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Ideals generated by submaximal minors

Jan O. Kleppe and Rosa M. Miró-Roig

The goal of this paper is to study irreducible families $W^{t-1}_{t,t}(\underline{b};\underline{a})$ of codimension 4, arithmetically Gorenstein schemes $X \subset \mathbb{P}^n$ defined by the submaximal minors of a $t \times t$ homogeneous matrix $\mathcal{A}$ whose entries are homogeneous forms of degree $a_j - b_i$. Under some numerical assumption on a_j and b_i, we prove that the closure of $W^{t-1}_{t,t}(\underline{b};\underline{a})$ is an irreducible component of $\mathrm{Hilb}^{p(x)}(\mathbb{P}^n)$, show that $\mathrm{Hilb}^{p(x)}(\mathbb{P}^n)$ is generically smooth along $W^{t-1}_{t,t}(\underline{b};\underline{a})$, and compute the dimension of $W^{t-1}_{t,t}(\underline{b};\underline{a})$ in terms of a_j and b_i. To achieve these results we first prove that X is determined by a regular section of $\mathcal{I}_Y/\mathcal{I}_Y^2(s)$ where $s = \deg(\det \mathcal{A})$ and $Y \subset \mathbb{P}^n$ is a codimension-2, arithmetically Cohen–Macaulay scheme defined by the maximal minors of the matrix obtained deleting a suitable row of $\mathcal{A}$.

1. Introduction

In this paper we deal with determinantal schemes. A scheme $X \subset \mathbb{P}^n$ of codimension c is called *determinantal* if its homogeneous saturated ideal can be generated by the $r \times r$ minors of a homogeneous $p \times q$ matrix with $c = (p-r+1)(q-r+1)$. When $r = \min(p,q)$ we say that X is *standard determinantal*. Given integers $r \le p \le q$, $a_1 \le a_2 \le \ldots \le a_p$, and $b_1 \le b_2 \le \ldots \le b_q$, we denote by $W^r_{p,q}(\underline{b};\underline{a}) \subset \mathrm{Hilb}^{p(x)}(\mathbb{P}^n)$ the locus of determinantal schemes $X \subset \mathbb{P}^n$ of codimension $c = (p-r+1)(q-r+1)$ defined by the $r \times r$ minors of a $p \times q$ matrix $(f_{ji})^{i=1,\ldots,q}_{j=1,\ldots,p}$ where $f_{ji} \in k[x_0, x_1, \ldots, x_n]$ is a homogeneous polynomial of degree $a_j - b_i$.

The study of determinantal schemes has received considerable attention in the literature [Bruns and Vetter 1988; Hochster and Eagon 1971; Eagon and Northcott 1962; Miró-Roig 2008]. Some classical schemes that can be constructed in this way are the Segre varieties, rational normal scrolls, and the Veronese varieties. This paper contributes to the classification of determinantal schemes, and addresses, in the case $p = q = t,\ r = t-1$, three fundamental problems:

(1) determining the dimension of $W^r_{p,q}(\underline{b};\underline{a})$ in terms of a_j and b_i,

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(2) determining whether the closure of $W^r_{p,q}(\underline{b};\underline{a})$ is an irreducible component of $\mathrm{Hilb}^{p(x)}(\mathbb{P}^n)$, and

(3) determining when $\mathrm{Hilb}^{p(x)}(\mathbb{P}^n)$ is generically smooth along $W^r_{p,q}(\underline{b};\underline{a})$.

The first important contribution to these problems was made by Ellingsrud [1975], who proved that every arithmetically Cohen–Macaulay, closed subscheme X of codimension 2 of $\mathbb{P}^n$ is unobstructed (that is, the corresponding point in the Hilbert scheme $\mathrm{Hilb}^{p(x)}(\mathbb{P}^n)$ is smooth) provided $n \geq 3$. He also computed the dimension of the Hilbert scheme at (X).

Recall that the homogeneous ideal of an arithmetically Cohen–Macaulay closed subscheme of codimension 2 of $\mathbb{P}^n$ is given by the maximal minors of a $(t-1)\times t$ homogeneous matrix, the Hilbert–Burch matrix; that is, such a scheme is standard determinantal. The purpose of this work is to extend Ellingsrud's Theorem, viewed as a statement on standard determinantal schemes of codimension 2, to arbitrary determinantal schemes. The case of codimension-3 standard determinantal schemes was mainly solved in [Kleppe et al. 2001, Proposition 1.12], and the case of standard determinantal schemes of arbitrary codimension was studied and partially solved in [Kleppe and Miró-Roig 2005]. In [Kleppe and Miró-Roig 2007], we treated the case of codimension-3 determinantal schemes $X \subset \mathbb{P}^n$ defined by the submaximal minors of a symmetric homogeneous matrix. In our opinion, it is difficult to solve the above three questions in full generality, and, in this paper, we will focus our attention on the first unsolved case; that is, we will deal with codimension-4 determinantal schemes $X \subset \mathbb{P}^n$, $n \geq 5$, defined by the submaximal minors of a homogeneous square matrix. As in [Kleppe et al. 2001; Kleppe and Miró-Roig 2005; Kleppe and Miró-Roig 2007], we prove our results by considering the smoothness of the Hilbert flag scheme of pairs, or, more generally, the Hilbert flag scheme of chains of closed subschemes obtained by deleting suitable rows, and its natural projections into the usual Hilbert schemes. We wonder if a similar strategy could facilitate the study of the general case.

Here we outline the structure of the paper. In Section 2, we recall the basic facts about local cohomology and deformation theory needed in what follows. In Section 3, we describe the deformations of the codimension-4 arithmetically Gorenstein schemes $X \subset \mathbb{P}^n$ defined as the degeneracy locus of a regular section of the twisted conormal sheaf $\mathcal{I}_Y/\mathcal{I}_Y^2(s)$ of a codimension-2, arithmetically Cohen–Macaulay scheme $Y \subset \mathbb{P}^n$ of dimension ≥ 3. Section 4 is the heart of the paper. There we determine the dimension of $W^{t-1}_{t,t}(\underline{b};\underline{a})$ in terms of b_i and a_j provided $a_i \geq b_{i+3}$ for $1 \leq i \leq t-3$ (and $a_1 \geq b_t$ if $t \leq 3$), $a_t > a_{t-1} + a_{t-2} - b_1$ and $\dim X \geq 1$. We also prove that, under this numerical restriction, $\mathrm{Hilb}^{p(x)}(\mathbb{P}^n)$ is generically smooth along $W^{t-1}_{t,t}(\underline{b};\underline{a})$, and that the closure of $W^{t-1}_{t,t}(\underline{b};\underline{a})$ is an irreducible component of $\mathrm{Hilb}^{p(x)}(\mathbb{P}^n)$ (see Theorem 4.6).

The key point in proving our result is the fact that any codimension-4, determinantal scheme $X \subset \mathbb{P}^n$ defined by the submaximal minors of a homogeneous square matrix $\mathscr{A}$ is arithmetically Gorenstein and determined by a regular section of $\mathscr{I}_Y/\mathscr{I}_Y^2(s)$ where $s = \deg(\det \mathscr{A})$ and $Y \subset \mathbb{P}^n$ is a codimension-2, arithmetically Cohen–Macaulay scheme defined by the maximal minors of the matrix $\mathscr{N}$ obtained deleting a suitable row of $\mathscr{A}$ (see Proposition 4.3). Conversely, any codimension-4, arithmetically Gorenstein scheme $X = \operatorname{Proj} A \subset \mathbb{P}^n$ defined by a regular section σ of $\mathscr{I}_Y/\mathscr{I}_Y^2(s)$ where $Y = \operatorname{Proj} B \subset \mathbb{P}^n$ is a codimension-2, arithmetically Cohen–Macaulay scheme, fits into an exact sequence

$$0 \longrightarrow K_B(n+1-2s) \longrightarrow N_B(-s) \xrightarrow{\sigma^*} B \longrightarrow A \longrightarrow 0,$$

and is determined by the submaximal minors of a $t \times t$ homogeneous matrix $\mathscr{A}$ obtained by adding a suitable row to the Hilbert–Burch matrix of Y (see Proposition 4.3). In Section 5, we include some examples which illustrate that the numerical hypothesis in Theorem 4.6, $a_t > a_{t-1} + a_{t-2} - b_1$, cannot be avoided.

Notation. Throughout this paper k will be an algebraically closed field k, $R = k[x_0, x_1, \ldots, x_n]$, $\mathfrak{m} = (x_0, \ldots, x_n)$ and $\mathbb{P}^n = \operatorname{Proj} R$. As usual, the sheafification of a graded R-module M will be denoted by $\widetilde{M}$ and the support of M by $\operatorname{Supp} M$.

Given a closed subscheme X of $\mathbb{P}^n$ of codimension c, we denote by $\mathscr{I}_X$ its ideal sheaf, by $\mathscr{N}_X$ its normal sheaf, and by $I(X) = H^0_*(\mathbb{P}^n, \mathscr{I}_X)$ its saturated homogeneous ideal unless $X = \varnothing$, in which case we let $I(X) = \mathfrak{m}$. If X is equidimensional and Cohen–Macaulay of codimension c, we set $\omega_X = \mathscr{E}xt^c_{\mathbb{O}_{\mathbb{P}^n}}(\mathbb{O}_X, \mathbb{O}_{\mathbb{P}^n})(-n-1)$ to be its canonical sheaf.

In the sequel, for any graded quotient A of R of codimension c, we let $I_A = \ker(R \twoheadrightarrow A)$, $N_A = \operatorname{Hom}_R(I_A, A)$ be the normal module. If A is Cohen–Macaulay of codimension c, we let $K_A = \operatorname{Ext}^c_R(A, R)(-n-1)$ be its canonical module. When we write $X = \operatorname{Proj} A$, we let $A = R/I(X)$ and $K_X = K_A$. If M is a finitely generated graded A-module, let $\operatorname{depth}_J M$ denote the length of a maximal M-sequence in a homogeneous ideal J and let $\operatorname{depth} M = \operatorname{depth}_{\mathfrak{m}} M$. If $\Gamma_J(-)$ is the functor of sections with support in $\operatorname{Spec}(A/J)$, we denote by $H^i_J(-)$ the right derived functor of $\Gamma_J(-)$.

Let $\operatorname{Hilb}^{p(x)}(\mathbb{P}^n)$ be the Hilbert scheme parameterizing closed subschemes X of $\mathbb{P}^n$ with Hilbert polynomial $p(x) \in \mathbb{Q}[x]$ [Grothendieck 1966]. By abuse of notation we will write $(X) \in \operatorname{Hilb}^{p(x)}(\mathbb{P}^n)$ for the k-point which corresponds to a closed subscheme $X \subset \mathbb{P}^n$. The Hilbert polynomial of X is sometimes denoted by p_X. By definition X is called unobstructed if $\operatorname{Hilb}^{p(x)}(\mathbb{P}^n)$ is smooth at (X).

The pullback of the universal family on $\operatorname{Hilb}^p(\mathbb{P}^n)$ via a morphism $\psi : W \longrightarrow \operatorname{Hilb}^p(\mathbb{P}^n)$ yields a flat family over W, and we will write $(X) \in W$ for a member of that family as well. Suppose that W is irreducible. Then, by definition, a general

$(X) \in W$ has a certain property if there is a nonempty open subset U of W such that all members of U have this property. Moreover, we say that (X) is general in W if it belongs to a sufficiently small open subset U of W (so any (X) in U has all the openness properties that we want to require).

Finally we let $D = D(p_X, p_Y)$ be the Hilbert flag scheme parameterizing pairs of closed subschemes $(X' \subset Y')$ of $\mathbb{P}^n$ with Hilbert polynomials $p_{X'} = p_X$ and $p_{Y'} = p_Y$, respectively.

2. Preliminaries

For the convenience of the reader we include in this section the background and basic results on local cohomology and deformation theory needed in the sequel.

2.1. ***Local cohomology.*** Let $B = R/I_B$ be a graded quotient of the polynomial ring R, let M and N be finitely generated graded B-modules and let $J \subset B$ be an ideal. We say that M (assumed nonzero) is *Cohen–Macaulay* if $\operatorname{depth} M = \dim M$ and *maximal Cohen–Macaulay* if $\operatorname{depth} M = \dim B$. Equivalently, since $\operatorname{depth}_J M \geq r$ is equivalent to $H^i_J(M) = 0$ for $i < r$, the module M is Cohen–Macaulay (resp. maximal Cohen–Macaulay) if $H^i_{\mathfrak{m}}(M) = 0$ for all $i \neq \dim M$ (resp. $i < \dim B$). If B is Cohen–Macaulay, we know by Gorenstein duality that the v-graded piece of $H^i_{\mathfrak{m}}(M)$ satisfies

$$ {}_vH^i_{\mathfrak{m}}(M) \simeq {}_{-v}\operatorname{Ext}^{\dim B - i}_B(M, K_B)^\vee.$$

Let Z be closed in $Y := \operatorname{Proj} B$ and let $U = Y - Z$. Then we have an exact sequence

$$0 \to H^0_{I(Z)}(M) \to M \to H^0_*(U, \widetilde{M}) \to H^1_{I(Z)}(M) \to 0$$

and isomorphisms $H^i_{I(Z)}(M) \simeq H^{i-1}_*(U, \widetilde{M})$ for $i \geq 2$, where as usual we write $H^i_*(U, \widetilde{M}) = \bigoplus_t H^i(U, \widetilde{M}(t))$. More generally, if $\operatorname{depth}_{I(Z)} N \geq i + 1$, there is an exact sequence

$$\begin{aligned} {}_0\operatorname{Ext}^i_B(M, N) \hookrightarrow \operatorname{Ext}^i_{\mathcal{O}_U}(\widetilde{M}|_U, \widetilde{N}|_U) \\ \to {}_0\operatorname{Hom}_B(M, H^{i+1}_{I(Z)}(N)) \to {}_0\operatorname{Ext}^{i+1}_B(M, N) \to \cdots \end{aligned} \quad (2\text{-}1)$$

by [Grothendieck 1968, exposé VI], where the middle form comes from a spectral sequence also treated in the same source.

2.2. ***Basic deformation theory.*** To use deformation theory, we will need to consider the (co)homology groups of algebras $H_2(R, B, B)$ and $H^2(R, B, B)$. Let us recall their definition. We consider

$$\cdots \to F_2 := \bigoplus_{j=1}^{\mu_2} R(-n_{2,j}) \to F_1 := \bigoplus_{i=1}^{\mu} R(-n_{1,i}) \to R \to B \to 0, \quad (2\text{-}2)$$

a minimal graded free R-resolution of B and let $H_1 = H_1(I_B)$ be the first Koszul homology built on a set of minimal generators of I_B. Then we may take the exact sequence

$$0 \to H_2(R, B, B) \to H_1 \to F_1 \otimes_R B \to I_B/{I_B}^2 \to 0 \tag{2-3}$$

as a definition of the second algebra homology $H_2(R, B, B)$ [Vasconcelos 1994], and the dual sequence

$$\to {}_v\mathrm{Hom}_B(F_1 \otimes B, B) \to {}_v\mathrm{Hom}_B(H_1, B) \to {}_vH^2(R, B, B) \to 0,$$

as a definition of graded second algebra cohomology $H^2(R, B, B)$. If B is generically a complete intersection, then it is well known that [André 1974, Proposition 16.1]:

$$\mathrm{Ext}^1_B(I_B/I_B^2, B) \simeq H^2(R, B, B).$$

We also know that $H^0(Y, \mathcal{N}_Y)$ is the tangent space of $\mathrm{Hilb}^{p(x)}(\mathbb{P}^n)$ in general, while $H^1(Y, \mathcal{N}_Y)$ contains the obstructions of deforming $Y \subset \mathbb{P}^n$ in the case in which Y is locally a complete intersection (l.c.i.) [Grothendieck 1966]. If ${}_0\mathrm{Hom}_R(I_B, H^1_{\mathfrak{m}}(B))$ vanishes (for example if $\mathrm{depth}_{\mathfrak{m}} B \geq 2$), we have by (2-1) that ${}_0\mathrm{Hom}_B(I_B/I_B^2, B) \simeq H^0(Y, \mathcal{N}_Y)$ and ${}_0H^2(R, B, B) \hookrightarrow H^1(Y, \mathcal{N}_Y)$ is injective in the l.c.i. case, and that ${}_0H^2(R, B, B)$ contains the obstructions of deforming $Y \subset \mathbb{P}^n$ [Kleppe 1979, Remark 3.7]. Thus ${}_0H^2(R, B, B) = 0$ suffices for the unobstructedness of an l.c.i. arithmetically Cohen–Macaulay subscheme Y of $\mathbb{P}^n$ of $\dim Y \geq 1$. For this conclusion we may even entirely skip "l.c.i." by slightly extending the argument, as done in [Kleppe 1979].

2.3. ***Useful exact sequences.*** In the last part of this section, we collect some exact sequences frequently used in this paper, in the case that $B = R/I_B$ is a generically complete intersection codimension-2 CM quotient of R. First, applying $\mathrm{Hom}_R(-, R)$ to the minimal graded free R-resolution of B,

$$0 \to F_2 := \bigoplus_{j=1}^{\mu-1} R(-n_{2,j}) \to F_1 := \bigoplus_{i=1}^{\mu} R(-n_{1,i}) \to R \to B \to 0, \tag{2-4}$$

we get a minimal graded free R-resolution of K_B:

$$0 \to R \to \bigoplus R(n_{1,i}) \to \bigoplus R(n_{2,j}) \to K_B(n+1) \to 0. \tag{2-5}$$

If we apply $\mathrm{Hom}(-, B)$ to (2-5) we get the exactness on the left in the exact sequence

$$0 \to K_B(n+1)^* \to \bigoplus B(-n_{2,j}) \to \bigoplus B(-n_{1,i}) \to I_B/I_B^2 \to 0, \tag{2-6}$$

which splits into two short exact sequences via $\bigoplus B(-n_{2,j}) \twoheadrightarrow H_1 \hookrightarrow \bigoplus B(-n_{1,i})$, one of which is (2-3) with $H_2(R, B, B) = 0$. Indeed since H_1 is Cohen–Macaulay by [Avramov and Herzog 1980], we get $H_2(R, B, B) = 0$ by (2-3). Moreover since

$\operatorname{Ext}^1_R(I_B, I_B) \simeq N_B$ we showed in [Kleppe and Peterson 2001, page 788] that there is an exact sequence of the form

$$0 \to F_1^* \otimes_R F_2 \to \big((F_1^* \otimes_R F_1) \oplus (F_2^* \otimes_R F_2)\big)/R \to F_2^* \otimes_R F_1 \to N_B \to 0, \quad (2\text{-}7)$$

where $F_i^* = \operatorname{Hom}_R(F_i, R)$. Indeed this sequence is deduced from the exact sequence

$$0 \to R \to \bigoplus I_B(n_{1,i}) \to \bigoplus I_B(n_{2,j}) \to N_B \to 0,$$

which we get by applying $\operatorname{Hom}_R(-, I_B)$ to (2-4) [Kleppe and Peterson 2001, (26)]. Similarly applying $\operatorname{Hom}_R(-, I_B/I_B^2)$ to (2-4), and noting that

$$\operatorname{Hom}_R(I_B, I_B/I_B^2) \simeq \operatorname{Hom}_B(I_B/I_B^2, I_B/I_B^2),$$

we get the exact sequence

$$0 \to \operatorname{Hom}_B(I_B/I_B^2, I_B/I_B^2) \to \bigoplus I_B/I_B^2(n_{1,i}) \to \bigoplus I_B/I_B^2(n_{2,j}) \to N_B \to 0. \quad (2\text{-}8)$$

Finally we recall the following frequently used exact sequence [Vasconcelos 1994]:

$$0 \to \bigwedge^2\big(\bigoplus R(-n_{2,j})\big) \to \big(\bigoplus R(-n_{1,i})\big) \otimes \big(\bigoplus R(-n_{2,j})\big) \to S^2\big(\bigoplus R(-n_{1,i})\big) \to I_B^2 \to 0\,. \quad (2\text{-}9)$$

3. Deformations of quotients of regular sections

In [Kleppe 2007] the first author studied deformations of a scheme $X := \operatorname{Proj} A$ defined as the degeneracy locus of a regular section of a "nice" sheaf $\widetilde{M}$ on an arithmetically Cohen–Macaulay (ACM) scheme $Y = \operatorname{Proj} B$. Recall that if we take a regular section of the anticanonical sheaf $\widetilde{K}_B^*(s)$ and Y is an l.c.i. of positive dimension, then we get an exact sequence

$$0 \to K_B(-s) \to B \to A \to 0,$$

in which A is Gorenstein. Indeed the mapping cone construction leads to a resolution of A from which we easily see that A is Gorenstein. In [Kleppe and Peterson 2001], we generalized this way of constructing Gorenstein algebras to sheaves of higher rank and, in [Kleppe 2007], we studied the deformations of this "construction", notably in the rank 2 case which we now recall.

Let M be a maximal Cohen–Macaulay B-module of rank $r = 2$ such that $\widetilde{M}|_U$ is locally free and $\bigwedge^2 \widetilde{M}|_U \simeq \widetilde{K_B}(t)|_U$ in an open set $U := Y - Z$ of Y satisfying $\operatorname{depth}_{I(Z)} B \geq 2$. Then a regular section σ of $\widetilde{M}^*(s)|_U$ defines an arithmetically Gorenstein scheme $X = \operatorname{Proj} A$ given by the exact sequence

$$0 \to K_B(t-2s) \to M(-s) \xrightarrow{\sigma^*} B \to A \to 0, \quad (3\text{-}1)$$

and $M \simeq \mathrm{Hom}_B(M, K_B(t))$ by Theorem 8 of [Kleppe and Peterson 2001]. In this paper we consider and further develop the case where $M = N_B$ and $\dim B = n-1$ ($n+1 = \dim R$, $n \geq 5$). By Proposition 13 of the same reference, N_B is a maximal Cohen–Macaulay B-module and we have the exact sequence

$$0 \to K_B(n+1-2s) \to N_B(-s) \to I_{A/B} \to 0, \quad \text{where } I_{A/B} := \ker(B \to A). \tag{3-2}$$

Example 3.1. Set $R = k[x_0, \ldots, x_5]$ and let $B = R/I_B$ be a codimension-2 quotient with minimal resolution

$$0 \to R(-3)^2 \to R(-2)^3 \to R \to B \to 0,$$

and suppose $Y = \mathrm{Proj}\, B$ is an l.c.i. in $\mathbb{P}^5$. Let A be given by a regular section of $\widetilde{I_B/I_B^2}(s)$, $s \geq 3$. Thanks to the exact sequences (2-5) and (2-7) and the mapping cone construction applied to both (3-2) and $0 \to I_{A/B} \to B \to A \to 0$, we get the following resolution of the Gorenstein algebra A:

$$0 \to R(-2s) \to R(2-2s)^3 \oplus R(-1-s)^6 \to R(3-2s)^2 \oplus R(-s)^{12} \oplus R(-3)^2 \\ \to R(1-s)^6 \oplus R(-2)^3 \to R \to A \to 0\,.$$

Indeed $X = \mathrm{Proj}\, A$ is an arithmetically Gorenstein curve of degree $d = 3s^2-10s+9$ and arithmetic genus $g = 1 + d(s-3)$ in $\mathbb{P}^5$ [Kleppe 2007, Example 43].

With M and A as above, it turns out that [Kleppe 2007, Theorems 1 and 25] describes the deformations space, GradAlg R, of the graded quotient A and computes the dimension of GradAlg R in terms of a number $\delta := \delta(K_B)_{t-2s} - \delta(M)_{-s}$, where

$$\delta(N)_v := {}_v\mathrm{hom}_B(I_B/I_B^2, N) - {}_v\mathrm{ext}^1_B(I_B/I_B^2, N). \tag{3-3}$$

Here we have used small letters for the k-dimension of ${}_v\mathrm{Ext}^i_B(-,-)$ and of similar groups. If we suppose $M = N_B$, $\mathrm{depth}_{I(Z)} B \geq 4$ and $\mathrm{char}\, k \neq 2$, then the conditions of parts A and B of [Kleppe 2007, Theorem 25] are satisfied provided ${}_0\mathrm{Ext}^2_B(N_B, N_B) = 0$ or ${}_{-s}\mathrm{Ext}^1_B(I_B/I_B^2, N_B) = 0$, respectively. In both cases X is unobstructed and

$$\dim_{(X)} \mathrm{Hilb}^{p(x)}(\mathbb{P}^n) = \\ \dim(N_B)_0 + \dim(I_B/I_B^2)_s - {}_0\mathrm{hom}_B(I_B/I_B^2, I_B/I_B^2) + \dim(K_B)_{t-2s} + \delta, \tag{3-4}$$

where $t = n+1$ [Kleppe 2007, Corollary 41 and its proof and Remark 42]. Using the exact sequence (2-7) we get ${}_{-s}\mathrm{Ext}^1_B(I_B/I_B^2, N_B) = 0$ for $s > 2\max n_{2,j} - \min n_{1,i}$ which led to Corollary 41 of [Kleppe 2007] which we slightly generalize in Corollary 3.2(i) below. The A-part was considered in [Kleppe 2007, Remark 42]. By the proof of [Kleppe 2007, Theorem 25] we may replace the vanishing of ${}_0\mathrm{Ext}^2_B(N_B, N_B)$ by the vanishing of the subgroup ${}_t\mathrm{Ext}^2_B(S^2(I_{A/B}(s)), K_B)$ and still

get all conclusions of the A-part. Therefore, we can also prove (ii) of the following corollary to [Kleppe 2007, Theorem 25].

Corollary 3.2. *Let $B = R/I_B$ be a codimension-2 CM quotient of R, let $U = \operatorname{Proj} B - Z \hookrightarrow \mathbb{P}^n$ be an l.c.i., and suppose $\operatorname{depth}_{I(Z)} B \geq 4$. Let A be given by a regular section of $\widetilde{N_B}^*(s)$ on U, let $\eta(v) := \dim(I_B/I_B^2)_v$, and put*

$$\epsilon := \eta(s) + \sum_{j=1}^{\mu-1} \eta(n_{2,j}) - \sum_{i=1}^{\mu} \eta(n_{1,i}).$$

(i) *Let j_0 satisfy $n_{2,j_0} = \max n_{2,j}$. If $s > n_{2,j_0} + \max_{j\neq j_0} n_{2,j} - \min n_{1,i}$ and $\operatorname{char} k \neq 2$, then X is a p_Y-generic unobstructed arithmetically Gorenstein subscheme of $\mathbb{P}^n$ of codimension 4 and $\dim_{(X)} \operatorname{Hilb}^{p(x)}(\mathbb{P}^n) = \epsilon$.*

(ii) *If ${}_s\operatorname{Ext}^1_B(N_B, A) = 0$, $\operatorname{char} k = 0$, $s > \max n_{2,j}/2$ and $(X \subset Y)$ is general, then X is unobstructed, $\dim_{(X)} \operatorname{Hilb}^{p(x)}(\mathbb{P}^n) = \epsilon + \delta$ and the codimension of the stratum in $\operatorname{Hilb}^{p(x)}(\mathbb{P}^n)$ of subschemes given by (3-1) is ${}_0\operatorname{ext}^1_B(I_B/I_B^2, I_{A/B})$. Moreover if $s > \max n_{2,j} + \max n_{1,i} - \min n_{1,i}$ we have ${}_0\operatorname{ext}^1_B(I_B/I_B^2, I_{A/B}) = {}_{-s}\operatorname{ext}^1_B(I_B/I_B^2, N_B) = \delta$, while if $s > \max n_{2,j}$ we have ${}_0\operatorname{ext}^1_B(I_B/I_B^2, I_{A/B}) = {}_{-s}\operatorname{ext}^1_B(I_B/I_B^2, N_B)$.*

Here $I_{A/B} = \ker(B \to A)$ and "X is p_Y-generic" if there is an open subset of $\operatorname{Hilb}^{p(x)}(\mathbb{P}^n)$ containing (X) whose members X' are subschemes of some closed Y' with Hilbert polynomial p_Y. The stratum in $\operatorname{Hilb}^{p(x)}(\mathbb{P}^n)$ of subschemes given by (3-1) around (X) is defined by functorially varying both B, M and the regular section around $(B \to A)$ [Kleppe 2007, the definition before Theorem 25]. Indeed it is proved in [Kleppe 2007, Lemma 2.9] that pairs of closed subschemes $(X' \subset Y')$ of $\mathbb{P}^n$, $X' = \operatorname{Proj} A'$ and $Y' = \operatorname{Proj} B'$, obtained as in (3-1), contain *an open subset* $U \ni (X \subset Y)$ in the Hilbert flag scheme D, and taking such a U small enough, we may define the mentioned stratum to be $p(U)$ where $p : D \to \operatorname{Hilb}^{p(x)}(\mathbb{P}^n)$ is the projection morphism induced by $(X' \subset Y') \to (X')$. Thus "X is p_Y-generic" essentially means that the codimension of the stratum of subschemes given by (3-1) around (X) is zero.

Note also that "$(X \subset Y)$ is general" means that it is the general member of an irreducible (nonembedded) component of the Hilbert flag scheme D. Since we, in Corollary 3.2 suppose $\operatorname{depth}_{I(Z)} B \geq 4$ and hence $\operatorname{depth}_{\mathfrak{m}} A \geq 2$, this is equivalent to saying that $(B \to A)$ is the general member of an irreducible (nonembedded) component of the "Hilbert flag scheme" parameterizing pairs of quotients of R with fixed Hilbert functions. Indeed we can replace the schemes $\operatorname{GradAlg} R$ of [Kleppe 2007] by $\operatorname{Hilb}^{p(x)}(\mathbb{P}^n)$ because we work with algebras of depth *at least* 2 at $\mathfrak{m}$ [Ellingsrud 1975; Kleppe 1979, Remark 3.7].

Proof. By the text before (3-4), in order to prove (i) it is sufficient to show that ${}_{-s}\mathrm{Ext}^1_B(I_B/I_B^2, N_B) = 0$. To see it we observe that

$$\mathrm{Ext}^1_B(I_B/I_B^2, N_B) \simeq \mathrm{Ext}^1_B(T_B, K_B(n+1)),$$

where $T_B := \mathrm{Hom}_B(I_B/I_B^2, I_B/I_B^2)$ by [Kleppe 2007, Remark 42]. We consider the exact sequence (2-8) and we define $F := \ker(\bigoplus I_B/I_B^2(n_{2,j}) \to N_B)$. Since N_B is a maximal CM B-module and I_B/I_B^2 has codepth 1 (that is, $\mathrm{Ext}^i_B(I_B/I_B^2, K_B) = 0$ for $i \geq 2$) by [Avramov and Herzog 1980] or (2-9), we get $\mathrm{Ext}^2_B(F, K_B) = 0$. It follows that

$$\mathrm{Ext}^1_B\big(\bigoplus I_B/I_B^2(n_{1,i}), K_B(n+1)\big) \to \mathrm{Ext}^1_B(T_B, K_B(n+1))$$

is surjective. Since

$$\mathrm{Ext}^1_B(I_B/I_B^2, K_B(n+1)) \simeq \mathrm{Ext}^3_R(I_B/I_B^2, R) \simeq \mathrm{Ext}^2_R(I_B^2, R)\ ,$$

it suffices to show ${}_{-s}\mathrm{Ext}^2_R(I_B^2(n_{1,i}), R) = 0$ for any i. Looking to (2-9) it is enough to see that ${}_{-s}\mathrm{Hom}\big(\bigwedge^2\big(\bigoplus R(-n_{2,j})\big)(n_{1,i}), R\big) = 0$. Since, however, $n_{2,j} + n_{2,j'} - n_{1,i} - s < 0$ for any $i, j, j', j \neq j'$ by assumption, we easily get this vanishing for any i and hence ${}_{-s}\mathrm{Ext}^1_B(I_B/I_B^2, N_B) = 0$. Finally, the dimension formula follows from (3-4) and (2-8) since we get $(K_B)_{t-2s} = 0$ and $\delta = 0$ from the proof of (ii).

(ii) By (2-5) we have $(K_B)_{t-2s} = 0$, provided $2s > \max n_{2,j}$. By the discussion before Corollary 3.2 we must prove ${}_t\mathrm{Ext}^2_B(S^2(I_{A/B}(s)), K_B) = 0$. Using the proof of [Kleppe 2007, Lemma 28] there is an exact sequence

$$0 \to {}_t\mathrm{Ext}^2_B(S^2(I_{A/B}(s)), K_B) \to {}_t\mathrm{Ext}^2_B(S^2(N_B), K_B) \to {}_s\mathrm{Ext}^2_B(N_B, B),$$

induced by (3-2), where we have

$${}_t\mathrm{Ext}^2_B(S^2(N_B), K_B) \simeq {}_0\mathrm{Ext}^2_B(N_B, N_B) \simeq {}_0\mathrm{Ext}^2_B(N_B, I_{A/B}(s)),$$

by (2-1), (3-2), and the fact that N_B is a maximal CM B-module. Indeed

$$\begin{aligned} {}_t\mathrm{Ext}^2_B(S^2(N_B), K_B) &\simeq \mathrm{Ext}^2_{\mathbb{O}_U}(\widetilde{S^2(N_B)}|_U, \widetilde{K_B}|_U(t)) \\ &\simeq \mathrm{Ext}^2_{\mathbb{O}_U}(\widetilde{N_B}|_U, \widetilde{N_B}^* \otimes \widetilde{K_B}|_U(t)) \simeq {}_0\mathrm{Ext}^2_B(N_B, N_B), \end{aligned}$$

by (2-1). Since $\mathrm{Ext}^1_B(N_B, B) = 0$ by (2-1) and (2-9), it follows that

$${}_t\mathrm{Ext}^2_B(S^2(I_{A/B}(s)), K_B) \simeq {}_s\mathrm{Ext}^1_B(N_B, A),$$

which vanishes by assumption.

It remains to prove the final statement. If we apply $\mathrm{Hom}(-, K_B)$ to (2-2) and we use (2-5), we get ${}_{-2s}\mathrm{Ext}^i_R(I_B, K_B(t)) = 0$ and hence ${}_{-2s}\mathrm{Ext}^i_B(I_B/I_B^2, K_B(t)) = 0$ for $i = 0, 1$ provided $s > \max n_{2,j}$. Similarly we use $\mathrm{Hom}(-, N_B)$ and (2-7) to

show that ${}_{-s}\mathrm{Hom}(I_B, N_B) = 0$ provided $s > \max n_{2,j} + \max n_{1,i} - \min n_{1,i}$. We conclude by applying $\mathrm{Hom}_B(I_B/I_B^2, -)$ to (3-2). □

Remark 3.3. If $\mathrm{depth}_{I(Z)} B \geq 4$ and $\mathrm{char}\, k \neq 2$, we showed in [Kleppe 2007, Remark 42] that

$${}_0\mathrm{Ext}^2_B(N_B, N_B) \simeq {}_0\mathrm{Hom}_B(I_B/I_B^2, H^3_{I(Z)}(I_B/I_B^2)) \simeq {}_0\mathrm{Hom}_B(I_B/I_B^2, H^4_{I(Z)}(I_B^2)) .$$

In a similar way one can show that $\mathrm{Ext}^2_B(N_B, B) \simeq H^4_{I(Z)}(I_B^2)$. Hence the group ${}_s\mathrm{Ext}^1_B(N_B, A)$ of Corollary 3.2 is isomorphic to the kernel of the natural map

$${}_0\mathrm{Hom}_B(I_B/I_B^2, H^4_{I(Z)}(I_B^2)) \to {}_sH^4_{I(Z)}(I_B^2),$$

induced by the regular section σ. This sometimes allows us to verify that

$${}_s\mathrm{Ext}^1_B(N_B, A) = 0.$$

Remark 3.4. The first author takes the opportunity to point out a missing assumption in [Kleppe 2007] as well as in [Kleppe 2006]. In these papers there are several theorems involving the *codimension of a stratum* in which the assumption "$(B \to A)$ is general" or "(B) general" is missing. The main result [Kleppe 2006, Theorem 5] (and hence [Kleppe 2007, Theorem 15]) uses generic smoothness in its proof and refers to [Kleppe et al. 2001, Proposition 9.14] where the generality assumption occurs, as it should. In the proof of [Kleppe 2006, Theorem 5] we need $(B \to A)$ to be general to compute the dimension of the stratum. It is easily seen from the proof that what we really need is that $(B \to A)$ be general, in the sense that, for a given $(B \to A)$, ${}_0\mathrm{hom}_R(I_B, I_{A/B})$ attains its least possible value in the irreducible components of $\mathrm{GradAlg}(H_B, H_A)$ to which $(B \to A)$ belongs. Thus in [Kleppe 2006, Theorem 5, Proposition 13, Theorem 16] (and hence [Kleppe 2007, Theorem 23]), for the codimension statement we should assume that (B) is general or at least that ${}_{-s}\mathrm{hom}_R(I_B, K_B)$ attains its least possible value in the irreducible component of $\mathrm{GradAlg}(H_B)$ to which (B) belongs. If we apply our results in a setting where these hom-numbers vanish (this is what we almost always do), we don't need to assume that (B) or $(B \to A)$ is general.

So Remark 3.4 gives the reason for including the assumption that $(X \subset Y)$ is general in Corollary 3.2(ii), even though this assumption does not occur in the codimension statements of the A-part of [Kleppe 2007, Theorems 1 and 25].

4. Ideals generated by submaximal minors of square matrices

Let $X = \mathrm{Proj}\, A \subset \mathbb{P}^n$ be a codimension-4, determinantal scheme defined by the submaximal minors of a $t \times t$ homogeneous matrix. In this section we compute the dimension of $\mathrm{Hilb}^{p(x)}(\mathbb{P}^n)$ for $n \geq 5$ at (X) in terms of the corresponding degree matrix. The proof requires a proposition (valid for $n \geq 3$) on how A is determined

by a locally regular section of $I_B/I_B^2(s)$ where $B = R/I_B$ is a codimension-2 CM quotient. Let us first fix the notation we will use throughout this section.

Given a homogeneous matrix $\mathscr{A}$, that is, a matrix representing a degree 0 morphism ϕ of free graded R-modules, we denote by $I(\mathscr{A})$ (or $I(\phi)$) the ideal of R generated by the maximal minors of $\mathscr{A}$ and by $I_j(\mathscr{A})$ (or $I_j(\phi)$) the ideal generated by the $j \times j$ minors of $\mathscr{A}$.

Definition 4.1. A codimension-c subscheme $X \subset \mathbb{P}^n$ is called a *determinantal* scheme if there exist integers r, p and q such that $c = (p-r+1)(q-r+1)$ and $I(X) = I_r(\mathscr{A})$ for some $p \times q$ homogeneous matrix $\mathscr{A}$. $X \subset \mathbb{P}^n$ is called a *standard determinantal* scheme if $r = \min(p, q)$. The corresponding rings $R/I_r(\mathscr{A})$ are called determinantal (resp. standard determinantal) rings.

Let $X \subset \mathbb{P}^n$ be a codimension-4, determinantal scheme defined by the vanishing of the submaximal minors of a $t \times t$ homogeneous matrix $\mathscr{A} = (f_{ji})_{i,j=1,\dots,t}$ where $f_{ji} \in k[x_0, \dots, x_n]$ are homogeneous polynomials of degree $a_j - b_i$ with $b_1 \le b_2 \le \dots \le b_t$ and $a_1 \le a_2 \le \dots \le a_t$. We assume without loss of generality that $\mathscr{A}$ is minimal; that is, $f_{ji} = 0$ for all i, j with $b_i = a_j$. If we let $u_{ji} = a_j - b_i$ for all $j = 1, \dots, t$ and $i = 1, \dots, t$, the matrix $\mathscr{U} = (u_{ji})_{i,j=1,\dots,t}$ is called the *degree matrix* associated to X.

We denote by $W_{t,t}^{t-1}(\underline{b}; \underline{a}) \subset \mathrm{Hilb}^{p(x)}(\mathbb{P}^n)$ the locus of determinantal schemes $X \subset \mathbb{P}^n$ of codimension 4 defined by the submaximal minors of a homogeneous square matrix $\mathscr{A} = (f_{ji})_{i,j=1,\dots,t}$ as above. Notice that $W_{t,t}^{t-1}(\underline{b}; \underline{a}) \neq \varnothing$ if and only if $u_{i-1,i} = a_{i-1} - b_i > 0$ for $i = 2, \dots, t$.

Let $\mathscr{N}$ be the matrix obtained by deleting the last row, let $I_B = I_{t-1}(\mathscr{N})$ be the ideal defined by the maximal minors of $\mathscr{N}$, and let $I_A = I_{t-1}(\mathscr{A})$ be the ideal generated by the submaximal minors of $\mathscr{A}$. Set $A = R/I_A = R/I(X)$ and $B = R/I_B$.

Remark 4.2. If the entries of $\mathscr{A}$ and $\mathscr{N}$ are sufficiently general polynomials of degree $a_i - b_j$, $1 \le i, j \le t$, and $a_{i-1} - b_i > 0$ for $2 \le i \le t$, then B is a graded Cohen–Macaulay quotient of codimension 2 and A is a graded Gorenstein quotient of codimension 4.

The goal of this section is to compute, in terms of a_j and b_i, the dimension of the determinantal locus $W_{t,t}^{t-1}(\underline{b}; \underline{a}) \subset \mathrm{Hilb}^{p(x)}(\mathbb{P}^n)$, where $p(x) \in \mathbb{Q}[x]$ is the Hilbert polynomial of X. Note that the Hilbert polynomial of X can be computed explicitly using the minimal free R-resolution of $R/I(X)$ given by Gulliksen and Negård [1972], see (4-5). We will also analyze whether the closure of $W_{t,t}^{t-1}(\underline{b}; \underline{a})$ in $\mathrm{Hilb}^{p(x)}(\mathbb{P}^n)$ is a generically smooth, irreducible component of $\mathrm{Hilb}^{p(x)}(\mathbb{P}^n)$. To this end, we consider

$$F := \bigoplus_{i=1}^{t} R(b_i) \xrightarrow{\phi} G := \bigoplus_{j=1}^{t} R(a_j),$$

the morphism induced by the matrix $\mathscr{A}$, and

$$F \xrightarrow{\phi_t} G_t := \bigoplus_{j=1}^{t-1} R(a_j),$$

the morphism induced by the matrix $\mathcal{N}$ obtained by deleting the last row of $\mathscr{A}$. The determinant of $\mathscr{A}$ is a homogeneous polynomial of degree $s = \sum_{j=1}^{t} a_j - \sum_{i=1}^{t} b_i$, and the degrees of the maximal minors of $\mathcal{N}$ are $s + b_i - a_t$; that is, I_B has the minimal free R-resolution

$$0 \longrightarrow G_t^*(a_t - s) \xrightarrow{{}^t\mathcal{N}} F^*(a_t - s) \xrightarrow{\beta} I_B \longrightarrow 0. \tag{4-1}$$

Proposition 4.3. *Suppose* $\operatorname{char} k = 0$.

(i) *Let* $A = R/I_{t-1}(\mathscr{A})$ *be a determinantal ring of codimension* 4 *where* $\mathscr{A}$ *is a* $t \times t$ *homogeneous matrix, and let* $B = R/I_{t-1}(\mathcal{N})$ *be the standard determinantal ring associated to* $\mathcal{N}$ *where* $\mathcal{N}$ *is the matrix obtained by deleting the last row of* $\mathscr{A}$. *Moreover, let* $Z \subset \operatorname{Proj} B$ *be a closed subset such that* $\operatorname{Proj} B - Z \hookrightarrow \mathbb{P}^n$ *is an l.c.i., and suppose* $\operatorname{depth}_{I(Z)} B \geq 2$. *Then there is a regular section* σ *of* $(\widetilde{I_B}/\widetilde{I_B^2}(s))_{|\operatorname{Proj} B - Z}$, *where* $s = \deg \det \mathscr{A}$, *whose zero locus precisely defines* A *as a quotient of* B (*that is,* σ *extends to a map* $\sigma : B \longrightarrow I_B/I_B^2(s)$ *such that* $A = B/\operatorname{im} \sigma^*$).

(ii) *Conversely, let* $B = R/I_{t-1}(\mathcal{N})$ *be a standard determinantal ring of codimension* 2, *let* $Z \subset \operatorname{Proj} B$ *be a closed subset such that* $\operatorname{Proj} B - Z \hookrightarrow \mathbb{P}^n$ *is an l.c.i. and* $\operatorname{depth}_{I(Z)} B \geq 2$, *and furthermore let* A' *be defined by a regular section* σ *of* $(\widetilde{I_B}/\widetilde{I_B^2}(s))_{|\operatorname{Proj} B - Z}$, *that is, given by*

$$0 \longrightarrow K_B(n + 1 - 2s) \longrightarrow N_B(-s) \xrightarrow{\sigma^*} B \longrightarrow A' \longrightarrow 0 \tag{4-2}$$

for some integer s. *Then, there is a* $t \times t$ *homogeneous matrix* $\mathscr{A}'$ *obtained by adding a row to* $\mathcal{N}$ *such that* $I_{A'} = I_{t-1}(\mathscr{A}')$.

Proof. To define σ, we consider the commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & & & & & \\
 & & \downarrow & & & & & & \\
0 & \longrightarrow & G_t^*(a_t - s) & \xrightarrow{{}^t\mathcal{N}} & F^*(a_t - s) & \xrightarrow{\beta} & I_B & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha} & & \| & & \downarrow & & \\
 & & G^*(a_t - s) & \xrightarrow{\phi^*(a_t - s)} & F^*(a_t - s) & \longrightarrow & (\operatorname{coker} \phi^*)(a_t - s) & \longrightarrow & 0 \\
 & & \downarrow & & & & & & \\
 & & R(-s) & & & & & & \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}$$

where $\alpha : G_t^*(a_t - s) \hookrightarrow G^*(a_t - s)$ is the natural inclusion defined by

$$\alpha \begin{pmatrix} f_1 \\ \vdots \\ f_{t-1} \end{pmatrix} = \begin{pmatrix} f_1 \\ \vdots \\ f_{t-1} \\ 0 \end{pmatrix},$$

and β is given by multiplication with the maximal minors of the matrix $\mathcal{N}$. The snake Lemma yields the exact sequence

$$R(-s) \xrightarrow{\cdot \det \phi} I_B \longrightarrow (\operatorname{coker} \phi^*)(a_t - s) \longrightarrow 0, \tag{4-3}$$

and hence

$$(\operatorname{coker} \phi^*)(a_t) \simeq I_B(s)/\det \phi. \tag{4-4}$$

If we tensor $R(-s) \xrightarrow{\cdot \det \phi} I_B$ with $B(s)$, we get a section σ of $I_B/I_B^2(s)$. Before proving that the zero locus of σ defines precisely A as a quotient of B via $\operatorname{im} \sigma^* = I_{A/B}$, we claim that any locally regular section σ' of $I_B/I_B^2(s)$ defining A' via $A' = B/\operatorname{im} \sigma'^*$ gives rise to a homogeneous matrix $\mathcal{A}'$ and a corresponding map ϕ' such that (4-3) and (4-4) hold with ϕ' instead of ϕ. Indeed, given a section σ' of $I_B/I_B^2(s)$, there exists a map σ'' fitting into a commutative diagram

$$\begin{array}{ccc} & & F^*(a_t) \otimes B \\ & \overset{\sigma''}{\nearrow} & \downarrow \\ B & \xrightarrow[\sigma']{} & I_B/I_B^2(s) \end{array}$$

and we denote by $\sigma_R \in \operatorname{Hom}_R(F, R(a_t))$ the map which corresponds to $\sigma''(1)$. Since $\operatorname{Hom}_R(F, R(a_t)) = \operatorname{Hom}\big(\bigoplus_{i=1}^t R(b_i), R(a_t)\big)$, the morphism σ_R determines a $1 \times t$ row $\underline{g} = (g_1, \ldots, g_t)$ where g_i is a homogeneous form of degree $a_t - b_i$, $1 \le i \le t$ and we define

$$\mathcal{A}' = \begin{pmatrix} \mathcal{N} \\ \underline{g} \end{pmatrix}.$$

Since the vertical map in the above diagram is induced by β described above, we may assume that $\det(\phi') = \sigma'(1)$ modulo $I_B^2(s)$ and we get the claim.

It remains to show that $\operatorname{im} \sigma^* = I_{A/B}$, where $I_A = I_{t-1}(\mathcal{A})$, and that σ is a locally regular section. Note that this will also show that $\operatorname{im} \sigma'^* = I_{A'/B}$, where $I_{A'} = I_{t-1}(\mathcal{A}')$; that is, we get the converse. Moreover, looking at the exact sequence (4-2) with A instead of A' and recalling that

$$N_B \simeq K_B(n+1) \otimes I_B/I_B^2,$$

we see that $\operatorname{im}\sigma^* = \operatorname{coker}(\sigma(-2s)\otimes \mathrm{id})$ where $\mathrm{id}: K_B(n+1) \longrightarrow K_B(n+1)$ is the identity map and σ is induced by $\det\phi$. Since we get

$$F(s-a_t) \longrightarrow G_t(s-a_t) \longrightarrow K_B(n+1) \longrightarrow 0$$

by dualizing the exact sequence (4-1), we see that the cokernel above is the same as the twisted cokernel of the composition

$$\gamma : G_t(-a_t) \longrightarrow K_B(n+1-s) \overset{\sigma(-s)\otimes \mathrm{id}}{\longrightarrow} N_B.$$

Hence, we must prove that $\operatorname{coker}(\gamma) = I_{A/B}(s)$ where $I_A = I_{t-1}(\mathscr{A})$.

