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We develop the theory of weak bimonoids in braided monoidal categories and show that they are in one-to-one correspondence with quantum categories with a separable Frobenius object-of-objects. Weak Hopf monoids are shown to be quantum groupoids. Each separable Frobenius monoid R leads to a weak Hopf monoid $R \otimes R$.

Introduction

Weak Hopf algebras, introduced by Böhm, Nill, and Szlachányi in the papers [Böhm and Szlachónyi 1996; Nill 1998; Szlachányi 1997; Böhm et al. 1999], are generalizations of Hopf algebras and were proposed as an alternative to weak quasi-Hopf algebras. A weak bialgebra is both an associative algebra and a coassociative coalgebra, but instead of requiring that the multiplication and unit morphism are

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coalgebra morphisms (or equivalently that the comultiplication and the counit are algebra morphisms), other "weakened" axioms are imposed. The multiplication is still required to be comultiplicative (equivalently, the comultiplication is still required to be multiplicative), but the counit is no longer required to be an algebra morphism and the unit is no longer required to be a coalgebra morphism. Instead, these requirements are replaced by weakened versions (see Equations (v) and (w) below). As the name suggests, any bialgebra satisfies these weakened axioms and is therefore a weak bialgebra.

For a given a weak bialgebra A, one may define source and target morphisms $s, t : A \to A$ whose images $s(A)$ and $t(A)$ are called the source and target (counital) subalgebras. Nill [1998] has shown that Hayashi's face algebras [1998] are special cases of weak bialgebras for which the, say, target subalgebra is commutative.

A weak Hopf algebra is a weak bialgebra H that is equipped with an antipode $\nu : H \to H$ satisfying the axioms[1]

$$\mu(\nu \otimes 1)\delta = t, \quad \mu(1 \otimes \nu)\delta = s, \quad \text{and} \quad \mu_3(\nu \otimes 1 \otimes \nu)\delta_3 = \nu,$$

where $\mu_3 = \mu(\mu \otimes 1)$ and $\delta_3 = (\delta \otimes 1)\delta$. Again, any Hopf algebra satisfies these weakened axioms and so is a weak Hopf algebra. Nill [1998] has also shown that the (finite-dimensional) generalized Kac algebras of Yamanouchi [1994] are examples of weak Hopf algebras with involutive antipode. Weak Hopf algebras have also been called "quantum groupoids" [Nikshych and Vainerman 2002], but in this paper this is *not* what we mean by quantum groupoid.

Perhaps the simplest examples of weak bialgebras and weak Hopf algebras are category algebras and groupoid algebras, respectively. Suppose that k is a field, and let $\mathscr{C}$ be a category with set of objects $\mathscr{C}_0$ and set of morphisms $\mathscr{C}_1$. The *category algebra* $k[\mathscr{C}]$ is the vector space $k[\mathscr{C}_1]$ over k with basis $\mathscr{C}_1$. Elements are formal linear combinations of the elements of $\mathscr{C}_1$ with coefficients in k, that is,

$$\alpha f + \beta g + \cdots \quad \text{with } \alpha, \beta \in k \text{ and } f, g \in \mathscr{C}_1.$$

An associative multiplication on $k[\mathscr{C}]$ is defined by

$$\mu(f, g) = f \cdot g = \begin{cases} g \circ f & \text{if } g \circ f \text{ exists,} \\ 0 & \text{otherwise} \end{cases}$$

and extended by linearity to $k[\mathscr{C}]$. This algebra does not have a unit unless $\mathscr{C}_0$ is finite, in which case the unit is

$$\eta(1) = e = \sum_{A \in \mathscr{C}_0} 1_A,$$

[1]There may be some discrepancy with what we call the source and target morphisms and what exists in the literature. This arises from our convention of taking multiplication in the groupoid algebra to be $f \cdot g = g \circ f$ (whenever $g \circ f$ is defined).

making $k[\mathscr{C}]$ into a unital algebra; all algebras (monoids) considered in this paper will be unital. A comultiplication and counit may be defined on $k[\mathscr{C}]$ as

$$\delta(f) = f \otimes f, \qquad \epsilon(f) = 1,$$

making $k[\mathscr{C}]$ into a coalgebra. Note that $k[\mathscr{C}]$ equipped with this algebra and coalgebra structure will not satisfy any of the following usual bialgebra axioms:

$$\epsilon\mu = \epsilon \otimes \epsilon, \qquad \delta\eta = \eta \otimes \eta, \qquad \epsilon\eta = 1_k.$$

The one bialgebra axiom that does hold is $\delta\mu = (\mu \otimes \mu)(1 \otimes c \otimes 1)(\delta \otimes \delta)$. Equipped with this algebra and coalgebra structure, $k[\mathscr{C}]$ does, however, satisfy the axioms of a weak bialgebra. Furthermore, if $\mathscr{C}$ is a groupoid, then $k[\mathscr{C}]$, which is then called the *groupoid algebra*, is an example of a weak Hopf algebra with antipode $\nu : k[\mathscr{C}] \to k[\mathscr{C}]$ defined by $\nu(f) = f^{-1}$ and extended by linearity. If $f : A \to B \in \mathscr{C}$, the source and target morphisms $s, t : k[\mathscr{C}] \to k[\mathscr{C}]$ are given by $s(f) = 1_A$ and $t(f) = 1_B$, as one would expect.

In this paper we define weak bialgebras and weak Hopf algebras in a braided monoidal category $\mathscr{V}$, where we prefer to call them "weak bimonoids" and "weak Hopf monoids". The only difference between our definition of a weak bimonoid in $\mathscr{V}$ and the one given by Böhm, Nill, and Szlachányi [Böhm et al. 1999] is that a choice of "crossing" must be made in the axioms. Our definition is not as general as the one given by J. N. Alonso, J. M. Fernández, and R. González in [Alonso Álvarez et al. 2008a; 2008b], but, in the case that their weak Yang–Baxter operator $t_{A,A}$ is the braiding $c_{A,A}$ and their idempotent $\nabla_{A\otimes A} = 1_{A\otimes A}$, our choices of crossings are the same. Our difference in defining weak bimonoids occurs in the choice of source and target morphisms. We have chosen $s : A \to A$ and $t : A \to A$ so that

(1) the "globular" identities $ts = s$ and $st = t$ hold,

(2) the source subcomonoid and target subcomonoid coincide (up to isomorphism) and are denoted by C; and

(3) $s : A \to C^{\circ}$ and $t : A \to C$ are comonoid morphisms.

These properties of the source and target morphisms are essential for our point of view of quantum categories. These are $s = \overline{\Pi}^L_A$ and $t = \Pi^R_A$ in the notation of [Alonso Álvarez et al. 2008a; 2008b] and $s = \epsilon'_s$ and $t = \epsilon_s$ in the notation of [Schauenburg 2003], with the appropriate choice of crossings.

We choose to work in the Cauchy completion $\mathscr{Q}\mathscr{V}$ of $\mathscr{V}$. The category $\mathscr{Q}\mathscr{V}$ is also called the "completion under idempotents" of $\mathscr{V}$ or the "Karoubi envelope" of $\mathscr{V}$. We do this rather than assume that idempotents split in $\mathscr{V}$. Suppose A is a weak bimonoid in $\mathscr{Q}\mathscr{V}$. In this case we find C by splitting either the source or target morphism. As in [Schauenburg 2003, Proposition 4.2], C is a separable Frobenius monoid in $\mathscr{Q}\mathscr{V}$, meaning that $(C, \mu, \eta, \delta, \epsilon)$ is a Frobenius monoid with $\mu\delta = 1_C$.

Our definition of weak Hopf monoid is the same as the one proposed in [Böhm et al. 1999] for the symmetric case and as in [Alonso Álvarez et al. 2008a; 2008b] when restricted to the braided case. A weak bimonoid H is a weak Hopf monoid if it is equipped with an antipode $\nu : H \to H$ satisfying

$$\mu(\nu \otimes 1)\delta = t, \qquad \mu(1 \otimes \nu)\delta = r, \quad \text{and} \quad \mu_3(\nu \otimes 1 \otimes \nu)\delta_3 = \nu,$$

where $r = \nu s$. This $r : H \to H$ turns out to be the "usual" source morphism, that is, Π^L_H in the notation of [Alonso Álvarez et al. 2008a; 2008b]. Ignoring crossings r is ϵ_t in the notation of [Schauenburg 2003], and our r and t correspond respectively to $\sqcap^L$ and $\sqcap^R$ in the notation of [Böhm et al. 1999], wherein the morphism s does not appear. Usually, in the second axiom above, $\mu(1 \otimes \nu)\delta = r$, the right side is equal to the chosen source map s of the weak bimonoid H. The reason that this r does not work for us as a source morphism is that it does not satisfy all three requirements for the source morphism mentioned above. This choice of r allows us to show that any Frobenius monoid in $\mathscr{V}$ yields a weak Hopf monoid $R \otimes R$ with bijective antipode; see [Böhm et al. 1999, example in the appendix].

There are a number of generalizations of bialgebras and Hopf algebras to their "many object" versions, for example, Sweedler's generalized bialgebras [1974], which were later generalized by Takeuchi to $\times_R$-bialgebras [1977]; the quantum groupoids of Lu [1996] and Xu [2001]; Schauenburg's $\times_R$-Hopf algebras [2000]; the bialgebroids and Hopf algebroids of Böhm and Szlachányi [2004]; the face algebras [Hayashi 1998] and generalized Kac algebras [Yamanouchi 1994]; and the ones of interest in this paper, the quantum categories and quantum groupoids of Day and Street [2004]. Brzeziński and Militaru [2002, Theorem 3.1] have shown that the quantum groupoids of Lu and Xu are equivalent to Takeuchi's $\times_R$-bialgebras. Schauenburg [1998] has shown that face algebras are an example of $\times_R$-bialgebras for which R is commutative and separable. In [2003, Theorem 5.1], Schauenburg shows that weak bialgebras are also examples of $\times_R$-bialgebras for which R is separable Frobenius (there called Frobenius-separable). Schauenburg also shows in [2003, Theorem 6.1] that a weak Hopf algebra may be characterized as a weak bialgebra H for which a certain canonical map $H \otimes_C H \to \mu(\delta(\eta(1)), H \otimes H)$ is a bijection. As a corollary he shows that the $\times_R$-bialgebra associated to the weak Hopf algebra is actually a $\times_R$-Hopf algebra.

While bialgebras are self dual, bialgebroids are not. The dual of a bialgebroid is called a "bicoalgebroid" by Brzeziński and Militaru [2002] and further studied by Bálint [2008b]. In the terminology of [Day and Street 2004], these structures are quantum categories in the monoidal category of vector spaces.

According to [Day and Street 2004], a quantum category in $\mathscr{V}$ consists of two comonoids A and C in $\mathscr{V}$, with A playing the role of the object of morphisms, and C the object-of-objects. There are source and target morphisms $s, t : A \to C$, a "composition" morphism $\mu : A \otimes_C A \to A$, and a "unit" morphism $\eta : C \to A$, all in $\mathcal{V}$. These data must satisfy a number of axioms. Indeed, ordinary categories are quantum categories in the category of sets.

Motivated by the duality present in $*$-autonomous categories [Barr 1995], Day and Street define a quantum groupoid to be a quantum category equipped with a generalized antipode coming from a $*$-autonomous structure.

In this paper we show there is a bijection between weak bimonoids and quantum categories for which the object-of-objects is a separable Frobenius monoid. In the case that the weak bimonoid is equipped with an invertible antipode, making it a weak Hopf monoid, we show how to yield a quantum groupoid.

The outline of this paper is as follows: In Section 1, we provide the definition of a weak bimonoid A in a braided monoidal category $\mathcal{V}$ and define the source and target morphisms. We then move to the Cauchy completion $\mathfrak{Q}\mathcal{V}$ and prove the three required properties of our source and target morphisms. In this section we also prove that C, the object-of-objects of A, is a separable Frobenius monoid.

In Section 2, we introduce Weak Hopf monoids in braided monoidal categories.

In Section 3, we describe a monoidal structure on the categories $\mathbf{Bicomod}(C)$ of C-bicomodules in $\mathcal{V}$, and $\mathbf{Comod}(A)$ of right A-comodules in $\mathcal{V}$, such that the underlying functor $U : \mathbf{Comod}(A) \to \mathbf{Bicomod}(C)$ is strong monoidal. If H is a weak Hopf monoid, we are then able to show that the category $\mathbf{Comod}_f(H)$, consisting of the dualizable objects of $\mathbf{Comod}(H)$, is left autonomous.

In Section 4, we prove that any separable Frobenius monoid R in a braided monoidal category $\mathcal{V}$ yields an example of a weak Hopf monoid $R \otimes R$ with invertible antipode in $\mathcal{V}$.

In Section 5, we recall the definitions of quantum categories and of quantum groupoids, and in Section 6, we show the correspondence between weak bimonoids and quantum categories with separable Frobenius object-of-objects. In Section 6, we also show that a weak Hopf monoid with invertible antipode yields a quantum groupoid.

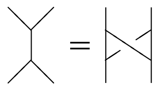
This paper depends heavily on the string diagrams in braided monoidal categories of Joyal and Street [1993], which were shown to be rigorous in [1991]. The reader unfamiliar with string diagrams may first want to read Appendix A, where we review some preliminary concepts using these diagrams. Many string proofs also appear in Appendix B.

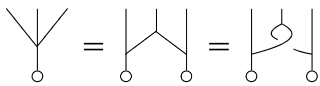
1. Weak bimonoids

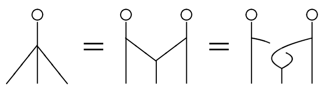
A weak bialgebra [Böhm and Szlachónyi 1996; Nill 1998; Szlachányi 1997; Böhm et al. 1999] is a generalization of a bialgebra with weakened axioms. These weakened axioms replace the three that follow by requiring that the unit be a coalgebra morphism and the counit be an algebra morphism. With the appropriate choices of under and over crossings, the definition of a weak bialgebra carries over rather straightforwardly into braided monoidal categories, where we prefer to call it a "weak bimonoid".

1.1. *Weak bimonoids.* Suppose that $\mathcal{V} = (\mathcal{V}, \otimes, I, c)$ is a braided monoidal category.

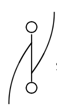
Definition 1.1. A *weak bimonoid* $A = (A, \mu, \eta, \delta, \epsilon)$ in $\mathcal{V}$ is an object $A \in \mathcal{V}$ equipped with the structure of a monoid (A, μ, η) and a comonoid (A, δ, ϵ) satisfying the following equations.

 (b)

 (v)

 (w)

Suppose A is a weak bimonoid, and define the *source* and *target* morphisms $s, t : A \to A$ of A as

$s =$ , and $t =$  .

Notice that $s : A \to A$ is invariant under rotation by π, while $t : A \to A$ is invariant under horizontal reflection and the inverse braiding. Importantly, under either of these transformations

- (m) and (c) are interchanged,[2]
- (b) is invariant, and
- (v) and (w) are interchanged.

Note that these are not the "usual" source and target morphisms. They were chosen, as mentioned in the introduction, precisely because we need them to satisfy the following three properties:

(i) the "globular" identities $ts = s$ and $st = t$ hold;

(ii) the source subcomonoid and target subcomonoid coincide (up to isomorphism), and are denoted by C;

(iii) $s : A \to C^{\circ}$ and $t : A \to C$ are comonoid morphisms.

[2] The (m) and (c) refer to the monoid and comonoid identities found in Appendix A.

These properties will be proved in this section. Note that we will run into the usual source morphism (which we call r) in Definition 2.1, which defines weak Hopf monoids.

We tabulate properties of the source morphism s in Figure 1, properties of the target morphism t in Figure 2, and properties involving the interaction of s and t in Figure 3. Proofs of these properties may be found in Appendix B.

Suppose A and B are weak bimonoids in $\mathscr{V}$. A *morphism of weak bimonoids* $f : A \to B$ is a morphism $f : A \to B$ in $\mathscr{V}$ that is both a monoid morphism and a comonoid morphism.

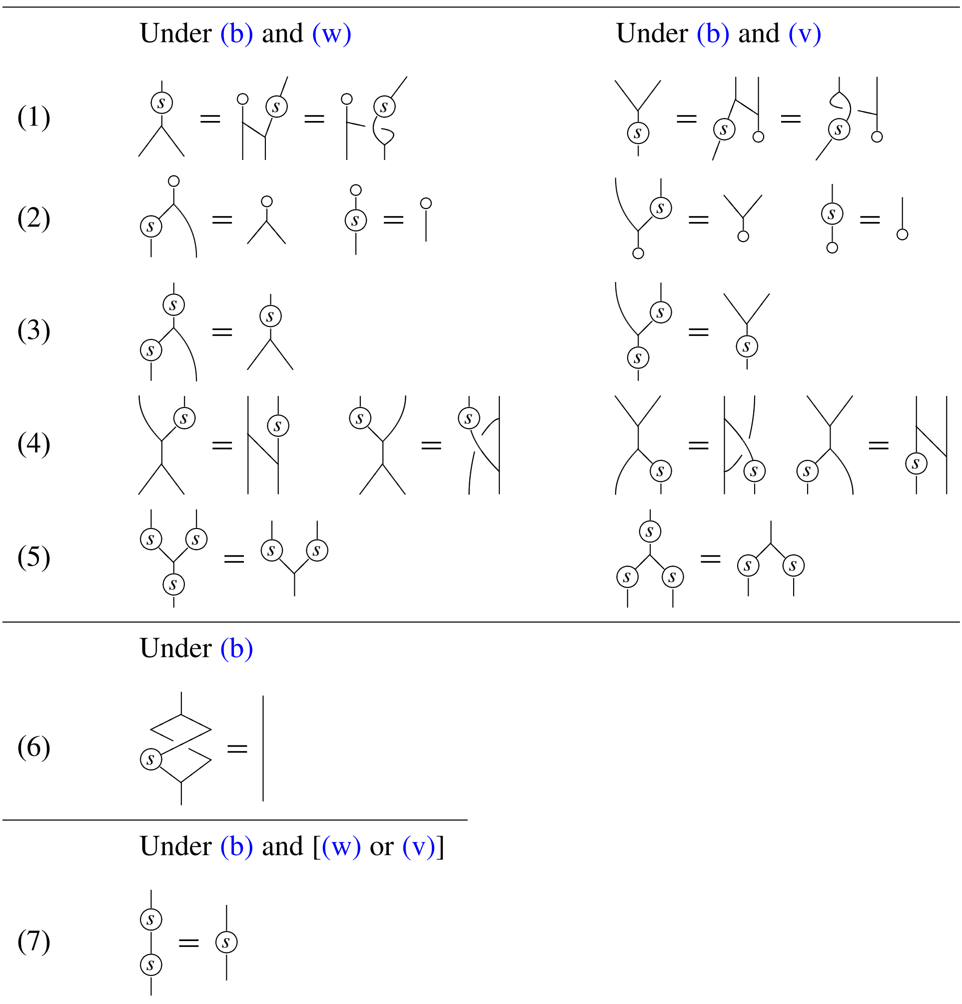


Figure 1. Properties of s.

