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T-adic exponential sums over finite fields

Chunlei Liu and Daqing Wan

We introduce T-adic exponential sums associated to a Laurent polynomial f. They interpolate all classical p^m-power order exponential sums associated to f. We establish the Hodge bound for the Newton polygon of L-functions of T-adic exponential sums. This bound enables us to determine, for all m, the Newton polygons of L-functions of p^m-power order exponential sums associated to an f that is ordinary for $m = 1$. We also study deeper properties of L-functions of T-adic exponential sums. Along the way, we discuss new open problems about the T-adic exponential sum itself.

1. Introduction

Classical exponential sums. We first recall the definition of classical exponential sums over finite fields of characteristic p with values in a p-adic field.

Let p be a fixed prime number, $\mathbb{Z}_p$ the ring of p-adic integers, $\mathbb{Q}_p$ the field of p-adic numbers, and $\overline{\mathbb{Q}}_p$ a fixed algebraic closure of $\mathbb{Q}_p$. Let $q = p^a$ be a power of p, $\mathbb{F}_q$ the finite field of q elements, $\mathbb{Q}_q$ the unramified extension of $\mathbb{Q}_p$ with residue field $\mathbb{F}_q$, and $\mathbb{Z}_q$ the ring of integers of $\mathbb{Q}_q$.

Fix a positive integer n. Let $f(x) \in \mathbb{Z}_q[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}]$ be a Laurent polynomial in n variables of the form

$$f(x) = \sum_u a_u x^u, \quad \text{where } a_u \in \mu_{q-1} \text{ and } x^u = x_1^{u_1} \cdots x_n^{u_n};$$

here μ_k denotes the group of k-th roots of unity in $\overline{\mathbb{Q}}_p$.

Definition 1.1. Let ψ be a locally constant character of $\mathbb{Z}_p$ of order p^m with values in $\overline{\mathbb{Q}}_p$, and let $\pi_\psi = \psi(1) - 1$. The sum

$$S_{f,\psi}(k) = \sum_{x \in \mu^n_{q^k-1}} \psi(\mathrm{Tr}_{\mathbb{Q}_{q^k}/\mathbb{Q}_p}(f(x)))$$

is called a p^m-power order exponential sum on the n-torus $\mathbb{G}_m^n$ over $\mathbb{F}_{q^k}$. The

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generating function

$$L_{f,\psi}(s) = L_{f,\psi}(s;\mathbb{F}_q) = \exp\left(\sum_{k=1}^{\infty} S_{f,\psi}(k)\frac{s^k}{k}\right) \in 1 + s\mathbb{Z}_p[\pi_\psi][\![s]\!]$$

is the L-function of p^m-power order exponential sums over $\mathbb{F}_q$ associated to $f(x)$.

For $m \geq 1$ this is still an exponential sum over a finite field, since we are just summing over the subset of roots of unity (corresponding to the elements of a finite field via the Teichmüller lifting), not over the whole finite residue ring $\mathbb{Z}_q/p^m\mathbb{Z}_q$. The exponential sum over the whole finite ring $\mathbb{Z}_q/p^m\mathbb{Z}_q$ and its generating function as m varies is the subject of Igusa's zeta function [1978].

In general, $L_{f,\psi}(s)$ is rational in s. However, $L_{f,\psi}(s)^{(-1)^{n-1}}$ is a polynomial if f is nondegenerate, as shown in [Adolphson and Sperber 1989; 1987] for ψ of order p, and in [Liu and Wei 2007] for all ψ. By a result of [Gel'fand et al. 1994], if p is large enough, then f is generically nondegenerate. For nondegenerate f, the location of the zeros of $L_{f,\psi}(s)^{(-1)^{n-1}}$ becomes an important issue. The p-adic theory of such L-functions was developed by Dwork [1960], Bombieri [1966], Adolphson and Sperber [1989; 1987], the second author [Wan 1993; 2004], and Blache [2008] for ψ of order p. Recently, the initial part of the theory was extended to all ψ by Liu and Wei [2007] and Liu [2007].

The p-adic theory of the above exponential sum for $n = 1$ and ψ of order p has a long history and has been studied extensively in the literature. For instance, in the simplest case that $f(x) = x^d$, the exponential sum was studied by Gauss; see [Berndt and Evans 1981] for a comprehensive survey. By the Hasse–Davenport relation for Gauss sums, the L-function is a polynomial whose zeros are given by roots of Gauss sums. Thus, the slopes of the L-function are completely determined by the Stickelberger theorem for Gauss sums. The roots of the L-function have explicit p-adic formulas in terms of p-adic Γ-function via the Gross–Koblitz formula [1979]. These ideas can be extended to treat the so-called diagonal f case for general n; see [Wan 2004]. These elementary cases have been used as building blocks to study the deeper nondiagonal $f(x)$ via various decomposition theorems, which are the main ideas of Wan [1993; 2004]. In the case $n = 1$ and ψ of order p, more facts about the slopes of the L-function were found in [Zhu 2003; 2004a; Blache and Férard 2007; Liu 2008].

T-adic exponential sums. We now define the T-adic exponential sum, state our main results, and put forward some new questions.

Definition 1.2. For a positive integer k, the T-adic exponential sum of f over $\mathbb{F}_{q^k}$ is the sum

$$S_f(k,T) = \sum_{x\in\mu^n_{q^k-1}} (1+T)^{\mathrm{Tr}_{\mathbb{Q}_{q^k}/\mathbb{Q}_p}(f(x))} \in \mathbb{Z}_p[\![T]\!].$$

The T-adic L-function of f over $\mathbb{F}_q$ is the generating function

$$L_f(s,T) = L_f(s,T;\mathbb{F}_q) = \exp\left(\sum_{k=1}^{\infty} S_f(k,T)\frac{s^k}{k}\right) \in 1 + s\mathbb{Z}_p[\![T]\!][\![s]\!].$$

The T-adic exponential sum interpolates classical exponential sums of p^m-order over finite fields for all positive integers m. In fact, we have

$$S_f(k,\pi_\psi) = S_{f,\psi}(k).$$

Similarly, one can recover the classical L-function of the p^m-order exponential sum from the T-adic L-function by the formula

$$L_f(s,\pi_\psi) = L_{f,\psi}(s).$$

We view $L_f(s,T)$ as a power series in the single variable s with coefficients in the complete discrete valuation ring $\mathbb{Q}_p[\![T]\!]$ with uniformizer T.

Definition 1.3. The T-adic characteristic function of f over $\mathbb{F}_q$, or C-function of f for short, is the generating function

$$C_f(s,T) = \exp\left(\sum_{k=1}^{\infty} -(q^k-1)^{-n} S_f(k,T)\frac{s^k}{k}\right) \in 1 + s\mathbb{Z}_p[\![T]\!][\![s]\!].$$

The C-function $C_f(s,T)$ and the L-function $L_f(s,T)$ determine each other. They are related by

$$L_f(s,T) = \prod_{i=0}^{n} C_f(q^i s,T)^{(-1)^{n-i-1}\binom{n}{i}}, \quad C_f(s,T)^{(-1)^{n-1}} = \prod_{j=0}^{\infty} L_f(q^j s,T)^{\binom{n+j-1}{j}}.$$

In Section 4, we prove:

Theorem 1.4 (analytic continuation). *The C-function $C_f(s,T)$ is T-adic entire in s. As a consequence, the L-function $L_f(s,T)$ is T-adic meromorphic in s.*

This theorem tells us that the C-function behaves T-adically better than the L-function. In fact, in the T-adic setting, the C-function is a more natural object than the L-function. Thus, we shall focus more on the C-function.