By [Gulliksen and Negård 1972, Theorem 2] and [Ile 2004, Theorem 2], we have an exact sequence

$$\ker\big[\operatorname{Hom}(F,F)\oplus\operatorname{Hom}(G,G) \xrightarrow{j} R\big] \longrightarrow \operatorname{Hom}(F,G) \longrightarrow I_A(s) \longrightarrow 0, \quad (4\text{-}5)$$

where $j(\rho_0,\rho_1) = \operatorname{tr}\rho_0 - \operatorname{tr}\rho_1$ is the difference between trace maps. The map

$$\operatorname{Hom}(F,G) \longrightarrow I_A(s)$$

is given by $\gamma \mapsto \operatorname{tr}(\gamma\psi)$, where ψ is the matrix of cofactors; that is, this map is given by the submaximal minors of $\mathscr{A}$ while the map $\operatorname{Hom}(F,F)\oplus\operatorname{Hom}(G,G) \xrightarrow{\eta}$ $\operatorname{Hom}(F,G)$ is given as a difference of the obvious compositions with ϕ, that is, $\eta(\rho_0,\rho_1) = \rho_1\phi - \phi\rho_0$. Since we have

$$\begin{array}{ccccc} \operatorname{Hom}(F,F)\oplus\operatorname{Hom}(G,G) & \xrightarrow{\eta} & \operatorname{Hom}(F,G) & \longrightarrow & I_A(s) \\ (\mathrm{id},0) & & \longmapsto & & t\cdot\det\phi \end{array}$$

and since there is a commutative diagram

$$\begin{array}{ccccccc} & & 0 & & & & \\ & & \downarrow & & & & \\ & & \ker j & \longrightarrow & \operatorname{Hom}(F,G) & \longrightarrow & I_A(s) \longrightarrow 0 \\ & & \downarrow & & \| & & \downarrow \\ \operatorname{Hom}(F,F)\oplus\operatorname{Hom}(G,G) & \xrightarrow{\eta} & \operatorname{Hom}(F,G) & \longrightarrow & \operatorname{coker}\eta & \longrightarrow & 0 \\ \downarrow & & & & & & \\ R & & & & & & \end{array}$$

we get an exact sequence

$$R \overset{\cdot t\det\phi}{\longrightarrow} I_A(s) \longrightarrow \operatorname{coker}\eta \longrightarrow 0.$$

Hence, $\operatorname{coker}(\eta) \simeq I_A(s)/\det\phi$ ($\operatorname{char} k = 0$) and the following sequence is exact:

$$\operatorname{Hom}(F,F)\oplus\operatorname{Hom}(G,G) \xrightarrow{\eta} \operatorname{Hom}(F,G) \longrightarrow I_A(s)/\det\phi \longrightarrow 0.$$

Now we look at the commutative diagram

$$\begin{array}{ccccccc}
\operatorname{Hom}(R(-a_t), G^*) & \longrightarrow & \operatorname{Hom}(R(-a_t), F^*) & \longrightarrow & I_B(s)/\det\phi & \longrightarrow & 0 \\
\downarrow\simeq & & \downarrow\simeq & & \downarrow & & \\
\operatorname{Hom}(G, R(a_t)) & \longrightarrow & \operatorname{Hom}(F, R(a_t)) & & & & \\
\downarrow{\scriptstyle (0,\cdot)} & & \downarrow & & \downarrow & & \\
\operatorname{Hom}(F,F)\oplus\operatorname{Hom}(G,G) & \xrightarrow{\eta} & \operatorname{Hom}(F,G) & \longrightarrow & I_A(s)/\det\phi & \longrightarrow & 0 \\
\downarrow{\scriptstyle (\mathrm{id},\alpha_1^*)} & & \downarrow{\scriptstyle \alpha_2^*} & & \downarrow & & \\
\operatorname{Hom}(F,F)\oplus\operatorname{Hom}(G,G_t) & \xrightarrow{\eta_t} & \operatorname{Hom}(F,G_t) & \longrightarrow & \operatorname{coker}(\eta_t) & \longrightarrow & 0
\end{array}$$

where α_1^* and α_2^* are induced by α in a natural way and η_t is a difference of the obvious compositions, that is, $\eta_t(\rho_0, \rho_1') = \rho_1'\phi - \phi_t\rho_0$. We see, in particular, that the ideal $I_{A/B} = I_A/I_B$ is given by an exact sequence

$$\operatorname{Hom}(F,F)\oplus\operatorname{Hom}(G,G_t) \xrightarrow{\eta_t} \operatorname{Hom}(F,G_t) \longrightarrow I_{A/B}(s) \longrightarrow 0,$$

where the rightmost map is given by the submaximal minors of the matrix $\mathcal{A}$ which do not belong to I_B.

On the other hand, by (2-7), there is an exact sequence

$$\operatorname{Hom}(G_t^*, G_t^*)\oplus\operatorname{Hom}(F^*,F^*) \longrightarrow \operatorname{Hom}(G_t^*, F^*) \longrightarrow N_B \longrightarrow 0,$$

or, equivalently,

$$\operatorname{Hom}(F,F)\oplus\operatorname{Hom}(G_t,G_t) \xrightarrow{\eta'} \operatorname{Hom}(F,G_t) \longrightarrow N_B \longrightarrow 0,$$

where η' is given by $\eta'(\rho_0,\rho_2) = \rho_2\phi_t - \phi_t\rho_0$. Using again the exact sequence

$$0 \longrightarrow R(a_t) \longrightarrow G \xrightarrow{\alpha^*} G_t \longrightarrow 0,$$

we get a commutative diagram

$$\begin{array}{ccccccc}
\operatorname{Hom}(F,F)\oplus\operatorname{Hom}(G_t,G_t) & \xrightarrow{\eta'} & \operatorname{Hom}(F,G_t) & \longrightarrow & N_B & \longrightarrow & 0 \\
\downarrow{\scriptstyle (\mathrm{id},\alpha_3)} & & \| & & \downarrow & & \\
\operatorname{Hom}(F,F)\oplus\operatorname{Hom}(G,G_t) & \xrightarrow{\eta_t} & \operatorname{Hom}(F,G_t) & \longrightarrow & I_{A/B}(s) & \longrightarrow & 0 \\
\downarrow & & & & & & \\
\operatorname{Hom}(R(a_t), G_t) & & & & & &
\end{array}$$

where α_3 is induced by α. Hence we get an exact sequence

$$\operatorname{Hom}(R(a_t), G_t) \xrightarrow{\gamma} N_B \longrightarrow I_{A/B}(s) \longrightarrow 0.$$

This proves that $\operatorname{coker}(\gamma) = I_{A/B}(s)$, that is, $\operatorname{im}\sigma^* = I_{A/B}$ as required. Finally note that the above codimension and depth relations imply that σ is a *regular* section on $U := \operatorname{Proj} B - Z$ because $(\operatorname{im}\tilde{\sigma}^*)_{|U}$ must locally on U be generated by two regular elements (to get that $(B/\operatorname{im}\tilde{\sigma}^*)_{|U}$ is a codimension-2 Cohen–Macaulay quotient of $\widetilde{B}|U$). This completes the proof of Proposition 4.3. □

This proposition seems to be known in special cases. For instance, Ellingsrud and Peskine [1993, before Proposition 6] state that the Artinian Gorenstein ring associated to an invertible sheaf $\mathcal{O}_S(C)$ on a surface S in $\mathbb{P}^3$, where C is an arithmetically CM curve, is given by the submaximal minors of a square matrix which extends the Hilbert–Burch matrix associated to C in $\mathbb{P}^3$. Since we get (3-1) with $M = N_B$ by applying $H^0_*(-)$ to the exact sequence

$$0 \to \mathcal{N}_{C/S}(-s) \to \mathcal{N}_C(-s) \to \mathcal{N}_S|_C(-s) \simeq \mathcal{O}_C \to 0,$$

of normal sheaves, it is clear that their Gorenstein ring (see their construction 2) is essentially the same as ours in the Artinian case. However, we have given a proof of the proposition suited to our applications.

As a nice application of Proposition 4.3 we have:

Proposition 4.4. *Let $X \subset \mathbb{P}^n$, $n \geq 4$, be a codimension-4 scheme defined by the submaximal minors of a $t \times t$ homogeneous square matrix $\mathcal{A}$. Then X is in the Gorenstein liaison class of a complete intersection, that is, X is glicci.*

Proof. By [Gulliksen and Negård 1972, Theorem 2] (see also Proposition 4.3), X is arithmetically Gorenstein and hence glicci [Casanellas et al. 2005, Theorem 7.1]. □

Remark 4.5. This proposition has been recently generalized by Gorla, who proved [2008, Theorem 3.1] that any codimension-$(t-r+1)^2$ ACM scheme $X \subset \mathbb{P}^n$ defined by the $r \times r$ minors of a $t \times t$ homogeneous square matrix $\mathcal{A}$ is glicci.

For an introduction to glicciness, see [Kleppe et al. 2001].

We are now ready to compute $\dim W^{t-1}_{t,t}(\underline{b};\underline{a})$ and $\dim_{(X)} \operatorname{Hilb}^{p(x)} \mathbb{P}^n$, $n \geq 5$, in terms of $a_1, \ldots, a_t$ and $b_1, \cdots, b_t$. Note that if $t = 2$ then a general X is a complete intersection in which case these dimensions are well known.

Theorem 4.6. *Assume $\operatorname{char} k = 0$. Fix integers $a_1 \leq a_2 \leq \cdots \leq a_t$ and $b_1 \leq b_2 \leq \cdots \leq b_t$. Assume $t > 2$, $a_i \geq b_{i+3}$ for $1 \leq i \leq t-3$ (and $a_1 \geq b_t$ if $t = 3$), $a_t > a_{t-1} + a_{t-2} - b_1$ and $n \geq 5$. Then $W^{t-1}_{t,t}(\underline{b};\underline{a})$ is irreducible. Moreover,*

if (X) *is general in* $W^{t-1}_{t,t}(\underline{b};\underline{a})$, *then* X *is unobstructed, and*

$$\begin{aligned}\dim W^{t-1}_{t,t}(\underline{b};\underline{a}) &= \dim_{(X)} \mathrm{Hilb}^{p(x)}(\mathbb{P}^n)\\ &= \sum_{1\le i,j\le t}\binom{a_j-b_i+n}{n} - \sum_{\substack{1\le i\le t-1\\ 1\le j\le t}}\binom{a_j-a_i+n}{n}\\ &\quad - \sum_{1\le i,j\le t}\binom{b_i-b_j+n}{n} + \sum_{\substack{1\le i\le t\\ 1\le j\le t-1}}\binom{b_i-a_j+n}{n}\\ &\quad - \sum_{\substack{1\le j\le t\\ 1\le i\le k\le t}}\binom{a_t-s-b_i-b_k+a_j+n}{n} + \sum_{\substack{1\le i,j\le t\\ 1\le k\le t-1}}\binom{a_t-s-b_i-a_k+a_j+n}{n}\\ &\quad - \sum_{\substack{1\le i<k\le t-1\\ 1\le j\le t}}\binom{a_t-s-a_i-a_k+a_j+n}{n} + \sum_{2\le i\le t}\binom{a_t-s+b_i-2b_1+n}{n}. \quad (4\text{-}6)\end{aligned}$$

Proof. Let $X\subset\mathbb{P}^n$ be an arithmetically Gorenstein scheme of codimension 4 defined by the submaximal minors of a homogeneous square matrix $\mathscr{A}=(f_{ji})^{i=1,\dots,t}_{j=1,\dots,t}$ where $f_{ji}\in k[x_0,\dots,x_n]$ is a sufficiently general homogeneous polynomial of degree a_j-b_i, and let $Y\subset\mathbb{P}^n$ be a codimension-2 subscheme defined by the maximal minors of the matrix $\mathscr{N}$ obtained deleting the last row of $\mathscr{A}$ (see Remark 4.2). So, the homogeneous ideal $I_B=I(Y)$ of Y has the minimal free R-resolution

$$0\longrightarrow F_2=\bigoplus_{j=1}^{t-1}R(a_t-s-a_j)\xrightarrow{{}^t\mathscr{N}}F_1=\bigoplus_{i=1}^{t}R(a_t-s-b_i)\longrightarrow I_B\longrightarrow 0. \quad (4\text{-}7)$$

By Proposition 4.3, X is the zero locus of a suitable regular section of $\widetilde{I_B}/\widetilde{I_B^2}(s)$ where $s=\deg(\det\mathscr{A})$ and $W^{t-1}_{t,t}(\underline{b};\underline{a})$ is irreducible by [Kleppe 2007, Corollary 41]. Since the hypothesis $a_t>a_{t-1}+a_{t-2}-b_1$ is equivalent to

$$s>s+a_{j_0}-a_t+\max_{\substack{1\le j\le t-1\\ j\ne j_0}}(s+a_j-a_t)-\min_{1\le i\le t}(s+b_i-a_t),$$

where $s+a_{j_0}-a_t=\max_{1\le j\le t-1}(s+a_j-a_t)$; and since $a_i\ge b_{i+3}$ for $1\le i\le t-3$ (and $a_1\ge b_t$ if $t=3$) implies that $B:=R/I_B$ given by (4-7) satisfies $\mathrm{depth}_{I(Z)}B\ge 4$ [Kleppe and Miró-Roig 2005, Remark 2.7], we can apply Corollary 3.2 and we get that X is unobstructed and

$$\dim W^{t-1}_{t,t}(\underline{b};\underline{a})=\dim_{(X)}\mathrm{Hilb}^{p(x)}(\mathbb{P}^n)=\eta(s)+\sum_{j=1}^{t-1}\eta(n_{2,j})-\sum_{i=1}^{t}\eta(n_{1,i}),$$

where $\eta(t)=\dim(I(Y)/I(Y)^2)_t=\dim I(Y)_t-\dim I(Y)^2_t$, $n_{2,j}=s+a_j-a_t$, $1\le j\le t-1$, and $n_{1,i}=s+b_i-a_t$, $1\le i\le t$. By (2-9), $I(Y)^2$ has a minimal free

R-resolution of the following type:

$$0 \longrightarrow \bigwedge^2 F_2 = \bigoplus_{1\le i<j\le t-1} R(-a_i-a_j+2a_t-2s)$$
$$\longrightarrow F_1\otimes F_2 = \bigoplus_{\substack{1\le i\le t\\ 1\le j\le t-1}} R(-b_i-a_j+2a_t-2s)$$
$$\longrightarrow S^2F_1 = \bigoplus_{1\le i\le j\le t} R(-b_i-b_j+2a_t-2s) \longrightarrow I(Y)^2 \longrightarrow 0. \quad (4\text{-}8)$$

Using (4-7) and (4-8), we obtain

$$\eta(s) = \sum_{1\le i\le t}\binom{a_t-b_i+n}{n} - \sum_{1\le i\le t-1}\binom{a_t-a_i+n}{n} - \sum_{1\le i\le j\le t}\binom{2a_t-s-b_i-b_j+n}{n}$$
$$+ \sum_{\substack{1\le i\le t\\ 1\le j\le t-1}}\binom{2a_t-s-b_i-a_j+n}{n} - \sum_{1\le i<j\le t-1}\binom{2a_t-s-a_i-a_j+n}{n}.$$

Using again (4-7) and (4-8), we get

$$\sum_{j=1}^{t-1}\eta(n_{2,j}) - \sum_{i=1}^{t}\eta(n_{1,i}) = \sum_{\substack{1\le i\le t\\ 1\le j\le t-1}}\binom{a_j-b_i+n}{n} - \sum_{\substack{1\le i\le t-1\\ 1\le j\le t-1}}\binom{a_j-a_i+n}{n}$$
$$- \sum_{\substack{1\le j\le t-1\\ 1\le i\le k\le t}}\binom{a_t-s-b_i-b_k+a_j+n}{n} + \sum_{\substack{1\le i\le t\\ 1\le j,k\le t-1}}\binom{a_t-s-b_i-a_k+a_j+n}{n}$$
$$- \sum_{\substack{1\le i<k\le t-1\\ 1\le j\le t-1}}\binom{a_t-s-a_i-a_k+a_j+n}{n} - \sum_{\substack{1\le i\le t\\ 1\le j\le t}}\binom{b_i-b_j+n}{n}$$
$$+ \sum_{\substack{1\le i\le t\\ 1\le j\le t-1}}\binom{b_i-a_j+n}{n} + \sum_{\substack{1\le i\le t\\ 1\le j\le k\le t}}\binom{a_t-s+b_i-b_j-b_k+n}{n}$$
$$- \sum_{\substack{1\le i,k\le t\\ 1\le j\le t-1}}\binom{a_t-s+b_i-b_k-a_j+n}{n} + \sum_{\substack{1\le k<j\le t-1\\ 1\le i\le t}}\binom{a_t-s+b_i-a_k-a_j+n}{n}.$$

Since $a_{i-1} > b_i$ and $a_i \ge b_{i+3}$ for $1 \le i \le t-3$ (and $a_1 \ge b_t$ if $t = 3$), by hypothesis, the last two sums of binomials vanish. Indeed, to see that $a_t - s + b_i - b_k - a_j < 0$ for $1 \le i, k \le t$ and $1 \le j \le t-1$, it suffices to show that $b_t - b_1 - a_1 < s - a_t = a_1 + a_2 + \cdots + a_{t-1} - b_1 - b_2 - \cdots - b_t$, which is straightforward. Similarly, showing that $a_t - s + b_i - a_k - a_j < 0$ for $1 \le i \le t$ and $1 \le k < j \le t-1$ reduces to showing that $b_t - a_1 - a_1 < s - a_t = a_1 + a_2 + \cdots + a_{t-1} - b_1 - b_2 - \cdots - b_t$, which is straightforward too.

The same type of argument applies to see that $a_t - s + b_i - b_j - b_k < 0$ for all $1 \le i \le t$ and $1 \le j < k \le t$ and we can replace the summand

$$\sum_{\substack{1\le i\le t\\ 1\le j\le k\le t}} \binom{a_t - s + b_i - b_j - b_k + n}{n} \quad \text{by} \quad \sum_{2\le i\le t} \binom{a_t - s + b_i - 2b_1 + n}{n}.$$

Putting all together we obtain (4-6). □

5. Examples

We will end this work with some examples where we use Theorem 4.6. Moreover, these examples show that the hypothesis $a_t > a_{t-1} + a_{t-2} - b_1$ cannot be avoided! To handle such cases, we state a proposition which estimates the codimension of the stratum in $\mathrm{Hilb}^{p(x)}(\mathbb{P}^n)$ of subschemes given by the exact sequence (3-1).

Example 5.1. Let $R = k[x_0, \ldots, x_5]$ and let $X = \mathrm{Proj}\, A \subset \mathbb{P}^5 = \mathrm{Proj}\, R$ be a general arithmetically Gorenstein curve defined by the submaximal minors of a 4×4 matrix whose first 3 rows are linear forms and whose last row are forms of degree $s-3$ ($s \ge 4$), that is, $b_i = 0$ for $1 \le i \le 4$, $a_j = 1$ for $1 \le j \le 3$ and $a_4 = s-3$. Then, Theorem 4.6 applies provided $s > 5$ and we get that X is unobstructed and

$$\begin{aligned}
\dim W^3_{4,4}(\underline{0}; 1, 1, 1, s-3) &= \dim_{(X)} \mathrm{Hilb}^{p(x)}(\mathbb{P}^5)\\
&= 12\tbinom{6}{5} + 4\tbinom{s+2}{5} - 9\tbinom{5}{5} - 3\tbinom{s+1}{5} - 16\tbinom{5}{5} - 10\tbinom{s-1}{5} + 12\tbinom{s-2}{5} - 3\tbinom{s-2}{5}\\
&= 2s^3 - 10s^2 + 13s + 48.
\end{aligned}$$

Moreover, deleting the last row and taking maximal minors, we get a threefold $Y = \mathrm{Proj}\, B$ with resolution

$$0 \longrightarrow R(-4)^3 \longrightarrow R(-3)^4 \longrightarrow R \longrightarrow B \longrightarrow 0, \tag{5-1}$$

leading to

$$H_B(\nu) = \tbinom{\nu+3}{3} + 2\tbinom{\nu+2}{3} + 3\tbinom{\nu+1}{3} = p_Y(\nu) \quad \text{for } \nu \ge 0.$$

Since A is given by (3-1) with $t = 6$ and $M = N_B$, we get $\mathbb{O}_X \simeq \omega_X(2s-6)$. Hence $h^1(\mathbb{O}_X(s-3)) = h^0(\mathbb{O}_X(s-3))$ and the Hilbert polynomial of X must be of the form $p_X(\nu) = d\nu + 1 - g = d(\nu - s + 3)$. Looking to (5-1) we get

$$\begin{aligned}
p_X(s-2) &= h^0(\mathbb{O}_X(s-2)) - h^0(\mathbb{O}_X(s-4))\\
&= h^0(\mathbb{O}_Y(s-2)) - h^0(\mathbb{O}_Y(s-4)) = 6s^2 - 28s + 36,
\end{aligned}$$

that is, $d = \deg X = 6s^2 - 28s + 36$ and $g = 1 + d(s-3)$.

Note that Theorem 4.6 takes care of all cases except for $s=4$ and $s=5$. For these two values of s, we can, however, use Corollary 3.2(ii) to find $\dim_{(X)} \mathrm{Hilb}^{p(x)}(\mathbb{P}^5)$ because

$$ {}_0\mathrm{Ext}^2_B(N_B, N_B) \simeq {}_0\mathrm{Hom}(I_B/I_B^2, H^4_{\mathfrak{m}}(I_B^2)) = 0, $$

by (5-1) and Remark 3.3. Indeed ${}_3H^4_{\mathfrak{m}}(I_B^2) \hookrightarrow {}_3H^6_{\mathfrak{m}}(R(-8)^3) = 0$ by (2-9). Hence X is unobstructed,

$$ \dim_{(X)} \mathrm{Hilb}^{p(x)}(\mathbb{P}^5) = 2s^3 - 10s^2 + 13s + 48 + \delta, $$

where $\delta = \delta(K_B)_{6-2s} - \delta(N_B)_{-s}$, and moreover, if $s=5$, then δ is the codimension of the closure of $W^3_{4,4} := W^3_{4,4}(\underline{0}; 1, 1, 1, s-3)$ in $\mathrm{Hilb}^{p(x)}(\mathbb{P}^5)$. We claim that

$$ (\delta(K_B)_{6-2s}, \delta(N_B)_{-s}) = \begin{cases} (-3, -15) & \text{for } s=4, \\ (0, -12) & \text{for } s=5, \end{cases} $$

that is, $\delta = 12$ in both cases.

To find $\delta(K_B)_{6-2s}$ we apply $\mathrm{Hom}_B(-, K_B(6))$ to (2-3) and we get

$$ {}_{-2s}\mathrm{Hom}_B(I_B/I_B^2, K_B(6)) = 0, \quad {}_{-2s}\mathrm{Ext}^1_B(I_B/I_B^2, K_B(6)) = {}_{-2s}\mathrm{Hom}(H_1, K_B(6)). $$

Since the rank of H_1 is 2, we have

$$ \mathrm{Hom}(H_1, K_B(6)) \simeq H_1\big(\sum_i n_{1,i}\big) = H_1(12) \tag{5-2} $$

by [Avramov and Herzog 1980] or [Kleppe and Peterson 2001, Theorem 8], see the isomorphism accompanying (3-1). Using (2-6) or, more precisely, the exactness of

$$ \textstyle\bigwedge^2 (R(-3)^4) \longrightarrow R(-4)^3 \longrightarrow H_1 \longrightarrow 0 \tag{5-3} $$

[Avramov and Herzog 1980], we get

$$ \delta(K_B)_{6-2s} = -\dim H_1(12)_{-2s} = \begin{cases} -3 & \text{for } s=4, \\ 0 & \text{for } s=5. \end{cases} $$

It remains to compute $\delta(N_B)_{-s}$. If we dualize the exact sequence (2-3) we get

$$ 0 \longrightarrow N_B \longrightarrow B(3)^4 \longrightarrow H_1^* \longrightarrow 0, $$

to which we apply ${}_{-s}\mathrm{Hom}(I_B/I_B^2, -)$. Combining with

$$ \begin{aligned} {}_{-s}\mathrm{Hom}(I_B/I_B^2, H_1^*) &\simeq {}_{-s}\mathrm{Hom}(I_B/I_B^2 \otimes K_B(6), H_1^* \otimes K_B(6)) \\ &\simeq {}_{-s}\mathrm{Hom}(N_B, H_1(12)), \end{aligned} $$

where again we have used (5-2), we get

$$ \delta(N_B)_{-s} = 4\dim(N_B)_{3-s} - \dim({}_{-s}\mathrm{Hom}(N_B(-12), H_1)). $$

Using (2-6), we see that

$$0 \to {}_{-s}\mathrm{Hom}(N_B(-12), K_B(6)^*) \\ \to {}_{-s}\mathrm{Hom}(N_B(-12), B(-4)^3) \to {}_{-s}\mathrm{Hom}(N_B(-12), H_1) \to 0$$

is exact because we have $\mathrm{Ext}^1_B(I_B/I_B^2 \otimes K_B, K_B^*) = 0$ by Lemma 4.9 of [Kleppe and Miró-Roig 2005]. Using (4.17) of the same reference we also get the surjectivity of the natural map $K_B^* \otimes B(-4)^4 \longrightarrow \mathrm{Hom}_B(I_B/I_B^2 \otimes K_B, K_B^*)$. Since we may use (5-3) to see that $(H_1)_\nu \simeq R(-4)^3_\nu \simeq B(-4)^3_\nu$ for $\nu \leq 5$, we get $K_B(6)^*_\nu = 0$ for $\nu \leq 5$ by (2-6) and hence

$${}_{-s}\mathrm{Hom}(N_B(-12), K_B^*(-6)) \simeq {}_{-s}\mathrm{Hom}(I_B/I_B^2 \otimes K_B(6), K_B^*(6)) = 0,$$

for $s \geq 4$. It follows that

$${}_{-s}\mathrm{Hom}(N_B(-12), H_1) \simeq (I_B/I_B^2)^3_{8-s},$$

for $s \geq 4$, which implies (by (2-7) and (2-9)) that

$$\delta(N_B)_{-s} = \begin{cases} 4\dim(N_B)_{-2} - 3\dim(I_B)_3 = -12 & \text{for } s = 5, \\ 4\dim(N_B)_{-1} - 3\dim(I_B)_4 = -15 & \text{for } s = 4. \end{cases}$$

Putting all together we get

$$\dim_{(X)} \mathrm{Hilb}^{p(x)}(\mathbb{P}^5) = \begin{cases} 2s^3 - 10s^2 + 13s + 48 = \dim W^3_{4,4} & \text{for } s > 5, \\ 125 & \text{for } s = 5, \\ 80 & \text{for } s = 4. \end{cases}$$

Moreover, applying Corollary 3.2(ii), we get $\mathrm{codim}_{\mathrm{Hilb}^{p(x)}(\mathbb{P}^5)} W^3_{4,4}(\underline{0}; 1, 1, 1, 2) = 12$ in the case $s = 5$. Finally, for $s = 4$, using a Macaulay 2 program [Grayson and Stillman] we have computed the dimension ${}_0\mathrm{hom}(I_B, I_{A/B}) = 3$ for $(B \to A)$ general and hence $\mathrm{codim}_{\mathrm{Hilb}^{p(x)}(\mathbb{P}^5)} W^3_{4,4}(\underline{0}; \underline{1}) = {}_0\mathrm{hom}(I_B, I_{A/B}) + \delta = 15$.

If $a_t \leq a_{t-1} + a_{t-2} - b_1$ we see in the example above that $W^{t-1}_{t,t}(\underline{b}; \underline{a})$ is a proper closed irreducible subset, that is, the generic curve of the component of $\mathrm{Hilb}^{p(x)}(\mathbb{P}^5)$ to which $W^{t-1}_{t,t}(\underline{b}; \underline{a})$ belongs is not defined by submaximal minors of a matrix of forms of degree $a_j - b_i$. The converse inequality always implies $\dim W^{t-1}_{t,t}(\underline{b}; \underline{a}) = \dim_{(X)} \mathrm{Hilb}^{p(x)}(\mathbb{P}^n)$ by Theorem 4.6. The pattern above for small a_t may be typical, but is in general rather difficult to prove. We illustrate this by two more examples.

Example 5.2. Let $X = \mathrm{Proj}\, A \subset \mathbb{P}^5$ be a general arithmetically Gorenstein curve defined by the submaximal minors of a 3×3 matrix whose first 2 rows are linear forms and whose last row are forms of degree $s - 2$ ($s \geq 3$), that is, $b_i = 0$ for $1 \leq i \leq 3$, $a_j = 1$ for $1 \leq j \leq 2$ and $a_3 = s - 2$. Thanks to Proposition 4.3, the

analysis of [Kleppe 2007, Example 43] immediately transfers to our case. Hence, for $s > 4$ ($a_t > a_{t-1} + a_{t-2} - b_1$), we see that X is unobstructed and

$$\dim W^2_{3,3}(\underline{0}; 1, 1, s-2) = \dim_{(X)} \mathrm{Hilb}^{p(x)}(\mathbb{P}^5) = (s+1)(s-1)^2 + 23.$$

Since by deleting the last row and taking maximal minors we get a threefold $Y = \mathrm{Proj}\, B$ for which ${}_0\mathrm{Ext}^2_B(N_B, N_B) = 0$, we have the unobstructedness of X also for $s = 3, 4$, and

$$(\delta(K_B)_{6-2s}, \delta(N_B)_{-s}) = \begin{cases} (-1, 2) & \text{for } s = 3, \\ (0, -3) & \text{for } s = 4. \end{cases}$$

That is, $\delta = -3$ when $s = 3$, and $\delta = 3$ when $s = 4$. In both cases,

$$\dim_{(X)} \mathrm{Hilb}^{p(x)}(\mathbb{P}^5) = (s+1)(s-1)^2 + 23 + \delta.$$

Thus

$$\dim_{(X)} \mathrm{Hilb}^{p(x)}(\mathbb{P}^5) = \begin{cases} 36 & \text{for } s = 3, \\ 71 & \text{for } s = 4; \end{cases}$$

see [Kleppe 2007, Example 43] for the computations. Now, applying Corollary 3.2(ii), we get $\mathrm{codim}_{\mathrm{Hilb}^{p(x)}(\mathbb{P}^5)} W^2_{3,3}(\underline{0}; 1, 1, 2) = 3$ in the case $s = 4$. Finally, for $s = 3$, a Macaulay 2 computation shows ${}_0\mathrm{hom}(I_B, I_{A/B}) = 3$ and hence

$$\mathrm{codim}_{\mathrm{Hilb}^{p(x)}(\mathbb{P}^5)} W^2_{3,3}(\underline{0}; \underline{1}) = {}_0\mathrm{hom}(I_B, I_{A/B}) + \delta = 0\,!$$

In the above examples we were able to analyze the case $a_t \le a_{t-1} + a_{t-2} - b_1$ through Corollary 3.2(ii) because ${}_s\mathrm{Ext}^1_B(N_B, A) = 0$. Since this vanishing may be rare, we want to improve upon Corollary 3.2(ii), at least to get estimates of the codimension of the stratum. We prefer to do it in the generality of [Kleppe 2007, Theorem 25] to extend Theorem 25 in this direction. This leads to the proposition below. Indeed with assumptions as in Proposition 5.3, one knows that the projection morphism $q : D \to \mathrm{Hilb}^{p_Y}(\mathbb{P}^n)$ induced by $(X' \subset Y') \to (Y')$ is smooth at $(X \subset Y)$ [Kleppe 2007, Theorem 47]. Using the fact that the corresponding tangent map is surjective, we get Proposition 5.3 and Remark 5.4(a). Since we only use these results in Example 5.6 and Remark 5.5, we skip the details of the proof which are rather straightforward once we have the results and proofs of [Kleppe 2007]. Put

$$c(I_{A/B}) := {}_0\mathrm{ext}^1_B(I_B/I_B^2, I_{A/B}) - {}_t\mathrm{ext}^2_B(S^2(I_{A/B}(s)), K_B).$$

Proposition 5.3. *Let $B = R/I_B$ be a graded licci quotient of R, let M be a graded maximal Cohen–Macaulay B-module, and suppose $\widetilde{M}$ is locally free of rank 2 in $U := \mathrm{Proj}\, B - Z$, that $\dim B - \dim B/I(Z) \ge 2$ and that $\bigwedge^2 \widetilde{M}|_U \simeq \widetilde{K_B}(t)|_U$. Let A be defined by a regular section σ of $\widetilde{M}^*(s)$ on U, as given by* (3-1), *let $X = \mathrm{Proj}\, A$, and suppose ${}_s\mathrm{Ext}^1_B(M, B) = 0$ and $\dim B \ge 4$. Moreover let char $(k) = 0$, let $(B \to A)$ be general and suppose (M, B) is unobstructed along any graded*

deformation of B *and* ${}_{-s}\mathrm{Ext}^2_B(I_B/I_B^2, M) = 0$. *Then the codimension* codi *of the stratum in* $\mathrm{Hilb}^{p(x)}(\mathbb{P}^n)$ *of subschemes given by* (3-1) *around* (X) *satisfies*

$$c(I_{A/B}) \le \mathrm{codi} \le c(I_{A/B}) + {}_0h^2(R, A, A) \le {}_0\mathrm{ext}^1_B(I_B/I_B^2, I_{A/B}),$$

and $\mathrm{codi} = c(I_{A/B}) + {}_0h^2(R, A, A)$ *if and only if* X *is unobstructed.*

Here "(M, B) unobstructed along any graded deformation of B" means that for every graded deformation (M_S, B_S) of (M, B), S local and Artinian with residue field k, there is a graded deformation of M_S to any graded deformation B_T of B_S for any small Artin surjection $T \to S$ [Kleppe 2007, Definition 11]. The important remark for our application $M = N_B$ where the codimension-2 CM quotient B satisfies $\mathrm{depth}_{I(Z)} B \ge 4$, is that *all* assumptions of the proposition are satisfied provided $\mathrm{char}\, k = 0$ and $(B \to A)$ is general (see proof of Corollary 41 and Remark 42 of [Kleppe 2007]).

Moreover recall that if we put

$$\delta := -\delta(I_{A/B})_0 = {}_0\mathrm{ext}^1_B(I_B/I_B^2, I_{A/B}) - {}_0\mathrm{hom}_R(I_B, I_{A/B}),$$

and use the exact sequence (3-2), we get $\delta = \delta(K_B)_{t-2s} - \delta(N_B)_{-s}$, as previously.

Remark 5.4. (a) With assumptions as in Proposition 5.3, except for $(B \to A)$ being general, we can also show

$$\delta - {}_t\mathrm{ext}^2_B(S^2(I_{A/B}(s)), K_B) \le \mathrm{codi}.$$

(b) If $\mathrm{depth}_{I(Z)} B \ge 4$, then we show

$${}_t\mathrm{Ext}^2_B(S^2(I_{A/B}(s)), K_B) \simeq {}_s\mathrm{Ext}^1_B(M, A),$$

exactly as we did for $M = N_B$ in the proof of Corollary 3.2. Thus if ${}_s\mathrm{Ext}^1_B(M, A) = 0$, then the lower bound $c(I_{A/B})$ of Proposition 5.3 is equal to the upper bound and we essentially get Corollary 3.2(ii)! Moreover, since codi ≥ 0, Corollary 3.2(i) corresponds to the case where the upper bound is zero!

Remark 5.5. In the case $s > \max n_{2,j}/2$, $\mathrm{depth}_{I(Z)} B \ge 4$ and $\mathrm{char}\,(k) = 0$, the inequalities of Proposition 5.3 lead to

$$\epsilon + \delta - {}_0\mathrm{ext}^1_B(N_B, A) \le \dim_{(X)} \mathrm{Hilb}^{p(x)}(\mathbb{P}^n) \le \epsilon + \delta\,,$$

with ϵ as in Corollary 3.2.

Example 5.6. Now let $X = \mathrm{Proj}\, A \subset \mathbb{P}^5$ be a general arithmetically Gorenstein curve defined by the submaximal minors of a 3×3 matrix whose first (resp. second) row consists of linear (resp. quadratic) forms and whose last row are forms of degree $s-3$ ($s \ge 5$), that is, $b_i = 0$ for $1 \le i \le 3$, $a_1 = 1$, $a_2 = 2$ and $a_3 = s-3$. In the following we skip a few details which we leave to the reader. Note that the case

$a_t > a_{t-1} + a_{t-2} - b_1$ or equivalently $s > 6$, is taken care of by Theorem 4.6. So we concentrate on the cases $s = 5$ and 6, which we analyze by using Proposition 5.3 and Remark 5.4. First, we use Remark 3.3 to compute ${}_0\mathrm{ext}^2_B(N_B, N_B)$ where B is obtained by deleting the last row and taking maximal minors. We easily get ${}_0\mathrm{ext}^2_B(N_B, N_B) = {}_s\mathrm{ext}^1_B(N_B, A) = 3$ by using

$$0 \longrightarrow R(-5) \oplus R(-4) \longrightarrow R(-3)^3 \longrightarrow R \longrightarrow B \longrightarrow 0,$$

(2-9), and

$${}_0\mathrm{Ext}^2_B(N_B, N_B) \simeq {}_0\mathrm{Hom}_B(I_B/I_B^2, H^4_{\mathfrak{m}}(I_B^2)).$$

Moreover, $\dim(K_B)_{6-2s} = 0$ by (2-5). Now if we apply ${}_{-2s}\mathrm{Hom}(-, K_B(6))$ to (2-3) we get $\delta(K_B)_{6-2s} = 0$ and ${}_{-2s}\mathrm{Ext}^1_B(I_B/I_B^2, K_B(6)) = 0$ for $s \geq 5$ provided we can show ${}_{-2s}\mathrm{Hom}(H_1, K_B(6)) = 0$. Using (2-3) we get that H_1 has rank 1 and $H_1 \simeq K_B(-3)$. Hence ${}_{-2s}\mathrm{Hom}(H_1, K_B(6)) \simeq B(9)_{-2s} = 0$ for $s \geq 5$.

It remains to compute $\delta(N_B)_{-s}$. We claim that $\delta(N_B)_{-s} = -8$ for $s = 5$ and $\delta(N_B)_{-s} = -3$ for $s = 6$. Indeed, dualizing the exact sequence (2-3), we get

$$0 \longrightarrow N_B \longrightarrow B(3)^3 \longrightarrow H_1^* \longrightarrow 0\,.$$

If we apply ${}_{-s}\mathrm{Hom}(I_B/I_B^2, -)$ to this sequence, recalling $H_1 \simeq K_B(-3)$ and hence ${}_{-s}\mathrm{Hom}(I_B/I_B^2, H_1^*) \simeq (I_B/I_B^2)_{9-s}$, we get an exact sequence which rather easily proves the claim. It follows that the numbers $\delta = \delta(K_B)_{6-2s} - \delta(N_B)_{-s}$ and ${}_s\mathrm{ext}^1_B(N_B, A)$ appearing in Remark 5.4 are computed. We conclude, for $s = 5$, that the codimension codi of the stratum in $\mathrm{Hilb}^{p(x)}(\mathbb{P}^5)$ of subschemes given by (3-1) around (X) is at least 5-dimensional. In fact a Macaulay 2 computation shows ${}_0h^2(R, A, A) = 0$ and ${}_0\mathrm{hom}(I_B, I_{A/B}) = 1$ and hence we have codi $= c(I_{A/B}) + {}_0h^2(R, A, A) = 6$ by Proposition 5.3. For $s = 6$ the lower bound for codi of Remark 5.4(a) is 0. Since a Macaulay 2 computation shows ${}_0\mathrm{hom}(I_B, I_{A/B}) = 0$ the better lower bound of Proposition 5.3 is also 0 while the smallest upper bound of Proposition 5.3 is 3. The latter is the correct bound for the codimension of the stratum, provided X is unobstructed. In conclusion, if X belongs to a reduced component V of $\mathrm{Hilb}^{p(x)}(\mathbb{P}^5)$, then codi $= 3$, but codi $= 0$ is possible, in which case V is nonreduced. We have not been able to fully tell what happens, but we expect V to be reduced and codi $= 3$.

The last case of the preceding example illustrates how difficult the analysis of when codi is positive could be. Cases where a_t is close to $a_{t-1} + a_{t-2} - b_1$ seem especially difficult to handle. Since it turns out that the lower bounds of Proposition 5.3 and Remark 5.4(a) are often negative (also in the case $a_t > a_{t-1} + a_{t-2} - b_1$ treated in Theorem 4.6), they are not very helpful. This, however, also indicates that the conclusions of Theorem 4.6 are rather strong.

References

[André 1974] M. André, *Homologie des algèbres commutatives*, Grundlehren der Math. Wissenschaften **206**, Springer, Berlin, 1974. MR 50 #4707 Zbl 0284.18009

[Avramov and Herzog 1980] L. Avramov and J. Herzog, "The Koszul algebra of a codimension 2 embedding", *Math. Z.* **175**:3 (1980), 249–260. MR 82g:13011 Zbl 0461.14014

[Bruns and Vetter 1988] W. Bruns and U. Vetter, *Determinantal rings*, Lecture Notes in Mathematics **1327**, Springer, Berlin, 1988. MR 89i:13001 Zbl 0673.13006

[Casanellas et al. 2005] M. Casanellas, E. Drozd, and R. Hartshorne, "Gorenstein liaison and ACM sheaves", *J. Reine Angew. Math.* **584** (2005), 149–171. MR 2006c:14072 Zbl 1095.14045

[Eagon and Northcott 1962] J. A. Eagon and D. G. Northcott, "Ideals defined by matrices and a certain complex associated with them.", *Proc. Roy. Soc. Ser. A* **269** (1962), 188–204. MR 26 #161 Zbl 0106.25603

[Ellingsrud 1975] G. Ellingsrud, "Sur le schéma de Hilbert des variétés de codimension 2 dans $\mathbf{P}^e$ à cône de Cohen-Macaulay", *Ann. Sci. École Norm. Sup.* (4) **8**:4 (1975), 423–431. MR 52 #13831 Zbl 0325.14002

[Ellingsrud and Peskine 1993] G. Ellingsrud and C. Peskine, "Anneau de Gorenstein associé à un fibré inversible sur une surface de l'espace et lieu de Noether-Lefschetz", pp. 29–42 in *Proceedings of the Indo-French Conference on Geometry* (Bombay, 1989), Hindustan Book Agency, Delhi, 1993. MR 95e:14025 Zbl 0846.14029

[Gorla 2008] E. Gorla, "A generalized Gaeta's theorem", *Compos. Math.* **144**:3 (2008), 689–704. MR 2009c:14098 Zbl 1147.14022

[Grayson and Stillman] D. Grayson and M. Stillman, "Macaulay 2: a software system for algebraic geometry and commutative algebra", Available at http://www.math.uiuc.edu/Macaulay2.

[Grothendieck 1966] A. Grothendieck, "Techniques de construction et théorèmes d'existence en géométrie algébrique, IV: Les schémas de Hilbert", in *Séminaire Bourbaki 1960/1961* (exposé 221), W. A. Benjamin, New York, 1966. Reprinted as pp. 249–276 in *Séminaire Bourbaki* **6**, Soc. Math. France, Paris, 1995. MR 1611822 Zbl 0236.14003

[Grothendieck 1968] A. Grothendieck, *Cohomologie locale des faisceaux cohérents et théorèmes de Lefschetz locaux et globaux* (SGA2), Masson, Paris, 1968. MR 57 #16294 Zbl 0197.47202

[Gulliksen and Negård 1972] T. H. Gulliksen and O. G. Negård, "Un complexe résolvant pour certains idéaux déterminantiels", *C. R. Acad. Sci. Paris Sér. A-B* **274** (1972), A16–A18. MR 45 #5124 Zbl 0238.13015

[Hochster and Eagon 1971] M. Hochster and J. A. Eagon, "Cohen-Macaulay rings, invariant theory, and the generic perfection of determinantal loci", *Amer. J. Math.* **93** (1971), 1020–1058. MR 46 #1787 Zbl 0244.13012

[Ile 2004] R. Ile, "Deformation theory of rank one maximal Cohen–Macaulay modules on hypersurface singularities and the Scandinavian complex", *Compos. Math.* **140**:2 (2004), 435–446. MR 2005e:13023 Zbl 1069.13010

[Kleppe 1979] J. O. Kleppe, "Deformations of graded algebras", *Math. Scand.* **45**:2 (1979), 205–231. MR 82j:13017 Zbl 0436.14004

[Kleppe 2006] J. O. Kleppe, "Maximal families of Gorenstein algebras", *Trans. Amer. Math. Soc.* **358**:7 (2006), 3133–3167. MR 2007e:13034 Zbl 1103.14005

[Kleppe 2007] J. O. Kleppe, "Unobstructedness and dimension of families of Gorenstein algebras", *Collect. Math.* **58**:2 (2007), 199–238. MR 2008g:14003 Zbl 1128.13015

[Kleppe and Miró-Roig 2005] J. O. Kleppe and R. M. Miró-Roig, "Dimension of families of determinantal schemes", *Trans. Amer. Math. Soc.* **357**:7 (2005), 2871–2907. MR 2006b:14086 Zbl 1073.14063

[Kleppe and Miró-Roig 2007] J. O. Kleppe and R. M. Miró-Roig, "Unobstructedness and dimension of families of codimension 3 ACM algebras", pp. 141–164 in *Algebra, geometry and their interactions*, Contemp. Math. **448**, Amer. Math. Soc., Providence, RI, 2007. MR 2009c:14097 Zbl 1128.13015

[Kleppe and Peterson 2001] J. O. Kleppe and C. Peterson, "Maximal Cohen-Macaulay modules and Gorenstein algebras", *J. Algebra* **238**:2 (2001), 776–800. MR 2002e:13029 Zbl 1028.13007

[Kleppe et al. 2001] J. O. Kleppe, J. C. Migliore, R. Miró-Roig, U. Nagel, and C. Peterson, *Gorenstein liaison, complete intersection liaison invariants and unobstructedness*, Mem. Amer. Math. Soc. **732**, 2001. MR 2002e:14083 Zbl 1006.14018

[Miró-Roig 2008] R. M. Miró-Roig, *Determinantal ideals*, Progress in Mathematics **264**, Birkhäuser Verlag, Basel, 2008. MR 2008k:14098 Zbl 1131.14002

[Vasconcelos 1994] W. V. Vasconcelos, *Arithmetic of blowup algebras*, London Math. Soc. Lecture Note Series **195**, Cambridge University Press, Cambridge, 1994. MR 95g:13005 Zbl 0813.13008

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The structure of the group $G(k[t])$: Variations on a theme of Soulé

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Following Soulé's ideas from 1979, we give a presentation of the abstract group $\mathrm{G}(k[t])$ for any semisimple (connected) simply connected absolutely almost simple k-group $\mathrm{G}(k[t])$. As an application, we give a description of $\mathrm{G}(k[t])$ in terms of direct limits, and show that the Whitehead group and the naive group of connected components of $\mathrm{G}(k[t])$ coincide.

1. Introduction

Let k be a field, and let $\boldsymbol{G}$ be a semisimple simply connected absolutely almost simple k-group. In the case that $\boldsymbol{G}$ is split, Soulé [1979] has given a presentation of the group $\boldsymbol{G}(k[t])$, thus extending a theorem of Nagao [1959] for $\boldsymbol{SL}_2$ (see also [Serre 1977, II.1.6]). The goal of this note is to provide a presentation of $\boldsymbol{G}(k[t])$ in the general case.

We will follow Soulé's original ideas and study the action of $\boldsymbol{G}(k[t])$ on the Bruhat–Tits building [1984] of $\boldsymbol{G}$ corresponding to the field $K = k((1/t))$, where K is viewed as the completion of $k(t)$ with respect to the valuation at ∞. As an application, we show that the Whitehead group of $\boldsymbol{G}$ coincides with the naive group of connected components of $\boldsymbol{G}$.

2. Structure of the group $G(k[t])$

Throughout k and $\boldsymbol{G}$ will be as above. For convenience the group $\boldsymbol{G}(k[t])$ will be denoted by Γ.

Notation and statement of the main theorem. Let $\boldsymbol{S}$ be a maximal k-split torus of $\boldsymbol{G}$, and let $\boldsymbol{T}$ be a maximal torus of $\boldsymbol{G}$ containing $\boldsymbol{S}$. Recall that $\boldsymbol{S}_K$ is a maximal K-split torus of $\boldsymbol{G}_K$. Let $\tilde{k}/k$ be a finite Galois extension that splits $\boldsymbol{T}$ (hence also $\boldsymbol{G}$). Set $\mathcal{G} = \mathrm{Gal}(\tilde{k}/k)$ and $\widetilde{\boldsymbol{T}} = \boldsymbol{T} \times_k \tilde{k}$.

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Let $\widetilde{\boldsymbol{G}} = \boldsymbol{G} \times_k \tilde{k}$ and $\widetilde{\boldsymbol{S}} = \boldsymbol{S} \times_k \tilde{k}$. We choose compatible orderings on the root systems $\Phi = \Phi(\boldsymbol{G}, \boldsymbol{S})$ and $\widetilde{\Phi} = \Phi(\widetilde{\boldsymbol{G}}, \widetilde{\boldsymbol{T}})$; see [Borel 1991]. We then have a set Δ of relative simple roots and a set $\widetilde{\Delta}$ of absolute simple roots.

It will be convenient to maintain essentially the same notation as in Soulé's paper:

- $A = k[t]$, $K = k((1/t))$ and $G = \boldsymbol{G}(K)$.
- ω is the valuation on K at ∞, that is, the valuation on K having $\mathcal{O} = k[\![1/t]\!]$ as its ring of integers.

We also have the analogues of the above objects for $\tilde{k}$:

- $\widetilde{A} = \tilde{k}[t]$, $\widetilde{K} = \tilde{k}((1/t))$, $\widetilde{\Gamma} = \boldsymbol{G}(\widetilde{A})$, and $\widetilde{\mathcal{O}} = \tilde{k}[\![1/t]\!]$.

At the level of buildings we set [Bruhat and Tits 1984, section 4.2]

- $\mathcal{T}$ the (affine) Bruhat–Tits building of the K-group $\boldsymbol{G}_K := \boldsymbol{G} \times_k K$, and
- $\widetilde{\mathcal{T}}$ the Bruhat–Tits building of the $\widetilde{K}$-group $\boldsymbol{G}_{\widetilde{K}} := \boldsymbol{G} \times_k \widetilde{K}$.[1]

Both $\mathcal{T}$ and $\widetilde{\mathcal{T}}$ have a natural simplicial complex structure [ibidem, section 4.2.23].

Recall that $\mathcal{T}$ is equipped with an action of $\boldsymbol{G}(K)$ and that $\widetilde{\mathcal{T}}$ is equipped with an action of $\boldsymbol{G}(\widetilde{K}) \rtimes \mathcal{G}$. We have an isometric embedding $j : \mathcal{T} \to \widetilde{\mathcal{T}}$ that identifies $\mathcal{T}$ with $\widetilde{\mathcal{T}}^{\mathcal{G}}$. The hyperspecial group $\boldsymbol{G}(\widetilde{O})$ of $\boldsymbol{G}(\widetilde{K})$ fixes a unique point $\widetilde{\phi}$ of $\widetilde{\mathcal{T}}$ [Bruhat and Tits 1972, section 9.1.9.c]. This point descends to a point ϕ of $\mathcal{T}$.

We denote by $\mathcal{A}$ the standard apartment of $\mathcal{T}$ associated to $\boldsymbol{S}$ (this is a real affine space) and similarly by $\widetilde{\mathcal{A}}$ the standard apartment associated to $\widetilde{\boldsymbol{T}}$. The point $\widetilde{\phi}$ belongs to $\widetilde{\mathcal{A}}$ (ibidem). Since

$$\mathrm{Hom}_{k-gr}(\boldsymbol{G}_m, \boldsymbol{S}) \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathrm{Hom}_{k-gr}(\boldsymbol{G}_m, \boldsymbol{T}) \otimes_{\mathbb{Z}} \mathbb{R} \cong (\mathrm{Hom}_{\tilde{k}-gr}(\boldsymbol{G}_{m,\tilde{k}}, \widetilde{\boldsymbol{T}}) \otimes_{\mathbb{Z}} \mathbb{R})^{\mathcal{G}}$$

[Bruhat and Tits 1984, section 4.2], we have $j(\mathcal{A}) = \widetilde{\mathcal{A}}^{\mathcal{G}}$, so ϕ belongs to $\mathcal{A}$ and

$$\mathcal{A} = \phi + \mathrm{Hom}_{k-gr}(\boldsymbol{G}_m, \boldsymbol{S}) \otimes_{\mathbb{Z}} \mathbb{R}.$$

By means of the canonical pairing $\langle \cdot, \cdot \rangle : \mathrm{Hom}_{k-gr}(\boldsymbol{S}, \boldsymbol{G}_m) \times \mathrm{Hom}_{k-gr}(\boldsymbol{S}, \boldsymbol{G}_m) \to \mathbb{Z}$ we can then define the *sector* (quartier)

$$\mathfrak{Q} := \phi + D, \quad \text{where } D := \left\{ v \in \mathrm{Hom}_{k-gr}(\boldsymbol{S}, \boldsymbol{G}_m) \otimes_{\mathbb{Z}} \mathbb{R} \;\middle|\; \langle b, \lambda \rangle \geq 0, \ \ \forall b \in \Delta \right\}.$$

The following result generalizes Soulé's theorem [1979].

