Algebra & Number Theory

Volume 3
2009
No. 4

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mathematical sciences publishers

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Following Soulé's ideas from 1979, we give a presentation of the abstract group $\mathrm{G}(k[t])$ for any semisimple (connected) simply connected absolutely almost simple k-group $\mathrm{G}(k[t])$. As an application, we give a description of $\mathrm{G}(k[t])$ in terms of direct limits, and show that the Whitehead group and the naive group of connected components of $\mathrm{G}(k[t])$ coincide.

1. Introduction

Let k be a field, and let $\boldsymbol{G}$ be a semisimple simply connected absolutely almost simple k-group. In the case that $\boldsymbol{G}$ is split, Soulé [1979] has given a presentation of the group $\boldsymbol{G}(k[t])$, thus extending a theorem of Nagao [1959] for $\boldsymbol{SL}_2$ (see also [Serre 1977, II.1.6]). The goal of this note is to provide a presentation of $\boldsymbol{G}(k[t])$ in the general case.

We will follow Soulé's original ideas and study the action of $\boldsymbol{G}(k[t])$ on the Bruhat–Tits building [1984] of $\boldsymbol{G}$ corresponding to the field $K = k((1/t))$, where K is viewed as the completion of $k(t)$ with respect to the valuation at ∞. As an application, we show that the Whitehead group of $\boldsymbol{G}$ coincides with the naive group of connected components of $\boldsymbol{G}$.

2. Structure of the group $\boldsymbol{G}(k[t])$

Throughout k and $\boldsymbol{G}$ will be as above. For convenience the group $\boldsymbol{G}(k[t])$ will be denoted by Γ.

Notation and statement of the main theorem. Let $\boldsymbol{S}$ be a maximal k-split torus of $\boldsymbol{G}$, and let $\boldsymbol{T}$ be a maximal torus of $\boldsymbol{G}$ containing $\boldsymbol{S}$. Recall that $\boldsymbol{S}_K$ is a maximal K-split torus of $\boldsymbol{G}_K$. Let $\tilde{k}/k$ be a finite Galois extension that splits $\boldsymbol{T}$ (hence also $\boldsymbol{G}$). Set $\mathcal{G} = \mathrm{Gal}(\tilde{k}/k)$ and $\widetilde{\boldsymbol{T}} = \boldsymbol{T} \times_k \tilde{k}$.

MSC2000: primary 20G10; secondary 20F05.

Keywords: Bruhat–Tits buildings, arithmetic groups, simplicial fundamental domain.

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Let $\widetilde{\boldsymbol{G}} = \boldsymbol{G} \times_k \tilde{k}$ and $\widetilde{\boldsymbol{S}} = \boldsymbol{S} \times_k \tilde{k}$. We choose compatible orderings on the root systems $\Phi = \Phi(\boldsymbol{G}, \boldsymbol{S})$ and $\widetilde{\Phi} = \Phi(\widetilde{\boldsymbol{G}}, \widetilde{\boldsymbol{T}})$; see [Borel 1991]. We then have a set Δ of relative simple roots and a set $\widetilde{\Delta}$ of absolute simple roots.

It will be convenient to maintain essentially the same notation as in Soulé's paper:

- $A = k[t]$, $K = k((1/t))$ and $G = \boldsymbol{G}(K)$.
- ω is the valuation on K at ∞, that is, the valuation on K having $\mathbb{O} = k[\![1/t]\!]$ as its ring of integers.

We also have the analogues of the above objects for $\tilde{k}$:

- $\widetilde{A} = \tilde{k}[t]$, $\widetilde{K} = \tilde{k}((1/t))$, $\widetilde{\Gamma} = \boldsymbol{G}(\widetilde{A})$, and $\widetilde{\mathbb{O}} = \tilde{k}[\![1/t]\!]$.

At the level of buildings we set [Bruhat and Tits 1984, section 4.2]

- $\mathcal{T}$ the (affine) Bruhat–Tits building of the K-group $\boldsymbol{G}_K := \boldsymbol{G} \times_k K$, and
- $\widetilde{\mathcal{T}}$ the Bruhat–Tits building of the $\widetilde{K}$-group $\boldsymbol{G}_{\widetilde{K}} := \boldsymbol{G} \times_k \widetilde{K}$.[1]

Both $\mathcal{T}$ and $\widetilde{\mathcal{T}}$ have a natural simplicial complex structure [ibidem, section 4.2.23].

Recall that $\mathcal{T}$ is equipped with an action of $\boldsymbol{G}(K)$ and that $\widetilde{\mathcal{T}}$ is equipped with an action of $\boldsymbol{G}(\widetilde{K}) \rtimes \mathcal{G}$. We have an isometric embedding $j : \mathcal{T} \to \widetilde{\mathcal{T}}$ that identifies $\mathcal{T}$ with $\widetilde{\mathcal{T}}^{\mathcal{G}}$. The hyperspecial group $\boldsymbol{G}(\widetilde{O})$ of $\boldsymbol{G}(\widetilde{K})$ fixes a unique point $\tilde{\phi}$ of $\widetilde{\mathcal{T}}$ [Bruhat and Tits 1972, section 9.1.9.c]. This point descends to a point ϕ of $\mathcal{T}$.

We denote by $\mathcal{A}$ the standard apartment of $\mathcal{T}$ associated to $\boldsymbol{S}$ (this is a real affine space) and similarly by $\widetilde{\mathcal{A}}$ the standard apartment associated to $\widetilde{\boldsymbol{T}}$. The point $\tilde{\phi}$ belongs to $\widetilde{\mathcal{A}}$ (ibidem). Since

$$\mathrm{Hom}_{k-gr}(\boldsymbol{G}_m, \boldsymbol{S}) \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathrm{Hom}_{k-gr}(\boldsymbol{G}_m, \boldsymbol{T}) \otimes_{\mathbb{Z}} \mathbb{R} \cong (\mathrm{Hom}_{\tilde{k}-gr}(\boldsymbol{G}_{m,\tilde{k}}, \widetilde{\boldsymbol{T}}) \otimes_{\mathbb{Z}} \mathbb{R})^{\mathcal{G}}$$

[Bruhat and Tits 1984, section 4.2], we have $j(\mathcal{A}) = \widetilde{\mathcal{A}}^{\mathcal{G}}$, so ϕ belongs to $\mathcal{A}$ and

$$\mathcal{A} = \phi + \mathrm{Hom}_{k-gr}(\boldsymbol{G}_m, \boldsymbol{S}) \otimes_{\mathbb{Z}} \mathbb{R}.$$

By means of the canonical pairing $\langle \cdot, \cdot \rangle : \mathrm{Hom}_{k-gr}(\boldsymbol{S}, \boldsymbol{G}_m) \times \mathrm{Hom}_{k-gr}(\boldsymbol{S}, \boldsymbol{G}_m) \to \mathbb{Z}$ we can then define the *sector* (quartier)

$$\mathcal{Q} := \phi + D, \quad \text{where } D := \left\{ v \in \mathrm{Hom}_{k-gr}(\boldsymbol{S}, \boldsymbol{G}_m) \otimes_{\mathbb{Z}} \mathbb{R} \;\middle|\; \langle b, \lambda \rangle \geq 0, \ \forall b \in \Delta \right\}.$$

The following result generalizes Soulé's theorem [1979].

Theorem 2.1. *The set $\mathcal{Q}$ is a simplicial fundamental domain for the action of $\boldsymbol{G}(k[t])$ on $\mathcal{T}$. In other words, any simplex of $\mathcal{T}$ is equivalent under the action of $\boldsymbol{G}(k[t])$ to a unique simplex of $\mathcal{Q}$.*

[1] Since $\boldsymbol{G} \times_K \widetilde{K}$ is split, the assumptions of [Bruhat and Tits 1984, section 5.1.1.1] are satisfied. This allows us to do away with the "standard" assumption that the base field k be perfect.

