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The semigroup of Betti diagrams

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The recent proof of the Boij–Söderberg conjectures reveals new structure about Betti diagrams of modules, giving a complete description of the cone of Betti diagrams. We begin to expand on this new structure by investigating the semigroup of such diagrams. We prove that this semigroup is finitely generated, and answer several other fundamental questions about it.

1. Introduction

Recent work of a number of authors [Boij and Söderberg 2008b; 2008a; Eisenbud et al. 2007; Eisenbud and Schreyer 2009], completely characterizes the structure of Betti diagrams of graded modules, but only if one is allowed to take arbitrary rational multiples of the diagrams. This Boij–Söderberg theory shows that the rational cone of Betti diagrams is a simplicial fan whose rays and facet equations have a remarkably simple description.[1]

In this note, we consider the integral structure of Betti diagrams from the perspective of Boij–Söderberg theory, and we begin to survey this new landscape. In particular, we replace the cone by the semigroup of Betti diagrams (see Definition 1.1 below) and answer several fundamental questions about the structure of this semigroup.

We first use the results of Boij–Söderberg theory to draw conclusions about the semigroup of Betti diagrams. This comparison leads to Theorem 1.3, that the semigroup of Betti diagrams is finitely generated.

We then seek conditions which prevent a diagram from being the Betti diagram of an actual module. Using these conditions, we build families of diagrams which are *not* the Betti diagram of any module. For instance, consider the family

$$E_\alpha := \begin{pmatrix} 2+\alpha & 3 & 2 & - \\ - & 5+6\alpha & 7+8\alpha & 3+3\alpha \end{pmatrix}, \quad \alpha \in \mathbb{N}.$$

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[1] See [Boij and Söderberg 2008b] for the original conjecture, [Eisenbud and Schreyer 2009] for the Cohen–Macaulay case, and [Boij and Söderberg 2008a] for the general case. The introduction of [Eisenbud and Schreyer 2009] includes a particularly clear exposition of the main results.

We will use the theory of Buchsbaum–Eisenbud multiplier ideals to conclude that no member of this family can be the Betti diagram of a module. Yet each E_α belongs to the cone of Betti diagrams, and in fact, if we were to multiply any diagram E_α by 3, then the result *would* equal the Betti diagram of a module.

We produce further examples of obstructed diagrams by using properties of the Buchsbaum–Rim complex. Based on our examples, we establish several negative results about the semigroup of Betti diagrams. These negative results are summarized in Theorem 1.6.

To state our results more precisely, we introduce notation. Let S be the polynomial ring $S = k[x_1, \dots, x_n]$ where k is any field. If M is any finitely generated graded S-module, we can take a minimal free resolution

$$0 \to F_p \to \cdots \to F_1 \to F_0 \to M \to 0$$

with $F_i = \bigoplus_j S(-j)^{\beta_{i,j}(M)}$. We write $\beta(M)$ for the Betti diagram of M, thought of as an element of the vector space $\bigoplus_{j=-\infty}^{\infty} \bigoplus_{i=0}^{p} \mathbb{Q}$ with coordinates $\beta_{i,j}(M)$. The set of graded S-modules is a semigroup under the operation of direct sum, and the vector space is a semigroup under addition. By observing that $\beta(M \oplus M') = \beta(M) + \beta(M')$, we can think of β as a map of semigroups:

$$\{\text{ finitely generated graded } S\text{-modules}\} \xrightarrow{\beta} \bigoplus_{j=-\infty}^{\infty} \bigoplus_{i=0}^{p} \mathbb{Q}.$$

The image of this map is thus a semigroup. Furthermore, if we restrict β to any subsemigroup of S-modules, then the image of the restricted map is also a semigroup.

A *degree sequence* will mean an integral sequence $d = (d_0, \dots, d_p) \in \mathbb{N}^{p+1}$ where $d_i < d_{i+1}$. If there exists a Cohen–Macaulay module M of codimension p with all Betti numbers equal to zero except for $\beta_{i,d_i}(M)$, then we say that $\beta(M)$ is a pure diagram of type d. It was first shown in [Herzog and Kühl 1984] that any two pure diagrams of type d would be scalar multiples of one another. The existence of modules whose Betti diagrams are pure diagrams of type d was conjectured by [Boij and Söderberg 2008b] and proved by [Eisenbud et al. 2007] in characteristic 0 and by [Eisenbud and Schreyer 2009] in arbitrary characteristic. These pure diagrams play a central role in the Boij–Söderberg theorems.

Fix two degree sequences $\underline{d}$ and $\bar{d}$ of length p and such that $\underline{d}_i \le \bar{d}_i$ for all i. Consider the semigroup $\mathfrak{X}$ of graded modules M such that

- M has projective dimension $\le p$, and
- the Betti number $\beta_{i,j}(M)$ is nonzero only if $i \le p$ and $\underline{d}_i \le j \le \bar{d}_i$.

Our choice of $\mathfrak{X}$ is meant to match the simplicial structure of the cone of Betti diagrams. We may now define our main objects of study.

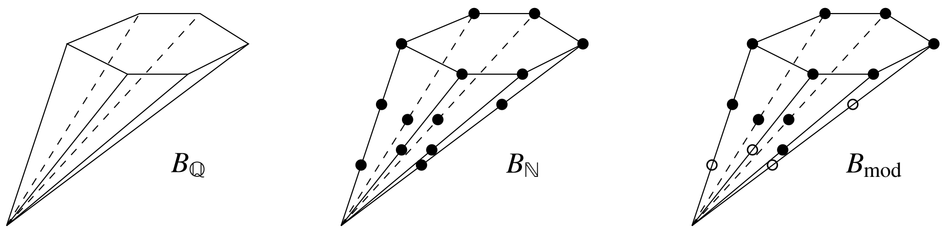

Figure 1. The cone of Betti diagrams $B_{\mathbb{Q}}$ is a simplicial fan which is described explicitly in [Eisenbud and Schreyer 2009] and [Boij and Söderberg 2008a]. This description can be used to understand the integral structure of the semigroup of virtual Betti diagrams $B_{\mathbb{N}}$. The semigroup of Betti diagrams B_{mod} is more mysterious.

Definition 1.1. The *semigroup of Betti diagrams* B_{mod} is defined as

$$B_{\mathrm{mod}} = B_{\mathrm{mod}}(\underline{d}, \bar{d}) := \operatorname{im} \beta|_{\mathfrak{X}}.$$

To study this object, it will be useful to consider two related ones:

Definition 1.2. The *cone of Betti diagrams* $B_{\mathbb{Q}}$ is the positive rational cone over the semigroup of Betti diagrams. The *semigroup of virtual Betti diagrams* $B_{\mathbb{N}}$ is the semigroup of lattice points in $B_{\mathbb{Q}}$.

One could define a cone of Betti diagrams without restricting which Betti numbers can be nonzero. This is the choice that [Eisenbud and Schreyer 2009] make, and our cone of Betti diagrams equals their big cone restricted to an interval. We choose to work with a finite dimensional cone in order to discuss the finiteness properties of B_{mod}.

A naive hope would be that the semigroups $B_{\mathbb{N}}$ and B_{mod} are equal. But a quick search yields virtual Betti diagrams which cannot equal the Betti diagram of module. Take for example the following pure diagram of type (0, 1, 3, 4):

$$D_1 := \pi_{(0,1,3,4)} = \begin{pmatrix} 1 & 2 & - & - \\ - & - & 2 & 1 \end{pmatrix}. \tag{1}$$

This diagram belongs to the semigroup of virtual Betti diagrams. However, D_1 cannot equal the Betti diagram of an actual module as the two first syzygies would satisfy a linear Koszul relation which does not appear in the diagram D_1.

It is thus natural to compare B_{mod} and $B_{\mathbb{N}}$, and we will consider some questions about the semigroup of Betti diagrams:

(Q1) Is B_{mod} finitely generated?

(Q2) Does $B_{\mathrm{mod}} = B_{\mathbb{N}}$ in some special cases?

(Q3) Is B_{mod} a saturated semigroup?

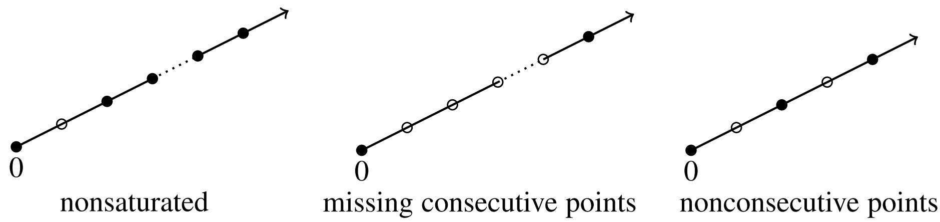

Figure 2. There exist rays that exhibit each of these behaviors.