Theorem 2.1. *The set $\mathfrak{Q}$ is a simplicial fundamental domain for the action of $\boldsymbol{G}(k[t])$ on $\mathcal{T}$. In other words, any simplex of $\mathcal{T}$ is equivalent under the action of $\boldsymbol{G}(k[t])$ to a unique simplex of $\mathfrak{Q}$.*

[1] Since $\boldsymbol{G} \times_K \widetilde{K}$ is split, the assumptions of [Bruhat and Tits 1984, section 5.1.1.1] are satisfied. This allows us to do away with the "standard" assumption that the base field k be perfect.

Buildings and valuations. Let $\boldsymbol{P}$ be the minimal parabolic k-subgroup of $\boldsymbol{G}$ defined by $\boldsymbol{S}$ and Δ. We denote by $\boldsymbol{U} = R_u(\boldsymbol{P})$ the unipotent radical of $\boldsymbol{P}$.

We denote by $\widetilde{\boldsymbol{U}}_{\tilde{a}}$ the split unipotent subgroup associated to a root $\tilde{a} \in \widetilde{\Phi}$, and by $\tilde{a}^\vee : \boldsymbol{SL}_2 \to \boldsymbol{G}$ the corresponding standard homomorphism; see [Springer 1979, Section 2.2].

The set of positive and negative roots with respect to the basis Δ of Φ will be denoted by Φ^+ and Φ^-, respectively. Given $b \in \Phi$, the subset of absolute roots

$$\widetilde{\Phi}^b := \big\{\tilde{a} \in \widetilde{\Phi} \mid \tilde{a}|_{S\times_k \tilde{k}} = b \text{ or } 2b\big\}$$

is positively closed in $\widetilde{\Phi}$. It defines then a split $\tilde{k}$-unipotent subgroup $\widetilde{\boldsymbol{U}}_b$ of $\widetilde{\boldsymbol{G}}$ that descends to a split k-unipotent subgroup $\boldsymbol{U}_b$ of $\boldsymbol{G}$. As in [Bruhat and Tits 1972], we make the convention that $\boldsymbol{U}_{2b} = 1$ if $2b \notin \Phi$.

For $I \subset \Delta$, we define along standard lines

$$\boldsymbol{S}_I = \Big(\bigcap_{b\in I} \ker(b)\Big)^0 \subset \boldsymbol{S}, \quad \boldsymbol{L}_I = \mathscr{Z}_{\boldsymbol{G}}(\boldsymbol{S}_I), \quad \boldsymbol{P}_I = \boldsymbol{U}_I \rtimes \boldsymbol{L}_I.$$

Thus $\boldsymbol{P}_I$ is the standard parabolic subgroup of $\boldsymbol{G}$ of type I and $\boldsymbol{L}_I$ is its standard Levi subgroup (see [Borel 1991, Section 21.11]). Recall that the root system $\Phi(\boldsymbol{L}_I, \boldsymbol{S}) = [I]$ is the subroot system of Φ consisting of roots that are linear combinations of I; the split unipotent k-group $\boldsymbol{U}_I$ is the subgroup of $\boldsymbol{U}$ generated by the $\boldsymbol{U}_b$ with b running over $\Phi^+ \setminus [I]$.

Given $\tilde{a} \in \widetilde{\Phi}$, the group $\widetilde{U}_{\tilde{a}} := \widetilde{\boldsymbol{U}}_{\tilde{a}}(\widetilde{K}) = \widetilde{K}$ is equipped with the valuation ω, which we denote by $\widetilde{\varphi}_a : \widetilde{U}_a \to \mathbb{R} \cup \{\infty\}$. This defines the Chevalley–Steinberg "donnée radicielle valuée"

$$(T(\widetilde{K}), (\widetilde{U}_{\tilde{a}}, M_{\tilde{a}})_{\tilde{a}\in\widetilde{\Phi}}), \quad \text{where } M_{\tilde{a}} = T(\widetilde{K})\ \tilde{a}^\vee\big(\left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]\big)$$

[Bruhat and Tits 1972, exemple 6.2.3.b], and also a filtration $(\widetilde{U}_{\tilde{a},m})_{m\in\mathbb{Z}}$ of $\widetilde{U}_{\tilde{a}}$ where $\widetilde{U}_{\tilde{a},m} := \widetilde{\varphi}^{-1}([m, +\infty[)$. Note that $\widetilde{U}_{\tilde{a},0} = \widetilde{\boldsymbol{U}}_{\tilde{a}}(\widetilde{\mathbb{O}})$.

A crucial point of Bruhat–Tits theory is the descent of this data to $G = \boldsymbol{G}(K)$ [1984, section 5.1]. Given $b \in \Phi$, the commutative group $U_b := \boldsymbol{U}_b(K)$ is equipped with the descended valuation $\varphi_b : U_b \to \mathbb{R} \cup \{\infty\}$. The definition of φ_b is delicate, and is given as follows [Bruhat and Tits 1984, section 5.1.16]. Define

$$\widetilde{U}_{b,m} := \prod_{\substack{\tilde{a}\in\widetilde{\Phi}^b,\\ \tilde{a}|_{S\times_k\tilde{k}}=b}} \widetilde{U}_{\tilde{a},m} \,.\, \prod_{\substack{\tilde{a}\in\widetilde{\Phi}^b,\\ \tilde{a}|_{S\times_k\tilde{k}}=2b}} \widetilde{U}_{\tilde{a},2m} \quad \text{for } m \in \mathbb{R}.$$

Then U_b is a subgroup of $\boldsymbol{U}_b(\widetilde{K}) = \widetilde{U}_b = \bigcup_{m\in\mathbb{R}} \widetilde{U}_{b,m}$ and the descended valuation is defined by

$$\varphi_b(u) := \mathrm{Sup}\{m \in \mathbb{R} \mid u \in \widetilde{U}_{b,m}\}.$$

Note that[2] $\Theta_b := \varphi_b\big(U_b \setminus \{e\}\big)$ is either $\mathbb{Z}$ or $\frac{1}{2}\mathbb{Z}$. As above, it gives rises to a filtration $(U_{b,m})_{m\in\Theta_b}$ of U_b such that $U_{b,0} = \boldsymbol{U}_b(\mathcal{O})$.

Again we make the convention that $U_{2b} = 1$ if $2b \notin \Phi$.

Description of the isotropy group of a vertex. Given $\Omega \subset \mathcal{Q}$, we denote by Γ_Ω the corresponding isotropy subgroup, namely the elements of Γ that fix all elements of Ω. We introduce an analogous definition and notation for $j(\Omega) \in \widetilde{\mathcal{A}}$. By Galois descent we have

$$\Gamma_\Omega = \big(\widetilde{\Gamma}_{j(\Omega)}\big)^{\mathcal{G}}. \tag{2-1}$$

In particular, since $\widetilde{\Gamma}_{\widetilde{\phi}} = \boldsymbol{G}(\widetilde{\mathcal{O}}) \cap \widetilde{\Gamma} = \boldsymbol{G}(\tilde{k})$ [Soulé 1973, section 1.1], we have $\Gamma_\phi = (\widetilde{\Gamma}_{\widetilde{\phi}})^{\mathcal{G}} = \boldsymbol{G}(\tilde{k})^{\mathcal{G}} = \boldsymbol{G}(k)$.

If $x \in \mathcal{Q} \setminus \{\phi\}$ and if $[x[$ is the halfline of origin x and direction $\overrightarrow{\phi x}$, we claim that $\Gamma_x = \Gamma_{[x[}$. If $\boldsymbol{G}$ is split, this is proven in Soulé's paper by reduction to the case of $\boldsymbol{SL}_n$. By applying the identity (2-1) to x and $[x[$, our claim now readily follows from the absolute case.

The isotropy of $[x[$ in $G = \boldsymbol{G}(K)$ is the Bruhat–Tits abstract parahoric group $P_{[x[}$. See [Bruhat and Tits 1972, section 7.1]. We have

$$P_{[x[} = U_{[x[} \,.\, H, \quad \text{where } H = \mathrm{Fix}_G(\mathcal{A}).$$

By [Bruhat and Tits 1984, section 5.2.2], we have $H = \mathscr{Z}_{\boldsymbol{G}}(\boldsymbol{S})(\mathcal{O})$. The group $U_{[x[}$ is defined by means of the function [Bruhat and Tits 1972, section 6.4.2]

$$f_{[x[} : \Phi \to \mathbb{R} \cup \{\infty\}, \quad b \mapsto \inf\{s \in \mathbb{R} \mid b(y) + s \geq 0 \text{ for all } y \in [x[\}.$$

Hence

$$f_{[x[}(b) = \begin{cases} 0 & \text{if } b(x) = 0, \\ -b(x) & \text{if } b(x) > 0, \\ \infty & \text{if } b(x) < 0. \end{cases}$$

The group $U_{[x[} \subset G$ is then the subgroup of G generated by the $U_{b,m}$ for $b \in \Phi^+$ and $m \geq -b(x)$ ($m \in \Theta_b$), together with the $\boldsymbol{U}_b(\mathcal{O})$ for $b \in \Phi^-$ such that $b(x) = 0$. In other words, by distinguishing positive roots that vanish at x, we see that $U_{[x[}$ is the subgroup of G generated by subgroups of the following three "shapes":

(I) $U_{b,m}$ for $b \in \Phi^+$ such that $b(x) > 0$ and $m \in \Theta_b$ such that $m \geq -b(x)$;

(II) $\boldsymbol{U}_b(\mathcal{O})$ for $b \in \Phi^+$ such that $b(x) = 0$;

(III) $\boldsymbol{U}_b(\mathcal{O})$ for $b \in \Phi^-$ such that $b(x) = 0$.

Define $U^{\pm}_{[x[} := U_{[x[} \cap \boldsymbol{U}^{\pm}(K)$ as in [Bruhat and Tits 1972, section 6.4.2]. These by definition generate $U_{[x[}$. On the other hand, $U^{+}_{[x[}$ (respectively $U^{-}_{[x[}$) is the subgroup

[2] We use the notation Θ_b rather than the more standard Γ_b found in [Bruhat and Tits 1972] to avoid any possible confusion with the notation used in Soulé's paper.

of $U_{[x[}$ generated by the subgroups of type (I) and (II) (respectively (III)); see [Bruhat and Tits 1972, proposition 6.4.9]. Define the subset of roots

$$I_x = \{b \in \Delta \mid b(x) = 0\}.$$

This definition makes sense if x is an element of $\mathscr{A}$, and we then have $I_\phi = \Delta$.

Lemma 2.2. *We have*

$$[I_x] \cap \Phi^+ = \{b \in \Phi^+ \mid b(x) = 0\}, \tag{2-2}$$

$$\Phi^+ \setminus [I_x] = \{b \in \Phi^+ \mid b(x) > 0\}, \tag{2-3}$$

$$[I_x] \cap \Phi^- = \{b \in \Phi^- \mid b(x) = 0\}. \tag{2-4}$$

Proof. Observe that if $b \in [I_x]$, then b is a linear combination of elements of I_x; hence $b(x) = 0$. This implies that $[I_x] \cap \Phi^+ \subset \{b \in \Phi^+ \mid b(x) = 0\}$. Conversely, let b be a positive root such that $b(x) = 0$. Then $b = \sum_{c\in\Delta} n_c c$, where the n_c are nonnegative integers. Hence $\sum_{c\in\Delta} n_c c(x) = 0$. Since $x \in \mathfrak{D}$, we have $c(x) \geq 0$. Therefore $n_c c(x) = 0$ and b is a linear combination of elements of I_x, proving (2-2). Since

$$\{b \in \Phi^+ \mid b(x) \neq 0\} = \{b \in \Phi^+ \mid b(x) > 0\},$$

we get also (2-3). Similar considerations apply to (2-4). □

It follows from (2-2) and (2-4) respectively that the subgroups of shape (II) and (III) are subgroups of $\boldsymbol{L}_{I_x}(\mathcal{O})$, and (2-3) shows that the subgroups of shape (I) are subgroups of $\boldsymbol{U}_{I_x}(K)$. Hence we get the inclusion

$$U_{[x[} \subset (U_{[x[} \cap \boldsymbol{U}_{I_x}(K)) \rtimes \boldsymbol{L}_{I_x}(\mathcal{O}) \subset \boldsymbol{P}_{I_x}(K). \tag{2-5}$$

Lemma 2.3. (1) $\boldsymbol{L}_{I_x}(\mathcal{O}) \subset P_{[x[} \subset \boldsymbol{U}_{I_x}(K) \rtimes \boldsymbol{L}_{I_x}(\mathcal{O}) \subset \boldsymbol{P}_{I_x}(K)$;

(2) $\boldsymbol{U}_{I_x}(K) \cap P_{[x[} \subset U^+_{[x[}$;

(3) $\bigcup_{z\geq 1}(U^+_{[zx[} \cap \boldsymbol{U}_{I_x}(K)) = \boldsymbol{U}_{I_x}(K)$.

Proof. Let $I = I_x$.

(1) Since $U_{[x[} \subset \boldsymbol{U}_I(K) \rtimes \boldsymbol{L}_I(\mathcal{O})$ and $\mathscr{Z}_{\boldsymbol{G}}(\boldsymbol{S}) \subset \boldsymbol{L}_I$, it follows that $P_{[x[} = U_{[x[} \,.\, H = U_{[x[} \,.\, \mathscr{Z}_{\boldsymbol{G}}(\boldsymbol{S})(\mathcal{O})$ is a subgroup of $\boldsymbol{U}_I(K) \rtimes \boldsymbol{L}_I(\mathcal{O})$.

Let us show that $\boldsymbol{L}_I(\mathcal{O}) \subset P_{[x[}$. Let $\boldsymbol{V}_I$ be the unipotent radical of the minimal standard parabolic subgroup of $\boldsymbol{L}_I$, namely the k-subgroup of $\boldsymbol{U}$ generated by the $\boldsymbol{U}_b$ such that $b \in \Phi^+$ and $b(x) = 0$. We have [SGA3 1962/1964, théorème XXVI.5.1]

$$\bigcup_{g\in\boldsymbol{V}_I(k)} g\Omega = \boldsymbol{L}_I,$$

where Ω stands for the big cell $\boldsymbol{V}_I^- \times_k \mathscr{Z}_{\boldsymbol{G}}(S) \times_k \boldsymbol{V}_I$ of $\boldsymbol{L}_I$. Since $\mathcal{O}$ is local, it follows that

$$\boldsymbol{L}_I(\mathcal{O}) = \boldsymbol{V}_I(k) \,.\, \Omega(\mathcal{O}) = \boldsymbol{V}_I(k) \,.\, \boldsymbol{V}_I^-(\mathcal{O}) \,.\, H \,.\, \boldsymbol{V}_I(\mathcal{O}).$$

We conclude that $\boldsymbol{L}_I(\mathcal{O}) \subset P_{[x[}$.

(2) We claim that $\boldsymbol{U}(K) \cap P_{[x[} = U^+_{[x[}$. This establishes (2) since $\boldsymbol{U}_I(K) \subset \boldsymbol{U}(K)$. To prove the claim, we need to show that $\boldsymbol{U}(K) \cap P_{[x[} \subset U^+_{[x[}$ (the reversed inclusion is obvious). With the notations of [Bruhat and Tits 1972, section 7], we have $\boldsymbol{U}(K) = U_D^+$ where D is the direction of the sector $\mathscr{Q}$. By [ibidem, 7.1.4], we have

$$P_{[x[} \cap \boldsymbol{U}(K) = U_{[x[+D},$$

where $U_{[x[+D}$ is the subgroup of $\boldsymbol{G}(K)$ attached to the subset $[x[+ D = x + D$ of $\mathscr{A}$. This group is defined by means of the function [ibidem, section 6.4.2]

$$f_{x+D} : \Phi \to \mathbb{R} \cup \{\infty\}, \quad b \mapsto \inf\big\{s \in \mathbb{R} \;\big|\; b(y) + s \geq 0 \text{ for all } y \in x + D\big\}.$$

Hence

$$f_{x+D}(b) = \begin{cases} -b(x) & \text{if } b > 0, \\ \infty & \text{if } b < 0, \end{cases}$$

so $U_{x+D} = U^+_{[x[}$ as desired.

(3) If $b \in \Phi^+$ satisfies $b(x) > 0$, then the number $\mathrm{Inf}\{m \in \Theta_b \mid m + b(zx) \geq 0\}$ tends to $-\infty$ as z tends to ∞. This readily yields $\bigcup_{z \geq 1}(U^+_{[zx[} \cap \boldsymbol{U}_I(K)) = \boldsymbol{U}_I(K)$. □

Remark 2.4. Geometrically speaking, the K-parabolic $\boldsymbol{P}_{I_x} \times_k K$ is attached to the extremity of the halfline $[x[$ in the spherical building at infinity; see [Garrett 1997, Section 16.9]. Since $P_{[x[}$ is the isotropy group of the half line $[x[$, it fixes its extremity. This point of view yields another way to prove the inclusion $P_{[x[} \subset \boldsymbol{P}_{I_x}(K)$ which is part of Lemma 2.3(1).

Given $b \in \Phi$, we set

$$m_x(b) := \mathrm{Inf}\{m \in \Theta_b \mid m + b(x) \geq 0\}.$$

Since $\Gamma_x = P_{[x[} \cap \Gamma$, we have the inclusion

$$\big\langle (U_{b,m_x(b)} \,.\, U_{2b,m_x(2b)}) \cap \Gamma,\ b \in \Phi,\ b(x) \geq 0 \big\rangle \subset \Gamma_x. \tag{2-6}$$

Proposition 2.5. (1) $\Gamma_x = \big(\Gamma_x \cap \boldsymbol{U}_{I_x}(K)\big) \rtimes \boldsymbol{L}_{I_x}(k)$;

(2) $\Gamma_x = \big\langle (U_{b,m_x(b)} \,.\, U_{2b,m_x(2b)}) \cap \Gamma,\ b(x) > 0 \big\rangle \rtimes \boldsymbol{L}_{I_x}(k)$;

(3) $\bigcup_{z \geq 1} \Gamma_{zx} = \boldsymbol{U}_{I_x}(k[t]) \rtimes \boldsymbol{L}_{I_x}(k)$.

Proof. To lighten the notation we set $I = I_x$.

(1) According to Lemma 2.3(1), $\boldsymbol{L}_I(k) = \Gamma \cap \boldsymbol{L}_I(\mathbb{O})$ fixes the point x. Hence the inclusion

$$(\Gamma_x \cap \boldsymbol{U}_I(K)) \rtimes \boldsymbol{L}_I(k) \subset \Gamma_x.$$

To prove the reverse inclusion we use the projection $\boldsymbol{P}_I(K) \to \boldsymbol{L}_I(K)$. The image of Γ_x inside $\boldsymbol{L}_I(K)$ is a subgroup of $\boldsymbol{L}_I(A)$. On the other hand, by Lemma 2.3(1), the image of P_x inside $\boldsymbol{L}_I(K)$ is the subgroup $\boldsymbol{L}_I(\mathbb{O})$. Hence the image of Γ_x inside $\boldsymbol{L}_I(K)$ is a subgroup of $\boldsymbol{L}_I(A) \cap \boldsymbol{L}_I(\mathbb{O}) = \boldsymbol{L}_I(k)$. We thus have an exact sequence

$$1 \to (\Gamma_x \cap \boldsymbol{U}_I(K)) \to \Gamma_x \to \boldsymbol{L}_I(k)$$

which is a split surjection.

(2) Put $V := \langle (U_{b,m_x(b)} \,.\, U_{2b,m_x(2b)}) \cap \Gamma,\ b \in \Phi,\ b(x) > 0 \rangle$. This is a subgroup of Γ_x by (2-6) and of $U_I(K)$ by (2-5). So $V \subset \Gamma_x \cap U_I(K)$. For showing the reverse inclusion, it suffices to show that

$$\Gamma_x \cap \boldsymbol{U}_I(K) \subset \left\langle (U_{b,m_x(b)} \,.\, U_{2b,m_x(2b)}) \cap \Gamma,\ b(x) \geq 0 \right\rangle. \tag{2-7}$$

From Lemma 2.3(3) we have $\Gamma_x \cap \boldsymbol{U}_I(K) \subset \Gamma \cap U^+_{[x[}$. Accordingly, it will suffice to show that $\Gamma_x \cap U^+_{[x[}$ is a subgroup of the right side of (2-7). Let $\Phi^+_{\mathrm{red}} = \{b_1, \ldots, b_N\}$ be the subset of reduced positive roots (with an arbitrary order). The product induces a isomorphism of k-varieties $\prod_{j=1}^N \boldsymbol{U}_{b_j} \xrightarrow{\sim} \boldsymbol{U}$ by [Borel 1991, Proposition 21.9]. In particular, we have compatible bijections

$$\begin{array}{ccc} \prod_{j=1}^N \boldsymbol{U}_{b_j}(K) & \xrightarrow{\sim} & \boldsymbol{U}(K) \\ \cup & & \cup \\ \prod_{j=1}^N \boldsymbol{U}_{b_j}(A) & \xrightarrow{\sim} & \boldsymbol{U}(A). \end{array}$$

By comparing these with the bijection [Bruhat and Tits 1972, section 6.4.9]

$$\prod_{j=1}^N U_{b_j,m_x(b_j)} \,.\, U_{2b_j,m_x(2b_j)} \xrightarrow{\sim} U^+_{[x[},$$

we can see that $\Gamma_x \cap \boldsymbol{U}_I(K) \subset U^+_{[x[} \cap \boldsymbol{U}(A)$ consists of products of elements $(U_{b_j,m_x(b_j)} \,.\, U_{2b_j,m_x(2b_j)}) \cap \Gamma$ with $b_j(x) \geq 0$.

(3) This follows from (1) and Lemma 2.3(3). □

Action on the star of certain points. We will now make use of the spherical building $\mathcal{B}(\boldsymbol{G})$ of $\boldsymbol{G}$ from [Tits 1974, Section 5]. Recall that $\mathcal{B}(\boldsymbol{G})$ is a simplicial complex whose simplexes are the k-parabolic subgroups of $\boldsymbol{G}$. If $\boldsymbol{Q}$ is such a parabolic subgroup, the faces of its associated simplex are the simplexes associated to the maximal proper k-parabolic subgroups of $\boldsymbol{Q}$. The standard apartment $\mathfrak{A}$ of $\mathcal{B}(\boldsymbol{G})$ is the subcomplex of k-parabolic subgroups containing $\boldsymbol{S}$, and the standard chamber $\mathfrak{C}$ is the simplex associated to the minimal k-parabolic subgroup $\boldsymbol{P}$. We denote by $\boldsymbol{W} = \boldsymbol{N}_{\boldsymbol{G}}(\boldsymbol{S})/\mathcal{Z}_{\boldsymbol{G}}(\boldsymbol{S})$ the relative Weyl group of $\boldsymbol{G}$.

If $x \in \mathcal{T}$, we denote by $\mathcal{L}_x$ the *star* of x (étoile in French),[3] that is, the subspace of $\mathcal{T}$ consisting of facets F such that $x \in \overline{F}$ [Bruhat and Tits 1984, section 4.6.33].

We denote by $\boldsymbol{S}_* = \mathrm{Hom}_{k-gr}(\boldsymbol{G}_m, \boldsymbol{S})$ the group of cocharacters of $\boldsymbol{S}$. Inside the apartment $\mathcal{A} = \phi + \boldsymbol{S}_* \otimes_{\mathbb{Z}} \mathbb{R}$, this corresponds to the lattice of points having type 0, that is, the type of ϕ. The action of $\boldsymbol{S}(K)$ on $\mathcal{T}$ preserves $\mathcal{A}$. More precisely, the element $s \in S(K)$ acts on $\mathcal{A}$ as the translation by the vector v_s defined by the property [Bruhat and Tits 1984, section 5.1.22]

$$\langle v_s, b\rangle = -\omega(b(s)) \quad \text{for all } b \in \Phi. \tag{2-8}$$

We denote by $\mathcal{C} \subset S_* \otimes_{\mathbb{Z}} \mathbb{R}$ the vector chamber such that $\phi + \mathcal{C}$ is the unique chamber of the sector $\mathcal{Q}$ that contains the special point ϕ in its adherence; see [Bruhat and Tits 1972, section 1.3.11].

Lemma 2.6. *Let x be a point of $\boldsymbol{S}_* \cap \mathcal{Q}$. Then the chambers of $\mathcal{L}_x \cap \mathcal{Q}$ are the $x + w\mathcal{C}$ for $w \in \boldsymbol{W}(k)$ satisfying $I_x \subset w \,.\, \Phi^+$.*

Proof. Set $I = I_x$. The chambers of $\mathcal{L}_x$ are the $x + w\mathcal{C}$ with $w \in \boldsymbol{W}(k)$. Let $y \in \mathcal{C}$. If $x + w\mathcal{C} \subset \mathcal{Q}$, then

$$b(x + w \,.\, y) = b(x) + (w^{-1} \,.\, b)(y) \geq 0 \quad \text{for all } b \in \Delta\,.$$

It follows that if $b \in I$, that is, $b(x) = 0$, then $(w^{-1} \,.\, b)(y) \geq 0$, and therefore $b \in w(\Phi_+)$. Conversely, if $w \in \boldsymbol{W}(k)$ satisfies $I \subset w(\Phi_+)$, then the inequality above holds for ϵy for all $b \in \Delta$ for $\epsilon > 0$ small enough. Thus $x + w \,.\, (\epsilon y) \in \mathcal{Q}$ and $x + w\mathcal{C} \subset \mathcal{Q}$. □

Lemma 2.7. *Let I be a subset of Δ, and set $\boldsymbol{W}_I := \boldsymbol{N}_{\boldsymbol{L}_I}(\boldsymbol{S})/\mathcal{Z}_{\boldsymbol{G}}(\boldsymbol{S})$. Let $\mathfrak{A}_I$ be the union of the $w\overline{\mathfrak{C}}$ for w running over the elements of $\boldsymbol{W}(k)$ satisfying $I \subset w \,.\, \Phi^+$.*

(1) $\boldsymbol{W}_I(k) \,.\, \mathfrak{A}_I = \mathfrak{A}$.

(2) $\boldsymbol{P}_I(k) \,.\, \mathfrak{A}_I = \mathcal{B}(\boldsymbol{G})$.

Proof. (1) We reason by induction on the cardinality of I. If $I = \varnothing$, then $\mathfrak{A}_I = \mathfrak{A}$ and there is nothing to prove. Assume that $I = I' \cup \{b\}$. We are given a chamber $w\mathfrak{C}$ of $\mathfrak{A}$ with $w \in \boldsymbol{W}(k)$. We want to show that $w\mathfrak{C}$ is equivalent under $\boldsymbol{W}_I(k)$ to a

[3] The terminology *link* is also used in the literature.

chamber of $\mathfrak{A}_I$. Since $\boldsymbol{W}_{I'}(k) \subset \boldsymbol{W}_I(k)$, we can assume by the induction hypothesis that $I' \subset w \,.\, \Phi^+$. If $b \in w \,.\, \Phi^+$, we have $I \subset w \,.\, \Phi^+$. The other case is when $-b \in w \,.\, \Phi^+$. Let $s_b \in \boldsymbol{W}_I(k)$ be the reflection associated to b. Then $s_b(b) = -b$; hence $b \subset s_b w \,.\, \Phi^+$. For $b' \in I'$, we have $s_b(b') = b' + mb$, where m is nonnegative. Therefore

$$b' = s_b^2(b') = s_b(b' + mb) = s_b(b') - mb \in s_b w \,.\, \Phi^+.$$

We conclude that $I \subset s_b w \,.\, \Phi^+$ and $s_b \,.\, (w\mathfrak{C}) \subset \mathfrak{A}_I$.

(2) Again it suffices to prove that any chamber of $\mathcal{B}(\boldsymbol{G})$ is equivalent under $\boldsymbol{P}_I(k)$ to a chamber of $\mathfrak{A}_I$. Let $\mathfrak{C}'$ be a chamber of $\mathcal{B}(\boldsymbol{G})$. Let $\boldsymbol{P}'$ be the underlying minimal k-parabolic subgroup. By [Borel and Tits 1965, Proposition 4.4.b], $\boldsymbol{P}_I \cap \boldsymbol{P}'$ contains a maximal k-split torus of $\boldsymbol{P}_I$. Since maximal k-split tori of $\boldsymbol{P}_I$ are conjugate under $\boldsymbol{U}_I(k)$, it follows that there exists $u \in \boldsymbol{U}_I(k)$ such that $u\boldsymbol{S}u^{-1} \subset \boldsymbol{P}_I \cap \boldsymbol{P}'$; hence $\boldsymbol{S} \subset u^{-1}\boldsymbol{P}'u$. So we can assume that $\boldsymbol{S} \subset \boldsymbol{P}'$, that is, that $\mathfrak{C}' \subset \mathfrak{A}$. Then $\mathfrak{C}' = w\mathfrak{C}$ for some $w \in \boldsymbol{W}(k)$. By (1), $\mathfrak{C}'$ is then equivalent under $\boldsymbol{W}_I(k)$ to a chamber of $\mathfrak{A}_I$. Since $\boldsymbol{N}_{\boldsymbol{L}_I}(\boldsymbol{S})(k)$ maps onto $\boldsymbol{W}_I(k)$, we conclude that $\mathfrak{C}'$ is then equivalent under $\boldsymbol{P}_I(k)$ to a chamber of $\mathfrak{A}_I$. □

We come now to the following important step in Soulé's proof.

Lemma 2.8. *Let $x \in \boldsymbol{S}_* \cap \mathfrak{Q}$. Then $\Gamma_x \,.\, (\mathscr{L}_x \cap \mathfrak{Q}) = \mathscr{L}_x$.*

Proof. We will make use of the canonical smooth model $\mathfrak{P}_x/\mathbb{O}$ of the parahoric subgroup associated to x [Bruhat and Tits 1984, section 5.2]. As an $\mathbb{O}$-group scheme, $\mathfrak{P}_x$ is isomorphic to $\boldsymbol{G} \times_k \mathbb{O}$, and we have an identification $\mathfrak{P}_x(\mathbb{O}) = P_x$. The star $\mathscr{L}_x$ is the spherical building of $\mathfrak{P}_x \times_{\mathbb{O}} k \cong \boldsymbol{G}$; see [Bruhat and Tits 1984, section 5.1.32]. Set for convenience $I = I_x$. By Lemma 2.6, $\mathscr{L}_x \cap \mathfrak{Q}$ is identified with $\mathfrak{A}_I$ in the spherical building $\mathcal{B}(\boldsymbol{G})$. Furthermore, the chamber $x + \mathfrak{C}$ identifies with $\mathfrak{C}$.

The inclusion $\Gamma_x \,.\, (\mathscr{L}_x \cap \mathfrak{Q}) \subset \mathscr{L}_x$ is clear. Let us prove the reverse inclusion. By definition, there exists $\lambda \in \boldsymbol{S}_* \cap \mathfrak{Q}$ such that $x = \lambda$. Define $g_\lambda = \lambda(1/t)^{-1} = \lambda(t) \in \boldsymbol{S}(K)$. Since $x = g_\lambda \,.\, \phi$ by (2-8) above, we have

$$P_x = g_\lambda P_\phi g_\lambda^{-1}. \tag{2-9}$$

Thus $\mathfrak{P}_x(\mathbb{O}) \cong P_x = g_\lambda \boldsymbol{G}(\mathbb{O}) g_\lambda^{-1} \subset \boldsymbol{G}(K)$. In view of Lemma 2.7(2), it will suffice to establish the following.

Claim 2.9. *The image of the composite map*

$$\Gamma_x \subset P_x \longrightarrow (\mathfrak{P}_x \times_{\mathbb{O}} k)(k) \cong \boldsymbol{G}(k)$$

contains $\boldsymbol{P}_I(k)$.

The group $\boldsymbol{L}_I(k)$ commutes with g_λ inside $\boldsymbol{G}(k(t))$, and it is therefore included in the image in question (as we have already observed in Proposition 2.5). So it

is enough to check that $g_\lambda \boldsymbol{U}(k) g_\lambda^{-1} \subset \Gamma_x$, or equivalently that $g_\lambda \boldsymbol{U}(k) g_\lambda^{-1} \subset \Gamma$. This can be verified by working over the field $\tilde{k}$ and checking the inclusion for the subgroups $\boldsymbol{U}_b(\tilde{k})$ of $\boldsymbol{U}(\tilde{k})$ for $b \in \Phi^+$. To verify this, we use that the product map induces a decomposition (with the notation of page 395)

$$\prod_{\substack{\tilde{a}\in\widetilde{\Phi}^b,\\ \tilde{a}|_{S\times_k\tilde{k}}=b}} \widetilde{\boldsymbol{U}}_{\tilde{a}}(\tilde{k}) \,.\, \prod_{\substack{\tilde{a}\in\widetilde{\Phi}^b,\\ \tilde{a}|_{S\times_k\tilde{k}}=2b}} \widetilde{\boldsymbol{U}}_{\tilde{a}}(\tilde{k}) \xrightarrow{\sim} \boldsymbol{U}_b(\tilde{k}).$$

For $\tilde{a} \in \widetilde{\Phi}^b$ and $s \in \tilde{k}$, we have

$$g_\lambda \, \boldsymbol{U}_{\tilde{a}}(s) \, g_\lambda^{-1} = \begin{cases} \widetilde{\boldsymbol{U}}_{\tilde{a}}(t^{\langle b,\lambda\rangle} s) & \text{if } \tilde{a}|_{S\times_k\tilde{k}} = b, \\ \widetilde{\boldsymbol{U}}_{\tilde{a}}(t^{2\,\langle b,\lambda\rangle} s) & \text{if } \tilde{a}|_{S\times_k\tilde{k}} = 2b. \end{cases}$$

Hence $g_\lambda \, \boldsymbol{U}_{\tilde{a}}(s) \, g_\lambda^{-1} \subset \widetilde{\Gamma}$. This establishes Claim 2.9. The proof of Lemma 2.8 is now complete. □

End of the proof of Theorem 2.1.

Two distinct points of $\mathfrak{Q}$ are not equivalent under Γ. Since two different points of $\widetilde{\mathfrak{Q}}$ are not equivalent under $\widetilde{\Gamma}$ [Soulé 1979, 1.3], it follows that two distinct points in $\mathfrak{Q}$ are not equivalent under Γ.

A point of $\mathcal{T}$ of type 0 is equivalent to a point of $\mathfrak{Q}$. We denote by $M \subset \boldsymbol{S}(K) = \boldsymbol{S}_* \otimes K^\times$ the subgroup generated by the $\lambda(t)$ for λ running over $\boldsymbol{S}_*$. We denote by $M_+ \subset M$ the semigroup generated by the $\lambda(t)$ for λ satisfying $\langle b, \lambda\rangle \geq 0$ for all $b \in \Delta$. By a result of Raghunathan [1994, Theorem 3.4],[4] we have the decomposition

$$\boldsymbol{G}(K) = \Gamma \,.\, M \,.\, \boldsymbol{G}(\mathcal{O}).$$

Again, since $\boldsymbol{N}_{\boldsymbol{G}}(\boldsymbol{S})(k)$ maps onto $\boldsymbol{W}(k)$ and $\boldsymbol{W}(k).M_+ = M$, we have actually a decomposition

$$\boldsymbol{G}(K) = \Gamma \,.\, M_+ \,.\, \boldsymbol{G}(\mathcal{O}).$$

Since $\boldsymbol{G}(K)/\boldsymbol{G}(\mathcal{O})$ is the set of points of type 0 of $\mathcal{T}$, this shows that every such point of $\mathcal{T}$ is Γ-conjugated to a point of $M \,.\, \phi$. But $M_+ \,.\, \phi \subset \mathfrak{Q}$, so we conclude that every such point of $\mathcal{T}$ is Γ-conjugated to a point of $\mathfrak{Q}$.

Every point of $\mathcal{T}$ is equivalent to a point of $\mathfrak{Q}$. Let y be a point of $\mathcal{T}$. Let F be a chamber of $\mathcal{T}$ containing y. Then $\overline{F}$ contains a (unique) point x whose type is that of ϕ. By the preceding step, we can assume that $x \in \mathfrak{Q}$. Then y belongs to $\mathcal{L}_x$ and Lemma 2.8 shows that y is equivalent under Γ to a point of $\mathfrak{Q}$.

From the above it follows that $\mathcal{T} = \Gamma \,.\, \mathfrak{Q}$, as stated in Theorem 2.1. □

[4] This reference presupposes that the base field k is infinite, but this assumption is not necessary; see [Gille 1994, III.3.4.2] for details.

3. Applications

We give two applications of Theorem 2.1. The notation and assumptions are as in the previous section. We begin by recalling some basic facts about direct limits of groups.

Direct limits of groups. Direct limits of groups occur in geometric group theory [Serre 1977]. In what follows we will repeatedly encounter the following situation: We are given a family of subgroups $(H_\lambda)_{\lambda\in\Lambda}$ of a group H (indexed by some set Λ) and we wish to consider the group that is the direct limit of the groups $(H_\lambda, H_\lambda\cap H_\mu)_{\lambda,\mu\in\Lambda}$ where the only transition maps are the inclusions $H_\lambda\cap H_\mu\subset H_\lambda$ and $H_\lambda\cap H_\mu\subset H_\mu$. We call the resulting group *the direct limit of the family* $(H_\lambda)_{\lambda\in\Lambda}$ *with respect to their intersections.*[5]

Let T be an abstract simplicial complex, E the set of its vertices, and Φ the set of its simplexes. Denote by X the geometric realization of T. Let H be a group that acts in a simplicial way on T, and for which there exists a simplicial fundamental domain T'. Recall that T' is a subcomplex of T such that if E' (respectively Φ') denotes the set of vertices (respectively simplexes) of T', then for every $s\in\Phi$, there exists a unique $s'\in\Phi'$ such that $s\in H\,.\,s'$.

The isotropy subgroup of H corresponding to an element z (respectively a subset M) of either T or X will be denoted by H_z (respectively H_M).

Theorem 3.1 [Soulé 1973]. *Let T, X, H, T' be as above. Assume that X is connected and simply connected and that the geometric realization X' of T' is connected. Then the group H is the direct limit of the family of isotropy subgroups $(H_M)_{M\in E'}$ with respect to their intersections.*

Chebotarëv [1982] has established higher-dimensional generalizations of this result. As pointed out by one of the referees, when X has additional structures there are other presentations, which are useful in practice.

Proposition 3.2. *Under the hypothesis of Theorem 3.1, assume that X is equipped with a distance d such that*

(i) *for any two points x and y, there is a unique geodesic linking x and y;*

(ii) *for any $x\in X$, there is an open neighborhood D_x of x such that $D_x\cap F\neq\varnothing$ implies $x\in\overline{F}$ for any simplex F of X;*

(iii) *H acts isometrically on X.*

Furthermore, we assume that

[5] Another terminology, which is a slight abuse of language, is that H is the sum of the H_M amalgamated over their intersections [Serre 1977, II.1.7].

(iv) *for each simplex F of X, the stabilizer of F (as a set) coincides with the isotropy group (pointwise stabilizer) of $\overline{F}$.*

Then

(1) *The group H is the direct limit of the family $(H_M \cap H_N)_{M,N\in E'}$ with transition maps $H_M \cap H_N \to H_M$ and $H_M \cap H_N \to H_N$ for M, N belonging to an edge of X'.*[6]

(2) *The group H is the direct limit of the family of isotropy subgroups $(H_x)_{x\in X'}$ with respect to their intersections.*

Note that when X is a tree, the first statement of the proposition allows us to recover a classical result [Serre 1977, section 4.5, théorème 10].

Remark 3.3. Note that the first statement of the proposition is different than that of Theorem 3.1. The point is that two vertices of X' do not necessarily belong to a common edge. In other words, the presentation of H given by Proposition 3.2(1) has fewer relations than the one given by Theorem 3.1.

Proof. We prove both statements at the same time. We denote by $H^\dagger$ the first limit and by $H^\sharp$ the second one. We have an obvious surjective map $H^\dagger \to H$, while the inclusion $E' \subset X$ gives rise to a map $H \to H^\sharp$. We denote by $\xi : H^\dagger \to H \to H^\sharp$ the composition of these two maps. It is enough then to show that $H \to H^\sharp$ is surjective, and to produce a section $\theta : H^\sharp \to H^\dagger$ of ξ.

If $x \in X$, we denote by $F_x \subset X$ the (open) simplex attached to x. Since every F_x contains in its closure a vertex M, our hypothesis on stabilizers implies that $H_x \subset H_M$. It follows that $H \to H^\sharp$ is surjective.

To define the splitting $\theta : H^\sharp \to H^\dagger$, we proceed as follows. We are given $x \in X$, and $M \in E'$ such that $M \in \overline{F}_x$. Since the action is simplicial, we have $H_x = H_{F_x}$. By our hypothesis on the stabilizers, we have then the inclusion $H_x \subset H_M \subset H$.

Step 1: The composite map $\theta_{x,M} : H_x \to H_M \to H^\dagger$ does not depend of the choice of M. We note that two distinct choices M and N of vertices of $\overline{F}_x$ define an edge of X', so that the maps $H_x \to H_M \to H^\dagger$ and $H_x \to H_M \to H^\dagger$ agree since they agree on $H_M \cap H_N$. This establishes this step and defines a map $\theta_x : H_x \to H^\sharp$.

Step 2: If $y \in \overline{F}_x$, then θ_x and θ_y agree on the subgroup H_x of H_y. Since $\overline{F}_y \subset \overline{F}_x$, we can pick a vertex $M \in \overline{F}_y$. By definition $\theta_{x,M}$ and $\theta_{y,M}$ agree on H_y. Hence θ_x and θ_y agree on H_y by the first step.

Step 3: Connectedness argument. We are given x, $y \in X$ and we want to show that θ_x and θ_y agree on $H_x \cap H_y$. Since $H_x \cap H_y$ acts trivially on the geodesic $[x, y]$,

[6] By taking $M = N$ in E' we see that the groups H_M are part of our family. Observe that if M, N are vertices of a common edge F, then $H_N \cap H_M$ is nothing but the isotropy group of $\overline{F}$.

we have $H_x \cap H_y \subset H_z$ for all $z \in [x, y]$. We consider then the restrictions $\Theta_z : H_x \cap H_y \subset H_z \to H^\dagger$ of θ_z to $H_x \cap H_y$ for z running over $[x, y]$.

Recall that D_z is the open neighborhood of $z \in X$ given by hypothesis (ii).
Step 4: If $z \in [x, y]$, then $\Theta_z = \Theta_{z'}$ for all $z' \in D_z \cap [x, y]$. Since $z' \in F_{z'} \cap D_z$, assumption (ii) implies that $z \in \overline{F}_{z'}$. Step 2 shows that θ_z and $\theta_{z'}$ agree on $H_{z'} \subset H_z$; hence $\Theta_z = \Theta_{z'}$.

We now finish the proof of the proposition. Since the $D_z \cap [x, y]$ define an open covering of the connected space $[x, y]$, Step 3 implies that Θ_z does not depend on z. In particular θ_x and θ_y agree on $H_x \cap H_y$. By the universal property defining $H^\sharp$, we obtain a map $\theta : H^\sharp \to H^\dagger$. By construction $\theta \circ \xi = \mathrm{id}_{H^\dagger}$. □

For future use we record the following.

Lemma 3.4. *Let H be a group that is the direct limit of a family of subgroups $(H_\alpha)_{\alpha\in\Lambda}$ of H with respect to their intersections.*

(1) *Let $\Lambda' \subset \Lambda$ be a directed subset, that is, for all $\alpha, \beta \in \Lambda'$, there exists $\gamma \in \Lambda'$ such that $H_\alpha \subset H_\gamma$ and $H_\beta \subset H_\gamma$. Then the direct limit of the family $(H_\alpha)_{\alpha\in\Lambda'}$ with respect to their intersections is canonically isomorphic to the subgroup $\bigcup_{\alpha\in\Lambda'} H_\alpha$ of H.*

(2) *Let $\Lambda = \bigsqcup_{j\in J} \Lambda_j$ be a partition of Λ in directed subsets. For $j \in J$, denote by $H_j := \bigcup_{\alpha\in\Lambda_j} H_\alpha$ the subgroup of H associated to Λ_j. Then H is the direct limit of the family of subgroups $(H_j)_{j\in J}$ of H with respect to their intersections.*

Proof. (1) Note that $\bigcup_{\alpha\in\Lambda'} H_\alpha$ is a subgroup of H since Λ' is directed. For any group M we have

$$\mathrm{Hom}_{\mathrm{gr}}(H', M) = \varprojlim_{\alpha\in\Lambda'} \mathrm{Hom}_{\mathrm{gr}}(H_\alpha, M),$$

whence the statement.

(2) Denote by $\widetilde{H}$ the direct limit of the family of subgroups $(H_j)_{j\in J}$ of H with respect to their intersections. The inclusion maps $H_j \subset H$ agree over their intersections and hence give rise to a natural map $\xi : \widetilde{H} \to H$. For defining the reverse map, denote by $\alpha \mapsto j(\alpha)$ the map $\Lambda \to J$ that maps α to the unique index j such that $\alpha \in \Lambda_j$. We then get maps

$$H_\alpha \hookrightarrow H_{j(\alpha)} \to \widetilde{H} \quad \text{for } \alpha \in \Lambda.$$

Since these maps agree over their intersections, they yield a map $\eta : H \to \widetilde{H}$. Given that the images of the H_α generate H (respectively $\widetilde{H}$), we get that $\eta \circ \xi = \mathrm{id}_{\widetilde{H}}$ and $\xi \circ \eta = \mathrm{id}_H$. □

The group $G(k[t])$ as a direct limit. Theorem 3.1 yields this:

Corollary 3.5. *Let V be the set of vertices of $\mathfrak{Q}$. The group $\Gamma = \boldsymbol{G}(k[t])$ is the direct limit of the family $(\Gamma_x)_{x\in V}$ with respect to their intersections.* □

From the corollary we see that Γ is generated by the Γ_x. By Proposition 2.5(1), Γ_x consists of products of elements of $\boldsymbol{G}(k)$ and elements of $\boldsymbol{U}(k[t])$, where $\boldsymbol{U}$ stands for the unipotent radical of the minimal parabolic subgroup attached to $\boldsymbol{S}$ and Δ.

Corollary 3.6. $\boldsymbol{G}(k[t]) = \langle \boldsymbol{G}(k), \boldsymbol{U}(k[t])\rangle$. □

Another presentation of Γ is given by means of Proposition 3.2(2).

Corollary 3.7. *The group $\Gamma = \boldsymbol{G}(k[t])$ is the direct limit of the family $(\Gamma_x)_{x\in\mathfrak{Q}}$ with respect to their intersections.*

Proof. We have to check that hypotheses (i) through (iv) of Proposition 3.2 are satisfied for the action of Γ on the Bruhat–Tits building $\mathcal{T}$, which is a metric space.

(i) Any two points of $\mathcal{T}$ are linked by a unique geodesic [Bruhat and Tits 1972, section 2.5].

(ii) By [ibidem, lemme 2.5.11], for any $x \in X$ there exists an open ball D_x of center x such that for any simplex F of X, $D_x \cap F \neq \varnothing$ implies $x \in \overline{F}$.

(iii) The group $\boldsymbol{G}(K)$ acts isometrically on $\mathcal{T}$ (ibidem).

(iv) Since $\boldsymbol{G}$ is simply connected, the stabilizer of a simplex F of $\mathcal{T}$ (or facet with the terminology of Bruhat and Tits) under $\Gamma \subset \boldsymbol{G}(K)$ is also its pointwise stabilizer [Bruhat and Tits 1984, proposition 4.6.32] and also of $\overline{F}$ [Bruhat and Tits 1972, proposition 2.4.13].

The corollary now follows from Proposition 3.2. □

We shall now give a nicer presentation of Γ. Given a subset $I \subset \Delta$, define $\mathfrak{Q}_I := \{x \in \mathfrak{Q} \mid I_x = I\}$. It is a subcone of $\mathfrak{Q}$, that is, $z\mathfrak{Q}_I \subset \mathfrak{Q}_I$ for all $z > 0$. Define the subgroup $\Gamma_I = \boldsymbol{U}_I(k[t]) \rtimes \boldsymbol{L}_I(k)$.

Lemma 3.8. (1) *The $(\Gamma_x)_{x\in\mathfrak{Q}_I}$ form a directed family of subgroups of Γ.*

(2) *Γ_I is the direct limit of the Γ_x for $x \in \mathfrak{Q}_I$.*

Proof. (1) The sector $\mathfrak{Q}$ is equipped with the partial order $x \le y$ if $y - x \in \mathfrak{Q}$. By restriction, we get a partial order on $\mathfrak{Q}_I$ that is directed. Indeed, given $x, y \in \mathfrak{Q}_I$, we have $x + y \in \mathfrak{Q}_I$ and $x + y \ge x$ and $x + y \ge y$.

Let x, y be elements of $\mathfrak{Q}_I$ such that $x \le y$. Then $b(y) \ge b(x)$ for all $b \in [I]^+$; hence $m_y(b) \le m_x(b)$ for all $b \in [I]^+$. It follows that for $b \in [I]^+$ we have

$$U_{b,m_x(b)} \cdot U_{2b,m_x(2b)} \subset U_{b,m_y(b)} \cdot U_{2b,m_y(2b)}.$$

Now Proposition 2.5(2) shows that $\Gamma_x \subset \Gamma_y$. Since $\mathfrak{D}_I$ is a directed subset of $\mathfrak{D}$, we conclude that the $(\Gamma_x)_{x\in\mathfrak{D}_I}$ form a directed family of subgroups of Γ.