Buildings and valuations. Let $\boldsymbol{P}$ be the minimal parabolic k-subgroup of $\boldsymbol{G}$ defined by $\boldsymbol{S}$ and Δ. We denote by $\boldsymbol{U} = R_u(\boldsymbol{P})$ the unipotent radical of $\boldsymbol{P}$.

We denote by $\widetilde{\boldsymbol{U}}_{\tilde{a}}$ the split unipotent subgroup associated to a root $\tilde{a} \in \widetilde{\Phi}$, and by $\tilde{a}^\vee : \boldsymbol{SL}_2 \to \boldsymbol{G}$ the corresponding standard homomorphism; see [Springer 1979, Section 2.2].

The set of positive and negative roots with respect to the basis Δ of Φ will be denoted by Φ^+ and Φ^-, respectively. Given $b \in \Phi$, the subset of absolute roots

$$\widetilde{\Phi}^b := \left\{\tilde{a} \in \widetilde{\Phi} \;\middle|\; \tilde{a}|_{S\times_k \tilde{k}} = b \text{ or } 2b\right\}$$

is positively closed in $\widetilde{\Phi}$. It defines then a split $\tilde{k}$-unipotent subgroup $\widetilde{\boldsymbol{U}}_b$ of $\widetilde{\boldsymbol{G}}$ that descends to a split k-unipotent subgroup $\boldsymbol{U}_b$ of $\boldsymbol{G}$. As in [Bruhat and Tits 1972], we make the convention that $\boldsymbol{U}_{2b} = 1$ if $2b \notin \Phi$.

For $I \subset \Delta$, we define along standard lines

$$\boldsymbol{S}_I = \Big(\bigcap_{b\in I} \ker(b)\Big)^0 \subset \boldsymbol{S}, \quad \boldsymbol{L}_I = \mathscr{Z}_{\boldsymbol{G}}(\boldsymbol{S}_I), \quad \boldsymbol{P}_I = \boldsymbol{U}_I \rtimes \boldsymbol{L}_I.$$

Thus $\boldsymbol{P}_I$ is the standard parabolic subgroup of $\boldsymbol{G}$ of type I and $\boldsymbol{L}_I$ is its standard Levi subgroup (see [Borel 1991, Section 21.11]). Recall that the root system $\Phi(\boldsymbol{L}_I, \boldsymbol{S}) = [I]$ is the subroot system of Φ consisting of roots that are linear combinations of I; the split unipotent k-group $\boldsymbol{U}_I$ is the subgroup of $\boldsymbol{U}$ generated by the $\boldsymbol{U}_b$ with b running over $\Phi^+ \setminus [I]$.

Given $\tilde{a} \in \widetilde{\Phi}$, the group $\widetilde{U}_{\tilde{a}} := \widetilde{\boldsymbol{U}}_{\tilde{a}}(\widetilde{K}) = \widetilde{K}$ is equipped with the valuation ω, which we denote by $\widetilde{\varphi}_a : \widetilde{U}_a \to \mathbb{R} \cup \{\infty\}$. This defines the Chevalley–Steinberg "donnée radicielle valuée"

$$(T(\widetilde{K}), (\widetilde{U}_{\tilde{a}}, M_{\tilde{a}})_{\tilde{a}\in\widetilde{\Phi}}), \quad \text{where } M_{\tilde{a}} = T(\widetilde{K})\ \tilde{a}^\vee\left(\left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]\right)$$

[Bruhat and Tits 1972, exemple 6.2.3.b], and also a filtration $(\widetilde{U}_{\tilde{a},m})_{m\in\mathbb{Z}}$ of $\widetilde{U}_{\tilde{a}}$ where $\widetilde{U}_{\tilde{a},m} := \widetilde{\varphi}^{-1}([m, +\infty[)$. Note that $\widetilde{U}_{\tilde{a},0} = \widetilde{\boldsymbol{U}}_{\tilde{a}}(\widetilde{\mathbb{O}})$.

A crucial point of Bruhat–Tits theory is the descent of this data to $G = \boldsymbol{G}(K)$ [1984, section 5.1]. Given $b \in \Phi$, the commutative group $U_b := \boldsymbol{U}_b(K)$ is equipped with the descended valuation $\varphi_b : U_b \to \mathbb{R} \cup \{\infty\}$. The definition of φ_b is delicate, and is given as follows [Bruhat and Tits 1984, section 5.1.16]. Define

$$\widetilde{U}_{b,m} := \prod_{\substack{\tilde{a}\in\widetilde{\Phi}^b,\\ \tilde{a}|_{S\times_k\tilde{k}}=b}} \widetilde{U}_{\tilde{a},m} \cdot \prod_{\substack{\tilde{a}\in\widetilde{\Phi}^b,\\ \tilde{a}|_{S\times_k\tilde{k}}=2b}} \widetilde{U}_{\tilde{a},2m} \quad \text{for } m \in \mathbb{R}.$$

Then U_b is a subgroup of $\boldsymbol{U}_b(\widetilde{K}) = \widetilde{U}_b = \bigcup_{m\in\mathbb{R}} \widetilde{U}_{b,m}$ and the descended valuation is defined by

$$\varphi_b(u) := \operatorname{Sup}\{m \in \mathbb{R} \mid u \in \widetilde{U}_{b,m}\}.$$

Note that[2] $\Theta_b := \varphi_b\big(U_b \setminus \{e\}\big)$ is either $\mathbb{Z}$ or $\frac{1}{2}\mathbb{Z}$. As above, it gives rises to a filtration $(U_{b,m})_{m\in\Theta_b}$ of U_b such that $U_{b,0} = \boldsymbol{U}_b(\mathbb{O})$.

Again we make the convention that $U_{2b} = 1$ if $2b \notin \Phi$.

Description of the isotropy group of a vertex. Given $\Omega \subset \mathfrak{Q}$, we denote by Γ_Ω the corresponding isotropy subgroup, namely the elements of Γ that fix all elements of Ω. We introduce an analogous definition and notation for $j(\Omega) \in \widetilde{\mathscr{A}}$. By Galois descent we have

$$\Gamma_\Omega = \big(\widetilde{\Gamma}_{j(\Omega)}\big)^{\mathscr{G}}. \tag{2-1}$$

In particular, since $\widetilde{\Gamma}_{\tilde{\phi}} = \boldsymbol{G}(\widetilde{\mathbb{O}}) \cap \widetilde{\Gamma} = \boldsymbol{G}(\tilde{k})$ [Soulé 1973, section 1.1], we have $\Gamma_\phi = (\widetilde{\Gamma}_{\tilde{\phi}})^{\mathscr{G}} = \boldsymbol{G}(\tilde{k})^{\mathscr{G}} = \boldsymbol{G}(k)$.

If $x \in \mathfrak{Q} \setminus \{\phi\}$ and if $[x[$ is the halfline of origin x and direction $\overrightarrow{\phi x}$, we claim that $\Gamma_x = \Gamma_{[x[}$. If $\boldsymbol{G}$ is split, this is proven in Soulé's paper by reduction to the case of $\boldsymbol{SL}_n$. By applying the identity (2-1) to x and $[x[$, our claim now readily follows from the absolute case.