Knowing the analytic continuation of $C_f(s,T)$, we are then interested in the location of its zeros. More precisely, we would like to determine the T-adic Newton polygon of this entire function $C_f(s,T)$. This is expected to be a complicated problem in general. It is open even in the simplest case $n=1$, and $f(x)=x^d$ is a monomial if $p \not\equiv 1 \pmod{d}$. What we can do is to give an explicit combinatorial lower bound depending only on q and Δ, called the q-Hodge bound $\mathrm{HP}_q(\Delta)$. This polygon will be described in detail in Section 3.

Let $\mathrm{NP}_T(f)$ denote the T-adic Newton polygon of the C-function $C_f(s,T)$. In Section 5, we prove this:

Theorem 1.5 (Hodge bound). $\mathrm{NP}_T(f) \geq \mathrm{HP}_q(\Delta)$.

This theorem shall give several new results on classical exponential sums, as we shall see in Section 2. In particular, this extends in one stroke all known ordinariness results for ψ of order p to all ψ of any p-power order. It demonstrates the significance of the T-adic L-function. It also gives rise to a definition:

Definition 1.6. A Laurent polynomial f that satisfies $\mathrm{NP}_T(f) = \mathrm{HP}_q(\Delta)$ is called T-adically ordinary.

We shall show that a classically ordinary f is T-adically ordinary, but it is possible that a nonordinary f is T-adically ordinary. Thus, it remains interesting to study exactly when f is T-adically ordinary. For this reason, in Section 6, we extend the facial decomposition theorem in [Wan 1993] to the T-adic case. Let Δ be the convex closure in $\mathbb{R}^n$ of the origin and the exponents of the nonzero monomials in the Laurent polynomial $f(x)$. For any closed face σ of Δ, we let f_σ denote the sum of monomials of f whose exponent vectors lie in σ.

Theorem 1.7 (T-adic facial decomposition). *A Laurent polynomial f is T-adically ordinary if and only if for every closed face σ of Δ of codimension* 1 *not containing the origin*, *the restriction f_σ is T-adically ordinary.*

In Section 7, we briefly discuss the variation of the C-function $C_f(s,T)$ and its Newton polygon when the reduction of f moves in an algebraic family over a finite field. The main questions concern generic ordinariness, the generic Newton polygon, the analogue of the Adolphson–Sperber conjecture [1989], Wan's limiting conjecture [2004], and Dwork's unit root conjecture [1973] in the T-adic and π_ψ-adic case. We shall give an overview about what can be proved and what is unknown, including a number of conjectures. In summary, a lot can be proved in the ordinary case, and a lot remain to be proved in the nonordinary case.

2. Applications

In this section, we give several applications of the T-adic exponential sum to classical exponential sums.

Theorem 2.1 (integrality theorem). *We have*

$$L_f(s,T) \in 1 + s\mathbb{Z}_p[\![T]\!][\![s]\!] \quad \textit{and} \quad C_f(s,T) \in 1 + s\mathbb{Z}_p[\![T]\!][\![s]\!].$$

Proof. Let $|\mathbb{G}_m^n|$ be the set of closed points of $\mathbb{G}_m^n$ over $\mathbb{F}_q$, and let $a \mapsto \hat{a}$ be the Teichmüller lifting. It is easy to check that the T-adic L-function has the Euler product expansion

$$L_f(s,T)=\prod_{x\in|\mathbb{G}_m^n|}\frac{1}{\left(1-(1+T)^{\mathrm{Tr}_{\mathbb{Q}_{q^{\deg x}}/\mathbb{Q}_p}(f(\hat{x}))}s^{\deg x}\right)}\in 1+s\mathbb{Z}_p[\![T]\!][\![s]\!],$$

where $\hat{x}=(\hat{x}_1,\ldots,\hat{x}_n)$. The theorem now follows. $\square$

This proof shows that the L-function $L_f(s,T)$ is the L-function $L(s,\rho_f)$ of the continuous (p,T)-adic representation of the arithmetic fundamental group given by

$$\rho_f:\pi_1^{\mathrm{arith}}(\mathbb{G}_m^n/\mathbb{F}_q)\to \mathrm{GL}_1(\mathbb{Z}_p[\![T]\!]),\quad \mathrm{Frob}_x\mapsto (1+T)^{\mathrm{Tr}_{\mathbb{Q}_{q^{\deg x}}/\mathbb{Q}_p}(f(\hat{x}))}.$$

The rank one representation ρ_f is transcendental in nature. Its L-function $L(s,\rho_f)$ seems to be beyond the reach of ℓ-adic cohomology, where ℓ is a prime different from p. However, the specialization of ρ_f at the special point $T=\pi_\psi$ is a character of finite order. Thus, the specialization

$$L(s,\rho_f)|_{T=\pi_\psi}=L_{f,\psi}(s)$$

can indeed be studied using Grothendieck's ℓ-adic trace formula [1965]. This gives another proof that the L-function $L_{f,\psi}(s)$ is a rational function in s. But the T-adic L-function $L_f(s,T)$ itself is certainly out of the reach of ℓ-adic cohomology as it is truly transcendental.

Let $\mathrm{NP}_{\pi_\psi}(f)$ denote the π_ψ-adic Newton polygon of the C-function $C_f(s,\pi_\psi)$. The integrality of $C_f(s,T)$ immediately gives the following theorem, whose proof is obvious.

Theorem 2.2 (rigidity bound). *If ψ is nontrivial, then* $\mathrm{NP}_{\pi_\psi}(f)\geq \mathrm{NP}_T(f)$.

A natural question is to ask when $\mathrm{NP}_{\pi_\psi}(f)$ coincides with its rigidity bound.

Theorem 2.3 (transfer theorem). *If $\mathrm{NP}_{\pi_\psi}(f)=\mathrm{NP}_T(f)$ holds for one nontrivial ψ, then it holds for all nontrivial ψ.*

Proof. By the integrality of $C_f(s,T)$, the T-adic Newton polygon of $C_f(s,T)$ coincides with the π_ψ-adic Newton polygon of $C_f(s,\pi_\psi)$ if and only if for every vertex (i,e) of the T-adic Newton polygon of $C_f(s,T)$, the coefficients of s^i in $C_f(s,T)$ differs from T^e by a unit in $\mathbb{Z}_p[\![T]\!]^\times$. It follows that if the coincidence happens for one nontrivial ψ, it happens for all nontrivial ψ. $\square$

Definition 2.4. We call f rigid if $\mathrm{NP}_{\pi_\psi}(f)=\mathrm{NP}_T(f)$ for one (and hence for all) nontrivial ψ.

In [Liu et al. 2008], the first author showed in cooperation with his students that f is generically rigid if $n=1$ and p is sufficiently large. So the rigid bound is the best possible bound. In contrast, the weaker Hodge bound $\mathrm{HP}_q(\Delta)$ is only the best possible if $p \equiv 1 \pmod d$, where d is the degree of f.

We now pause to describe the relationship between the Newton polygons of $C_f(s, \pi_\psi)$ and $L_{f,\psi}(s)^{(-1)^{n-1}}$. We need the following definitions.

Definition 2.5. A convex polygon with initial point $(0, 0)$ is called algebraic if it is the graph of a $\mathbb{Q}$-valued function defined on $\mathbb{N}$ or on an interval of $\mathbb{N}$, and its slopes are of finite multiplicity and of bounded denominator.

Definition 2.6. For an algebraic polygon with slopes $\{\lambda_i\}$, we define its slope series to be $\sum_i t^{\lambda_i}$.

It is clear that an algebraic polygon is uniquely determined by its slope series. So the slope series embeds the set of algebraic polygons into the ring $\lim_{\vec{d}} \mathbb{Z}[\![t^{1/d}]\!]$. The image is $\lim_{\vec{d}} \mathbb{N}[\![t^{1/d}]\!]$ and is closed under addition and multiplication. Therefore one can define addition and multiplication on the set of algebraic polygons.