(2) By Lemma 3.4(1), it is enough to show that

$$\bigcup_{x\in\mathfrak{D}_I} \Gamma_x = \Gamma_I. \tag{3-1}$$

Proposition 2.5(1) shows that the inclusion $\subset$ holds. Conversely, suppose we are given an element $g \in \Gamma_I$. Let $x \in \mathfrak{D}_I$. By Proposition 2.5(3) there is a real number $z \geq 1$ such that $g \in \Gamma_{zx}$. Since $zx \in \mathfrak{D}_I$, g belongs to the left side of (3-1). □

Theorem 3.9. *The group $\Gamma = \boldsymbol{G}(k[t])$ is the direct limit of the family of subgroups $(\Gamma_I)_{I\subset\Delta}$ with respect to their intersections*

Proof. Lemma 3.8(2) shows that Γ_I is the limit of the directed family of subgroups $(\Gamma_x)_{x\in\mathfrak{D}_I}$. To finish the proof we apply Lemma 3.4(2) to the decomposition $\mathfrak{D} = \bigsqcup_{I\subset\Delta} \mathfrak{D}_I$ of $\mathfrak{D}$ into directed subsets. □

Application to Whitehead groups. Let $\boldsymbol{G}(k)^+$ be the (normal) subgroup of $\boldsymbol{G}(k)$ generated by the $(R_u\boldsymbol{P})(k)$ for $\boldsymbol{P}$ running over all parabolic k-subgroups of $\boldsymbol{G}$. If $\mathrm{card}(k) \geq 4$, Tits [1964] has shown that every proper normal subgroup of $\boldsymbol{G}(k)^+$ is central. The quotient $W(k, \boldsymbol{G}) = \boldsymbol{G}(k)/\boldsymbol{G}(k)^+$ is the Whitehead group of $\boldsymbol{G}$ by [Tits 1978]. By Tits's result this group detects whether $\boldsymbol{G}(k)$ is projectively simple.

It turns out that the Whitehead group admits another characterization. Denote by $H\boldsymbol{G}(k)$ the (normal) subgroup of $\boldsymbol{G}(k)$ composed of elements $g \in \boldsymbol{G}(k)$ for which there exists an element $h \in \Gamma = \boldsymbol{G}(k[t])$ such that $h(0) = e$ and $h(1) = g$. We denote by $\pi_0(k, \boldsymbol{G}) = \boldsymbol{G}(k)/H\boldsymbol{G}(k)$ this naive group of connected components of $\boldsymbol{G}$.

Theorem 3.10. *There is a canonical isomorphism $W(k, \boldsymbol{G}) \xrightarrow{\sim} \pi_0(k, \boldsymbol{G})$.*

Proof. The unipotent radical $\boldsymbol{V}$ of a k-parabolic subgroup $\boldsymbol{Q}$ of $\boldsymbol{G}$ is a split unipotent group, so it satisfies $H(\boldsymbol{V})(k) = \boldsymbol{V}(k)$. Hence we have $\boldsymbol{G}(k)^+ \subset H\boldsymbol{G}(k)$ and a surjection $\boldsymbol{G}(k)/\boldsymbol{G}(k)^+ \to \pi_0(k, \boldsymbol{G}) = \boldsymbol{G}(k)/H\boldsymbol{G}(k)$. It remains to show that $H\boldsymbol{G}(k) \subset \boldsymbol{G}(k)^+$. Let $g \in H\boldsymbol{G}(k)$, and choose $h \in \boldsymbol{G}(k[t])$ satisfying $h(0) = e$ and $h(1) = g$. According to Corollary 3.6, the element h can be written in the form

$$h = g_1u_1g_2u_2\cdots g_nu_n$$

with $g_i \in \boldsymbol{G}(k)$ and $u_i \in \boldsymbol{U}(k[t])$, where $\boldsymbol{U}$ is the unipotent radical of a minimal parabolic k-subgroup of $\boldsymbol{G}$. We can assume that $u_i(0) = e$, so the condition $h(0) = e$ reads $g_1\cdots g_n = e$. It follows that

$$h = g_1'u_1g_1'^{-1}\cdots g_n'u_ng_n'^{-1},$$

with $g'_1 = g_1$, $g'_2 = g_1 g_2$ and so on up to $g'_n = g_1 \cdots g_n = e \in \boldsymbol{G}(k)$. Hence, as desired

$$g = h(1) = g'_1 u_1(1) {g'}_1^{-1} \cdots {g'}_n u_n(1) {g'}_n^{-1} \in \boldsymbol{G}(k)^+. \qquad \square$$

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References

[Borel 1991] A. Borel, *Linear algebraic groups*, 2nd ed., Graduate Texts in Mathematics **126**, Springer, New York, 1991. MR 92d:20001 Zbl 0726.20030

[Borel and Tits 1965] A. Borel and J. Tits, "Groupes réductifs", *Inst. Hautes Études Sci. Publ. Math.* 27 (1965), 55–150. MR 34 #7527 Zbl 0145.17402

[Bruhat and Tits 1972] F. Bruhat and J. Tits, "Groupes réductifs sur un corps local", *Inst. Hautes Études Sci. Publ. Math.* 41 (1972), 5–251. MR 48 #6265 Zbl 0254.14017

[Bruhat and Tits 1984] F. Bruhat and J. Tits, "Groupes réductifs sur un corps local, II: Schémas en groupes. Existence d'une donnée radicielle valuée", *Inst. Hautes Études Sci. Publ. Math.* 60 (1984), 197–376. MR 86c:20042 Zbl 0254.14017

[Chebotarëv 1982] N. G. Chebotarëv, "The decomposition of groups acting on complexes with a fundamental domain", *Uspekhi Mat. Nauk* **37**:4 (1982), 191–192. In Russian; translated in *Russian Math Surveys* **37**:4 (1982), 103–105. MR 83k:20038

[Garrett 1997] P. Garrett, *Buildings and classical groups*, Chapman & Hall, London, 1997. MR 98k:20081 Zbl 0933.20019

[Gille 1994] P. Gille, *Torseurs sur la droite affine et R-équivalence*, thesis, Université de Paris–Sud, 1994, Available at http://www.dma.ens.fr/~gille/prepublis/doct.pdf.

[Nagao 1959] H. Nagao, "On GL(2, $K[x]$)", *J. Inst. Polytech. Osaka City Univ. Ser. A* **10** (1959), 117–121. MR 22 #5684 Zbl 0092.02504

[Raghunathan 1994] M. S. Raghunathan, "Principal bundles admitting a rational section", *Invent. Math.* **116**:1-3 (1994), 409–423. MR 95f:14093 Zbl 0807.14012

[Serre 1977] J.-P. Serre, *Arbres, amalgames,* SL_2, Astérisque **46**, Société Mathématique de France, Paris, 1977. MR 57 #16426 Zbl 0369.20013

[SGA3 1962/1964] M. Demazure and A. Grothendieck (editors), *Schémas en groupes, III: Structure des schémas en groupes réductifs*, Lecture Notes in Mathematics **153**, Springer, Berlin, 1962/1964. MR 43 #223c

[Soulé 1973] C. Soulé, "Groupes opérant sur un complexe simplicial avec domaine fondamental", *C. R. Acad. Sci. Paris Sér. A-B* **276** (1973), 607–609. MR 47 #5854

[Soulé 1979] C. Soulé, "Chevalley groups over polynomial rings", pp. 359–367 in *Homological group theory* (Durham, 1977), edited by C. T. C. Wall, London Math. Soc. Lecture Note Ser. **36**, Cambridge Univ. Press, 1979. MR 81g:20080 Zbl 0437.20036

[Springer 1979] T. A. Springer, "Reductive groups", pp. 3–27 in *Automorphic forms, representations and L-functions, I* (Corvallis, OR, 1977), edited by A. Borel and W. Casselman, Proc. Sympos. Pure Math. **33**, Amer. Math. Soc., Providence, R.I., 1979. MR 80h:20062 Zbl 0416.20034

[Tits 1964] J. Tits, "Algebraic and abstract simple groups", *Ann. of Math.* (2) **80** (1964), 313–329. MR 29 #2259 Zbl 0131.26501

[Tits 1974] J. Tits, *Buildings of spherical type and finite BN-pairs*, Lecture Notes in Mathematics **386**, Springer, Berlin, 1974. MR 57 #9866 Zbl 0295.20047

[Tits 1978] J. Tits, "Groupes de Whitehead de groupes algébriques simples sur un corps (d'après V. P. Platonov et al.)", pp. 218–236 in *Séminaire Bourbaki, 29e année* (*1976/77*), Lecture Notes in Math. **677**, Springer, Berlin, 1978. MR 80d:12008

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On some crystalline representations of $\mathrm{GL}_2(\mathbb{Q}_p)$

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We show that the universal unitary completion of certain locally algebraic representation of $G := \mathrm{GL}_2(\mathbb{Q}_p)$ with $p > 2$ is nonzero, topologically irreducible, admissible and corresponds to a 2-dimensional crystalline representation with nonsemisimple Frobenius via the p-adic Langlands correspondence for G.

1. Introduction

Let $G := \mathrm{GL}_2(\mathbb{Q}_p)$ and B be the subgroup of upper-triangular matrices in G. Let L be a finite extension of $\mathbb{Q}_p$.

Theorem 1.1. *Assume that $p > 2$, let $k \geq 2$ be an integer, and let $\chi : \mathbb{Q}_p^\times \to L^\times$ be a smooth character with $\chi(p)^2 p^{k-1} \in \mathfrak{o}_L^\times$. Assume there exists a G-invariant norm $\|\cdot\|$ on $(\mathrm{Ind}_B^G \chi \otimes \chi|\cdot|^{-1}) \otimes \mathrm{Sym}^{k-2} L^2$. Then the completion E is a topologically irreducible, admissible Banach space representation of G. If we let E^0 be the unit ball in E, then*

$$V_{k,2\chi(p)^{-1}} \otimes (\chi|\chi|) \cong L \otimes_{\mathfrak{o}_L} \varprojlim \mathbf{V}(E^0/\varpi_L^n E^0),$$

where $\mathbf{V}$ is Colmez's Montreal functor and $V_{k,2\chi(p)^{-1}}$ is a 2-dimensional irreducible crystalline representation of $\mathcal{G}_{\mathbb{Q}_p}$, the absolute Galois group of $\mathbb{Q}_p$, with Hodge–Tate weights $(0, k-1)$ and the trace of crystalline Frobenius equal to $2\chi(p)^{-1}$.

As we explain in Section 5, the existence of such G-invariant norm follows from [Colmez 2008]. Our result addresses [Berger and Breuil 2007, remarque 5.3.5]. In other words, the completion E fits into the p-adic Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$.

The idea is to approximate $(\mathrm{Ind}_B^G \chi \otimes \chi|\cdot|^{-1}) \otimes \mathrm{Sym}^{k-2} L^2$ with representations $(\mathrm{Ind}_B^G \chi\delta_x \otimes \chi\delta_{x^{-1}}|\cdot|^{-1}) \otimes \mathrm{Sym}^{k-2} L^2$, where $\delta_x : \mathbb{Q}_p^\times \to L^\times$ is an unramified character with $\delta_x(p) = x \in 1 + \mathfrak{p}_L$. If $x^2 \neq 1$, then $\chi\delta_x \neq \chi\delta_{x^{-1}}$ and the analogue of Theorem 1.1 is a result of Berger and Breuil [2007]. This allows to deduce admissibility. This approximation process relies on the results of [Vignéras 2008].

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Using Colmez's functor **V**, we may then transfer the question of irreducibility to the Galois side. Here, we use the fact that for $p > 2$ the representation $V_{k,\pm 2p^{(k-1)/2}}$ sits in the p-adic family studied by Berger, Li and Zhu [2004].

2. Notation

We fix an algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$. We let val be the valuation on $\overline{\mathbb{Q}}_p$ such that $\mathrm{val}(p) = 1$, and we set $|x| := p^{-\mathrm{val}(x)}$. Let L be a finite extension of $\mathbb{Q}_p$ contained in $\overline{\mathbb{Q}}_p$, let $\mathfrak{o}_L$ be the ring of integers of L, let ϖ_L be a uniformizer, and let $\mathfrak{p}_L$ be the maximal ideal of $\mathfrak{o}_L$. Given a character $\chi : \mathbb{Q}_p^\times \to L^\times$, we consider χ as a character of the absolute Galois group $\mathcal{G}_{\mathbb{Q}_p}$ of $\mathbb{Q}_p$ via the local class field theory by sending the geometric Frobenius to p.

Let $G := \mathrm{GL}_2(\mathbb{Q}_p)$, and let B be the subgroup of upper-triangular matrices. Given two characters $\chi_1, \chi_2 : \mathbb{Q}_p^\times \to L^\times$, we consider $\chi_1 \otimes \chi_2$ as a character of B sending a matrix $\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)$ to $\chi_1(a)\chi_2(d)$. Let Z be the centre of G. Define

$$K := \mathrm{GL}_2(\mathbb{Z}_p), \qquad K_m := \begin{pmatrix} 1+p^m\mathbb{Z}_p & p^m\mathbb{Z}_p \\ p^m\mathbb{Z}_p & 1+p^m\mathbb{Z}_p \end{pmatrix} \quad \text{for } m \geq 1,$$

$$I := \begin{pmatrix} \mathbb{Z}_p^\times & \mathbb{Z}_p \\ p\mathbb{Z}_p & \mathbb{Z}_p^\times \end{pmatrix}, \quad I_m := \begin{pmatrix} 1+p^m\mathbb{Z}_p & p^{m-1}\mathbb{Z}_p \\ p^m\mathbb{Z}_p & 1+p^m\mathbb{Z}_p \end{pmatrix} \quad \text{for } m \geq 1.$$

Let $\mathfrak{K}_0$ be the G-normalizer of K, so that $\mathfrak{K}_0 = KZ$, and let $\mathfrak{K}_1$ be the G-normalizer of I, so that $\mathfrak{K}_1$ is generated as a group by I and $\Pi := \left(\begin{smallmatrix} 0 & 1 \\ p & 0 \end{smallmatrix}\right)$. We note that if $m \geq 1$, then K_m is normal in $\mathfrak{K}_0$ and I_m is normal in $\mathfrak{K}_1$. We denote $s := \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.

3. Diagrams

Let R be a commutative ring, (typically $R = L$, $\mathfrak{o}_L$ or $\mathfrak{o}_L/\mathfrak{p}_L^n$). By a diagram D of R-modules, we mean the data (D_0, D_1, r), where D_0 is an $R[\mathfrak{K}_0]$-module, D_1 is an $R[\mathfrak{K}_1]$-module and $r : D_1 \to D_0$ is a $\mathfrak{K}_0 \cap \mathfrak{K}_1 = IZ$-equivariant homomorphism of R-modules. A morphism α between two diagrams D and D' is given by (α_0, α_1), where $\alpha_0 : D_0 \to D_0'$ is a morphism of $R[\mathfrak{K}_0]$-modules, $\alpha_1 : D_1 \to D_1'$ is a morphism of $R[\mathfrak{K}_1]$-modules, and the diagram

$$\begin{array}{ccc} D_0 & \xrightarrow{\alpha_0} & D_0' \\ {\scriptstyle r}\uparrow & & \uparrow{\scriptstyle r'} \\ D_1 & \xrightarrow{\alpha_1} & D_1' \end{array} \tag{1}$$

commutes in the category of $R[IZ]$-modules. The condition (1) is important, since one can have two diagrams of R-modules D and D', such that $D_0 \cong D_0'$ as $R[\mathfrak{K}_0]$-modules and $D_1 \cong D_1'$ as $R[\mathfrak{K}_1]$-modules, but $D \ncong D'$ as diagrams. The diagrams

of R-modules with the above morphisms form an abelian category. To a diagram D one may associate a complex

$$\text{c-Ind}_{\mathfrak{K}_1}^{G} D_1 \otimes \delta \xrightarrow{\partial} \text{c-Ind}_{\mathfrak{K}_0}^{G} D_0 \tag{2}$$

of G-representations, where $\delta : \mathfrak{K}_1 \to R^\times$ is the character $\delta(g) := (-1)^{\mathrm{val}(\det g)}$; $\text{c-Ind}_{\mathfrak{K}_i}^{G} D_i$ denotes the space of functions $f : G \to D_i$ such that $f(kg) = kf(g)$ for $k \in \mathfrak{K}_i$ and $g \in G$, and f is supported only on finitely many cosets $\mathfrak{K}_i g$. To describe ∂, we note that Frobenius reciprocity gives

$$\mathrm{Hom}_G(\text{c-Ind}_{\mathfrak{K}_1}^{G} D_1 \otimes \delta, \text{c-Ind}_{\mathfrak{K}_0}^{G} D_0) \cong \mathrm{Hom}_{\mathfrak{K}_1}(D_1 \otimes \delta, \text{c-Ind}_{\mathfrak{K}_0}^{G} D_0);$$

now $\mathrm{Ind}_{IZ}^{\mathfrak{K}_1} D_0$ is a direct summand of the restriction of $\text{c-Ind}_{\mathfrak{K}_0}^{G} D_0$ to $\mathfrak{K}_1$, and

$$\mathrm{Hom}_{\mathfrak{K}_1}(D_1 \otimes \delta, \mathrm{Ind}_{IZ}^{\mathfrak{K}_1} D_0) \cong \mathrm{Hom}_{IZ}(D_1, D_0),$$

since δ is trivial on IZ. Composition of the maps above yields a map

$$\mathrm{Hom}_{IZ}(D_1, D_0) \to \mathrm{Hom}_G(\text{c-Ind}_{\mathfrak{K}_1}^{G} D_1 \otimes \delta, \text{c-Ind}_{\mathfrak{K}_0}^{G} D_0).$$

We let ∂ be the image of r. We define $H_0(D)$ to be the cokernel of ∂ and $H_1(D)$ to be the kernel of ∂. So we have this exact sequence of G-representations:

$$0 \to H_1(D) \to \text{c-Ind}_{\mathfrak{K}_1}^{G} D_1 \otimes \delta \xrightarrow{\partial} \text{c-Ind}_{\mathfrak{K}_0}^{G} D_0 \to H_0(D) \to 0 \tag{3}$$

Further, if r is injective then one may show that $H_1(D) = 0$; see [Vignéras 2008, Proposition 0.1]. To a diagram D one may associate a G-equivariant coefficient system $\mathscr{V}$ of R-modules on the Bruhat–Tits tree; see [Paškūnas 2004, Section 5]. Then $H_0(D)$ and $H_1(D)$ compute the homology of the coefficient system $\mathscr{V}$, and the map ∂ has a natural interpretation. Assume that $R = L$ (or any field of characteristic 0), and let π be a smooth irreducible representation of G on an L-vector space, so that for all $v \in \pi$ the subgroup $\{g \in G : gv = v\}$ is open in G. Since the action of G is smooth, there exists an $m \geq 0$ such that $\pi^{I_m} \neq 0$. To π we may associate a diagram $D := (\pi^{I_m} \hookrightarrow \pi^{K_m})$. As a very special case of a result by Schneider and Stuhler [1997, Theorem V.1; 1993, Section 3], we obtain that $H_0(D) \cong \pi$.

We are going to compute such diagrams D, attached to smooth principal series representations of G on L-vector spaces. Given smooth characters $\theta_1, \theta_2 : \mathbb{Z}_p^\times \to L^\times$ and $\lambda_1, \lambda_2 \in L^\times$, we define a diagram $D(\lambda_1, \lambda_2, \theta_1, \theta_2)$ as follows. Let $c \geq 1$ be an integer such that θ_1 and θ_2 are trivial on $1 + p^c\mathbb{Z}_p$. Set $J_c := (K \cap B)K_c = (I \cap B)K_c$, so that J_c is a subgroup of I. Let $\theta : J_c \to L^\times$ be the character $\theta\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) := \theta_1(a)\theta_2(d)$. Let $D_0 := \mathrm{Ind}_{J_c}^{K} \theta$, and let $p \in Z$ act on D_0 by a scalar $\lambda_1\lambda_2$, so that D_0 is a representation of $\mathfrak{K}_0$. Set $D_1 := D_0^{I_c}$, so that D_1 is naturally a representation of IZ.

We are going to put an action of Π on D_1, so that D_1 is a representation of $\mathfrak{K}_1$. Let

$$V_1 := \{f \in D_1 : \operatorname{Supp} f \subseteq I\}, \quad V_s := \{f \in D_1 : \operatorname{Supp} f \subseteq J_c s I\}. \tag{4}$$

Since I contains K_1, we have $J_c s I = (B \cap K) s I = I s I$; hence $D_1 = V_1 \oplus V_s$. For all $f_1 \in V_1$ and $f_s \in V_s$, we define $\Pi \cdot f_1 \in V_s$ and $\Pi \cdot f_s \in V_1$ such that

$$[\Pi \cdot f_1](sg) := \lambda_1 f_1(\Pi^{-1} g \Pi), \quad [\Pi \cdot f_s](g) = \lambda_2 f_s(s \Pi g \Pi^{-1}) \quad \text{for all } g \in I. \tag{5}$$

Every $f \in D_1$ can be written uniquely as $f = f_1 + f_s$, with $f_1 \in V_1$ and $f_s \in V_s$, and we define $\Pi \cdot f := \Pi \cdot f_1 + \Pi \cdot f_s$.

Lemma 3.1. *Equation* (5) *defines an action of* $\mathfrak{K}_1$ *on* D_1. *We denote the diagram* $D_1 \hookrightarrow D_0$ *by* $D(\lambda_1, \lambda_2, \theta_1, \theta_2)$. *Let* $\pi := \operatorname{Ind}_B^G \chi_1 \otimes \chi_2$ *be a smooth principal series representation of* G, *with*

$$\chi_1(p) = \lambda_1, \quad \chi_2(p) = \lambda_2, \quad \chi_1|_{\mathbb{Z}_p^\times} = \theta_1, \quad \chi_2|_{\mathbb{Z}_p^\times} = \theta_2.$$

There exists an isomorphism of diagrams $D(\lambda_1, \lambda_2, \theta_1, \theta_2) \cong (\pi^{I_c} \hookrightarrow \pi^{K_c})$. *In particular*, *we have a* G*-equivariant isomorphism* $H_0(D(\lambda_1, \lambda_2, \theta_1, \theta_2)) \cong \pi$.

Proof. We note that $p \in Z$ acts on π by a scalar $\lambda_1 \lambda_2$. Since $G = BK$, we have $\pi|_K \cong \operatorname{Ind}_{B \cap K}^K \theta$, and so the map $f \mapsto [g \mapsto f(g)]$ induces an isomorphism $\iota_0 : \pi^{K_c} \cong \operatorname{Ind}_{J_c}^K \theta = D_0$. Let

$$\mathscr{F}_1 := \{f \in \pi : \operatorname{Supp} f \subseteq BI\} \quad \text{and} \quad \mathscr{F}_s := \{f \in \pi : \operatorname{Supp} f \subseteq BsI\}.$$

Iwasawa decomposition gives $G = BI \cup BsI$; hence $\pi = \mathscr{F}_1 \oplus \mathscr{F}_s$. If $f_1 \in \mathscr{F}_1$, then $\operatorname{Supp}(\Pi f_1) = (\operatorname{Supp} f_1)\Pi^{-1} \subseteq BI\Pi^{-1} = BsI$. Moreover,

$$\begin{aligned} [\Pi f_1](sg) = f_1(sg\Pi) &= f_1(s\Pi(\Pi^{-1} g \Pi)) \\ &= \chi_1(p) f_1(\Pi^{-1} g \Pi) \quad \text{for all } g \in I. \end{aligned} \tag{6}$$

Similarly, if $f_s \in \mathscr{F}_s$, then $\operatorname{Supp}(\Pi f_s) = (\operatorname{Supp} f_s)\Pi^{-1} \subseteq BsI\Pi^{-1} = BI$, and

$$\begin{aligned} [\Pi f_s](g) = f_1(g\Pi) &= f_1((\Pi s) s(\Pi^{-1} g \Pi)) \\ &= \chi_2(p) f_s(s(\Pi^{-1} g \Pi)) \quad \text{for all } g \in I. \end{aligned} \tag{7}$$

Now $\pi^{I_c} = \mathscr{F}_1^{I_c} \oplus \mathscr{F}_s^{I_c} \subset \pi^{K_c}$. Let ι_1 be the restriction of ι_0 to π^{I_c}. Then it is immediate that $\iota_1(\mathscr{F}_1^{I_c}) = V_1$ and $\iota_1(\mathscr{F}_s^{I_1}) = V_s$, where V_1 and V_s are as above. Moreover, if $f \in D_1$ and $\Pi \cdot f$ is given by (5), then $\Pi \cdot f = \iota_1(\Pi \iota_1^{-1}(f))$. Since $\mathfrak{K}_1$ acts on π^{I_c}, Equation (5) defines an action of $\mathfrak{K}_1$ on D_1 such that ι_1 is $\mathfrak{K}_1$-equivariant. Hence, (ι_0, ι_1) is an isomorphism of diagrams $(\pi^{I_c} \hookrightarrow \pi^{K_c}) \cong (D_1 \hookrightarrow D_0)$. □

4. The main result

Lemma 4.1. *Let U be a finite dimensional L-vector space with subspaces U_1, U_2 such that $U = U_1 \oplus U_2$. For $x \in L$ define a map $\phi_x : U \to U$ by $\phi_x(v_1+v_2) = xv_1+v_2$ for all $v_1 \in U_1$ and $v_2 \in U_2$. Let M be an $\mathfrak{o}_L$-lattice in V. Then there exists an integer $a \geq 1$ such that $\phi_x(M) = M$ for $x \in 1 + \mathfrak{p}_L^a$.*

Proof. Let N denote the image of M in U/U_2. Then N contains $(M \cap U_1) + U_2$, and both are lattices in U/U_2. Define $a \geq 1$ to be the smallest integer such that $\mathfrak{p}_L^{-a}(M \cap U_1) + U_2$ contains N. Suppose that $x \in 1 + \mathfrak{p}_F^a$ and $v \in M$. We may write $v = \lambda v_1 + v_2$, with $v_1 \in M \cap U_1$, $v_2 \in U_2$ and $\lambda \in \mathfrak{p}_L^{-a}$. Now $\phi_x(v) = v + \lambda(x-1)v_1 \in M$. Hence we get $\phi_x(M) \subseteq M$ and $\phi_{x^{-1}}(M) \subseteq M$. Applying $\phi_{x^{-1}}$ to the first inclusion gives $M \subseteq \phi_{x^{-1}}(M)$. □

We fix an integer $k \geq 2$ and set $W := \mathrm{Sym}^{k-2} L^2$, an algebraic representation of G. Let $\pi := \pi(\chi_1, \chi_2) := \mathrm{Ind}_B^G \chi_1 \otimes \chi_2$ be a smooth principal series L-representation of G. We say that $\pi \otimes W$ admits a G-invariant norm if there exists a norm $\|\cdot\|$ on $\pi \otimes W$ with respect to which $\pi \otimes W$ is a normed L-vector space such that $\|gv\| = \|v\|$ for all $v \in \pi \otimes W$ and $g \in G$.

Let $c \geq 1$ be an integer such that both χ_1 and χ_2 are trivial on $1 + p^c\mathbb{Z}_p$. Let D be the diagram $\pi^{I_c} \otimes W \hookrightarrow \pi^{K_c} \otimes W$. Since $H_0(\pi^{I_c} \hookrightarrow \pi^{K_c}) \cong \pi$, by tensoring (2) with W we obtain $H_0(D) \cong \pi \otimes W$. Assume that $\pi \otimes W$ admits a G-invariant norm $\|\cdot\|$, and set $(\pi \otimes W)^0 := \{v \in \pi \otimes W : \|v\| \leq 1\}$. Then we may define a diagram $\mathscr{D} = (\mathscr{D}_1 \hookrightarrow \mathscr{D}_0)$ of $\mathfrak{o}_L$-modules by

$$\mathscr{D} := ((\pi^{I_c} \otimes W) \cap (\pi \otimes W)^0 \hookrightarrow (\pi^{K_c} \otimes W) \cap (\pi \otimes W)^0).$$

In this case Vignéras [2008] has shown that the inclusion $\mathscr{D} \hookrightarrow D$ induces a G-equivariant injection $H_0(\mathscr{D}) \hookrightarrow H_0(D)$ such that $H_0(\mathscr{D}) \otimes_{\mathfrak{o}_L} L = H_0(D)$ and $H_1(\mathscr{D}) = 0$. Moreover, $H_0(\mathscr{D})$ does not contain an $\mathfrak{o}_L$-submodule isomorphic to L; see [Vignéras 2008, Proposition 0.1]. Since $H_0(D)$ is an L-vector space of countable dimension, this implies that $H_0(\mathscr{D})$ is a free $\mathfrak{o}_L$-module. By tensoring (2) with $\mathfrak{o}_L/\mathfrak{p}_L^n$, we obtain

$$H_0(\mathscr{D}) \otimes_{\mathfrak{o}_L} \mathfrak{o}_L/\mathfrak{p}_L^n \cong H_0(\mathscr{D} \otimes_{\mathfrak{o}_L} \mathfrak{o}_L/\mathfrak{p}_L^n). \tag{8}$$

Proposition 4.2. *Let $\pi = \pi(\chi_1, \chi_2)$ be a smooth principal series representation, assume that $\pi \otimes W$ admits a G-invariant norm, and let $\mathscr{D}$ be as above. Then there exists an integer $a \geq 1$ such that for all $x \in 1 + \mathfrak{p}_F^b$, with $b \geq a$, there exists both a finitely generated $\mathfrak{o}_L[G]$-module M in $\pi(\chi_1\delta_{x^{-1}}, \chi_2\delta_x) \otimes W$ that is free as an $\mathfrak{o}_L$-module, and a G-equivariant isomorphism*

$$M \otimes_{\mathfrak{o}_L} \mathfrak{o}_L/\mathfrak{p}_L^b \cong H_0(\mathscr{D}) \otimes_{\mathfrak{o}_L} \mathfrak{o}_L/\mathfrak{p}_L^b,$$

where $\delta_x : \mathbb{Q}_p^\times \to L^\times$ is an unramified character with $\delta_x(p) = x$.

Proof. Apply Lemma 4.1 to $U = D_1,\ U_1 = V_1 \otimes W,\ U_2 = V_s \otimes W$ and $M = \mathscr{D}_1$, where V_1 and V_s are given by (4). We get an integer $a \geq 1$ such that $\phi_x(\mathscr{D}_1) = \mathscr{D}_1$ for all $x \in 1 + \mathfrak{p}_L^a$. It is immediate that ϕ_x is IZ-equivariant. We define a new action $\star$ of Π on D_1 by $\Pi \star v := \phi_x(\Pi \phi_x^{-1}(v))$. This gives us a new diagram $D(x)$, so that $D(x)_0 = D_0$ as a representation of $\mathfrak{K}_0$, $D(x)_1 = D_1$ as a representation of IZ, the IZ-equivariant injection $D(x)_1 \hookrightarrow D(x)_0$ is equal to the IZ-equivariant injection $D_1 \hookrightarrow D_0$, but the action of Π on D_1 is given by $\star$, (here by $=$ we really mean an equality, not an isomorphism). If $f_1 \in V_1$ and $f_s \in V_s$ then

$$\Pi \star (f_1 \otimes w) = f_s' \otimes (\Pi w), \quad \Pi \star (f_s \otimes w) = f_1' \otimes (\Pi w) \quad \text{for all } w \in W,$$

where $f_s' \in V_s$, $f_1' \in V_1$ and for all $g \in I$ we have

$$f_s'(sg) = x^{-1}[\Pi \cdot f_1](sg) = x^{-1}\lambda_1 f_1(\Pi^{-1} g \Pi), \tag{9}$$

$$f_1'(g) = x[\Pi \cdot f_s](g) = x\lambda_2 f_s(s\Pi g \Pi^{-1}). \tag{10}$$

Hence, we have an isomorphism of diagrams $D(x) \cong D(x^{-1}\lambda_1, x\lambda_2, \theta_1, \theta_2)$, and so Lemma 3.1 gives $H_0(D(x)) \cong \pi(\chi_1 \delta_{x^{-1}}, \chi_2 \delta_x) \otimes W$. Now let $b \geq a$ be an integer and suppose that $x \in 1 + \mathfrak{p}_L^b$. Since $\Pi \cdot \mathscr{D}_1 = \phi_x(\mathscr{D}_1) = \phi_x^{-1}(\mathscr{D}_1) = \mathscr{D}_1$, we get

$$\Pi \star (\mathscr{D}_0 \cap D_1) = \Pi \star \mathscr{D}_1 = \phi_x(\Pi \phi_x^{-1}(\mathscr{D}_1)) = \mathscr{D}_1.$$

So if we let $\mathscr{D}(x)_0 := \mathscr{D}_0$ and $\mathscr{D}(x)_1 := \mathscr{D}(x)_0 \cap D(x)_1$, where Π acts on $\mathscr{D}(x)_1$ by $\star$, then the diagram $\mathscr{D}(x) := (\mathscr{D}(x)_1 \hookrightarrow \mathscr{D}(x)_0)$ is an integral structure in $D(x)$ in the sense of [Vignéras 2008]. The results of Vignéras cited above imply that $M := H_0(\mathscr{D}(x))$ is a finitely generated $\mathfrak{o}_L[G]$-submodule of $\pi(\chi_1 \delta_{x^{-1}}, \chi_2 \delta_x) \otimes W$, which is free as an $\mathfrak{o}_L$-module, and $M \otimes_{\mathfrak{o}_L} L \cong \pi(\chi_1 \delta_{x^{-1}}, \chi_2 \delta_x) \otimes W$. Moreover, since ϕ_x is the identity modulo $\mathfrak{p}_L^b$, we have $\Pi \star v \equiv \Pi \cdot v \pmod{\varpi_L^b \mathscr{D}_1}$ for all $v \in \mathscr{D}_1$, and so the identity map $\mathscr{D}(x)_0 \to \mathscr{D}_0$ induces an isomorphism of diagrams $\mathscr{D}(x) \otimes_{\mathfrak{o}_L} \mathfrak{o}_L/\mathfrak{p}_L^b \cong \mathscr{D} \otimes_{\mathfrak{o}_L} \mathfrak{o}_L/\mathfrak{p}_L^b$. Now (8) gives $H_0(\mathscr{D}) \otimes_{\mathfrak{o}_L} \mathfrak{o}_L/\mathfrak{p}_L^b \cong M \otimes_{\mathfrak{o}_L} \mathfrak{o}_L/\mathfrak{p}_L^b$. □

Let $k \geq 2$ be an integer and $a_p \in \mathfrak{p}_L$. Following [Breuil 2003] we define a filtered φ-module D_{k,a_p} as the following data: a 2-dimensional L-vector space D with basis $\{e_1, e_2\}$, an L-linear automorphism $\varphi : D \to D$ given by

$$\varphi(e_1) = p^{k-1} e_2 \quad \text{and} \quad \varphi(e_2) = -e_1 + a_p e_2,$$

and a decreasing filtration $(\mathrm{Fil}^i D)_{i \in \mathbb{Z}}$ by L-subspaces such that if $i \leq 0$ then $\mathrm{Fil}^i D = D$, if $1 \leq i \leq k-1$ then $\mathrm{Fil}^i D = Le_1$, and if $i \geq k$ then $\mathrm{Fil}^i D = 0$. We set $V_{k,a_p} := \mathrm{Hom}_{\varphi, \mathrm{Fil}^\cdot}(D_{k,a_p}, B_{cris})$. Then V_{k,a_p} is a 2-dimensional L-linear absolutely irreducible crystalline representation of $\mathscr{G}_{\mathbb{Q}_p} := \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ with Hodge–Tate weights 0 and $k-1$. We denote by χ_{k,a_p} the trace character of V_{k,a_p}. Since $\mathscr{G}_{\mathbb{Q}_p}$ is

compact and the action is continuous, $\mathcal{G}_{\mathbb{Q}_p}$ stabilizes some $\mathfrak{o}_L$-lattice in V_{k,a_p}, and so χ_{k,a_p} takes values in $\mathfrak{o}_L$.

Proposition 4.3. *Let m be the largest integer such that $m \le (k-2)/(p-1)$. Let $a_p, a_p' \in \mathfrak{p}_L$, and assume that $\mathrm{val}(a_p) > m$ and $\mathrm{val}(a_p') > m$. Let $n \ge em$ be an integer, where $e := e(L/\mathbb{Q}_p)$ is the ramification index. Suppose $a_p \equiv a_p' \pmod{\mathfrak{p}_L^n}$. Then $\chi_{k,a_p}(g) \equiv \chi_{k,a_p'}(g) \pmod{\mathfrak{p}_L^{n-em}}$ for all $g \in \mathcal{G}_{\mathbb{Q}_p}$.*

Proof. This a consequence of a result of Berger, Li and Zhu [Berger et al. 2004], where the authors construct $\mathcal{G}_{\mathbb{Q}_p}$-invariant lattices T_{k,a_p} in V_{k,a_p}. The assumption $a_p \equiv a_p' \pmod{\mathfrak{p}_L^n}$ implies $T_{k,a_p} \otimes_{\mathfrak{o}_L} \mathfrak{o}_L/\mathfrak{p}_L^{n-em} \cong T_{k,a_p'} \otimes_{\mathfrak{o}_L} \mathfrak{o}_L/\mathfrak{p}_L^{n-em}$; see their [Remark 4.1.2(2)]. This implies the congruences of characters. □

Let $k \ge 2$ be an integer and choose $\lambda_1, \lambda_2 \in L$ such that $\lambda_1 + \lambda_2 = a_p$ and $\lambda_1\lambda_2 = p^{k-1}$ (enlarge L if necessary). Assume $\mathrm{val}(\lambda_1) \ge \mathrm{val}(\lambda_2) > 0$. Let $\chi_1, \chi_2 : \mathbb{Q}_p^\times \to L^\times$ be unramified characters, with $\chi_1(p) = \lambda_1^{-1}$ and $\chi_2(p) = \lambda_2^{-1}$. Let M be a finitely generated $\mathfrak{o}_L[G]$-module in $\pi(\chi_1, \chi_2|\cdot|^{-1}) \otimes W$, where $W := \mathrm{Sym}^{k-2} L^2$. In the case $\lambda_1 \ne \lambda_2$, Berger and Breuil have shown that the unitary L-Banach space representation

$$E_{k,a_p} := L \otimes_{\mathfrak{o}_L} \varprojlim M/\varpi_L^n M$$

of G is nonzero, topologically irreducible, admissible in the sense of [Schneider and Teitelbaum 2002], and contains $\pi(\chi_1, \chi_2|\cdot|^{-1}) \otimes W$ as a dense G-invariant subspace [Berger and Breuil 2007, Section 5.3]. Moreover, the dual of E_{k,a_p} is isomorphic to the representation of Borel subgroup B constructed from the (φ, Γ)-module of V_{k,a_p}.

Let $\mathrm{Rep}_{\mathfrak{o}_L} G$ be the category of finite length $\mathfrak{o}_L[G]$-modules with a central character such that the action of G is smooth (that is, the stabilizer of a vector is an open subgroup of G). Let $\mathrm{Rep}_{\mathfrak{o}_L} \mathcal{G}_{\mathbb{Q}_p}$ be the category of continuous representations of $\mathcal{G}_{\mathbb{Q}_p}$ on $\mathfrak{o}_L$-modules of finite length. Colmez [2008, IV.2.14] has defined an exact covariant functor $\mathbf{V} : \mathrm{Rep}_{\mathfrak{o}_L} G \to \mathrm{Rep}_{\mathfrak{o}_L} \mathcal{G}_{\mathbb{Q}_p}$. The constructions in [Berger and Breuil 2007] and [Colmez 2008] are mutually inverse to one another. This means if we assume $\lambda_1 \ne \lambda_2$ and let M be as above, then

$$V_{k,a_p} \cong L \otimes_{\mathfrak{o}_L} \varprojlim \mathbf{V}(M/\varpi_L^n M). \tag{11}$$

That $M/\varpi_L^n M$ is an $\mathfrak{o}_L[G]$-module of finite length follows from [Berger 2005, Theorem A].

Theorem 4.4. *Assume that $p > 2$. Let $\lambda = \pm p^{(k-1)/2}$, and let $\chi : \mathbb{Q}_p^\times \to L^\times$ be a smooth character with $\chi(p) = \lambda^{-1}$. Assume there exists a G-invariant norm $\|\cdot\|$ on $\pi(\chi, \chi|\cdot|^{-1}) \otimes W$, where $W := \mathrm{Sym}^{k-2} L^2$. Let E be the completion of $\pi(\chi, \chi|\cdot|^{-1}) \otimes W$ with respect to $\|\cdot\|$. Then E is a nonzero, topologically*

irreducible, admissible Banach space representation of G. If we let E^0 be the unit ball in E, then $V_{k,2\lambda} \otimes (\chi|\chi|) \cong L \otimes_{\mathfrak{o}_L} \varprojlim \mathbf{V}(E^0/\varpi_L^n E^0)$.

Proof. Since the character $\chi|\chi|$ is integral, by twisting we may assume that χ is unramified. We denote the diagram

$$\pi(\chi, \chi|\cdot|^{-1})^{I_1} \otimes W \hookrightarrow \pi(\chi, \chi|\cdot|^{-1})^{K_1} \otimes W$$

by $D = (D_1 \hookrightarrow D_0)$. Let $\mathscr{D} = (\mathscr{D}_1 \hookrightarrow \mathscr{D}_0)$ be the diagram of $\mathfrak{o}_L$-modules with $\mathscr{D}_1 = D_1 \cap E^0$ and $\mathscr{D}_0 = D_0 \cap E^0$. Let $a \geq 1$ be the integer Proposition 4.2 gives. For each $j \geq 0$, we fix $x_j \in 1 + \mathfrak{p}_L^{a+j}$ with $x_j \neq 1$ and a finitely generated $\mathfrak{o}_L[G]$-submodule M_j in $\pi(\chi\delta_{x_j^{-1}}, \chi\delta_{x_j}|\cdot|^{-1}) \otimes W$ (which is then a free $\mathfrak{o}_L$-module) such that

$$H_0(\mathscr{D}) \otimes_{\mathfrak{o}_L} \mathfrak{o}_L/\mathfrak{p}_L^{a+j} \cong M_j \otimes_{\mathfrak{o}_L} \mathfrak{o}_L/\mathfrak{p}_L^{a+j}.$$

This is possible by Proposition 4.2. To ease the notation we set $M := H_0(\mathscr{D})$. Let $a_p(j) := \lambda x_j^{-1} + \lambda x_j$, let $a_p := 2\lambda$, and let m be the largest integer such that $m \leq (k-2)/(p-1)$. Since $p > 2$, $x_j + x_j^{-1}$ is a unit in $\mathfrak{o}_L$, we have $\mathrm{val}(a_p(j)) = \mathrm{val}(a_p) = (k-1)/2 > m$. (Here we really need $p > 2$.) Moreover, we have $a_p \equiv a_p(j) \pmod{\mathfrak{p}_L^{j+a+em}}$, where $e := e(L/\mathbb{Q}_p)$ is the ramification index. Now since $x_j \neq 1$ we get that $\lambda x_j \neq \lambda x_j^{-1}$, and hence we may apply the results of Berger and Breuil to $\pi(\chi\delta_{x_j^{-1}}, \chi\delta_{x_j}|\cdot|^{-1}) \otimes W$. By (11),

$$T_{k,a_p(j)} := \varprojlim \mathbf{V}(M_j/\varpi_L^n M_j)$$

is a $\mathscr{G}_{\mathbb{Q}_p}$-invariant lattice in $V_{k,a_p(j)}$. Since $M \otimes_{\mathfrak{o}_L} \mathfrak{o}_L/\mathfrak{p}_L^{a+j} \cong M_j \otimes_{\mathfrak{o}_L} \mathfrak{o}_L/\mathfrak{p}_L^{a+j}$ we get

$$\mathbf{V}(M/\varpi_L^{a+j}M) \cong \mathbf{V}(M_j/\varpi_L^{a+j}M_j) \cong T_{k,a_p(j)} \otimes_{\mathfrak{o}_L} \mathfrak{o}_L/\mathfrak{p}_L^{a+j}. \tag{12}$$

Set $V := L \otimes_{\mathfrak{o}_L} \varprojlim \mathbf{V}(M/\varpi_L^n M)$. Then (12) implies that V is a 2-dimensional L-vector space. Let χ_V be the trace character of V. Then it follows from (12) that $\chi_V \equiv \chi_{k,a_p(j)} \pmod{\mathfrak{p}_L^{a+j}}$. Since $a_p \equiv a_p(j) \pmod{\mathfrak{p}_L^{a+j+em}}$, Proposition 4.3 says that $\chi_{k,a_p} \equiv \chi_{k,a_p(j)} \pmod{\mathfrak{p}_L^{a+j}}$. We obtain $\chi_V \equiv \chi_{k,a_p} \pmod{\mathfrak{p}_L^{a+j}}$ for all $j \geq 0$. This gives us $\chi_V = \chi_{k,a_p}$. Since V_{k,a_p} is irreducible, the equality of characters implies $V \cong V_{k,a_p}$.

Set $\widehat{M} := \varprojlim M/\varpi_L^n M$, and $E' := \widehat{M} \otimes_{\mathfrak{o}_L} L$. Since M is a free $\mathfrak{o}_L$-module, we get an injection $M \hookrightarrow \widehat{M}$. In particular, E' contains $\pi(\chi, \chi|\cdot|^{-1}) \otimes W$ as a dense G-invariant subspace. We claim that E' is a topologically irreducible and admissible G-representation. Now Theorem 4.1.1 and Proposition 4.1.4 of [Berger et al. 2004] say that the semisimplification of $T_{k,a_p(j)} \otimes_{\mathfrak{o}_L} k_L$ is irreducible if $p+1 \nmid k-1$ and is otherwise isomorphic to

$$\begin{pmatrix} \mu_{\sqrt{-1}} & 0 \\ 0 & \mu_{-\sqrt{-1}} \end{pmatrix} \otimes \omega^{(k-1)/(p+1)},$$

where $\mu_{\pm\sqrt{-1}}$ is the unramified character sending arithmetic Frobenius to $\pm\sqrt{-1}$, and ω is the cyclotomic character. Then [Berger 2005, Theorem A] implies that if $p+1 \nmid k-1$, then $M_j \otimes_{\mathfrak{o}_L} k_L$ is an irreducible supersingular representation of G, and if $p+1 \mid k-1$, then the semisimplification of $M_j \otimes_{\mathfrak{o}_L} k_L$ is a direct sum of two irreducible principal series. The irreducibility of principal series follows from [Barthel and Livné 1994, Theorem 33], since $\sqrt{-1} \neq \pm 1$, as $p > 2$. Since $M \otimes_{\mathfrak{o}_L} k_L \cong M_j \otimes_{\mathfrak{o}_L} k_L$, we get that $M \otimes_{\mathfrak{o}_L} k_L$ is an admissible representation of G (so that for every open subgroup $\mathcal{U}$ of G, the space of $\mathcal{U}$-invariants is finite dimensional). This implies that E' is admissible.

Suppose that E_1 is a closed G-invariant subspace of E' with $E' \neq E_1$. Let $E_1^0 := E_1 \cap \widehat{M}$. We obtain a G-equivariant injection $E_1^0 \otimes_{\mathfrak{o}_L} k_L \hookrightarrow M \otimes_{\mathfrak{o}_L} k_L$. If $E_1^0 \otimes_{\mathfrak{o}_L} k_L = 0$ or $M \otimes_{\mathfrak{o}_L} k_L$, then Nakayama's lemma gives $E_1^0 = 0$ or $E_1^0 = \widehat{M}$, respectively. If $p+1 \nmid k-1$, then $M \otimes_{\mathfrak{o}_L} k_L$ is irreducible and we are done. If $p+1 \mid k-1$, then $E_1^0 \otimes_{\mathfrak{o}_L} k_L$ is an irreducible principal series, and so $\mathbf{V}(E_1^0 \otimes_{\mathfrak{o}_L} k_L)$ is one-dimensional [Colmez 2008, IV.4.17]. But then $V_1 := L \otimes_{\mathfrak{o}_L} \varprojlim \mathbf{V}(E_1^0/\varpi_L^n E_1^0)$ is a 1-dimensional subspace of V_{k,a_p} stable under the action of $\mathcal{G}_{\mathbb{Q}_p}$. Since V_{k,a_p} is irreducible we obtain a contradiction.

Since E' is a completion of $\pi(\chi, \chi|\cdot|^{-1}) \otimes W$ with respect to a finitely generated $\mathfrak{o}_L[G]$-submodule, E' is in fact the universal completion; see for example [Emerton 2005, Proposition 1.17]. In particular, we obtain a nonzero G-equivariant map of L-Banach space representations $E' \to E$, but since E' is irreducible and $\pi(\chi, \chi|\cdot|^{-1}) \otimes W$ is dense in E, this map is an isomorphism. □

Corollary 4.5. *Assume that $p > 2$, and let $\chi : \mathbb{Q}_p^\times \to L^\times$ be a smooth character such that $\chi(p)^2 p^{k-1} = 1$. Assume that there is a G-invariant norm $\|\cdot\|$ on $\pi(\chi, \chi|\cdot|^{-1}) \otimes W$, where $W := \mathrm{Sym}^{k-2} L^2$. Then every bounded G-invariant $\mathfrak{o}_L$-lattice in $\pi(\chi, \chi|\cdot|^{-1}) \otimes W$ is finitely generated as an $\mathfrak{o}_L[G]$-module.*

Proof. The existence of a G-invariant norm implies that the universal completion is nonzero. It follows from Theorem 4.4 that the universal completion is topologically irreducible and admissible. The assertion follows from the proof of [Berger and Breuil 2007, Corollary 5.3.4]. □

For the purposes of [Paškūnas 2008] we record the following corollary to the proof of Theorem 4.4.

Corollary 4.6. *Assume $p > 2$, and let $\chi : \mathbb{Q}_p^\times \to L^\times$ be a smooth character such that $\chi^2(p) p^{k-1}$ is a unit in $\mathfrak{o}_L$. Assume there exists a unitary L-Banach space representation $(E, \|\cdot\|)$ of G containing $(\mathrm{Ind}_B^G \chi \otimes \chi|\cdot|^{-1}) \otimes \mathrm{Sym}^{k-2} L^2$ as a dense G-invariant subspace and satisfying $\|E\| \subseteq |L|$. Then there exists $x \in 1 + \mathfrak{p}_L$*

with $x^2 \neq 1$ and a unitary completion E_x of $(\mathrm{Ind}_B^G \chi\delta_x \otimes \chi\delta_{x^{-1}}|\cdot|^{-1}) \otimes \mathrm{Sym}^{k-2} L^2$ *such that $E^0 \otimes_{\mathfrak{o}_L} k_L \cong E_x^0 \otimes_{\mathfrak{o}_L} k_L$, where E_x^0 is the unit ball in E_x and E^0 is the unit ball in E.*

Proof. Let $\pi := \mathrm{Ind}_B^G \chi \otimes \chi|\cdot|^{-1}$ and $M := (\pi \otimes W) \cap E^0$. Now $M \cap \varpi_L E^0 = (\pi \otimes W) \cap \varpi_L E^0 = \varpi_L M$. So $\iota : M/\varpi_L M \hookrightarrow E^0/\varpi_L E^0$ is a G-equivariant injection. We claim that ι is a surjection. Let $v \in E^0$. Since $\pi \otimes W$ is dense in E, there exists a sequence $\{v_n\}_{n\geq 1}$ in $\pi \otimes W$ such that $\lim v_n = v$. We also have $\lim\|v_n\| = \|v\|$. Since $\|E\| \subseteq |L| \cong \mathbb{Z}$, there exists an $m \geq 0$ such that $v_n \in M$ for all $n \geq m$. This implies the surjectivity of ι. So we get $M \otimes_{\mathfrak{o}_L} k_L \cong E^0 \otimes_{\mathfrak{o}_L} k_L$.