The isotropy of $[x[$ in $G = \boldsymbol{G}(K)$ is the Bruhat–Tits abstract parahoric group $P_{[x[}$. See [Bruhat and Tits 1972, section 7.1]. We have

$$P_{[x[} = U_{[x[}\,.\,H, \quad \text{where } H = \mathrm{Fix}_G(\mathscr{A}).$$

By [Bruhat and Tits 1984, section 5.2.2], we have $H = \mathscr{Z}_{\boldsymbol{G}}(\boldsymbol{S})(\mathbb{O})$. The group $U_{[x[}$ is defined by means of the function [Bruhat and Tits 1972, section 6.4.2]

$$f_{[x[} : \Phi \to \mathbb{R} \cup \{\infty\}, \quad b \mapsto \inf\{s \in \mathbb{R} \mid b(y) + s \geq 0 \text{ for all } y \in [x[\}.$$

Hence

$$f_{[x[}(b) = \begin{cases} 0 & \text{if } b(x) = 0, \\ -b(x) & \text{if } b(x) > 0, \\ \infty & \text{if } b(x) < 0. \end{cases}$$

The group $U_{[x[} \subset G$ is then the subgroup of G generated by the $U_{b,m}$ for $b \in \Phi^+$ and $m \geq -b(x)$ ($m \in \Theta_b$), together with the $\boldsymbol{U}_b(\mathbb{O})$ for $b \in \Phi^-$ such that $b(x) = 0$. In other words, by distinguishing positive roots that vanish at x, we see that $U_{[x[}$ is the subgroup of G generated by subgroups of the following three "shapes":

(I) $U_{b,m}$ for $b \in \Phi^+$ such that $b(x) > 0$ and $m \in \Theta_b$ such that $m \geq -b(x)$;

(II) $\boldsymbol{U}_b(\mathbb{O})$ for $b \in \Phi^+$ such that $b(x) = 0$;

(III) $\boldsymbol{U}_b(\mathbb{O})$ for $b \in \Phi^-$ such that $b(x) = 0$.

Define $U^{\pm}_{[x[} := U_{[x[} \cap \boldsymbol{U}^{\pm}(K)$ as in [Bruhat and Tits 1972, section 6.4.2]. These by definition generate $U_{[x[}$. On the other hand, $U^{+}_{[x[}$ (respectively $U^{-}_{[x[}$) is the subgroup

[2] We use the notation Θ_b rather than the more standard Γ_b found in [Bruhat and Tits 1972] to avoid any possible confusion with the notation used in Soulé's paper.

of $U_{[x[}$ generated by the subgroups of type (I) and (II) (respectively (III)); see [Bruhat and Tits 1972, proposition 6.4.9]. Define the subset of roots

$$I_x = \{b \in \Delta \mid b(x) = 0\}.$$

This definition makes sense if x is an element of $\mathscr{A}$, and we then have $I_\phi = \Delta$.

Lemma 2.2. *We have*

$$[I_x] \cap \Phi^+ = \{b \in \Phi^+ \mid b(x) = 0\}, \tag{2-2}$$

$$\Phi^+ \setminus [I_x] = \{b \in \Phi^+ \mid b(x) > 0\}, \tag{2-3}$$

$$[I_x] \cap \Phi^- = \{b \in \Phi^- \mid b(x) = 0\}. \tag{2-4}$$

Proof. Observe that if $b \in [I_x]$, then b is a linear combination of elements of I_x; hence $b(x) = 0$. This implies that $[I_x] \cap \Phi^+ \subset \{b \in \Phi^+ \mid b(x) = 0\}$. Conversely, let b be a positive root such that $b(x) = 0$. Then $b = \sum_{c \in \Delta} n_c c$, where the n_c are nonnegative integers. Hence $\sum_{c \in \Delta} n_c c(x) = 0$. Since $x \in \mathscr{Q}$, we have $c(x) \geq 0$. Therefore $n_c c(x) = 0$ and b is a linear combination of elements of I_x, proving (2-2). Since

$$\{b \in \Phi^+ \mid b(x) \neq 0\} = \{b \in \Phi^+ \mid b(x) > 0\},$$

we get also (2-3). Similar considerations apply to (2-4). □

It follows from (2-2) and (2-4) respectively that the subgroups of shape (II) and (III) are subgroups of $\boldsymbol{L}_{I_x}(\mathbb{O})$, and (2-3) shows that the subgroups of shape (I) are subgroups of $\boldsymbol{U}_{I_x}(K)$. Hence we get the inclusion

$$U_{[x[} \subset (U_{[x[} \cap \boldsymbol{U}_{I_x}(K)) \rtimes \boldsymbol{L}_{I_x}(\mathbb{O}) \subset \boldsymbol{P}_{I_x}(K). \tag{2-5}$$

Lemma 2.3. (1) $\boldsymbol{L}_{I_x}(\mathbb{O}) \subset P_{[x[} \subset \boldsymbol{U}_{I_x}(K) \rtimes \boldsymbol{L}_{I_x}(\mathbb{O}) \subset \boldsymbol{P}_{I_x}(K)$;

(2) $\boldsymbol{U}_{I_x}(K) \cap P_{[x[} \subset U^+_{[x[}$;

(3) $\bigcup_{z \geq 1} (U^+_{[zx[} \cap \boldsymbol{U}_{I_x}(K)) = \boldsymbol{U}_{I_x}(K)$.

Proof. Let $I = I_x$.

(1) Since $U_{[x[} \subset \boldsymbol{U}_I(K) \rtimes \boldsymbol{L}_I(\mathbb{O})$ and $\mathscr{Z}_{\boldsymbol{G}}(\boldsymbol{S}) \subset \boldsymbol{L}_I$, it follows that $P_{[x[} = U_{[x[} \,.\, H = U_{[x[} \,.\, \mathscr{Z}_{\boldsymbol{G}}(\boldsymbol{S})(\mathbb{O})$ is a subgroup of $\boldsymbol{U}_I(K) \rtimes \boldsymbol{L}_I(\mathbb{O})$.

Let us show that $\boldsymbol{L}_I(\mathbb{O}) \subset P_{[x[}$. Let $\boldsymbol{V}_I$ be the unipotent radical of the minimal standard parabolic subgroup of $\boldsymbol{L}_I$, namely the k-subgroup of $\boldsymbol{U}$ generated by the $\boldsymbol{U}_b$ such that $b \in \Phi^+$ and $b(x) = 0$. We have [SGA3 1962/1964, théorème XXVI.5.1]

$$\bigcup_{g \in \boldsymbol{V}_I(k)} g\Omega = \boldsymbol{L}_I,$$

where Ω stands for the big cell $V_I^- \times_k \mathscr{Z}_G(S) \times_k V_I$ of L_I. Since $\mathbb{O}$ is local, it follows that

$$L_I(\mathbb{O}) = V_I(k) \,.\, \Omega(\mathbb{O}) = V_I(k) \,.\, V_I^-(\mathbb{O}) \,.\, H \,.\, V_I(\mathbb{O}).$$

We conclude that $L_I(\mathbb{O}) \subset P_{[x[}$.

(2) We claim that $U(K) \cap P_{[x[} = U^+_{[x[}$. This establishes (2) since $U_I(K) \subset U(K)$. To prove the claim, we need to show that $U(K) \cap P_{[x[} \subset U^+_{[x[}$ (the reversed inclusion is obvious). With the notations of [Bruhat and Tits 1972, section 7], we have $U(K) = U^+_D$ where D is the direction of the sector $\mathscr{Q}$. By [ibidem, 7.1.4], we have

$$P_{[x[} \cap U(K) = U_{[x[+D},$$

where $U_{[x[+D}$ is the subgroup of $G(K)$ attached to the subset $[x[+ D = x + D$ of $\mathscr{A}$. This group is defined by means of the function [ibidem, section 6.4.2]

$$f_{x+D} : \Phi \to \mathbb{R} \cup \{\infty\}, \quad b \mapsto \inf\bigl\{s \in \mathbb{R} \bigm| b(y) + s \geq 0 \text{ for all } y \in x + D\bigr\}.$$

Hence

$$f_{x+D}(b) = \begin{cases} -b(x) & \text{if } b > 0, \\ \infty & \text{if } b < 0, \end{cases}$$

so $U_{x+D} = U^+_{[x[}$ as desired.