Lemma 2.7. *Suppose that f is nondegenerate. Then the q-adic Newton polygon of $C_f(s, \pi_\psi; \mathbb{F}_q)$ is the product of the q-adic Newton polygon of $L_{f,\psi}(s; \mathbb{F}_q)^{(-1)^{n-1}}$ and the algebraic polygon $1/(1-t)^n$.*

Proof. The C-value $C_f(s, \pi_\psi)$ and the L-function $L_{f,\psi}(s)$ determine each other. They are related by

$$L_{f,\psi}(s) = \prod_{i=0}^{n} C_f(q^i s, \pi_\psi)^{(-1)^{n-i-1}\binom{n}{i}}, \quad C_f(s, \pi_\psi)^{(-1)^{n-1}} = \prod_{j=0}^{\infty} L_{f,\psi}(q^j s)^{\binom{n+j-1}{j}}.$$

Suppose that $L_{f,\psi}(s)^{(-1)^{n-1}} = \prod_{i=1}^{d}(1 - \alpha_i s)$. Then

$$C_f(s, \pi_\psi) = \prod_{j=0}^{\infty} \prod_{i=1}^{d} (1 - \alpha_i q^j s)^{\binom{n+j-1}{j}}.$$

Let λ_i be the q-adic order of α_i. Then the q-adic order of $\alpha_i q^j$ is $\lambda_i + j$. So the slope series of the q-adic Newton polygon of $L_{f,\psi}(s)^{(-1)^{n-1}}$ is $S(t) = \sum_{i=1}^{d} t^{\lambda_i}$, and the slope series of the q-adic Newton polygon of $C_f(s, \pi_\psi)$ is

$$\sum_{j=0}^{+\infty} \sum_{i=0}^{d} \binom{n+j-1}{j} t^{\lambda_i + j} = \frac{1}{(1-t)^n} S(t). \qquad \square$$

The next theorem, whose proof is obvious, combines the rigidity bound and the Hodge bound.

Theorem 2.8. *If ψ is nontrivial, then* $\mathrm{NP}_{\pi_\psi}(f) \geq \mathrm{NP}_T(f) \geq \mathrm{HP}_q(\Delta)$.

If we drop the middle term, we arrive at the Hodge bound

$$\mathrm{NP}_{\pi_\psi}(f) \geq \mathrm{HP}_q(\Delta)$$

of [Adolphson and Sperber 1987] and [Liu and Wei 2007].

Theorem 2.9. *If* $\mathrm{NP}_{\pi_\psi}(f) = \mathrm{HP}_q(\Delta)$ *holds for one nontrivial* ψ, *then* f *is rigid*, T*-adically ordinary*, *and the equality holds for all nontrivial* ψ.

Proof. Suppose that $\mathrm{NP}_{\pi_{\psi_0}}(f) = \mathrm{HP}_q(\Delta)$ for a nontrivial ψ_0. Then, by the last theorem, we have

$$\mathrm{NP}_{\pi_{\psi_0}}(f) = \mathrm{NP}_T(f) = \mathrm{HP}_q(\Delta).$$

So f is rigid and T-adically ordinary, and $\mathrm{NP}_{\pi_\psi}(f) = \mathrm{NP}_T(f) = \mathrm{HP}_q(\Delta)$ holds for all nontrivial ψ. □

Definition 2.10. We call f ordinary if $\mathrm{NP}_{\pi_\psi}(f) = \mathrm{HP}_q(\Delta)$ holds for one (and hence for all) nontrivial ψ.

The notion of ordinariness now carries much more information than we had known before. From this, we see that the T-adic exponential sum provides a new framework to study all p^m-power order exponential sums simultaneously. Instead of the usual way of extending the methods for ψ of order p to cases of higher order, the T-adic exponential sum has the novel feature that it can sometimes transfer a known result for one nontrivial ψ to all nontrivial ψ. This philosophy is carried out further in [Liu et al. 2008].

Example 2.11. Let

$$f(x) = x_1 + x_2 + \cdots + x_n + \frac{\alpha}{x_1 x_2 \cdots x_n} \quad \text{for } \alpha \in \mu_{q-1}.$$

Then, by the result of [Sperber 1980] and our new information on ordinariness, we have $\mathrm{NP}_{\pi_\psi}(f) = \mathrm{HP}_q(\Delta)$ for all nontrivial ψ.

3. The q-Hodge polygon

Here, we describe explicitly the q-Hodge polygon mentioned in the introduction. Recall that $f(x) \in \mathbb{Z}_q[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}]$ is a Laurent polynomial in n variables of the form

$$f(x) = \sum_{u \in \mathbb{Z}^n} a_u x^u, \quad \text{where } a_u \in \mathbb{Z}_q \text{ and } a_u^q = a_u.$$

We stress that the nonzero coefficients of $f(x)$ are roots of unity in $\mathbb{Z}_q$, and thus correspond uniquely to Teichmüller liftings of elements of the finite field $\mathbb{F}_q$. If the coefficients of $f(x)$ are arbitrary elements in $\mathbb{Z}_q$, much of the theory still holds, but it is more complicated to describe the results. In this paper, we make the simplifying assumption that the nonzero coefficients are always roots of unity.

Let Δ be the convex polyhedron in $\mathbb{R}^n$ associated to f, which is generated by the origin and the exponent vectors of the nonzero monomials of f. Let $C(\Delta)$ be the cone in $\mathbb{R}^n$ generated by Δ. Define the degree function $u \mapsto \deg u$ on $C(\Delta)$ so that $\deg u = 1$ when u lies on a codimensional 1 face of Δ that does not contain the origin, and so that $\deg(ru) = r \deg u$ for $r \in \mathbb{R}_{\geq 0}$ and $u \in C(\Delta)$; we call it the degree function associated to Δ. We have $\deg(u+v) \leq \deg u + \deg v$ for $u, v \in C(\Delta)$, and the equality holds if and only if u and v are cofacial. In other words, the number

$$c(u, v) := \deg u + \deg v - \deg(u+v)$$

is 0 if $u, v \in C(\Delta)$ are cofacial, and is positive otherwise. We call $c(u, v)$ the cofacial defect of u and v. Let

$$M(\Delta) := C(\Delta) \cap \mathbb{Z}^n$$

be the set of lattice points in the cone $C(\Delta)$. Let D be the denominator of the degree function, which is the smallest positive integer such that $\deg M(\Delta) \subset (1/D)\mathbb{Z}$. For every natural number k, we define

$$W(k) := W_\Delta(k) = \#\{u \in M(\Delta) \mid \deg u = k/D\}$$

to be the number of lattice points of degree k/D in $M(\Delta)$. For prime power $q = p^a$, the q-Hodge polygon of f is the polygon with vertices (0, 0) and

$$\left(\sum_{j=0}^{i} W(j),\, a(p-1)\sum_{j=0}^{i} \frac{j}{D} W(j)\right) \quad \text{for } i = 0, 1, \ldots .$$

It is also called the q-Hodge polygon of Δ and denoted by $\mathrm{HP}_q(\Delta)$. It depends only on q and Δ. It has a side of slope $a(p-1)(j/D)$ with horizontal length $W(j)$ for each nonnegative integer j.

4. Analytic continuation

Here we prove the T-adic analytic continuation of the C-function $C_f(s, T)$. The idea is to employ Dwork's trace formula in the T-adic case.

Note that the Galois group $\mathrm{Gal}(\mathbb{Q}_q/\mathbb{Q}_p)$ is cyclic of order $a = \log_p q$. There is an element σ in the Galois group whose restriction to μ_{q-1} is the p-power morphism. It is of order a, and is called the Frobenius element.