By Corollary 4.5 we may find $u_1, \ldots, u_n \in M$ that generate M as an $\mathfrak{o}_L[G]$-module. Further, $u_i = \sum_{j=1}^{m_i} v_{ij} \otimes w_{ij}$ with $v_{ij} \in \pi$ and $w_{ij} \in W$. Since π is a smooth representation of G, there exists an integer $c \geq 1$ such that v_{ij} is fixed by K_c for all $1 \leq i \leq n$ and $1 \leq j \leq m_i$. Set

$$\mathscr{D} := \big((\pi^{I_c} \otimes W) \cap M \hookrightarrow (\pi^{K_c} \otimes W) \cap M\big), \quad D := \big(\pi^{I_c} \otimes W \hookrightarrow \pi^{K_c} \otimes W\big)$$

and let M' be the image of $H_0(\mathscr{D}) \hookrightarrow H_0(D) \cong \pi \otimes W$. It follows from (3) that M' is generated by $(\pi^{K_c} \otimes W) \cap M$ as an $\mathfrak{o}_L[G]$-module. Hence, $M' \subseteq M$. By construction $(\pi^{K_c} \otimes W) \cap M$ contains $u_1, \ldots u_n$, and so $M \subseteq M'$. In particular, $H_0(\mathscr{D}) \otimes_{\mathfrak{o}_L} k_L \cong M \otimes_{\mathfrak{o}_L} k_L$. The claim follows from the proof of Theorem 4.4. □

5. Existence

Recent results of Colmez, which appeared after the first version of this note, imply the existence of a G-invariant norm on $(\mathrm{Ind}_B^G \chi \otimes \chi|\cdot|^{-1}) \otimes \mathrm{Sym}^{k-2} L^2$ for $\chi^2(p)p^{k-1} \in \mathfrak{o}_L^\times$, thus making our results unconditional. We briefly explain this.

We continue to assume that $p > 2$, that $k \geq 2$ is an integer and that $a_p = 2p^{(k-1)/2}$. The representation V_{k,a_p} of $\mathscr{G}_{\mathbb{Q}_p}$ sits in the p-adic family of Berger, Li and Zhu, [2004, 3.2.5]. Moreover, all the other points in the family correspond to the crystalline representations with distinct Frobenius eigenvalues, to which the theory of [Berger and Breuil 2007] applies. Hence [Colmez 2008, II.3.1 and IV.4.11] imply that there exists an irreducible unitary L-Banach space representation Π of $\mathrm{GL}_2(\mathbb{Q}_p)$ such that $\mathbf{V}(\Pi) \cong V_{k,a_p}$. If $p \geq 5$ or $p = 3$ and $k \not\equiv 3 \pmod 8$ and $k \not\equiv 7 \pmod 8$, the existence of such Π also follows from [Kisin 2008]. It follows from [Colmez 2008, VI.6.46] that the set of locally algebraic vectors Π^{alg} of Π is isomorphic to

$$(\mathrm{Ind}_B^G \chi \otimes \chi|\cdot|^{-1}) \otimes \mathrm{Sym}^{k-2} L^2,$$

where $\chi : \mathbb{Q}_p^\times \to L^\times$ is an unramified character with $\chi(p) = p^{-(k-1)/2}$. The restriction of the G-invariant norm of Π to Π^{alg} solves the problem. Also, if $\delta : \mathbb{Q}_p^\times \to L^\times$ is a unitary character, then we also obtain a G-invariant norm on $\Pi^{\mathrm{alg}} \otimes \delta \circ \det$.

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References

[Barthel and Livné 1994] L. Barthel and R. Livné, "Irreducible modular representations of GL_2 of a local field", *Duke Math. J.* **75**:2 (1994), 261–292. MR 95g:22030 Zbl 0826.22019

[Berger 2005] L. Berger, "Représentations modulaires de $\mathrm{GL}_2(\mathbb{Q}_p)$ et représentations galoisiennes de dimension 2", preprint, 2005. To appear in *Astérisque*. arXiv math/0510090

[Berger and Breuil 2007] L. Berger and C. Breuil, "Sur quelques représentations potentiellement cristallines de $\mathrm{GL}_2(\mathbb{Q}_p)$", preprint, 2007, Available at http://tinyurl.com/qtbvtz. To appear in *Astérisque*.

[Berger et al. 2004] L. Berger, H. Li, and H. J. Zhu, "Construction of some families of 2-dimensional crystalline representations", *Math. Ann.* **329**:2 (2004), 365–377. MR 2005k:11104 Zbl 1085.11028

[Breuil 2003] C. Breuil, "Sur quelques représentations modulaires et p-adiques de $\mathrm{GL}_2(\mathbb{Q}_p)$, II", *J. Inst. Math. Jussieu* **2**:1 (2003), 23–58. MR 2005d:11079

[Colmez 2008] P. Colmez, "Représentations de $\mathrm{GL}_2(\mathbb{Q}_p)$ et (φ, Γ)-modules", preprint, 2008, Available at http://www.math.jussieu.fr/~colmez/kirilov.pdf.

[Emerton 2005] M. Emerton, "p-adic L-functions and unitary completions of representations of p-adic reductive groups", *Duke Math. J.* **130**:2 (2005), 353–392. MR 2007e:11058 Zbl 1092.11024

[Kisin 2008] M. Kisin, "Deformations of $G_{\mathbb{Q}_p}$ and $\mathrm{GL}_2(\mathbb{Q}_p)$ representations", preprint, 2008, Available at http://www.math.uchicago.edu/~kisin/dvifiles/dense.dvi.

[Paškūnas 2004] V. Paškūnas, *Coefficient systems and supersingular representations of* $\mathrm{GL}_2(F)$, Mém. Soc. Math. Fr. (N.S.) **99**, Société Mathématique de France, Paris, 2004. MR 2005m:22017

[Paškūnas 2008] V. Paškūnas, "Admissible unitary completions of locally $\mathbb{Q}_p$-rational representations of $\mathrm{GL}_2(F)$", preprint, 2008. arXiv 0805.1006

[Schneider and Stuhler 1993] P. Schneider and U. Stuhler, "Resolutions for smooth representations of the general linear group over a local field", *J. Reine Angew. Math.* **436** (1993), 19–32. MR 94a:22039 Zbl 0780.20026

[Schneider and Stuhler 1997] P. Schneider and U. Stuhler, "Representation theory and sheaves on the Bruhat–Tits building", *Inst. Hautes Études Sci. Publ. Math.* 85 (1997), 97–191. MR 98m:22023 Zbl 0892.22012

[Schneider and Teitelbaum 2002] P. Schneider and J. Teitelbaum, "Banach space representations and Iwasawa theory", *Israel J. Math.* **127** (2002), 359–380. MR 2003c:22026 Zbl 1006.46053

[Vignéras 2008] M.-F. Vignéras, "A criterion for integral structures and coefficient systems on the tree of PGL(2, F)", *Pure Appl. Math. Q.* **4**:4, part 1 (2008), 1291–1316. MR 2009e:20059

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ALGEBRA AND NUMBER THEORY 3:4(2009)

Equivariant Hilbert series

Frank Himstedt and Peter Symonds

We consider a finite group acting on a graded module and define an equivariant degree that generalizes the usual nonequivariant degree. The value of this degree is a module for the group, up to a rational multiple. We investigate how this behaves when the module is a ring and apply our results to reprove some results of Kuhn on the cohomology of groups.

1. Introduction

We consider a finitely generated graded module M over a graded ring R that is finitely generated over some base field k and such that R_0 is finite-dimensional over k. We suppose that there is a finite group G that acts on M, preserving the grading and commuting with R.

To this data we associate a formal Laurent series $[M]$ in t in which the coefficient of t^r is the homogeneous part M_r, *considered as a kG-module*. The difficulty of the theory depends on whether we wish to keep track of these modules up to isomorphism (that is, in the Green ring) or only up to composition factors (in the representation ring). We develop both cases.

This series $[M]$ is shown to satisfy a form of the Hilbert–Serre Theorem (in particular it is a rational function, or at least a sum of them in the Green ring case). We define the equivariant degree $\deg_G M$ to be the coefficient of the leading term when we expand $[M]$ as a Laurent series in $1-t$. This is a kG-module up to rational multiple, although there is sometimes a problem of whether it is well defined in the Green ring case. The dimension of this module agrees with the usual definition of the degree in the nonequivariant case.

We investigate various properties of the equivariant degree; Theorem 6.4, in particular, lists several equivalent characterizations.

In Section 7, we go on to consider the case of the homogeneous coordinate ring on a projective variety and show that in this case the degree is always defined and it is a permutation module that can be easily described in terms of the geometry.

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Finally, in Section 8, this theory is applied to the variety associated to the cohomology of a group to reprove a result of Nick Kuhn on the action of the outer automorphism group of a p-group G on the cohomology $H^*(G; \mathbb{F}_p)$.

2. General setup

Let $R = \bigoplus_{j=0}^{\infty} R_j$ be a commutative graded algebra over a field k. We suppose that R is a finitely generated k-algebra and that R_0 is finite-dimensional over k, so all the homogeneous components R_j are also finite-dimensional vector spaces over k. Let G be a finite group and let $M = \bigoplus_{i=N}^{\infty} M_i$ a finitely generated graded left RG-module, where the action of G preserves the grading and each M_i is a finite-dimensional k-vector space.

We recall some facts about the Hilbert series $H(M,t) = \sum_{i=N}^{\infty} \dim_k(M_i)\, t^i$ of M. The graded version of Noether normalization [Benson 1993, Theorem 2.2.7] guarantees the existence of homogeneous elements $d_1, d_2, \ldots, d_n$ of positive degrees in R that generate a polynomial subring $k[d_1, \ldots, d_n]$ of R and such that R is finitely generated as a $k[d_1, \ldots, d_n]$-module. We write $|d_i| := \deg d_i$ for the degree of d_i. The number n is equal to the Krull dimension of R. By the Hilbert–Serre Theorem [Benson 1993, 2.1.1] the Hilbert series $H(M,t)$ is of the form

$$H(M,t) = \frac{f(M,t)}{\prod_{i=1}^{n}(1-t^{|d_i|})},$$

where $f(M,t)$ is a Laurent polynomial with integer coefficients. As in [Benson 1993, Section 2.4], for example, the rational number $\deg M$ is defined by the Laurent expansion of $H(M,t)$ about $t = 1$:

$$H(M,t) = \frac{\deg M}{(1-t)^n} + O\left(\frac{1}{(1-t)^{n-1}}\right). \tag{2-1}$$

Obviously the definition of the degree $\deg M$ ignores the action of G on M. In the next two sections, we shall define an equivariant analogue $\deg_G M$, which also incorporates the group action.

First, we define the degree of certain Laurent series. Let $p(t)$ be a Laurent series of the form

$$p(t) = \sum_{i=N}^{\infty} a_i t^i = \frac{g(t)}{\prod_{i=1}^{n}(1-t^{|d_i|})},$$

where the a_i are rational numbers and $g(t)$ is a Laurent polynomial with rational coefficients. We define the rational number $\deg p(t)$ to be the coefficient of $\frac{1}{(1-t)^n}$ in the Laurent expansion of $p(t)$ about $t = 1$ and we call $\deg p(t)$ the *degree* of $p(t)$. If we want to emphasize the dependency on n, we write $\deg^n p(t)$ instead of $\deg p(t)$. In particular, we have $\deg H(M,t) = \deg M$ with $\deg M$ as in (2-1).

3. Equivariant degree over the Green ring

As usual, the Green ring $a(kG)$ is defined to be the ring with generators the isomorphism classes $|V|$ of kG-modules, and relations $|V| + |W| = |V \oplus W|$, $|V| \cdot |W| = |V \otimes_k W|$. We set $a(kG)_{\mathbb{Q}} := \mathbb{Q} \otimes_{\mathbb{Z}} a(kG)$. The representation ring $\mathcal{R}(kG)$ is defined to be the quotient of $a(kG)$ by the ideal generated by the elements $|V_2| - |V_1| - |V_3|$, where $0 \to V_1 \to V_2 \to V_3 \to 0$ is a short exact sequence of kG-modules. We set $\mathcal{R}(kG)_{\mathbb{Q}} := \mathbb{Q} \otimes_{\mathbb{Z}} \mathcal{R}(kG)$.

We will consider two versions of the equivariant degree: one is an element of $a(kG)_{\mathbb{Q}}$, but is not always defined; the other is a weaker one, which is an element of $\mathcal{R}(kG)_{\mathbb{Q}}$, but it is always defined. The main tool used in the definition of the former is the following Weak Structure Theorem 3.1, so-called because it is a generalization of the Structure Theorem of [Symonds 2007].

Theorem 3.1. *For any finitely generated graded $k[d_1, \dots, d_n]G$-module M,*

$$M \cong \bigoplus_{U \in \mathrm{Indecomp}(M)} \bigoplus_{I \subseteq \{1,\dots,n\}} k[d_I] \otimes_k X_{U,I},$$

as a kG-module, where $X_{U,I}$ is a finite-dimensional graded kG-module that is a sum of U's (ignoring grading) and $k[d_I] = k[d_i \mid i \in I]$. The map from right to left is given by multiplication.

Proof. The only difference between this theorem and Proposition 4.4 of [Symonds 2007] is that there $\mathrm{Indecomp}(M)$ is supposed to be finite. But the same proof works, although it is better to keep the different indecomposables separate by using the double summation, as in the statement above, rather than combining them as $\bar{X}_I = \bigoplus_{U \in \mathrm{Indecomp}(M)} X_{U,I}$ as in [Symonds 2007]. □

Next we describe the definition of the degree with values in $a(kG)_{\mathbb{Q}}$. For each i, the kG-module M_i defines an element $|M_i|$ of $a(kG)$.

Definition 3.2. We call the Laurent series

$$[M] := \sum_{i=N}^{\infty} |M_i|\, t^i \tag{3-1}$$

with coefficients in $a(kG)$ the *equivariant Hilbert series of M with coefficients in the Green ring*.

Clearly, if $G = \{1\}$ is the trivial group, we can identify $|M_i|$ with the dimension of M_i as a k-vector space. So in this situation $[M]$ coincides with the usual Hilbert series of M. The equivariant Hilbert series has the following basic properties:

Lemma 3.3. *Suppose $M' = \bigoplus_{i=N'}^{\infty} M'_i$ is another finitely generated graded left RG-module, such that the action of G preserves the grading and every M'_i is a finite-dimensional k-vector space. Then*

$$[M \oplus M'] = [M] + [M'] \quad \textit{and} \quad [M \otimes_k M'] = [M] \cdot [M'].$$

Proof. Clear. □

Besides the Hilbert series $H(M,t)$, we can consider a Hilbert series that counts the multiplicity of some isomorphism class of indecomposable summands. Let $\mathrm{Indecomp}(M)$ be a set of representatives for the isomorphism classes of all indecomposable kG-modules which occur as a direct summand of some M_i and let $m_{U,i}$ be the multiplicity of $U \in \mathrm{Indecomp}(M)$ as a direct summand of M_i. We set $H_U(M,t) := \sum_{i=N}^{\infty} m_{U,i}\, t^i$. The Laurent series $H_U(M,t)$ can be written as a rational function too.

Proposition 3.4. *For each $U \in \mathrm{Indecomp}(M)$, the Laurent series $H_U(M,t)$ can be written as*

$$H_U(M,t) = \frac{f_U(M,t)}{\prod_{i=1}^{n}(1-t^{|d_i|})},$$

where $f_U(M,t)$ is a Laurent polynomial in t with integer coefficients.

Proof. This is a consequence of the Weak Structure Theorem 3.1. □

Let F be an arbitrary finite subset of $\mathrm{Indecomp}(M)$. We consider the Laurent series with integer coefficients $q(t) := H(M,t) - \sum_{U\in F} \dim_k(U) H_U(M,t)$. By definition of the Hilbert series, all the coefficients of $q(t)$ are nonnegative integers, and $q(t)$ is of the form

$$q(t) = \frac{g(t)}{\prod_{i=1}^{n}(1-t^{|d_i|})}$$

for some Laurent polynomial $g(t)$ with integer coefficients since something similar holds for $H(M,t)$ and $H_U(M,t)$ by Proposition 3.4. So we can take degrees and obtain

$$\deg M = \left(\sum_{U\in F} \dim_k(U) \deg H_U(M,t)\right) + \deg q(t). \tag{3-2}$$

It turns out that all the degrees occurring in (3-2) are nonnegative with bounded denominators by the following result.

Lemma 3.5. *Suppose that*

$$p(t) = \frac{h(t)}{\prod_{i=1}^{n}(1-t^{|d_i|})} = \sum_{i=N}^{\infty} a_i\, t^i,$$

where $h(t)$ is a Laurent polynomial with rational coefficients and the a_i's are nonnegative integers. Then $\deg p(t) \geq 0$. *If all the coefficients of $h(t)$ are integers then* $\deg p(t)$ *is of the form* $\deg p(t) = d \big/ \prod_{i=1}^{n} |d_i|$ *for some nonnegative integer d.*

Proof. We compute

$$\deg p(t) = \lim_{t\to 1}(1-t)^n p(t) = \lim_{t\to 1} \frac{h(t)}{\prod_{i=1}^n (1+t+\cdots+t^{|d_i|-1})} = \frac{h(1)}{\prod_{i=1}^n |d_i|}\,.$$

We still have to show that $\deg p(t) \geq 0$. Since multiplication with

$$\prod_{i=1}^n (1+t+\cdots+t^{|d_i|-1})$$

and a suitable power of t does not affect the sign of the degree or the sign of the a_i, we may assume that $p(t)$ is a Laurent *polynomial* in $1-t$ with rational coefficients, that is that

$$p(t) = \frac{b_{-n}}{(1-t)^n} + \frac{b_{1-n}}{(1-t)^{n-1}} + \cdots + b_{m-1}(1-t)^{m-1} + b_m(1-t)^m$$

for some rational numbers b_i and a nonnegative integer m. In particular, $b_{-n} = \deg p(t)$. Expanding the negative powers $(1-t)^{-j}$ as power series in t and comparing the coefficients of t^i we see that there exists a polynomial $r(i)$ in i of degree at most $n-2$ (or $r(i) = 0$ if $n = 1$) with coefficients depending on n and the b_j's such that $a_i = (1/(n-1)!)\, b_n\, i^{n-1} + r(i)$ for all large enough i. So the condition $a_i \geq 0$ implies that $\deg p(t) = b_{-n} \geq 0$. □

Corollary 3.6. *There are only finitely many $U \in \mathrm{Indecomp}(M)$ with*

$$\deg H_U(M,t) \neq 0$$

and we have

$$\sum_U \dim_k(U) \deg H_U(M,t) \leq \deg M,$$

where the sum means the sum over all $U \in \mathrm{Indecomp}(M)$ with $\deg H_U(M,t) \neq 0$.

Proof. This follows from (3-2) and Lemma 3.5. □

We can now define the equivariant degree with values in the Green ring.

Definition 3.7. We say that $\deg_{a(kG)} M$ *is defined* if

$$\sum_U \dim_k(U) \deg H_U(M,t) = \deg M.$$

In this case we call $\deg_{a(kG)} M := \sum_U \deg(H_U(M,t))\,|U| \in a(kG)_{\mathbb{Q}}$ the *equivariant degree of M* (*in the Green ring*). If we want to emphasize the dependency on n, we write $\deg^n_{a(kG)} M$ instead of $\deg_{a(kG)} M$.

The existence of the degree in the Green ring can be characterized as follows.

Lemma 3.8. *For R, G, M as above the following statements are equivalent.*

(1) $\deg_{a(kG)} M$ *is defined.*

(2) *There is a finite set* F *of indecomposable* kG*-modules such that*

$$\sum_{U\in F} \dim_k(U) \deg H_U(M,t) = \deg M.$$

(3) *There is a finite set* F *of indecomposable* kG*-modules such that*

$$\deg \sum_{U\notin F} \dim_k(U) H_U(M,t) = 0.$$

Here we have set $\sum_{U\notin F} \dim_k(U) H_U(M,t) := H(M,t) - \sum_{U\in F} \dim_k(U) H_U(M,t)$.

Proof. This is clear from the definition of $\deg_{a(kG)} M$. □

Certainly the equivariant degree $\deg_{a(kG)} M$ is defined if M has only finitely many isomorphism types of indecomposable summands. For example, this is the case if k is a finite field, M a polynomial ring in n variables over k, G a finite group acting on this polynomial ring by homogeneous linear substitutions and $R = M^G$ the ring of invariants [Karagueuzian and Symonds 2007, Theorem 17.1]. The following example shows that there are situations where $\deg_{a(kG)} M$ is not defined:

Example (see Example 4.4 in [Karagueuzian and Symonds 2004]). Let k be a field of two elements and $R = k[x, y]$ a polynomial ring in two variables over k. The Klein four group $G = \langle \alpha, \beta\rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ acts on $M = k[x, y]\langle 1, z\rangle$ by $\alpha : z \mapsto z + x$ and $\beta : z \mapsto z + y$. We can regard M as a subset of $k[x, y, z]$ or as a free R-module of rank two.

If we attach a grading to R and the module M by assigning x, y and z grading 1, then M is the direct sum $M = \bigoplus_{i=0}^{\infty} M_i$. It is shown in [Karagueuzian and Symonds 2004] that $M_i \cong \Omega^i k$ as kG-modules, where $\Omega^i k$ is the i-th Heller translate of the trivial kG-module k. In particular, the M_i's are indecomposable and pairwise nonisomorphic.

We have $n = 2$, $\mathrm{Indecomp}(M) = \{\Omega^i k \mid i \in \mathbb{N}_0\}$ and $H_{\Omega^i k}(M, t) = t^i$. So we obtain $\deg H_{\Omega^i k}(M, t) = 0$ for all i. On the other hand we have

$$H(M,t) = \sum_{i=0}^{\infty} \dim_k(M_i)\, t^i = \sum_{i=0}^{\infty} (2i+1) t^i = \frac{2}{(1-t)^2} - \frac{1}{1-t},$$

and thus $\deg M = 2$. So $\deg_{a(kG)} M$ is not defined in this example.

4. Equivariant degree in the representation ring

One way to construct an equivariant degree which is defined for every module M (satisfying the assumptions in Section 2) is to work over the representation ring. In

this section we will define the equivariant degree with values in the representation ring.

The first steps are very similar to those for the Green ring. Let R, G and M be as in Section 2. For each i, the kG-module M_i defines an element $|M_i|$ of $\mathcal{R}(kG)$.

Definition 4.1. We call the Laurent series $[M] := \sum_{i=N}^{\infty} |M_i|\, t^i$ with coefficients in $\mathcal{R}(kG)$ the *equivariant Hilbert series of M with coefficients in the representation ring*.

Clearly Lemma 3.3 carries over to equivariant Hilbert series with coefficients in the representation ring.

For each irreducible kG-module V, let $m_{V,i}$ be the multiplicity of V as a composition factor of M_i. We set $H_V(M,t) := \sum_{i=N}^{\infty} m_{V,i}\, t^i$. We choose a polynomial subring $k[d_1, \ldots, d_n]$ of R as in Section 2. In fact the Laurent series $H_V(M,t)$ can be written as a rational function.

Lemma 4.2. *For each irreducible kG-module V, the Laurent series $H_V(M,t)$ can be written as*

$$H_V(M,t) = \frac{f_V(M,t)}{\prod_{i=1}^{n}(1-t^{|d_i|})},$$

where $f_V(M,t)$ is a Laurent polynomial in t with rational coefficients. If k is a splitting field for V, then all the coefficients of $f_V(M,t)$ are integers.

Proof. Let P_V be a projective cover of V. The graded $k[d_1, \ldots, d_n]$-module

$$\mathrm{Hom}_{kG}(P_V, M)$$

is a direct summand of the graded $k[d_1, \ldots, d_n]$-module $\mathrm{Hom}_{kG}(kG, M) \cong M$. This implies that $\mathrm{Hom}_{kG}(P_V, M)$ is finitely generated as a $k[d_1, \ldots, d_n]$-module. Therefore, by the Hilbert–Serre Theorem [Benson 1993, 2.1.1], the Hilbert series $H(\mathrm{Hom}_{kG}(P_V, M), t)$ has the form

$$\frac{\tilde{f}_V(t)}{\prod_{i=1}^{n}(1-t^{|d_i|})}$$

for some Laurent polynomial $\tilde{f}_V(t)$ with integer coefficients. Since

$$\dim_k(\mathrm{Hom}_{kG}(P_V, M_i)) = \dim_k(\mathrm{End}_{kG}(V)) \cdot m_{V,i}$$

we get

$$\begin{aligned} H_V(M,t) &= \frac{1}{\dim_k(\mathrm{End}_{kG}(V))} H(\mathrm{Hom}_{kG}(P_V, M), t) \\ &= \frac{1}{\dim_k(\mathrm{End}_{kG}(V))} \cdot \frac{\tilde{f}_V(t)}{\prod_{i=1}^{n}(1-t^{|d_i|})}. \end{aligned} \tag{4-1}$$

If k is a splitting field for V then $\dim_k(\mathrm{End}_{kG}(V)) = 1$. □

Corollary 4.3. *The equivariant Hilbert series $[M]$ with coefficients in the representation ring is of the form*

$$[M] = \frac{[f](M,t)}{\prod_{i=1}^{n}(1-t^{|d_i|})},$$

where $[f](M,t)$ is a Laurent polynomial with coefficients in $\mathcal{R}(kG)_{\mathbb{Q}}$. If k is a splitting field for G then all the coefficients of $[f](M,t)$ are elements of $\mathcal{R}(kG)$.

Proof. This follows from Lemma 4.2. □

Now we can define the equivariant degree with values in the representation ring:

Definition 4.4. We call $\deg_{\mathcal{R}(kG)} M := \sum_V \deg(H_V(M,t))\,|V| \in \mathcal{R}(kG)_{\mathbb{Q}}$ the *equivariant degree of M (in the representation ring)*. Here the sum varies over a set of representatives for the isomorphism classes of irreducible kG-modules. If we want to emphasize the dependency on n, we also write $\deg^n_{\mathcal{R}(kG)} M$ instead of $\deg_{\mathcal{R}(kG)} M$.

We use the same notation for the two degrees, specifying the ring in which the values lie explicitly when necessary. In any case the two versions are compatible in the following sense. Let $\pi : a(kG)_{\mathbb{Q}} \twoheadrightarrow \mathcal{R}(kG)_{\mathbb{Q}}$ denote the canonical map.

Proposition 4.5. *The map π takes the equivariant degree of M in the Green ring to the equivariant degree of M in the representation ring whenever the former is defined.*

Proof. Suppose that $\deg_{a(kG)} M$ is defined. For each $U \in \mathrm{Indecomp}(M)$ and each irreducible kG-module V, let $\mu_{U,V}$ be the multiplicity of V as a composition factor of U and choose a finite subset F of $\mathrm{Indecomp}(M)$ as in Lemma 3.8. We set

$$\sum_{U\notin F} \dim_k(U) H_U(M,t) := H(M,t) - \sum_{U\in F} \dim_k(U) H_U(M,t),$$

$$\sum_{U\notin F} \mu_{U,V} H_U(M,t) := H_V(M,t) - \sum_{U\in F} \mu_{U,V} H_U(M,t).$$

By Lemma 3.8 we get

$$\sum_V \left(\dim_k(V) \deg \sum_{U\notin F} \mu_{U,V} H_U(M,t) \right) = \deg \sum_{U\notin F} \dim_k(U) H_U(M,t) = 0, \quad (4\text{-}2)$$

where the first sum runs over a set of representatives for the isomorphism classes of the irreducible kG-modules. By Lemma 3.5 all degrees occurring in (4-2) are non-negative. Hence $\deg\left(\sum_{U\notin F} \mu_{U,V} H_U(M,t)\right) = 0$ for all irreducible kG-modules V. The epimorphism π maps the equivariant degree

$$\deg_{a(kG)} M = \sum_{U\in F} \deg(H_U(M,t))|U| \in a(kG)_{\mathbb{Q}}$$

to

$$\begin{aligned}\sum_V \sum_{U\in F} &\deg(H_U(M,t))\mu_{U,V}|V| \\ &= \sum_V \sum_{U\in F} \deg(\mu_{U,V} H_U(M,t))\,|V| \\ &= \sum_V \Bigg(\sum_{U\in F} \deg(\mu_{U,V} H_U(M,t)) + \deg \sum_{U\notin F} \mu_{U,V} H_U(M,t) \Bigg) |V| \\ &= \sum_V \deg(H_V(M,t))\,|V|,\end{aligned}$$

which is, by definition, the equivariant degree of M over the representation ring. □

5. Basic properties of the equivariant degree

In this section we collect some of the basic properties of the equivariant degree. We always assume that R, G and M are as in Section 2 and that M' and M'' are finitely generated graded left RG-modules, where the action of G preserves the grading and every homogeneous component is finite dimensional as a k-vector space. We choose a polynomial subring $k[d_1, \ldots, d_n]$ of R as in Section 2.

We begin with a trivial observation showing that the equivariant degree coincides with the usual degree if there is "no group action":

Lemma 5.1. *If $G = \{1\}$ is the trivial group then $\deg_{a(kG)} M$ is defined and*

$$\deg_{a(kG)} M = \deg(M)\,|k|$$

where k is the trivial kG-module. A similar statement holds for $\deg_{\mathcal{R}(kG)} M$.

Proof. This is clear from the definition of the degree. □

From now on G is again an arbitrary finite group. The next lemma holds both for the equivariant degree taking values in the Green ring as well as for the degree taking values in the representation ring.

Lemma 5.2. *If $\deg_{a(kG)} M$ is defined, then there is a positive integer c such that $c\cdot\deg_{a(kG)} M$ is a genuine module, that is, it is of the form $|V|$ for some kG-module V. A similar statement holds for $\deg_{\mathcal{R}(kG)} M$.*

Proof. By the definition of $\deg_{a(kG)} M$ and Lemma 3.5, we can take $c := \prod_{i=1}^n |d_i|$ for the degree with values in the Green ring. In the case of the representation ring, $c := \big(\prod_V \dim_k \operatorname{End}_{kG}(V)\big) \cdot \big(\prod_{i=1}^n |d_i|\big)$ does the job (where V runs through a set of representatives for the isomorphism classes of irreducible kG-modules). □

Lemma 5.3. *If $M' \hookrightarrow M \twoheadrightarrow M''$ is a short exact sequence of finitely generated graded RG-modules that is split over kG, then $\deg_{a(kG)} M$ is defined if and only if both $\deg_{a(kG)} M'$ and $\deg_{a(kG)} M''$ are defined. If this is the case then*

$$\deg_{a(kG)} M = \deg_{a(kG)} M' + \deg_{a(kG)} M''.$$

The same formula with $\deg_{a(kG)}$ replaced by $\deg_{\mathcal{R}(kG)}$ holds for any short exact sequence.

Proof. Let $U \in \mathrm{Indecomp}(M)$. Then $H_U(M,t) = H_U(M',t) + H_U(M'',t)$ because of the splitting. Thus

$$\sum_{U\in F} \dim_k(U) \deg H_U(M,t) = \sum_{U\in F} \dim_k(U) \deg H_U(M',t) + \sum_{U\in F} \dim_k(U) \deg H_U(M'',t) \quad (5\text{-}1)$$

with F as in Lemma 3.8. By additivity of the nonequivariant degree we have $\deg M = \deg M' + \deg M''$. Since all these degrees are nonnegative by Lemma 3.5 we get that $\deg_{a(kG)} M$ is defined if and only if $\deg_{a(kG)} M'$ and $\deg_{a(kG)} M''$ are defined. In this case we get

$$\begin{aligned} \deg_{a(kG)} M &= \sum_{U\in F} \deg(H_U(M,t))\,|U| \\ &= \sum_{U\in F} \deg(H_U(M',t))\,|U| + \sum_{U\in F} \deg(H_U(M'',t))\,|U| \\ &= \deg_{a(kG)} M' + \deg_{a(kG)} M''. \end{aligned} \quad (5\text{-}2)$$

The statement about the degree over the representation ring follows from

$$H_V(M,t) = H_V(M',t) + H_V(M'',t)$$

for every irreducible kG-module V. □

For $W, W' \in \mathcal{R}(kG)_{\mathbb{Q}}$ we write $W \le W'$ if $W' - W$ is a linear combination of isomorphism classes of kG-modules with nonnegative rational coefficients. We write $W \ge W'$ if $W' \le W$.

Corollary 5.4. *For a finitely generated graded RG-module M, as at the beginning of this section, the following properties hold for the degree with values in the representation ring.*

(1) *If M' is a graded RG-submodule of M then $\deg_{\mathcal{R}(kG)} M' \le \deg_{\mathcal{R}(kG)} M$.*

(2) *If M' is a graded RG-epimorphic image of M then $\deg_{\mathcal{R}(kG)} M \ge \deg_{\mathcal{R}(kG)} M'$.*

Proof. This follows from Lemmas 5.2 and 5.3. □

For an integer d we write $M[d]$ for M with a degree shift of d, so that $M[d]_i = M_{i+d}$. For a positive integer q let $R^{[q]}$ be the graded k-algebra obtained from R by multiplying all degrees by q, that is, $(R^{[q]})_{iq} = R_i$ and $(R^{[q]})_i = 0$ for all i not divisible by q. Analogously, we can construct a graded $R^{[q]}G$-module $M^{[q]}$ with G-action from M by multiplying all degrees by q, that is, $(M^{[q]})_{iq} = M_i$ and $(M^{[q]})_i = 0$ for all i not divisible by q.

Lemma 5.5. *With the above notation, the equivariant degree has the following properties.*

(1) *If the Krull dimension of M is at most $n-1$ then $\deg_{a(kG)} M$ is defined and both $\deg_{a(kG)} M$ and $\deg_{\mathcal{R}(kG)} M$ are equal to 0.*

(2) $\deg_{a(kG)}(M[d])$ *is defined if and only if $\deg_{a(kG)} M$ is defined. If this is the case then $\deg_{a(kG)}(M[d]) = \deg_{a(kG)} M$. We always have $\deg_{\mathcal{R}(kG)}(M[d]) = \deg_{\mathcal{R}(kG)} M$.*

(3) $\deg_{a(kG)}(M^{[q]})$ *is defined if and only if $\deg_{a(kG)} M$ is defined. If this is the case then $\deg_{a(kG)}(M^{[q]}) = q^{-n} \deg_{a(kG)} M$. We always have $\deg_{\mathcal{R}(kG)}(M^{[q]}) = q^{-n} \deg_{\mathcal{R}(kG)} M$.*

Proof. (1) follows from the corresponding property of the nonequivariant degree [Benson 1993, 2.4.1]. (2) and (3) are clear. □

Sometimes it is convenient to add an element z in degree 1 to R. Then $R[z] \otimes_k M$ is finitely generated over $R[z]$, which has dimension $n+1$.

Lemma 5.6. *The degree $\deg^{n+1}_{a(kG)}(R[z] \otimes_R M)$ is defined if and only if $\deg^n_{a(kG)} M$ is defined, and if this is the case then they are both equal. Equality always holds when $\deg_{a(kG)}$ is replaced by $\deg_{\mathcal{R}(kG)}$.*

Proof. Clear. □

Sometimes it is convenient to change the field k.

Lemma 5.7. *Let ℓ be a field extension of k.*

(1) *If $\deg_{a(kG)} M$ is defined then so is $\deg_{a(\ell G)}(\ell \otimes_k M)$ and $\deg_{a(\ell G)}(\ell \otimes_k M) = \ell \otimes_k \deg_{a(kG)} M$.*

(2) *If ℓ/k is finite and L is a finitely generated graded $(\ell \otimes_k RG)$-module such that $\deg_{a(\ell G)} L$ is defined then $\deg_{a(kG)}(L \downarrow^\ell_k) = (\deg_{a(\ell G)} L) \downarrow^\ell_k$.*

(3) *If ℓ/k is finite and if $\deg_{a(\ell G)}(\ell \otimes_k M)$ is defined then so is $\deg_{a(kG)} M$ and we have $\deg_{a(kG)} M = |\ell : k|^{-1}(\deg_{a(\ell G)}(\ell \otimes_k M)) \downarrow^\ell_k$.*

Proof. Only (3) needs any comment. Since $(\ell \otimes_k M) \downarrow^\ell_k \cong M^{|\ell/k|}$, then by (2) we get $(\deg_{a(\ell G)}(\ell \otimes_k M)) \downarrow^\ell_k = \deg_{a(kG)}\big(M^{|\ell/k|}\big)$. But then $\deg_{a(kG)} M$ is defined and the formula holds, by 5.3. □

6. Further results

In this section $R = k[d_1, \dots, d_n]$ is a graded polynomial ring with generators in positive degrees. Unless otherwise stated the degree will always take values in the Green ring.

We say that a map of R-modules *dominates* when the cokernel has dimension strictly less than n. This is not consistent with the customary use of *dominant* in algebraic geometry, but it is very convenient for us here.

Proposition 6.1. *The degree* $\deg_{a(kG)} M$ *of a finitely generated graded* RG*-module* M *is defined if and only if there is a finite-dimensional graded* kG*-submodule* $X \subseteq M$ *such that the multiplication map* $R \otimes_k X \to M$ *is injective and dominant and the image is a summand over* kG*. If this holds then* $\deg_{a(kG)} M = \deg R \cdot |X|$.

Proof. Suppose that such an X exists; then $\deg_{a(kG)}(M/(R \otimes_k X))$ is defined and equal to 0 by hypothesis (take $F = \varnothing$). We claim that $\deg_{a(kG)}(R \otimes_k X) = \deg R \cdot |X|$.

It is easy to see that $H(R \otimes X, t) = H(R, t)H(X, t)$, so

$$\deg_{a(kG)}(R \otimes_k X) = \lim_{t \to 1} \big((1-t)^n H(R, t)H(X, t)\big) = \deg R \cdot |X|.$$

By additivity (Lemma 5.3), $\deg_{a(kG)} M$ is defined and is equal to $\deg R \cdot |X|$. Conversely, suppose that $\deg_{a(kG)} M$ is defined using a finite set $F \subseteq \operatorname{Indecomp} M$. Then, using the notation of the Weak Structure Theorem 3.1, we must have

$$\deg\Big(\bigoplus_{U \notin F} \bigoplus_{I \subseteq \{1,\dots,n\}} k[d_I] \otimes_k X_{U,I}\Big) = 0.$$

Thus we can take $X = \bigoplus_{U \in F} X_{U,\{1,\dots,n\}}$. □

A lot of our work is made easier by the next easy, but surprising, result.

Proposition 6.2. *If* M *is a finitely generated graded* RG*-module and* X *is a finite dimensional graded* kG*-submodule such that the multiplication map* $R \otimes_k X \to M$ *is injective and dominates then the image is a summand over* kG*, so in particular* $\deg_{a(kG)} M$ *is defined and is equal to* $\deg R \cdot |X|$.

Proof. There is a homogeneous element $z \in R$ that annihilates the cokernel. Consider the composition of maps

$$R \otimes_k X \longrightarrow M \xrightarrow{z} zM \subseteq R \otimes_k X.$$

The image is $zR \otimes_k X$, and since zR is a k-summand of R it follows that the image is a kG-summand of $R \otimes_k X$. Thus the image of $R \otimes_k X$ in M is also a summand. □

Given a graded commutative ring S, let $Q(S)$ denote the graded ring of fractions, where we invert all the homogeneous elements. It is a $\mathbb{Z}$-graded ring and $Q(S)_0$ is a field. $Q(S) = Q(S)_0[z, z^{-1}]$, where z is an element of $Q(S)$ of least positive degree. $Q(S)$ is flat over S.

Notice that if M is a finitely generated graded RG-module then $Q(R) \otimes_R M$ is a finitely generated $Q(R)$-module and in each degree it is a finite-dimensional vector space over $Q(R)_0$. In addition, $Q(R) \otimes_R M = 0$ if and only if $\dim M < \dim R$.

Proposition 6.3. *Let M be a finitely generated graded RG-module. Then the degree $\deg_{a(kG)} M$ is defined if and only if there is a finite-dimensional graded kG-submodule $X \subseteq Q(R) \otimes_R M$ such that $Q(R) \otimes_R M = Q(R) \otimes_k X$. If this is the case then $\deg_{a(kG)} M = \deg R \cdot |X|$.*

Proof. If $\deg_{a(kG)} M$ is defined, we have a short exact sequence $R \otimes_k X \hookrightarrow M \twoheadrightarrow M/(R \otimes_k X)$ with $\dim(M/(R \otimes_k X)) < \dim R$, by Proposition 6.1. If we tensor this with $Q(R)$, we obtain $Q(R) \otimes_k X \hookrightarrow Q(R) \otimes_R M \twoheadrightarrow Q(R) \otimes_R (M/(R \otimes_k X))$. But the last term must be 0.

Conversely, suppose that we have an X satisfying the conditions of the statement of the proposition. Let $\{x_i\}$ be a k-basis for X and write $x_i = \sum_j (a_{i,j}/b_{i,j}) m_j$, where $a_{i,j}$, $b_{i,j} \in R$ and $m_j \in M$, all homogeneous. Let $\bar{b}$ be the product of all the $b_{i,j}$. Then $\bar{b}X \subseteq M$, and we have a short exact sequence

$$R \otimes_k \bar{b}X \hookrightarrow M \twoheadrightarrow M/(R \otimes_k \bar{b}X).$$

But when we tensor with $Q(R)$ the first arrow becomes an isomorphism, so we must have $Q(R) \otimes_R (M/(R \otimes_k \bar{b}X)) = 0$ and thus $\dim(M/(R \otimes_k \bar{b}X)) < \dim R$, as required by 6.2. □

We now summarize the equivalent characterizations of the equivariant degree.

Theorem 6.4. *Let M be a finitely generated graded RG-module. The following conditions on M are equivalent.*

(1) *$\deg_{a(kG)} M$ is defined.*

(2) *There is a finite-dimensional graded kG-submodule $X \subseteq M$ such that the multiplication map $R \otimes_k X \to M$ dominates and is split injective over kG.*

(3) *There is a finite-dimensional graded kG-submodule $Y \subseteq M$ such that the multiplication map $R \otimes_k Y \to M$ dominates and is injective.*

(4) *There is a finite-dimensional graded kG-submodule $Z \subseteq Q(R) \otimes_R M$ such that $Q(R) \otimes_R M = Q(R) \otimes_k Z$.*

When these conditions hold we have $|X| = |Y| = |Z| = \dfrac{1}{\deg R} \deg_{a(kG)} M$.

Proof. Just combine 6.1, 6.2 and 6.3. □

Lemma 6.5. *Let R and R' be polynomial rings in n and n' variables respectively and let M and M' be finitely generated graded RG- and $R'G$-modules respectively. Let L be a finitely generated graded RH-module and let H be a subgroup of G. The degree commutes with the following operations* (*when the quantity on the right hand side is defined*):

(1) tensor product: $\deg^{n+n'}_{a(kG)}(M \otimes_k M') = \deg^{n}_{a(kG)}(M) \cdot \deg^{n'}_{a(kG)}(M')$.

(2) restriction: $\deg_{a(kH)}(M \downarrow^G_H) = (\deg_{a(kG)} M) \downarrow^G_H$.

(3) induction: $\deg_{a(kG)}(L \uparrow^G_H) = (\deg_{a(kH)} L) \uparrow^G_H$.

(4) fixed points: $\deg_{a(kG/H)} M^H = (\deg_{a(kG)} M)^H$ *if H is a normal subgroup of G.*

Proof. These all follow easily from property 6.4(3). □

In the remainder of this section we consider how Theorem 6.4 and Lemma 6.5 can be reformulated for the degree with values in the representation ring. Clearly if one of the conditions in Theorem 6.4 is satisfied then Proposition 4.5 implies that $|X| = |Y| = |Z| = (1/\deg R)\deg_{a(kG)} M$ also holds for the degree over the representation ring. The analogue of Lemma 6.5 is the following lemma.

Lemma 6.6. *With the same hypotheses as in the previous lemma, the degree with values in the representation ring commutes with the following operations*:

(1) tensor product: $\deg^{n+n'}_{a(kG)}(M \otimes_k M') = \deg^{n}_{a(kG)}(M) \cdot \deg^{n'}_{a(kG)}(M')$.

(2) restriction: $\deg_{a(kH)}(M \downarrow^G_H) = (\deg_{a(kG)} M) \downarrow^G_H$.

(3) induction: $\deg_{a(kG)}(L \uparrow^G_H) = (\deg_{a(kH)} L) \uparrow^G_H$.

Proof. This is straightforward and left to the reader. □

7. Rings

Throughout this section, S will be a graded ring in nonnegative degrees that is finitely generated over the field k and such that S_0 is finite-dimensional over k. We suppose that a finite group G acts on S by graded k-algebra automorphisms.

Geometrically, G acts as a group of automorphisms of the projective variety $V = \mathrm{Proj}(S)$, defined over k. Conversely, S could be the homogeneous coordinate ring of a variety over k on which G acts.

The invariant subring S^G is necessarily Noetherian and S is finitely generated over S^G [Benson 1993, 1.3.1]. By Noether normalization, we can find a graded polynomial subring $R \leq S^G$ such that S^G is finitely generated over R [Benson 1993, 2.2.7]. Thus S is finitely generated over R, and S and R have the same dimension. We need this ring R to exist in order for the preceding theory to apply, but it does not matter which ring R we choose.

Proposition 7.1. *If S is an integral domain and G acts faithfully, then* $\deg_{a(kG)} S$ *is defined and*

$$\deg_{a(kG)} S = \frac{\deg S}{|G|} \cdot kG$$

and the same equality holds with $\deg_{a(kG)} S$ *replaced by* $\deg_{\mathcal{R}(kG)} S$.

Proof. In [Symonds 2000], a graded submodule $F \leq S$ is produced such that $F \cong kG$ and such that the multiplication map $S^G \otimes_k F \hookrightarrow S$ dominates and is split over kG. It follows from the Additivity Lemma 5.3 that

$$\deg_{a(kG)} S = \deg_{a(kG)}(S^G \otimes F) = \deg S^G \cdot kG.$$

The proof for the degree with values in the representation ring is analogous.

There is an alternative proof that we sketch here. By Lemma 5.6, we may assume that R contains an element z of degree 1. But S is an integral domain, so it injects into $Q(S)$, thus G acts faithfully on $Q(S)$. Since $Q(S) = Q(S)_0[z, z^{-1}]$ and G acts trivially on z, G must act faithfully on $Q(S)_0$. By the Normal Basis Theorem there is a basis $\{x_g\}_{g\in G}$ for $Q(S)_0$ over $Q(S)_0^G$ that is freely permuted by G.

But $Q(S)_0$ is a finite-dimensional vector space over $Q(R)$; let $\{y_i\}$ be a basis. If we let X be the k-span of the set $\{y_i x_g\}$, then this is the module that we require. □

Let $\mathcal{P}_0$ denote the (finite) set of prime ideals in S of height 0.

Lemma 7.2. *The natural map* $S \to \bigoplus_{\mathfrak{p}\in\mathcal{P}_0} S/\mathfrak{p}$ *dominates and has* $\operatorname{rad} S$ *as kernel.*

Proof. The radical is equal to the intersection of all the prime ideals, which is equal to the intersection of the minimal ones.

We prove the claim of domination by labeling the distinct prime ideals of height 0 as $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$ and showing by induction on r that the map $S \to \bigoplus_{i=1}^r S/\mathfrak{p}_i$ dominates.

This is clearly true when $r = 1$, and the induction step follows from considering the following diagram with exact rows and columns.

$$\begin{array}{ccccc}
S\Big/\bigcap_{i=1}^{r+1}\mathfrak{p}_i & \longrightarrow & S\Big/\bigcap_{i=1}^{r}\mathfrak{p}_i \oplus S/\mathfrak{p}_{r+1} & \longrightarrow & S\Big/\Big(\bigcap_{i=1}^{r}\mathfrak{p}_i + \mathfrak{p}_{r+1}\Big) \\
\| & & \downarrow & & \downarrow \\
S\Big/\bigcap_{i=1}^{r+1}\mathfrak{p}_i & \longrightarrow & \bigoplus_{i=1}^{r+1} S/\mathfrak{p}_i & \longrightarrow & X \\
 & & \downarrow & & \downarrow \\
 & & Y & = & Y
\end{array}$$

The induction hypothesis applied to the middle column shows that $\dim Y < \dim S$, and $\dim S/\big(\bigcap_{i=1}^{r}\mathfrak{p}_i + \mathfrak{p}_{r+1}\big) < \dim S$ by construction. Thus $\dim X < \dim S$ and the middle row yields the next stage in the induction. □

Given a prime $\mathfrak{p} < S$, let $G_\mathfrak{p}$ denote the stabilizer in G of $\mathfrak{p}$ and let $\bar{G}_\mathfrak{p}$ be the pointwise stabilizer of $S/\mathfrak{p}$. We can now state a decomposition theorem for the degree of S.

Theorem 7.3. *If S contains no nilpotent elements then* $\deg_{a(kG)} S$ *is defined and*

$$\deg_{a(kG)} S = \sum_{\substack{\mathfrak{p}\in\mathcal{P}_0/G \\ \dim S/\mathfrak{p}=\dim S}} \frac{\deg S/\mathfrak{p}}{|G_\mathfrak{p}/\bar{G}_\mathfrak{p}|}\cdot k[G/\bar{G}_\mathfrak{p}]$$

and the same equality holds with $\deg_{a(kG)} S$ *replaced by* $\deg_{\mathcal{R}(kG)} S$.

Proof. In view of Proposition 7.1, Lemma 7.2 and Theorem 6.4, all we need to do is to show that $\deg_{a(kG)}\big(\bigoplus_{\mathfrak{p}\in\mathcal{P}_0} S/\mathfrak{p}\big)$ is equal to the expression shown.

But

$$\bigoplus_{\mathfrak{p}\in\mathcal{P}_0} S/\mathfrak{p} \cong \bigoplus_{\mathfrak{p}\in\mathcal{P}_0/G}\ \bigoplus_{\mathfrak{q}\sim_G\mathfrak{p}} S/\mathfrak{q} \cong \bigoplus_{\mathfrak{p}\in\mathcal{P}_0/G} \operatorname{Ind}_{G_\mathfrak{p}}^G S/\mathfrak{p}.$$

So

$$\begin{aligned}
\deg_{a(kG)}\Big(\bigoplus_{\mathfrak{p}\in\mathcal{P}_0} S/\mathfrak{p}\Big) &\cong \bigoplus_{\mathfrak{p}\in\mathcal{P}_0/G} \deg_G \operatorname{Ind}_{G_\mathfrak{p}}^G S/\mathfrak{p} \\
&\cong \bigoplus_{\mathfrak{p}\in\mathcal{P}_0/G} \operatorname{Ind}_{G_\mathfrak{p}}^G \deg_{G_\mathfrak{p}} S/\mathfrak{p} && \text{by Lemma 6.5(3)} \\
&\cong \bigoplus_{\mathfrak{p}\in\mathcal{P}_0/G} \operatorname{Ind}_{G_\mathfrak{p}}^G \frac{\deg S/\mathfrak{p}}{|G_\mathfrak{p}/\bar{G}_\mathfrak{p}|}\cdot k[G_\mathfrak{p}/\bar{G}_\mathfrak{p}] && \text{by Proposition 7.1} \\
&\cong \bigoplus_{\mathfrak{p}\in\mathcal{P}_0/G} \frac{\deg S/\mathfrak{p}}{|G_\mathfrak{p}/\bar{G}_\mathfrak{p}|}\cdot k[G/\bar{G}_\mathfrak{p}].
\end{aligned}$$

We can omit from the sum the primes $\mathfrak{p}$ for which $\dim S/\mathfrak{p} \neq \dim S$, since for these $\deg S/\mathfrak{p} = 0$. □

Geometrically, the permutation modules that occur in the statement of the theorem correspond to the way that the group permutes the irreducible components of maximum dimension of the projective variety $\operatorname{Proj}(S)$.