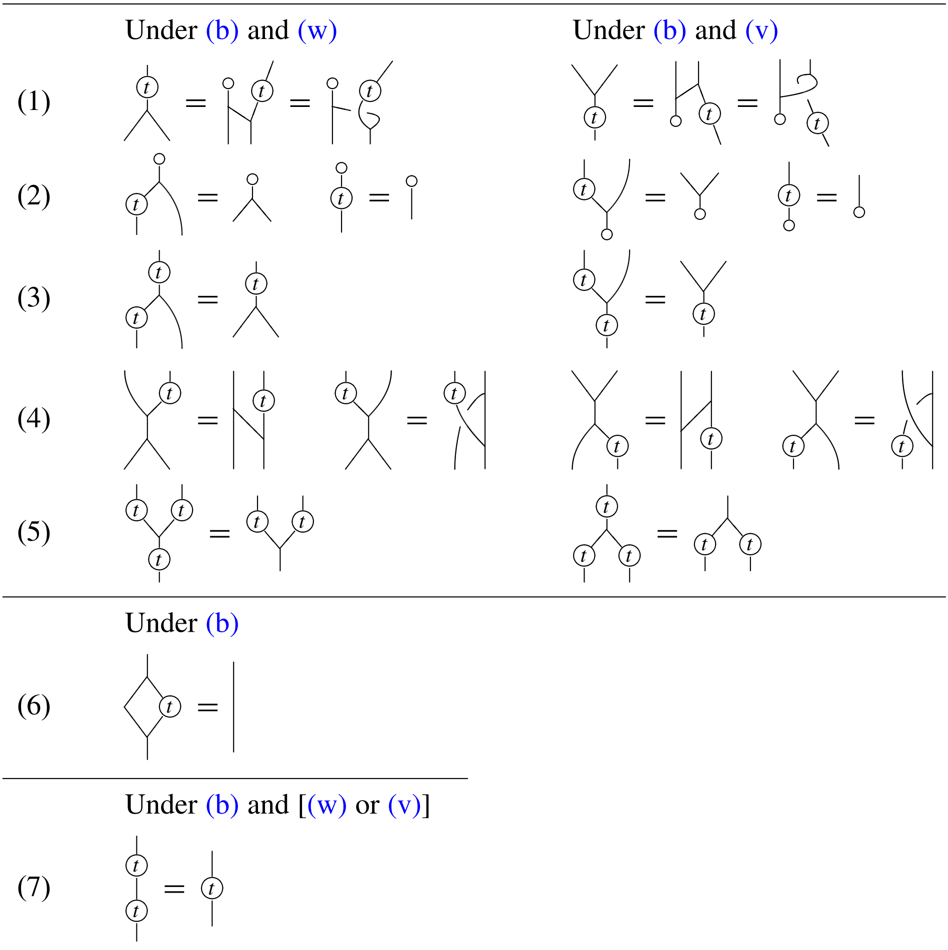
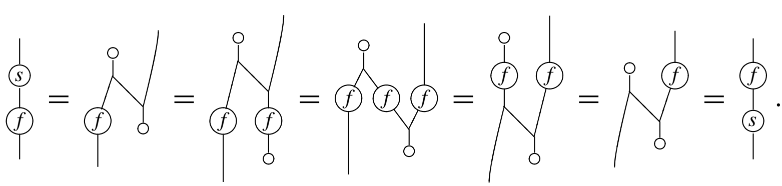


Figure 2. Properties of t.

Lemma 1.2. *Suppose A and B are weak bimonoids in $\mathcal{V}$ each with source and target morphisms s and t. If $f : A \to B$ is a morphism of weak bimonoids, then*

$$fs = sf \quad \text{and} \quad ft = tf.$$

Proof. The proof of the first statement is



The second statement follows from a similar proof. □

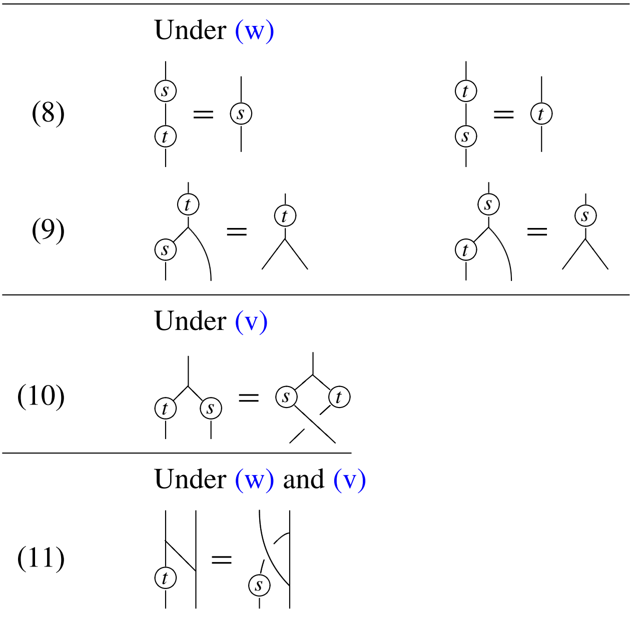


Figure 3. Interactions of s and t.

In what follows $A = (A, \mu, \eta, \delta, \epsilon)$ will always denote a weak bimonoid and $s, t : A \to A$ will always denote the source and target morphisms.

From properties (7) and (7) in Figures 1 and 2 respectively, s and t are idempotents. In the following we will work in the Cauchy completion (= completion under idempotents = Karoubi envelope) $\mathcal{Q}\mathcal{V}$ of $\mathcal{V}$. We do this rather than assume that idempotents split in $\mathcal{V}$.

1.2. *Cauchy completion.* Given a category $\mathcal{V}$, its *Cauchy completion* $\mathcal{Q}\mathcal{V}$ is the category whose objects are pairs (X, e) with $X \in \mathcal{V}$ and $e : X \to X \in \mathcal{V}$ an idempotent. A morphism $(X, e) \to (X', e')$ in $\mathcal{Q}\mathcal{V}$ is a morphism $f : X \to X' \in \mathcal{V}$ such that $e' f e = f$. Note that the identity morphism of (X, e) is e itself.

The point of working in the Cauchy completion is that every idempotent $f : (X, e) \to (X, e)$ in $\mathcal{Q}\mathcal{V}$ has a splitting, namely,

$$\begin{tikzcd} (X, e) \arrow[rr, "f"] \arrow[dr, "f"'] & & (X, e) \\ & (X, f). \arrow[ur, "f"'] & \end{tikzcd}$$

If $\mathcal{V}$ is a monoidal category, then $\mathcal{Q}\mathcal{V}$ is a monoidal category via

$$(X, e) \otimes (X', e') = (X \otimes X', e \otimes e').$$

The category $\mathcal{V}$ may be fully embedded in $\mathcal{Q}\mathcal{V}$ by sending $X \in \mathcal{V}$ to $(X, 1) \in \mathcal{Q}\mathcal{V}$ and $f : X \to Y \in \mathcal{V}$ to $f : (X, 1) \to (Y, 1)$, which is obviously a morphism in $\mathcal{Q}\mathcal{V}$. When working in $\mathcal{Q}\mathcal{V}$ we will often identify an object $X \in \mathcal{V}$ with $(X, 1) \in \mathcal{Q}\mathcal{V}$.

1.3. ***Properties of the source and target morphisms.*** Let $A = (A, 1)$ be a weak bimonoid in $\mathcal{Q}\mathcal{V}$. From the definition of the Cauchy completion, the result of splitting the source morphism s is (A, s), and similarly, the result of splitting the target morphism t is (A, t). The following proposition shows that these two objects are isomorphic.

Proposition 1.3. *The idempotent $t : (A, 1) \to (A, 1)$ has the two splittings*

$$\begin{tikzcd}
(A,1) \arrow[rr, "t"] \arrow[dr, "t"'] & & (A,1) \\
& (A,t) \arrow[ur, "t"'] &
\end{tikzcd}
\quad \textit{and} \quad
\begin{tikzcd}
(A,1) \arrow[rr, "t"] \arrow[dr, "t"'] & & (A,1) \\
& (A,s). \arrow[ur, "s"'] &
\end{tikzcd}$$

In this case $s : (A, s) \to (A, t)$ and $t : (A, t) \to (A, s)$ are inverse morphisms, and hence $(A, t) \cong (A, s)$.

Proof. This result follows from the identities $ts = s$ and $st = t$ (property (8) in Figure 3). □

We will denote this object by $C = (A, t)$ and call it the *object-of-objects* of A. In the propositions next we will show that C is a comonoid and that it is a separable Frobenius monoid; this is similar to what was done in [Schauenburg 2003] (where it was called Frobenius-separable).

Proposition 1.4. *The object $C = (A, t)$ equipped with*

$$\delta = \left(C \xrightarrow{\delta} C \otimes C \xrightarrow{t \otimes t} C \otimes C\right) \quad \textit{and} \quad \epsilon = C \xrightarrow{\epsilon} I$$

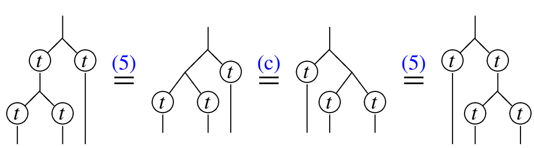
is a comonoid in $\mathcal{Q}\mathcal{V}$. If C is furthermore equipped with

$$\mu = \left(C \otimes C \xrightarrow{t \otimes t} C \otimes C \xrightarrow{\mu} C\right) \quad \textit{and} \quad \eta = I \xrightarrow{\eta} C,$$

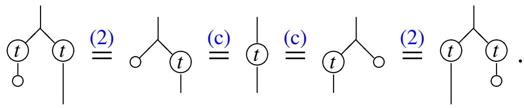
then C is a separable Frobenius monoid in $\mathcal{Q}\mathcal{V}$ (see Definition A.5).

Proof. We first observe that $(t \otimes t)\delta : C \to C \otimes C$ and $\epsilon : C \to I$ are in $\mathcal{Q}\mathcal{V}$, which follows respectively from (5) and (2).

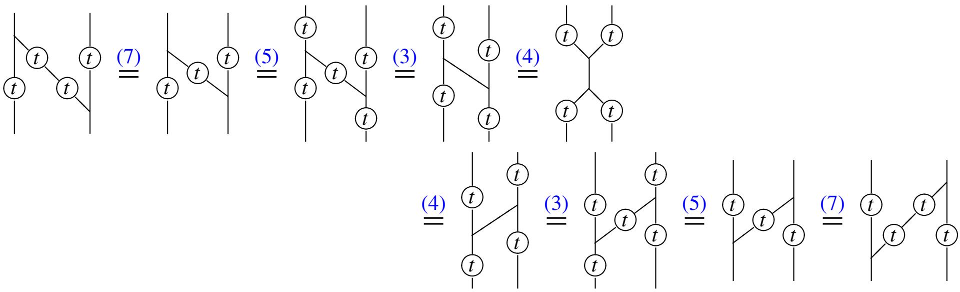
The comonoid identities are given as



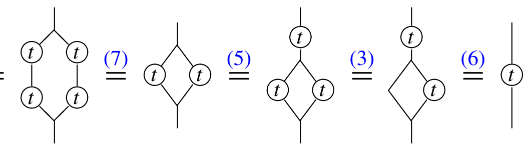
and



To see that C is a separable Frobenius monoid we first observe that μ and η are morphisms in $\mathfrak{Q}\mathscr{V}$ from (5) and (2), and the monoid identities are dual to the comonoid identities. The following calculation proves that the Frobenius condition holds.



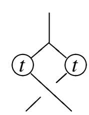
Finally, that this is a separable Frobenius monoid follows from

$\mu\delta =$  $= 1_C$. $\square$

Corollary 1.5. *Every morphism of weak bimonoids induces an isomorphism on the objects-of-objects. That is, if* $(A, 1)$ *and* $(B, 1)$ *are weak bimonoids, and* $f : (A, 1) \to (B, 1)$ *is a morphism of weak bimonoids, then the induced morphism* $tft : (A, t) \to (B, t)$ *is an isomorphism.*

Proof. If $f : A \to B$ is a morphism of weak bimonoids, then by Lemma 1.2 $ft = tf$ and $fs = st$. The corollary now follows from Propositions 1.4 and A.3. $\square$

Proposition 1.6. *If we write* C° *for the comonoid* C *with the "opposite" comultiplication defined via*

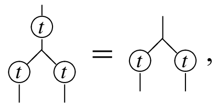
$C \xrightarrow{\delta} C \otimes C \xrightarrow{t \otimes t} C \otimes C \xrightarrow{c} C \otimes C =$  ,

then $s : A \to C^\circ$ *and* $t : A \to C$ *are comonoid morphisms. That is, the diagrams*

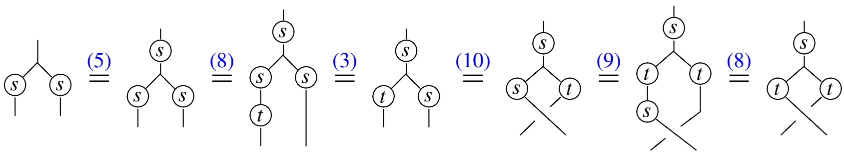
$$\begin{tikzcd} A \arrow[r, "s"] \arrow[d, "\delta"'] & C \arrow[d, "c(t\otimes t)\delta"] \\ A \otimes A \arrow[r, "s\otimes s"] & C \otimes C \end{tikzcd} \quad \textit{and} \quad \begin{tikzcd} A \arrow[r, "t"] \arrow[d, "\delta"'] & C \arrow[d, "(t\otimes t)\delta"] \\ A \otimes A \arrow[r, "t\otimes t"] & C \otimes C \end{tikzcd}$$

commute.

Proof. The second diagram expresses



which is exactly (5), and the calculation



shows that the first diagram commutes. □

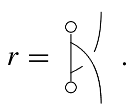
2. Weak Hopf monoids

In this section we introduce weak Hopf monoids. A weak Hopf monoid is a weak bimonoid H equipped with an antipode $\nu : H \to H$ satisfying the three axioms

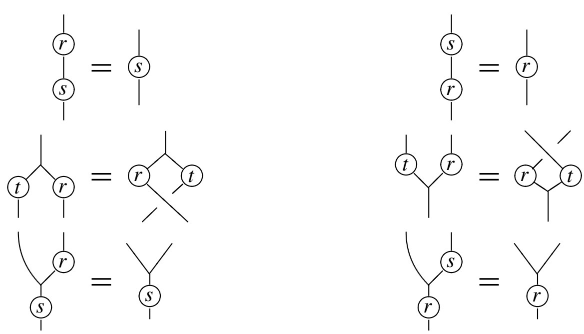
$$\nu * 1 = t, \qquad 1 * \nu = r, \quad \text{and} \quad \nu * 1 * \nu = \nu,$$

where $f * g = \mu(f \otimes g)\delta$ is the convolution product, and the morphism $r : H \to H$ is introduced below. This turns out to be the usual definition of weak Hopf monoids as found in the literature; in the symmetric case see [Böhm et al. 1999], and in the braided case see [Alonso Álvarez et al. 2008a; 2008b]. Note property (15), which says that $r = \nu s$.

2.1. ***The endomorphism r and weak Hopf monoids.*** Define an endomorphism $r : A \to A$ by rotating the target morphism $t : A \to A$ by π, that is,



Since r is just t rotated by π, all the identities for t in Figure 2 rotated by π hold for r. We list some additional identities of r interacting with s and t.



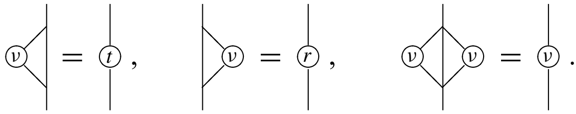
(12)

(13)

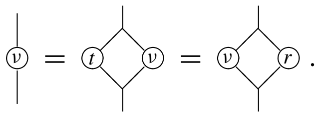
(14)

The proofs of these properties may also be found in Appendix B.

Definition 2.1. A weak bimonoid H is called a *weak Hopf monoid* if it is equipped with an endomorphism $\nu : H \to H$, called the *antipode*, satisfying



The axioms of a weak Hopf monoid immediately imply the identities



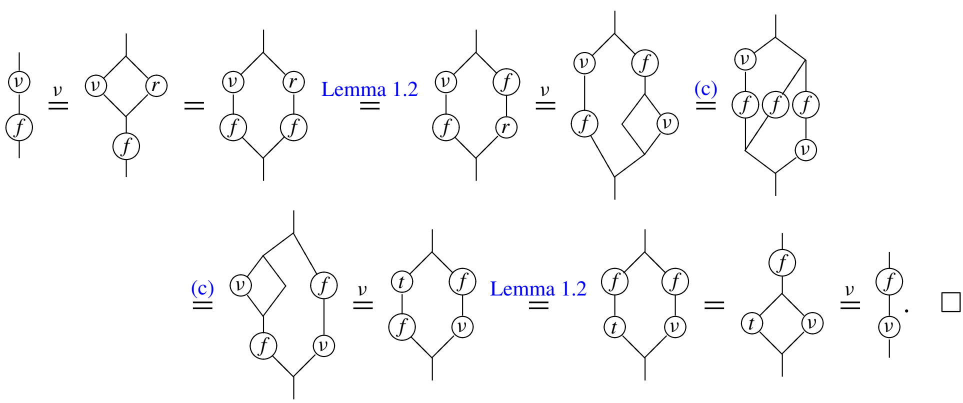
The antipode is unique since if ν' is another, then

$$\nu' = \nu' * 1 * \nu' = t * \nu' = \nu * 1 * \nu' = \nu * r = \nu * 1 * \nu = \nu.$$

Proposition 2.2 [Alonso Álvarez et al. 2003, Proposition 1.4]. *Suppose H and K are weak Hopf monoids in $\mathscr{V}$ and that $f : H \to K$ is both a monoid and comonoid morphism. Then f preserves the antipode, that is, $f\nu = \nu f$.*

The proof is also due to the authors of [Alonso Álvarez et al. 2003], where a similar proof may be found. We include it here for completeness.

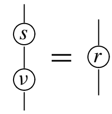
Proof. Recall from Lemma 1.2 that $tf = ft$, from which we may easily conclude that $rf = fr$. The proposition is then established by the calculation



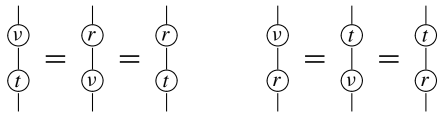
Therefore, if H and K are weak Hopf monoids in $\mathscr{V}$, then a *morphism of weak Hopf monoids* $f : H \to K$ is a morphism $f : H \to K$ in $\mathscr{V}$ that is a monoid and a comonoid morphism.