We define a new variable π by the relation $E(\pi) = 1 + T$, where

$$E(\pi) = \exp\left(\sum_{i=0}^{\infty} \frac{\pi^{p^i}}{p^i}\right) \in 1 + \pi\mathbb{Z}_p[\![\pi]\!]$$

is the Artin–Hasse exponential series. Thus, π and T are two different uniformizers of the T-adic local ring $\mathbb{Q}_p[\![T]\!]$. It is clear that

$$E(\pi\alpha) \in 1+\pi\mathbb{Z}_q[\![\pi]\!] \quad \text{for } \alpha \in \mathbb{Z}_q,$$
$$E(\pi)^{\beta} \in 1+\pi\mathbb{Z}_p[\![\pi]\!] \quad \text{for } \beta \in \mathbb{Z}_p.$$

The Galois group $\mathrm{Gal}(\mathbb{Q}_q/\mathbb{Q}_p)$ can act on $\mathbb{Z}_q[\![\pi]\!]$ but keep π fixed. The Artin–Hasse exponential series has a kind of commutativity, which we express through the following lemma.

Lemma 4.1 (commutativity). *We have the following commutative diagram*:

$$\begin{array}{ccc} \mu_{q-1} & \xrightarrow{E(\pi\,\cdot)} & \mathbb{Z}_q[\![\pi]\!] \\ {\scriptstyle \mathrm{Tr}}\downarrow & & \downarrow{\scriptstyle \mathrm{Norm}} \\ \mu_{p-1} & \xrightarrow{E(\pi)^{\bullet}} & \mathbb{Z}_p[\![\pi]\!] \end{array}$$

That is, if $x \in \mu_{q-1}$, *then* $E(\pi)^{x+x^p+\cdots+x^{p^{(a-1)}}} = E(\pi x)E(\pi x^p)\cdots E(\pi x^{p^{a-1}})$.

Proof. Since $\sum_{j=0}^{a-1} x^{p^j} = \sum_{j=0}^{a-1} x^{p^{j+i}}$ for $x \in \mu_{q-1}$, we have

$$\begin{aligned} E(\pi)^{x+x^p+\cdots+x^{p^{(a-1)}}} &= \exp\left(\sum_{i=0}^{\infty}\frac{\pi^{p^i}}{p^i}\sum_{j=0}^{a-1}x^{p^{j+i}}\right) \\ &= E(\pi x)E(\pi x^p)\cdots E(\pi x^{p^{a-1}}). \qquad \square \end{aligned}$$

Definition 4.2. Let $\pi^{1/D}$ be a fixed D-th root of π. Define

$$L(\Delta) = \left\{ \sum_{u\in M(\Delta)} b_u \pi^{\deg u} x^u : b_u \in \mathbb{Z}_q[\![\pi^{1/D}]\!] \right\},$$
$$B = \left\{ \sum_{u\in M(\Delta)} b_u \pi^{\deg u} x^u \in L(\Delta),\ \mathrm{ord}_T(b_u) \to +\infty \text{ if } \deg u \to +\infty \right\}.$$

The spaces $L(\Delta)$ and B are T-adic Banach algebras over the ring $\mathbb{Z}_q[\![\pi^{1/D}]\!]$. The monomials $\pi^{\deg u}x^u$ for $u \in M(\Delta)$ form an orthonormal basis of B and a formal basis $L(\Delta)$. The algebra B is contained in the larger Banach algebra $L(\Delta)$. If $u \in \Delta$, it is clear that $E(\pi x^u) \in L(\Delta)$. Write

$$E_f(x) := \prod_{a_u \neq 0} E(\pi a_u x^u) \quad \text{if } f(x) = \sum_{u\in\mathbb{Z}^n} a_u x^u.$$

This is an element of $L(\Delta)$ since $L(\Delta)$ is a ring.

The Galois group $\mathrm{Gal}(\mathbb{Q}_q/\mathbb{Q}_p)$ can act on $L(\Delta)$, while keeping $\pi^{1/D}$ as well as the x_i fixed. From the commutativity of the Artin–Hasse exponential series, one can infer the following lemma.

Lemma 4.3 (Dwork's splitting lemma). *If* $x \in \mu_{q^k-1}$, *then*

$$E(\pi)^{\mathrm{Tr}_{\mathbb{Q}_{q^k}/\mathbb{Q}_p}(f(x))} = \prod_{i=0}^{ak-1} E_f^{\sigma^i}(x^{p^i}),$$

where a *is the order of* $\mathrm{Gal}(\mathbb{Q}_q/\mathbb{Q}_p)$.

Proof. We have

$$\begin{aligned} E(\pi)^{\mathrm{Tr}_{\mathbb{Q}_{q^k}/\mathbb{Q}_p}(f(x))} &= \prod_{a_u \neq 0} E(\pi)^{\mathrm{Tr}_{\mathbb{Q}_{q^k}/\mathbb{Q}_p}(a_u x^u)} \\ &= \prod_{a_u \neq 0} \prod_{i=0}^{ak-1} E(\pi(a_u x^u)^{p^i}) = \prod_{i=0}^{ak-1} E_f^{\sigma^i}(x^{p^i}). \end{aligned}$$ □

Definition 4.4. We define a map

$$\psi_p : L(\Delta) \to L(\Delta), \qquad \sum_{u \in M(\Delta)} b_u x^u \mapsto \sum_{u \in M(\Delta)} b_{pu} x^u.$$

It is clear that the composition map $\psi_p \circ E_f$ sends B to B.

Lemma 4.5. *Write* $E_f(x) = \sum_{u \in M(\Delta)} \alpha_u(f) \pi^{\deg u} x^u$. *Then*

$$\psi_p \circ E_f(\pi^{\deg u} x^u) = \sum_{w \in M(\Delta)} \alpha_{pw-u}(f) \pi^{c(pw-u,u)} \pi^{(p-1)\deg w} \pi^{\deg w} x^w$$

for $u \in M(\Delta)$, *where* $c(pw-u, u)$ *is the cofacial defect of* $pw-u$ *and* u.

Proof. This follows directly from the definition of ψ_p and $E_f(x)$. □

Definition 4.6. Define $\psi := \sigma^{-1} \circ \psi_p \circ E_f : B \to B$, and its a-th iterate

$$\psi^a = \psi_p^a \circ \prod_{i=0}^{a-1} E_f^{\sigma^i}(x^{p^i}).$$

Note that ψ is linear over $\mathbb{Z}_p[[\pi^{1/D}]]$, but semilinear over $\mathbb{Z}_q[[\pi^{1/D}]]$. On the other hand, ψ^a is linear over $\mathbb{Z}_q[[\pi^{1/D}]]$. By the last lemma, ψ^a is completely continuous in the sense of [Serre 1962].

Theorem 4.7 (Dwork's trace formula). *For every positive integer* k,

$$(q^k - 1)^{-n} S_f(k, T) = \mathrm{Tr}_{B/\mathbb{Z}_q[[\pi^{1/D}]]}(\psi^{ak}).$$

Proof. Let $g(x) \in B$. We have

$$\psi^{ak}(g) = \psi_p^{ak}(g \prod_{i=0}^{ak-1} E_f^{\sigma^i}(x^{p^i})).$$

Write $\prod_{i=0}^{ak-1} E_f^{\sigma^i}(x^{p^i}) = \sum_{u\in M(\Delta)} \beta_u x^u$. One computes that

$$\psi^{ak}(\pi^{\deg v} x^v) = \sum_{u\in M(\Delta)} \beta_{q^k u - v} \pi^{\deg v} x^u.$$

Thus, $\mathrm{Tr}(\psi^{ak} \mid B/\mathbb{Z}_q[\![\pi^{1/D}]\!]) = \sum_{u\in M(\Delta)} \beta_{(q^k-1)u}$. But, by Dwork's splitting lemma, we have

$$(q^k-1)^{-n} S_f(k,T) = (q^k-1)^{-n} \sum_{x\in\mu^n_{q^k-1}} \prod_{i=0}^{ak-1} E_f^{\sigma^i}(x^{p^i}) = \sum_{u\in M(\Delta)} \beta_{(q^k-1)u}. \qquad \square$$

Theorem 4.8 (analytic trace formula). *We have*

$$C_f(s,T) = \det(1-\psi^a s \mid B/\mathbb{Z}_q[\![\pi^{1/D}]\!]).$$

In particular, the T-adic C-function $C_f(s,T)$ is T-adic analytic in s.