Now suppose that the action of G on S can be written over a finite field $\mathbb{F}_q$. Recall from Lemma 5.5 that the operation of multiplying all degrees by q gives us a new ring $S^{[q]}$ with G-action and $\deg_{\mathcal{R}(kG)} S^{[q]} = q^{-n}\deg_{\mathcal{R}(kG)} S$ and $\deg_{a(kG)} S^{[q]} = q^{-n}\deg_{a(kG)} S$. Let $S^q < S$ denote the subring of q-th powers. There is a surjection $S^{[q]} \to S^q$ and this is an isomorphism if $\operatorname{rad} S = 0$.

Lemma 7.4. *We have* $\deg_{\mathcal{R}(kG)} S^q \le q^{-n}\deg_{\mathcal{R}(kG)} S$ *and if S contains no nilpotents then* $\deg_{a(kG)} S^q = q^{-n}\deg_{a(kG)} S$.

Proof. This follows from the preceding remarks and the Additivity Lemma 5.3. □

8. Group cohomology

In this section we apply some of the theory that we have developed to a problem in group cohomology considered by Nick Kuhn [2008]. We fix a prime p and a finite group P (we do not yet require P to be a p-group). Then $G = \mathrm{Aut}(P)$ acts on the graded commutative ring $H^*(P) = H^*(P; \mathbb{F}_p)$.

By the Evens–Venkov theorem (see [Benson 1991, 3.10, 4.2], for example), $H^*(P)$ is Noetherian, hence so is $H^*(P)^G$, thus $H^*(P)$ is certainly finitely generated over some commutative polynomial ring R such that the action of G commutes with that of R; we can assume that $\dim R = \dim H^*(P)$.

Given a p-group P and a simple G-module V, Martino and Priddy [1992] asked whether the dimension of V as a composition factor of $H^*(P)$ is equal to $\dim H^*(P)$ (see also [Kuhn 2008]). It was already known from [Diethelm and Stammbach 1984; Harris and Kuhn 1988; Symonds 1999] that V does occur in $H^*(P)$.

Theorem 8.1 [Kuhn 2008]. *For p odd, the dimension of V as a composition factor of $H^*(P)$ is equal to the dimension of $H^*(P)$.*

The case of $p = 2$ is still undecided. Kuhn's methods used the nilpotent filtration of the category of unstable modules over the Steenrod algebra. We will show how this theorem can be proved using the equivariant degree. Clearly what we need to do is to show that V occurs as a composition factor of $\deg_{\mathcal{R}(\mathbb{F}_p G)} H^*(P)$.

For any finite elementary abelian p-group E, let $F^*(E) = H^*(E)/\operatorname{rad}$, which is just the symmetric algebra $\mathbb{F}_p[E] = S^*(E^*)$, where $E^* = \mathrm{Hom}(E, \mathbb{F}_p)$ is in degree 2 (or degree 1 if $p = 2$).

In general, let

$$F^*(P) = \varprojlim_{E \in \mathcal{A}_p(P)} F^*(E),$$

where $\mathcal{A}_p(P)$ denotes the category with objects the elementary abelian subgroups of P and morphisms the inclusions between them. G acts naturally on this.

Quillen [1971] (see also [Benson 1991, 5.6]) showed that the natural map induced by restrictions, $r : H^*(P) \to F^*(P)^P$ is a purely inseparable isogeny (or uniform F-isomorphism): that is that the kernel is nilpotent and there is an integer N such that $(F^*(P)^P)^{p^N} \subseteq \mathrm{Im}(r)$. From this he deduced that $\dim H^*(P)$ is equal to the p-rank of P, which we will denote by n.

Consider what this means for the degree with values in the representation ring. We have $\deg_{\mathcal{R}(\mathbb{F}_p G)} H^*(P) \geq \deg_{\mathcal{R}(\mathbb{F}_p G)} \mathrm{Im}(r) \geq \deg_{\mathcal{R}(\mathbb{F}_p G)}\big((F^*(P)^P)^{p^N}\big)$ using Lemma 5.3. By Lemma 7.4 we have

$$\deg_{a(\mathbb{F}_p G)}\big((F^*(P)^P)^{p^N}\big) = \frac{1}{p^{Nn}} \deg_{a(\mathbb{F}_p G)} F^*(P)^P,$$

since $F^*(P)$ contains no nilpotent elements.

Now we see that $\deg_{a(\mathbb{F}_pG)} F^*(P)^P = \left(\deg_{a(\mathbb{F}_pG)} F^*(P)\right)^P$ by Lemma 6.5(4).

We conclude that it is sufficient to show that $\deg_{a(\mathbb{F}_pG)} F^*(P)$ contains every simple G-module as a submodule. But the Decomposition Theorem 7.3 tells us that

$$\deg_{a(\mathbb{F}_pG)} F^*(P) = \sum_{\substack{E \in \mathcal{A}_p(P)/G \\ \text{rank } E = n}} \frac{\deg S/\mathfrak{p}_E}{|N_G(E)/C_G(E)|} \cdot \mathbb{F}_p[G/C_G(E)],$$

where $\mathfrak{p}_E$ denotes the ideal corresponding to E. Since each E in the sum has maximal rank, $\deg(S/\mathfrak{p}_E) \neq 0$. Suppose that some $C_G(E)$ is a p-group. Then

$$\mathrm{Hom}_G(V, \mathbb{F}_p[G/C_G(E)]) \cong \mathrm{Hom}_{C_G(E)}(V, \mathbb{F}_p) \neq 0,$$

so V does occur in $\deg_{a(\mathbb{F}_pG)} F^*(P)$ and we are done. That this always happens when p is odd is the content of the next lemma, which appears as [Kuhn 2008, 2.3], although we first learnt it from Benson (private communication) in 1996. We include the proof for the convenience of the reader.

Lemma 8.2. *If p is odd and E is maximal then $C_G(E)$ is a p-group.*

Proof. Consider the composition of homomorphisms

$$C_G(E) \xrightarrow{\alpha} \mathrm{Aut}(C_P(E)) \xrightarrow{\beta} \mathrm{Aut}(E).$$

The composition is trivial, so it suffices to prove that the kernel of each map is a p-group. For β we use the result that if p is odd and Q is a p-group then the kernel of the map $\mathrm{Aut}(Q) \to \mathrm{Aut}(\Omega_1(Q))$ is a p-group [Gorenstein 1968, 5.3.10]. (This is the only place in this section where the argument requires p to be odd.)

For α we use Thompson's $A \times B$ Lemma [Gorenstein 1968, 5.3.4], which states that for any p-group P, if $A \times B \subseteq \mathrm{Aut}(P)$ with A a p'-group and B a p-group such that A acts trivially on $C_P(B)$, then $A = 1$. We apply this with A some p'-subgroup of $\mathrm{Ker}(\alpha)$ and B the image of E in G. □

9. Further results on the degree with values in the representation ring

We assume that k is a splitting field for the group G, but we do not need R to be polynomial.

Let V be a simple kG-module and let M be a finitely generated graded RG-module. Let M_V denote the part of M that is generated by submodules isomorphic to V.

Lemma 9.1. $\mathrm{Hom}_{kG}(V, M) \otimes_k V \cong M_V$ *by the map* $f \otimes v \mapsto f(v)$.

Proof. Since $\mathrm{Hom}_{kG}(V, M) \cong \mathrm{Hom}_{kG}(V, M_V)$ we may assume that $M = M_V$. But now the claimed isomorphism is additive in M_V, and M_V is just a direct sum of submodules isomorphic to V, so we are reduced to the case where $M_V = V$. But now it holds by the assumption that k is a splitting field, so $\mathrm{End}_{kG}(V) \cong k$ [Curtis and Reiner 1981, 7.14]. □

The next result is an equivariant analogue of [Hartshorne 1977, I 7.4].

Proposition 9.2. *Let M be a finitely generated graded RG-module. Then M has a finite filtration $0 = M_0 \leq M_1 \leq \cdots \leq M_m = M$ by graded RG-submodules such that $M_i/M_{i-1} \cong R/\mathfrak{p}_i[\ell_i] \otimes_k V_i$, where $\mathfrak{p}_i$ is a homogeneous prime ideal of R and V_i is a simple kG-module.*

(1) *The minimal elements among the $\mathfrak{p}_i$ occurring are the minimal primes for M.*

(2) *For each minimal prime $\mathfrak{p}$ of M, let $k(\mathfrak{p})$ denote the quotient field of $R/\mathfrak{p}$. For each simple kG-module V, the number of times that $R/\mathfrak{p} \otimes_k V$ occurs as a composition factor of the filtration is equal to the number of times that the simple $R_{\mathfrak{p}}G$-module $k(\mathfrak{p}) \otimes_k V$ occurs as a composition factor of the localization $M_{\mathfrak{p}}$, hence is independent of the filtration.*

Proof. Let $\mathfrak{p}$ be an associated prime of $\mathrm{Hom}_{RG}(V, M)$, so it is the annihilator of some $\phi : V \to M$. Thus we have an injection of graded RG-modules $R/\mathfrak{p} \hookrightarrow \mathrm{Hom}_{RG}(V, M)$, $\quad r \mapsto r\phi$ and hence an injection $R/\mathfrak{p} \otimes_k V \hookrightarrow \mathrm{Hom}_{RG}(V, M) \otimes V$.

By Lemma 9.1 this leads to an injection $R/\mathfrak{p} \otimes_k V \hookrightarrow M$; denote its image by M_1.

Now repeat the process with M/M_1, and let M_2 be the inverse image in M of the resulting submodule. In this way we obtain an ascending sequence of graded RG-submodules of M, which must terminate since M is Noetherian.

Notice that this filtration can be refined to a nonequivariant one by filtering the V. Thus (1) follows from the nonequivariant case.

For (2), let $\mathfrak{q}$ be a minimal prime and consider what happens when we localize at $\mathfrak{q}$. If $\mathfrak{p}_i \neq \mathfrak{q}$ then $(R/\mathfrak{p}_i)_{\mathfrak{q}} = 0$, since $\mathfrak{q}$ is minimal in $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_m\}$. If $\mathfrak{p}_i = \mathfrak{q}$ then $(R/\mathfrak{q})_{\mathfrak{q}} = k(\mathfrak{q})$ and $(R/\mathfrak{q} \otimes_k V)_{\mathfrak{q}} = k(\mathfrak{q}) \otimes_k V$. This is a simple $S_{\mathfrak{q}}G$-module since k is a splitting field. □

Write $m(\mathfrak{p}, V, M)$ for the number of times that $R/\mathfrak{p} \otimes_k V$ occurs as a factor in a filtration of M of the type considered in the proposition above.

Corollary 9.3. $\deg_{\mathcal{R}(kG)} M = \sum_{\dim R/\mathfrak{p} = \dim M} m(\mathfrak{p}, V, M) \deg(R/\mathfrak{p}) \cdot |V|.$

There are some straightforward reduction methods for calculating the degree with values in the representation ring.

Lemma 9.4. *Let $f \in R$ be homogeneous and let M be a finitely generated graded RG-module of dimension m. Suppose that the dimension of the kernel of the multiplication map $\phi_f : M \to M, \quad m \mapsto fm$, has dimension at most $m-2$. Then* $\deg^{m-1}_{\mathcal{R}(kG)}(M/fM) = |f| \deg^m_{\mathcal{R}(kG)} M$.

Proof. There is a short exact sequence $\ker(\phi_f) \to M \stackrel{\phi_f[|f|]}{\to} fM\big[|f|\big]$, where $\big[|f|\big]$ denotes the degree shift needed to make all the maps degree preserving. Thus $\big[fM[|f|]\big] = [M] + O((1-t)^{-(m-2)})$ as a Laurent series in $1-t$ and so $[fM] = t^{|f|}[M] + O((1-t)^{-(m-2)})$.

There is also a short exact sequence $fM \to M \to M/fM$, so $[M/fM] = [M] - [fM]$.

Combining, we find that $[M/fM] = [M] - t^{|f|}[M] + O((1-t)^{-(m-2)})$. Thus

$$\begin{aligned}\deg^{m-1}_{\mathcal{R}(kG)}[M/fM] &= \lim_{t\to 1}(1-t)^{m-1}\cdot(1-t^{|f|})[M] \\ &= \lim_{t\to 1}\frac{1-t^{|f|}}{1-t}\cdot(1-t)^m[M] = |f|\deg^m_{\mathcal{R}(kG)} M. \qquad \square\end{aligned}$$

Our last result follows by repeated use of this lemma.

Proposition 9.5. *Let M be a finitely generated graded RG-module of dimension m and suppose that $f_1, \ldots, f_r \in R$ is an M-regular sequence of homogeneous elements. Then*

$$\deg^m_{\mathcal{R}(kG)} M = \prod |f_i| \cdot \deg^{m-r}_{\mathcal{R}(kG)}\big(M/(f_1, \ldots, f_r)M\big).$$

References

[Benson 1991] D. J. Benson, *Representations and cohomology, II: Cohomology of groups and modules*, Cambridge Studies in Advanced Mathematics **31**, Cambridge Univ. Press, Cambridge, 1991. MR 93g:20099 Zbl 0731.20001

[Benson 1993] D. J. Benson, *Polynomial invariants of finite groups*, London Mathematical Society Lecture Note Series **190**, Cambridge University Press, 1993. MR 94j:13003 Zbl 0864.13001

[Curtis and Reiner 1981] C. W. Curtis and I. Reiner, *Methods of representation theory, with applications to finite groups and orders*, vol. I, Wiley, New York, 1981. MR 82i:20001 Zbl 0469.20001

[Diethelm and Stammbach 1984] T. Diethelm and U. Stammbach, "On the module structure of the mod p cohomology of a p-group", *Arch. Math.* (*Basel*) **43**:6 (1984), 488–492. MR 86k:20047 Zbl 0535.20029

[Gorenstein 1968] D. Gorenstein, *Finite groups*, Harper & Row, New York, 1968. MR 38 #229 Zbl 0185.05701

[Harris and Kuhn 1988] J. C. Harris and N. J. Kuhn, "Stable decompositions of classifying spaces of finite abelian p-groups", *Math. Proc. Cambridge Philos. Soc.* **103** (1988), 427–449. MR 89d:55021 Zbl 0686.55007

[Hartshorne 1977] R. Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics **52**, Springer, New York, 1977. MR 57 #3116 Zbl 0367.14001

[Karagueuzian and Symonds 2004] D. B. Karagueuzian and P. Symonds, "The module structure of a group action on a polynomial ring: examples, generalizations, and applications", pp. 139–158 in *Invariant theory in all characteristics*, edited by H. E. A. Eddy Campbell and D. L. Wehlau, CRM Proc. Lecture Notes **35**, Amer. Math. Soc., Providence, RI, 2004. MR 2005g:13011 Zbl 1069.20501

[Karagueuzian and Symonds 2007] D. B. Karagueuzian and P. Symonds, "The module structure of a group action on a polynomial ring: a finiteness theorem", *J. Amer. Math. Soc.* **20**:4 (2007), 931–967. MR 2328711 Zbl 1138.13004

[Kuhn 2008] N. J. Kuhn, "The nilpotent filtration and the action of automorphisms on the cohomology of finite p-groups", *Math. Proc. Cambridge Philos. Soc.* **144**:3 (2008), 575–602. MR 2418705 Zbl 05288437

[Martino and Priddy 1992] J. Martino and S. Priddy, "On the dimension theory of dominant summands", pp. 281–292 in *Adams Memorial Symposium on Algebraic Topology, 1* (Manchester, 1990), edited by N. Ray and G. Walker, London Math. Soc. Lecture Note Ser. **175**, Cambridge Univ. Press, 1992. MR 93h:55009 Zbl 0770.55011

[Quillen 1971] D. Quillen, "The spectrum of an equivariant cohomology ring, I, II", *Ann. of Math.* (2) **94** (1971), 549–572 and 573–602. MR 45 #7743 Zbl 0247.57013

[Symonds 1999] P. Symonds, "The action of automorphisms on the cohomology of a p-group", *Math. Proc. Cambridge Philos. Soc.* **127**:3 (1999), 495–496. MR 2000g:20104 Zbl 0947.20035

[Symonds 2000] P. Symonds, "Group action on polynomial and power series rings", *Pacific J. Math.* **195**:1 (2000), 225–230. MR 2001h:13008 Zbl 1030.13002

[Symonds 2007] P. Symonds, "Structure theorems over polynomial rings", *Adv. Math.* **208**:1 (2007), 408–421. MR 2008b:20008 Zbl 1108.13007

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Syzygies of the secant variety of a curve

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We show the secant variety of a linearly normal smooth curve of degree at least $2g+3$ is arithmetically Cohen–Macaulay, and we use this information to study the graded Betti numbers of the secant variety.

1. Introduction

We work throughout over an algebraically closed field of characteristic 0. A well-known result dating back to Castelnuovo states that if $C \subset \mathbb{P}^n$ is a linearly normal curve of genus g with $\deg C \geq 2g+1$, then C is projectively normal and hence is arithmetically Cohen–Macaulay (ACM). Our main result is this:

Theorem 1. *If $C \subset \mathbb{P}^n$ is a smooth linearly normal curve of genus g and degree $d \geq 2g+3$, then its secant variety Σ is ACM.*

Using the Auslander–Buschbaum theorem [Eisenbud 1995, §19], this tells us that a minimal free resolution of the coordinate ring of Σ, S_Σ, has length equal to $\operatorname{codim} \Sigma$, and the remainder of this paper is devoted to studying the syzygies among the defining equations of Σ.

To describe our results on syzygies more precisely, we set up some notation. Let $S = k[x_0, \ldots, x_n]$. Any finitely generated S-module M has a minimal free resolution

$$0 \to \bigoplus S(-j)^{\beta_{r,j}} \to \cdots \to \bigoplus S(-j)^{\beta_{1,j}} \to \bigoplus S(-j)^{\beta_{0,j}} \to M \to 0,$$

where the *graded Betti numbers* $\beta_{i,j}$ are uniquely determined by minimality. It is convenient to display the $\beta_{i,j}$ in a graded Betti diagram in which the (i, j) entry is $\beta_{i,i+j}$.

	0	1	2	3
0	$\beta_{0,0}$	$\beta_{1,1}$	$\cdots$	
1	$\beta_{0,1}$	$\beta_{1,2}$	$\cdots$	
2	$\beta_{0,2}$	$\beta_{1,3}$	$\cdots$	

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As in [Eisenbud 2005] we say that the Betti numbers $\beta_{i,i+k}$ in the ith row of the Betti diagram form the degree $k+1$ *linear strand* if $M = S/I$ for some homogeneous ideal I. In this case, $\beta_{1,k+1}$ is the number of minimal generators of I in degree $k+1$. (For an arbitrary module, M, it might make more sense to call this the degree k linear strand.)

It is useful to recast several notions from the geometric literature in terms of the graded Betti diagram. Suppose that a variety $X \subset \mathbb{P}^n$ is projectively normal with an ideal generated by quadrics. Then for $p \geq 1$ it satisfies Green's condition N_p [Green 1984] if for all $i \leq p$, $\beta_{ij} = 0$ unless $j = i+1$. Eisenbud et al. [2005] extended this notion to a variety X with ideal generated in degree k, so that X satisfies $N_{k,p}$ if for all $i \leq p$, $\beta_{ij} = 0$ unless $j = i+k-1$. Thus, the only nonzero entries in columns one through p of a Betti diagram of a variety satisfying $N_{k,p}$ are in row $k-1$. The Castelnuovo–Mumford regularity [Mumford 1966], or simply *regularity*, of a module can also be defined in terms of graded Betti numbers. A module is *m-regular* if $\beta_{i,m+i} = 0$ for all $i > 0$, which is equivalent to stating that its Betti diagram is zero in all rows greater than m.

If $C \subset \mathbb{P}^n$ is a linearly normal curve of genus g and degree $d \geq 2g+3$, we obtain several results as consequences of the Cohen–Macaulay condition. In Corollary 3.9 we show that if $\operatorname{reg} I_\Sigma < 5$, then C is rational and $\operatorname{reg} I_\Sigma = 3$. We give explicit formulas for several graded Betti numbers in Corollary 4.1 and Proposition 4.4, showing that

- $\beta_{1,3} = \binom{n+1}{3} - (d-2)n - 3g + 1$.
- $\beta_{2,4} = \beta_{1,4} + \beta_{1,3}(n+1) - \binom{n+4}{n} + P_\Sigma(4)$.
- $\beta_{n-3,n+1} = \binom{g+1}{2}$.

Note that via Theorem 1 there are exactly $n-3$ syzygy modules in the resolution of S_Σ, and if $g \geq 1$, then Corollary 3.9 implies that the final syzygy module is generated by elements of degree $\leq n+1$. Thus, $\beta_{n-3,n+1}$ is the bottom right corner of the graded Betti diagram, and it depends only on the genus of the curve.

We compute the Hilbert polynomial of S_Σ by relating it to the Hilbert polynomial of a curve of degree D and genus G gotten by intersecting Σ with a plane of codimension 2.

Theorem 1.1. *The Hilbert polynomial of S_Σ agrees with its Hilbert function for all positive integers and is given by*

$$D\binom{m+2}{3} + (1-G)\binom{m+1}{2} + \alpha_1 m + \alpha_0,$$

where $\alpha_1 = \binom{n+2}{2} - (n+1) - 3D - 2(1-G)$ *and* $\alpha_0 = -\binom{n+2}{2} + 2(n+1) + 2D + 1 - G$.

We also obtain a nonvanishing result on the graded Betti numbers of higher secant varieties.

Theorem 1.2. *Let C be a smooth curve of genus g embedded into $\mathbb{P}^n$ via a line bundle L of degree $d \geq 2g+2k+p+1$ and Σ_k be its variety of secant k-planes. Suppose that $L = L_1 \otimes L_2$ where $|L_1| = s \leq |L_2| = t$. If $s+1 \geq k+2$, then the length of the degree $k+2$ linear strand of S_{Σ_k} is at least $s+t-2k-1$. In particular, if L is a general line bundle of degree $d \geq 2g+2k+p+1$, then $\beta_{s+t-2k-1,s+t-k}(\Sigma_k) \neq 0$.*

We briefly sketch part of the picture of what is known about syzygies of high degree curves to put our results in context. The homogeneous coordinate ring of a curve of degree at least $2g+1$ is 1-regular if $g = 0$ and has regularity two otherwise. From [Green 1984; Green and Lazarsfeld 1985; 1988] we know that if $d \geq 2g+1+p$, then the curve satisfies N_p. By a result of Schreyer [Eisenbud 2005, Theorem 8.17], we know that $\beta_{p+\lfloor g/2 \rfloor, p+\lfloor g/2 \rfloor+1}$ is nonzero. Furthermore, as a consequence of duality, the "last" graded Betti number is $\beta_{n-1,n+1} = g$. (See [Eisenbud 2005, Chapter 8] for a nice discussion.)

Based on what we have seen we extend and refine the conjectures in [Vermeire 2008b] as follows:

Conjecture 1.3. Suppose that $C \subset \mathbb{P}^n$ is a smooth linearly normal curve of genus g and degree $d \geq 2g+2k+1+p$, where $p, k \geq 0$. Then

(1) Σ_k is ACM and has regularity $(2k+2)$ unless $g = 0$, in which case the regularity is $k+1$.

(2) $\beta_{n-2k-1,n+1} = \binom{g+k}{k+1}$.

(3) Σ_k satisfies $N_{k+2,p}$.

As described above, the full conjecture is known to hold for $k = 0$. Further, by [Graf von Bothmer and Hulek 2004] and [Fisher 2006] it holds for $g \leq 1$. In this work, we show that parts (1) and (2) hold for $k = 1$. After the completion of this work, progress on part (3) was made for $k = 1$ [Vermeire 2008a]. We illustrate the behavior that we have seen with the example below.

Example 1.4. At the suggestion of D. Eisenbud we used ideas of F. Schreyer to compute the ideal of a genus 2 curve embedded in $\mathbb{P}^7$. Let $\overline{C}$ be a plane curve of degree 5 with 4 nodes. If we blow up the four nodes in $\mathbb{P}^2$ and consider the linear system $|5H - 2\Sigma E_i|$, where H is the proper transform of a hyperplane and the E_i are the exceptional divisors of the blow-up, the restriction of this system to the proper transform C of $\overline{C}$ has degree $9 = 2g+5$, and embeds $C \subset \mathbb{P}^7$ as a smooth curve of genus 2. Using Macaulay 2 [Grayson and Stillman] we can write down a basis of $|5H - 2\Sigma E_i|$ over the rationals. We display the Betti diagram of the

coordinate ring, where "–" denotes a zero entry.

	0	1	2	3	4	5	6
0	1	–	–	–	–	–	–
1	–	19	58	75	44	5	–
2	–	–	–	–	–	6	2

Note that the quadratic strand of the resolution has length 5 but that the curve satisfies N_4 but not N_5.

Using code developed for [Sidman and Sullivant 2006], we computed the ideal of Σ. From the Betti diagram we see that the cubic strand of the resolution has length 2 and that $\beta_{4,8} = 3$ as predicted by Conjecture 1.3.

	0	1	2	3	4
0	1	–	–	–	–
1	–	–	–	–	–
2	–	12	16	–	–
3	–	–	–	4	–
4	–	–	–	4	3

Comparing the diagram to the statement of Corollary 4.7, we see that the three unknowns at the tail of the resolution are all zero here as in Example 4.8. □

We give a brief outline of the structure of the paper. The ACM condition is treated in §3. To understand the ACM condition, we work geometrically to show that cohomology groups vanish. The key observation is that there is a desingularization $\widetilde{\Sigma} \to \Sigma$ such that $\widetilde{\Sigma}$ is a $\mathbb{P}^1$-bundle over the symmetric square of C, which we denote by S^2C, and hence the cohomology of the structure sheaf of $\widetilde{\Sigma}$ is the same as that of S^2C, which is easier to understand. As Σ has nonrational singularities, the higher direct image sheaves of the ideal of $\widetilde{\Sigma}$ do not vanish, but there is another divisor whose ideal sheaf has the same direct image and whose higher direct images do vanish. (See Lemma 2.4). Making the exact relationships between these objects precise is the bulk of our work. The technical preliminaries are summarized in Section 2. We examine the graded Betti diagram of S_Σ in Section 4.

To improve readability we have written out some arguments which are surely well-known to experts, but are perhaps not easily available in the standard references.

2. Setup and notation

Suppose that $X \subset \mathbb{P}^n$ is a variety. We let $\mathcal{O}_X$ and $\mathcal{I}_X$ denote the structure sheaf and ideal sheaf of X. The homogeneous coordinate ring of $\mathbb{P}^n$ is $S = k[x_0, \ldots, x_n]$. We

let $I_X = \bigoplus H^0(\mathbb{P}^n, \mathcal{I}_X(d))$ and $S_X = S/I_X$. We let H denote a general hyperplane in $\mathbb{P}^n$ and its pullback under a morphism. We write $\mathcal{O}(k)$ for $\mathcal{O}(kH)$ when no confusion will arise. We may write $H^i(\mathcal{F})$ for $H^i(X, \mathcal{F})$ and $h^i(\mathcal{F})$ for $h^i(X, \mathcal{F})$ if the meaning is clear.

Let C be a smooth curve of genus g. Throughout, L is a very ample line bundle on C embedding it as a linearly normal curve in $\mathbb{P}^n = \mathbb{P}(H^0(C, L))$ with degree $d = \deg L$.

A line bundle L on a smooth curve C is said to *separate k points* if

$$h^0(C, L(-Z)) = h^0(C, L) - k \quad \text{for all } Z \in S^k C,$$

where $S^k C$ is the kth symmetric product of C. We let Σ_k denote the variety of $(k+1)$-secant k-planes to C and write Σ for the variety Σ_1.

We recall the first stages of a construction of Aaron Bertram which provides the geometric framework for our results.

Theorem 2.1 [Bertram 1992, Theorem 1]. *Suppose that L separates 4 points. Let $g : B_1 \to B_0 = \mathbb{P}^n$ be the blowup of B_0 along C with $\widetilde{\Sigma}$ the proper transform of Σ. Let $h : B_2 \to B_1$ be the blowup of B_1 along $\widetilde{\Sigma}$ and E_i be the proper transform in B_i of each exceptional divisor. We further let $f = g \circ h$.*

Then $\widetilde{\Sigma} \subset B_1$ is smooth and irreducible, and transverse to E_1, so in particular B_2 is smooth. Moreover, by Terracini recursiveness, if $x \in \Sigma \setminus C$, then $f^{-1}(x) \cong \mathbb{P}(H^0(C, L(-2V)))$, where V is the unique divisor of degree 2 whose span contains x. If $x \in C$, then $f^{-1}(x)$ is isomorphic to the blowup of $\mathbb{P}(H^0(C, L(-2x)))$ along the image of C embedded by $L(-2x)$. □

Remark 2.2. Bertram's construction continues, blowing up the strict transform of each Σ_k successively, so that a fiber over a point of C of the composition is $\mathbb{P}^{n-2}$ in which we have blown up copies of Σ_i for $i = 0, \ldots, k-1$ and the degree of $\Sigma_0 = C$ is two less than the degree of the original embedding. We will abuse notation in the hopes of highlighting the recursive nature of the construction and denote the restriction of E_i to a fiber F of the composition using the notation of our setup relative to the blowing up that has occurred within F. For example, if $x \in C$ and $F = f^{-1}(x)$, we will write $\mathcal{O}_{B_2}(E_2)|_F = \mathcal{O}_F(E_1)$, keeping in mind that "$E_1 \subset F$" is the exceptional divisor of $\mathbb{P}^{n-2}$ blown up at C where the degree has already dropped by two.

A key point in what follows is that $\widetilde{\Sigma}$ is a resolution of singularities of Σ, and is a $\mathbb{P}^1$-bundle over $S^2 C$ in a natural way. We summarize this relationship:

Lemma 2.3. *The variety $\widetilde{\Sigma} \subset B_1$ is a resolution of singularities $g : \widetilde{\Sigma} \to \Sigma$ with the following properties*:

(1) $g_* \mathcal{O}_{\widetilde{\Sigma}} = \mathcal{O}_\Sigma$.

(2) $Z := E_1 \cap \widetilde{\Sigma} \cong C \times C$.

(3) *The restriction* $g : C \times C \to C$ *is projection onto one factor.*

(4) *The restriction of the linear system* $|2H - E_1|$ *to* $\widetilde{\Sigma}$ *yields a morphism* $\pi : \widetilde{\Sigma} \to S^2C$ *realizing* $\widetilde{\Sigma}$ *as a* $\mathbb{P}^1$*-bundle over* S^2C*. The restriction of this morphism to* $Z \cong C \times C$ *is the canonical double cover* $d : C \times C \to S^2C$.

(5) *If we define* δ *by* $d^*\mathcal{O}_{S^2C}\left(\frac{\delta}{2}\right) = \mathcal{O}_{C\times C}(\Delta)$*, then* $d_*\mathcal{O}_{C\times C} = \mathcal{O}_{S^2C} \oplus \mathcal{O}_{S^2C}\left(-\frac{\delta}{2}\right)$.

(6) *If* F *is a fiber of the* $\mathbb{P}^1$*-bundle* $\pi : \widetilde{\Sigma} \to S^2C$*, then*

$$\mathcal{O}_F(aH - bE_1) = \mathcal{O}_{\mathbb{P}^1}(a - 2b).$$

Proof. The first is [Vermeire 2002, 3.2], the second and third are [Vermeire 2001, 3.7], the fourth is [Vermeire 2001, 3.8]. Part (5) follows from [Barth et al. 2004, V.22]. For (6), note that each fiber F is the proper transform of a secant line, hence the intersection with a hyperplane is 1, while the intersection with the exceptional divisor is 2 (since each secant or tangent line intersects C in a scheme of length two). □

Lemma 2.4. *With hypotheses and notation as above*:

(1) $\Sigma \subset B_0$ *is normal and is smooth away from* C.

(2) $f_*\mathcal{O}_{B_2} = \mathcal{O}_{B_0}$ *and* $R^j f_*\mathcal{O}_{B_2} = 0$ *for* $j \geq 1$.

(3) $R^i f_*\mathcal{O}_{B_2}(-E_2) = \begin{cases} \mathcal{I}_\Sigma & i = 0, \\ H^1(C, \mathcal{O}_C) \otimes \mathcal{O}_C & i = 2, \\ 0 & i \neq 0, 2. \end{cases}$

(4) $R^i g_*\mathcal{O}_{B_1}(-mE_1) = R^i h_*\mathcal{O}_{B_2}(-mE_2) = 0$ *for* $i > 0$ *and* $m \geq 0$.

(5) $R^i g_*\mathcal{I}_{\widetilde{\Sigma}} = R^i f_*\mathcal{O}_{B_2}(-E_2)$.

(6) $R^i f_*\mathcal{O}_{B_2}(-E_1 - E_2) = \mathcal{I}_{\Sigma/\mathbb{P}^n}$ *for* $i = 0$ *and is zero otherwise.*

Proof. The first two can be found in [Vermeire 2002, 3.2], while the third is [Vermeire 2008b, Proposition 9] and the fourth is [Lazarsfeld 2004, Lemma 4.3.16]. Part (5) follows immediately from (4) and a degenerate case of Grothendieck's composition of functors spectral sequence [Grothendieck 1957].

For the sixth item, we compute sheaves $R^i f_*\mathcal{O}_{E_1}(-E_2)$ and use them to show the claim via

$$0 \to \mathcal{O}_{B_2}(-E_1 - E_2) \to \mathcal{O}_{B_2}(-E_2) \to \mathcal{O}_{E_1}(-E_2) \to 0.$$

Since $E_1 \to C$ is flat, the locally free sheaf $\mathcal{O}_{E_1}(-E_2)$ is also flat over C. Thus, we can compute higher direct images via cohomology along the fibers of f restricted to E_1 by [Hartshorne 1977, Corollary III.12.9]. By the Terracini recursiveness portion of Theorem 2.1, if $x \in C$, a fiber $F = f^{-1}(x)$ is the blowup of C in

$\mathbb{P}H^0(C, L(-2x))$ and E_2 intersects F in the exceptional divisor E_1 of this blowup. As $H^i(F, \mathcal{O}_F(-E_1)) = H^i(\mathbb{P}(H^0(C, L(-2x))), \mathcal{I}_C)$, $H^i(F, \mathcal{O}_F(-E_1))$ vanishes for $i = 0, 1$, and $h^2(\mathbb{P}(H^0(C, L(-2x))), \mathcal{I}_C) = h^1(C, \mathcal{O}_C) = g$. We conclude that $R^i f_* \mathcal{O}_{E_1}(-E_2) = 0$ for $i = 0, 1$ and that for $i = 2$ it is locally free of rank g. Note that by part (5), $R^2 f_* \mathcal{O}_{B_2}(-E_2)$ is also locally free of rank g. Therefore, if the map between them is a surjection, it is an isomorphism.

To get the surjectivity above we show $R^3 f_* \mathcal{O}_{B_2}(-E_1 - E_2) = 0$ by looking at

$$0 \to \mathcal{O}_{B_2}(-E_1 - E_2) \to \mathcal{O}_{B_2}(-E_1) \to \mathcal{O}_{E_2}(-E_1) \to 0. \tag{1}$$

Applying h_*, the projection formula and the observation that $E_2 \to \widetilde{\Sigma}$ is a projective bundle, we see that

$$0 \to \mathcal{I}_{\widetilde{\Sigma}}(-E_1) \to \mathcal{O}_{B_1}(-E_1) \to \mathcal{O}_{\widetilde{\Sigma}}(-E_1) \to 0$$

is exact and all higher direct images vanish. If we apply g_* we get

$$\to R^2 g_* \mathcal{O}_{\widetilde{\Sigma}}(-E_1) \to R^3 g_* \mathcal{I}_{\widetilde{\Sigma}}(-E_1) \to R^3 g_* \mathcal{O}_{B_1}(-E_1) \to,$$

where the left-hand term vanishes because $\widetilde{\Sigma} \to \Sigma$ has fibers of dimension at most one, and the right-hand term vanishes by (4). □

We will use Lemma 2.5 to show that $H^1(\Sigma, \mathcal{O}_\Sigma(2)) = 0$ in Theorem 3.3.

Lemma 2.5. *Let L be a very ample line bundle on a variety X with $H^i(X, L) = 0$ for $i > 0$, E a locally free sheaf on X. Let $\varphi : X \to \mathbb{P}^n = \mathbb{P}(H^0(X, L))$ be the induced morphism.*

(1) $H^i(X \times X, (L \boxtimes E) \otimes \mathcal{I}_\Delta) = H^i(X, \varphi^* \Omega^1_{\mathbb{P}^n} \otimes L \otimes E)$.

(2) $H^i(X \times X, (L \boxtimes E) \otimes \mathcal{I}_\Delta^2) = H^i(X, N^*_{X/\mathbb{P}^n} \otimes L \otimes E)$.

Proof. Applying $(\pi_2)_*$ to the exact sequence

$$0 \to (L \boxtimes E) \otimes \mathcal{I}_\Delta \to L \boxtimes E \to (L \boxtimes E) \otimes \mathcal{O}_\Delta \to 0$$

yields a twist of the Euler sequence on X:

$$0 \to \varphi^* \Omega^1_{\mathbb{P}^n} \otimes L \otimes E \to H^0(X, L) \otimes E \to L \otimes E \to 0$$

Note that the hypothesis $H^i(X, L) = 0$ and the fact that L is globally generated imply that all higher direct images vanish, and part (1) follows immediately.

As $\mathcal{O}_\Delta \otimes \mathcal{I}_\Delta = N^*_\Delta = \Omega^1_X$; applying $(\pi_2)_*$ to the exact sequence

$$0 \to (L \boxtimes E) \otimes \mathcal{I}_\Delta^2 \to (L \boxtimes E) \otimes \mathcal{I}_\Delta \to (L \boxtimes E) \otimes N^*_\Delta \to 0$$

yields a twist of the conormal sequence on X:

$$0 \to N^*_{X/\mathbb{P}^n} \otimes L \otimes E \to \varphi^* \Omega^1_{\mathbb{P}^n} \otimes L \otimes E \to \Omega^1_X \otimes L \otimes E \to 0$$

Note that the hypothesis $H^i(X, L) = 0$ and the fact that L is very ample imply that all higher direct images vanish, and part (2) follows similarly. □

3. Σ is ACM

The main goal of this section is the proof of Theorem 1. As a consequence of our work we get Corollary 3.4 showing that Σ is projectively normal. We will work throughout with the following hypothesis.

Hypothesis 3.1. *Let $C \subset \mathbb{P}^n$ be a smooth linearly normal curve of genus g and degree $d \geq 2g + 3$.*

Using the Serre–Grothendieck correspondence between local and global cohomology, the depth of the maximal ideal on the homogeneous coordinate ring of $\Sigma \subset \mathbb{P}^n$ can be measured by vanishings of global cohomology groups. We see that Σ is ACM if and only if $H^i(\mathbb{P}^n, \mathcal{I}_\Sigma(k)) = 0$ for all k and for $0 < i \leq \dim \Sigma$ (for example, [Eisenbud 1995, Example 18.16]). In light of [Vermeire 2008b] where it is shown that $\mathcal{I}_\Sigma$ is 5-regular, in order to show that Σ is ACM we are left to show that $H^i(\Sigma, \mathcal{O}_\Sigma(k)) = 0$ for $i = 1, 2$ and all $k \leq 3 - i$. In what follows we handle the required cohomological vanishing cases individually.

3A. ***Vanishings for k < 0.*** The vanishings needed for $k < 0$ follow easily from Kawamata–Viehweg vanishing together with part (3) of Lemma 2.4. We write the 5-term sequence associated to the Leray spectral sequence (applying Theorem 2.1) to the map $g : \widetilde{\Sigma} \to \Sigma$ as it will be crucial in what follows (note that the first and fourth terms follow by part (1) of Lemma 2.3).

$$0 \to H^1(\Sigma, \mathcal{O}_\Sigma(k)) \to H^1(\widetilde{\Sigma}, \mathcal{O}_{\widetilde{\Sigma}}(k)) \to H^0(\Sigma, R^1 g_* \mathcal{O}_{\widetilde{\Sigma}}(k)) \to H^2(\Sigma, \mathcal{O}_\Sigma(k)) \to H^2(\widetilde{\Sigma}, \mathcal{O}_{\widetilde{\Sigma}}(k)) \quad (2)$$

Theorem 3.2. *If C satisfies Hypothesis 3.1, then $H^i(\Sigma, \mathcal{O}_\Sigma(k)) = 0$ for $k < 0$ and $i = 1, 2$.*

Proof. We know that $g^*\mathcal{O}_\Sigma(1) = \mathcal{O}_{\widetilde{\Sigma}}(1)$ is big and nef on $\widetilde{\Sigma}$; hence $H^i(\widetilde{\Sigma}, \mathcal{O}_{\widetilde{\Sigma}}(k))$ is 0 for $k < 0$ and $i < 3$ by Kawamata–Viehweg vanishing. Using the sequence (2), we have the claimed vanishing for $i = 1$ immediately. As $R^1 g_* \mathcal{O}_{\widetilde{\Sigma}} \cong H^1(C, \mathcal{O}_C) \otimes \mathcal{O}_C$ by Lemma 2.4 (3–5), we have $H^0(\Sigma, R^1 g_* \mathcal{O}_{\widetilde{\Sigma}}(k)) = H^1(C, \mathcal{O}_C) \otimes H^0(C, \mathcal{O}_C(k)) = 0$, and the vanishing for $i = 2$ also follows. □

3B. ***Vanishings of $H^1(\Sigma, \mathcal{O}_\Sigma(k))$ for k > 0.*** All of the remaining vanishings exploit the structure of $\widetilde{\Sigma}$ as a $\mathbb{P}^1$-bundle over S^2C. Given work of the second author in [Vermeire 2008b], the projective normality of Σ follows by exploiting Terracini recursion as a corollary of the next result.

Theorem 3.3. *If C satisfies Hypothesis 3.1, then $H^1(\Sigma, \mathcal{O}_\Sigma(2)) = 0$.*

Proof. We show that $H^2(\mathbb{P}^n, \mathcal{I}_\Sigma(2)) = 0$.

Since $\mathcal{O}(2H-E)$ is trivial along the fibers of $\pi : \widetilde{\Sigma} \to S^2C$, $\mathcal{O}_{\widetilde{\Sigma}}(2H-E) = \pi^*M$ for some line bundle M on S^2C [Hartshorne 1977, Exercise III.12.4]. We know from [Vermeire 2002, 3.6] that

$$\mathcal{O}_{\widetilde{\Sigma}}(2H-E) \otimes \mathcal{O}_Z \cong \pi^*M \otimes \mathcal{O}_Z \cong L \boxtimes L \otimes \mathcal{O}_Z(-2\Delta).$$

Further restricting π to the double cover $d : C \times C \to S^2C$, by the projection formula and part (5) of Lemma 2.3 we have

$$H^i(Z, L \boxtimes L \otimes \mathcal{O}_Z(-2\Delta)) = H^i(S^2C, M) \oplus H^i\Big(S^2C, M \otimes \mathcal{O}_{S^2C}\Big(-\frac{\delta}{2}\Big)\Big).$$

Again by the projection formula, we know that $H^i(\widetilde{\Sigma}, \mathcal{O}(2H-E)) = H^i(S^2C, M)$. By Lemma 2.5, we have $H^i(Z, L \boxtimes L \otimes \mathcal{O}_Z(-2\Delta)) \cong H^i(C, N_C^*(2))$. Thus we immediately have $H^2(Z, L \boxtimes L \otimes \mathcal{O}_Z(-2\Delta)) = 0$, but this in turn implies $H^2(S^2C, M) = H^2(\widetilde{\Sigma}, \mathcal{O}(2H-E)) = 0$.

Let $\mathcal{L}_L$ be the line bundle on S^2C such that $d^*\mathcal{L}_L = L \boxtimes L$ (see [Kouvidakis 2002, §2.1], for example). Now, as $L \boxtimes L \otimes \mathcal{O}_Z(-\Delta) = d^*\big(\mathcal{L}_L \otimes \mathcal{O}_{S^2C}\big(-\frac{\delta}{2}\big)\big)$, we know that

$$\begin{aligned} d_*\big((L \boxtimes L) \otimes \mathcal{O}_Z(-\Delta)\big) &= \Big[\mathcal{L}_L \otimes \mathcal{O}_{S^2C}\Big(-\frac{\delta}{2}\Big)\Big] \oplus \Big[\mathcal{L}_L \otimes \mathcal{O}_{S^2C}\Big(-2\frac{\delta}{2}\Big)\Big] \\ &= \Big[\mathcal{L}_L \otimes \mathcal{O}_{S^2C}\Big(-\frac{\delta}{2}\Big)\Big] \oplus M. \end{aligned}$$

Again by Lemma 2.5 we know that

$$H^1(C \times C, L \boxtimes L \otimes \mathcal{O}_Z(-\Delta)) = H^1(C, \Omega^1_{\mathbb{P}^n}(2) \otimes \mathcal{O}_C) = 0,$$

where the vanishing comes from quadratic normality of the embedding of C. Thus $H^1(S^2C, M) = H^1(\widetilde{\Sigma}, \mathcal{O}_{\widetilde{\Sigma}}(2H-E)) = 0$.

We see immediately that $H^2(B_1, \mathcal{I}_{\widetilde{\Sigma}}(2H)) = H^1(\widetilde{\Sigma}, \mathcal{O}_{\widetilde{\Sigma}}(2H))$, and from the sequence

$$0 \to \mathcal{O}_{\widetilde{\Sigma}}(2H-E) \to \mathcal{O}_{\widetilde{\Sigma}}(2H) \to \mathcal{O}_{\widetilde{\Sigma}}(2H) \otimes \mathcal{O}_E \to 0$$

and the (just proved) fact that $H^i(\widetilde{\Sigma}, \mathcal{O}_{\widetilde{\Sigma}}(2H-E)) = 0$ for $i = 1, 2$ implies further that $H^2(B_1, \mathcal{I}_{\widetilde{\Sigma}}(2)) = H^1(\widetilde{\Sigma}, \mathcal{O}_{\widetilde{\Sigma}}(2) \otimes \mathcal{O}_E)$. A straightforward computation gives

$$\begin{aligned} h^1(\widetilde{\Sigma}, \mathcal{O}_{\widetilde{\Sigma}}(2H) \otimes \mathcal{O}_E) &= h^1(C \times C, L^2 \boxtimes \mathcal{O}_C) \\ &= h^0(C, L^2) \cdot h^1(C, \mathcal{O}_C) \\ &= h^0(C, H^1(C, \mathcal{O}_C) \otimes L^2) \\ &= h^0(\mathbb{P}^n, R^2 g_* \mathcal{I}_{\widetilde{\Sigma}}(2)). \end{aligned}$$

Therefore, $h^2(B_1, \mathcal{I}_{\widetilde{\Sigma}}(2)) = h^0(\mathbb{P}^n, R^2 g_* \mathcal{I}_{\widetilde{\Sigma}}(2))$.

Interpreting what we have just shown in terms of the Leray–Serre spectral sequence associated to $g_*\mathcal{I}_{\widetilde{\Sigma}}(2)$, we have $h^2(B_1, \mathcal{I}_{\widetilde{\Sigma}}(2)) = \dim E_2^{0,2}$. We also know that $R^1 g_* \mathcal{I}_{\widetilde{\Sigma}}(2) = 0$ by the projection formula and Lemma 2.4 (3) and (5). Thus, at the E_2 level, where we have

$$0 \to E_2^{0,1} \xrightarrow{d_2} E_2^{2,0} \to 0 \quad \text{and} \quad 0 \to E_2^{0,2} \xrightarrow{d_2} E_2^{2,1} \to 0,$$

we see that $E_2^{2,0} = E_\infty^{2,0}$ and $E_2^{0,2} = E_\infty^{0,2}$ because $H^i(\mathbb{P}^n, R^1 g_* \mathcal{I}_{\widetilde{\Sigma}}(2)) = 0$. Recall [Weibel 1994, 5.2.6] that $H^2 := H^2(B_1, \mathcal{I}_{\widetilde{\Sigma}}(2))$ has a finite filtration

$$0 = F^3 H^2 \subseteq F^2 H^2 \subseteq F^1 H^2 \subseteq F^0 H^2 = H^2,$$

where $F^2 H^2 \cong E_\infty^{2,0}$ and $H^2/F^1 H^2 \cong E_\infty^{0,2}$.

Now, because $\dim H^2 = \dim E_2^{0,2} = \dim E_\infty^{0,2}$, we have $F^1 H^2 = 0$, but this implies that $F^2 H^2 = E_\infty^{2,0} = 0$, and hence that $E_2^{2,0} = 0$. □

In [Vermeire 2008b] it was shown that for the general embedding of degree at least $2g+3$, Σ is projectively normal; the only vanishing that could not be shown to always hold was $H^1(\mathbb{P}^n, \mathcal{I}_\Sigma(2)) = 0$. Theorem 3.3 allows us remove the hypothesis that the embedding must be general. The idea in [Vermeire 2008b] was to obtain a vanishing statement for direct image sheaves, and then to use those vanishings along with [Mumford 1966, p. 52, Corollary $1\frac{1}{2}$] to show that the cohomology groups along the fibers vanish. Of course, to make this work, we must find a flat morphism and a locally free sheaf so that the restriction of the sheaf to the fiber is precisely the vanishing statement we want. This is done using Theorem 2.1. However, note that in the proof we need to increase the degree of the embedding to at least $2g + 5$, so that curves of degree $2g + 3$ occur in the fibers.

Corollary 3.4. *Let $C \subset \mathbb{P}^n$ be a smooth curve embedded by a line bundle L of degree at least $2g + 3$. Then Σ is projectively normal.*

Proof. We know by combining [Vermeire 2008b, Proposition 12] with [Wahl 1997, 1.16] that $H^1(\mathbb{P}^n, \mathcal{I}_\Sigma(k)) = 0$ for $k = 1, 3$, and by [Vermeire 2008b, Corollary 11] that $H^1(\mathbb{P}^n, \mathcal{I}_\Sigma(k)) = 0$ for $k \geq 4$. Clearly, $H^1(\Sigma, \mathcal{O}_\Sigma(2)) = H^2(\mathbb{P}^n, \mathcal{I}_\Sigma(2))$. As these vanish by Theorem 3.3, we note that by Lemma 2.4 we have

$$H^2(B_2, \mathcal{O}(2H - E_1 - E_2)) = 0.$$

We further have $H^i(B_2, \mathcal{O}(2H - E_1 - E_2)) = 0$ for $i \geq 3$ by 5-regularity of $\mathcal{I}_\Sigma$.

