem 9.1.** Fix a $d$-complete poset. For which classes of tableaux $\mathscr{C} = \{C_\mu\}$ (indexed by lower order ideals $\mu$ of $D$) is it true that an analogous Theorem 1.2 holds (that is, if $K\mathtt{rect}(T) = C \in \mathscr{C}$ under one rectification order, this holds for any rectification order)?

It seems plausible that good classes $\mathscr{C}$ that play the role of the superstandard tableaux of Theorem 1.2 always exist. As we have said, for the minuscule cases, we believe that the superstandard tableaux suffice. Perhaps this also holds more generally.

Assuming this plausible claim holds, one would also like to find a geometric origin to the ring $K(D)$ (outside of the cases where it should be isomorphic to the $K$-theory ring of a minuscule $G/P$).

***A product-differences conjecture.*** Let $\lambda, \mu \in \mathbb{Y}$. Since this poset is in fact a lattice, we can speak of their *meet* $\lambda \wedge \mu$ and *join* $\lambda \vee \mu$.

**Conjecture 9.2.** *Suppose $\lambda, \mu \subseteq \Lambda$. Let*

$$[\mathbb{O}_{X_{\lambda\wedge\mu}}][\mathbb{O}_{X_{\lambda\vee\mu}}] - [\mathbb{O}_{X_\lambda}][\mathbb{O}_{X_\mu}] = \sum_\nu d_\nu [\mathbb{O}_{X_\nu}].$$

*Then*

$$(-1)^{|\nu|-|\lambda|-|\mu|} d_\nu \geq 0.$$

This conjecture generalizes a theorem in the cohomological case [Lam et al. 2007]; see related work [Okounkov 2003; Fomin et al. 2005; Chindris et al. 2007]. (We also know of no counterexample for the corresponding minuscule conjecture, even in the cohomology case.)

**Example 9.3.** Let

$$\lambda = (4, 2, 1), \qquad \mu = (3, 3, 2) \subseteq \Lambda = 4 \times 5.$$

The join is the unique minimal shape that contains $\lambda$ and $\mu$, that is, $\lambda \vee \mu = (4, 3, 2)$. Similarly, the meet is the unique maximal shape contained in $\lambda$ and $\mu$. Hence $\lambda \wedge \mu = (3, 2, 1)$. One computes using Theorem 1.4 (or otherwise), preferably with the help of a computer, that

$$\begin{aligned}
&[\mathbb{O}_{X_{(4,3,2)}}] \cdot [\mathbb{O}_{X_{(3,2,1)}}] - [\mathbb{O}_{X_{(4,2,1)}}] \cdot [\mathbb{O}_{X_{(3,3,2)}}] \\
&\qquad = \big([\mathbb{O}_{X_{(5,5,3,2)}}] + 2[\mathbb{O}_{X_{(5,5,4,1)}}] + [\mathbb{O}_{X_{5,5,5}}] + [\mathbb{O}_{X_{5,4,4,2}}]\big) \\
&\qquad\quad - \big(3[\mathbb{O}_{X_{(5,5,5,1)}}] + [\mathbb{O}_{X_{(5,5,3,3)}}] + 5[\mathbb{O}_{X_{(5,5,4,2)}}] + [\mathbb{O}_{X_{(5,4,4,3)}}]\big) \\
&\qquad\quad + \big(3[\mathbb{O}_{X_{(5,5,5,2)}}] + 3[\mathbb{O}_{X_{(5,5,4,3)}}]\big) \\
&\qquad\quad - \big([\mathbb{O}_{X_{(5,5,5,3)}}]\big),
\end{aligned}$$

in agreement with Conjecture 9.2.

***Hecke insertion and factor sequence formulae.*** In [Buch et al. 2008] a generalization of the Robinson–Schensted and Edelman–Greene insertion algorithms was given. In fact, increasing tableaux also play a prominent role there, although in a different, but related way. As we have mentioned in Section 1, this is explored, in part, in [Thomas and Yong 2008b], in connection to longest strictly increasing subsequences in random words. There we show that the insertion tableau of a word under Hecke insertion can be alternatively computed as a $K$-rectification of a permutation tableau (for a particular choice of rectification order).

A related question: is there a "plactification map" in the sense of [Reiner and Shimozono 1995]?

We believe that further developing this connection may allow one to, for example, prove a $K$-theory analogue of the "factor sequence formula" conjectured in [Buch and Fulton 1999] and proved in [Knutson et al. 2006], which is a problem that has remained open in this topic; see [Buch 2002a; 2005a]. (In [Buch et al. 2008] a different factor sequence formula, generalizing the one given in [Buch 2005a], was given.)