Proof. It follows from the last theorem and the well-known identity

$$\det(1-\psi^a s) = \exp\biggl(-\sum_{k=1}^{\infty} \mathrm{Tr}(\psi^{ak}) \frac{s^k}{k}\biggr). \qquad \square$$

This theorem gives another proof that the coefficients of $C_f(s,T)$ and $L_f(s,T)$ as power series in s are T-adically integral.

Corollary 4.9. *For each nontrivial ψ, the C-value $C_f(s,\pi_\psi)$ is p-adic entire in s and the L-function $L_{f,\psi}(s)$ is rational in s.*

5. The Hodge bound

The analytic trace formula in the previous section reduces the study of $C_f(s,T)$ to the study of the operator ψ^a. We consider ψ first. Note that ψ operates on B and is linear over $\mathbb{Z}_p[\![\pi^{1/D}]\!]$.

Theorem 5.1. *The T-adic Newton polygon of* $\det(1-\psi s \mid B/\mathbb{Z}_p[\![\pi^{1/D}]\!])$ *lies above the polygon with vertices* $(0,0)$ *and*

$$\biggl(a\sum_{k=0}^{i} W(k),\ a(p-1)\sum_{k=0}^{i} \frac{k}{D} W(k)\biggr) \quad \textit{for } i = 0,1,\ldots.$$

Proof. Let $\xi_1,\xi_2,\ldots,\xi_a$ be a normal basis of $\mathbb{Q}_q$ over $\mathbb{Q}_p$. Write

$$(\xi_j \alpha_{pw-u}(f))^{\sigma^{-1}} = \sum_{i=0}^{a-1} \alpha_{(i,w),(j,u)}(f)\xi_i \quad \text{for } \alpha_{(i,w),(j,u)}(f) \in \mathbb{Z}_p[\![\pi^{1/D}]\!].$$

Then

$$\psi(\xi_j\pi^{\deg u}x^u)=\sum_{i=0}^{a-1}\sum_{w\in M(\Delta)}\alpha_{(i,w),(j,u)}(f)\pi^{c(pw-u,u)}\pi^{(p-1)\deg w}\xi_i\pi^{\deg w}x^w.$$

That is, the matrix of ψ over $\mathbb{Z}_p[\![\pi^{1/D}]\!]$ with respect to the orthonormal basis $\{\xi_j\pi^{\deg u}x^u\}_{0\le j<a,u\in M(\Delta)}$ is

$$A=\left(\alpha_{(i,w),(j,u)}(f)\pi^{c(pw-u,u)}\pi^{(p-1)\deg w}\right)_{(i,w),(j,u)}.$$

The claim follows. □

We are now ready to prove the Hodge bound for the Newton polygon.

Theorem 5.2. $\mathrm{NP}_T(f)\ge \mathrm{HP}_q(\Delta)$.

Proof. By the theorem above, it suffices to prove that the T-adic Newton polygon of $\det\left(1-\psi^a s^a \mid B/\mathbb{Z}_q[\![\pi^{1/D}]\!]\right)$ coincides with that of $\det\left(1-\psi s\mid B/\mathbb{Z}_p[\![\pi^{1/D}]\!]\right)$. Note that

$$\det\left(1-\psi^a s\mid B/\mathbb{Z}_p[\![\pi^{1/D}]\!]\right)=\mathrm{Norm}\left(\det\left(1-\psi^a s\mid B/\mathbb{Z}_q[\![\pi^{1/D}]\!]\right)\right),$$

where the norm map is the norm from $\mathbb{Z}_q[\![\pi^{1/D}]\!]$ to $\mathbb{Z}_p[\![\pi^{1/D}]\!]$. The theorem now follows from the equality

$$\prod_{\zeta^a=1}\det\left(1-\psi\zeta s\mid B/\mathbb{Z}_p[\![\pi^{1/D}]\!]\right)=\det\left(1-\psi^a s^a\mid B/\mathbb{Z}_p[\![\pi^{1/D}]\!]\right).\qquad □$$

6. Facial decomposition

In this section, we extend the facial decomposition theorem in [Wan 1993]. Recall that the operator $\psi=\sigma^{-1}\circ(\psi_p\circ E_f)$ is only semilinear over $\mathbb{Z}_q[\![\pi^{1/D}]\!]$. But its second factor $\psi_p\circ E_f$ is clearly linear, and so $\det\left(1-(\psi_p\circ E_f)s\mid B/\mathbb{Z}_q[\![\pi^{1/D}]\!]\right)$ is well defined. We begin with the following theorem.

Theorem 6.1. *The T-adic Newton polygon of $C_f(s,T)$ coincides with* $\mathrm{HP}_q(\Delta)$ *if and only if the T-adic Newton polygon of* $\det\left(1-(\psi_p\circ E_f)s\mid B/\mathbb{Z}_q[\![\pi^{1/D}]\!]\right)$ *coincides with the polygon with vertices* $(0,0)$ *and*

$$\left(\sum_{k=0}^{i}W(k),\ (p-1)\sum_{k=0}^{i}\frac{k}{D}W(k)\right)\quad \textit{for } i=0,1,\dots.$$

Proof. In the proof of Theorem 5.2, we showed that the T-adic Newton polygon of $C_f(s^a,T)$ coincides with that of $\det\left(1-\psi s\mid B/\mathbb{Z}_p[\![\pi^{1/D}]\!]\right)$. Note that

$$\det\left(1-(\psi_p\circ E_f)s\mid B/\mathbb{Z}_p[\![\pi^{1/D}]\!]\right)=\mathrm{Norm}\left(\det\left(1-(\psi_p\circ E_f)s\mid B/\mathbb{Z}_q[\![\pi^{1/D}]\!]\right)\right),$$

where the norm map is the norm from $\mathbb{Z}_q[\![\pi^{1/D}]\!]$ to $\mathbb{Z}_p[\![\pi^{1/D}]\!]$. The theorem is equivalently stated that the T-adic Newton polygon of $\det\bigl(1-\psi s \mid B/\mathbb{Z}_p[\![\pi^{1/D}]\!]\bigr)$ coincides with the polygon with vertices $(0,0)$ and

$$\left(\sum_{k=0}^{i} aW(k),\, a(p-1)\sum_{k=0}^{i}\frac{k}{D}W(k)\right) \quad \text{for } i=0,1,\ldots$$

if and only if the T-adic Newton polygon of $\det\bigl(1-(\psi_p\circ E_f)s \mid B/\mathbb{Z}_p[\![\pi^{1/D}]\!]\bigr)$ does. Therefore it suffices to show that the determinant of the matrix

$$(\alpha_{(i,w),(j,u)}(f)\pi^{c(pw-u,u)})_{0\le i,j<a,\,\deg w,\deg u\le k/D}$$

is not divisible by T in $\mathbb{Z}_p[\![\pi^{1/D}]\!]$ if and only if the determinant of the matrix

$$(\alpha_{pw-u}(f)\pi^{c(pw-u,u)})_{\deg w,\deg u\le k/D}$$

is not divisible by T in $\mathbb{Z}_q[\![\pi^{1/D}]\!]$. The theorem now follows from the fact that the former determinant is the norm of the latter from $\mathbb{Q}_q[\![\pi^{1/D}]\!]$ to $\mathbb{Q}_p[\![\pi^{1/D}]\!]$ up to a sign. □

We now define the open facial decomposition $F(\Delta)$. It is the decomposition of $C(\Delta)$ into a disjoint union of relatively open cones generated by the relatively open faces of Δ whose closure does not contain the origin. Note that every relatively open cone generated by cofacial vectors in $C(\Delta)$ is contained in a unique element of $F(\Delta)$.