(Q4) Is $B_{\mathbb{N}} \setminus B_{\text{mod}}$ a finite set?

(Q5) On a single ray, can we have consecutive points of $B_{\mathbb{N}}$ which fail to belong to B_{mod}? Nonconsecutive points?

In Section 2, we answer (Q1) affirmatively:

Theorem 1.3. *The semigroup of Betti diagrams B_{mod} is finitely generated.*

Sections 3 and 4 of this paper develop obstructions which prevent a virtual Betti diagram from being the diagram of some module. These obstructions are our tools for answering the other questions above. In Section 5, we consider (Q2), and prove the following:

Proposition 1.4. $B_{\mathbb{N}} = B_{\text{mod}}$ *for projective dimension* 1 *and for projective dimension* 2 *level modules.*

Our proof of Proposition 1.4 rests heavily on [Söderberg 2006], which shows the existence of level modules of embedding dimension 2 and with a given Hilbert function by constructing these modules as quotients of monomial ideals.

In [Erman $\geq$ 2009] we verify that, in a certain sense, projective dimension 2 diagrams generated in a single degree are "unobstructed." This leads us to:

Conjecture 1.5. $B_{\mathbb{N}} = B_{\text{mod}}$ for projective dimension 2 diagrams.

In the final section, we will consider questions (Q3)–(Q5). Here we show that the semigroup of Betti diagrams can have rather complicated behavior (see also Figure 2):

Theorem 1.6. *Each of the following occurs in the semigroup of Betti diagrams*:

(1) B_{mod} *is not necessarily a saturated semigroup.*

(2) *The set* $|B_{\mathbb{N}} \setminus B_{\text{mod}}|$ *is not necessarily finite.*

(3) *There exist rays of* B_{mod} *missing at least* $\dim S - 2$ *consecutive lattice points.*

(4) *There exist rays of* $B_{\mathbb{N}}$ *where the points of* B_{mod} *are nonconsecutive lattice points.*

Remark 1.7. Almost nothing in this paper would be changed if we swapped the semigroup $\mathscr{X}$ for some subsemigroup of $\mathscr{X}$ which respects the simplicial structure of $B_{\mathbb{Q}}$. For instance, we could consider the subsemigroup of Cohen–Macaulay modules of codimension e. The analogous statements of Theorems 1.3 and 1.6 and Proposition 1.4 all remain true in the Cohen–Macaulay case; one can even use the same proofs.

This paper is organized as follows. In Section 2, we prove that the semigroup of Betti diagrams is finitely generated. Sections 3 and 4 introduce obstructions for a virtual Betti diagram to be the Betti diagram of some module. The obstructions in Section 3 are based on properties of the Buchsbaum–Rim complex, and the obstruction in Section 4 focuses on the linear strand of a resolution and is based on the properties of Buchsbaum–Eisenbud multiplier ideals. Section 5 deals with the semigroup of Betti diagrams for small projective dimension, and contains the proof of Proposition 1.4. In Section 6 we prove Theorem 1.6 by constructing explicit examples based on our obstructions. Section 7 offers some open questions.

2. Finite generation of the semigroup of Betti diagrams

We fix a pair of degree sequences $\bar{d}, \underline{d} \in \mathbb{N}^{p+1}$ and work with the corresponding semigroup of Betti diagrams B_{mod}. Our proof of the finite generation of the semigroup of Betti diagrams uses the structure of the cone of Betti diagrams, so we begin by reviewing the relevant results. This structure was first proved in [Eisenbud and Schreyer 2009] for the Cohen–Macaulay case; the general case is similar, and was worked out in [Boij and Söderberg 2008a].

If d is any degree sequence then we set π_d to be the first lattice point on the ray corresponding to d. As illustrated in Figure 3, the cone $B_{\mathbb{Q}}$ is a rational simplicial fan whose defining rays correspond to rays of pure diagrams. To describe the simplicial structure, we recall the following partial ordering on degree sequences, introduced in [Boij and Söderberg 2008a]:

Definition 2.1. Let $d \in \mathbb{N}^{t+1}$ and $d' \in \mathbb{N}^{u+1}$. Then $d \leq d'$ if $t \geq u$ and $d_i \leq d'_i$ for all $i \leq u$.

The simplices of the fan $B_{\mathbb{Q}}$ correspond to maximal chains

$$d^0 < d^1 < \cdots < d^{s-1} < d^s$$

of degree sequences, where if $d^j \in \mathbb{N}^{t+1}$ then $\underline{d}_i \leq d^j_i \leq \bar{d}_i$ for all $i \leq t$. There are thus $s+1$ positions which may be nonzero for a Betti diagram in B_{mod} [Boij and Söderberg 2008a, Example 1]. In particular, $s+1 = \sum_{i=0}^{p} \bar{d}_i - \underline{d}_i + 1$.

Before proving Theorem 1.3, we first prove a simpler analog for the semigroup of virtual Betti diagrams $B_{\mathbb{N}}$.

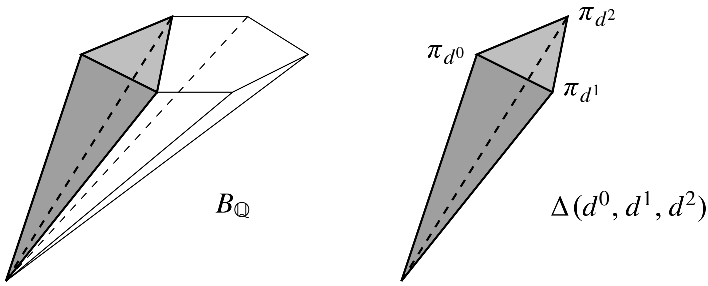

Figure 3. The cone $B_{\mathbb{Q}}$ is a simplicial fan. The simplex corresponding to a maximal sequence d^0, d^1, d^2 is highlighted in gray. The extremal rays of a simplex correspond to pure diagrams.

Lemma 2.2. *The semigroup $B_{\mathbb{N}}$ is finitely generated. There exists an integer m such that every virtual Betti diagram can be written as a $(1/m)\mathbb{N}$-combination of pure diagrams.*

Proof. Since $B_{\mathbb{N}}$ consists of the lattice points of the simplicial fan $B_{\mathbb{Q}}$, it is sufficient to prove this lemma after restricting to a single simplex Δ. Let $\pi_{d^0}, \dots, \pi_{d^s}$ be the pure diagrams defining Δ. Then the semigroup $B_{\mathbb{N}} \cap \Delta$ is generated by pure diagrams spanning Δ and by the lattice points inside the fundamental parallelepiped of Δ. This proves the first claim.

For the second claim of the lemma, let $P_1, \dots, P_N$ be the minimal generators of $B_{\mathbb{N}} \cap \Delta$. Every generator can be written as a positive rational sum:

$$P_i = \sum_j \frac{p_{ij}}{q_{ij}} \pi_{d^j}, \qquad p_{ij}, q_{ij} \in \mathbb{N}.$$

We set m_Δ to be the least common multiple of all the q_{ij}. Then we set m to be the least common multiple of the m_Δ for all Δ. □

We refer to m_Δ as a *universal denominator* for $B_{\mathbb{N}} \cap \Delta$. The existence of this universal denominator is central to our proof of the finite generation of B_{mod}.

Proof of Theorem 1.3. It is sufficient to prove the theorem for $B_{\text{mod}} \cap \Delta$ where Δ is a simplex of $B_{\mathbb{Q}}$. Let $\pi_{d^0}, \dots, \pi_{d^s}$ be the pure diagrams defining Δ, and let m_Δ be the universal denominator for $B_{\mathbb{N}} \cap \Delta$.