By Lemma 2.4, along the fibers of $E_1 \to C$ we are computing $H^i(\mathbb{P}^{n-2}, \mathcal{I}_C(1))$; thus $R^i f_* \mathcal{O}_{E_1}(2H - E_1 - E_2) = 0$ for $i \geq 0$. Hence $H^i(B_2, \mathcal{O}_{E_1}(2H - E_1 - E_2))$ vanishes, showing that $H^i(B_2, \mathcal{O}(2H - 2E_1 - E_2)) = 0$ for $i \geq 2$.

Fixing a point $p \in C$, and applying an extension of Theorem 2.1 to $L(2p)$ (which now separates 6 points as L is nonspecial), we may blow up three times to get a resolution of Σ_2. In the notation of [Vermeire 2008b, Theorem 15], the

previous paragraph gives $R^i f_* \mathbb{O}_{E_1}(kH - 2E_1 - 2E_2 - E_3) = 0$ for $i \geq 2$, since the restriction of $\mathbb{O}_{E_1}(kH - 2E_1 - 2E_2 - E_3)$ to a fiber of $E_1 \to C$ is $\mathbb{O}(2H - 2E_1 - E_2)$ using the convention of Remark 2.2. It was shown in [Vermeire 2008b, Theorem 15] that $R^1 f_* \mathbb{O}_{E_1}(kH - 2E_1 - 2E_2 - E_3) = 0$, and so we know that H^1 along the fibers vanishes by [Mumford 1966, page 52, Corollary $1\frac{1}{2}$]. Thus we have $H^1(B_2, \mathbb{O}(2H - 2E_1 - E_2)) = 0$ and so, as above,

$$H^1(B_2, \mathbb{O}(2H - E_1 - E_2)) = H^1(\mathbb{P}^n, \mathcal{I}_\Sigma(2)) = 0. \qquad \square$$

Theorem 3.5. *If C satisfies Hypothesis 3.1, then $H^i(\Sigma, \mathbb{O}_\Sigma(1)) = 0$ for $i = 1, 2$.*

Proof. For $i = 1, 2$, we have $h^i(\Sigma, \mathbb{O}_\Sigma(1)) = h^{i+1}(\mathbb{P}^n, \mathcal{I}_\Sigma(1))$ and $H^{i+1}(\mathbb{P}^n, \mathcal{I}_\Sigma(1))$ is isomorphic to $H^{i+1}(B_2, \mathbb{O}_{B_2}(H - E_1 - E_2))$ by the last part of Lemma 2.4. Using Equation (1) twisted by H, the projection formula gives $R^i h_*(\mathbb{O}_{E_2}(H - E_1)) = R^i h_*(\mathbb{O}_{E_2}) \otimes \mathbb{O}_{\widetilde{\Sigma}}(H - E_1)$. By part (6) of Lemma 2.3 the restriction of $\mathbb{O}(H - E_1)$ to the fibers of $\widetilde{\Sigma} \to S^2 C$ is isomorphic to $\mathbb{O}_{\mathbb{P}^1}(-1)$, hence $h^i(\widetilde{\Sigma}, \mathbb{O}(H - E_1)) = 0$ for all i, which implies that $h^i(E_2, \mathbb{O}_{E_2}(H - E_1)) = 0$. We therefore have $h^{i+1}(B_2, \mathbb{O}_{B_2}(H - E_1 - E_2)) = h^{i+1}(B_2, \mathbb{O}_{B_2}(H - E_1))$.

We see that $R^i f_*(\mathbb{O}_{B_2}(H - E_1)) = 0$ for $i \geq 1$ and $f_*(\mathbb{O}_{B_2}(H - E_1)) = \mathcal{I}_C(1)$ by [Bertram et al. 1991, 1.2,1.4]. Thus $h^{i+1}(B_2, \mathbb{O}_{B_2}(H - E_1)) = h^{i+1}(\mathbb{P}^n, \mathcal{I}_C(1)) = 0$. $\square$

Remark 3.6. In the case of a canonical curve, we have

$$h^0(\Sigma, R^1 g_* \mathbb{O}_{\widetilde{\Sigma}}(H)) = h^1(C, \mathbb{O}_C) \cdot h^0(C, \mathbb{O}_C(1)) = g^2$$

while

$$h^1(\widetilde{\Sigma}, \mathbb{O}_{\widetilde{\Sigma}}(H)) = h^1(\mathbb{O}_C) \cdot h^0(\mathbb{O}_C(1)) + h^0(\mathbb{O}_C) \cdot h^1(\mathbb{O}_C(1)) = g^2 + 1.$$

Therefore using the 5-term sequence (2) again we see that $h^1(\Sigma, \mathbb{O}_\Sigma(1)) \geq 1$ (in fact, equality can be shown to hold). Thus the secant variety to a canonical curve of Clifford index at least 3 (for example, the generic curve of genus ≥ 7) is *never* ACM.

Note the secant variety of a canonical curve $C \subset \mathbb{P}^4$ is a hypersurface of degree 16, hence is ACM, but such curves have Clifford index ≤ 2.

3C. *Vanishings for $k = 0$.* We now consider the vanishing of $H^i(\Sigma, \mathbb{O}_\Sigma)$ where $i = 1, 2$.

Proposition 3.7. *If C satisfies Hypothesis 3.1, then $H^1(\Sigma, \mathbb{O}_\Sigma) = 0$.*

Proof. Associated to the morphism $g : B_1 \to \mathbb{P}^n$ we have

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H^1(g_*\mathbb{O}_{\widetilde{\Sigma}}) & \longrightarrow & H^1(\mathbb{O}_{\widetilde{\Sigma}}) & \xrightarrow{\gamma} & H^0(R^1g_*\mathbb{O}_{\widetilde{\Sigma}}) & \longrightarrow & \\
 & & \downarrow & & \downarrow{\scriptstyle \alpha} & & \downarrow & & \\
0 & \longrightarrow & H^1(g_*\mathbb{O}_Z) & \longrightarrow & H^1(\mathbb{O}_Z) & \xrightarrow{\beta} & H^0(R^1g_*\mathbb{O}_Z) & \longrightarrow & 0
\end{array}$$

where the horizontal maps come from 5-term exact sequences.

As $Z \cong C \times C$, we see that the inclusion and projection in the bottom row come from the Künneth formula. The map $\alpha : H^1(\mathbb{O}_{\widetilde{\Sigma}}) \to H^1(\mathbb{O}_Z)$ is an inclusion because it is the diagonal mapping $\alpha : H^1(\mathbb{O}_{S^2C}) \to H^1(C, \mathbb{O}_C) \oplus H^1(C, \mathbb{O}_C)$ induced by the pull-back of $d : Z \to S^2C$ to $\widetilde{\Sigma}$. We conclude that the composition $\beta \circ \alpha$ is an isomorphism. Moreover, as $H^0(R^1g_*\mathbb{O}_{\widetilde{\Sigma}}) \to H^0(R^1g_*\mathbb{O}_Z)$ is an isomorphism, we see that γ is an isomorphism by commutativity of the diagram. Hence, $H^1(g_*\mathbb{O}_{\widetilde{\Sigma}}) = H^1(\Sigma, \mathbb{O}_\Sigma) = 0$. □

Proposition 3.8. *If C satisfies Hypothesis 3.1, then $H^2(\Sigma, \mathbb{O}_\Sigma) = 0$.*

Proof. We note that $h^i(\Sigma, \mathbb{O}_\Sigma) = h^{i+1}(\mathbb{P}^n, \mathcal{I}_\Sigma)$ for $i = 1, 2$. Moreover, by part (6) of Lemma 2.4 we have $h^j(\mathbb{P}^n, \mathcal{I}_\Sigma) = h^j(B_2, \mathbb{O}_{B_2}(-E_1 - E_2))$. Therefore, the result follows if we can show that $h^2(B_2, \mathbb{O}_{B_2}(-E_1 - E_2)) = h^3(B_2, \mathbb{O}_{B_2}(-E_1 - E_2))$, since we know by Proposition 3.7 that $h^2(B_2, \mathbb{O}_{B_2}(-E_1 - E_2)) = 0$.

To this end, consider the long exact sequence associated to Equation (1) on page 451. The result will follow if $h^2(B_2, \mathbb{O}_{B_2}(-E_1)) = h^2(E_2, \mathbb{O}_{E_2}(-E_1))$ is equal to g and $h^1(E_2, \mathbb{O}_{E_2}(-E_1)) = h^3(B_2, \mathbb{O}_{B_2}(-E_1)) = 0$.

From the sequence $0 \to \mathbb{O}_{B_2}(-E_1) \to \mathbb{O}_{B_2} \to \mathbb{O}_{E_1} \to 0$ we see immediately that $h^i(B_2, \mathbb{O}_{B_2}(-E_1)) = g$ if $i = 2$ and is zero otherwise as $R^j f_*\mathbb{O}_{B_2} = 0$ for $j > 0$ from Lemma 2.4 (2) and $h^j(\mathbb{O}_{E_1}) = h^j(\mathbb{O}_C)$ for all j.

We compute the cohomology of $\mathbb{O}_{E_2}(-E_1)$ using Equation (1). Using the projection formula and part (4) of Lemma 2.4, we see that $R^i h_*\mathbb{O}_{E_2}(-E_1) = 0$ for $i > 0$. Thus, $H^i(\mathbb{O}_{E_2}(-E_1)) \cong H^i(\mathbb{O}_{\widetilde{\Sigma}}(-E_1))$.

To compute $H^i(\mathbb{O}_{\widetilde{\Sigma}}(-E_1))$, observe that

$$0 \to \pi_*\mathbb{O}_{\widetilde{\Sigma}}(-E_1) \to \pi_*\mathbb{O}_{\widetilde{\Sigma}} \to \pi_*\mathbb{O}_Z \to R^1\pi_*\mathbb{O}_{\widetilde{\Sigma}}(-E_1) \to 0$$

with all remaining higher direct images vanishing by parts (2) and (4) of Lemma 2.3 and $\pi_*\mathbb{O}_{\widetilde{\Sigma}}(-E_1) = 0$ by part (6).

As $\mathrm{Hom}_{\mathbb{O}_{S^2C}}\left(\mathbb{O}_{S^2C}, \mathbb{O}_{S^2C}\left(-\frac{\delta}{2}\right)\right)$ is trivial, this gives rise to the natural inclusion

$$\pi_*\mathbb{O}_{\widetilde{\Sigma}} \cong \mathbb{O}_{S^2C} \hookrightarrow \mathbb{O}_{S^2C} \oplus \mathbb{O}_{S^2C}\left(-\frac{\delta}{2}\right) \cong \pi_*\mathbb{O}_Z,$$

and we see that $H^i(\widetilde{\Sigma}, \mathbb{O}_{\widetilde{\Sigma}}) \hookrightarrow H^i(Z, \mathbb{O}_Z)$. In fact, using the long exact sequence on $\widetilde{\Sigma}$, these inclusions imply that $H^i(Z, \mathbb{O}_Z) \cong H^i(\widetilde{\Sigma}, \mathbb{O}_{\widetilde{\Sigma}}) \oplus H^{i+1}(\widetilde{\Sigma}, \mathbb{O}_{\widetilde{\Sigma}}(-E_1))$.

As $h^1(S^2C, \mathbb{O}_{S^2C}) = g$ and $h^2(S^2C, \mathbb{O}_{S^2C}) = \binom{g}{2}$ by [Macdonald 1962], using the sequence $0 \to \mathbb{O}_{\widetilde{\Sigma}}(-E_1) \to \mathbb{O}_{\widetilde{\Sigma}} \to \mathbb{O}_Z \to 0$ together with the Künneth formula and that $H^i(\widetilde{\Sigma}, \mathbb{O}_{\widetilde{\Sigma}}) \cong H^i(S^2C, \mathbb{O}_{S^2C})$, implies that $h^2(E_2, \mathbb{O}_{E_2}(-E_1)) = g$, and that $h^3(E_2, \mathbb{O}_{E_2}(-E_1)) = \binom{g+1}{2}$. Further, we see immediately that $H^1(\widetilde{\Sigma}, \mathbb{O}_{\widetilde{\Sigma}}(-E_1))$ is 0, since $H^0(Z, \mathbb{O}_Z) \cong H^0(\widetilde{\Sigma}, \mathbb{O}_{\widetilde{\Sigma}}) \oplus H^1(\widetilde{\Sigma}, \mathbb{O}_{\widetilde{\Sigma}}(-E_1))$. □

Proof of Theorem 1. As explained at the beginning of the section, in order to show that Σ is ACM we are left to show that $H^i(\Sigma, \mathbb{O}_\Sigma(k)) = 0$ for $i = 1, 2$ and all $k \le 3 - i$.

The vanishings for $k < 0$ were shown in Theorem 3.2. The vanishing for $i = 1$ and $k = 0$ is Proposition 3.7, while $i = 2$ and $k = 0$ is Proposition 3.8. Both vanishings for $k = 1$ are found in Theorem 3.5. Finally, the vanishing for $i = 1$ and $k = 2$ is found in Theorem 3.3. □

As an immediate consequence of the proof of Proposition 3.8 we get a sharpening of the regularity result of the second author in [Vermeire 2008b].

Corollary 3.9. *If C satisfies Hypothesis 3.1, then $\mathcal{I}_\Sigma$ has regularity* 3 *if C is rational and regularity* 5 *otherwise.*

Proof. Running the long exact sequence associated to Equation (1) in the proof of Proposition 3.8 shows that $h^4(\mathbb{P}^n, \mathcal{I}_\Sigma) = \binom{g+1}{2}$. □

4. Betti diagrams

In this section we attempt to paint a picture of the shape of the Betti diagram of S_Σ that parallels the discussion of the Betti diagram of a high degree curve in Chapter 8 of [Eisenbud 2005]. In Section 4A we use the fact that Σ is ACM to use duality and algebraic techniques to compute the extremal nontrival Betti numbers, $\beta_{1,3}$ (Proposition 4.4) and $\beta_{n-3,n+1}$ (Corollary 4.1) as well as the Hilbert polynomial. Independent of the Cohen–Macaulay property, we prove a nonvanishing result about the length of the degree $(k+2)$ linear strand of S_{Σ_k} using determinantal methods and Koszul homology (Proposition 4.10 and Theorem 1.2) in Section 4B.

4A. ***Computing Betti numbers.*** We begin with a simple consequence of duality. As Σ is ACM, dualizing a resolution of S_Σ and shifting by $-n-1$ gives a resolution of the canonical module, which is defined to be $\omega_\Sigma = \operatorname{Ext}^{n-3}(S_\Sigma, S(-n-1)) = \bigoplus_{d\in\mathbb{Z}} H^0(\Sigma, \omega_\Sigma^\circ \otimes L^d)$ where $\omega_\Sigma^\circ = \mathscr{E}xt^{n-3}_{\mathbb{P}^n}(\mathbb{O}_\Sigma, \mathbb{O}_{\mathbb{P}^n}(-n-1))$ is the dualizing sheaf of Σ. Therefore, the last few Betti numbers of S_Σ are the first few of ω_Σ. As an immediate consequence of Corollary 3.9 we see that the number of minimal generators of ω_Σ in degree 0 is $\binom{g+1}{2}$ and hence depends only on g, independent of the embedding (as long as the degree is at least $2g+3$).

Corollary 4.1. *If C satisfies Hypothesis 3.1, then $\beta_{n-3,n+1} = \binom{g+1}{2}$.*

Proof. If $g = 0$, we know that $\beta_{n-3,n+1} = 0$. If $g > 0$, then Corollary 3.9 shows that $\operatorname{reg} S_\Sigma = 4$. Hence, the a-invariant of S_Σ is 0, so $h^0(\omega_\Sigma^\circ) = \beta_{0,0}(\omega_\Sigma) = \beta_{n-3,n+1}(S_\Sigma)$. By Serre duality,

$$h^0(\Sigma, \omega_\Sigma^\circ) = h^3(\Sigma, \mathbb{O}_\Sigma) = h^4(\mathbb{P}^n, \mathcal{I}_\Sigma) = \binom{g+1}{2}. \qquad \square$$

Knowing $\beta_{n-3,n+1}$ allows us to compute the Hilbert polynomial of S_Σ and to gather information about other Betti numbers inductively. To begin this process, fix general linear forms $H_1, H_2, H_3, H_4 \in S$. Let X be the intersection of Σ with the hyperplanes determined by H_1 and H_2 and $M = S_\Sigma/\langle H_1, H_2, H_3, H_4\rangle$. Using Corollary 4.1 we may compute the genus of X from which formulae for the Hilbert polynomial of S_Σ and $\beta_{1,3}$ follow. First we gather together basic facts about X.

Lemma 4.2. *If C satisfies Hypothesis 3.1, the variety X is a smooth curve of degree $D = \binom{d-1}{2} - g$ embedded in $\mathbb{P}^{n-2}$ via the complete linear series associated to a line bundle A and $S_X = S_\Sigma/\langle H_1, H_2\rangle$.*

Proof. All the statements follow immediately from the fact that Σ is ACM. The only thing that may not be immediate to the reader is that $\deg(\Sigma) = \binom{d-1}{2} - g$, though this is certainly well-known to experts.

To see this, take a generic $L = \mathbb{P}^{n-3} \subset \mathbb{P}^n$ and consider the induced projection $\pi : \mathbb{P}^n \dashrightarrow \mathbb{P}^2$. Every point of intersection of L with Σ corresponds to a node of $\pi(C)$. It is well-known that the number of nodes is $\binom{d-1}{2} - g$. $\square$

We will denote the genus of X by G. To compute G we compare the Hilbert function of S_X to that of successive quotients by H_1 and H_2.

Proposition 4.3. *If C satisfies Hypothesis 3.1, the genus of X is*

$$G = \tfrac{1}{2}(d-2)(d+2g-3).$$

Proof. Since S_X is 4-regular, $h^0(X, A^m) = mD - G + 1$ for $m \geq 3$. We also know that the ideal of Σ is empty in degree less than three, since a quadric hypersurface vanishing on Σ must vanish twice on C, but this is not possible since C is nondegenerate. Therefore, we can fill in the table of Hilbert functions below where each entry in the first two columns of the table is the sum of the entries directly above and to the right.

	$S_\Sigma/\langle H_1, H_2\rangle$	$S_\Sigma/\langle H_1, H_2, H_3\rangle$	M
2	$\binom{n}{2}$	$\binom{n-1}{2}$	$\binom{n-2}{2}$
3	$3D-G+1$	$3D-G+1-\binom{n}{2}$	$3D-G+1-\binom{n}{2}-\binom{n-1}{2}$
4	$4D-G+1$	D	$G-2D-1+\binom{n}{2}$

But computing graded Betti numbers via Koszul homology as in Proposition 2.7 in [Eisenbud 2005] shows that $\dim M_4 = \beta_{n-3,n+1} = \binom{g+1}{2}$. Substituting $n = d-g$ and simplifying $G = 2D+1-\binom{d-g}{2}+\binom{g+1}{2}$ gives the desired result. □

The computation of the Hilbert polynomial $P_\Sigma(m)$ follows easily.

Proof of Theorem 1.1. Using [Eisenbud 2005, Theorem 4.2], the Hilbert polynomial and Hilbert function of S_Σ agree for $m \geq \operatorname{reg} S_\Sigma + \text{proj-dim}\, S_\Sigma - n \geq 4-3 = 1$. Write

$$P_\Sigma(m) = \sum_{i=0}^{3} \alpha_i \binom{m+i-1}{i}.$$

As X is gotten by cutting down by a regular sequence of two hyperplanes, $P_X(m) = P_\Sigma(m) - P_\Sigma(m-1) - P_\Sigma(m-2) = \alpha_3 m + \alpha_2$. Since X is a curve of degree D and genus, G, we see that $\alpha_3 = D$ and $\alpha_2 = 1-G$. Since the ideal of Σ is empty in degrees 1and 2, we see that $P_\Sigma(1) = n+1$ and $P_\Sigma(2) = \binom{n+2}{2}$ and the result follows. □

We compute $\beta_{1,3}$ and get a relationship on Betti numbers at the beginning of the resolution.

Proposition 4.4. *If C satisfies Hypothesis 3.1, we have*

$$\beta_{1,3} = \binom{n+1}{3} - (d-2)n - 3g + 1 \quad \textit{and} \quad \beta_{2,4} = \beta_{1,4} + \beta_{1,3}(n+1) - \binom{n+4}{n} + P_\Sigma(4).$$

Proof. As observed above, the Hilbert polynomial and function of S_X agree in degree 3 and higher. Since $\beta_{1,3} = \binom{n+1}{3} - (S_X)_3$ we get $\beta_{1,3} = \binom{n+1}{3} - 3D + G - 1$, and this simplifies to the given formula.

By [Eisenbud 2005, Corollary 1.10] we get a formula for the Hilbert function of S_Σ in terms of graded Betti numbers:

$$(S_\Sigma)_m = \sum_{i \geq 0,\, j \in \mathbb{Z}} (-1)^i \beta_{i,j} \binom{n+m-j}{n}.$$

When $m = 4$, we must have $j \leq 4$ for $\beta_{i,j}$ to contribute to the sum. As we know that the ideal of X does not contain any forms of degree < 3, the result follows. □

Remark 4.5. In the formula for $\beta_{2,4}$ we have an explicit formula for each term except $\beta_{1,4}$, which is the number of quartic minimal generators of I_Σ. For large d, we know $\beta_{1,4} = 0$, as the ideal of Σ is generated by cubics [Vermeire 2008a].

Using duality, we get a similar result for the tail of the resolution.

Theorem 4.6. *If C satisfies Hypothesis 3.1, the tail of the graded Betti diagram of S_Σ has the form*

	$n-5$	$n-4$	$n-3$
0	$-$	$-$	$-$
1	$-$	$-.$	$-$
2	$*$	$*$	A
3	$*$	$A+B+\binom{g+1}{2}\binom{n}{2}-\binom{g}{2}(n-3)(n-1)-G$	C
4	B	$C+\binom{g}{2}(n-3)$	$\binom{g+1}{2}$

Proof. Let $A=\beta_{n-3,n-1}$, $B=\beta_{n-5,n-1}$ and $C=\beta_{n-3,n}$. We know that the canonical module ω_X is $\bigoplus_{n\in\mathbb{Z}} H^0(K_X\otimes A^n)$, where K_X is the canonical line bundle of X. By duality, $\beta_{i,j}(\omega_X)=\beta_{n-3-i,n-1-j}(S_\Sigma)$.

By [Eisenbud 2005, Corollary 1.10] we get a formula for the Hilbert function of ω_X in terms of graded Betti numbers:

$$\mathrm{h}^0(K_X\otimes A^m)=\sum_{i\geq 0,\, j\in\mathbb{Z}}(-1)^i\beta_{i,j}(\omega_X)\binom{n-2+m-j}{n-2}.$$

By Serre duality and Riemann–Roch $h^0(K_X\otimes A^{-1})=h^1(A)=g(d-2)$. Thus, $g(d-2)=(n-1)\binom{g+1}{2}+C-\beta_{n-4,n}$, which gives the desired satement. The second statement follows from the equation

$$G=\binom{g+1}{2}\binom{n}{2}-\binom{g}{2}(n-3)(n-1)+B-\beta_{n-4,n-1}+A. \qquad \square$$

In particular, if $g=2$, we have the following immediate corollary.

Corollary 4.7. *If C satisfies Hypothesis 3.1 and $g=2$, the tail has the form*

	$n-5$	$n-4$	$n-3$
0	$-$	$-$	$-$
1	$-$	$-$	$-$
2	$*$	$*$	A
3	$*$	$A+B+d-5$	C
4	B	$C+d-5$	$\binom{g+1}{2}$

Based on Example 1.4 and the following example, we expect $A=B=C=0$.

Example 4.8. Suppose C is a genus 2 curve of degree 12 in $\mathbb{P}^{10}$. We use Example (c) of [Eisenbud et al. 1988] to compute the ideal of the curve determinantally in Macaulay 2 over the field of rational numbers. We then used the code created to implement ideas in [Sidman and Sullivant 2006] to compute the least degree pieces of the ideals of the secant varietes. Computing the degree, dimension, and

projective dimension of the resulting ideals showed that we had actually computed the secant ideals.

```
            0  1   2   3   4   5   6   7  8 9
      total: 1 43 222 558 840 798 468 147 17 2
          0: 1  -   -   -   -   -   -   -  - -
          1: - 43 222 558 840 798 468 147  8 -
          2: -  -   -   -   -   -   -   -  9 2
```

The Betti diagrams for S_{Σ_1} and S_{Σ_2} are

```
       0  1   2   3   4   5  6 7          0  1  2  3  4 5
total: 1 70 283 483 413 155 14 3  total: 1 41 94 61 11 4
    0: 1  -   -   -   -   -  - -      0: 1  -  -  -  - -
    1: -  -   -   -   -   -  - -      1: -  -  -  -  - -
    2: - 70 283 483 413 155  - -      2: -  -  -  -  - -
    3: -  -   -   -   -   -  7 -      3: - 41 94 61  - -
    4: -  -   -   -   -   -  7 3      4: -  -  -  -  - -
                                      5: -  -  -  -  6 -
                                      6: -  -  -  -  5 4
```

4B. ***The length of the first nonzero strand.*** We now turn to the consideration of a lower bound on the length of the minimal degree linear strand of the ideal of Σ_k, essentially following Chapter 8B.2 of [Eisenbud 2005], building on Green and Lazarsfeld's proof of nonvanishing for curves. In this section we will assume the following:

Hypothesis 4.9. *C is a smooth curve of genus g and degree d embedded into $\mathbb{P}^n$ via a line bundle L that factors as $L = L_1 \otimes L_2$, where $|L_1| = s$ and $|L_2| = t$, with $1 \le s \le t$.*

First note that part of the proof of [Eisenbud 2005, Theorem 8.12] which is given in the case $k = 0$ goes through for arbitrary k and allows us to see easily that the degree $k+2$ linear strand of the Betti diagram of Σ_k has length at least p.

Proposition 4.10. *Under the conditions of Hypothesis 4.9, if $d \ge 2g + 2k + 1 + p$, then $\beta_{p,k+1+p} \ne 0$.*

Proof. Factor L so that $\deg L_1 \ge g+k+1$ and $\deg L_2 = g+k+p$. By Riemann–Roch $h^0(C, L_1) \ge k+2$ and $h^0(C, L_2) \ge k+p+1$. Thus multiplication of sections gives rise to a 1-generic matrix of linear forms with at least $(k+2)$ rows and $(k+1+p)$ columns. Delete rows and columns to get a $(k+2) \times (k+1+p)$ matrix which is still 1-generic as an equation making a generalized entry of the smaller matrix zero also makes a generalized entry of the larger matrix zero. The maximal minors of the smaller matrix are resolved by an Eagon–Northcott complex of length p. The resolution of this ideal is a subcomplex of the ideal of Σ_k. The result follows. □

We can get a better lower bound by exhibiting an explicit nontrivial cycle in the Koszul homology of S_{Σ_k} to show that $\beta_{s+t-2k-1,s+t-k}$ does not vanish.

In [Eisenbud 2005, Theorem 8.15], the following result is stated for $k = 1$:

Theorem 4.11 [Eisenbud 2005, Theorem 8.15]. *If $I \subset S$ is a homogenous ideal which contains no forms of degree less than or equal to k, then $\beta_{i,i+k} \neq 0$ if and only if there exists $\gamma \in \bigwedge^i S^{n+1}(-i)$ of degree $i+k$ whose image under the differential of the Koszul complex is nonzero and lies in $I \otimes \bigwedge^{i-1} S^{n+1}(-i+1)$.*

Proof. The proof goes through as in [Eisenbud 2005], replacing one by k everywhere. □

We show that [Eisenbud 2005, Theorem 8.13] can be extended to the case of minors of arbitrary size.

Theorem 4.12. *Suppose that A is an $(s+1) \times (t+1)$ matrix of linear forms with $s+1 \geq k+2$. If the $s+t+1$ elements in the union of the entries of the zeroth row and column are linearly independent and some $(k+2)$ minor involving the zeroth row or column does not vanish, then $\beta_{s+t-2k-1,s+t-k}(S/I_{k+2})$ does not vanish.*

Proof. By Theorem 4.11 it suffices to construct an explicit cycle

$$\gamma \in \bigwedge^{s+t-2k-1} S^{n+1}(-s-t+2k+1)$$

of degree $s+t-k$ whose image under the differential is a nonzero element of $I_{k+1} \otimes \bigwedge^{s+t-2k-2} S^{n+1}(-s-t+2k+2)$. To do this we set some notation.

By our hypotheses, the matrix A has the form

$$A = \begin{pmatrix} a_{0,0} & a_{0,1} & \cdots & a_{0,t} \\ a_{1,0} & a_{1,1} & \cdots & a_{1,t} \\ \cdot & & & \\ a_{s,0} & a_{s,1} & \cdots & a_{s,t} \end{pmatrix} = \begin{pmatrix} x_0 & x_1 & \cdots & x_t \\ x_{1+t} & a_{1,1} & \cdots & x_{1,t} \\ \cdot & & & \\ x_{s+t} & a_{s,1} & \cdots & a_{s,t} \end{pmatrix}.$$

Since the x_i are linearly independent they may be chosen as part of a basis for S_1, and we may choose a basis $\{e_i\}$ for S^{n+1} so that $\partial(e_i) = x_i$ for $i = 0, \ldots, s+t$.

Let $\sigma \subset \{1, \ldots, s\}$ and $\tau \subset \{0, \ldots, t\}$ be sets of size $k+1$ and σ_t denote the set gotten by adding t to each element of σ. Let $e_{\sigma_t,\tau}$ be the wedge product of $\{e_0, \ldots, e_{s+t}\} \backslash (\sigma_t \cup \tau)$ in the standard order. Note that $e_{\sigma_t,\tau} \in \bigwedge^{s+t-2k-2} S^{n+1}$.

We define an element γ which will serve as our nonzero cycle. Informally, it is the signed sum of all of the $(k+1)$-minors of A which do not involve the top row, each indexed by an element $e_{\sigma_t,\tau}$ in a natural way. More precisely,

$$\gamma = \sum_{\sigma,\tau} (-1)^{(\sigma+\tau)+t(k+1)} \det(\sigma \,|\, \tau) e_{\sigma_t,\tau},$$

where we define $\sigma + \tau$ to be the sum of the union of the elements in σ and τ and $\det(\sigma \,|\, \tau)$ is the minor of A gotten by using the rows in σ and the columns in τ.

To complete the proof we need to show that the coefficients of $\partial(\gamma)$ are all of the $(k+2)$-minors of A involving the zeroth row or column. The only basis elements which can have nonzero coefficients are $e_{\sigma'_t,\tau}$, where $\sigma' \subset \{1,\dots,s\}$ and $|\sigma'| = k+2$ and $e_{\sigma_t,\tau'}$ where $\tau' \subset \{0,\dots,t\}$ also has size $k+2$.

To understand the coefficient of $e_{\sigma_t,\tau'}$, note that there are $k+2$ basis elements $e_{\sigma_t,\tau}$ whose images under the differential could contain $e_{\sigma_t,\tau'}$ with nonzero coefficient. Since $\partial(e_i) = x_i$ for $i = 0,\dots,t$, we see that the coefficient of $e_{\sigma_t,\tau'}$, is $\pm\det(\sigma_t \cup \{0\} \mid \tau')$ where the differential expands the determinant along the zeroth row.

Similarly, the coefficient of $e_{\sigma'_t,\tau}$, is $\pm\det(\sigma'_t \mid \tau \cup \{0\})$, the differential expands the determinant along the zeroth column. (If $0 \in \tau$, we repeat the zeroth column twice and get coefficient zero.) □

We are now ready to prove Theorem 1.2, which is analogous to [Eisenbud 2005, Theorem 8.12].

Proof of Theorem 1.2. We will construct a matrix A corresponding to the factorization of $L = L_1 \otimes L_2$ by choosing bases carefully as in the proof of [Eisenbud 2005, Theorem 8.12]. Let B_i be the base locus of L_i. Fix a basis $\beta_0,\dots,\beta_t$ of $H^0(L_2)$ so that the divisor of β_i is $B_2 + D_i$ where D_i and B_2 have disjoint support. Let D be the divisor consisting of the union of the points in the divisors determined by $\beta_0,\dots,\beta_t$. Since $L_1(-B_1)$ is base-point free, a general element is disjoint from D and from B_1. Therefore we can pick a basis $\alpha_0,\dots,\alpha_s$ so that the divisor of each α_i is $B_1 + E_i$ where E_i is disjoint from D and from B_1.

We will show that the $s+t+1$ elements in the union of any row and any column of the corresponding matrix A are linearly independent. Without loss of generality, consider the top row and leftmost column. We know that the elements of the column $\alpha_0\beta_0, \alpha_1\beta_0, \dots, \alpha_s\beta_0$ are linearly independent, as are the elements $\alpha_0\beta_0, \alpha_0\beta_1, \dots, \alpha_0, \beta_t$. Suppose γ is an element in the intersection of the two vector spaces with these bases. This implies that the divisor of γ contains the divisor of α_0 and of β_0. This implies that it must contain D_0 and E_0 as well as the base loci B_1 and B_2. Since $\gamma \in H^0(L)$ and $\alpha_0\beta_0 \in H^0(L)$, then one is a scalar multiple of the other. Therefore, we conclude that the union of the elements in the top row and first column form a set of $s+t+1$ linearly independent elements.

As the matrix A is 1-generic, we know that the ideal generated by its maximal minors has the expected codimension and hence some $(k+2)$-minor does not vanish. Permuting rows and columns we can assume it is in the upper lefthand corner. Since $I_{k+2} \subseteq I_{\Sigma_k}$, the result follows from Theorems 4.11 and 4.12.

If $\deg L \geq 2g+2k+p+1$, then L can be factored as the product of line bundles L_1 with degree at least $g+k+\lfloor(1+p)/2\rfloor$ and L_2 with degree greater than or equal to $\deg L_1$. If L_1 and L_2 are generic, then each has at least $k+2$ sections. □

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References

[Barth et al. 2004] W. P. Barth, K. Hulek, C. A. M. Peters, and A. Van de Ven, *Compact complex surfaces*, 2nd ed., Ergebnisse der Math. (3) **4**, Springer, Berlin, 2004. MR 2004m:14070 Zbl 1036.14016

[Bertram 1992] A. Bertram, "Moduli of rank-2 vector bundles, theta divisors, and the geometry of curves in projective space", *J. Differential Geom.* **35**:2 (1992), 429–469. MR 93g:14037 Zbl 0787.14014

[Bertram et al. 1991] A. Bertram, L. Ein, and R. Lazarsfeld, "Vanishing theorems, a theorem of Severi, and the equations defining projective varieties", *J. Amer. Math. Soc.* **4**:3 (1991), 587–602. MR 92g:14014 Zbl 0762.14012

[Graf von Bothmer and Hulek 2004] H.-C. Graf von Bothmer and K. Hulek, "Geometric syzygies of elliptic normal curves and their secant varieties", *Manuscripta Math.* **113**:1 (2004), 35–68. MR 2006b:14009 Zbl 1053.14032

[Eisenbud 1995] D. Eisenbud, *Commutative algebra: with a view toward algebraic geometry*, Graduate Texts in Math. **150**, Springer, New York, 1995. MR 97a:13001 Zbl 0819.13001

[Eisenbud 2005] D. Eisenbud, *The geometry of syzygies: a second course in commutative algebra and algebraic geometry*, Graduate Texts in Math. **229**, Springer, New York, 2005. MR 2005h:13021 Zbl 1066.14001

[Eisenbud et al. 1988] D. Eisenbud, J. Koh, and M. Stillman, "Determinantal equations for curves of high degree", *Amer. J. Math.* **110**:3 (1988), 513–539. MR 89g:14023 Zbl 0681.14027

[Eisenbud et al. 2005] D. Eisenbud, M. Green, K. Hulek, and S. Popescu, "Restricting linear syzygies: algebra and geometry", *Compos. Math.* **141**:6 (2005), 1460–1478. MR 2006m:14072 Zbl 1086.14044

[Fisher 2006] T. Fisher, "The higher secant varieties of an elliptic normal curve", preprint, 2006, Available at http://www.dpmms.cam.ac.uk/~taf1000/papers/hsecenc.html.

[Grayson and Stillman] D. R. Grayson and M. E. Stillman, Macaulay 2, a software system for research in algebraic geometry, Available at http://www.math.uiuc.edu/Macaulay2.

[Green 1984] M. L. Green, "Koszul cohomology and the geometry of projective varieties", *J. Differential Geom.* **19**:1 (1984), 125–171. MR 85e:14022 Zbl 0559.14008

[Green and Lazarsfeld 1985] M. Green and R. Lazarsfeld, "On the projective normality of complete linear series on an algebraic curve", *Invent. Math.* **83**:1 (1985), 73–90. MR 87g:14022 Zbl 0594.14010

[Green and Lazarsfeld 1988] M. Green and R. Lazarsfeld, "Some results on the syzygies of finite sets and algebraic curves", *Compositio Math.* **67**:3 (1988), 301–314. MR 90d:14034 Zbl 0671.14010

[Grothendieck 1957] A. Grothendieck, "Sur quelques points d'algèbre homologique", *Tôhoku Math. J.* (2) **9** (1957), 119–221. MR 21 #1328 Zbl 0118.26104

[Hartshorne 1977] R. Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics **52**, Springer, New York, 1977. MR 57 #3116 Zbl 0531.14001

[Kouvidakis 2002] A. Kouvidakis, "On some results of Morita and their application to questions of ampleness", *Math. Z.* **241**:1 (2002), 17–33. MR 2003i:14006 Zbl 1031.14010

[Lazarsfeld 2004] R. Lazarsfeld, *Positivity in algebraic geometry, I: Classical setting: line bundles and linear series*, Ergebnisse der Math. (3) **48**, Springer, Berlin, 2004. MR 2005k:14001a Zbl 1093.14501

[Macdonald 1962] I. G. Macdonald, "Symmetric products of an algebraic curve", *Topology* **1** (1962), 319–343. MR 27 #1445 Zbl 0121.38003

[Mumford 1966] D. Mumford, *Lectures on curves on an algebraic surface*, Annals of Mathematics Studies **59**, Princeton University Press, 1966. MR 35 #187 Zbl 0187.42701

[Sidman and Sullivant 2006] J. Sidman and S. Sullivant, "Prolongations and computational algebra", preprint, 2006. to appear in Canadian Math. J. arXiv arxiv:math.AC/0611696

[Vermeire 2001] P. Vermeire, "Some results on secant varieties leading to a geometric flip construction", *Compositio Math.* **125**:3 (2001), 263–282. MR 2002c:14027 Zbl 1056.14016

[Vermeire 2002] P. Vermeire, "On the regularity of powers of ideal sheaves", *Compositio Math.* **131**:2 (2002), 161–172. MR 2003f:14016 Zbl 1083.14503

[Vermeire 2008a] P. Vermeire, "Generation and syzygies of the first secant variety", preprint, 2008. arXiv 0809.4463

[Vermeire 2008b] P. Vermeire, "Regularity and normality of the secant variety to a projective curve", *J. Algebra* **319**:3 (2008), 1264–1270. MR 2009b:14099 Zbl 1132.14027

[Wahl 1997] J. Wahl, "On cohomology of the square of an ideal sheaf", *J. Algebraic Geom.* **6**:3 (1997), 481–511. MR 99b:14046 Zbl 0892.14022

[Weibel 1994] C. A. Weibel, *An introduction to homological algebra*, Cambridge Studies in Advanced Mathematics **38**, Cambridge University Press, 1994. MR 95f:18001 Zbl 0797.18001

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The essential dimension of the normalizer of a maximal torus in the projective linear group

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Let p be a prime, k a field of characteristic $\neq p$ and N the normalizer of the maximal torus in the projective linear group PGL_n. We compute the exact value of the essential dimension $\mathrm{ed}_k(N; p)$ of N at p for every $n \geq 1$.

1. Introduction

Let k be a field, Fields_k the category of field extensions K/k, and F a covariant functor from Fields_k into the category of sets. As usual, for a field extension L/K, we denote the image of $a \in F(K)$ under the natural map $F(K) \to F(L)$ by a_L.

Given a field extension L/k, an object $a \in F(L)$ is said to *descend* to an intermediate field $k \subseteq K \subseteq L$ if a is in the image of the induced map $F(K) \to F(L)$. The *essential dimension* $\mathrm{ed}(a)$ of $a \in F(L)$ is the minimum of the transcendence degrees $\mathrm{trdeg}_k(K)$ taken over all fields $k \subseteq K \subseteq L$ such that a descends to K. The essential dimension $\mathrm{ed}(a; p)$ of a at a prime integer p is the minimum of $\mathrm{ed}(a_{L'})$, taken over all finite field extensions L'/L such that the degree $[L' : L]$ is prime to p.

The essential dimension $\mathrm{ed}(F)$ of the functor F (respectively, the essential dimension $\mathrm{ed}(F; p)$ of F at a prime p) is the supremum of $\mathrm{ed}(a)$ (respectively, of $\mathrm{ed}(a; p)$) taken over all $a \in F(L)$ and over all field extensions L/k. Informally speaking, the essential dimension of $a \in F(L)$ can be thought of as the minimal number of parameters one needs to define a, and $\mathrm{ed}(F)$ as the minimal number of parameters required to define any object in F.

An important example is the Galois cohomology functor $F_G = H^1(*, G)$ sending a field K/k to the set $H^1(K, G)$ of isomorphism classes of G-torsors over

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$\mathrm{Spec}(K)$, in the fppf topology. Here G is an algebraic group defined over k. The essential dimension of this functor is a numerical invariant of G, which, informally speaking, measures the complexity of G-torsors over fields. This number is usually denoted by $\mathrm{ed}_k(G)$ or, if k is fixed throughout, simply by $\mathrm{ed}(G)$. The notion of essential dimension was originally introduced and has since been extensively studied in this context; see for example [Buhler and Reichstein 1997; Reichstein 2000; Reichstein and Youssin 2000; Lemire 2004; Chernousov and Serre 2006]. The theory of essential dimension of algebraic groups may be viewed as a natural extension of the theory of *special groups* initiated in [Serre 1958]. Over an algebraically closed field k special groups are precisely those of essential dimension 0, these groups were classified in [Grothendieck 1958]. The more general definition of essential dimension for a covariant functor given above is due to Merkurjev [Berhuy and Favi 2003; Merkurjev 2007].

The purpose of this paper is to compute the relative essential dimension $\mathrm{ed}(N; p)$, where N is the normalizer of the (split) maximal torus in the projective linear group PGL_n. Before proceeding to state our main result, we would like to explain why we are interested in the essential dimension of N.

We begin by recalling that elements of $H^1(K, G)$ can often be naturally identified with K-forms of a single "split" algebraic object over k. Here by an algebraic object we mean a tensor t defined on a finite-dimensional k-vector space V; the group $G \subset \mathrm{GL}(V)$ then naturally arises as the automorphism group of t [Serre 1997, Chapter III]. Two examples will be of primary interest in the sequel:

$$H^1(*, \mathrm{PGL}_n) : K \mapsto \left\{ \begin{array}{c} \text{degree } n \text{ central simple algebras } A/K, \\ \text{up to } K\text{-isomorphism} \end{array} \right\} \tag{1}$$

and

$$H^1(*, N) : K \mapsto \big\{K\text{-isomorphism classes of pairs } (A, L)\big\}, \tag{2}$$

where K is a field extension of k, A is a degree n central simple algebra over K, L is a maximal étale subalgebra of A, and N is the normalizer of a split maximal torus in PGL_n, as above. For the functor (1) the split central simple k-algebra of degree n is M_n, its automorphism group is PGL_n. Similarly, in the case of the functor (2) the split pair (A, L) is $(\mathrm{M}_n, \mathrm{Diag}_n)$, where Diag_n denotes the subalgebra of diagonal matrices in $\mathrm{M}_n(k)$. The automorphism group of this split pair is N.

Computing the essential dimension of the projective linear group PGL_n, or equivalently, of the functor (1), is a fundamental problem in the theory of central simple algebras. To the best of our knowledge, it was first raised by C. Procesi, who showed (using different terminology) that $\mathrm{ed}(\mathrm{PGL}_n) \le n^2$ [Procesi 1967, Theorem 2.1]. This problem and the related question of computing the relative essential dimension $\mathrm{ed}(\mathrm{PGL}_n; p)$ at a prime p remain largely open. The best currently

known lower bound [Reichstein 1999, Theorem 16.1(b); Reichstein and Youssin 2000, Theorem 8.6] is

$$\operatorname{ed}(\mathrm{PGL}_{p^r};\, p) \geq 2r,$$

and it falls far below the best known upper bound [Lorenz and Reichstein 2000; Lorenz et al. 2003, Theorem 1.1; Lemire 2004, Proposition 1.6; Favi and Florence 2008], given by

$$\operatorname{ed}(\mathrm{PGL}_n) \leq \begin{cases} \frac{1}{2}(n-1)(n-2) & \text{for every odd } n \geq 5, \\ n^2 - 3n + 1 & \text{for every } n \geq 4. \end{cases} \tag{3}$$

We remark that the primary decomposition theorem reduces the computation of $\operatorname{ed}(\mathrm{PGL}_n;\, p)$ to the case where n is a power of p. That is, if $n = p_1^{r_1} \dots p_s^{r_s}$ then $\operatorname{ed}(\mathrm{PGL}_n;\, p_i) = \operatorname{ed}(\mathrm{PGL}_{p_i^{r_i}};\, p_i)$. The computation of $\operatorname{ed}(\mathrm{PGL}_n)$ also partially reduces to the prime power case, because

$$\operatorname{ed}(\mathrm{PGL}_{p_i^{r_i}}) \leq \operatorname{ed}(\mathrm{PGL}_n) \leq \operatorname{ed}(\mathrm{PGL}_{p_1^{r_1}}) + \dots + \operatorname{ed}(\mathrm{PGL}_{p_s^{r_s}})$$

for every $i = 1, \dots, s$ [Reichstein 2000, Proposition 9.8].

Note that the proofs of the upper bounds (3) are not based on a direct analysis of the functor $H^1(*, \mathrm{PGL}_n)$. Instead, one works with the related functor $H^1(*, N)$ of (2). This functor is often more accessible than $H^1(*, \mathrm{PGL}_n)$ because many of the standard constructions in the theory of central simple algebras depend on the choice of a maximal subfield L in a given central simple algebra A/K. Projecting a pair (A, L) to the first component, we obtain a surjective morphism of functors $H^1(*, N) \to H^1(*, \mathrm{PGL}_n)$, [Rowen 1980, Corollary 3.1.11]. The surjectivity of this morphism leads to the inequalities

$$\operatorname{ed}(N) \geq \operatorname{ed}(\mathrm{PGL}_n) \quad \text{and} \quad \operatorname{ed}(N;\, p) \geq \operatorname{ed}(\mathrm{PGL}_n;\, p); \tag{4}$$

see [Merkurjev 2007, Proposition 1.3], [Berhuy and Favi 2003, Lemma 1.9] or [Reichstein 2000, Proposition 4.3].

The inequalities (3) were, in fact, proved as upper bounds on $\operatorname{ed}(N)$ [Lorenz et al. 2003; Lemire 2004]. It is thus natural to try to determine the exact values of $\operatorname{ed}(N)$ and $\operatorname{ed}(N;\, p)$. In addition to being of independent interest, these numbers represent a limitation on the techniques used in [Lorenz et al. 2003] and [Lemire 2004]. This brings us to the main result of this paper.

Theorem 1.1. *Let N the normalizer of a maximal torus in the projective linear group PGL_n defined over a field k with $\operatorname{char}(k) \neq p$. Then*:

(a) $\operatorname{ed}_k(N;\, p) = [n/p]$, *if n is not divisible by p.*

(b) $\operatorname{ed}_k(N;\, p) = 2$, *if $n = p$.*

(c) $\operatorname{ed}_k(N;\, p) = n^2/p - n + 1$, *if $n = p^r$ for some $r \geq 2$.*

(d) $\operatorname{ed}_k(N;\, p) = p^e(n - p^e) - n + 1$, *in all other cases.*

Here $[n/p]$ is the integer part of n/p and p^e is the highest power of p dividing n.

In each part we will prove an upper bound and a lower bound on $\mathrm{ed}(N)$ separately. We do not have an a priori reason why the two should match, thus yielding an exact value of $\mathrm{ed}(N; p)$; the fact that this happens may be viewed as a lucky coincidence. We also remark that our proof of the upper bounds on $\mathrm{ed}_k(N; p)$ in part (c) and (d) does not use the assumption that $\mathrm{char}(k) \neq p$. These bounds are valid for every base field k.

As we mentioned above, the computation of $\mathrm{ed}(\mathrm{PGL}_n; p)$ reduces to the case where n is a power of p. A quick glance at the statement of Theorem 1.1 shows that the computation of $\mathrm{ed}(N; p)$ does not. On the other hand, the proof of part (c), where $n = p^r$ and $r \geq 2$, requires the most intricate arguments. Another reason for our special interest in part (c) is that it leads to a new upper bound on $\mathrm{ed}(\mathrm{PGL}_n; p)$. More precisely, combining the upper bound in part (c) with (4), and remembering that the upper bound in part (c) is valid for any the ground field k, we obtain the following inequality.

Corollary 1.2. *Let $n = p^r$ be a prime power. Then*

$$\mathrm{ed}_k(\mathrm{PGL}_n; p) \leq p^{2r-1} - p^r + 1$$

for any field k and for any $r \geq 2$. □

Corollary 1.2 fails for $r = 1$ because

$$\mathrm{ed}_k(\mathrm{PGL}_p; p) = 2; \tag{5}$$

see [Reichstein 2000, Corollary 5.7] or [Reichstein and Youssin 2000, Lemma 8.5.7]. For $r = 2$, Corollary 1.2 is valid but is not optimal. Indeed, in this case L. H. Rowen and D. J. Saltman showed that, after a prime-to-p extension L/K, every degree p^2 central simple algebra A/K becomes a $(\mathbb{Z}/p\mathbb{Z})^2$-crossed product [Rowen and Saltman 1992, Corollary 1.3]. The upper bound on the essential dimension of a crossed product given by [Lorenz et al. 2003, Corollary 3.10] then yields the inequality $\mathrm{ed}(\mathrm{PGL}_{p^2}; p) \leq p^2 + 1$, which is stronger than Corollary 1.2 for any $p \geq 3$. Merkurjev [2008] recently showed that in fact, $\mathrm{ed}_k(\mathrm{PGL}_{p^2}; p) = p^2 + 1$ for any field k of characteristic different from p. For $r \geq 3$ Corollary 1.2 gives the best currently known upper bound on $\mathrm{ed}(\mathrm{PGL}_{p^r}; p)$.