We list some properties of the antipode $\nu : H \to H$.

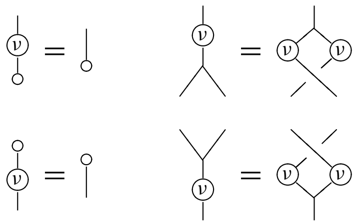
Proposition 2.3.



(15)



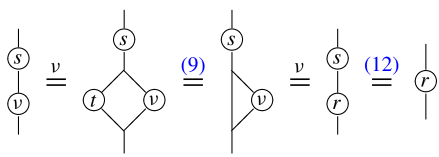
(16)



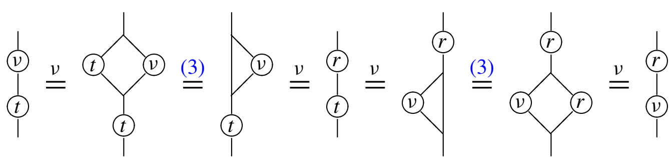
(17)

The last identity (17) states that $\nu : H \to H$ is both an anticomonoid morphism and an antimonoid morphism.

Proof. The calculation



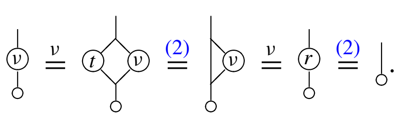
verifies the identity (15), and



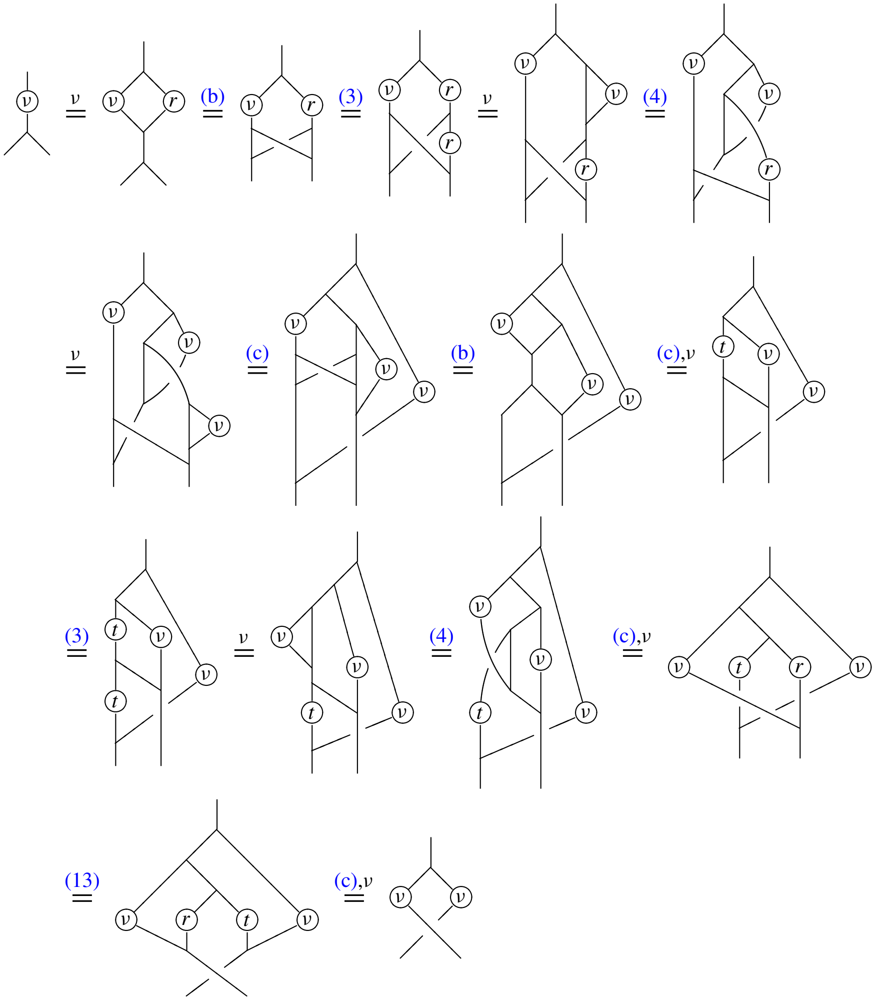
verifies the first identity of (16). The second follows from a similar calculation.

We prove the first two properties of (17), which show that ν is an anticomonoid morphism. The remaining two properties of (17), which show that ν is an antimonoid mophism, following from rotating the diagrams by π.

The proof of the counit property is easy enough:



The following calculation proves that the antipode is anticomultiplicative.



□

3. The monoidal category of A-comodules

Suppose $A = (A, 1)$ is a weak bimonoid in $\mathfrak{Q}\mathscr{V}$ and let $C = (A, t)$, which we recall is a separable Frobenius monoid. In this section we describe a monoidal structure on the categories **Bicomod**(C) of C-bicomodules in $\mathfrak{Q}\mathscr{V}$, and **Comod**(A) of right A-comodules in $\mathfrak{Q}\mathscr{V}$, such that the underlying functor

$$U : \mathbf{Comod}(A) \to \mathbf{Bicomod}(C)$$

is strong monoidal. If A is a weak Hopf monoid then we show that $\mathbf{Comod}_f(A)$, the subcategory consisting of the dualizable objects, is left autonomous.

This section is fairly standard in the $\mathcal{V} = \mathbf{Vect}$ case — see for example [Böhm and Szlachányi 2000; Nill 1998; Nikshych and Vainerman 2002] — and carries over rather straightforwardly to the general braided $\mathcal{V}$ case; see [Day et al. 2003].

3.1. ***The monoidal structure on C-bicomodules.*** Suppose, for this section, that idempotents split in $\mathcal{V}$, that $C \in \mathcal{V}$ is a separable Frobenius monoid, and that $M \in \mathcal{V}$ is a C-bicomodule with coaction

$$\gamma : M \to C \otimes M \otimes C.$$

A left C-coaction and a right C-coaction are obtained from γ by involving the counit ϵ as follows:

$$\gamma_l = \left(M \xrightarrow{\gamma} C \otimes M \otimes C \xrightarrow{1\otimes 1\otimes \epsilon} C \otimes M\right),$$

$$\gamma_r = \left(M \xrightarrow{\gamma} C \otimes M \otimes C \xrightarrow{\epsilon\otimes 1\otimes 1} M \otimes C\right).$$

Suppose now that N is another C-bicomodule. We now wish to define the tensor product of M and N. Before doing so we will need the following definition.

Definition 3.1. Let $f, g : X \to Y$ be a parallel pair in $\mathcal{V}$. This pair is called *cosplit* when there is an arrow $d : Y \to X$ such that

$$df = 1_X \quad \text{and} \quad fdg = gdg.$$

It is not hard to see that, in this case, $dg : X \to X$ is an idempotent and a splitting of dg, that is,

$$\begin{tikzcd} X \arrow[rr, "dg"] \arrow[dr, "x"'] & & X \\ & Q \arrow[ur, "y"'] & \end{tikzcd} \qquad \begin{tikzcd} Q \arrow[rr, "1"] \arrow[dr, "y"'] & & Q \\ & X \arrow[ur, "x"'] & \end{tikzcd}$$

provides an absolute equalizer (Q, y) for f and g.

Now M and N are C-bicomodules and we have two morphisms

$$\begin{tikzcd} M \otimes N \arrow[r, shift left, "\gamma_r\otimes 1"] \arrow[r, shift right, "1\otimes\gamma_l"'] & M \otimes C \otimes N. \end{tikzcd}$$

Proposition 3.2. *The pair $\gamma_r \otimes 1$ and $1 \otimes \gamma_l$ are cosplit by*

$$d = (1 \otimes \epsilon \otimes 1)(1 \otimes \mu \otimes 1)(\gamma_r \otimes 1 \otimes 1) : M \otimes C \otimes N \to M \otimes N.$$

Proof. Here we barely sketch the proof and note that we prove a very similar statement in greater detail in Proposition 3.3.

That $d(\gamma_r \otimes 1) = 1$ follows from the separable property of the Frobenius monoid, and $(\gamma_r \otimes 1)d(1 \otimes \gamma_l) = (1 \otimes \gamma_l)d(1 \otimes \gamma_l)$ follows from the Frobenius property. □

Thus, the equalizer of the two morphisms $\gamma_r \otimes 1$ and $1 \otimes \gamma_l$ is found by splitting the idempotent $d(1 \otimes \gamma_l)$, which is possible from our assumption that idempotents split in $\mathscr{V}$. So the equalizer exists and is absolute. This equalizer is then defined to be the tensor product of M and N over C, denoted $M \otimes_C N$.

That this defines a monoidal structure on the category $\mathbf{Bicomod}(C)$ with tensor product $\otimes_C$ and unit C is yet to be proved. However, we thought it better to write the next section more explicitly from which the details here may be filled in.

3.2. ***The tensor product of A-comodules.*** Let $A = (A, 1)$ be a weak bimonoid in $\mathscr{Q}\mathscr{V}$. Let $C = (A, t)$. It will be shown that the monoidal structure on the category of right A-comodules is $\otimes_C$, the tensor product over C, with unit C.

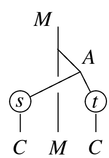
Suppose that M is a right A-comodule. We know that $s : A \to C^\circ$ and $t : A \to C$ are comonoid morphisms and that property (10) holds, recalling that property (10) expresses the commutativity of the diagram

$$\begin{tikzcd} & A \otimes A \arrow[r, "s\otimes t"] & C \otimes C \arrow[dd, "c"] \\ A \arrow[ur, "\delta"] \arrow[dr, "\delta"'] & & \\ & A \otimes A \arrow[r, "t\otimes s"] & C \otimes C. \end{tikzcd}$$

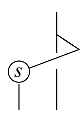
Therefore M may be made into a C-bicomodule via

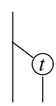
$$\gamma = \left(M \xrightarrow{\gamma} M \otimes A \xrightarrow{1\otimes\delta} M \otimes A \otimes A \xrightarrow{c^{-1}\otimes 1} A \otimes M \otimes A \xrightarrow{s\otimes 1\otimes t} C \otimes M \otimes C\right),$$

which is

$\gamma =$ 

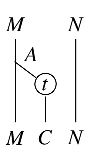
in strings. The left and right C-coactions are

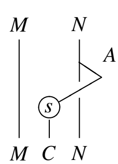
$\gamma_l =$ 

and $\gamma_r =$ 

.

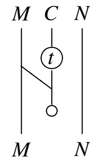
The tensor product of two A-comodules M and N over C then may be defined as in Section 3.1. We derive an explicit description of $M \otimes_C N$. Suppose M and N are A-comodules. Two morphisms $M \otimes N \to M \otimes C \otimes N$ are given as

$\gamma_r \otimes 1 =$ 

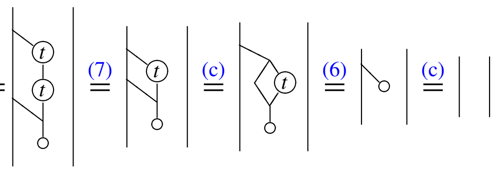
and $1 \otimes \gamma_l =$ 

.

Proposition 3.3. *The pair $\gamma_r \otimes 1$ and $1 \otimes \gamma_l$ are cosplit by*

$$d = $$  .

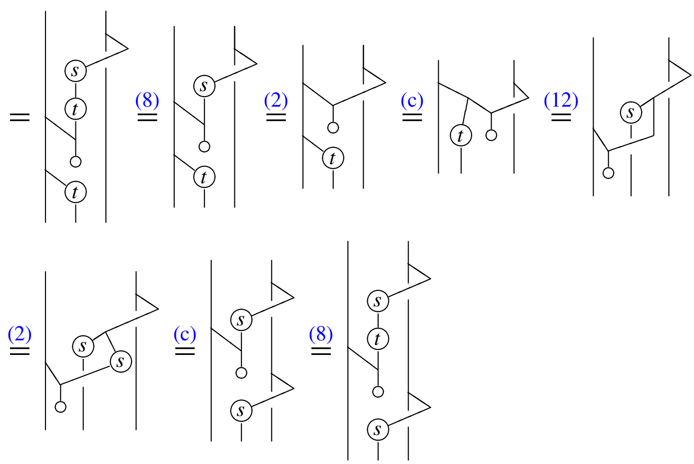
Proof. That d is a morphism in $\mathfrak{Q}\mathscr{V}$ follows immediately as t is idempotent. The calculation

$$d(\gamma_r \otimes 1) =$$  $$= 1_{M\otimes N}$$

shows that $d(\gamma_r \otimes 1) = 1$, and the identity

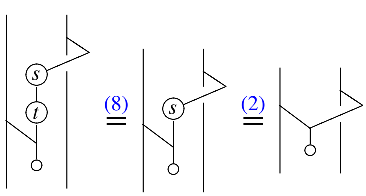
$$(\gamma_r \otimes 1)d(1 \otimes \gamma_l) = (1 \otimes \gamma_l)d(1 \otimes \gamma_l)$$

follows from

$$(\gamma_r \otimes 1)d(1 \otimes \gamma_l) =$$ 

$$= (1 \otimes \gamma_l)d(1 \otimes \gamma_l).$$ □

The idempotent $d(1 \otimes \gamma_l)$ will be denoted by m, which gains a simpler representation from the calculation

 $$= m.$$

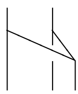
A splitting of m, that is,

$$\begin{tikzcd}[column sep=small] (M\otimes N,1) \arrow[rr,"m"] \arrow[dr,"m"'] & & (M\otimes N,1) \\ & (M\otimes N,m) \arrow[ur,"m"'] & \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] (M\otimes N,m) \arrow[rr,"m"] \arrow[dr,"m"'] & & (M\otimes N,m) \\ & (M\otimes N,1), \arrow[ur,"m"'] & \end{tikzcd}$$

provides an absolute equalizer $(M \otimes N, m)$ of $(\gamma_r \otimes 1)$ and $(1 \otimes \gamma_l)$. Thus, the tensor product of M and N over C is

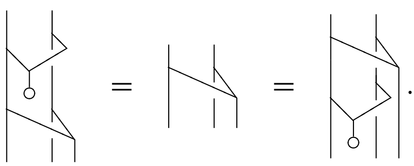
$$M \otimes_C N = (M \otimes N, m).$$

3.3. ***The coaction on the tensor product.*** If $\mathbf{Comod}(A)$ is to be a monoidal category with underlying functor $U : \mathbf{Comod}(A) \to \mathbf{Bicomod}(C)$ strong monoidal, then the tensor product of two A-comodules must also be an A-comodule. In this section we show that the obvious coaction on $M \otimes_C N$, namely,

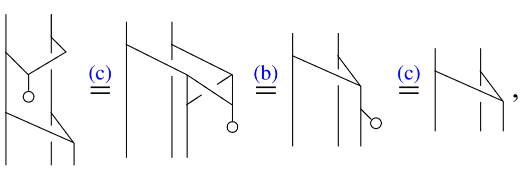
$\gamma =$  $: M \otimes_C N \to M \otimes_C N \otimes A,$

does the job.

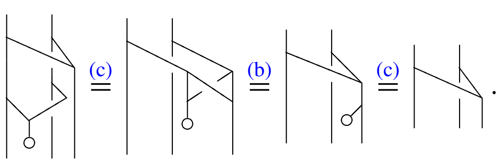
Lemma 3.4. *The coaction $\gamma : M \otimes_C N \to M \otimes_C N \otimes A$, as defined above, is a morphism in $\mathfrak{Q}\mathscr{V}$. That is,*



Proof. The first equality is given by



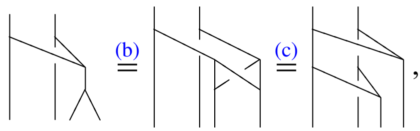
and the second by the similar calculation:



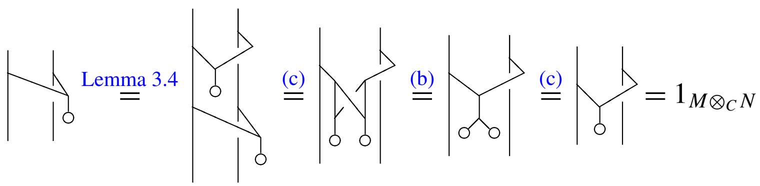
□

Proposition 3.5. *$(M \otimes_C N, \gamma)$ is an A-comodule.*

Proof. Coassociativity is proved as usual, by



and the counit condition is proved as



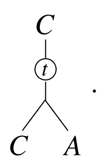
□

3.4. Comod(A) *is a monoidal category.* We now set out to prove the claim, at the beginning of this section, that $(\mathbf{Comod}(A), \otimes_C, C)$ is a monoidal category. It will turn out that associativity is a strict equality (if it is so in $\mathscr{V}$) and the unit conditions are only up to isomorphism.

We state this as a theorem and devote the remainder of this section to its proof.

Theorem 3.6. $\mathbf{Comod}(A) = (\mathbf{Comod}(A), \otimes_C, C)$ *is a monoidal category.*

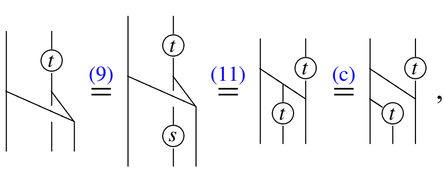
First note that C itself is an A-comodule with coaction



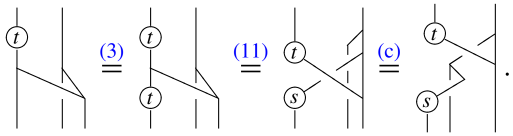
The following lemma will be useful.

Lemma 3.7. *The following identities hold.*

Proof. The first identity is proved by

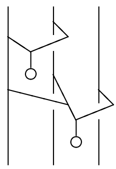
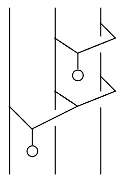


and the second is proved by

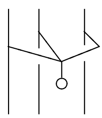


□

Proof of Theorem 3.6. Consider $(M \otimes_C N) \otimes_C P$ and $M \otimes_C (N \otimes_C P)$ in $\mathfrak{Q}\mathscr{V}$. The former is $(M \otimes N \otimes P, u)$ and the latter is $(M \otimes N \otimes P, v)$, where

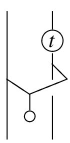
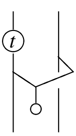
$u =$  and $v =$  .

Since, by Lemma 3.4, γ is a morphism in $\mathfrak{Q}\mathscr{V}$, both u and v may be rewritten as

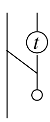
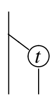
 ,

proving the (strict) equality $(M \otimes_C N) \otimes_C P = M \otimes_C (N \otimes_C P)$ in $\mathfrak{Q}\mathscr{V}$ (since we are writing as if $\mathscr{V}$ were strict).