For $i = 0, \dots, s$, let $c_i \in \mathbb{N}$ be minimal such that $c_i \pi_{d^i}$ belongs to B_{mod}. The existence of such a c_i is guaranteed by Theorems 0.1 and 0.2 of [Eisenbud et al. 2007] and Theorem 5.1 of [Eisenbud and Schreyer 2009]. Let $\mathcal{S}_1$ be the semigroup generated by the pure diagrams $c_i \pi_{d^i}$. Let $\mathcal{S}_0$ be the semigroup generated by the pure diagrams $(1/m_\Delta)\pi_{d^i}$. Then we have the inclusions of semigroups

$$\mathcal{S}_1 \subseteq (B_{\text{mod}} \cap \Delta) \subseteq (B_{\mathbb{N}} \cap \Delta) \subseteq \mathcal{S}_0.$$

Passing to semigroup rings gives

$$k[\mathcal{S}_1] \subseteq k[B_{\text{mod}} \cap \Delta] \subseteq k[B_{\mathbb{N}} \cap \Delta] \subseteq k[\mathcal{S}_0].$$

Observe that $k[\mathcal{S}_1]$ and $k[\mathcal{S}_0]$ are both polynomial rings of dimension $s+1$, and that $k[\mathcal{S}_1] \subseteq k[\mathcal{S}_0]$ is a finite extension of rings. This implies that $k[\mathcal{S}_1] \subseteq k[B_{\text{mod}} \cap \Delta]$ is also a finite extension, and hence $k[B_{\text{mod}} \cap \Delta]$ is a finitely generated k-algebra. We conclude that $B_{\text{mod}} \cap \Delta$ is a finitely generated semigroup. □

Computing generators of $B_{\mathbb{N}}$. Minimal generators of $B_{\mathbb{N}} \cap \Delta$ can be computed explicitly as the generators of the $\mathbb{N}$-solutions to a certain linear $\mathbb{Z}$-system defined by the π_{d^i} and by m_Δ. For an overview of relevant algorithms, see the introduction of [Pisón-Casares and Vigneron-Tenorio 2004]. The following example illustrates the method.

Consider $S = k[x, y]$, $\underline{d} = (0, 1, 4)$, $\bar{d} = (0, 3, 4)$. The corresponding cone of Betti diagrams has several simplices and we choose the simplex Δ spanned by the maximal chain of degree sequences

$$(0) > (0, 3) > (0, 3, 4) > (0, 2, 4) > (0, 1, 4).$$

The corresponding pure diagrams are

$$\begin{pmatrix} 1 & - & - \\ - & - & - \\ - & - & - \end{pmatrix}, \quad \begin{pmatrix} 1 & - & - \\ - & - & - \\ - & 1 & - \end{pmatrix}, \quad \begin{pmatrix} 1 & - & - \\ - & - & - \\ - & 4 & 3 \end{pmatrix}, \quad \begin{pmatrix} 1 & - & - \\ - & 2 & - \\ - & - & 1 \end{pmatrix}, \quad \begin{pmatrix} 3 & 4 & - \\ - & - & - \\ - & - & 1 \end{pmatrix}. \tag{2}$$

First we must compute m_Δ. To do this, we consider the square matrix Φ whose columns correspond to the pure diagrams above:

$$\Phi = \begin{pmatrix} 1 & 1 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 1 & 4 & 0 & 0 \\ 0 & 0 & 3 & 1 & 1 \end{pmatrix}. \tag{3}$$

The columns of Φ are indexed by the pure diagrams in (2) and the rows of Φ are indexed by the Betti numbers $\beta_{0,0}, \beta_{1,1}, \beta_{1,2}, \beta_{1,3}$ and $\beta_{2,4}$ respectively. Since the columns of Φ are $\mathbb{Q}$-linearly independent, it follows that the cokernel of Φ is entirely torsion. Note that each minimal generator of $B_{\mathbb{N}} \cap \Delta$ is either a pure diagram or corresponds to a unique nonzero torsion element of coker(Φ). The annihilator of coker Φ is thus the universal denominator for Δ. A computation in Macaulay2 shows that $m_\Delta = 12$ in this case.

We next compute minimal generators of the $\mathbb{N}$-solutions of the linear $\mathbb{Z}$-system

$$\mathbb{Z}^{10} \xrightarrow{\begin{pmatrix} -12 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 3 \\ 0 & -12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & -12 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -12 & 0 & 0 & 1 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & -12 & 0 & 0 & 3 & 1 & 1 \end{pmatrix}} \mathbb{Z}^5.$$

The $\mathbb{N}$-solutions of the above system correspond to elements of $B_{\mathbb{N}} \cap \Delta$ under the correspondence

$$(b_1, b_2, b_3, b_4, b_5, a_1, a_2, a_3, a_4, a_5) \\ \mapsto \frac{a_1}{12}\pi_{(0)} + \frac{a_2}{12}\pi_{(0,3)} + \frac{a_3}{12}\pi_{(0,3,4)} + \frac{a_4}{12}\pi_{(0,2,4)} + \frac{a_5}{12}\pi_{(0,1,4)}.$$

Computation yields that $B_{\mathbb{N}} \cap \Delta$ has 14 minimal semigroup generators.[2] These consist of the 5 pure diagrams from line (2) plus the following 9 diagrams:

$$\begin{pmatrix} 1 & 1 & - \\ - & - & - \\ - & 1 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 2 & - \\ - & 1 & - \\ - & - & 1 \end{pmatrix}, \begin{pmatrix} 1 & - & - \\ - & 1 & - \\ - & 2 & 2 \end{pmatrix}, \begin{pmatrix} 1 & - & - \\ - & - & - \\ - & 2 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 2 & - \\ - & - & - \\ - & 1 & 1 \end{pmatrix}, \\ \begin{pmatrix} 3 & 3 & - \\ - & - & - \\ - & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & - & - \\ - & - & - \\ - & 3 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 1 & - \\ - & 1 & - \\ - & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & - & - \\ - & 1 & - \\ - & 1 & 1 \end{pmatrix}.$$

It is not difficult to verify that each of these generators is the Betti diagram of some module. Thus in this case we have $B_{\mathbb{N}} \cap \Delta = B_{\text{mod}} \cap \Delta$.

Remark 2.3. We can easily bound the number of generators of $B_{\mathbb{N}} \cap \Delta$ from above. Let Δ be a simplex spanned by $d^0, \dots, d^s$. Let Φ be the square matrix

$$\Phi : \mathbb{Z}^{s+1} \to \bigoplus_{i=0}^{n} \bigoplus_{j=\underline{d}_i}^{\bar{d}_i} \mathbb{Z},$$

which sends the ℓth generator to the pure diagram π_{d^ℓ}. As in line (3), the cokernel of Φ will be entirely torsion (this follows from [Boij and Söderberg 2008a, Proposition 1].) Each minimal generator of $B_{\mathbb{N}} \cap \Delta$ will correspond to either a pure diagram or a unique nonzero element of coker Φ. Since the order of coker Φ equals the determinant of Φ, the number of generators of $B_{\mathbb{N}} \cap \Delta$ is bounded above by $\det(\Phi) + s$.

We know of no effective upper bound for the number of generators of $B_{\text{mod}} \cap \Delta$.

[2] We use [Sturmfels 1993, Algorithm 2.7.3] for this computation. Also, see [Pisón-Casares and Vigneron-Tenorio 2004] for other relevant algorithms.

Remark 2.4. Although the semigroup $B_{\mathbb{N}}$ is saturated, the map $k[B_{\text{mod}}] \to k[B_{\mathbb{N}}]$ may not be the normalization map. For instance, if there is a ray r such that $r \cap B_{\text{mod}}$ only contains every other lattice point, then the saturation of $r \cap B_{\text{mod}}$ will not equal $r \cap B_{\mathbb{N}}$. Eisenbud et al. [2007] conjecture that there are no rays corresponding to pure diagrams which have this property .

3. Buchsbaum–Rim obstructions to existence of Betti diagrams

In Proposition 3.1 we illustrate obstructions which prevent a virtual Betti diagram from being the Betti diagram of an actual module. To yield information not contained in the main results of [Eisenbud and Schreyer 2009; Boij and Söderberg 2008a], these obstructions must be sensitive to scalar multiplication of diagrams. For simplicity, we restrict to the case that M is generated in degree 0, though all of these obstructions can be extended to the general case.

We say that a diagram D is a *Betti diagram* if D equals the Betti diagram of some module M, and we say that D is a *virtual* Betti diagram if D belongs to the semigroup of virtual Betti diagrams $B_{\mathbb{N}}$. Many properties of modules (for example, codimension, Hilbert function) can be computed directly from the Betti diagram. We extend such properties to virtual diagrams in the obvious way. Proposition 3.1 only involves quantities which can be determined entirely from the Betti diagram; thus we may easily test whether an arbitrary virtual Betti diagram is "obstructed" in the sense of this proposition.

Proposition 3.1 (Buchsbaum–Rim obstructions). *Let M a graded module of codimension $e \geq 2$ with minimal presentation*

$$\bigoplus_{\ell=1}^{b} S(-j_\ell) \xrightarrow{\ \phi\ } S^a \longrightarrow M \longrightarrow 0.$$

Assume that $j_1 \leq j_2 \leq \cdots \leq j_b$. Then we have the following obstructions, which are independent of one another, and each of which occurs for some virtual Betti diagram.