(3) If $b \in \Phi^+$ satisfies $b(x) > 0$, then the number $\operatorname{Inf}\{m \in \Theta_b \mid m + b(zx) \geq 0\}$ tends to $-\infty$ as z tends to ∞. This readily yields $\bigcup_{z \geq 1} (U^+_{[zx[} \cap U_I(K)) = U_I(K)$. $\square$

Remark 2.4. Geometrically speaking, the K-parabolic $P_{I_x} \times_k K$ is attached to the extremity of the halfline $[x[$ in the spherical building at infinity; see [Garrett 1997, Section 16.9]. Since $P_{[x[}$ is the isotropy group of the half line $[x[$, it fixes its extremity. This point of view yields another way to prove the inclusion $P_{[x[} \subset P_{I_x}(K)$ which is part of Lemma 2.3(1).

Given $b \in \Phi$, we set

$$m_x(b) := \operatorname{Inf}\{m \in \Theta_b \mid m + b(x) \geq 0\}.$$

Since $\Gamma_x = P_{[x[} \cap \Gamma$, we have the inclusion

$$\bigl\langle (U_{b,m_x(b)} \,.\, U_{2b,m_x(2b)}) \cap \Gamma,\ b \in \Phi,\ b(x) \geq 0 \bigr\rangle \subset \Gamma_x. \tag{2-6}$$

Proposition 2.5. (1) $\Gamma_x = \bigl(\Gamma_x \cap U_{I_x}(K)\bigr) \rtimes L_{I_x}(k)$;

(2) $\Gamma_x = \bigl\langle (U_{b,m_x(b)} . U_{2b,m_x(2b)}) \cap \Gamma,\ b(x) > 0 \bigr\rangle \rtimes L_{I_x}(k)$;

(3) $\bigcup_{z \geq 1} \Gamma_{zx} = U_{I_x}(k[t]) \rtimes L_{I_x}(k)$.

Proof. To lighten the notation we set $I = I_x$.

(1) According to Lemma 2.3(1), $\boldsymbol{L}_I(k) = \Gamma \cap \boldsymbol{L}_I(\mathbb{O})$ fixes the point x. Hence the inclusion

$$(\Gamma_x \cap \boldsymbol{U}_I(K)) \rtimes \boldsymbol{L}_I(k) \subset \Gamma_x.$$

To prove the reverse inclusion we use the projection $\boldsymbol{P}_I(K) \to \boldsymbol{L}_I(K)$. The image of Γ_x inside $\boldsymbol{L}_I(K)$ is a subgroup of $\boldsymbol{L}_I(A)$. On the other hand, by Lemma 2.3(1), the image of P_x inside $\boldsymbol{L}_I(K)$ is the subgroup $\boldsymbol{L}_I(\mathbb{O})$. Hence the image of Γ_x inside $\boldsymbol{L}_I(K)$ is a subgroup of $\boldsymbol{L}_I(A) \cap \boldsymbol{L}_I(\mathbb{O}) = \boldsymbol{L}_I(k)$. We thus have an exact sequence

$$1 \to (\Gamma_x \cap \boldsymbol{U}_I(K)) \to \Gamma_x \to \boldsymbol{L}_I(k)$$

which is a split surjection.

(2) Put $V := \langle (U_{b,m_x(b)} \,.\, U_{2b,m_x(2b)}) \cap \Gamma,\ b \in \Phi,\ b(x) > 0 \rangle$. This is a subgroup of Γ_x by (2-6) and of $U_I(K)$ by (2-5). So $V \subset \Gamma_x \cap U_I(K)$. For showing the reverse inclusion, it suffices to show that

$$\Gamma_x \cap \boldsymbol{U}_I(K) \subset \big\langle (U_{b,m_x(b)} \,.\, U_{2b,m_x(2b)}) \cap \Gamma,\ b(x) \geq 0 \big\rangle. \tag{2-7}$$

From Lemma 2.3(3) we have $\Gamma_x \cap \boldsymbol{U}_I(K) \subset \Gamma \cap U^+_{[x[}$. Accordingly, it will suffice to show that $\Gamma_x \cap U^+_{[x[}$ is a subgroup of the right side of (2-7). Let $\Phi^+_{\mathrm{red}} = \{b_1, \ldots, b_N\}$ be the subset of reduced positive roots (with an arbitrary order). The product induces a isomorphism of k-varieties $\prod_{j=1}^N \boldsymbol{U}_{b_j} \xrightarrow{\sim} \boldsymbol{U}$ by [Borel 1991, Proposition 21.9]. In particular, we have compatible bijections

$$\begin{array}{ccc} \prod_{j=1}^N \boldsymbol{U}_{b_j}(K) & \xrightarrow{\sim} & \boldsymbol{U}(K) \\ \cup & & \cup \\ \prod_{j=1}^N \boldsymbol{U}_{b_j}(A) & \xrightarrow{\sim} & \boldsymbol{U}(A). \end{array}$$

By comparing these with the bijection [Bruhat and Tits 1972, section 6.4.9]

$$\prod_{j=1}^N U_{b_j,m_x(b_j)} \,.\, U_{2b_j,m_x(2b_j)} \xrightarrow{\sim} U^+_{[x[},$$

we can see that $\Gamma_x \cap \boldsymbol{U}_I(K) \subset U^+_{[x[} \cap \boldsymbol{U}(A)$ consists of products of elements $(U_{b_j,m_x(b_j)} \,.\, U_{2b_j,m_x(2b_j)}) \cap \Gamma$ with $b_j(x) \geq 0$.

(3) This follows from (1) and Lemma 2.3(3). $\square$

Action on the star of certain points. We will now make use of the spherical building $\mathscr{B}(\boldsymbol{G})$ of $\boldsymbol{G}$ from [Tits 1974, Section 5]. Recall that $\mathscr{B}(\boldsymbol{G})$ is a simplicial complex whose simplexes are the k-parabolic subgroups of $\boldsymbol{G}$. If $\boldsymbol{Q}$ is such a parabolic subgroup, the faces of its associated simplex are the simplexes associated to the maximal proper k-parabolic subgroups of $\boldsymbol{Q}$. The standard apartment $\mathfrak{A}$ of $\mathscr{B}(\boldsymbol{G})$ is the subcomplex of k-parabolic subgroups containing $\boldsymbol{S}$, and the standard chamber $\mathfrak{C}$ is the simplex associated to the minimal k-parabolic subgroup $\boldsymbol{P}$. We denote by $\boldsymbol{W} = \boldsymbol{N_G}(\boldsymbol{S})/\mathscr{Z}_{\boldsymbol{G}}(\boldsymbol{S})$ the relative Weyl group of $\boldsymbol{G}$.

If $x \in \mathscr{T}$, we denote by $\mathscr{L}_x$ the *star* of x (étoile in French),[3] that is, the subspace of $\mathscr{T}$ consisting of facets F such that $x \in \overline{F}$ [Bruhat and Tits 1984, section 4.6.33].

We denote by $\boldsymbol{S}_* = \mathrm{Hom}_{k-gr}(\boldsymbol{G}_m, \boldsymbol{S})$ the group of cocharacters of $\boldsymbol{S}$. Inside the apartment $\mathscr{A} = \phi + \boldsymbol{S}_* \otimes_{\mathbb{Z}} \mathbb{R}$, this corresponds to the lattice of points having type 0, that is, the type of ϕ. The action of $\boldsymbol{S}(K)$ on $\mathscr{T}$ preserves $\mathscr{A}$. More precisely, the element $s \in S(K)$ acts on $\mathscr{A}$ as the translation by the vector v_s defined by the property [Bruhat and Tits 1984, section 5.1.22]

$$\langle v_s, b\rangle = -\omega(b(s)) \quad \text{for all } b \in \Phi. \tag{2-8}$$

We denote by $\mathscr{C} \subset S_* \otimes_{\mathbb{Z}} \mathbb{R}$ the vector chamber such that $\phi + \mathscr{C}$ is the unique chamber of the sector $\mathscr{Q}$ that contains the special point ϕ in its adherence; see [Bruhat and Tits 1972, section 1.3.11].