Lemma 6.2. *Let $\sigma\in F(\Delta)$, and $u\in\sigma$. Then $\alpha_u(f_{\bar\sigma})\equiv\alpha_u(f) \pmod{\pi^{1/D}}$, where $f_{\bar\sigma}$ is the sum of monomials of f whose exponent vectors lie in the closure $\bar\sigma$ of σ.*

Proof. Let $v_1,\ldots,v_j$ be exponent vectors of monomials of f such that $a_1v_1+\cdots+a_jv_j=u$, with $a_1>0,\ldots,a_j>0$. It suffices to show that either $v_1,\ldots v_j$ lie in the closure of σ, or their contribution to $\alpha_u(f)$ is $\equiv 0 \pmod{\pi^{1/D}}$. Suppose their contribution to $\alpha_u(f)$ is $\not\equiv 0 \pmod{\pi^{1/D}}$. Then $v_1,\ldots,v_j$ must be cofacial. So the interior of the cone generated by those vectors is contained in a unique element of $F(\Delta)$. Since that interior has a common point u with σ, it must be σ. It follows that $v_1,\ldots v_j$ lie in the closure of σ. □

Lemma 6.3. *Let $\sigma,\tau\in F(\Delta)$ be distinct. Let $w\in\sigma$ and $u\in\tau$. Suppose that the dimension of σ is no greater than that of τ. Then $pw-u$ and u are not cofacial, that is, $c(pw-u,u)>0$.*

Proof. Suppose that $pw-u$ and u are cofacial. Then the interior of the cone generated by $pw-u$ and u is contained in a unique element of $F(\Delta)$. Since that interior has a common point w with σ, it must be σ. It follows that u lies in the closure of σ. Since σ and τ are distinct, u lies in the boundary of σ. This implies that the dimension of τ is less than that of σ, which is a contradiction. Therefore $pw-u$ and u are not cofacial. $\square$

For $\sigma \in F(\Delta)$, we define $M(\sigma)=M(\Delta)\cap\sigma=\mathbb{Z}^n\cap\sigma$ to be the set of lattice points in the cone σ.

Theorem 6.4 (open facial decomposition). *The T-adic Newton polygon of $C_f(s,T)$ coincides with $\mathrm{HP}_q(\Delta)$ if and only if for every $\sigma \in F(\Delta)$, the determinants of the matrices*

$$\{\alpha_{pw-u}(f_{\bar\sigma})\pi^{c(pw-u,u)}\}_{w,u\in M(\sigma),\,\deg w,\deg u\le k/D} \quad \textit{for } k=0,1,\dots$$

are not divisible by T in $\mathbb{Z}_q[\![\pi^{1/D}]\!]$, where $\bar\sigma$ is the closure of σ.

Proof. By Theorem 6.1, the T-adic Newton polygon of $C_f(s,T)$ coincides with the q-Hodge polygon of f if and only if the determinants of the matrices

$$A^{(k)}=\{\alpha_{pw-u}(f)\pi^{c(pw-u,u)}\}_{w,u\in M(\Delta),\,\deg w,\deg u\le k/D} \quad \text{for } k=0,1,\dots$$

are not divisible by T in $\mathbb{Z}_q[\![\pi^{1/D}]\!]$. Write

$$A^{(k)}_{\sigma,\tau}=\{\alpha_{pw-u}(f)\pi^{c(pw-u,u)}\}_{w\in M(\sigma),\,u\in M(\tau),\,\deg w,\deg u\le k/D}.$$

The facial decomposition shows that $A^{(k)}$ has the block form $(A^{(k)}_{\sigma,\tau})_{\sigma,\tau\in F(\Delta)}$. The last lemma shows that the block form modulo $\pi^{1/D}$ is triangular if we order the cones in $F(\Delta)$ by increasing dimension. It follows that $\det A^{(k)}$ is not divisible by T in $\mathbb{Z}_q[\![\pi^{1/D}]\!]$ if and only if for all $\sigma\in F(\Delta)$, $\det A^{(k)}_{\sigma,\sigma}$ is not divisible by T in $\mathbb{Z}_q[\![\pi^{1/D}]\!]$. By Lemma 6.2, modulo $\pi^{1/D}$, $A^{(k)}_{\sigma,\sigma}$ is congruent to the matrix

$$\{\alpha_{pw-u}(f_{\bar\sigma})\pi^{c(pw-u,u)}\}_{w,u\in M(\sigma),\,\deg w,\deg u\le k/D}.$$

So $\det A^{(k)}_{\sigma,\sigma}$ is not divisible by T in $\mathbb{Z}_q[\![\pi^{1/D}]\!]$ if and only if the determinant of the matrix

$$\{\alpha_{pw-u}(f_{\bar\sigma})\pi^{c(pw-u,u)}\}_{w,u\in M(\sigma),\,\deg w,\deg u\le k/D}$$

is not divisible by T in $\mathbb{Z}_q[\![\pi^{1/D}]\!]$. $\square$

The closed facial decomposition Theorem 1.7 follows from the open decomposition theorem and the fact that

$$F(\Delta)=\bigcup_{\substack{\sigma\in F(\Delta)\\ \dim\sigma=\dim\Delta}} F(\bar\sigma).$$

A similar π_ψ-adic facial decomposition theorem for $C_f(s,\pi_\psi)$ can be proved in a similar way. Alternatively, it follows from the transfer theorem together with the π_ψ-adic facial decomposition in [Wan 1993] for ψ of order p.

7. Variation of C-functions in a family

Fix an n-dimensional integral convex polytope Δ in $\mathbb{R}^n$ containing the origin. For each prime p, let $P(\Delta, \mathbb{F}_p)$ denote the parameter space of all Laurent polynomials $f(x)$ over $\overline{\mathbb{F}}_p$ such that $\Delta(f) = \Delta$. This is a connected rational variety defined over $\mathbb{F}_p$. For each $f \in P(\Delta, \mathbb{F}_p)(\mathbb{F}_q)$, the Teichmüller lifting gives a Laurent polynomial $\tilde{f}$ whose nonzero coefficients are roots of unity in $\mathbb{Z}_q$. The C-function $C_{\tilde{f}}(s, T)$ is then defined and T-adically entire. For simplicity of notation, we shall just write $C_f(s, T)$ for $C_{\tilde{f}}(s, T)$ and similarly $L_f(s, T)$ for $L_{\tilde{f}}(s, T)$. Thus, our C-function and L-function are now defined for Laurent polynomials over finite fields via the Teichmüller lifting. We would like to study how $C_f(s, T)$ varies when f varies in the algebraic variety $P(\Delta, \mathbb{F}_p)$.

Recall that for a closed face $\sigma \in \Delta$, f_σ denotes the restriction of f to σ. That is, f_σ is the sum of those nonzero monomials in f whose exponents are in σ.

Definition 7.1. A Laurent polynomial $f \in P(\Delta, \mathbb{F}_p)$ is called nondegenerate if for every closed face σ of Δ of arbitrary dimension that does not contain the origin, the system

$$\frac{\partial f_\sigma}{\partial x_1} = \cdots = \frac{\partial f_\sigma}{\partial x_n} = 0$$

has no common zeros with $x_1 \cdots x_n \neq 0$ over the algebraic closure of $\mathbb{F}_p$.