(1) *Second syzygy obstruction*:

$$\underline{d}_2(M) \leq \sum_{\ell=1}^{a+1} j_\ell.$$

(2) *Codimension obstruction*:

$$b = \sum_j \beta_{1,j}(M) \geq e + a - 1.$$

If we have equality, $\beta(M)$ must equal the Betti diagram of the Buchsbaum–Rim complex of ϕ.

(3) *Regularity obstruction in Cohen–Macaulay case*: *If M is Cohen–Macaulay*,

$$\operatorname{reg}(M)+e=\bar{d}_e(M)\leq \sum_{\ell=b-e-a+2}^{b} j_\ell.$$

Note that both the weak and strong versions of the Buchsbaum–Eisenbud–Horrocks rank conjecture about minimal Betti numbers (see [Buchsbaum and Eisenbud 1977] or [Charalambous et al. 1990] for a description) would lead to similar obstructions. Since each Buchsbaum–Eisenbud–Horrocks conjecture imposes a condition on each column of the Betti diagram, the corresponding obstruction would greatly strengthen part (2) of Proposition 3.1.

Remark 3.2. For D a diagram, let $D^\vee$ be the diagram obtained by rotating D by 180 degrees. When D is the Betti diagram of a Cohen–Macaulay module M of codimension e, then $D^\vee$ is the Betti diagram of some twist of $M^\vee:=\operatorname{Ext}^e_S(M,S)$, which is also a Cohen–Macaulay module of codimension e. Thus, in the Cohen–Macaulay case, we may apply these obstructions to D or to $D^\vee$.

Given any map $\widetilde{\phi}$ between free modules F and G, we can construct the Buchsbaum–Rim complex on this map, which we denote as $\operatorname{Buchs}_\bullet(\widetilde{\phi})$. The Betti table of the complex $\operatorname{Buchs}_\bullet(\widetilde{\phi})$ will depend only on the Betti numbers of F and G, and it can be thought of as an approximation of the Betti diagram of the cokernel of $\widetilde{\phi}$.

As in the statement of Proposition 3.1, let M be a graded S-module of codimension ≥ 2 with minimal presentation

$$F_1:=\bigoplus_{\ell=1}^{b} S(-j_\ell)\xrightarrow{\ \phi\ } S^a \longrightarrow M \longrightarrow 0.$$

We will consider free submodules $\widetilde{F}_1\subseteq F_1$, the induced map $\widetilde{\phi}:\widetilde{F}_1\to S^a$, and the Buchsbaum–Rim complex on $\widetilde{\phi}$. By varying $\widetilde{\phi}$ we will produce the obstructions listed in Proposition 3.1.

To prove the first obstruction, we introduce some additional notation. Let the first syzygies of M be $\sigma_1,\dots,\sigma_b$ with degrees $\deg(\sigma_\ell)=j_\ell$. The first stage of the Buchsbaum–Rim complex on ϕ is the complex

$$\bigwedge^{a+1} F_1 \xrightarrow{\ \epsilon\ } F_1\to S^a.$$

A basis of $\bigwedge^{a+1} F_1$ is given by $e_{I'}$ where I' is a subset $I'\subseteq\{1,\dots,b\}$ with $|I'|=a+1$. Let $\det(\phi_{I'\setminus\{i\}})$ be the maximal minor corresponding to the columns $I'\setminus\{i\}$. Then the map ϵ sends $e_{I'}\mapsto\sum_{i\in I'} e_i\det(\phi_{I'\setminus\{i\}})$. We refer to $\epsilon(e_{I'})$ as a *Buchsbaum–Rim second syzygy*, and we denote it by $\rho_{I'}$. There are $\binom{b}{a+1}$

Buchsbaum–Rim second syzygies. It may happen that one of these syzygies specializes to 0 in the case of ϕ. But as we now prove, if $\rho_{I'}$ specializes to 0 then we can find another related syzygy in lower degree.

Lemma 3.3. *Let $I' = \{i_1, \dots, i_{a+1}\} \subseteq \{1, \dots, b\}$, and assume that $\rho_{I'}$ is a trivial second syzygy. Then M has a second syzygy of degree strictly less than $\sum_{i \in I'} j_i$ and supported on a subset of the columns corresponding to I'.*

Proof. Let A be an $a \times b$-matrix representing ϕ. Let $C = \{1, \dots, b\}$ index the columns of A, and let $W = \{1, \dots, a\}$ index the rows of A. If $I \subseteq C$ and $J \subseteq W$ then we write $A_{I,J}$ for the corresponding submatrix.

The Buchsbaum–Rim syzygy $\rho_{I'}$ is trivial if and only if all the $a \times a$ minors of $A_{I',W}$ are zero. Let $a' = \operatorname{rank} A_{I',W}$ which by assumption is strictly less than a. We may assume that the upper left $a' \times a'$ minor of $A_{I',W}$ is nonzero. We set $I'' = \{i_1, \dots, i_{a'+1}\}$ and $J'' = \{1, \dots, a'\}$. Let τ be the Buchsbaum–Rim syzygy of $A_{I'',J''}$. Then $\tau \neq 0$ because $\det(A_{I'' \setminus \{a'+1\}, J''}) \neq 0$. Also $(A_{I'',J''}) \cdot \tau = 0$. Thus,

$$\big(A_{I'',W}\big) \cdot \tau = \begin{pmatrix} A_{I'',J} \\ A_{I'',W-J''} \end{pmatrix} \cdot \tau = \begin{pmatrix} 0 \\ * \end{pmatrix}.$$

There exists an invertible matrix $B \in GL_a(k(x_1, \dots, x_n))$ such that

$$B \cdot A_{I'',W} = \begin{pmatrix} A_{I'',J''} \\ 0 \end{pmatrix}.$$

This gives

$$0 = (B \cdot A_{I'',W}) \cdot \tau = B \cdot (A_{I'',W} \cdot \tau).$$

Since B is invertible over $k(x_1, \dots, x_n)$, we conclude that $A_{I'',W} \cdot \tau = 0$. Thus τ is a syzygy on the columns of A indexed by I'', and therefore τ represents a second syzygy of M. The degree of τ is $\sum_{i \in I''} j_i$ which is strictly less than $\sum_{i \in I'} j_i$. □

We may now prove the second syzygy obstruction and the codimension obstruction.

Proof of the second syzygy obstruction in Proposition 3.1. Apply Lemma 3.3, choosing $I' = \{1, \dots, a+1\}$. □

Proof of codimension obstruction in Proposition 3.1. Recall that the module M has minimal presentation

$$\bigoplus_{\ell=1}^{b} S(-j_\ell) \xrightarrow{\ \phi\ } S^a \longrightarrow M \longrightarrow 0.$$

Let $\mathrm{Buchs}_\bullet(\phi)$ be the Buchsbaum–Rim complex of ϕ. Then we have

$$\operatorname{codim} M \leq \operatorname{pdim} M \leq \operatorname{pdim} \mathrm{Buchs}_\bullet(\phi) = b - a + 1 = \sum_j \beta_{1,j}(M) - a + 1.$$

Since M has codimension e, we obtain the desired inequality. In the case of equality, the maximal minors of ϕ contain a regular sequence of length e, so we may conclude that

$$\beta(M) = \beta(\text{Buchs}_\bullet(\phi)). \qquad \square$$

Proof of regularity obstruction in Proposition 3.1. Since M is Cohen–Macaulay of codimension e, we may assume by Artinian reduction that M is finite length. Recall that $b = \sum_j \beta_{1,j}(M)$ and let ϕ as in the proof of the codimension obstruction. If $b = e + a - 1$, then

$$\text{reg}(M) = \text{reg}(\text{Buchs}_\bullet(\phi)) = \sum_{\ell=1}^{b} j_\ell.$$

We are left with the case that $b > e + a - 1$. Recall that $\sigma_1, \dots, \sigma_b$ is a basis of the syzygies of M. We may change bases on the first syzygies by sending $\sigma_i \mapsto \sum p_{i\ell}\sigma_\ell$ where $\deg(p_{i\ell}) = \deg \sigma_i - \deg \sigma_\ell = j_i - j_\ell$, and where the matrix $(p_{i\ell})$ is invertible over the polynomial ring. We choose a generic $(p_{i\ell})$ which satisfies these conditions. Let $\widetilde{\phi}$ be the map defined by $\sigma_b, \sigma_{b-1}, \dots, \sigma_{b-e-a+2}$. Define $M' := \text{coker}\,\widetilde{\phi}$. By construction, M' has finite length, $\beta(M') = \beta(\text{Buchs}_\bullet(\widetilde{\phi}))$, and M' surjects onto M. Thus we have

$$\sum_{\ell=b-e-a+2}^{f} j_\ell = \text{reg}(M') \geq \text{reg}(M) = \bar{d}_n(M),$$

where the inequality follows from Corollary 20.19 of [Eisenbud 1995]. $\square$

Proof of independence of obstructions in Proposition 3.1. To show that the obstructions of Proposition 3.1 are independent, we construct an explicit example of a virtual Betti diagram with precisely one of the obstructions.