We remark that the inequalities of (4) have counterparts for algebraic groups other than PGL_n. Indeed, if G is a linear group defined over k, C is a Cartan subgroup of G and $N(C)$ is the normalizer of C then by a theorem of T. Springer the natural map $H^1(K, N(C)) \to H^1(K, G)$ is surjective for every perfect field extension K/k [Serre 1997, III.4.3, Lemma 6]. Consequently, $\mathrm{ed}_k(N(C)) \geq \mathrm{ed}_k(G)$ if $\mathrm{char}(k) = 0$ and $\mathrm{ed}_k(N(C); p) \geq \mathrm{ed}_k(G; p)$ if $\mathrm{char}(k) \neq p$; compare [Reichstein

2000, Proposition 4.3]. It would thus be of interest to prove an analogue of Theorem 1.1 in the more general setting, where N is the normalizer of a split maximal torus in an arbitrary simple (or semisimple) linear algebraic group G. The new technical difficulty one encounters in this more general setting is that the natural sequence

$$1 \to T \to N \to W \to 1,$$

may not split. Here T is a split maximal torus and $W = N/T$ is the Weyl group of G. The fact that this sequence splits for $G = \mathrm{PGL}_n$ is an important ingredient in our proof of the upper bound on $\mathrm{ed}(N; p)$.

A key ingredient in our proofs of the lower bounds in Theorem 1.1(c) and (d) is a recent theorem of Karpenko and Merkurjev [2008] on the essential dimension of a p-group, stated as Theorem 7.1 below. To the best of our knowledge, these lower bounds were not accessible by previous techniques. Corollary 1.2 and the other parts of Theorem 1.1 do not rely on the Karpenko–Merkurjev theorem.

2. A general strategy

Let G be an algebraic group defined over a field k. Recall that the action of G on an algebraic variety X defined over k is generically free if the stabilizer subgroup $\mathrm{Stab}_G(x)$ is trivial for $x \in X(\bar{k})$ in general position.

Remark 2.1. If G is a finite constant group and X is irreducible then the G-action on X is generically free if and only if it is faithful.

Indeed, the "only if" is obvious. Conversely, if the G-action on X is faithful then $\mathrm{Stab}_G(x) = \{1\}$ for any x outside of the closed subvariety $\bigcup_{1 \neq g \in G} X^{\langle g \rangle}$, whose dimension is at most $\dim X - 1$. □

Remark 2.2. Suppose k'/k is a field extension of degree prime to p. Then the essential dimension at p does not change if we replace k by k' [Merkurjev 2007, Proposition 1.5(2)]. This happens in particular if $\mathrm{char}(k) \neq p$ and k' is obtained from k by adjoining a primitive p-th root of unity. Thus in the course of proving Theorem 1.1 we may assume without loss of generality that k contains a primitive p-th root of unity.

In the sequel we will repeatedly encounter the following situation. Suppose we want to show that

$$\mathrm{ed}_k(G) = \mathrm{ed}_k(G; p) = d, \tag{6}$$

where G is a linear algebraic group defined over k. All such assertions will be proved in two steps:

(i) Construct a generically free linear representation of G over k of dimension $d + \dim G$. This implies that $\mathrm{ed}_k(G) \leq d$; see [Reichstein 2000, Theorem 3.4] or [Berhuy and Favi 2003, Proposition 4.11].

(ii) Prove the lower bound $\mathrm{ed}_k(G;\, p) \geq d$.

Since clearly $\mathrm{ed}(G;\, p) \leq \mathrm{ed}(G)$, equality (6) follows from (i) and (ii).

The group G will always be of the form $G = D \rtimes F$, where D is diagonalizable and F is finite. In the next section we will recall some known facts about representations of such groups. This will help us in carrying out step (i) and, in the most interesting cases, step (ii) as well, via the Karpenko–Merkurjev Theorem 7.1.

3. Representation-theoretic preliminaries

We will work over a ground field k which remains fixed throughout. Suppose that a linear algebraic k-group G contains a diagonalizable (over k) group D and the quotient G/D is a constant finite group F. Here diagonalizable over k means that D is a subgroup of the split torus $\mathbb{G}_m^d$ defined over k or, equivalently, that every linear representation of D defined over k decomposes as a direct sum of one-dimensional subrepresentations.

Denote the group of (multiplicative) characters of D by $X(D)$. Note that since D is diagonalizable over k, every multiplicative character of D is defined over k. Consider a linear k-representation $G \to \mathrm{GL}(V)$. Restricting this representation to D, we decompose V into a direct sum of one-dimensional character spaces. Let $\Lambda \subset X(D)$ be the set of characters (weights) of D which occur in this decomposition. Note that here $|\Lambda| \leq \dim V$, and equality holds if and only if each character from Λ occurs in V with multiplicity 1. The finite group F acts on $X(D)$ and Λ is invariant under this action. Moreover, if the G-action (and hence, the D-action) on V is generically free then Λ generates $X(D)$ as an abelian group. In summary, we have proved the following lemma; cf. [Serre 1977, Section 8.1].

Lemma 3.1. *Suppose that every F-invariant generating set Λ of $X(D)$ contains at least d elements. If $G \to \mathrm{GL}(V)$ is a generically free k-representation of G then* $\dim V \geq d$. □

As we explained in the previous section, we are interested in constructing low-dimensional generically free representations of G. In this section we will prove simple sufficient conditions for generic freeness for two particular families of representations.

Lemma 3.2. *Let W be a faithful representation of F and V be a representation of G whose restriction to D is generically free. Then $V \times W$ is a generically free representation of G.*

Here we view W as a representation of G via the natural projection

$$G \to G/D = F.$$

Proof. For $w \in W(\bar{k})$ in general position, we have $\mathrm{Stab}_G(w) = D$ by Remark 2.1. Choosing v in general position in $V(\bar{k})$, we see that $\mathrm{Stab}_G(v, w) = \mathrm{Stab}_G(v) \cap \mathrm{Stab}_G(w) = \mathrm{Stab}_D(v) = \{1\}$. □

From now on we will assume that $G = D \rtimes F$ is the semidirect product of D and F. In this case, given an F-invariant generating set $\Lambda \subset X(D)$, we can construct a linear k-representation V_Λ of G so that each character from Λ occurs in V_Λ exactly once. To do this, we associate a basis element v_λ to each $\lambda \in \Lambda$. The finite group F acts on

$$V_\Lambda = \mathrm{Span}(v_\lambda \mid \lambda \in \Lambda)$$

by permuting these basis elements in the natural way, that is, via

$$\sigma : v_\lambda \mapsto v_{\sigma(\lambda)} \tag{7}$$

for any $\sigma \in F$ and any $\lambda \in \Lambda$. The diagonalizable group D-acts by the character λ on each one-dimensional space $\mathrm{Span}(v_\lambda)$, that is, via

$$t : v_\lambda \mapsto \lambda(t) v_\lambda \tag{8}$$

for any $t \in D$ and $\lambda \in \Lambda$. Extending (7) and (8) linearly to all of V_Λ, we obtain a linear representation $G = D \rtimes F \to \mathrm{GL}(V_\Lambda)$. Note that by our construction $\dim V_\Lambda = |\Lambda|$.

Our second criterion for generic freeness is a variant of [Lorenz and Reichstein 2000, Lemma 3.1] or [Lemire 2004, Proposition 2.1]. For the sake of completeness we outline a characteristic-free proof.

Lemma 3.3. *Let Λ be an F-invariant subset of $X(D)$ and $\phi : \mathbb{Z}[\Lambda] \to X(D)$ be the natural morphism of $\mathbb{Z}[F]$-modules, taking $\lambda \in \Lambda$ to itself. Let V_Λ be the linear representation of $G = D \rtimes F$ defined by* (7) *and* (8)*, as above. The G-action on V_Λ is generically free if and only if*

(a) *Λ spans $X(D)$ (or equivalently, ϕ is surjective) and*

(b) *the F-action on* $\mathrm{Ker}\,\phi$ *is faithful.*

Proof. Let $U \simeq \mathbb{G}_m^n$ be the diagonal subgroup of $\mathrm{GL}(V_\Lambda)$, in the basis e_λ, where $\lambda \in \Lambda$. Here $n = |\Lambda| = \dim V_\Lambda$. The G-action on V induces an F-equivariant morphism $\rho : D \to U$, which is dual to ϕ under the usual (antiequivalence) Diag between finitely generated abelian groups and diagonalizable algebraic groups. Applying Diag to the exact sequence

$$(0) \longrightarrow \mathrm{Ker}\,\phi \longrightarrow \mathbb{Z}[\Lambda] \xrightarrow{\phi} X(D) \longrightarrow \mathrm{Coker}\,\phi \longrightarrow (0)$$

of finitely generated abelian $\mathbb{Z}[F]$-modules and setting

$$U = \mathrm{Diag}(\mathbb{Z}[\Lambda]), \quad N = \mathrm{Diag}(\mathrm{Coker}\,\phi), \quad Q = \mathrm{Diag}(\mathrm{Ker}\,\phi),$$

we obtain an F-equivariant exact sequence

$$1 \longrightarrow N \longrightarrow D \xrightarrow{\rho} U \longrightarrow Q \longrightarrow 1$$

of diagonalizable groups; see [Jantzen 2003, I 5.6] or [Demazure and Gabriel 1970, IV 1.1]. Since U is F-equivariantly isomorphic to a dense open subset of V, the G-action on V is generically free if and only if the G-action on U is generically free. On the other hand, the G-action on U is generically free if and only if the D-action on U is generically free and the F-action on Q is generically free. But the first of these conditions is equivalent to (a), while the second is equivalent to (b); see Remark 2.1. □

4. Subgroups of prime-to-p index

Lemma 4.1. *Let G' be a closed subgroup of a smooth algebraic group G defined over k. Assume that the index $[G : G'] := \dim_k k[G/G']$ is finite and prime to p. Then* $\operatorname{ed}(G; p) = \operatorname{ed}(G'; p)$.

In the case where G is finite a proof can be found in [Merkurjev 2007, Proposition 4.10]; the argument below proceeds along similar lines.

Proof. Recall that if G is a linear algebraic group and H is a closed subgroup then

$$\operatorname{ed}(G; p) \geq \operatorname{ed}(H; p) + \dim H - \dim G \tag{9}$$

for any prime p; see [Brosnan et al. 2008, Lemma 2.2] or [Merkurjev 2007, Corollary 4.3]. Since $\dim G' = \dim G$, this yields $\operatorname{ed}(G; p) \geq \operatorname{ed}(G'; p)$.

To prove the opposite inequality, it suffices to show that for any field K/k the map $H^1(K, G') \to H^1(K, G)$ induced by the inclusion $G' \subset G$ is p-surjective, meaning that for every $\alpha \in H^1(K, G)$ there is a finite field extension L/K of degree prime to p such that α_L is in the image of $H^1(L, G') \to H^1(L, G)$; see for example [Merkurjev 2007, Proposition 1.3].

Let $X \to \operatorname{Spec}(K)$ be a G-torsor and X/G' be the natural quotient of X by the action of G'. Recall that X/G' is a K-form of G/G' and that it is constructed by descent [Serre 1962, 1.3.2]. Alternatively, X/G' may be viewed as the Galois twist of G/G' by X with respect to the natural G-action on G/G' [Milne 1980, p. 134].

For a field L/K and an L-point $\operatorname{Spec}(L) \to X/G'$ we construct a G'-torsor Y as the pullback

$$\begin{array}{ccc} Y & \longrightarrow & X \\ \downarrow & & \downarrow \\ \operatorname{Spec}(L) & \longrightarrow & X/G' \\ & & \downarrow \\ & & \operatorname{Spec}(K) \end{array}$$

In this situation $Y \times^{G'} G \cong X_L$ as G-torsors. Thus we have the natural diagram

$$\begin{array}{ccc} H^1(L, G') & \longrightarrow & H^1(L, G) \\ [Y] & \longmapsto & [X]_L \\ & & \uparrow \\ & & [X] \\ & & \uparrow \\ & & H^1(K, G) \end{array}$$

where $[X]$ and $[Y]$ denote the classes of X and Y in $H^1(K, G)$ and $H^1(L, G')$, respectively. It remains to show the existence of such an L-point, with the degree $[L : K]$ prime to p.

Note that G/G' is affine, since G and G' are of the same dimension and hence $G/G' \cong (G/G^\circ)/(G'/G^\circ) = \operatorname{Spec} k[G/G^\circ]^{G'/G^\circ}$ where G° is the connected component of G (and G'). Furthermore G/G' is smooth [Demazure and Gabriel 1970, III 3.2.7]. Let K_s be a separable closure of K. Since X is a G-torsor, we have $X_{K_s} \cong G_{K_s}$ and $(X/G')_{K_s} \cong (G/G')_{K_s}$ which implies that X/G' is also affine [Demazure and Gabriel 1970, III 3.5.6 d)]. Thus, $K[X/G'] \otimes K_s \cong k[G/G'] \otimes K_s$ is reduced and its dimension $\dim_K K[X/G'] = [G : G']$ is not divisible by p by assumption.

Therefore $K[X/G']$ is étale or, equivalently, a product of separable field extensions of K

$$K[X/G'] = L_1 \times \cdots \times L_r;$$

see for example [Bourbaki 1990, V, Theorem 4]. For each L_j the projection $K[X/G'] \to L_j$ is an L_j-point of X/G' and since $\dim_K K[X/G'] = \sum_{j=1}^r [L_j : K]$ is prime to p, one of the fields L_j must be of degree prime to p over K. We now take $L = L_j$. □

Corollary 4.2. *Suppose k is a field of characteristic $\neq p$. Then* $\operatorname{ed}_k(\mathrm{S}_n; p) = [n/p]$.

Proof. Let $m = [n/p]$ and let $D \simeq (\mathbb{Z}/p\mathbb{Z})^m$ be the subgroup generated by the disjoint p-cycles

$$\sigma_1 = (1, \ldots, p), \ldots, \sigma_m = \big((m-1)p + 1, \ldots, mp\big).$$

The inequality $\operatorname{ed}(\mathrm{S}_n; p) \geq \operatorname{ed}_k(D; p) \geq [n/p]$ is well known; see any of [Buhler and Reichstein 1997, Section 6; Buhler and Reichstein 1999, Section 7; Berhuy and Favi 2003, Proposition 3.7].

To the best of our knowledge, the opposite inequality was first noticed by J.-P. Serre (private communication, May 2005) and independently by R. Lötscher [Lötscher 2008]. The proof is quite easy. However, since it has not previously appeared in print, we reproduce it below.

The semidirect product $D \rtimes \mathrm{S}_m$, where S_m permutes $\sigma_1, \dots, \sigma_m$, embeds in S_n with index prime to p. By Lemma 4.1, $\mathrm{ed}_k(D \rtimes \mathrm{S}_m;\, p) = \mathrm{ed}_k(\mathrm{S}_n;\, p)$ and it suffices to show that $\mathrm{ed}_k(D \rtimes \mathrm{S}_m) \le [n/p]$. As we mentioned in Section 2, in order to prove this, it is enough to construct a generically free m-dimensional representation of $D \rtimes \mathrm{S}_m$ defined over k. Moreover, by Remark 2.2 we may assume that $\zeta_p \in k$, where ζ_p denotes a primitive p-th root of unity.

To construct a generically free m-dimensional representation of $D \rtimes \mathrm{S}_m$, let $\sigma_1^*, \dots, \sigma_m^* \subset X(D)$ be the "basis" of D dual to $\sigma_1, \dots, \sigma_m$. That is,

$$\sigma_i^*(\sigma_j) = \begin{cases} \zeta_p & \text{if } i = j, \\ 1 & \text{otherwise.} \end{cases}$$

The S_m-invariant subset $\Lambda = \{\sigma_1^*, \dots, \sigma_m^*\}$ of $X(D)$ gives rise to the m-dimensional k-representation V_Λ of $D \rtimes \mathrm{S}_m$, as in Section 3. An easy application of Lemma 3.3 shows that this representation is generically free. □

5. First reductions and proof of Theorem 1.1 parts (a) and (b)

Let $T \simeq \mathbb{G}_m^n/\Delta$ be the diagonal maximal torus in PGL_n, where $\Delta = \mathbb{G}_m$ is diagonally embedded into $\mathbb{G}_m^n$. Recall that the normalizer N of T is isomorphic to $T \rtimes \mathrm{S}_n$, where we identify S_n with the subgroup of permutation matrices in PGL_n.

Let P_n be a Sylow p-subgroup of S_n. Lemma 4.1 tells us that

$$\mathrm{ed}_k(N;\, p) = \mathrm{ed}_k(T \rtimes P_n;\, p).$$

Note also that by Remark 2.2 we may assume without loss of generality that k contains a primitive p-th root of unity.

Thus in order to prove Theorem 1.1 it suffices to establish the following proposition.

Proposition 5.1. *Let $T \simeq \mathbb{G}_m^n/\Delta$, where $\Delta = \mathbb{G}_m$ is diagonally embedded into $\mathbb{G}_m^n$. Assume that k is of characteristic $\neq p$, containing a primitive p-th root of unity. Then*:

(a) $\mathrm{ed}_k(T \rtimes P_n) = \mathrm{ed}_k(T \rtimes P_n;\, p) = [n/p]$, *if n is not divisible by p.*

(b) $\mathrm{ed}_k(T \rtimes P_n) = \mathrm{ed}_k(T \rtimes P_n;\, p) = 2$, *if $n = p$.*

(c) $\mathrm{ed}_k(T \rtimes P_n) = \mathrm{ed}_k(T \rtimes P_n;\, p) = n^2/p - n + 1$, *if $n = p^r$ for some $r \ge 2$.*

(d) $\mathrm{ed}_k(T \rtimes P_n) = \mathrm{ed}_k(T \rtimes P_n;\, p) = p^e(n - p^e) - n + 1$, *in all other cases.*

Here P_n is a Sylow p-subgroup of S_n, *$[n/p]$ is the integer part of n/p and p^e is the highest power of p dividing n.*

The assumption that k contains a primitive p-th root of unity is only needed for the proof of the first equality in parts (a) and (b).

Our proof of each part of this proposition will be based on the strategy outlined in Section 2, with $G = T \rtimes P_n$. We start by recalling that the character lattice $X(T)$ is naturally isomorphic to

$$\big\{(a_1, \dots, a_n) \in \mathbb{Z}^n \mid a_1 + \cdots + a_n = 0\big\},$$

where we identify the character

$$(t_1, \dots, t_n) \to t_1^{a_1} \dots t_n^{a_n}$$

of $T = \mathbb{G}_m^n/\Delta$ with $(a_1, \dots, a_n) \in \mathbb{Z}^n$. Note that $(t_1, \dots, t_n)$ is viewed as an element of $\mathbb{G}_m^n$ modulo the diagonal subgroup Δ, so the above character is well defined if and only if $a_1 + \cdots + a_n = 0$. An element σ of S_n (and in particular, of $P_n \subset \mathrm{S}_n$) acts on $\mathbf{a} = (a_1, \dots, a_n) \in X(T)$ by naturally permuting $a_1, \dots, a_n$.

For notational convenience, we will denote by $\mathbf{a}_{i,j} = (a_1, \dots, a_n) \in X(T)$ the element such that $a_i = 1$, $a_j = -1$ and $a_h = 0$ for every $h \neq i, j$.

We also recall that for $n = p^r$ the Sylow p-subgroup P_n of S_n can be described inductively as the wreath product

$$P_{p^r} \cong P_{p^{r-1}} \wr \mathbb{Z}/p \cong (P_{p^{r-1}})^p \rtimes \mathbb{Z}/p.$$

For general n, P_n is the direct product of certain P_{p^r}; see Section 8.

Proof of Proposition 5.1(a). (i) Since n is not divisible by p, we may assume that P_n is contained in S_{n-1}, where we identify S_{n-1} with the subgroup of S_n consisting of permutations $\sigma \in \mathrm{S}_n$ such that $\sigma(1) = 1$.

We will now construct a generically free linear representation V of $T \rtimes P_n$ of dimension $n - 1 + [n/p]$. This will show that $\mathrm{ed}(T \rtimes P_n) \leq [n/p]$.

To construct V, let $\Lambda = \{\mathbf{a}_{1,i} \mid i = 2, \dots, n\}$ and V_Λ be as in Section 3 and let W be an $[n/p]$-dimensional faithful linear representation of P_n constructed in the proof of Corollary 4.2. Applying Lemma 3.2, we see that $V = V_\Lambda \times W$ is generically free.

(ii) Since the natural projection $p : T \rtimes P_n \to P_n$ has a section, so does the map $p^* : H^1(K, T \rtimes P_n) \to H^1(K, P_n)$ of Galois cohomology sets. Hence, p^* is surjective for every field K/k. This implies that

$$\mathrm{ed}(T \rtimes P_n) \geq \mathrm{ed}(P_n; p) = [n/p].$$

Here $\mathrm{ed}(P_n; p) = \mathrm{ed}(\mathrm{S}_n; p)$ by Lemma 4.1 and $\mathrm{ed}(\mathrm{S}_n; p) = [n/p]$ by Corollary 4.2. □

Remark 5.2. We will now outline a different and perhaps more conceptual proof of the upper bound $\mathrm{ed}(N; p) \leq [n/p]$ of Theorem 1.1(a). As we pointed out in the introduction, $\mathrm{ed}(N; p)$ is the essential dimension at p of the functor

$$H^1(*, N) : K \mapsto \big\{K\text{-isomorphism classes of pairs } (A, L)\big\},$$

where A is a degree n central simple algebra over K and L is a maximal étale subalgebra of A. Similarly, $\mathrm{ed}(\mathrm{S}_n;\, p)$ is the essential dimension at p of the functor

$$H^1(*, \mathrm{S}_n) : K \mapsto \{K\text{-isomorphism classes of } n\text{-dimensional étale algebras } L/K\}.$$

Let $\alpha : H^1(*, \mathrm{S}_n) \to H^1(*, N)$ be the map taking an n-dimensional étale algebra L/K to $(\mathrm{End}_K(L), L)$. Here we embed L in $\mathrm{End}_K(L) \simeq \mathrm{M}_n(K)$ via the regular action of L on itself.

It is easy to see that, in the terminology of [Merkurjev 2007, Section 1.3], α is p-surjective. That is, for any class (A, L) in $H^1(K, N)$ there exists a prime-to-p extension K'/K such that $(A \otimes_K K', L \otimes_K K')$ lies in the image of α. In fact, any K'/K of degree prime to p which splits A will do (such an extension exists because we are assuming that the degree n of A is not divisible by p). Indeed, by the Skolem–Noether theorem, any two embeddings of $L \otimes_K K'$ into $\mathrm{M}_n(K')$ are conjugate. By [Merkurjev 2007, Proposition 1.3], we conclude that $\mathrm{ed}(N;\, p) \le \mathrm{ed}(\mathrm{S}_n;\, p)$. Combining this with Corollary 4.2 yields the desired inequality $\mathrm{ed}(N;\, p) \le [n/p]$.
□

Proof of Proposition 5.1(b). Here $n = p$ and $P_n \simeq \mathbb{Z}/p$ is generated by the p-cycle $(1, 2, \dots, n)$. We follow the strategy outlined in Section 2.

(i) To show that $\mathrm{ed}_k(T \rtimes P_n) \le 2$, we construct a generically free k-representation of $T \rtimes P_n$ of dimension $2 + \dim(T \rtimes P_n) = n + 1$.

Let $\Lambda = \{\mathbf{a}_{1,2}, \dots, \mathbf{a}_{p-1,p}, \mathbf{a}_{p,1}\}$ and $V = V_\Lambda \times L$, where L is a one-dimensional faithful representation of $P_n \simeq \mathbb{Z}/p$ and $T \rtimes P_n$ acts on L via the natural projection $T \rtimes P_n \to P_n$. Note that $\dim V = |\Lambda| + 1 = n + 1$. Since Λ generates $X(T)$, Lemma 3.2 tells us that V is a generically free representation of $T \rtimes P_n$.

(ii) Recall that $\mathrm{ed}_k(T \rtimes P_n;\, p) = \mathrm{ed}_k(N;\, p)$ by Lemma 4.1. On the other hand, as we mentioned in the introduction,

$$\mathrm{ed}_k(N;\, p) \ge \mathrm{ed}_k(\mathrm{PGL}_p;\, p) = 2;$$

see (4) and (5). This completes the proof of Proposition 5.1(b) and of Theorem 1.1(b). □

6. Proof of Theorem 1.1(c): The upper bound

In the next two sections we will prove Proposition 5.1(c), and hence Theorem 1.1(c). We will assume that $n = p^r$ for some $r \ge 2$ and follow the strategy of Section 2. In this section we will carry out Step (i). That is, we will construct a generically free representation V of $T \rtimes P_n$ of dimension p^{2r-1}. This will show

that $\mathrm{ed}(T \rtimes P_n) \leq p^{2r-1} - p^r + 1$. Our V will be of the form V_Λ for a particular P_n-invariant $\Lambda \subset X(T)$, following the recipe of Section 3. Note that this construction (and thus the above inequality) will not require any assumption on the base field k.

For notational convenience, we will subdivide the integers $1, 2, \ldots, p^r$ into p big blocks $B_1, \ldots, B_p$, where each B_i consists of the p^{r-1} consecutive integers $(i-1)p^{r-1}+1$, $(i-1)p^{r-1}+2, \ldots, ip^{r-1}$.

We define $\Lambda \subset X(T)$ as the P_n-orbit of the element

$$\mathbf{a}_{1,p^{r-1}+1} = (\underbrace{1, 0, \ldots, 0}_{B_1}, \underbrace{-1, 0, \ldots, 0}_{B_2}, \underbrace{0, 0, \ldots, 0}_{B_3}, \ldots, \underbrace{0, 0, \ldots, 0}_{B_p})$$

in $X(T)$. Thus, Λ consists of elements $\mathbf{a}_{\alpha,\beta}$, subject to the condition that if α lies in the big block B_i then β has to lie in B_j, where $j - i \equiv 1$ modulo p. There are p^r choices for α. Once α is chosen, there are exactly p^{r-1} further choices for β. Thus

$$|\Lambda| = p^r \cdot p^{r-1} = p^{2r-1}.$$

As described in Section 3, we obtain a linear representation V_Λ of $T \rtimes P_n$ of the desired dimension

$$\dim V_\Lambda = |\Lambda| = p^{2r-1}.$$

It remains to prove that V_Λ is generically free. By Lemma 3.3 it suffices to show that

(i) Λ generates $X(T)$ as an abelian group and

(ii) the P_n action on the kernel of the natural morphism $\phi : \mathbb{Z}[\Lambda] \to X(T)$ is faithful.

The elements $\mathbf{a}_{\alpha,\beta}$ clearly generate $X(T)$ as an abelian group, as α and β range over $1, 2, \ldots, p^r$. Thus in order to prove (i) it suffices to show that $\mathrm{Span}_{\mathbb{Z}}(\Lambda)$ contains every element of this form. Suppose α lies in the big block B_i and β in B_j. If $j - i \equiv 1 \pmod{p}$, then $\mathbf{a}_{\alpha,\beta}$ lies in Λ and there is nothing to prove. If $j - i \equiv 2 \pmod{p}$ then choose some $\gamma \in B_{i+1}$ (where the subscript $i+1$ should be viewed modulo p) and write

$$\mathbf{a}_{\alpha,\beta} = \mathbf{a}_{\alpha,\gamma} + \mathbf{a}_{\gamma,\beta}.$$

Since both terms on the right are in Λ, we see that in this case $\mathbf{a}_{\alpha,\beta} \in \mathrm{Span}_{\mathbb{Z}}(\Lambda)$. Using this argument recursively, we see that $\mathbf{a}_{\alpha,\beta}$ also lies in $\mathrm{Span}_{\mathbb{Z}}(\Lambda)$ if $j - i \equiv 3, \ldots, p \pmod{p}$, i.e., for all possible i and j. This proves (i).

To prove (ii), denote the kernel of ϕ by M. Since P_n is a finite p-group, every normal subgroup of P_n intersects the center of P_n, which we shall denote by Z_n. Thus it suffices to show that Z_n acts faithfully on M.

Recall that Z_n is the cyclic subgroup of P_n of order p generated by the product of disjoint p-cycles

$$\sigma_1 \dots \sigma_{p^{r-1}} = (1 \ \cdots \ p)\,(p{+}1 \ \cdots \ 2p) \cdots (p^r{-}p{+}1 \ \cdots \ p^r).$$

Since $|Z_n| = p$, it either acts faithfully on M or it acts trivially, so we only need to check that the Z_n-action on M is nontrivial. Indeed, Z_n does not fix the nonzero element

$$\mathbf{a}_{1,p^{r-1}+1} + \mathbf{a}_{p^{r-1}+1,2p^{r-1}+1} + \cdots + \mathbf{a}_{(p-1)p^{r-1}+1,1} \in \mathbb{Z}[\Lambda]$$

which lies in M. This proves the upper bound of Proposition 5.1(c) and Theorem 1.1(c). □

7. Proof of Theorem 1.1(c): The lower bound

In this section we will continue to assume that $n = p^r$. We will show that

$$\operatorname{ed}(N;\, p) \geq p^{2r-1} - p^r + 1, \tag{10}$$

thus completing the proof of Proposition 5.1(c) and Theorem 1.1(c). Let

$$q := p^e, \text{ where } e \geq 1 \text{ if } p \text{ is odd and } e \geq 2 \text{ if } p = 2. \tag{11}$$

be a power of p. The specific choice of e will not be important in the sequel; in particular, the reader may assume that $q = p$ if p is odd and $q = 4$, if $p = 2$. Whatever e we choose, $q = p^e$ will remain unchanged for the rest of this section.

We now recall that if k'/k is a field extension then

$$\operatorname{ed}_k(N;\, p) \geq \operatorname{ed}_{k'}(N;\, p),$$

by [Merkurjev 2007, Proposition 1.5(1)]. Thus for the purpose of proving (10) we may replace k by k'. In particular, we may assume that k' contains a primitive q-th root of unity.

Let $T_{(q)} = \mu_q^n/\mu_q$ be the q-torsion subgroup of $T = \mathbb{G}_m^n/\Delta$. Applying the inequality (9) to $G = T \rtimes P_n$ and its finite subgroup $H = T_{(q)} \rtimes P_n$, we obtain

$$\operatorname{ed}(T \rtimes P_n;\, p) \geq \operatorname{ed}(T_{(q)} \rtimes P_n;\, p) - p^r + 1.$$

Thus it suffices to show that

$$\operatorname{ed}(T_{(q)} \rtimes P_n;\, p) \geq p^{2r-1}. \tag{12}$$

The advantage of replacing $T \rtimes P_n$ by $T_{(q)} \rtimes P_n$ is that $T_{(q)} \rtimes P_n$ is a finite p-group, so that we can apply the following result:

Theorem 7.1 [Karpenko and Merkurjev 2008]. *Let G be a finite p-group and k be a field containing a primitive p-th root of unity. Then* $\mathrm{ed}_k(G; p) = \mathrm{ed}_k(G)$ *equals the minimal value of* $\dim V$, *where V ranges over all faithful linear k-representations $G \to \mathrm{GL}(V)$.*

Now recall that we are assuming that k contains a primitive q-th root of unity and hence, a primitive p-th root of unity. Hence, Theorem 7.1 applies in our situation. That is, in order to prove (12) it suffices to show that $T_{(q)} \rtimes P_n$ does not have a faithful linear representation of dimension less than p^{2r-1}. Lemma 3.1 further reduces this representation-theoretic assertion to the combinatorial statement of Proposition 7.2 below. Before stating the proposition we recall that the character lattice of $T_{(q)} \simeq \mu_q^n/\mu_q$ is

$$X_n := \{(a_1, \dots, a_n) \in (\mathbb{Z}/q\mathbb{Z})^n \mid a_1 + \cdots + a_n = 0 \text{ in } \mathbb{Z}/q\mathbb{Z}\},$$

where we identify the character

$$(t_1, \dots, t_n) \to t_1^{a_1} \dots t_n^{a_n}$$

of $T_{(q)}$ with $(a_1, \dots, a_n) \in (\mathbb{Z}/q\mathbb{Z})^n$. Here $(t_1, \dots, t_n)$ stands for an element of μ_q^n, modulo the diagonally embedded μ_q, so the character above is well defined if and only if $a_1 + \cdots + a_n = 0$ in $\mathbb{Z}/q\mathbb{Z}$. (This is completely analogous to our description of the character lattice of T in the previous section.) Note that X_n depends on the integer $q = p^e$, which we assume to be fixed throughout this section.

Proposition 7.2. *Let $n = p^r$ and P_n be a Sylow p-subgroup of S_n. If Λ is a P_n-invariant generating subset of X_n then $|\Lambda| \geq p^{2r-1}$ for any $r \geq 1$.*

Our proof relies on the following special case of Nakayama's Lemma:

Lemma 7.3 [Atiyah and Macdonald 1969, Proposition 2.8]. *Let $q = p^e$ be a prime power, $M = (\mathbb{Z}/q\mathbb{Z})^d$ and Λ be a generating subset of M (as an abelian group). If we remove from Λ all elements that lie in pM, the remaining set, $\Lambda \setminus pM$, will still generate M.* □

Proof of Proposition 7.2. We argue by induction on r. For the base case, set $r = 1$. We need to show that $|\Lambda| \geq p$. Assume the contrary. In this case P_n is a cyclic p-group, and every nontrivial orbit of P_n has exactly p elements. Hence, $|\Lambda| < p$ is only possible if every element of Λ is fixed by P_n. Since we are assuming that Λ generates X_n as an abelian group, we conclude that P_n acts trivially on X_n. This can happen only if $p = q = 2$. Since these values are ruled out by our definition (11) of q, we have proved the proposition for $r = 1$.

In the previous section we subdivided the integers $1, 2, \dots, p^r$ into p big blocks $B_1, \dots, B_p$ of length p^{r-1}. Now we will now work with small blocks $b_1, \dots, b_{p^{r-1}}$,

where b_j consists of the p consecutive integers

$$(j-1)p+1,\ (j-1)p+2,\ \ldots,\ jp.$$

We can identify $P_{p^{r-1}}$ with the subgroup of P_{p^r} that permutes the small blocks $b_1, \ldots, b_{p^{r-1}}$ without changing the order of the elements in each block.

For the induction step, assume $r \geq 2$ and consider the homomorphism $\Sigma : X_{p^r} \to X_{p^{r-1}}$ given by

$$\mathbf{a} = (a_1, a_2, \ldots, a_{p^r}) \mapsto \mathbf{s} = (s_1, \ldots, s_{p^{r-1}}), \tag{13}$$

where $s_i = a_{(i-1)p+1} + a_{(i-1)p+2} + \cdots + a_{ip}$ is the sum of the entries of $\mathbf{a}$ in the i-th small block b_i. Thus

(i) if Λ generates X_{p^r} then $\Sigma(\Lambda)$ generates $X_{p^{r-1}}$.

(ii) if Λ is a P_{p^r}-invariant subset of X_{p^r} then $\Sigma(\Lambda)$ is a $P_{p^{r-1}}$-invariant subset of $X_{p^{r-1}}$.

Let us remove from $\Sigma(\Lambda)$ all elements which lie in $pX_{p^{r-1}}$. The resulting set, $\Sigma(\Lambda) \setminus pX_{p^{r-1}}$, is clearly $P_{p^{r-1}}$-invariant. By Lemma 7.3 this set generates $X_{p^{r-1}}$. Thus by the induction assumption $|\Sigma(\Lambda) \setminus pX_{p^{r-1}}| \geq p^{2r-3}$.

We claim that the fiber of each element $\mathbf{s} = (s_1, \ldots, s_{p^{r-1}})$ in $\Sigma(\Lambda) \setminus pX_{p^{r-1}}$ has at least p^2 elements in Λ. If we can show this, then we will be able to conclude that

$$|\Lambda| \geq p^2 \cdot |\Sigma(\Lambda) \setminus pX_{p^{r-1}}| \geq p^2 \cdot p^{2r-3} = p^{2r-1},$$

thus completing the proof of Proposition 7.2.

Let σ_i be the single p-cycle, cyclically permuting the elements in the small block b_i. To prove the claim, note that the subgroup

$$\langle \sigma_i \mid i = 1, \ldots, p^{r-1} \rangle \simeq (\mathbb{Z}/p\mathbb{Z})^{p^{r-1}}$$

of P_n acts on each fiber of Σ.

To simplify the exposition in the argument to follow, we introduce the following bit of terminology. Let us say that $\mathbf{a} \in (\mathbb{Z}/q\mathbb{Z})^n$ is *scalar in the small block* b_i if all the entries of $\mathbf{a}$ in the block b_i are the same, that is, if

$$a_{(i-1)p+1} = a_{(i-1)p+2} = \cdots = a_{ip}.$$

We are now ready to prove the claim. Suppose $\mathbf{a} = (a_1, \ldots, a_{p^r}) \in X_{p^r}$ lies in the preimage of $\mathbf{s} = (s_1, \ldots, s_{p^{r-1}})$, as in (13). If $\mathbf{a}$ is scalar in the small block b_i then clearly

$$s_i = a_{(i-1)p+1} + a_{(i-1)p+2} + \cdots + a_{ip} \in p\mathbb{Z}/q\mathbb{Z}.$$

Since we are assuming that $\mathbf{s}$ lies in

$$\Sigma(\Lambda) \setminus pX_{p^{r-1}},$$

s must have at least two entries that are not divisible by p, say, s_i and s_j. (Recall that $s_1+\cdots+s_{p^r}=0$ in $\mathbb{Z}/q\mathbb{Z}$, so **s** cannot have exactly one entry not divisible by p.) Thus **a** is nonscalar in the small blocks b_i and b_j. Consequently, the elements $\sigma_i^\alpha\sigma_j^\beta(a)$ are distinct, as α and β range between 0 and $p-1$. All of these elements lie in the fiber of **s** under Σ. Therefore we conclude that this fiber contains at least p^2 distinct elements. This completes the proof of the claim and thus of Proposition 7.2, Proposition 5.1(c) and Theorem 1.1(c). □

8. Proof of Theorem 1.1(d)

In this section we assume that n is divisible by p but is not a power of p. We will modify the arguments of the last two sections to show that

$$\operatorname{ed}(T\rtimes P_n)=\operatorname{ed}(T\rtimes P_n;\,p)=p^e(n-p^e)-n+1,$$

where p^e is the highest power of p dividing n. This will complete the proof of Proposition 5.1 and thus of Theorem 1.1.

Write out the p-adic expansion

$$n=n_1p^{e_1}+n_2p^{e_2}+\cdots+n_up^{e_u}, \tag{14}$$

of n, where $1\le e=e_1<e_2<\cdots<e_u$, and $1\le n_i<p$ for each i. Subdivide the integers $1,\dots,n$ into $n_1+\cdots+n_u$ blocks B_j^i of length p^{e_i}, for j ranging over $1,2,\dots,n_i$. By our assumption there are at least two such blocks. The Sylow subgroup P_n is a direct product

$$P_n=(P_{p^{e_1}})^{n_1}\times\cdots\times(P_{p^{e_u}})^{n_u}$$

where each $P_{p^{e_i}}$ acts on one of the blocks B_j^i.

Once again we will use the strategy outlined in Section 2.

(i) We will construct a generically free representation of $T\rtimes P_n$ of dimension $p^{e_1}(n-p^{e_1})$. This will prove the upper bound $\operatorname{ed}_k(T\rtimes P_n)\le p^{e_1}(n-p^{e_1})-n+1$. Note that this construction (and thus the above inequality) do not require any assumption on the field k.

To construct this representation, let $\Lambda\subset X(T)$ be the union of the P_n-orbits of the elements

$$\mathbf{a}_{1,j+1}\text{ where }j=p^{e_1},\dots,n_1p^{e_1},\,n_1p^{e_1}+p^{e_2},\dots,n-p^{e_u},$$

i.e., the union of the P_n-orbits of elements of the form $(1,0,\dots,0,-1,0,\dots,0)$, where 1 appears in the first position of the first block and -1 appears in the first position of one of the other blocks. For $\mathbf{a}_{\alpha,\beta}$ in Λ there are p^{e_1} choices for α and $n-p^{e_1}$ choices for β. Thus

$$\dim V_\Lambda=|\Lambda|=p^{e_1}(n-p^{e_1}).$$

It is not difficult to see that Λ generates $X(T)$ as an abelian group. To conclude with Lemma 3.3 that V_Λ is a generically free representation of $T \rtimes P_n$, it remains to show that the P_n-action on the kernel of the natural morphism $\phi : \mathbb{Z}[\Lambda] \to X(T)$ is faithful when $e_1 \geq 1$. As in Section 6 we only need to check that the center Z_n of P_n acts faithfully on the kernel. Let σ be a nontrivial element of $Z_n = (Z_{p^{e_1}})^{n_1} \times \cdots \times (Z_{p^{e_u}})^{n_u}$, with each $Z_{p^{e_i}}$ cyclic of order p. Let h, h' be in the first block B_1^1 and l, l' in some other block B_i^j (there are at least two blocks each of size at least p). The element

$$\mathbf{a} = \mathbf{a}_{h,l} - \mathbf{a}_{h,l'} + \mathbf{a}_{h',l'} - \mathbf{a}_{h',l}$$

lies in the kernel of ϕ. To fix $\mathbf{a}$, σ must either (1) fix all h, h', l, l' or (2) $\sigma(h) = h', \sigma(h') = h$ and $\sigma(l) = l', \sigma(l') = l$. Since σ is nontrivial we may choose B_i^j such that (1) is not possible and if $p \neq 2$, (2) is not possible either. If $p = 2$, by (14), B_i^j is at least of size 4 and we can choose l, l' within B_i^j such that (2) does not hold. Therefore σ does not fix a nonzero element of the kernel of ϕ.

(ii) We now want to prove the lower bound,

$$\operatorname{ed}(T \rtimes P_n; p) \geq p^{e_1}(n - p^{e_1}) - n + 1.$$

Arguing as in Section 7 (and using the same notation, with $q = p$), it suffices to show that $\operatorname{ed}(T_{(p)} \rtimes P_n; p) \geq p^{e_1}(n - p^{e_1})$. By the Karpenko–Merkurjev Theorem 7.1 this is equivalent to showing that every faithful representation of $T_{(p)} \rtimes P_n$ has dimension at least $p^{e_1}(n - p^{e_1})$. By Lemma 3.1 it now suffices to prove the following lemma.

Lemma 8.1. *Let n be a positive integer, P_n be the Sylow subgroup of S_n, p^e be the highest power of p dividing n, and*

$$X_n := \big\{(a_1, \ldots, a_n) \in (\mathbb{Z}/p\mathbb{Z})^n \mid a_1 + \cdots + a_n = 0 \text{ in } \mathbb{Z}/p\mathbb{Z}\big\}.$$

Then every P_n-invariant generating subset of X_n has at least $p^e(n - p^e)$ elements.

In the statement of the lemma we allow $e = 0$, to facilitate the induction argument. For the purpose of proving the lower bound in Proposition 5.1(d) we only need this lemma for $e \geq 1$.

Proof. Once again, we consider the p-adic expansion (14) of n with $0 \leq e_1 < e_2 < \cdots < e_u$ and $1 \leq n_i < p$. We may assume that n is not a power of p, since otherwise the lemma is vacuous.

We will argue by induction on $e = e_1$. For the base case, let $e_1 = 0$. Here the lemma is obvious: since X_n has rank $n - 1$, every generating set (P_n-invariant or not) has to have at least $n - 1$ elements.

For the induction step, we may suppose $e = e_1 \geq 1$; in particular, n is divisible by p. Define $\Sigma : X_n \to X_{n/p}$ by sending $(a_1, \ldots, a_n)$ to $(s_1, \ldots, s_{n/p})$, where

$$s_j = a_{(j-1)p+1} + \cdots + a_{jp}$$

for $j = 1, \ldots, n/p$. Arguing as in Section 7 we see that $\Sigma(\Lambda) \setminus pX_{n/p}$ is a $(P_{p^{e_1-1}})^{n_1} \times \cdots \times (P_{p^{e_u-1}})^{n_u}$-invariant generating subset of $X_{n/p}$ and that every

$$\mathbf{s} \in \Sigma(\Lambda) \setminus pX_{n/p}$$

has at least p^2 preimages in Λ. By the induction assumption,

$$|\Sigma(\Lambda) \setminus pX_{n/p}| \geq p^{e-1}\Big(\frac{n}{p} - p^{e-1}\Big)$$

and thus

$$|\Lambda| \geq p^2 \cdot p^{e-1}\Big(\frac{n}{p} - p^{e-1}\Big) = p^e(n - p^e)$$

This completes the proof of Lemma 8.1 and thus of parts (d) of Proposition 5.1 and of Theorem 1.1. □

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References

[Atiyah and Macdonald 1969] M. F. Atiyah and I. G. Macdonald, *Introduction to commutative algebra*, Addison-Wesley, Reading, MA, 1969. MR 39 #4129 Zbl 0175.03601

[Berhuy and Favi 2003] G. Berhuy and G. Favi, "Essential dimension: a functorial point of view (after A. Merkurjev)", *Doc. Math.* **8** (2003), 279–330. MR 2004m:11056 Zbl 1101.14324

[Bourbaki 1990] N. Bourbaki, *Algebra II: Chapters 4–7*, Springer, Berlin, 1990. MR 91h:00003 Zbl 0719.12001

[Brosnan et al. 2008] P. Brosnan, Z. Reichstein, and A. Vistoli, "Essential dimension, spinor groups and quadratic forms", preprint, 2008, http://www.math.ubc.ca/~reichst/pfister-numbers.pdf. To appear in *Annals of Math.*

[Buhler and Reichstein 1997] J. Buhler and Z. Reichstein, "On the essential dimension of a finite group", *Compositio Math.* **106**:2 (1997), 159–179. MR 98e:12004 Zbl 0905.12003

[Buhler and Reichstein 1999] J. Buhler and Z. Reichstein, "On Tschirnhaus transformations", pp. 127–142 in *Topics in number theory (University Park, PA, 1997)*, edited by D. Scott, Math. Appl. **467**, Kluwer Acad. Publ., Dordrecht, 1999. MR 2000b:12003 Zbl 0937.12001

[Chernousov and Serre 2006] V. Chernousov and J.-P. Serre, "Lower bounds for essential dimensions via orthogonal representations", *J. Algebra* **305**:2 (2006), 1055–1070. MR 2007i:20070 Zbl 05078318

[Demazure and Gabriel 1970] M. Demazure and P. Gabriel, *Groupes algébriques, I: Géométrie algébrique, généralités, groupes commutatifs*, Masson, 1970. MR 46 #1800 Zbl 0203.23401

[Favi and Florence 2008] G. Favi and M. Florence, "Tori and essential dimension", *J. Algebra* **319**:9 (2008), 3885–3900. MR 2009b:20085 Zbl 1141.14028

[Grothendieck 1958] A. Grothendieck, "Torsion homologique et sections rationnelles", in *Séminaire C. Chevalley, 2ème année: Anneaux de Chow et applications* (Exposé no. 5), Secrétariat de mathématique, ENS, Paris, 1958.

[Jantzen 2003] J. C. Jantzen, *Representations of algebraic groups*, 2nd. ed., Mathematical Surveys and Monographs **107**, American Mathematical Society, Providence, RI, 2003. MR 2004h:20061 Zbl 1034.20041

[Karpenko and Merkurjev 2008] N. A. Karpenko and A. S. Merkurjev, "Essential dimension of finite p-groups", *Invent. Math.* **172**:3 (2008), 491–508. MR 2009b:12009 Zbl 05279042

[Lemire 2004] N. Lemire, "Essential dimension of algebraic groups and integral representations of Weyl groups", *Transform. Groups* **9**:4 (2004), 337–379. MR 2005j:20056 Zbl 1076.14060

[Lorenz and Reichstein 2000] M. Lorenz and Z. Reichstein, "Lattices and parameter reduction in division algebras", preprint 2000-001, MSRI, 2000, Available at http://www.msri.org/publications/preprints/online/2000-001.html.

[Lorenz et al. 2003] M. Lorenz, Z. Reichstein, L. H. Rowen, and D. J. Saltman, "Fields of definition for division algebras", *J. London Math. Soc.* (2) **68**:3 (2003), 651–670. MR 2004j:16022 Zbl 1071.16012

[Lötscher 2008] R. Lötscher, "Application of multihomogeneous rational covariants to the determination of essential dimension of finite groups", preprint, 2008.

[Merkurjev 2007] A. S. Merkurjev, "Essential dimension", preprint, 2007, Available at http://www.math.ucla.edu/~merkurev/papers/review.dvi. To appear in *Algebraic and arithmetic theory of quadratic forms* (Llanquihue, Chile, 2007), Contemporary Mathematics, Amer. Math. Soc.

[Merkurjev 2008] A. S. Merkurjev, "Essential p-dimension of $\mathrm{PGL}(p^2)$", preprint, 2008, Available at http://www.math.ucla.edu/~merkurev/papers/pgl.dvi.

[Milne 1980] J. S. Milne, *Étale cohomology*, Princeton Mathematical Series **33**, Princeton University Press, Princeton, NJ, 1980. MR 81j:14002 Zbl 0433.14012

[Procesi 1967] C. Procesi, "Non-commutative affine rings", *Atti Accad. Naz. Lincei Mem. Cl. Sci. Fis. Mat. Natur. Sez. I* (8) **8** (1967), 239–255. MR 37 #256 Zbl 0204.04802

[Reichstein 1999] Z. Reichstein, "On a theorem of Hermite and Joubert", *Canad. J. Math.* **51**:1 (1999), 69–95. MR 2000h:12008 Zbl 0942.12001

[Reichstein 2000] Z. Reichstein, "On the notion of essential dimension for algebraic groups", *Transform. Groups* **5**:3 (2000), 265–304. MR 2001j:20073 Zbl 0981.20033

[Reichstein and Youssin 2000] Z. Reichstein and B. Youssin, "Essential dimensions of algebraic groups and a resolution theorem for G-varieties", *Canad. J. Math.* **52**:5 (2000), 1018–1056. MR 2001:14088 Zbl 1044.14023

[Rowen 1980] L. H. Rowen, *Polynomial identities in ring theory*, Pure and Applied Mathematics **84**, Academic Press, New York, 1980. MR 82a:16021 Zbl 0461.16001

[Rowen and Saltman 1992] L. H. Rowen and D. J. Saltman, "Prime-to-p extensions of division algebras", *Israel J. Math.* **78**:2-3 (1992), 197–207. MR 93k:12006 Zbl 0795.16012

[Serre 1958] J.-P. Serre, "Modules projectifs et espaces fibrés à fibre vectorielle", in *Séminaire P. Dubreil, M.-L. Dubreil-Jacotin et C. Pisot, 1957/58,* Fasc. 2, Exposé 23, Secrétariat mathématique, Paris, 1958. MR 31 #1277 Zbl 0132.41202

[Serre 1962] J.-P. Serre, "Cohomologie galoisienne des groupes algébriques linéaires", pp. 53–68 in *Colloq. Théorie des Groupes Algébriques* (Brussels, 1962), Librairie Universitaire, Louvain, 1962. MR 32 #4177 Zbl 0145.17501

[Serre 1977] J.-P. Serre, *Linear representations of finite groups*, Graduate Texts in Mathematics, Springer, New York, 1977. 42. MR 56 #8675 Zbl 0355.20006

[Serre 1997] J.-P. Serre, *Galois cohomology*, Springer, Berlin, 1997. MR 98g:12007 ZBL 0902. 12004

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