Lemma 2.6. *Let x be a point of $\boldsymbol{S}_* \cap \mathscr{Q}$. Then the chambers of $\mathscr{L}_x \cap \mathscr{Q}$ are the $x + w\mathscr{C}$ for $w \in \boldsymbol{W}(k)$ satisfying $I_x \subset w \,.\, \Phi^+$.*

Proof. Set $I = I_x$. The chambers of $\mathscr{L}_x$ are the $x + w\mathscr{C}$ with $w \in \boldsymbol{W}(k)$. Let $y \in \mathscr{C}$. If $x + w\mathscr{C} \subset \mathscr{Q}$, then

$$b(x + w \,.\, y) = b(x) + (w^{-1} \,.\, b)(y) \geq 0 \quad \text{for all } b \in \Delta.$$

It follows that if $b \in I$, that is, $b(x) = 0$, then $(w^{-1} \,.\, b)(y) \geq 0$, and therefore $b \in w(\Phi_+)$. Conversely, if $w \in \boldsymbol{W}(k)$ satisfies $I \subset w(\Phi_+)$, then the inequality above holds for ϵy for all $b \in \Delta$ for $\epsilon > 0$ small enough. Thus $x + w \,.\, (\epsilon y) \in \mathscr{Q}$ and $x + w\mathscr{C} \subset \mathscr{Q}$. □

Lemma 2.7. *Let I be a subset of Δ, and set $\boldsymbol{W}_I := \boldsymbol{N}_{\boldsymbol{L}_I}(\boldsymbol{S})/\mathscr{Z}_{\boldsymbol{G}}(\boldsymbol{S})$. Let $\mathfrak{A}_I$ be the union of the $w\overline{\mathfrak{C}}$ for w running over the elements of $\boldsymbol{W}(k)$ satisfying $I \subset w \,.\, \Phi^+$.*

(1) $\boldsymbol{W}_I(k) \,.\, \mathfrak{A}_I = \mathfrak{A}$.

(2) $\boldsymbol{P}_I(k) \,.\, \mathfrak{A}_I = \mathscr{B}(\boldsymbol{G})$.

Proof. (1) We reason by induction on the cardinality of I. If $I = \varnothing$, then $\mathfrak{A}_I = \mathfrak{A}$ and there is nothing to prove. Assume that $I = I' \cup \{b\}$. We are given a chamber $w\mathfrak{C}$ of $\mathfrak{A}$ with $w \in \boldsymbol{W}(k)$. We want to show that $w\mathfrak{C}$ is equivalent under $\boldsymbol{W}_I(k)$ to a

[3]The terminology *link* is also used in the literature.

chamber of $\mathfrak{A}_I$. Since $\boldsymbol{W}_{I'}(k) \subset \boldsymbol{W}_I(k)$, we can assume by the induction hypothesis that $I' \subset w\,.\,\Phi^+$. If $b \in w\,.\,\Phi^+$, we have $I \subset w\,.\,\Phi^+$. The other case is when $-b \in w\,.\,\Phi^+$. Let $s_b \in \boldsymbol{W}_I(k)$ be the reflection associated to b. Then $s_b(b) = -b$; hence $b \subset s_b w\,.\,\Phi^+$. For $b' \in I'$, we have $s_b(b') = b' + mb$, where m is nonnegative. Therefore

$$b' = s_b^2(b') = s_b(b' + mb) = s_b(b') - mb \in s_b w\,.\,\Phi^+.$$

We conclude that $I \subset s_b w\,.\,\Phi^+$ and $s_b\,.\,(w\mathfrak{C}) \subset \mathfrak{A}_I$.

(2) Again it suffices to prove that any chamber of $\mathscr{B}(\boldsymbol{G})$ is equivalent under $\boldsymbol{P}_I(k)$ to a chamber of $\mathfrak{A}_I$. Let $\mathfrak{C}'$ be a chamber of $\mathscr{B}(\boldsymbol{G})$. Let $\boldsymbol{P}'$ be the underlying minimal k-parabolic subgroup. By [Borel and Tits 1965, Proposition 4.4.b], $\boldsymbol{P}_I \cap \boldsymbol{P}'$ contains a maximal k-split torus of $\boldsymbol{P}_I$. Since maximal k-split tori of $\boldsymbol{P}_I$ are conjugate under $\boldsymbol{U}_I(k)$, it follows that there exists $u \in \boldsymbol{U}_I(k)$ such that $u\boldsymbol{S}u^{-1} \subset \boldsymbol{P}_I \cap \boldsymbol{P}'$; hence $\boldsymbol{S} \subset u^{-1}\boldsymbol{P}'u$. So we can assume that $\boldsymbol{S} \subset \boldsymbol{P}'$, that is, that $\mathfrak{C}' \subset \mathfrak{A}$. Then $\mathfrak{C}' = w\mathfrak{C}$ for some $w \in \boldsymbol{W}(k)$. By (1), $\mathfrak{C}'$ is then equivalent under $\boldsymbol{W}_I(k)$ to a chamber of $\mathfrak{A}_I$. Since $\boldsymbol{N}_{\boldsymbol{L}_I}(\boldsymbol{S})(k)$ maps onto $\boldsymbol{W}_I(k)$, we conclude that $\mathfrak{C}'$ is then equivalent under $\boldsymbol{P}_I(k)$ to a chamber of $\mathfrak{A}_I$. ☐

We come now to the following important step in Soulé's proof.

Lemma 2.8. *Let $x \in \boldsymbol{S}_* \cap \mathfrak{Q}$. Then $\Gamma_x\,.\,(\mathscr{L}_x \cap \mathfrak{Q}) = \mathscr{L}_x$.*

Proof. We will make use of the canonical smooth model $\mathfrak{P}_x/\mathbb{O}$ of the parahoric subgroup associated to x [Bruhat and Tits 1984, section 5.2]. As an $\mathbb{O}$-group scheme, $\mathfrak{P}_x$ is isomorphic to $\boldsymbol{G} \times_k \mathbb{O}$, and we have an identification $\mathfrak{P}_x(\mathbb{O}) = P_x$. The star $\mathscr{L}_x$ is the spherical building of $\mathfrak{P}_x \times_{\mathbb{O}} k \cong \boldsymbol{G}$; see [Bruhat and Tits 1984, section 5.1.32]. Set for convenience $I = I_x$. By Lemma 2.6, $\mathscr{L}_x \cap \mathfrak{Q}$ is identified with $\mathfrak{A}_I$ in the spherical building $\mathscr{B}(\boldsymbol{G})$. Furthermore, the chamber $x + \mathscr{C}$ identifies with $\mathfrak{C}$.

The inclusion $\Gamma_x\,.\,(\mathscr{L}_x \cap \mathfrak{Q}) \subset \mathscr{L}_x$ is clear. Let us prove the reverse inclusion. By definition, there exists $\lambda \in \boldsymbol{S}_* \cap \mathfrak{Q}$ such that $x = \lambda$. Define $g_\lambda = \lambda(1/t)^{-1} = \lambda(t) \in \boldsymbol{S}(K)$. Since $x = g_\lambda\,.\,\phi$ by (2-8) above, we have

$$P_x = g_\lambda P_\phi g_\lambda^{-1}. \tag{2-9}$$

Thus $\mathfrak{P}_x(\mathbb{O}) \cong P_x = g_\lambda \boldsymbol{G}(\mathbb{O}) g_\lambda^{-1} \subset \boldsymbol{G}(K)$. In view of Lemma 2.7(2), it will suffice to establish the following.