For Proposition 3.1(1), consider

$$2 \cdot \pi_{(0,1,5,6,7,8)} + \pi_{(0,5,6,7,8,9)} = \begin{pmatrix} 3 & 4 & - & - & - & - \\ - & - & - & - & - & - \\ - & - & - & - & - & - \\ - & 70 & 252 & 336 & 200 & 45 \end{pmatrix}.$$

Then $\underline{d}_2 = 5 > 4$ so this diagram has a Buchsbaum–Rim second syzygy obstruction.

For Proposition 3.1(2), consider

$$\pi_{(0,1,3,4)} = \begin{pmatrix} 1 & 2 & - & - \\ - & - & 2 & 1 \end{pmatrix}.$$

In this case $\sum \beta_{1,j}(\pi_{(0,1,3,4)}) = 2 < 3+1-1 = 3$. More generally, the pure diagram $\pi_{(0,1,\alpha,\alpha+1)}$ has a codimension obstruction for any $\alpha \geq 3$.

For the case of equality in Proposition 3.1(2), consider

$$\pi_{(0,1,6,10)} = \begin{pmatrix} 6 & 8 & - & - \\ - & - & - & - \\ - & - & - & - \\ - & - & 3 & - \\ - & - & - & - \\ - & - & - & - \\ - & - & - & - \\ - & - & - & 1 \end{pmatrix}.$$

Since we have $\sum \beta_{1,j}(\pi_{(0,1,6,10)}) = 8 = 3 + 6 - 1$, the diagram $\pi_{(0,1,6,10)}$ should equal the Betti table of the Buchsbaum–Rim complex on a map: $\phi : R(-1)^8 \to R^6$. This is not the case.

For Proposition 3.1(3), consider

$$2 \cdot \pi_{(0,1,4,9,10)} = \begin{pmatrix} 6 & 10 & - & - & - \\ - & - & - & - & - \\ - & - & 6 & - & - \\ - & - & - & - & - \\ - & - & - & - & - \\ - & - & - & - & - \\ - & - & - & 6 & 4 \end{pmatrix}.$$

Here we have $\bar{d}_4 = 10 > 9 = \sum_{j=1}^{9} 1$.

□

4. A linear strand obstruction in projective dimension 3

In this section, we build obstructions based on one of Buchsbaum and Eisenbud's structure theorems about free resolutions in the special case of codimension 3 [Buchsbaum and Eisenbud 1974]. The motivation of this section is to explain why the following virtual Betti diagrams do not belong to B_{mod}:

$$D = \begin{pmatrix} 2 & 4 & 3 & - \\ - & 3 & 4 & 2 \end{pmatrix}, \quad D' = \begin{pmatrix} 3 & 6 & 4 & - \\ - & 4 & 6 & 3 \end{pmatrix}, \quad D'' = \begin{pmatrix} 2 & 3 & 2 & - \\ - & 5 & 7 & 3 \end{pmatrix}. \tag{4}$$

Note that these diagrams do not have any of the Buchsbaum–Rim obstructions. In fact, there are virtual Betti diagrams similar to each of these which are Betti diagrams of modules. For instance, all of the following variants of D *are* Betti diagrams of modules:

$$\begin{pmatrix} 2 & 4 & 1 & - \\ - & 1 & 4 & 2 \end{pmatrix}, \quad \begin{pmatrix} 2 & 4 & 2 & - \\ - & 2 & 4 & 2 \end{pmatrix}, \quad \begin{pmatrix} 2 & 4 & 3 & 1 \\ - & 3 & 5 & 2 \end{pmatrix}, \quad \begin{pmatrix} 4 & 8 & 6 & - \\ - & 6 & 8 & 4 \end{pmatrix}.$$

The problem with D must therefore relate to the fact that it has too many linear second syzygies to *not* contain a Koszul summand. Yet whatever obstruction exists for D must disappear upon scaling from D to $2 \cdot D$. Incidentally, the theory of matrix pencils could be used to show that D and D'' are not Betti diagrams. We do not approach this problem via matrix pencils because we seek an obstruction which does not depend on the fact that $\beta_{0,0} = 2$.

Let $S = k[x, y, z]$ and let M be a graded S-module M of finite length. Further, let M be generated in degree 0 and with regularity 1, so that

$$\beta(M) = \begin{pmatrix} a & b & c & d \\ - & b' & c' & d' \end{pmatrix}.$$

Let T_i be the maps along the top row of the resolution of M so that we have a complex

$$0 \longrightarrow S(-3)^d \xrightarrow{(T_3)} S(-2)^c \xrightarrow{(T_2)} S(-1)^b \xrightarrow{(T_1)} S^a \longrightarrow 0.$$

Similarly, let U_j stand for matrices which give the maps along the bottom row of the resolution of M. Observe that each T_i and U_j consists entirely of linear forms, and that $U_1 = 0$. If $d \neq 0$, then the minimal resolution of M contains a copy of the Koszul complex as a free summand. Since we may split off this summand, we assume that $d = 0$.

Proposition 4.1 (Maximal minor, codimension 3 obstruction). *Let M be as defined above, and continue with the same notation. Then*

$$b' - a + \operatorname{rank} T_1 + \operatorname{rank} U_3 \le c'.$$

Equivalently $c - d' + \operatorname{rank} T_1 + \operatorname{rank} U_3 \le b$.

Proof. By assumption, M has a minimal free resolution given by

$$0 \longrightarrow S(-4)^{d'} \xrightarrow{\left(\begin{smallmatrix} Q_3 \\ U_3 \end{smallmatrix}\right)} S(-2)^c \oplus S(-3)^{c'} \xrightarrow{\left(\begin{smallmatrix} T_2 & Q_2 \\ 0 & U_2 \end{smallmatrix}\right)} S(-1)^b \oplus S(-2)^{b'} \xrightarrow{(T_1\ Q_1)} S^a \longrightarrow M.$$

Each Q_i stands for a matrix of degree 2 polynomials. By [Buchsbaum and Eisenbud 1974] we know that each maximal minor of the middle matrix is the product of a corresponding maximal minor from the first matrix and a corresponding maximal minor from the third matrix.

Let $\tau = \operatorname{rank} T_1$ and $\mu = \operatorname{rank} U_3$. Since $\operatorname{codim} M \neq 0$, the rank of the matrix $(T_1\ \ Q_1)$ equals a. By thinking of this matrix over the quotient field $k(x, y, z)$, we may choose a basis of the column space which contains τ columns from T_1 and $a - \tau$ columns from Q_1. Let Δ_1 be the determinant of the resulting $a \times a$ submatrix, and observe that Δ_1 is nonzero. Similarly, we may construct a $d' \times d'$

minor Δ_3 from the last matrix such that Δ_3 is nonzero and involves μ rows from U_3 and $d' - \mu$ rows from Q_3.

Now consider the middle matrix

$$\begin{array}{c} \\ b \\ b' \end{array} \begin{array}{c} \begin{array}{cc} c & c' \end{array} \\ \begin{pmatrix} T_2 & Q_2 \\ 0 & U_2 \end{pmatrix} \end{array}.$$

Note that the columns of this matrix are indexed by the rows of the third matrix, and the rows of this matrix are indexed by the columns of the first matrix. Choose the unique maximal submatrix such that the columns repeat none of the choices from Δ_3 and such that the rows repeat none of the choices from Δ_1. We obtain a matrix of the shape

$$\begin{array}{c} \\ b - \tau \\ b' - a + \tau \end{array} \begin{array}{c} \begin{array}{cc} c - d' + \mu & c' - \mu \end{array} \\ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \end{array}.$$

Since M has finite length, the Herzog–Kühl conditions [1984] imply that $c' + c - d' = b + b' - a$, and thus this is a square matrix. If Δ_2 is the determinant of the matrix constructed above, then $\Delta_2 = \Delta_1 \Delta_3$ by [Buchsbaum and Eisenbud 1974]. Since $\Delta_1 \neq 0$ and $\Delta_3 \neq 0$, this implies that the $(b' - a + \tau \times c - d' + \mu)$ block of zeroes in the lower left corner cannot be too large. In particular,

$$b' - a + \tau + c - d' + \mu \leq b' + b - a.$$

By applying the Herzog–Kühl equality $c' + c - d' = b + b' - a$, we obtain the desired results. □

We now prove a couple of lemmas which will allow us to use this obstruction to rule out the virtual Betti diagrams from (4). We continue with the same notation, but without the assumption that $d = 0$.