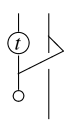
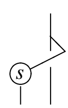
It remains to prove $M \otimes_C C \cong M \cong C \otimes_C M$. By definition

$M \otimes_C C = (M \otimes C,$  $)$ and $C \otimes_C M = (C \otimes M,$  $)$.

We will show that the morphisms

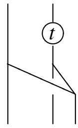
 $: M \otimes_C C \to M$ and  $: M \to M \otimes_C C$

will establish the isomorphism $M \otimes_C C \cong M$, and

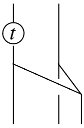
 $: C \otimes_C M \to M$ and  $: M \to C \otimes_C M$

will establish the isomorphism $M \cong C \otimes_C M$. These morphisms are easily seen to be in $\mathfrak{Q}\mathscr{V}$, and the fact that they are mutually inverse pairs is given in one direction by Lemma 3.7, and in the other by an easy string calculation making use of the identity (6) in Figure 1 or (6) in Figure 2.

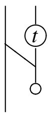
It now remains to show that these four morphisms are A-comodules morphisms, that is, that they are in **Comod**(A). Note that $M \otimes_C C$ and $C \otimes_C M$ are A-comodules via the coactions



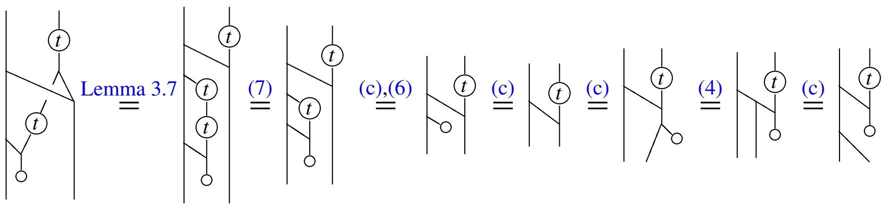
and

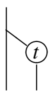


respectively. We then have these facts:

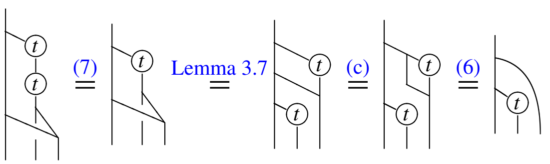
- 

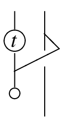
: $M \otimes_C C \to M$ is an A-comodule morphism since



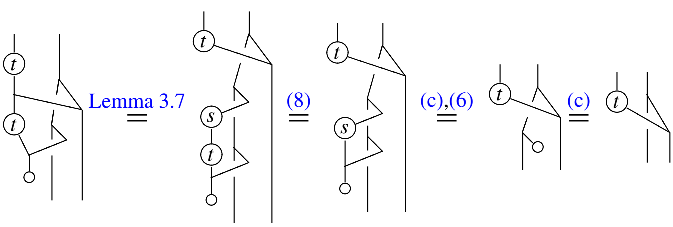
- 

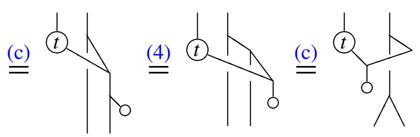
: $M \to M \otimes_C C$ is an A-comodule morphism since

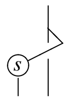


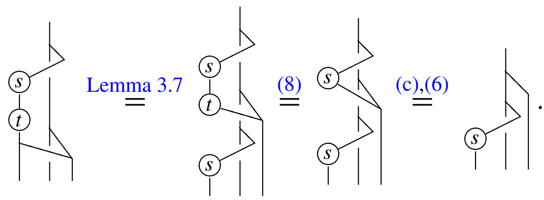
- 

: $C \otimes_C M \to M$ is an A-comodule morphism since





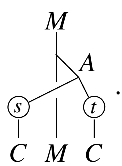
•  : $M \to C \otimes_C M$ is an A-comodule morphism since



Thus, $M \otimes_C C \cong M \cong C \otimes_C M$ in $\mathfrak{Q}\mathscr{V}$. □

Thus, $\mathbf{Comod}(A) = (\mathbf{Comod}(A), \otimes_C, C)$ is a monoidal category.

3.5. ***The forgetful functor from A-comodules to C-bicomodules.*** There is a forgetful functor $U : \mathbf{Comod}(A) \to \mathbf{Bicomod}(C)$ that assigns to each A-comodule M a C-bicomodule UM that is M itself with coaction



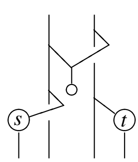
Obviously a morphism of A-comodules $f : M \to N$ is automatically a morphism of the underlying C-bicomodules $f : UM \to UN$.

Proposition 3.8. *The forgetful functor* $U : \mathbf{Comod}(A) \to \mathbf{Bicomod}(C)$ *is strong monoidal.*

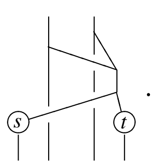
Proof. We must establish the C-bicomodule isomorphisms

$$C \cong UC \quad \text{and} \quad UM \otimes_C UN \cong U(M \otimes_C N).$$

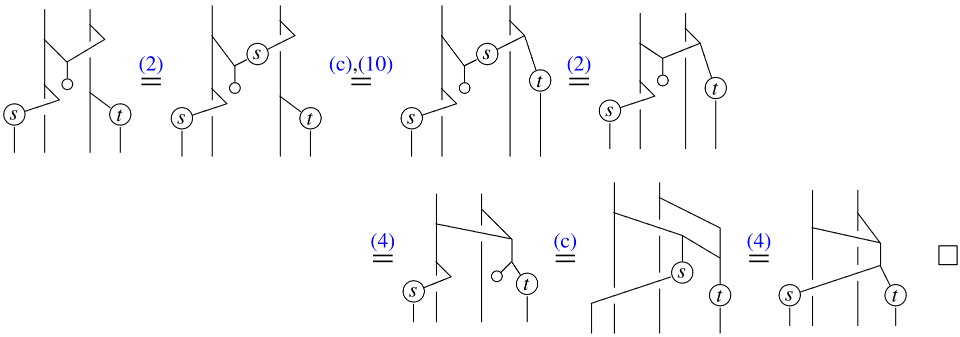
The first is obvious. To establish the second we observe that the object $UM \otimes_C UN$ is $(M \otimes_C N, m)$ with coaction



and $U(M \otimes_C N)$ is also $(M \otimes_C N, m)$ but with coaction



The following calculation shows that these two coactions are the same, and hence the isomorphism $U(M\otimes_C N)\cong UM\otimes_C UN$.



This may seem to be a strict equality, but as tensor products are really only defined up to isomorphism, we prefer "strong".

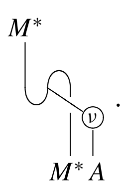
3.6. $\mathbf{Comod}_f(H)$ *is left autonomous.* Let $\mathscr{V}_f$ denote the subcategory of $\mathscr{V}$ consisting of the objects with a left dual (since $\mathscr{V}$ is braided, left duals are right duals), and suppose that H is a weak Hopf monoid. There is a forgetful functor $U_l : \mathbf{Comod}(H)\to\mathscr{V}$ defined as the composite of the two forgetful functors $\mathbf{Comod}(H)\to\mathbf{Bicomod}(C)$ and $\mathbf{Bicomod}(C)\to\mathscr{V}$. Sometimes this composite $U_l : \mathbf{Comod}(H)\to\mathscr{V}$ is called the *long forgetful functor*, as opposed to the *short forgetful functor* $U : \mathbf{Comod}(H)\to\mathbf{Bicomod}(C)$.

Let us say an object $M\in\mathbf{Comod}(H)$ is *dualizable* if U_lM has a left dual in $\mathscr{V}$, that is, $U_lM\in\mathscr{V}_f$. Denote by $\mathbf{Comod}_f(H)$ the subcategory of $\mathbf{Comod}(H)$ consisting of the dualizable objects.

The goal of this section is to prove the following proposition.

Proposition 3.9. *If H is a weak Hopf monoid, then the category $\mathbf{Comod}_f(H)$ is left autonomous (= left compact = left rigid).*

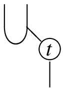
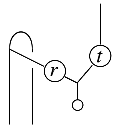
Suppose $M\in\mathbf{Comod}_f(H)$ has a left dual M^* in $\mathscr{V}$. Using the antipode of H, a coaction on M^* is defined as



.

By (17) it is easy to see that this defines a comodule structure on M^*. We claim that M^* is the left dual of M in $\mathbf{Comod}_f(H)$. Define morphisms $e : M^*\otimes_C M\to C$

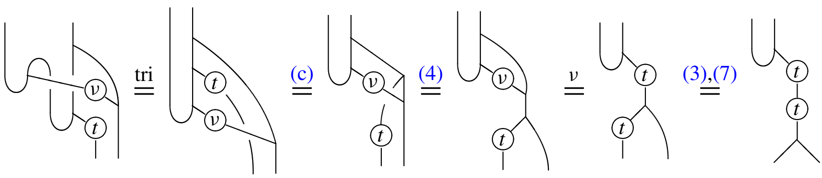
and $n : C \to M \otimes_C M^*$ via

$e =$  and $n =$  .

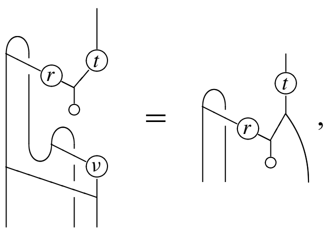
Proposition 3.10. *Suppose $M \in$ **Comod**$_f(H)$ with underlying left dual M^*. Then M^* with evaluation and coevaluation morphisms e and n respectively is the left dual of M in* **Comod**$_f(H)$*. That is,* **Comod**$_f(H)$ *is left autonomous.*

Proof. Let M, M^*, e, and n be as above. We will first show that e and n are comodule morphisms, and secondly that they satisfy the triangle identities.

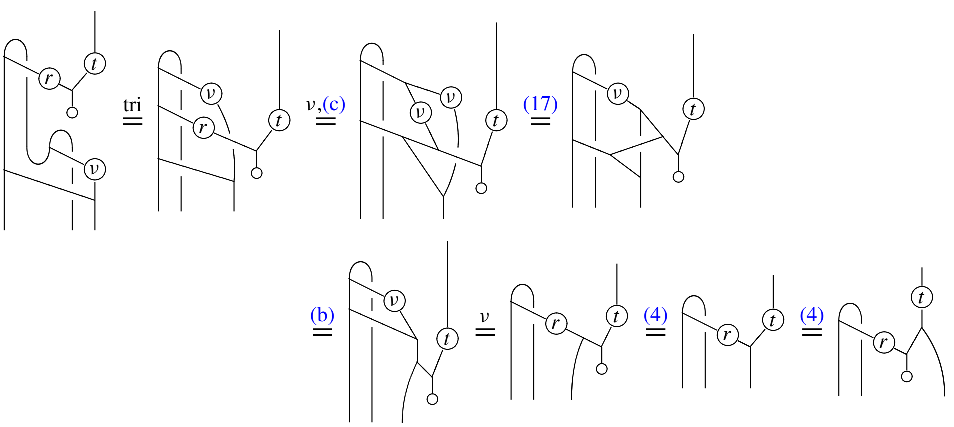
The following calculation shows that e is a comodule morphism.



To show that n is a comodule morphism, we must establish the equality



which is proved by the calculation

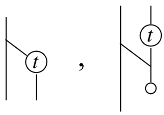


It remains to show that e and n satisfy the triangle identities, that is, that the following composites are the identity:

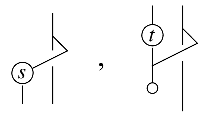
(i) $M \cong C \otimes_C M \xrightarrow{n\otimes 1} M \otimes_C M^* \otimes_C M \xrightarrow{1\otimes e} M \otimes_C C \cong M$;

(ii) $M^* \cong M^* \otimes_C C \xrightarrow{1\otimes n} M^* \otimes_C M \otimes_C M^* \xrightarrow{e\otimes 1} C \otimes_C M^* \cong M^*$.

Recall that $M \cong M \otimes_C C$ and $M \cong C \otimes_C M$ via

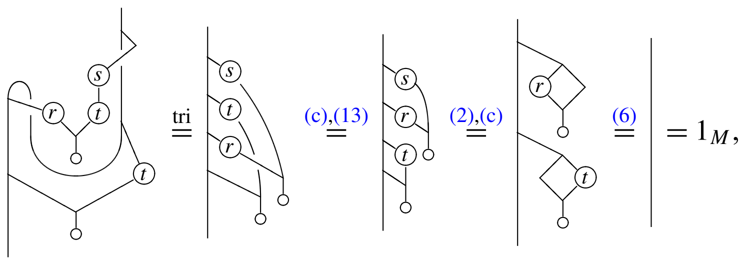


and

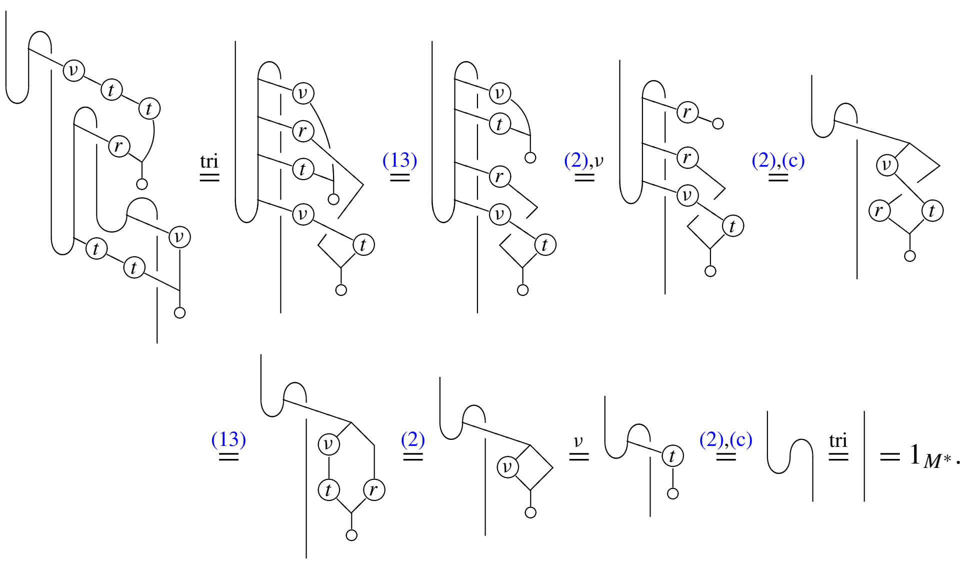


respectively.

The calculation



proves (i), and (ii) is given by

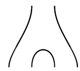


This completes the proof that M^* is the left dual of M in $\mathbf{Comod}_f(H)$, and hence that $\mathbf{Comod}_f(H)$ is left autonomous. $\square$

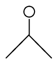
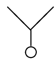
4. Frobenius monoid example

Let R be a separable Frobenius monoid in $\mathscr{V}$. In this section we prove that $R \otimes R$ is an example of a weak Hopf monoid with an invertible antipode. In the case $\mathscr{V} = \mathbf{Vect}$, this example is essentially the same as in [Böhm et al. 1999, Appendix].

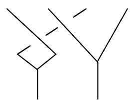
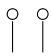
Let R be a Frobenius monoid in $\mathscr{V}$. Then $R \otimes R$ becomes a comonoid via

$\delta =$  and $\epsilon =$  .

where, for simplicity, in this section we will adopt the simpler notation

 $=$  and  $=$  ,

and $R \otimes R$ becomes a monoid via

$\mu =$  and $\eta =$  .

The comonoid structure is via the comonad generated by the adjunction $R \dashv R$. The monoid structure is the usual monoid structure (viewing R as a monoid) on the tensor product $R^\circ \otimes R$, where R° is the opposite monoid of R.

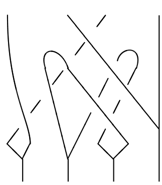
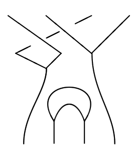
Proposition 4.1. *If R is separable, meaning $\mu\delta = 1_R$, then $R \otimes R$ is a weak bimonoid. An invertible antipode ν on $R \otimes R$ is given by*

$\nu =$  ,

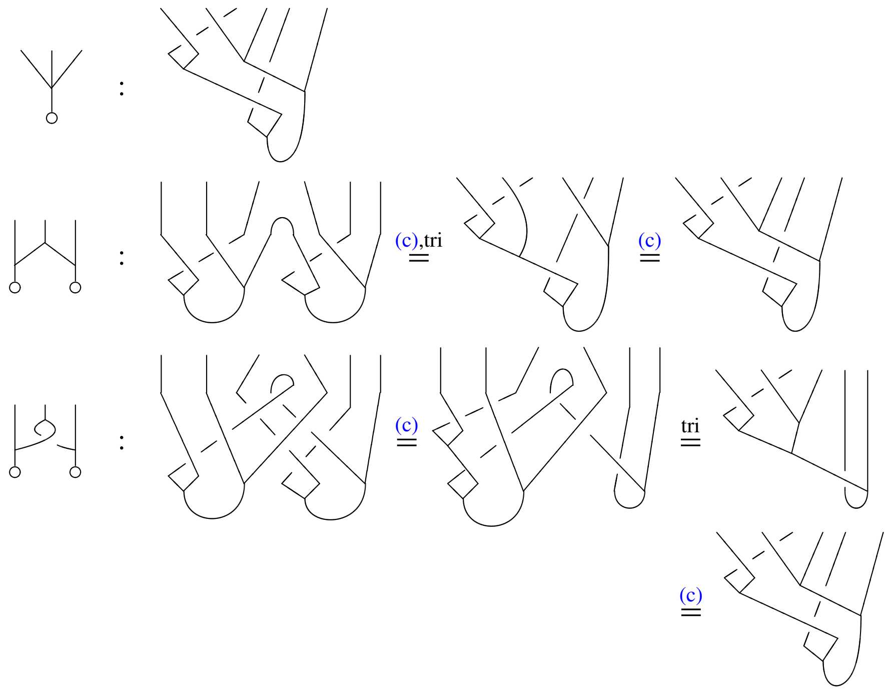
which makes $R \otimes R$ into a weak Hopf monoid.

The next three sets of calculations establish the axioms (b), (v), and (w), and hence the first claim.

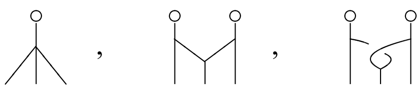
Axiom (b) is given by

$(\mu \otimes \mu)(1 \otimes c \otimes 1)(\delta \otimes \delta) =$  $\overset{\text{nat}}{=}$  $\overset{\text{sep}}{=}$  $= \delta\mu.$

Axiom (v) is seen from the diagrams



For (w), by the naturality of the braiding and the counit property of R, each equation in (w), that is,



is easily seen to be equal to the diagram

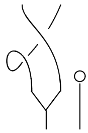
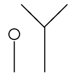
.