Claim 2.9. *The image of the composite map*

$$\Gamma_x \subset P_x \longrightarrow (\mathfrak{P}_x \times_{\mathbb{O}} k)(k) \cong \boldsymbol{G}(k)$$

contains $\boldsymbol{P}_I(k)$.

The group $\boldsymbol{L}_I(k)$ commutes with g_λ inside $\boldsymbol{G}(k(t))$, and it is therefore included in the image in question (as we have already observed in Proposition 2.5). So it is enough to check that $g_\lambda \boldsymbol{U}(k) g_\lambda^{-1} \subset \Gamma_x$, or equivalently that $g_\lambda \boldsymbol{U}(k) g_\lambda^{-1} \subset \Gamma$. This can be verified by working over the field $\tilde{k}$ and checking the inclusion for the subgroups $\boldsymbol{U}_b(\tilde{k})$ of $\boldsymbol{U}(\tilde{k})$ for $b \in \Phi^+$. To verify this, we use that the product map induces a decomposition (with the notation of page 395)

$$\prod_{\substack{\tilde{a} \in \widetilde{\Phi}^b, \\ \tilde{a}|_{S \times_k \tilde{k}} = b}} \widetilde{\boldsymbol{U}}_{\tilde{a}}(\tilde{k}) \,.\, \prod_{\substack{\tilde{a} \in \widetilde{\Phi}^b, \\ \tilde{a}|_{S \times_k \tilde{k}} = 2b}} \widetilde{\boldsymbol{U}}_{\tilde{a}}(\tilde{k}) \xrightarrow{\sim} \boldsymbol{U}_b(\tilde{k}).$$

For $\tilde{a} \in \widetilde{\Phi}^b$ and $s \in \tilde{k}$, we have

$$g_\lambda \, \boldsymbol{U}_{\tilde{a}}(s) \, g_\lambda^{-1} = \begin{cases} \widetilde{\boldsymbol{U}}_{\tilde{a}}(t^{\langle b, \lambda \rangle} s) & \text{if } \tilde{a}|_{S \times_k \tilde{k}} = b, \\ \widetilde{\boldsymbol{U}}_{\tilde{a}}(t^{2\,\langle b, \lambda \rangle} s) & \text{if } \tilde{a}|_{S \times_k \tilde{k}} = 2b. \end{cases}$$

Hence $g_\lambda \, \boldsymbol{U}_{\tilde{a}}(s) \, g_\lambda^{-1} \subset \widetilde{\Gamma}$. This establishes Claim 2.9. The proof of Lemma 2.8 is now complete. ☐

End of the proof of Theorem 2.1.

Two distinct points of $\mathfrak{Q}$ *are not equivalent under* Γ. Since two different points of $\widetilde{\mathfrak{Q}}$ are not equivalent under $\widetilde{\Gamma}$ [Soulé 1979, 1.3], it follows that two distinct points in $\mathfrak{Q}$ are not equivalent under Γ.

A point of $\mathcal{T}$ *of type* 0 *is equivalent to a point of* $\mathfrak{Q}$. We denote by $M \subset \boldsymbol{S}(K) = \boldsymbol{S}_* \otimes K^\times$ the subgroup generated by the $\lambda(t)$ for λ running over $\boldsymbol{S}_*$. We denote by $M_+ \subset M$ the semigroup generated by the $\lambda(t)$ for λ satisfying $\langle b, \lambda \rangle \geq 0$ for all $b \in \Delta$. By a result of Raghunathan [1994, Theorem 3.4],[4] we have the decomposition

$$\boldsymbol{G}(K) = \Gamma \,.\, M \,.\, \boldsymbol{G}(\mathbb{O}).$$

Again, since $\boldsymbol{N_G}(\boldsymbol{S})(k)$ maps onto $\boldsymbol{W}(k)$ and $\boldsymbol{W}(k).M_+ = M$, we have actually a decomposition

$$\boldsymbol{G}(K) = \Gamma \,.\, M_+ \,.\, \boldsymbol{G}(\mathbb{O}).$$

Since $\boldsymbol{G}(K)/\boldsymbol{G}(\mathbb{O})$ is the set of points of type 0 of $\mathcal{T}$, this shows that every such point of $\mathcal{T}$ is Γ-conjugated to a point of $M \,.\, \phi$. But $M_+ \,.\, \phi \subset \mathfrak{Q}$, so we conclude that every such point of $\mathcal{T}$ is Γ-conjugated to a point of $\mathfrak{Q}$.

Every point of $\mathcal{T}$ *is equivalent to a point of* $\mathfrak{Q}$. Let y be a point of $\mathcal{T}$. Let F be a chamber of $\mathcal{T}$ containing y. Then $\overline{F}$ contains a (unique) point x whose type is that of ϕ. By the preceding step, we can assume that $x \in \mathfrak{Q}$. Then y belongs to $\mathcal{L}_x$ and Lemma 2.8 shows that y is equivalent under Γ to a point of $\mathfrak{Q}$.

From the above it follows that $\mathcal{T} = \Gamma \,.\, \mathfrak{Q}$, as stated in Theorem 2.1. ☐

[4] This reference presupposes that the base field k is infinite, but this assumption is not necessary; see [Gille 1994, III.3.4.2] for details.

3. Applications

We give two applications of Theorem 2.1. The notation and assumptions are as in the previous section. We begin by recalling some basic facts about direct limits of groups.

Direct limits of groups. Direct limits of groups occur in geometric group theory [Serre 1977]. In what follows we will repeatedly encounter the following situation: We are given a family of subgroups $(H_\lambda)_{\lambda\in\Lambda}$ of a group H (indexed by some set Λ) and we wish to consider the group that is the direct limit of the groups $(H_\lambda, H_\lambda \cap H_\mu)_{\lambda,\mu\in\Lambda}$ where the only transition maps are the inclusions $H_\lambda \cap H_\mu \subset H_\lambda$ and $H_\lambda \cap H_\mu \subset H_\mu$. We call the resulting group *the direct limit of the family* $(H_\lambda)_{\lambda\in\Lambda}$ *with respect to their intersections.*[5]

Let T be an abstract simplicial complex, E the set of its vertices, and Φ the set of its simplexes. Denote by X the geometric realization of T. Let H be a group that acts in a simplicial way on T, and for which there exists a simplicial fundamental domain T'. Recall that T' is a subcomplex of T such that if E' (respectively Φ') denotes the set of vertices (respectively simplexes) of T', then for every $s \in \Phi$, there exists a unique $s' \in \Phi'$ such that $s \in H\,.\,s'$.

The isotropy subgroup of H corresponding to an element z (respectively a subset M) of either T or X will be denoted by H_z (respectively H_M).

Theorem 3.1 [Soulé 1973]. *Let T, X, H, T' be as above. Assume that X is connected and simply connected and that the geometric realization X' of T' is connected. Then the group H is the direct limit of the family of isotropy subgroups $(H_M)_{M\in E'}$ with respect to their intersections.*

Chebotarëv [1982] has established higher-dimensional generalizations of this result. As pointed out by one of the referees, when X has additional structures there are other presentations, which are useful in practice.

Proposition 3.2. *Under the hypothesis of Theorem 3.1, assume that X is equipped with a distance d such that*

(i) *for any two points x and y, there is a unique geodesic linking x and y;*

(ii) *for any $x \in X$, there is an open neighborhood D_x of x such that $D_x \cap F \neq \varnothing$ implies $x \in \overline{F}$ for any simplex F of X;*

(iii) *H acts isometrically on X.*

Furthermore, we assume that

[5] Another terminology, which is a slight abuse of language, is that H is the sum of the H_M amalgamated over their intersections [Serre 1977, II.1.7].