Definition 4.2. A matrix T is *decomposable* if there exists a change of coordinates on the source and target of T such that T becomes block diagonal or such that T contains a column or row of all zeroes. If T is not decomposable then we say that T is *indecomposable*.

Lemma 4.3. *If the Betti diagram*

$$\begin{pmatrix} a & b & c & d \\ - & b' & c' & d' \end{pmatrix}$$

is Cohen–Macaulay and is a minimal generator of B_{mod}, *then* T_1 *is indecomposable or* $b = 0$.

Proof. If we project the semigroup B_{mod} onto its linear strand via

$$\begin{pmatrix} a & b & c & d \\ - & b' & c' & d' \end{pmatrix} \mapsto \begin{pmatrix} a & b & c & d \end{pmatrix},$$

then the image equals the semigroup of linear strands in B_{mod}. By the Herzog–Kühl equations, the linear strand

$$\begin{pmatrix} a & b & c & d \end{pmatrix}$$

of such a Cohen–Macaulay module determines the entire Betti diagram. Hence the projection induces an isomorphism between the subsemigroup of Cohen–Macaulay modules of codimension 3 in B_{mod} and the semigroup of linear strands in B_{mod}. The modules with T_1 decomposable and $b \neq 0$ cannot be minimal generators of the semigroup of linear strands in B_{mod}. □

Lemma 4.4. *Let the notation be as above.*

(a) *If there exists a free submodule $F \subseteq S(-1)^b$ such that $F \cong S(-1)^3$ and such that the restricted map $T_1|_F$ has rank* 1, *then the minimal resolution of M contains a copy of the Koszul complex as a direct summand.*

(b) *If $a = 2$, $b \geq 3$, and T_1 is indecomposable then T_1 has rank* 2*.*

Proof. (a) Given the setup of the lemma, we have that $T_1|_F$ is an $a \times 3$ matrix of rank 1 with linearly independent columns over k. All matrices of linear forms of rank 1 are compression spaces by [Eisenbud and Harris 1988]. Since the columns of $T_1|_F$ are linearly independent, this means that we may choose bases such that

$$T_1|_F = \begin{pmatrix} x & y & z \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{pmatrix}. \tag{5}$$

The result follows immediately.

(b) Assume that T_1 has rank 1 and apply part (a) with F any free submodule isomorphic to $S(-1)^3$. We may then assume that the first three columns of T_1 look like (5), and whether $b = 3$ or $b > 3$, it quickly follows that T_1 is decomposable. □

Proposition 4.5. *The virtual Betti diagrams*

$$D = \begin{pmatrix} 2 & 4 & 3 & - \\ - & 3 & 4 & 2 \end{pmatrix}, \quad D' = \begin{pmatrix} 3 & 6 & 4 & - \\ - & 4 & 6 & 3 \end{pmatrix}, \quad D'' = \begin{pmatrix} 2 & 3 & 2 & - \\ - & 5 & 7 & 3 \end{pmatrix}$$

do not belong to B_{mod}.

Proof. Assuming D were a Betti diagram, Lemma 4.3 implies that the corresponding matrices T_1 and U_3 are indecomposable. Lemma 4.4(b) implies that for D as in (5), we have $\operatorname{rank} T_1 = \operatorname{rank} U_3 = 2$. Observe that D now has a maximal minor obstruction, as $c - d' + \tau + \mu = 5$ while $b = 4$.

Next we consider D'. If D' were a Betti diagram, the corresponding T_1 and U_3 would both have to be indecomposable. If also T_1 had rank 2, then Theorem 1.1 of [Eisenbud and Harris 1988] would imply that it is a compression space. In particular, T_1 would have one of the following forms:

$$\begin{pmatrix} 0&0&0&0&*&* \\ 0&0&0&0&*&* \\ 0&0&0&0&*&* \end{pmatrix}, \quad \begin{pmatrix} 0&0&0&0&0&* \\ 0&0&0&0&0&* \\ *&*&*&*&*&* \end{pmatrix}, \quad \text{or} \quad \begin{pmatrix} 0&0&0&0&0&0 \\ *&*&*&*&*&* \\ *&*&*&*&*&* \end{pmatrix}.$$

The matrix forms on the left and right fail to be indecomposable. The middle form could not have linearly independent columns, since each $*$ stands for a linear form, and we are working over $k[x, y, z]$. Thus T_1 and U_3 both have rank 3, and it follows that D' has a maximal minor obstruction.

In the case of D'', similar arguments show that the ranks of T_1 and U_3 must equal 2 and 3 respectively. Thus D'' also has a maximal minor obstruction. □

Example 4.6. The diagram $2\cdot D$ belongs to B_{mod}. In fact, if $N = k[x, y, z]/(x, y, z)^2$ and $N^\vee = Ext^3(N, S)$, then

$$\beta(N \oplus N^\vee(4)) = \begin{pmatrix} 1 & - & - & - \\ - & 6 & 8 & 3 \end{pmatrix} + \begin{pmatrix} 3 & 8 & 6 & - \\ - & - & - & 1 \end{pmatrix} = \begin{pmatrix} 4 & 8 & 6 & - \\ - & 6 & 8 & 4 \end{pmatrix} = 2\cdot D.$$

This diagram does not have a maximal minor obstruction as $\operatorname{rank} T_1 = \operatorname{rank} U_3 = 3$.

Conversely, up to isomorphism the direct sum $N \oplus N^\vee(4)$ is the only module M whose Betti diagram equals $2\cdot D$. The key observation is that for M to avoid having a maximal minor obstruction, we must have that $\operatorname{rank} T_1 + \operatorname{rank} U_3 \le 6$. Thus we may assume that M is determined by a 4×8 matrix of linear forms which has rank at most 3. Such matrices are completely classified in [Eisenbud and Harris 1988], and an argument such as that in Proposition 4.5 can rule out all possibilities except that $M \cong N \oplus N^\vee(4)$.

In the proof of Theorem 1.6(4), we will show that $3 \cdot D$ does not belong to B_{mod}.

5. Special cases when $B_{\mathbb{N}} = B_{\text{mod}}$

In this section we prove Proposition 1.4 in two parts. We first deal with projective dimension 1.

Proposition 5.1. *Let $S = k[x]$ and fix $\underline{d} \le \bar{d}$. Then $B_{\mathbb{N}} = B_{\text{mod}}$. The semigroup B_{mod} is minimally generated by pure diagrams.*

Proof. Let $D \in B_{\mathbb{N}}$ be a virtual Betti diagram of projective dimension 1. We may assume that D is a Cohen–Macaulay diagram of codimension 1. Then the Herzog–Kühl conditions [1984] imply that D has the same number of generators and first syzygies. List the degrees of the generators of D in increasing order $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_s$, and list the degrees of the syzygies of D in increasing order $\gamma_1 \leq \gamma_2 \leq \cdots \leq \gamma_s$. Then $D \in B_{\mathbb{N}}$ if and only if we have

$$\alpha_i + 1 \leq \gamma_i$$

for $i = 1, \ldots, s$. Choose M to be a direct sum of the modules

$$M_i := \operatorname{coker}(\phi_i : R(-\gamma_i) \to R(-\alpha_i)),$$

where ϕ_i is represented by any element of degree $\gamma_i - \alpha_i$ in R. Note that $\beta(M_i)$ equals the pure diagram $\pi_{(\alpha_i,\gamma_i)}$. Thus $D \in B_{\mathrm{mod}}$ and $D = \beta(M) = \sum_i \pi_{(\alpha_i,\gamma_i)}$. □

Definition 5.2 [Boij 2000]. A graded module M is a *level module* if its generators are concentrated in a single degree and its socle is concentrated in a single degree.

We now show that in the case of projective dimension 2 level modules, the semigroups $B_{\mathbb{N}}$ and B_{mod} are equal.