Thus, $R \otimes R$ is a weak bimonoid. We next prove that $R \otimes R$ is a weak Hopf monoid with invertible antipode

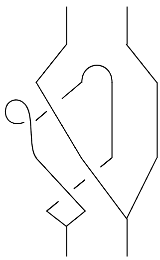
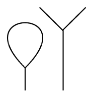
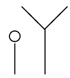
$\nu =$ .

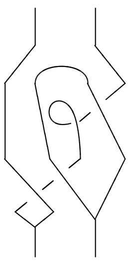
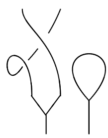
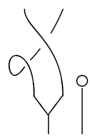
An inverse to ν is easily seen to be given by

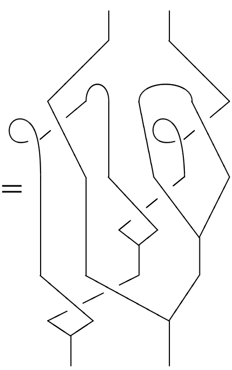
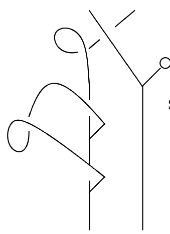
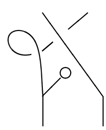
$\nu^{-1} =$ , and so the antipode is invertible. We note that (in simplified form)

$r =$  and $t =$  .

The following calculations then prove the antipode axioms.

$\mu(\nu \otimes 1)\delta =$  $\overset{\text{tri}}{=}$  $\overset{\text{sep}}{=}$  $= t$

$\mu(1 \otimes \nu)\delta =$  $\overset{\text{nat}}{=}$  $\overset{\text{sep}}{=}$  $= r$

$\mu_3(\nu \otimes 1 \otimes \nu)\delta_3 =$  $\overset{\text{sep}}{=}$  $\overset{\text{sep}}{=}$  $\overset{\text{(c)}}{=}$  $= \nu$

Thus, $R \otimes R$ is a weak Hopf monoid with invertible antipode.

5. Quantum groupoids

In this section we recall the quantum categories and quantum groupoids of Day and Street [2004], where there is a succinct definition on [page 216] in terms of "basic data" and "Hopf basic data". Here we give the unpacked definition of quantum category and quantum groupoid which is essentially found in [page 221]; however, we do make a correction.

Our setting is a braided monoidal category $\mathscr{V} = (\mathscr{V}, \otimes, I, c)$ in which the functors

$$A \otimes - : \mathscr{V} \to \mathscr{V}$$

with $A \in \mathscr{V}$, preserve coreflexive equalizers, that is, equalizers of pairs of morphisms with a common left inverse.

5.1. *Quantum categories.* Suppose A and C are comonoids in $\mathscr{V}$ and $s : A \to C^{\circ}$ and $t : A \to C$ are comonoid morphisms such that the diagram

$$\begin{tikzcd}
 & A \otimes A \arrow[r, "s\otimes t"] & C \otimes C \arrow[dd, "c"] \\
A \arrow[ur, "\delta"] \arrow[dr, "\delta"'] & & \\
 & A \otimes A \arrow[r, "t\otimes s"] & C \otimes C
\end{tikzcd}$$

commutes. Then A may be viewed as a C-bicomodule with left and right coactions defined respectively via

$$\gamma_l = \left(A \xrightarrow{\delta} A \otimes A \xrightarrow{1\otimes s} A \otimes C \xrightarrow{c^{-1}} C \otimes A\right),$$

$$\gamma_r = \left(A \xrightarrow{\delta} A \otimes A \xrightarrow{1\otimes t} A \otimes C\right).$$

Recall that the tensor product $P = A \otimes_C A$ of A with itself over C is defined as the equalizer

$$P \xrightarrow{\iota} A \otimes A \underset{1\otimes\gamma_l}{\overset{\gamma_r\otimes 1}{\rightrightarrows}} A \otimes C \otimes A\ .$$

The diagrams

$$P \xrightarrow{\iota} A \otimes A \xrightarrow{\gamma_l\otimes 1} C \otimes A \otimes A \underset{1\otimes 1\otimes\gamma_l}{\overset{1\otimes\gamma_r\otimes 1}{\rightrightarrows}} C \otimes A \otimes C \otimes A,$$

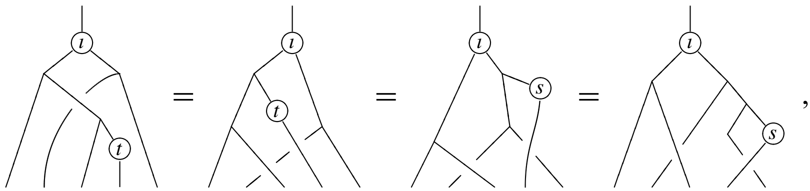
$$P \xrightarrow{\iota} A \otimes A \xrightarrow{1\otimes\gamma_r} A \otimes A \otimes C \underset{1\otimes\gamma_l\otimes 1}{\overset{\gamma_r\otimes 1\otimes 1}{\rightrightarrows}} A \otimes C \otimes A \otimes C$$

may be seen to commute and therefore induce respectively a left C- and right C-coaction on P. These coactions make P into a C-bicomodule.

The commutativity of the diagram

$$P \xrightarrow{\iota} A \otimes A \xrightarrow{\delta\otimes\delta} A^{\otimes 4} \xrightarrow{1\otimes c\otimes 1} A^{\otimes 4} \underset{1\otimes 1\otimes 1\otimes\gamma_l}{\overset{1\otimes 1\otimes\gamma_r\otimes 1}{\rightrightarrows}} A \otimes A \otimes A \otimes C \otimes A$$

may be seen from



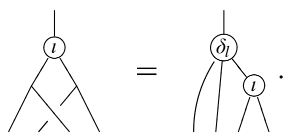
and since $1 \otimes 1 \otimes \iota$ is the equalizer of $1 \otimes 1 \otimes \gamma_r \otimes 1$ and $1 \otimes 1 \otimes 1 \otimes \gamma_l$, there is a unique morphism

$$\delta_l : P \to A \otimes A \otimes P$$

making the diagram

$$\begin{tikzcd}
P \arrow[r, "\iota"] \arrow[d, "\delta_l"'] & A \otimes A \arrow[r, "\delta\otimes\delta"] & A \otimes A \otimes A \otimes A \arrow[d, "1\otimes c\otimes 1"] \\
A \otimes A \otimes P \arrow[rr, "1\otimes 1\otimes\iota"'] & & A \otimes A \otimes A \otimes A
\end{tikzcd}$$

commute. In strings,



It is easy to see (postcompose with the monomorphism $1 \otimes 1 \otimes 1 \otimes 1 \otimes \iota$) that the morphism δ_l is the left coaction of the comonoid $A \otimes A$ on P that makes P into a (left) $A \otimes A$-comodule. This means that the diagrams

$$\begin{tikzcd}
P \arrow[rr, "\delta_l"] \arrow[d, "\delta_l"'] & & A \otimes A \otimes P \arrow[dd, "1\otimes 1\otimes\delta_l"] \\
A \otimes A \otimes P \arrow[d, "\delta\otimes\delta\otimes 1"'] & & \\
A \otimes A \otimes A \otimes A \otimes P \arrow[rr, "1\otimes c\otimes 1\otimes 1"] & & A \otimes A \otimes A \otimes A \otimes P
\end{tikzcd}
\qquad
\begin{tikzcd}
P \arrow[r, "\delta_l"] \arrow[dr, "1"'] & A \otimes A \otimes P \arrow[d, "\epsilon\otimes\epsilon\otimes 1"] \\
& P
\end{tikzcd}$$

commute.

5.2. ***The definition.*** We are now ready to state the definition. A *quantum category* in $\mathscr{V}$ consists of the data $\mathbf{A} = (A, C, s, t, \mu, \eta)$ where A, C, s, t are as above, and $\mu : P = A \otimes_C A \to A$ and $\eta : C \to A$ are morphisms in $\mathscr{V}$, called the *composition morphism* and *unit morphism*, respectively. This data must satisfy axioms (B1) through (B6) below.

(B1) (A, μ, η) is a monoid in $\mathbf{Bicomod}(C)$.

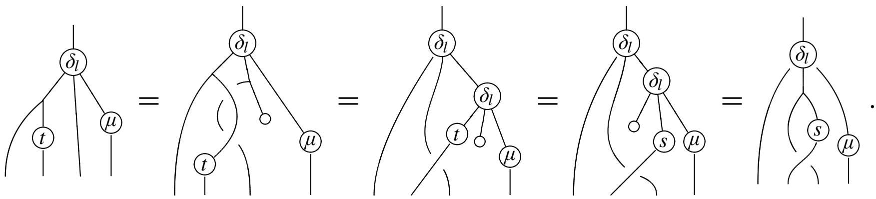
(B2) The following diagram commutes.

$$P \xrightarrow{\delta_l} A \otimes A \otimes P \overset{t\otimes\epsilon\otimes 1}{\underset{\epsilon\otimes s\otimes 1}{\rightrightarrows}} C \otimes P \xrightarrow{1\otimes\mu} C \otimes A$$

Before stating (B3), we use (B2) to show that the diagram

$$P \xrightarrow{\delta_l} A \otimes A \otimes P \xrightarrow{1\otimes 1\otimes\mu} A \otimes A \otimes A \overset{\gamma_r\otimes 1\otimes 1}{\underset{1\otimes\gamma_l\otimes 1}{\rightrightarrows}} A \otimes C \otimes A \otimes A$$

commutes, as seen by the calculation



Since $\iota \otimes 1$ is the equalizer of $\gamma_r \otimes 1 \otimes 1$ and $1 \otimes \gamma_l \otimes 1$, there is a unique morphism $\delta_r : P \to P \otimes A$ making the square

$$\begin{array}{ccc} P & \xrightarrow{\delta_l} & A \otimes A \otimes P \\ {\scriptstyle\delta_r}\downarrow & & \downarrow{\scriptstyle 1\otimes 1\otimes\mu} \\ P \otimes A & \xrightarrow{\iota\otimes 1} & A \otimes A \otimes A \end{array}$$

commute. We can now state (B3).

(B3) The following diagram commutes.

$$\begin{array}{ccc} P & \xrightarrow{\mu} & A \\ {\scriptstyle\delta_r}\downarrow & & \downarrow{\scriptstyle\delta} \\ P \otimes A & \xrightarrow{\mu\otimes 1} & A \otimes A \end{array}$$

(B4) The following diagram commutes.

$$\begin{array}{ccc} P & \xrightarrow{\mu} & A \\ {\scriptstyle\iota}\downarrow & & \downarrow{\scriptstyle\epsilon} \\ A \otimes A & \xrightarrow{\epsilon\otimes\epsilon} & I \end{array}$$

(B5) The following diagram commutes.

$$\begin{tikzcd} C \arrow[dr, "\epsilon"] \arrow[dd, "\eta"'] & \\ & I \\ A \arrow[ur, "\epsilon"'] & \end{tikzcd}$$

(B6) The following diagram commutes.

$$\begin{tikzcd} & A \arrow[r, "\delta"] & A\otimes A \arrow[r, "s\otimes 1"] & C\otimes A \arrow[dr, "\eta\otimes 1"] & \\ C \arrow[ur, "\eta"] \arrow[rr, "\eta"] \arrow[dr, "\eta"'] & & A \arrow[rr, "\delta"] & & A\otimes A \\ & A \arrow[r, "\delta"] & A\otimes A \arrow[r, "t\otimes 1"] & C\otimes A \arrow[ur, "\eta\otimes 1"'] & \end{tikzcd}$$

A consequence of these axioms is that P becomes a left $A\otimes A$-, right A-bicomodule.

The axiom (B6) makes C into a right A-comodule via

$$C \xrightarrow{\eta} A \xrightarrow{\delta} A\otimes A \xrightarrow{s\otimes 1} C\otimes A\ .$$

We refer to A as the *object-of-arrows* and C as the *object-of-objects*.

5.3. *Quantum groupoids.* Suppose we have comonoid isomorphisms

$$\upsilon : C^{\circ\circ} \xrightarrow{\cong} C \quad \text{and} \quad \nu : A^{\circ} \xrightarrow{\cong} A\ .$$

Denote by P_l the left $A^{\otimes 3}$-comodule P with coaction defined by

$$P \xrightarrow{\delta} A\otimes A\otimes P\otimes A \xrightarrow{1\otimes 1\otimes 1\otimes \nu} A\otimes A\otimes P\otimes A \xrightarrow{1\otimes 1\otimes c_{P,A}} A\otimes A\otimes A\otimes P\ ,$$

and by P_r the left $A^{\otimes 3}$-comodule P with coaction defined by

$$P \xrightarrow{\delta} A\otimes A\otimes P\otimes A \xrightarrow{1\otimes 1\otimes 1\otimes \nu^{-1}} A\otimes A\otimes P\otimes A \xrightarrow{c^{-1}_{A\otimes A\otimes P,A}} A\otimes A\otimes A\otimes P\ .$$

Furthermore, suppose that $\theta : P_l \to P_r$ is a left $A^{\otimes 3}$-comodule isomorphism. We define a *quantum groupoid* in $\mathscr{V}$ to be a quantum category $\mathbf{A}$ in $\mathscr{V}$ equipped with an υ, ν, and θ satisfying (G1) through (G3) below.

(G1) $s\nu = t$,

(G2) $t\nu = \upsilon s$, and

(G3) the diagram[3]

$$\begin{tikzcd} P \arrow[r, "\varsigma"] \arrow[d, "\theta"'] & C\otimes C\otimes C \arrow[r, "c_{C,C\otimes C}"] & C\otimes C\otimes C \arrow[d, "1\otimes 1\otimes \upsilon"] \\ P \arrow[rr, "\varsigma"] & & C\otimes C\otimes C \end{tikzcd}$$

[3]This corrects [Day and Street 2004, Section 12, page 223].

commutes, where the morphism $\varsigma : P \to C^{\otimes 3}$ is defined by taking either of the equal routes

$$P \xrightarrow{\iota} A \otimes A \underset{1\otimes\gamma_l}{\overset{\gamma_r\otimes 1}{\rightrightarrows}} A \otimes C \otimes A \xrightarrow{s\otimes 1\otimes t} C^{\otimes 3}.$$

6. Weak Hopf monoids and quantum groupoids

The goal of this section is to prove the following theorem.

Theorem 6.1. *There is a bijection in $\mathbb{Q}\mathscr{V}$ between weak bimonoids and quantum categories with separable Frobenius object-of-objects. Also, if the weak bimonoid is equipped with an invertible antipode, making it a weak Hopf monoid, then the quantum category becomes a quantum groupoid.*

In the case $\mathscr{V} = \mathbf{Vect}$, it has been shown in [Brzeziński and Militaru 2002, Proposition 5.2] that a weak Hopf monoid with invertible antipode yields a quantum groupoid.

Concerning the converse, we would like to warmly thank Gabriella Böhm for not only suggesting that it may be true, and pointing out that the $\mathscr{V} = \mathbf{Vect}$ case appears in the PhD thesis of Imre Bálint [2008a], but for also helping us with the proof.

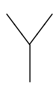
The theorem has an obvious corollary.

Corollary 6.2. *Any Frobenius monoid in $\mathfrak{Q}\mathscr{V}$ yields a quantum groupoid in $\mathfrak{Q}\mathscr{V}$.*

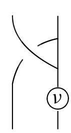
Proof. By Proposition 4.1, every Frobenius monoid R in $\mathfrak{Q}\mathscr{V}$ leads to a weak Hopf monoid with invertible antipode $R \otimes R$. Apply Theorem 6.1 to this weak Hopf monoid with invertible antipode to get a quantum groupoid in $\mathfrak{Q}\mathscr{V}$. □

Now let us discuss the necessary data for the theorem. The proof involves many string calculations, which may be found in Sections 6.1 and 6.2.

Suppose $A = (A, 1)$ is a weak bimonoid in $\mathfrak{Q}\mathscr{V}$ with source morphism s and target morphism t, and put $C = (A, t)$. Our claim is that this data along with

$\mu =$  $: P \to A, \quad \eta = t : C \to A$

forms a quantum category in $\mathfrak{Q}\mathscr{V}$. If moreover A is a weak Hopf monoid in $\mathfrak{Q}\mathscr{V}$ with an invertible antipode $\nu : A \to A$, then setting

$\upsilon = t\nu\nu t : C^{\circ\circ} \to C, \quad \nu = \nu : A^{\circ} \to A, \quad \theta =$  $: P \to P.$

yields a quantum groupoid in $\mathfrak{Q}\mathscr{V}$. The details will be proved in Section 6.1.

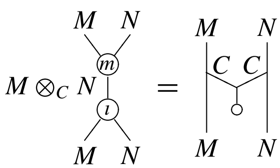
For the other direction, recall from Section 3.1 that, if C is a separable Frobenius monoid and M and N are C-comodules, we may form $M \otimes_C N$, the tensor product over C of M and N. Moreover, the tensor product over C is a retract of the tensor product in $\mathscr{V}$ so that

$$M \otimes_C N \xrightarrow{\iota} M \otimes N$$

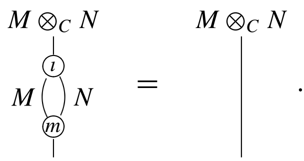
has a retraction $m : M \otimes N \to M \otimes_C N$. Again, from Section 3.1, we see that we may explicitly write ιm as

$$\iota m = \big(M \otimes N \xrightarrow{\delta_r \otimes \delta_l} M \otimes C \otimes C \otimes N \xrightarrow{1 \otimes \mu \otimes 1} M \otimes C \otimes N \xrightarrow{1 \otimes \epsilon \otimes 1} M \otimes N\big).$$

Graphically,

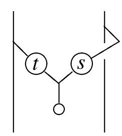


and



(@)

Now suppose $\mathbf{A} = (A, C, s, t, \mu, \eta)$ is a quantum category in $\mathscr{V}$, in which $C = (C, \mu, \eta, \delta, \epsilon)$ is a separable Frobenius monoid. The comonoid A is therefore a C-bicomodule, so that $\iota : P \to A \otimes A$ has a retraction $m : A \otimes A \to P$. This is such that

$\iota m =$ 

and $\quad m\iota = 1_P$.

If we then define a multiplication and unit for A as

$$\mu = \ (A \otimes A \xrightarrow{m} P \xrightarrow{\mu} A)\ , \quad \eta = \ (I \xrightarrow{\eta} C \xrightarrow{\eta} A),$$

where $\eta : I \to C$ comes from the fact that C is a Frobenius module, then we have the data for a weak bimonoid A. That this is actually a weak bimonoid will be proved in Section 6.2.