(iv) *for each simplex F of X, the stabilizer of F (as a set) coincides with the isotropy group (pointwise stabilizer) of $\overline{F}$.*

Then

(1) *The group H is the direct limit of the family $(H_M \cap H_N)_{M,N\in E'}$ with transition maps $H_M \cap H_N \to H_M$ and $H_M \cap H_N \to H_N$ for M, N belonging to an edge of X'.*[6]

(2) *The group H is the direct limit of the family of isotropy subgroups $(H_x)_{x\in X'}$ with respect to their intersections.*

Note that when X is a tree, the first statement of the proposition allows us to recover a classical result [Serre 1977, section 4.5, théorème 10].

Remark 3.3. Note that the first statement of the proposition is different than that of Theorem 3.1. The point is that two vertices of X' do not necessarily belong to a common edge. In other words, the presentation of H given by Proposition 3.2(1) has fewer relations than the one given by Theorem 3.1.

Proof. We prove both statements at the same time. We denote by $H^\dagger$ the first limit and by $H^\sharp$ the second one. We have an obvious surjective map $H^\dagger \to H$, while the inclusion $E' \subset X$ gives rise to a map $H \to H^\sharp$. We denote by $\xi : H^\dagger \to H \to H^\sharp$ the composition of these two maps. It is enough then to show that $H \to H^\sharp$ is surjective, and to produce a section $\theta : H^\sharp \to H^\dagger$ of ξ.

If $x \in X$, we denote by $F_x \subset X$ the (open) simplex attached to x. Since every F_x contains in its closure a vertex M, our hypothesis on stabilizers implies that $H_x \subset H_M$. It follows that $H \to H^\sharp$ is surjective.

To define the splitting $\theta : H^\sharp \to H^\dagger$, we proceed as follows. We are given $x \in X$, and $M \in E'$ such that $M \in \overline{F}_x$. Since the action is simplicial, we have $H_x = H_{F_x}$. By our hypothesis on the stabilizers, we have then the inclusion $H_x \subset H_M \subset H$.

Step 1: The composite map $\theta_{x,M} : H_x \to H_M \to H^\dagger$ does not depend of the choice of M. We note that two distinct choices M and N of vertices of $\overline{F}_x$ define an edge of X', so that the maps $H_x \to H_M \to H^\dagger$ and $H_x \to H_M \to H^\dagger$ agree since they agree on $H_M \cap H_N$. This establishes this step and defines a map $\theta_x : H_x \to H^\sharp$.

Step 2: If $y \in \overline{F}_x$, then θ_x and θ_y agree on the subgroup H_x of H_y. Since $\overline{F}_y \subset \overline{F}_x$, we can pick a vertex $M \in \overline{F}_y$. By definition $\theta_{x,M}$ and $\theta_{y,M}$ agree on H_y. Hence θ_x and θ_y agree on H_y by the first step.

Step 3: Connectedness argument. We are given $x, y \in X$ and we want to show that θ_x and θ_y agree on $H_x \cap H_y$. Since $H_x \cap H_y$ acts trivially on the geodesic $[x, y]$,

[6] By taking $M = N$ in E' we see that the groups H_M are part of our family. Observe that if M, N are vertices of a common edge F, then $H_N \cap H_M$ is nothing but the isotropy group of $\overline{F}$.

we have $H_x \cap H_y \subset H_z$ for all $z \in [x, y]$. We consider then the restrictions $\Theta_z : H_x \cap H_y \subset H_z \to H^\dagger$ of θ_z to $H_x \cap H_y$ for z running over $[x, y]$.

Recall that D_z is the open neighborhood of $z \in X$ given by hypothesis (ii).

Step 4: If $z \in [x, y]$, then $\Theta_z = \Theta_{z'}$ for all $z' \in D_z \cap [x, y]$. Since $z' \in F_{z'} \cap D_z$, assumption (ii) implies that $z \in \overline{F}_{z'}$. Step 2 shows that θ_z and $\theta_{z'}$ agree on $H_{z'} \subset H_z$; hence $\Theta_z = \Theta_{z'}$.

We now finish the proof of the proposition. Since the $D_z \cap [x, y]$ define an open covering of the connected space $[x, y]$, Step 3 implies that Θ_z does not depend on z. In particular θ_x and θ_y agree on $H_x \cap H_y$. By the universal property defining $H^\sharp$, we obtain a map $\theta : H^\sharp \to H^\dagger$. By construction $\theta \circ \xi = \mathrm{id}_{H^\dagger}$. □

For future use we record the following.

Lemma 3.4. *Let H be a group that is the direct limit of a family of subgroups $(H_\alpha)_{\alpha\in\Lambda}$ of H with respect to their intersections.*

(1) *Let $\Lambda' \subset \Lambda$ be a directed subset, that is, for all $\alpha, \beta \in \Lambda'$, there exists $\gamma \in \Lambda'$ such that $H_\alpha \subset H_\gamma$ and $H_\beta \subset H_\gamma$. Then the direct limit of the family $(H_\alpha)_{\alpha\in\Lambda'}$ with respect to their intersections is canonically isomorphic to the subgroup $\bigcup_{\alpha\in\Lambda'} H_\alpha$ of H.*

(2) *Let $\Lambda = \bigsqcup_{j\in J} \Lambda_j$ be a partition of Λ in directed subsets. For $j \in J$, denote by $H_j := \bigcup_{\alpha\in\Lambda_j} H_\alpha$ the subgroup of H associated to Λ_j. Then H is the direct limit of the family of subgroups $(H_j)_{j\in J}$ of H with respect to their intersections.*

Proof. (1) Note that $\bigcup_{\alpha\in\Lambda'} H_\alpha$ is a subgroup of H since Λ' is directed. For any group M we have

$$\mathrm{Hom}_{\mathrm{gr}}(H', M) = \varprojlim_{\alpha\in\Lambda'} \mathrm{Hom}_{\mathrm{gr}}(H_\alpha, M),$$

whence the statement.

(2) Denote by $\widetilde{H}$ the direct limit of the family of subgroups $(H_j)_{j\in J}$ of H with respect to their intersections. The inclusion maps $H_j \subset H$ agree over their intersections and hence give rise to a natural map $\xi : \widetilde{H} \to H$. For defining the reverse map, denote by $\alpha \mapsto j(\alpha)$ the map $\Lambda \to J$ that maps α to the unique index j such that $\alpha \in \Lambda_j$. We then get maps

$$H_\alpha \hookrightarrow H_{j(\alpha)} \to \widetilde{H} \quad \text{for } \alpha \in \Lambda.$$

Since these maps agree over their intersections, they yield a map $\eta : H \to \widetilde{H}$. Given that the images of the H_α generate H (respectively $\widetilde{H}$), we get that $\eta \circ \xi = \mathrm{id}_{\widetilde{H}}$ and $\xi \circ \eta = \mathrm{id}_H$. □

The group $\boldsymbol{G}(k[t])$ as a direct limit. Theorem 3.1 yields this:

Corollary 3.5. *Let V be the set of vertices of $\mathfrak{Q}$. The group $\Gamma = \boldsymbol{G}(k[t])$ is the direct limit of the family $(\Gamma_x)_{x \in V}$ with respect to their intersections.* $\square$

From the corollary we see that Γ is generated by the Γ_x. By Proposition 2.5(1), Γ_x consists of products of elements of $\boldsymbol{G}(k)$ and elements of $\boldsymbol{U}(k[t])$, where $\boldsymbol{U}$ stands for the unipotent radical of the minimal parabolic subgroup attached to $\boldsymbol{S}$ and Δ.

Corollary 3.6. $\boldsymbol{G}(k[t]) = \langle \boldsymbol{G}(k), \boldsymbol{U}(k[t]) \rangle$. $\square$

Another presentation of Γ is given by means of Proposition 3.2(2).

Corollary 3.7. *The group $\Gamma = \boldsymbol{G}(k[t])$ is the direct limit of the family $(\Gamma_x)_{x \in \mathfrak{Q}}$ with respect to their intersections.*

Proof. We have to check that hypotheses (i) through (iv) of Proposition 3.2 are satisfied for the action of Γ on the Bruhat–Tits building $\mathcal{T}$, which is a metric space.