The nondegeneracy condition is a geometric condition that insures the associated Dwork cohomology can be calculated. In particular, it implies that if ψ is of order p^m, then the L-function $L_{f,\psi}(s)^{(-1)^{n-1}}$ is a polynomial in s whose degree is precisely $n!\,\mathrm{Vol}(\Delta)\,p^{n(m-1)}$; see [Liu and Wei 2007]. Consequently:

Theorem 7.2. *Let $f \in P(\Delta, \mathbb{F}_p)(\mathbb{F}_q)$. Write*

$$L_f(s, T)^{(-1)^{n-1}} = \sum_{k=0}^{\infty} L_{f,k}(T) s^k \quad \text{for } L_{f,k}(T) \in \mathbb{Z}_p[\![T]\!].$$

Assume that f is nondegenerate. Then for every positive integer m and all positive integers $k > n!\,\mathrm{Vol}(\Delta)\,p^{n(m-1)}$, we have the congruence

$$L_{f,k}(T) \equiv 0 \pmod{((1+T)^{p^m} - 1)/T} \quad \text{in } \mathbb{Z}_p[\![T]\!].$$

Proof. Write $((1+T)^{p^m} - 1)/T = \prod (T - \xi)$. The nondegeneracy assumption implies that

$$L_f(s, \xi)^{(-1)^{n-1}} = \sum_{j=0}^{\infty} L_{f,j}(\xi) s^j,$$

is a polynomial in s of degree $\leq n!\,\mathrm{Vol}(\Delta)\,p^{n(m-1)} < k$. It follows that $L_{f,k}(\xi) = 0$ for all ξ. That is, $L_{f,k}(T)$ is divisible by $(T - \xi)$ for ξ. $\square$

Definition 7.3. Let $N(\Delta, \mathbb{F}_p)$ denote the subset of all nondegenerate Laurent polynomials $f \in P(\Delta, \mathbb{F}_p)$.

The subset $N(\Delta, \mathbb{F}_p)$ is Zariski open in $P(\Delta, \mathbb{F}_p)$. It can be empty for some pair $(\Delta, \mathbb{F}_p)$. But, $N(\Delta, \mathbb{F}_p)$ for a given Δ is Zariski open dense in $P(\Delta, \mathbb{F}_p)$ for all primes p except for possibly finitely many primes depending on Δ. It is an interesting and independent question to classify the primes p for which $N(\Delta, \mathbb{F}_p)$ is nonempty. This is related to the GKZ discriminant [Gel'fand et al. 1994]. For simplicity, we shall only consider nondegenerate f in the following.

Generic ordinariness. The first question is, How often f is T-adically ordinary when f varies in the nondegenerate locus $N(\Delta, \mathbb{F}_p)$? Let $U_p(\Delta, T)$ be the subset of $f \in N(\Delta, \mathbb{F}_p)$ such that f is T-adically ordinary, and $U_p(\Delta)$ the subset of $f \in N(\Delta, \mathbb{F}_p)$ such that f is ordinary. One can prove this:

Lemma 7.4. *The set $U_p(\Delta)$ is Zariski open in $N(\Delta, \mathbb{F}_p)$.*

Is $U_p(\Delta, T)$ also Zariski open in $N(\Delta, \mathbb{F}_p)$? We do not know the answer.

For which p are $U_p(\Delta)$ and $U_p(\Delta, T)$ Zariski dense in $N(\Delta, \mathbb{F}_p)$? The rigidity bound as well as the Hodge bound imply that $U_p(\Delta) \subseteq U_p(\Delta, T)$. It follows that if $U_p(\Delta)$ is Zariski dense in $N(\Delta, \mathbb{F}_p)$, then $U_p(\Delta, T)$ is also Zariski dense in $N(\Delta, \mathbb{F}_p)$.

The Adolphson–Sperber conjecture [1989] says that if $p \equiv 1 \pmod{D}$, then $U_p(\Delta)$ is Zariski dense in $N(\Delta, \mathbb{F}_p)$. This conjecture was proved to be true in [Wan 1993; 2004] if $n \leq 3$, which implies this:

Theorem 7.5. *If $p \equiv 1 \pmod{D}$ and $n \leq 3$, then $U_p(\Delta, T)$ is Zariski dense in $N(\Delta, \mathbb{F}_p)$.*

For $n \geq 4$, it was shown in [Wan 1993; 2004] that there is an effectively computable positive integer $D^*(\Delta)$ depending only on Δ such that $U_p(\Delta)$ is Zariski dense in $N(\Delta, \mathbb{F}_p)$ if $p \equiv 1 \pmod{D^*(\Delta)}$.

Theorem 7.6. *For each Δ, there exists an effectively computable positive integer $D^*(\Delta)$ such that $U_p(\Delta, T)$ is Zariski dense in $N(\Delta, \mathbb{F}_p)$ if $p \equiv 1 \pmod{D^*(\Delta)}$.*

The smallest possible $D^*(\Delta)$ is rather subtle to compute in general, and it can be much larger than D. We now state a conjecture giving reasonably precise estimates of $D^*(\Delta)$.

Definition 7.7. Let $S(\triangle)$ be the monoid generated by the degree 1 lattice points in $M(\Delta)$, that is, those lattice points on the codimension 1 faces of Δ not containing the origin. Define the exponent of $\triangle$ by

$$I(\triangle) = \inf\{d \in \mathbb{Z}_{>0} \mid dM(\Delta) \subseteq S(\triangle)\}.$$

If $u \in M(\Delta)$, then the degree of Du will be integral, but Du may not be a non-negative integral combination of degree 1 elements in $M(\Delta)$; thus $DM(\Delta)$ may not be a subset of $S(\Delta)$. It is not hard to show that $I(\Delta) \geq D$. In general they are different but they are equal if $n \leq 3$. This explains why the Adolphson–Sperber conjecture is true if $n \leq 3$ but may be false if $n \geq 4$. The following conjecture is a modified form, and it is a consequence of [Wan 1993, Conjecture 9.1].

Conjecture 7.8. *If $p \equiv 1 \mod I(\triangle)$, then $U_p(\Delta)$ is Zariski dense in $N(\Delta, \mathbb{F}_p)$. In particular, $U_p(\Delta, T)$ is Zariski dense in $N(\Delta, \mathbb{F}_p)$ for such p.*

By the facial decomposition theorem, it suffices in proving this conjecture to assume that Δ has only one codimension 1 face not containing the origin.

Generic Newton polygon. In the case that $U_p(\Delta, T)$ is empty, we expect the existence of a generic T-adic Newton polygon. For this purpose, we need to rescale the uniformizer. For $f \in N(\Delta, \mathbb{F}_p)(\mathbb{F}_{p^a})$, the $T^{a(p-1)}$-adic Newton polygon of $C_f(s, T; \mathbb{F}_{p^a})$ is independent of the choice of a for which f is defined over $\mathbb{F}_{p^a}$. We call this the absolute T-adic Newton polygon of f.

Conjecture 7.9. *There is a Zariski open dense subset $G_p(\Delta, T)$ of $N(\Delta, \mathbb{F}_p)$ such that the absolute T-adic Newton polygon of f is constant for all $f \in G_p(\Delta, T)$. Denote this common polygon by $\mathrm{GNP}_T(\Delta, p)$, and call it the generic Newton polygon of $(\triangle, T)$.*

More generally, one expects that much of the classical theory for finite rank F-crystals extends to a certain nuclear infinite rank setting. This includes the classical Dieudonne–Manin isogeny theorem, the Grothendieck specialization theorem, the Katz isogeny theorem [1979]. All these are essentially understood in the ordinary infinite rank case, but open in the nonordinary infinite rank case.