Proposition 5.3. *Let $S = k[x, y]$ and fix $\underline{d} \leq \bar{d}$ such that $\underline{d}_0 = \bar{d}_0$ and $\underline{d}_2 = \bar{d}_2$. Then $B_{\mathbb{N}} = B_{\mathrm{mod}}$.*

Proof. We may assume that $\underline{d}_0 = 0$, and then we are considering the semigroup of level modules of projective dimension 2 with socle degree $(\underline{d}_2 - 2)$. Let $D \in B_{\mathbb{N}}$ and let c be a positive integer such that $cD \in B_{\mathrm{mod}}$. Let $\vec{h}(D) = (h_0, h_1, \ldots)$ be the Hilbert function of D. The main result of [Söderberg 2006] shows that $\vec{h}(D)$ is the Hilbert function of some level module of embedding dimension 2 if and only if $h_{i-1} - 2h_i + h_i \leq 0$ for all $i \leq \underline{d}_2 - 2$.

Since $cD \in B_{\mathrm{mod}}$, we know that $\vec{h}(cD) = c\vec{h}(D)$ is the Hilbert function of a level module. Thus

$$ch_{i-1} - 2ch_i + ch_i \leq 0.$$

The same holds when we divide by c, and thus $\vec{h}(D)$ is the Hilbert function of some level module M. Since M is also a level module, its Betti diagram must equal D. □

Remark 5.4. We conjectured above that $B_{\mathbb{N}} = B_{\mathrm{mod}}$ in general in projective dimension 2. Some evidence for this conjecture is provided by computations by Erman [≥ 2009] which prove that all virtual Betti diagrams of projective dimension 2 and generated in a single degree are "unobstructed" in the sense of Proposition 3.1.

6. The structure of $B_{\mathbb{N}} \setminus B_{\mathbf{mod}}$

We are now prepared to prove Theorem 1.6 and thus show that for projective dimension greater than 2, the semigroups $B_{\mathbb{N}}$ and B_{mod} diverge.

The various pieces of the theorem follow from a collection of obstructed virtual Betti diagrams.

Proof of Theorem 1.6(1): B_{mod} is not necessarily a saturated semigroup. We will show that on the ray corresponding to

$$D_1 = \begin{pmatrix} 1 & 2 & - & - \\ - & - & 2 & 1 \end{pmatrix},$$

every lattice point except D_1 itself belongs to B_{mod}. We have seen in (1) that $D_1 \notin B_{\mathrm{mod}}$. Certainly $2 \cdot D_1 \in B_{\mathrm{mod}}$ as $2 \cdot D$ is the Buchsbaum–Rim complex on a generic 2×4 matrix of linear forms. We claim that $3 \cdot D_1$ also belongs to B_{mod}. In fact, if we set $S = k[x, y, z]$ and

$$M := \operatorname{coker} \begin{pmatrix} x & y & z & 0 & 0 & 0 \\ 0 & 0 & x & y & z & 0 \\ x+y & 0 & 0 & x & y & z \end{pmatrix},$$

then the Betti diagram of M is $3 \cdot D_1$. □

Proof of Theorem 1.6(2): $|B_{\mathbb{N}} \setminus B_{\mathrm{mod}}|$ may be infinite. We will show that for all $\alpha \in \mathbb{N}$, the virtual Betti diagram

$$E_\alpha := \begin{pmatrix} 2+\alpha & 3 & 2 & - \\ - & 5+6\alpha & 7+8\alpha & 3+3\alpha \end{pmatrix}$$

does not belong to B_{mod}.

Note that $E_0 \notin B_{\mathrm{mod}}$ by Proposition 4.5. Imagine now that $\beta(M) = E_\alpha$ for some α. Let T_1 be the linear part of the presentation matrix of M so that T_1 is an $(\alpha+2) \times 3$ matrix of linear forms. Let T_2 be the (3×2) matrix of linear second syzygies and write

$$T_1 \cdot T_2 = \begin{pmatrix} l_{1,1} & l_{1,2} & l_{1,3} \\ l_{2,1} & l_{2,2} & l_{2,3} \\ \vdots & \vdots & \vdots \end{pmatrix} \cdot \begin{pmatrix} s_{1,1} & s_{1,2} \\ s_{2,1} & s_{2,2} \\ s_{3,1} & s_{3,2} \end{pmatrix}.$$

By Lemma 4.4(a), the rank of T_1 must be at least 2. Let T_1' be the top two rows of T_1, and by shuffling the rows of T_1, we may assume that the rank of T_1' equals 2. So then may assume that $l_{1,1}$ and $l_{2,2}$ are nonzero. Since each column of T_2 has at least 2 nonzero entries, it follows that the syzygies represented by T_2 remain nontrivial syzygies on the columns of T_1'.

It is possible however that columns of T_1' are not k-linearly independent. But since the rank of T_1' equals 2, we know that at least two of the columns are linearly independent. Let C be the cokernel of T_1', and let $M' := C_{\leq 1}$ be the truncation of C in degrees greater than 1. Then we would have

$$\beta(M') = \begin{pmatrix} 2 & 3 & 2 & - \\ - & 5 & 7 & 3 \end{pmatrix} \text{ or } = \begin{pmatrix} 2 & 2 & 2 & - \\ - & * & * & * \end{pmatrix}.$$

The first case is impossible by Proposition 4.5, and the second case does not even belong to $B_{\mathbb{N}}$. □

Proof of Theorem 1.6(3): A ray of B_{mod} *can miss* $\dim S - 2$ *consecutive lattice points.* Fix some prime $P \geq 2$ and let

$$S = k[x_1, \ldots, x_{P+1}].$$

Consider the degree sequence

$$d = (0, 1, P+1, P+2, \ldots, 2P).$$

We will show that the first $P - 1$ lattice points of the ray r_d have a codimension obstruction.

Let $\overline{\pi}_d$ be the pure diagram of type d where we fix $\beta_{0,0}(\overline{\pi}_d) = 1$. We claim that

- $\beta_{1,1}(\overline{\pi}_d) = 2$, and
- all the entries of $\beta(\overline{\pi}_d)$ are positive integers.

For both claims we use the formula $\beta_{i,d_i}(\overline{\pi}_d) = \prod_{k \neq i} \frac{d_k}{(-1)^k (d_i - d_k)}$. We first compute

$$\beta_{1,1}(\overline{\pi}_d) = \frac{(P+1) \cdot \cdots \cdot (2P-1) \cdot (2P)}{(P \cdot (P+1) \ldots (2P-1))} = \frac{2P}{P} = 2.$$

For the other entries of $\overline{\pi}_d$ we compute

$$\begin{aligned} \beta_{i,d_i}(\overline{\pi}_d) &= \frac{2P \cdot (2P-1) \cdot \cdots \cdot (P+1)}{(i-2)!\,(P-i+1)!} \cdot \frac{1}{P+i-1} \cdot \frac{1}{P+i-2} \\ &= \frac{1}{P} \binom{P+i-3}{i-2} \binom{2P}{P-i+1}. \end{aligned}$$

Note that $\binom{2P}{P-i+1}$ is divisible by P for all $i \geq 2$ and thus $\beta_{i,d_i}(\overline{\pi}_d)$ is an integer as claimed.

Since $\beta_{0,0} = 1$ and $\beta_{1,1} = 2$, the diagram $c \cdot \overline{\pi}_d$ has a codimension obstruction for $c = 1, \ldots, P-1$. Thus the first $P - 1$ lattice points of the ray of π_d do not correspond to Betti diagrams. □

Proof of Theorem 1.6(4): There exist rays of $B_{\mathbb{N}}$ where the points of B_{mod} are nonconsecutive lattice points. Consider the ray corresponding to

$$D_2 = \begin{pmatrix} 2 & 4 & 3 & - \\ - & 3 & 4 & 2 \end{pmatrix}.$$

Proposition 4.5 shows that D_2 does not belong to B_{mod}. In Example 4.6 we showed that $2 \cdot D_2$ does belong to B_{mod}. Thus, it will be sufficient to show that

$$3 \cdot D_2 = \begin{pmatrix} 6 & 12 & 9 & - \\ - & 9 & 12 & 6 \end{pmatrix}$$

does not belong to B_{mod}.

We assume for a contradiction that there exists M such that $\beta(M) = 3 \cdot D_2$. Then the minimal free resolution of M is

$$0 \longrightarrow R(-4)^6 \xrightarrow{\begin{pmatrix} Q_3 \\ U_3 \end{pmatrix}} R(-2)^9 \oplus R(-3)^{12} \xrightarrow{\begin{pmatrix} T_2 & Q_2 \\ 0 & U_2 \end{pmatrix}} R(-1)^{12} \oplus R(-2)^9 \xrightarrow{\begin{pmatrix} T_1 & Q_1 \end{pmatrix}} R^6 \quad (6)$$

where T_1, T_2, U_2 and U_3 are matrices of linear forms. By Proposition 4.1 we have that $\operatorname{rank} T_1 + \operatorname{rank} U_3 \leq 9$. Since the diagram $3 \cdot D_2$ is Cohen–Macaulay and symmetric, we may use Remark 3.2 to assume that $\operatorname{rank} T_1 \leq 4$.