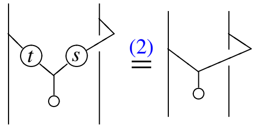
Let us see that this correspondence between weak bimonoids and quantum categories with separable Frobenius object-of-objects is one-to-one. Suppose we have a weak bimonoid $A = (A, \mu, \eta, \delta, \epsilon)$ in $\mathfrak{Q}\mathscr{V}$. It becomes a quantum category in $\mathfrak{Q}\mathscr{V}$ by setting

$$C := (A, t), \quad s := s, \quad t := t, \quad \eta := t, \quad \text{and} \quad \mu := \mu.$$

This quantum category then becomes a weak bimonoid by setting

$$\mu := \mu \circ m \quad \text{and} \quad \eta := t \circ \eta$$

where, in this case,

$m =$ 

and, as we see from the first paragraph of Section 6.1, $\mu \circ m = \mu$, and moreover, the morphism $t \circ \eta = \eta$ by axiom (2) for weak bimonoids. Thus, we have ended up with the weak bimonoid A that we started with.

Let us now go in the other direction. Suppose that we have a quantum category $\mathbf{A} = (A, C, s, t, \mu, \eta)$ with (A, δ, ϵ) a comonoid, and $C = (C, \mu_C, \eta_C, \delta_C, \epsilon_C)$ a separable Frobenius monoid. Then A becomes a weak bimonoid with

$$\mu' := \mu \circ m, \quad \text{where } m : A \otimes A \to P, \quad \text{and} \quad \eta' := \eta \circ \eta_C.$$

We note that the source and target morphisms for the weak bimonoid are given by

$$s' := \eta \circ s \quad \text{and} \quad t' := \eta \circ t.$$

First, we observe that $C \cong (A, t)$ and $P \cong (A \otimes A, \iota m)$ respectively via

$$\begin{tikzcd} (A, t') \arrow[r, "t"'] & (C, 1) \arrow[l, bend right, "\eta"'] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} (P, 1) \arrow[r, "\iota"'] & (A \otimes A, \iota m). \arrow[l, bend right, "m"'] \end{tikzcd}$$

The first isomorphism is established in one direction by definition and in the other by

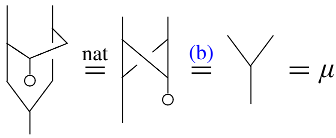
$$\begin{aligned} t\eta &= (1 \otimes t)(\epsilon \otimes 1)\delta\eta \\ &= (\epsilon \otimes 1)(\eta \otimes 1)\delta_C && \text{since } \eta \text{ is a } C\text{-comodule morphism} \\ &= (\epsilon_C \otimes 1)\delta_C && \text{by (B5)} \\ &= 1_C. \end{aligned}$$

The second isomorphism is again established in one direction by definition and in the other by the fact that m is a retract of ι. This weak bimonoid becomes a quantum category by stripping off the m from μ' so that we are left with the original μ. Since t' is a morphism from (A, t) to $(A, 1)$, if we wish to consider it as a morphism from C to A, we must precompose with $\eta : (C, 1) \to (A, t)$. This gives $t' \circ \eta = \eta \circ t \circ \eta = \eta$, and so we are left with the original η. We have already seen that $C \cong (A, t)$, so we are left with our original quantum category.

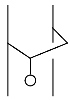
This establishes the bijection one-to-one correspondence between quantum categories and weak bimonoids, and moreover shows how to construct a quantum groupoid from a weak Hopf monoid.

The remainder of this section is devoted to proving the details of the theorem.

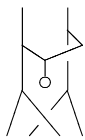
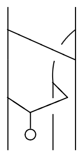
6.1. ***Weak bimonoids yield quantum categories.*** In this section, we prove that a weak bimonoid in $\mathfrak{Q}\mathcal{V}$ yields a quantum category in $\mathfrak{Q}\mathcal{V}$. Suppose that $A = (A, 1)$ is a weak bimonoid with source and target morphisms s and t, respectively. Set $C = (A, t)$ as usual. The morphisms s and t are obviously in $\mathfrak{Q}\mathcal{V}$, hence so is $\eta = t$, and



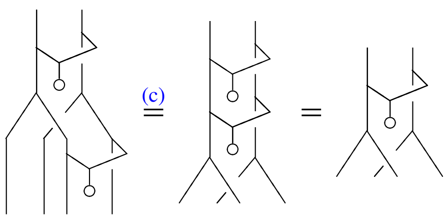
shows that μ is as well. Recall that

$P = (A \otimes A,$  $).$

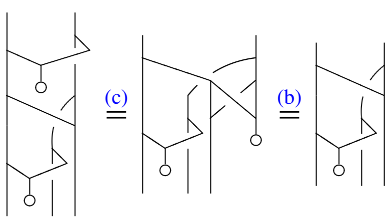
The morphisms $\delta_l : P \to A \otimes A \otimes P$ and $\delta_r : P \to P \otimes A$ are given by

$\delta_l =$  and $\delta_r =$  .

The two calculations



and

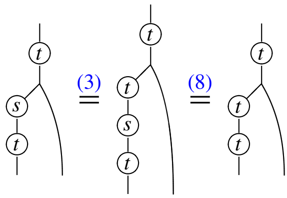


show that these are morphisms in $\mathfrak{Q}\mathcal{V}$.

To see that (A, μ, η) is a comonoid in $\mathbf{Bicomod}(C)$, notice that associativity follows from that of μ viewed as a weak bimonoid, and the counit property may be seen from property (6), that is,

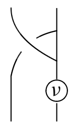
$$\mu(1 \otimes t)\delta = 1_A \quad \text{and} \quad \mu(s \otimes 1)c^{-1}\delta = 1_A,$$

and so (B1) holds. (B2) follows from one application of (12), (B3) follows from (b), (B4) from (c), and (B5) follows from (2). The calculation

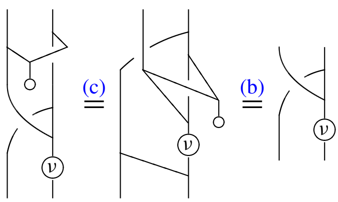


verifies (B6). Thus, $\mathbf{A} = (A, C, s, t, \mu, \eta)$ is a quantum category in $\mathscr{Q}\mathscr{V}$.

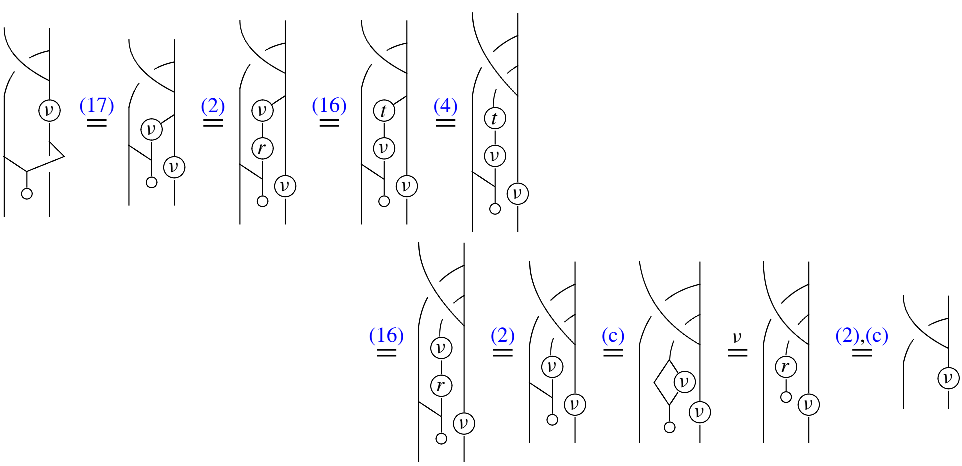
We now wish to show that a weak Hopf monoid with invertible antipode yields a quantum groupoid. So suppose that our weak bimonoid A is equipped with an invertible antipode $\nu : A \to A$, and set

$$\upsilon = t\nu\nu t : C^{\circ\circ} \to C, \quad \nu = \nu : A^{\circ} \to A, \quad \theta =$$  $$: P \to P.$$

The morphisms υ and ν are obviously morphisms in $\mathscr{Q}\mathscr{V}$, and the two calculations



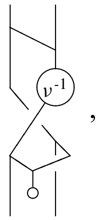
and



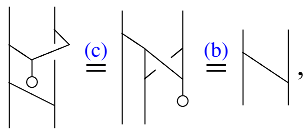
show that θ is as well.

Lemma 6.3. *An inverse for θ is given by*

$$\theta^{-1} =$$

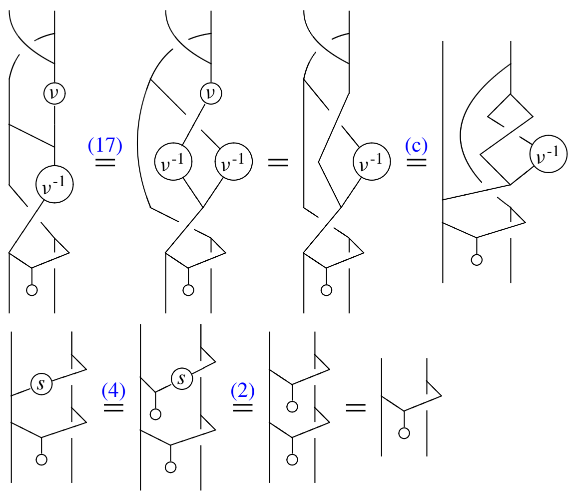


Proof. Since



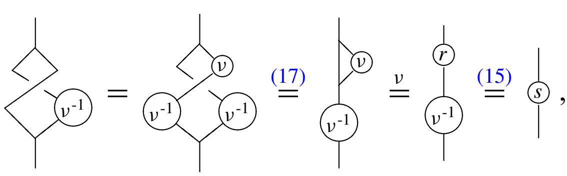
it is clear that θ^{-1} is a morphism in $\mathfrak{Q}\mathscr{V}$. That θ^{-1} is an inverse for θ may be seen in one direction from

$$\theta^{-1}\theta =$$

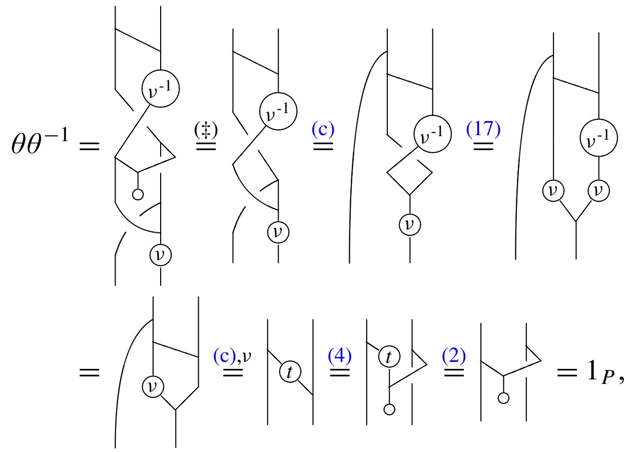


$$= 1_P,$$

where (†) is given by



and in the other direction by



for which the first step (‡) holds because θ is a morphism in $\mathfrak{Q}\mathscr{V}$. $\square$

That the antipode $\nu : A^\circ \to A$ is a comonoid isomorphism is our assumption. That $\upsilon : C^{\circ\circ} \to C$ is as well may be seen from the calculation

$$
\begin{aligned}
(t \otimes t)\delta\upsilon &= (t \otimes t)\delta t\nu\nu t \\
&= (t \otimes t)\delta\nu\nu t && \text{by (3)} \\
&= (t \otimes t)c(\nu \otimes \nu)\delta\nu t && \text{by (17)} \\
&= (t \otimes t)c(\nu \otimes \nu)c(\nu \otimes \nu)\delta t && \text{by (17)} \\
&= (t \otimes t)(\nu \otimes \nu)(\nu \otimes \nu)cc\delta t && \text{by nat} \\
&= (t \otimes t)(t \otimes t)(\nu \otimes \nu)(\nu \otimes \nu)cc\delta t && \text{by (7)} \\
&= (t \otimes t)(\nu \otimes \nu)(r \otimes r)(\nu \otimes \nu)cc\delta t && \text{by (16)} \\
&= (t \otimes t)(\nu \otimes \nu)(\nu \otimes \nu)(t \otimes t)cc\delta t && \text{by (16)} \\
&= (t \otimes t)(\nu \otimes \nu)(\nu \otimes \nu)(t \otimes t)cc\delta && \text{by (5)} \\
&= (\upsilon \otimes \upsilon)cc\delta && \text{by (5).}
\end{aligned}
$$

An inverse for υ is given by the morphism $\upsilon^{-1} = t\nu^{-1}\nu^{-1}t$, as may be seen in one direction by the calculation

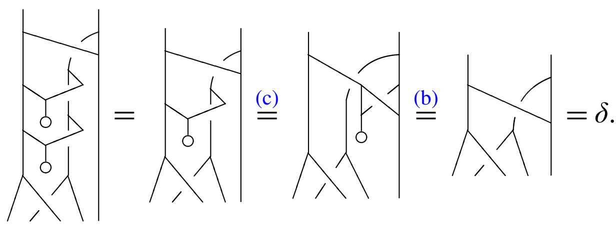
$$
\begin{aligned}
\upsilon^{-1}\upsilon &= t\nu^{-1}\nu^{-1}tt\nu\nu t \\
&= tt\nu^{-1}\nu^{-1}\nu\nu tt && \text{by (16)} \\
&= t\nu^{-1}\nu^{-1}\nu\nu t && \text{by (7)} \\
&= tt \\
&= t = 1_C && \text{by (7).}
\end{aligned}
$$

The other direction is similar.

Recall that the left $A \otimes A$-, right A-coaction δ on P is defined by taking the diagonal of the commutative square

$$\begin{tikzcd} P \arrow[r, "\delta_l"] \arrow[d, "\delta_r"'] & A \otimes A \otimes P \arrow[d, "1\otimes 1\otimes \delta_r"] \\ P \otimes A \arrow[r, "\delta_l \otimes 1"] & A \otimes A \otimes P \otimes A\,. \end{tikzcd}$$

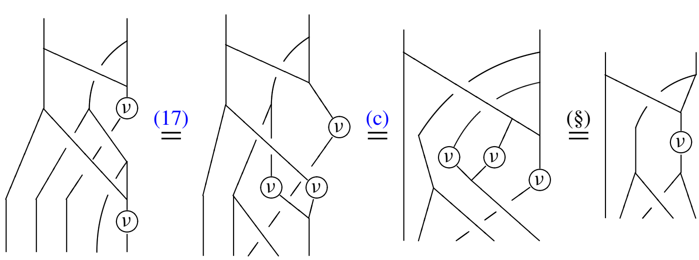
We note that δ may be written as



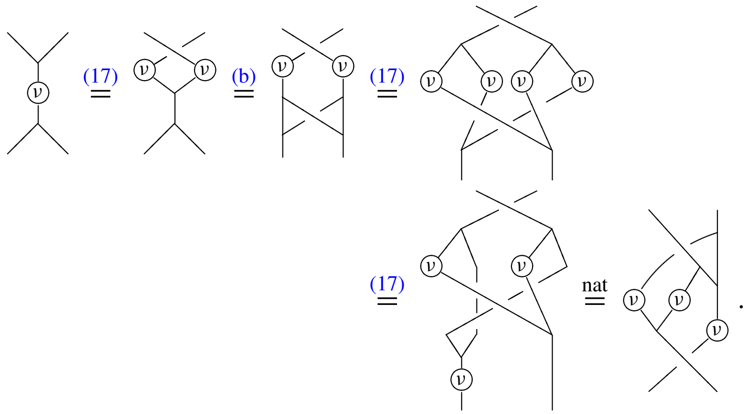
We must show that θ is a left $A^{\otimes 3}$-comodule isomorphism $P_l \to P_r$. That is, we must prove the commutativity of the square

$$\begin{tikzcd} P_l \arrow[r, "\gamma"] \arrow[d, "\theta"'] & A^{\otimes 3} \otimes P_l \arrow[d, "1\otimes\theta"] \\ P_r \arrow[r, "\gamma"] & A^{\otimes 3} \otimes P_r\,, \end{tikzcd}$$

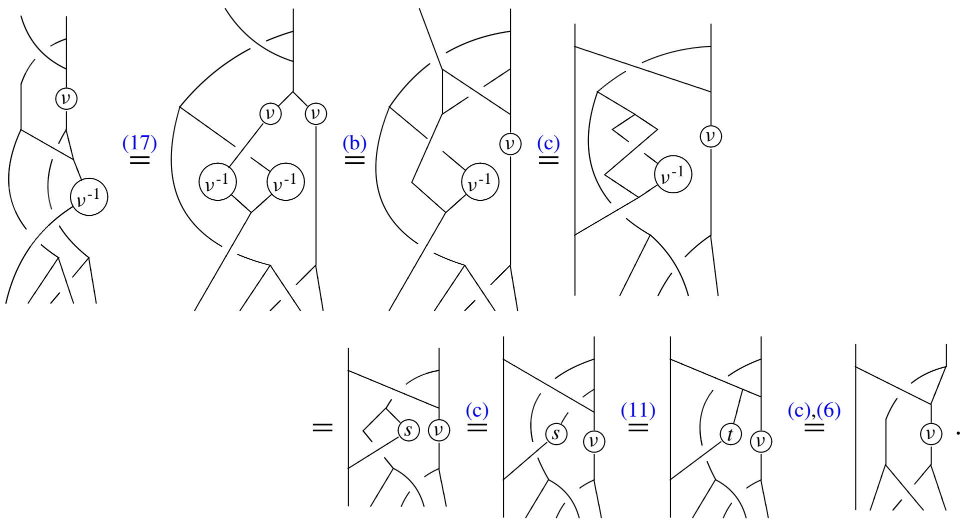
where the left $A^{\otimes 3}$-coactions on P_l and P_r were defined using δ (see Section 5.3). The clockwise direction around the square is



where the last step (§) is given by the calculation

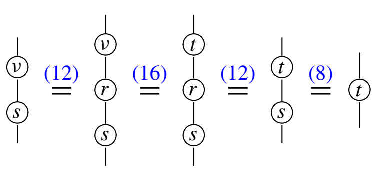


The counterclockwise direction is

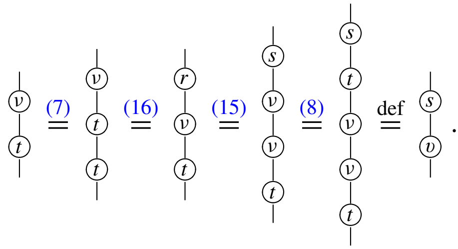


Thus, θ is a left $A^{\otimes 3}$-comodule morphism $P_l \to P_r$. The inverse of θ then is a left $A^{\otimes 3}$-comodule morphism $P_r \to P_l$.