(i) Any two points of $\mathcal{T}$ are linked by a unique geodesic [Bruhat and Tits 1972, section 2.5].

(ii) By [ibidem, lemme 2.5.11], for any $x \in X$ there exists an open ball D_x of center x such that for any simplex F of X, $D_x \cap F \neq \varnothing$ implies $x \in \overline{F}$.

(iii) The group $\boldsymbol{G}(K)$ acts isometrically on $\mathcal{T}$ (ibidem).

(iv) Since $\boldsymbol{G}$ is simply connected, the stabilizer of a simplex F of $\mathcal{T}$ (or facet with the terminology of Bruhat and Tits) under $\Gamma \subset \boldsymbol{G}(K)$ is also its pointwise stabilizer [Bruhat and Tits 1984, proposition 4.6.32] and also of $\overline{F}$ [Bruhat and Tits 1972, proposition 2.4.13].

The corollary now follows from Proposition 3.2. $\square$

We shall now give a nicer presentation of Γ. Given a subset $I \subset \Delta$, define $\mathfrak{Q}_I := \{x \in \mathfrak{Q} \mid I_x = I\}$. It is a subcone of $\mathfrak{Q}$, that is, $z\mathfrak{Q}_I \subset \mathfrak{Q}_I$ for all $z > 0$. Define the subgroup $\Gamma_I = \boldsymbol{U}_I(k[t]) \rtimes \boldsymbol{L}_I(k)$.

Lemma 3.8. (1) *The $(\Gamma_x)_{x \in \mathfrak{Q}_I}$ form a directed family of subgroups of Γ.*

(2) *Γ_I is the direct limit of the Γ_x for $x \in \mathfrak{Q}_I$.*

Proof. (1) The sector $\mathfrak{Q}$ is equipped with the partial order $x \leq y$ if $y - x \in \mathfrak{Q}$. By restriction, we get a partial order on $\mathfrak{Q}_I$ that is directed. Indeed, given $x, y \in \mathfrak{Q}_I$, we have $x + y \in \mathfrak{Q}_I$ and $x + y \geq x$ and $x + y \geq y$.

Let x, y be elements of $\mathfrak{Q}_I$ such that $x \leq y$. Then $b(y) \geq b(x)$ for all $b \in [I]^+$; hence $m_y(b) \leq m_x(b)$ for all $b \in [I]^+$. It follows that for $b \in [I]^+$ we have

$$U_{b,m_x(b)} \,.\, U_{2b,m_x(2b)} \subset U_{b,m_y(b)} \,.\, U_{2b,m_y(2b)}.$$

Now Proposition 2.5(2) shows that $\Gamma_x \subset \Gamma_y$. Since $\mathfrak{Q}_I$ is a directed subset of $\mathfrak{Q}$, we conclude that the $(\Gamma_x)_{x\in\mathfrak{Q}_I}$ form a directed family of subgroups of Γ.

(2) By Lemma 3.4(1), it is enough to show that

$$\bigcup_{x\in\mathfrak{Q}_I} \Gamma_x = \Gamma_I. \tag{3-1}$$

Proposition 2.5(1) shows that the inclusion $\subset$ holds. Conversely, suppose we are given an element $g \in \Gamma_I$. Let $x \in \mathfrak{Q}_I$. By Proposition 2.5(3) there is a real number $z \geq 1$ such that $g \in \Gamma_{zx}$. Since $zx \in \mathfrak{Q}_I$, g belongs to the left side of (3-1). □

Theorem 3.9. *The group $\Gamma = \boldsymbol{G}(k[t])$ is the direct limit of the family of subgroups $(\Gamma_I)_{I\subset\Delta}$ with respect to their intersections*

Proof. Lemma 3.8(2) shows that Γ_I is the limit of the directed family of subgroups $(\Gamma_x)_{x\in\mathfrak{Q}_I}$. To finish the proof we apply Lemma 3.4(2) to the decomposition $\mathfrak{Q} = \bigsqcup_{I\subset\Delta} \mathfrak{Q}_I$ of $\mathfrak{Q}$ into directed subsets. □

Application to Whitehead groups. Let $\boldsymbol{G}(k)^+$ be the (normal) subgroup of $\boldsymbol{G}(k)$ generated by the $(R_u\boldsymbol{P})(k)$ for $\boldsymbol{P}$ running over all parabolic k-subgroups of $\boldsymbol{G}$. If $\operatorname{card}(k) \geq 4$, Tits [1964] has shown that every proper normal subgroup of $\boldsymbol{G}(k)^+$ is central. The quotient $W(k, \boldsymbol{G}) = \boldsymbol{G}(k)/\boldsymbol{G}(k)^+$ is the Whitehead group of $\boldsymbol{G}$ by [Tits 1978]. By Tits's result this group detects whether $\boldsymbol{G}(k)$ is projectively simple.

It turns out that the Whitehead group admits another characterization. Denote by $H\boldsymbol{G}(k)$ the (normal) subgroup of $\boldsymbol{G}(k)$ composed of elements $g \in \boldsymbol{G}(k)$ for which there exists an element $h \in \Gamma = \boldsymbol{G}(k[t])$ such that $h(0) = e$ and $h(1) = g$. We denote by $\pi_0(k, \boldsymbol{G}) = \boldsymbol{G}(k)/H\boldsymbol{G}(k)$ this naive group of connected components of $\boldsymbol{G}$.

Theorem 3.10. *There is a canonical isomorphism $W(k, \boldsymbol{G}) \xrightarrow{\sim} \pi_0(k, \boldsymbol{G})$.*

Proof. The unipotent radical $\boldsymbol{V}$ of a k-parabolic subgroup $\boldsymbol{Q}$ of $\boldsymbol{G}$ is a split unipotent group, so it satisfies $H(\boldsymbol{V})(k) = \boldsymbol{V}(k)$. Hence we have $\boldsymbol{G}(k)^+ \subset H\boldsymbol{G}(k)$ and a surjection $\boldsymbol{G}(k)/\boldsymbol{G}(k)^+ \to \pi_0(k, \boldsymbol{G}) = \boldsymbol{G}(k)/H\boldsymbol{G}(k)$. It remains to show that $H\boldsymbol{G}(k) \subset \boldsymbol{G}(k)^+$. Let $g \in H\boldsymbol{G}(k)$, and choose $h \in \boldsymbol{G}(k[t])$ satisfying $h(0) = e$ and $h(1) = g$. According to Corollary 3.6, the element h can be written in the form

$$h = g_1 u_1 g_2 u_2 \cdots g_n u_n$$

with $g_i \in \boldsymbol{G}(k)$ and $u_i \in \boldsymbol{U}(k[t])$, where $\boldsymbol{U}$ is the unipotent radical of a minimal parabolic k-subgroup of $\boldsymbol{G}$. We can assume that $u_i(0) = e$, so the condition $h(0) = e$ reads $g_1 \cdots g_n = e$. It follows that

$$h = g'_1 u_1 {g'}_1^{-1} \cdots g'_n u_n {g'}_n^{-1},$$

with $g_1' = g_1$, $g_2' = g_1 g_2$ and so on up to $g_n' = g_1 \cdots g_n = e \in \boldsymbol{G}(k)$. Hence, as desired

$$g = h(1) = g_1' u_1(1) g'^{-1}_1 \cdots g'_n u_n(1) g'^{-1}_n \in \boldsymbol{G}(k)^+. \qquad \square$$

Acknowledgments

The author wishes to express his sincere gratitude to Gopal Prasad and the referees for their useful comments.

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Communicated by Brian Conrad
Received 2007-11-19 Revised 2008-11-05 Accepted 2008-11-11

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