We next use the fact that, after a change of coordinates, T_2 contains a second syzygy which involves only 2 of the variables of S. This is proved in Lemma 6.1 below. Change coordinates so that the first column of T_2 represents this second syzygy and equals

$$\begin{pmatrix} y \\ -x \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Since T_1 must be indecomposable, we may put T_1 into the form

$$T_1 = \begin{pmatrix} x & y & z & 0 & \dots & 0 \\ 0 & 0 & * & * & \dots & * \\ \vdots & & & & & \vdots \\ 0 & 0 & * & * & \dots & * \end{pmatrix}. \quad (7)$$

Now set $\widetilde{T_1}$ to be the lower right corner of $*$'s in T_1. Since $\operatorname{rank} T_1 \leq 4$ we have that $\operatorname{rank} \widetilde{T_1} \leq 3$. Matrices of rank ≤ 3 are fully classified, and by applying Corollary 1.4 of [Eisenbud and Harris 1988] we conclude that $\widetilde{T_1}$ is a compression space. We can rule out the compression spaces cases where $\widetilde{T_1}$ has a column or a row equal to zero, or else T_1 would have been decomposable. Thus $\widetilde{T}_1$ is equivalent to one of the two following forms:

$$\begin{pmatrix} 0&0&0&0&0&0&0&0&0&* \\ 0&0&0&0&0&0&0&0&0&* \\ 0&0&0&0&0&0&0&0&0&* \\ 0&0&0&0&0&0&0&0&0&* \\ *&*&*&*&*&*&*&*&*&* \\ *&*&*&*&*&*&*&*&*&* \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0&0&0&0&0&0&0&0&*&* \\ 0&0&0&0&0&0&0&0&*&* \\ 0&0&0&0&0&0&0&0&*&* \\ 0&0&0&0&0&0&0&0&*&* \\ 0&0&0&0&0&0&0&0&*&* \\ *&*&*&*&*&*&*&*&*&* \end{pmatrix}.$$

If we substitute the matrix on the left into the form for T_1 from (7), then we see that T_1 would have 8 k-linearly independent columns which are supported on only the bottom two rows. Since all entries of T_1 are linear forms in $k[x, y, z]$, this is impossible. We can similarly rule out the possibility of the matrix on the right. □

Lemma 6.1. *If there exists a minimal resolution as in* (6), *then the matrix T_2 contains a second syzygy involving only* 2 *variables of S.*

Proof. Assume that this is not the case and quotient by the variable z. Then the quotient matrices $\overline{T}_1$ and $\overline{T}_2$ still multiply to 0. It is possible that after quotienting, some of the columns of T_1 are dependent. However this is not possible for T_2. For if some combination went to 0 after quotienting by z, then there would exist a column of T_2, that is, a second syzygy of M, which involves only the variable z. This is clearly impossible. Thus the columns of $\overline{T}_2$ are linearly independent.

Nevertheless, we know that the columns of a 6×12 matrix of linear forms over $k[x, y]$ can satisfy at most 6 independent linear syzygies. By changing coordinates we may arrange that 3 of the columns of $\overline{T}_2$ are *trivial* syzygies on $\overline{T}_1$. By trivial syzygy, we mean a column of $\overline{T}_2$ where the nonzero entries of that columns multiply with zero entries of $\overline{T}_1$. For an example of how a nontrivial syzygy over $k[x, y, z]$ can become trivial after quotienting by z, consider

$$\begin{pmatrix} x & z & 0 \\ y & 0 & z \end{pmatrix} \begin{pmatrix} z \\ -x \\ -y \end{pmatrix} \to \begin{pmatrix} x & 0 & 0 \\ y & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ -x \\ -y \end{pmatrix}.$$

Change coordinates so that the first 3 columns of $\overline{T}_2$ represent the trivial syzygies and are in Kronecker normal form. By assumption, each column of $\overline{T}_2$ involves both x and y, so these first 3 columns must consist of combinations of the Kronecker blocks

$$B_1 = \begin{pmatrix} x \\ y \end{pmatrix}, \quad B_2 = \begin{pmatrix} x & 0 \\ y & x \\ 0 & y \end{pmatrix}, \quad B_3 = \begin{pmatrix} x & 0 & 0 \\ y & x & 0 \\ 0 & y & x \\ 0 & 0 & y \end{pmatrix}.$$

Since each nonzero entry in the trivial part of $\overline{T}_2$ must multiply with a 0 from $\overline{T}_1$, this forces certain columns of $\overline{T}_1$ to equal 0. More precisely, the number of nonzero rows in the trivial part of $\overline{T}_2$ is a lower bound for the number of columns of $\overline{T}_1$ which are identically zero. The block decomposition shows that the trivial part of $\overline{T}_2$ has at least 4 nonzero rows, and thus $\overline{T}_1$ has at least 4 columns which are identically zero.

But now the nonzero part of $\overline{T}_1$ is a 6×8 matrix of linear forms, and this can satisfy at most 4 linear syzygies. This forces two *additional* columns of $\overline{T}_2$ to be trivial syzygies which in turn forces more columns of $\overline{T}_1$ to equal zero, and so on.

Working through this iterative process, we eventually conclude that $\overline{T}_1$ contains 8 columns which are identically zero. This means that T_1 must have contained 8 columns which involved only z. But since T_1 is a 6×12 matrix of linear forms with linearly independent columns, this is impossible. □

Remark 6.2. Consider the diagram

$$D = \frac{a}{2}\pi_{(0,1,2,4)} + \frac{b}{2}\pi_{(0,2,3,4)} = \begin{pmatrix} \frac{3a+b}{2} & 4a & 3a & - \\ - & 3b & 4b & \frac{a+3b}{2} \end{pmatrix}.$$

Clearly $D \in B_{\mathbb{N}}$ if and only if $a+b$ is even. By an argument analogous to that in the proof of Theorem 1.6(2), one can show that $D \notin B_{\mathrm{mod}}$ if $a = 1$ or $b = 1$.

Recent unpublished work of Eisenbud and Schreyer uses this example to greatly strengthen parts (2) and (4) of Theorem 1.6. They show that $D \notin B_{\mathrm{mod}}$ whenever a is odd (or equivalently whenever b is odd). Furthermore, they show that if M is any module such that

$$\beta(M) = a'\pi_{(0,1,2,4)} + b'\pi_{(0,2,3,4)},$$

then the module M splits into a direct sum of the pure pieces. Namely, $M \cong M' \oplus M''$ where $\beta(M') = a'\pi_{(0,1,2,4)}$ and $\beta(M'') = b'\pi_{(0,2,3,4)}$. Similar results are shown to hold in codimension greater than 3.

Based on a generalization of Eisenbud and Schreyer's methods, we have recently computed all generators for B_{mod} when $\underline{d} = (0,1,2,3)$ and $\overline{d} = (1,2,3,4)$. This computation will appear in [Erman $\geq$ 2009].

7. Further questions

An ambitious question is whether we can find a better description of B_{mod} or compile a complete list of obstructions. Here are several more specific questions. A further list of questions is compiled in [Erman et al. 2008].

(1) Bounds on B_{mod}: Can we bound the number of generators of the semigroup of Betti diagrams? Can we bound the size of a minimal generator of the semigroup of Betti diagrams?

(2) The behavior of single rays: Given a degree sequence d, what is the minimal c_d such that $c_d \pi_d$ is the Betti diagram of some module? In many cases where computation is feasible, it is known that the examples produced by Eisenbud et al. [2007] and Eisenbud and Schreyer [2009] do not represent the first element of B_{mod} on the ray. In some other cases, it is known that π_d itself does not belong to B_{mod} so that c_d is greater than 1. Can we find better lower and upper bounds for the integer c_d?

(3) Dependence on the characteristic: Schreyer's conjecture that the semigroup of Betti diagrams depends on the characteristic of k has recently been proved by Kunte [2008, Corollary 2.4.10]. In particular, Kunte shows that the virtual Betti diagram

$$\begin{pmatrix} 1 & - & - & - & - & - \\ - & 10 & 16 & - & - & - \\ - & - & - & 16 & 10 & - \\ - & - & - & - & - & 1 \end{pmatrix}$$

is not the Betti diagram of a finite length algebra when the characteristic of k equals 2. It was previously known that this is a Betti diagram when the characteristic of k equals 0. To what extent does B_{mod} depend on the characteristic? Can we find obstructions which only live in specific characteristics?

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