We now prove the properties (G1) through (G3) required of a quantum groupoid. The calculation



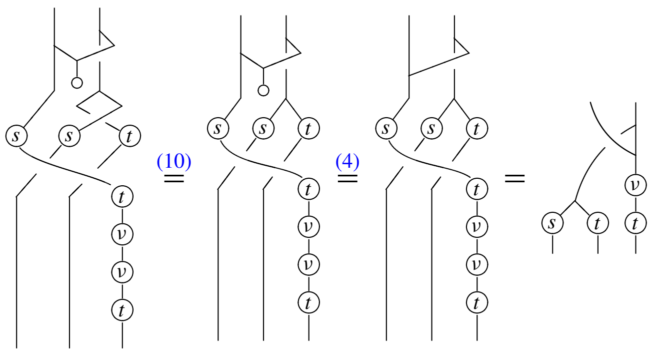
verifies (G1), and (G2) is established by



It remains to prove (G3), that is, we must show that θ makes the square

$$\begin{tikzcd} P \arrow[r, "\varsigma"] \arrow[d, "\theta"'] & C^{\otimes 3} \arrow[r, "c_{C,C\otimes C}"] & C^{\otimes 3} \arrow[d, "1\otimes 1\otimes \upsilon"] \\ P \arrow[rr, "\varsigma"] & & C^{\otimes 3} \end{tikzcd}$$

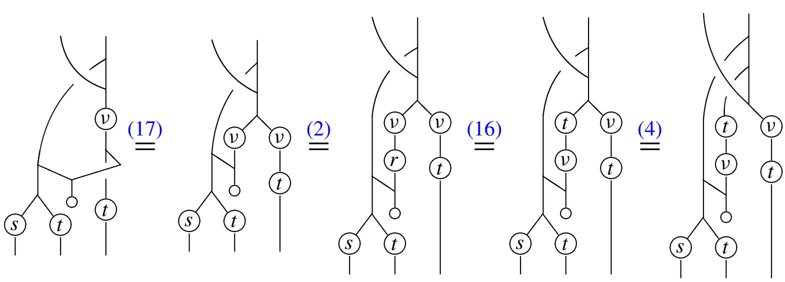
commute. The clockwise direction around the square is

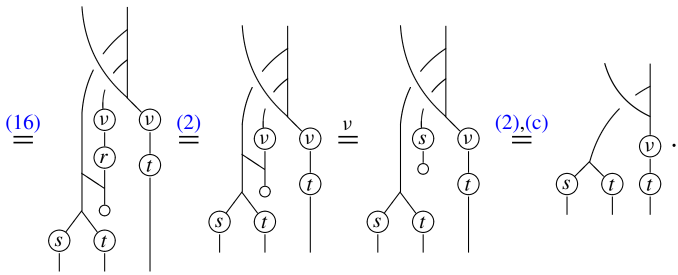


for which the last step holds since

$$\begin{aligned} t\nu\nu t s &= t\nu\nu s && \text{by (8)} \\ &= t\nu r && \text{by (15)} \\ &= t t\nu && \text{by (16)} \\ &= t\nu && \text{by (7).} \end{aligned}$$

The counterclockwise direction is





This establishes the commutativity of the square.

6.2. ***Quantum categories are weak bimonoids.*** In this section we prove that a quantum category with a separable Frobenius object-of-objects yields a weak bimonoid. Thus, suppose that $\mathbf{A} = (A, C, s, t, \mu, \eta)$ is a quantum category in $\mathcal{V}$ with $C = (C, \mu, \eta, \delta, \epsilon)$ a separable Frobenius monoid. The object A is a comonoid in $\mathcal{V}$, and our goal here is to show that equipping it with a multiplication and unit as

$$\mu = \ (A \otimes A \xrightarrow{\ m\ } P \xrightarrow{\ \mu\ } A)\ , \quad \eta = \ (I \xrightarrow{\ \eta\ } C \xrightarrow{\ \eta\ } A)$$

then yields a weak bimonoid A in $\mathcal{V}$ (and hence in $\mathfrak{Q}\mathcal{V}$).

Let us begin by establishing that the multiplication and unit defined here give a monoid structure on A. Note that the morphisms

$$\eta \otimes_C 1 : C \otimes_C A \to A \otimes_C A \quad \text{and} \quad 1 \otimes_C \eta : A \otimes_C C \to A \otimes_C A$$

are the unique morphisms such that $(\eta \otimes 1)\imath = \imath(\eta \otimes_C 1) : C \otimes_C A \to A \otimes A$ and $(1 \otimes \eta)\imath = \imath(1 \otimes_C \eta) : A \otimes_C C \to A \otimes A$, respectively. Thus, we have

$$\eta \otimes_C 1 = m(\eta \otimes 1)\imath \quad \text{and} \quad 1 \otimes_C \eta = m(1 \otimes \eta)\imath,$$

so that

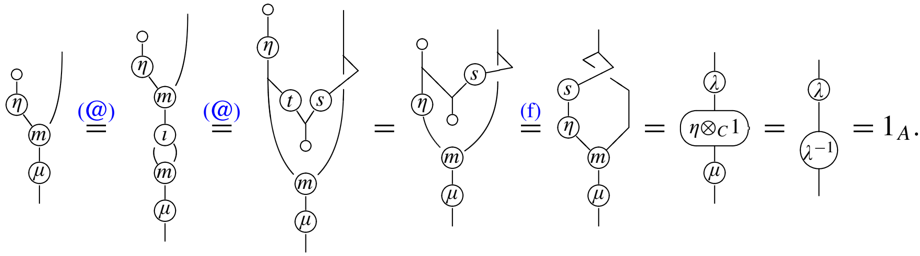
$$(\eta \otimes_C 1)\lambda = m(\eta \otimes 1)\delta_l \quad \text{and} \quad (1 \otimes_C \eta)\rho = m(1 \otimes \eta)\delta_r,$$

where $\lambda : A \to C \otimes_C A$ and $\rho : A \to A \otimes_C C$ are the left and right unit isomorphisms, respectively.

Also, $\mu \otimes_C 1$ and $1 \otimes_C \mu$ are the unique morphisms such that $(\mu \otimes 1)\imath = \imath(\mu \otimes_C 1)$ and $(1 \otimes \mu)\imath = \imath(1 \otimes_C \mu)$. Thus,

$$1 \otimes_C \mu = m(1 \otimes \mu)\imath \quad \text{and} \quad \mu \otimes_C 1 = m(\mu \otimes 1)\imath.$$

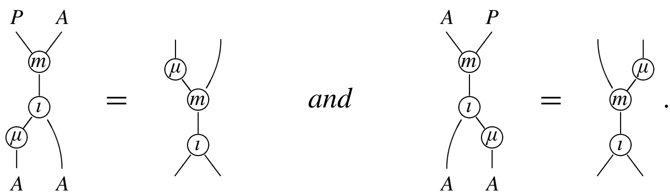
Given these identities, one of the unit conditions is seen from the calculation



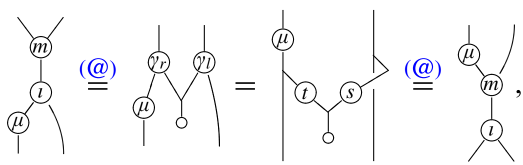
The third equality uses the fact that η is a C-comodule morphism. The other unit condition may be calculated similarly.

To establish associativity, we first prove the following lemma.

Lemma 6.4. *and*

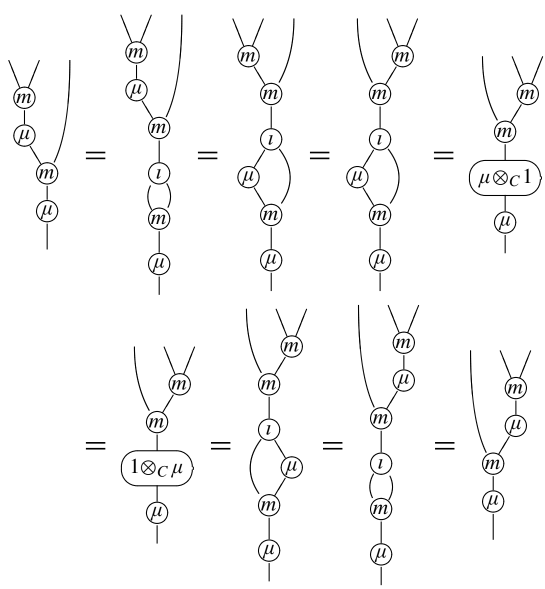


Proof. We prove the first equality. The second is similar.



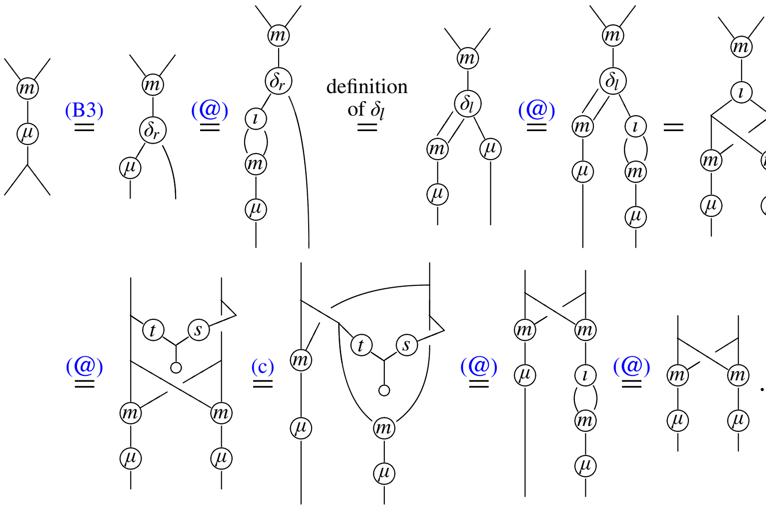
where the second step follows since μ is a C-comodule morphism. □

The following calculation then shows that associativity holds.

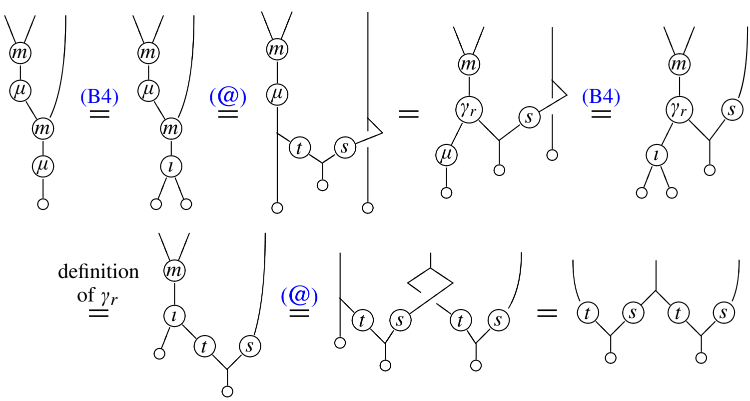


This then proves that (A, μ, η) is a monoid in $\mathscr{V}$. We now prove the remaining axioms for a weak bimonoid.

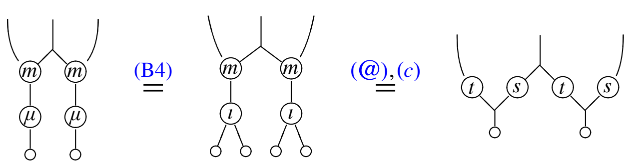
Axiom (b) is given by the calculation



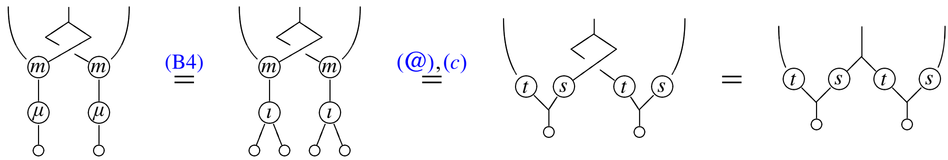
Axiom (v) is established with three calculations. The first is given as follows, where the third equality below follows from the fact that μ is a C-comodule morphism.



The second:

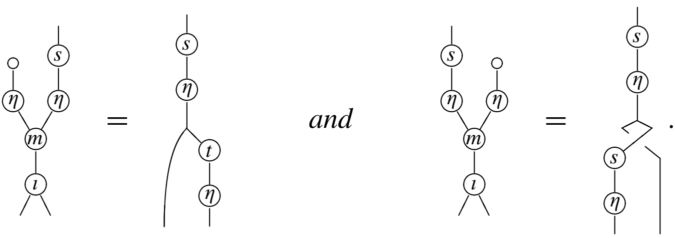


The third:

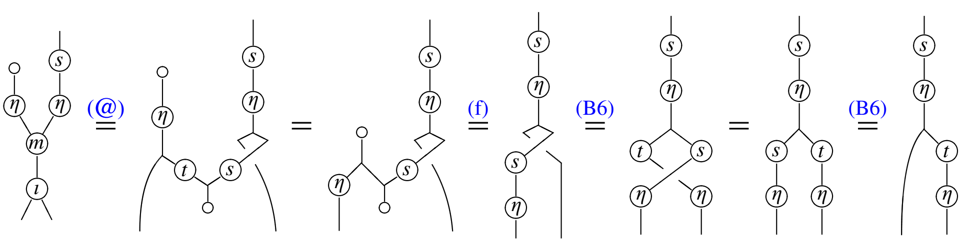


To prove the final axiom (w) for a weak bimonoid, we need a lemma.

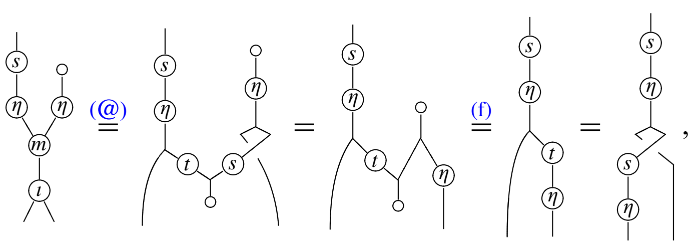
Lemma 6.5.



Proof. The first property of the lemma may be seen as follows. (The second equality below holds since η is a C-comodule morphism.)

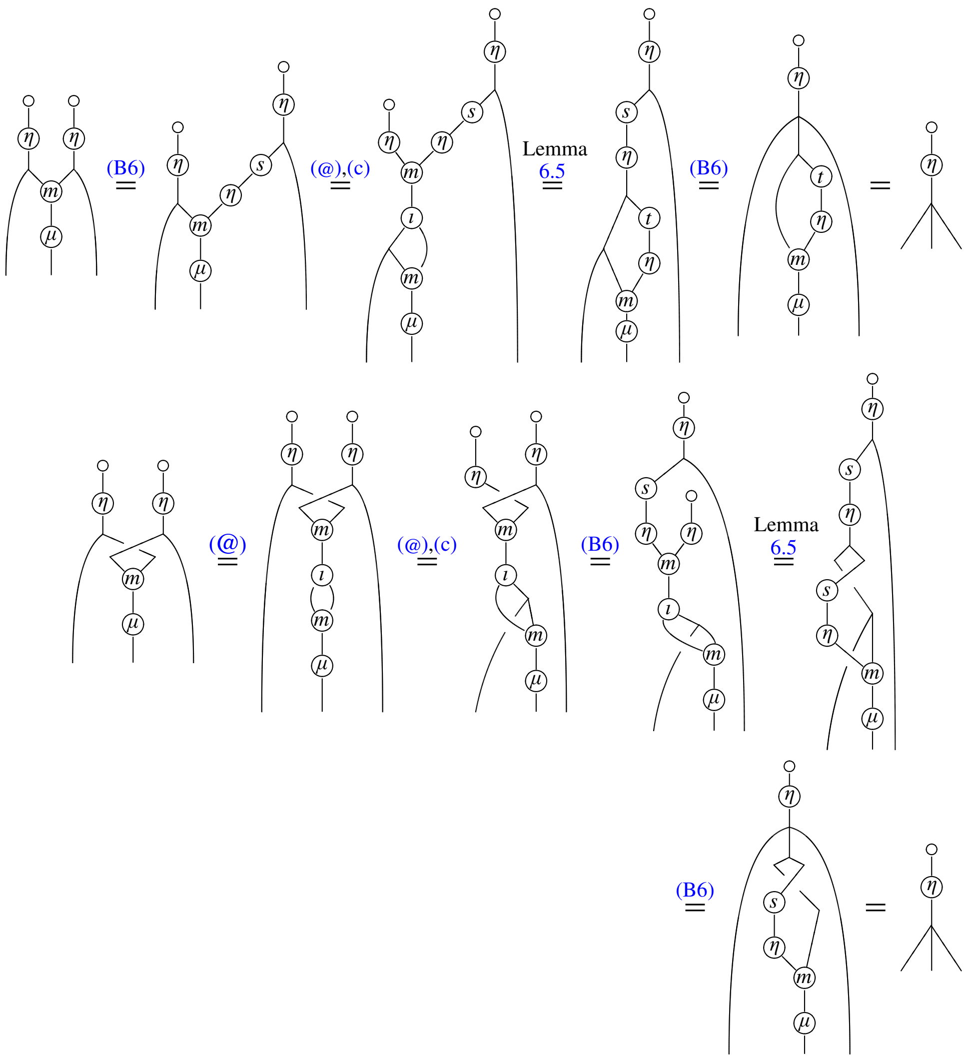


The second property of the lemma is proven as



where the second equality holds again since η is a C-comodule morphism and the last equality follows from the proof of the first part. $\square$

The following two calculations prove the axiom (w). In both calculations the last equality follows from the monoid structure on A.

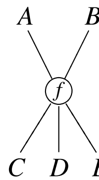


This completes the proof. Thus, a quantum category with a separable Frobenius object-of-objects yields a weak bimonoid.

Appendix A. String diagrams and basic definitions

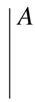
In this appendix we give a quick introduction to string diagrams in a braided monoidal category $\mathscr{V} = (\mathscr{V}, \otimes, I, c)$ [Joyal and Street 1993] and use these to define monoid, module, comonoid, comodule, and separable Frobenius monoid in $\mathscr{V}$. The string calculus was shown to be rigorous in [Joyal and Street 1991].

A.1. ***String diagrams.*** Suppose that $\mathcal{V} = (\mathcal{V}, \otimes, I, c)$ is a braided (strict) monoidal category. In a string diagram, objects label edges and morphisms label nodes. For example, if $f : A \otimes B \to C \otimes D \otimes E$ is a morphism in $\mathcal{V}$, it is represented as

$f =$ 

,

where this diagram is meant to be read top to bottom. The identity morphism on an object will be represented as the object itself, as in

$A =$ 

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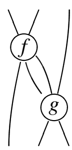
A special case is the object $I \in \mathcal{V}$, which is represented as the empty string.