## Appendix: Grothendieck polynomials

The goal of this appendix is to provide combinatorial background for the results of Sections 1–7, in terms of the Grothendieck polynomials of Lascoux and Schützenberger [1982]. This presentation is not needed for the paper.

Fix a shape $\lambda$ and define a *set-valued tableau* $T$ to be an assignment of nonempty sets of natural numbers to each box of $\lambda$ [Buch 2002b]. Such a tableau is *semistandard* if for every box, the largest entry is weakly smaller than the minimum entry of the box immediately to its right and strictly smaller than the minimum entry of the box immediately below it. The *ordinary* case is when $T$ assigns a singleton to each box. The following are examples of an ordinary and a set-valued semistandard tableau:

$$T_1 = \begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 4 & 4 & 6 \\ \hline 2 & 3 & 5 \\ \cline{1-3} 4 \\ \cline{1-1}\end{array}\,, \qquad T_2 = \begin{array}{|c|c|c|c|c|}\hline 1,2 & 2,3 & 4,5,6 & 6,7 & 7,8 \\ \hline 3,4 & 4,5 & 7 \\ \cline{1-3} 6,7,8 \\ \cline{1-1}\end{array}\,.$$

Associate to each semistandard tableau a *weight*

$$\omega(T) := (-1)^{|T|-|\lambda|} x^T$$

where here $x^T = x_1^{i_1} x_2^{i_2} \cdots$ if $i_j$ is the number of $j$'s appearing in $T$, and $|T|$ is the number of entries of $T$. For example, we have

$$\omega(T_1) = x_1 x_2^2 x_3 x_4^3 x_5 x_6 \quad \text{and} \quad \omega(T_2) = (-1)^{19-9} x_1 x_2^2 x_3^2 x_4^3 x_5^2 x_6^3 x_7^4 x_8^2.$$

The *Grothendieck polynomial* is defined as

$$G_\lambda(x_1, x_2, \ldots, x_k) := \sum_T \omega(T)$$

with the sum over all set-valued semistandard tableaux using the labels of size at most $k$. This is an inhomogeneous symmetric polynomial whose lowest degree $(= |\lambda|)$ homogeneous component is equal to the *Schur polynomial* $s_\lambda(x_1, x_2, \ldots x_k)$.

It is not immediately obvious from the definitions, but true [Buch 2002b] (for an alternative proof, see [Buch et al. 2008]) that the $G_\lambda(x_1, \ldots, x_k)$ (for $\lambda$ with at most $k$ parts) form a $\mathbb{Z}$-linear basis for the ring of symmetric polynomials in $x_1, \ldots, x_k$ (say, with coefficients in $\mathbb{Q}$). Thus we can write

$$G_\lambda(x_1, \ldots, x_k) G_\mu(x_1, \ldots, x_k) = \sum_\nu C_{\lambda,\mu}^\nu G_\nu(x_1, \ldots, x_k).$$

The coefficients $C_{\lambda,\mu}^\nu$ agree with the $K$-theory structure constants for $\mathrm{Gr}(k, \mathbb{C}^n)$ whenever $\nu \subseteq \Lambda$.

There are more general Grothendieck polynomials $\mathfrak{G}_\pi(x_1, \ldots, x_n)$ defined in [Lascoux and Schützenberger 1982] for any permutation $\pi \in S_n$. The polynomials $G_\lambda$ amount to the case that $\pi$ is *Grassmannian*: it has a unique descent at position $k$. In [Buch et al. 2005] a formula was first given that expresses any $\mathfrak{G}_\pi$ in terms of the $G_\lambda$'s. Other formulas for both $\mathfrak{G}_\pi$ and $G_\lambda$ are also available; see, for example, [Buch et al. 2008; Knutson and Yong 2004; Knutson et al. 2008; Lascoux 2001] and the references therein.

## Acknowledgments

This work was partially completed while Thomas was visiting the Norges Teknisk–Naturvitenskapelige Universitet; he would like to thank the Institutt for Matematiske Fag for its hospitality. Yong utilized the resources of the Fields Institute, Toronto, while a visitor there. We thank Allen Knutson, Victor Reiner and Mark Shimozono for helpful discussions, as well as Anders Buch for supplying us with software to independently compute the $K$-theoretic numbers $C_{\lambda,\mu}^{\nu}$ (for Grassmannians). We also thank an anonymous referee for comments that led to an improvement of this paper.


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Communicated by Ravi Vakil
Received 2007-11-04 Revised 2008-09-17 Accepted 2008-11-29

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