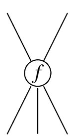
If, in $\mathcal{V}$, there are morphisms $f : A \otimes B \to C \otimes D \otimes E$ and $g : D \otimes E \otimes F \to G \otimes H$, then they may be composed as

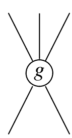
$$A \otimes B \otimes F \xrightarrow{f \otimes 1} C \otimes D \otimes E \otimes F \xrightarrow{1 \otimes g} C \otimes G \otimes H,$$

which may be represented as vertical concatenation

$(1 \otimes g)(f \otimes 1) =$ 

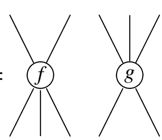
(where we have left off the objects). The tensor product of morphisms, say



and


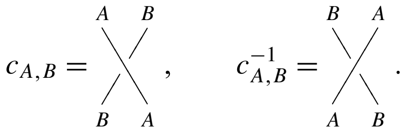
,

is represented by horizontal juxtaposition:

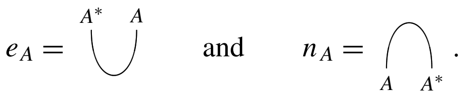
$f \otimes g =$ 

(again leaving off the objects).

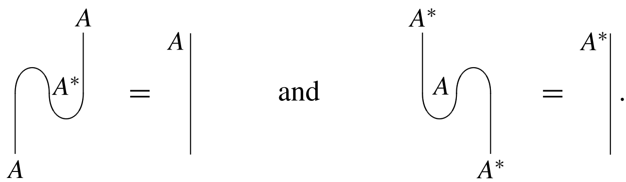
The braiding $c_{A,B} : A \otimes B \to B \otimes A$ is represented as a left-over-right crossing. The inverse braiding is then represented as a right-over-left crossing:



Suppose $A \in \mathcal{V}$ has a chosen left dual A^*, which we denote by $A^* \dashv A$ (it would be an adjunction if we were to view $\mathcal{V}$ as a one object bicategory). The evaluation and coevaluation morphisms $e_A : A^* \otimes A \to I$ and $n_A : I \to A \otimes A^*$ are represented as

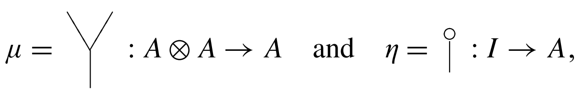


The triangle equalities become

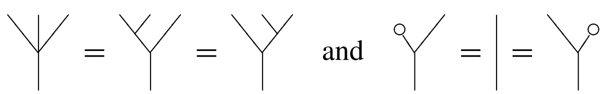


To simplify the string diagrams in what follows, we will omit the nodes from certain morphisms (for example, multiplication and comultiplication morphisms) or simplify them (for example, unit and counit morphisms).

A.2. ***Monoids and modules.*** A *monoid* $A = (A, \mu, \eta)$ in $\mathcal{V}$ is an object $A \in \mathcal{V}$ equipped with morphisms

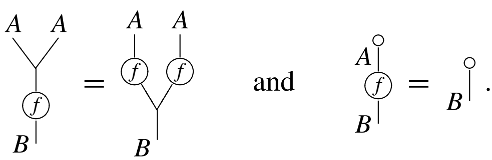


called the *multiplication* and *unit* of the monoid respectively, satisfying

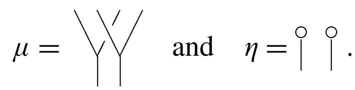


(m)

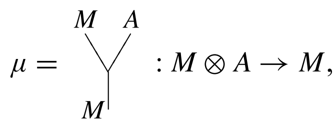
If A and B are monoids, a *monoid morphism* $f : A \to B$ is a morphism in $\mathcal{V}$ satisfying



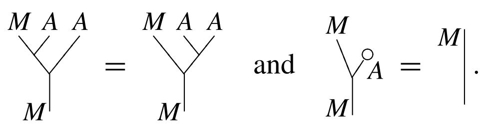
Monoids make sense in any monoidal category, however, in order that the tensor product $A \otimes B$ of monoids $A, B \in \mathcal{V}$ should again be a monoid, there must be a "switch" morphism $c_{A,B} : A \otimes B \to B \otimes A$ in $\mathcal{V}$ given by, say, a braiding. In this case, $A \otimes B$ becomes a monoid in $\mathcal{V}$ via



Suppose that A is a monoid in $\mathcal{V}$. A *right A-module* in $\mathcal{V}$ is an object $M \in \mathcal{V}$ equipped with a morphism



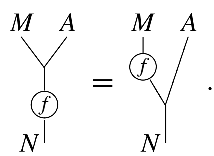
called the *action of A on M*, satisfying



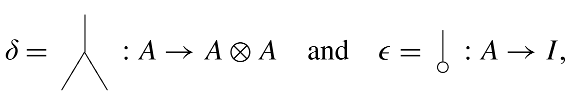
(m)

Notice that we use the same label "(m)" as in the monoid axioms (and "(c)" below for the comodule axioms). This should not cause any confusion as the labeling of strings disambiguates a multiplication and an action; however, the labeling will usually be left off.

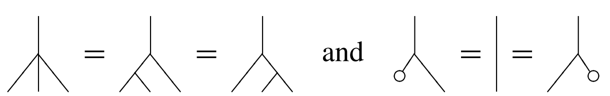
If M and N are modules, a *module morphism* $f : M \to N$ is a morphism in $\mathcal{V}$ satisfying



A.3. *Comonoids and comodules.* Comonoids and comodules are dual to monoids and modules. Explicitly, a *comonoid* $C = (C, \delta, \epsilon)$ in $\mathcal{V}$ is an object $C \in \mathcal{V}$ equipped with morphisms

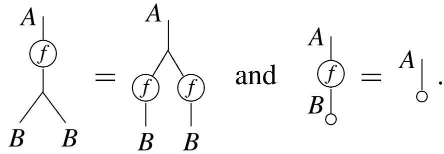


called the *comultiplication* and *counit* of the comonoid respectively, satisfying

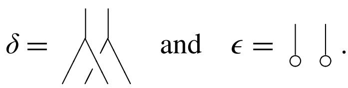


(c)

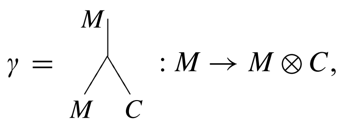
If C, D are comonoids, a *comonoid morphism* $f : C \to D$ is a morphism in $\mathcal{V}$ satisfying



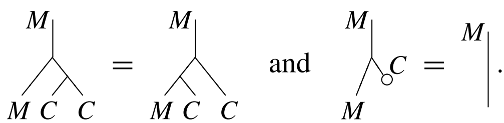
Similarly here, $\mathcal{V}$ must contain a switch morphism $c_{C,D} : C \otimes D \to D \otimes C$ in order that the tensor product $C \otimes D$ of comonoids $C, D \in \mathcal{V}$ should again be a comonoid. In this case the comultiplication and counit are given by



Suppose that C is a comonoid in $\mathcal{V}$. A *right C-comodule* in $\mathcal{V}$ is an object $M \in \mathcal{V}$ equipped with a morphism

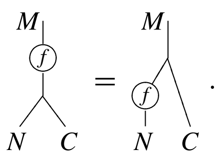


called the *coaction of A on M*, satisfying



(c)

If M and N are C-comodules, a *comodule morphism* $f : M \to N$ is a morphism in $\mathcal{V}$ satisfying



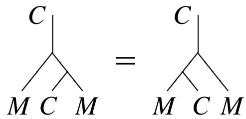
In this paper we also make use of C-bicomodules. Suppose that M is both a left C-comodule and a right C-comodule with coactions

$$\gamma_l : M \to C \otimes M \quad \text{and} \quad \gamma_r : M \to M \otimes C.$$

If the square

$$\begin{tikzcd}
M \arrow[r, "\gamma_l"] \arrow[d, "\gamma_r"'] & C \otimes M \arrow[d, "1\otimes\gamma_r"] \\
M \otimes C \arrow[r, "\gamma_l\otimes 1"] & C \otimes M \otimes C
\end{tikzcd}$$

commutes, meaning

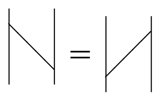


in string diagrams, then M is called a *C-bicomodule*. The diagonal of the square will be denoted by $\gamma : M \to C \otimes M \otimes C$.

A.4. *Frobenius monoids.* A *Frobenius monoid* R in $\mathscr{V}$ is both a monoid and a comonoid in $\mathscr{V}$ that additionally satisfies the "Frobenius condition":

$$\begin{tikzcd} R \otimes R \arrow[r, "\delta\otimes 1"] \arrow[d, "1\otimes\delta"'] & R \otimes R \otimes R \arrow[d, "1\otimes\mu"] \\ R \otimes R \otimes R \arrow[r, "\mu\otimes 1"] & R \otimes R. \end{tikzcd}$$

In strings the Frobenius condition is displayed as

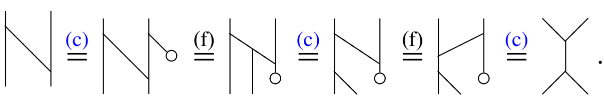


. (f)

We will now review some basic facts about Frobenius monoids.

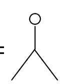
Lemma A.1. $(1 \otimes \mu)(\delta \otimes 1) = \delta\mu = (\mu \otimes 1)(1 \otimes \delta) : R \otimes R \to R \otimes R$.

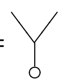
Proof. The left equality is proved by the string calculation



The right equality follows from a similar calculation. □

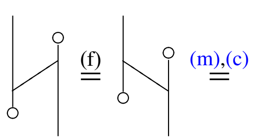
Define morphisms ρ and σ by

$$\rho = \left(I \xrightarrow{\eta} R \xrightarrow{\delta} R \otimes R\right) =$$  ,

$$\sigma = \left(R \otimes R \xrightarrow{\mu} R \xrightarrow{\epsilon} I\right) =$$  .

Proposition A.2. *The morphisms ρ and σ respectively form the unit and counit of an adjunction $R \dashv R$.*

Proof. One of the triangle identities is given as



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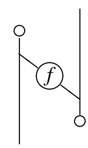
and the other should now be clear. □

A *morphism of Frobenius monoids* $f : R \to S$ is a morphism in $\mathscr{V}$ that is both a monoid and comonoid morphism.

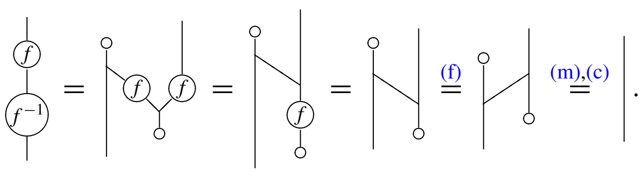
Proposition A.3. *Any morphism of Frobenius monoids $f : R \to S$ is an isomorphism.*

Proof. Given $f : R \to S$, define $f^{-1} : S \to R$ by

$$f^{-1} = S \xrightarrow{\rho\otimes 1} R\otimes R\otimes S \xrightarrow{1\otimes f\otimes 1} R\otimes S\otimes S \xrightarrow{1\otimes\sigma} R \;=$$



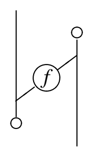
It is then an easy calculation to show that f^{-1} is the inverse of f, namely,



That $ff^{-1} = 1$ may be seen by viewing the previous calculation upside down. □

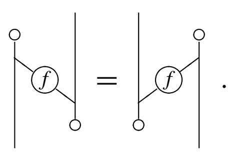
A similar calculation shows that

$$S \xrightarrow{1\otimes\rho} S\otimes R\otimes R \xrightarrow{1\otimes f\otimes 1} S\otimes S\otimes R \xrightarrow{\sigma\otimes 1} R \;=$$

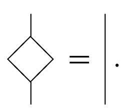


is also an inverse of f. Therefore:

Corollary A.4. *For any morphism of Frobenius monoids $f : R \to S$, we have*

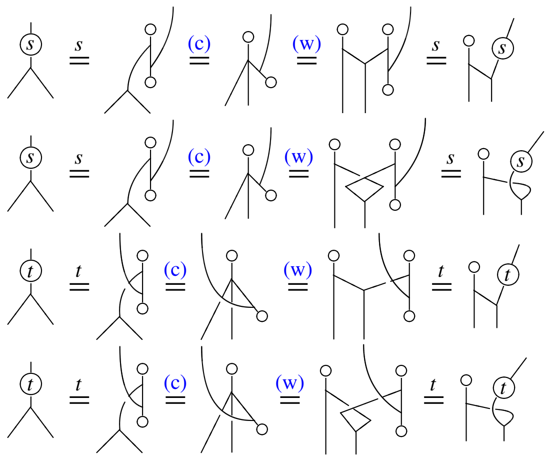


Definition A.5. A Frobenius monoid R is said to be *separable* if and only if $\mu\delta = 1$, that is,

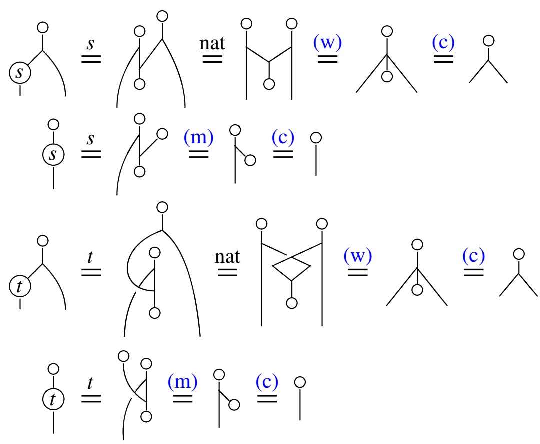


Appendix B. Proofs of the properties of s, t, and r

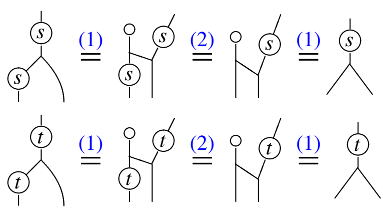
As we have noted in Section 1, the source morphism $s : A \to A$ is invariant under rotation by π, the target $t : A \to A$ is invariant under horizontal reflection, and the endomorphism r is t rotated by π. This reduces the number of proofs we present, since the others are derivable.



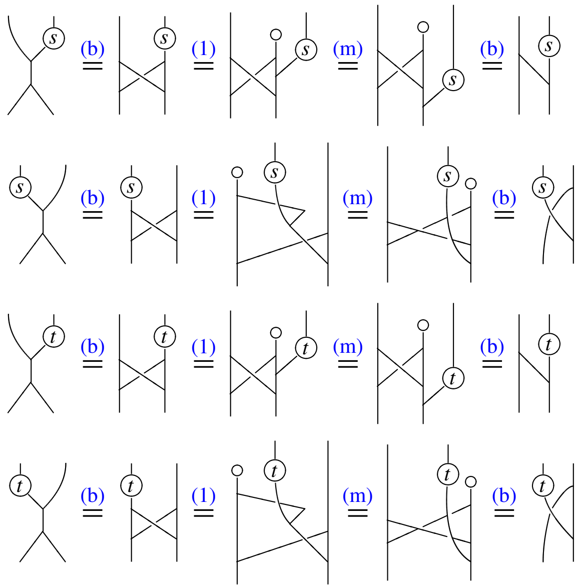
(1)



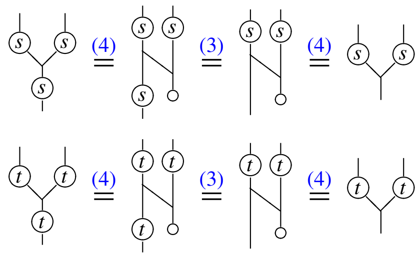
(2)



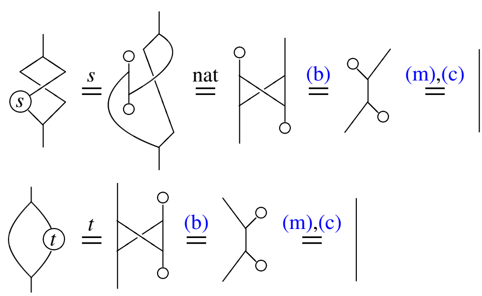
(3)



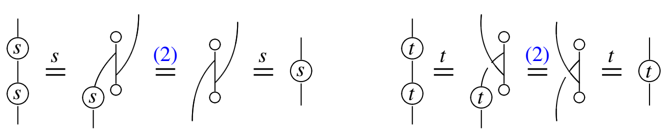
(4)



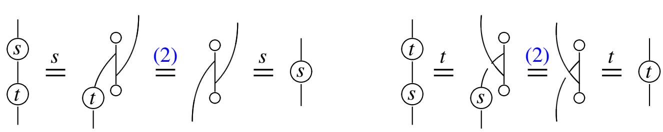
(5)



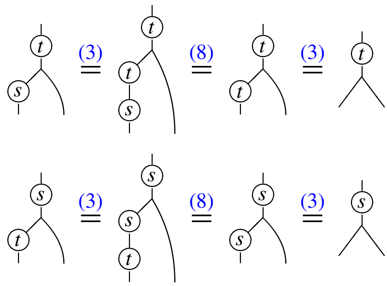
(6)



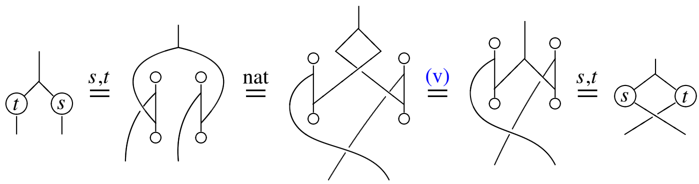
(7)



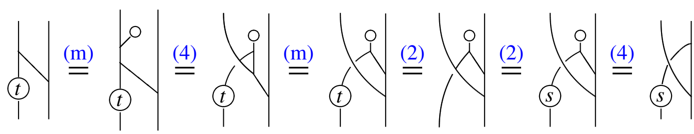
(8)



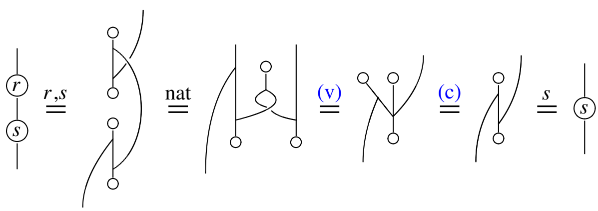
(9)



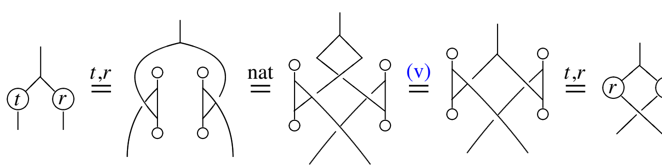
(10)



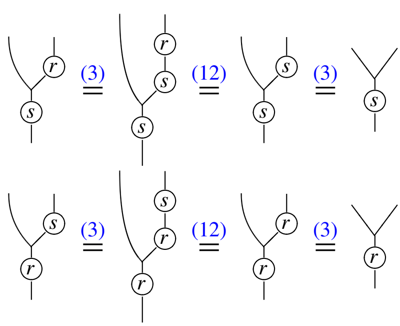
(11)



(12)



(13)



(14)

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