Similarly, for each nontrivial ψ there is a Zariski open dense subset $G_p(\Delta, \psi)$ of $N(\Delta, \mathbb{F}_p)$ with the property that the $\pi_\psi^{a(p-1)}$-adic Newton polygon of the C-value $C_f(s, \pi_\psi; \mathbb{F}_{p^a})$ is constant for all $f \in G_p(\Delta, \psi)$. Denote this common polygon by $\mathrm{GNP}_p(\Delta, \psi)$, and call it the generic Newton polygon of $(\triangle, \psi)$. The existence of $G_p(\Delta, \psi)$ can be proved, since the nondegeneracy assumption implies that the C-function $C_f(s, \pi_\psi)$ is determined by a single finite rank F-crystal via a Dwork type cohomological formula for $L_{f,\psi}(s)$. In the T-adic case, we are not aware of any such finite rank reduction.

Clearly, we have the relation

$$\mathrm{GNP}_p(\Delta, \psi) \geq \mathrm{GNP}_T(\Delta, p).$$

Conjecture 7.10. *If p is sufficiently large, then*

$$\mathrm{GNP}_p(\Delta, \psi) = \mathrm{GNP}_T(\Delta, p).$$

This conjecture is proved in the case $n = 1$ in [Liu et al. 2008].

Let $\mathrm{HP}(\Delta)$ denote the absolute Hodge polygon with vertices $(0, 0)$ and

$$\left(\sum_{k=0}^{i} W(k), \sum_{k=0}^{i} \frac{k}{D} W(k)\right) \quad \text{for } i = 0, 1, \dots.$$

Note that $\mathrm{HP}(\Delta)$ depends only on Δ, and no longer on q. It is rescaled from the q-Hodge polygon $\mathrm{HP}_q(\Delta)$. Clearly, we have

$$\mathrm{GNP}_p(\Delta, \psi) \geq \mathrm{GNP}_T(\Delta, p) \geq \mathrm{HP}(\Delta).$$

Conjecture 7.8 says that if $p \equiv 1 \pmod{I(\triangle)}$, then both $\mathrm{GNP}_p(\Delta, \psi)$ and $\mathrm{GNP}_T(\Delta, p)$ are equal to $\mathrm{HP}(\Delta)$. In general, the generic Newton polygon lies above $\mathrm{HP}(\Delta)$, but for many Δ it should get closer and closer to $\mathrm{HP}(\Delta)$ as p goes to infinity. We now make this more precise. Let $E(\Delta)$ be the monoid generated by the lattice points in Δ. This is a subset of $M(\Delta)$. We may generalize the limiting [Wan 2004, Conjecture 1.11] for ψ of order p:

Conjecture 7.11. *If the difference $M(\Delta) - E(\Delta)$ is a finite set, then for each nontrivial ψ, we have*

$$\lim_{p\to\infty} \mathrm{GNP}_p(\Delta, \psi) = \mathrm{HP}(\Delta).$$

In particular, $\lim_{p\to\infty} \mathrm{GNP}_T(\Delta, p) = \mathrm{HP}(\Delta)$.

This conjecture is equivalent to the existence of the limit. This is because for all primes $p \equiv 1 \pmod{D^*(\Delta)}$, we already have by Theorem 7.6 the equality $\mathrm{GNP}_p(\Delta, \psi) = \mathrm{HP}(\Delta)$. A stronger version of this conjecture (namely, [Wan 2004, Conjecture 1.12]) has been proved by Zhu [2003; 2004a; 2004b] in the case $m = 1$ and $n = 1$; see also [Blache and Férard 2007; Blache et al. 2008] and [Liu 2008] for related further work in the case $m = 1$ and $n = 1$; see [Hong 2001; 2002] and [Yang 2003] for more specialized one variable results. For $n \geq 2$, the conjecture is clearly true for any Δ for which both $D \leq 2$ and the Adolphson–Sperber conjecture holds, because then $\mathrm{GNP}_p(\Delta, \psi) = \mathrm{HP}(\Delta)$ for every $p > 2$. There are many such higher-dimensional examples [Wan 2004]. Using free products of polytopes and the examples above, one can construct further examples [Blache 2008].

***T*-adic Dwork conjecture.** In this final subsection, we describe the T-adic version of Dwork's conjecture [1973] on pure slope zeta functions.

Let Λ be a quasiprojective subvariety of $N(\Delta, \mathbb{F}_p)$ defined over $\mathbb{F}_p$. Let f_λ be a family of Laurent polynomials parameterized by $\lambda \in \Lambda$. For each closed point $\lambda \in \Lambda$, the Laurent polynomial f_λ is defined over the finite field $\mathbb{F}_{p^{\deg \lambda}}$. The T-adic entire function $C_{f_\lambda}(s,T)$ has the pure slope factorization

$$C_{f_\lambda}(s,T)=\prod_{\alpha\in\mathbb{Q}_{\geq 0}} P_\alpha(f_\lambda,s),$$

where each $P_\alpha(f_\lambda,s)\in 1+s\mathbb{Z}_p[\![T]\!][s]$ is a polynomial in s whose reciprocal roots all have $T^{\deg\lambda(p-1)}$-slope equal to α.

Definition 7.12. For $\alpha\in\mathbb{Q}_{\geq 0}$, the T-adic pure slope L-function of the family f_Λ is defined to be the infinite Euler product

$$L_\alpha(f_\Lambda,s)=\prod_{\lambda\in|\Lambda|}\frac{1}{P_\alpha(f_\lambda,s^{\deg\lambda})}\in 1+s\mathbb{Z}_p[\![T]\!][\![s]\!],$$

where $|\Lambda|$ denotes the set of closed points of Λ over $\mathbb{F}_p$.

Dwork's conjecture then has a T-adic version:

Conjecture 7.13. *For $\alpha\in\mathbb{Q}_{\geq 0}$, the T-adic pure slope L-function $L_\alpha(f_\Lambda,s)$ is T-adic meromorphic in s.*

In the ordinary case, this conjecture can be proved using the methods from [Wan 2000a; 2000b; 1999]. It would be interesting to prove this conjecture in the general case. The π_ψ-adic version of this conjecture is essentially Dwork's original conjecture, which can be proved as it reduces to finite rank F-crystals. The difficulty of the T-adic version is that we have to work with infinite rank objects, where much less is known in the nonordinary case.

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Dress induction and the Burnside quotient Green ring

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We define and study the Burnside quotient Green ring of a Mackey functor, introduced in our 1990 MSRI preprint. Some refinements of Dress induction theory are presented, together with applications to computation results for K-theory and L-theory of finite and infinite groups.

1. Introduction

Induction theory began with Artin and Brauer's work in representation theory, was continued by Swan [1970] and Lam [1968] for K-theory, and was put in its most abstract and elegant setting by Green [1971] and Dress [1973; 1975]. The theory sets up a convenient framework for computing the value of a Mackey functor on some finite group G, given suitable generation results for a Green ring which acts on the Mackey functor. (See [tom Dieck 1987; Lindner 1976; Thévenaz 1990; Thévenaz and Webb 1995] for some subsequent developments.)

The main examples in this theory are (i) the Swan Green ring $SW(G, \mathbf{Z})$ [Swan 1970], which leads to the Brauer–Berman–Witt induction theorem for representations of finite groups, and computation results for Quillen K-theory $K_n(RG)$, and (ii) the Dress Green ring $GU(G, \mathbf{Z})$ [Dress 1975], which leads to computation results for the oriented surgery obstruction groups $L_n(\mathbf{Z}G)$ of Wall [1976].

In Section 2 we define the Burnside quotient Green ring $\mathcal{A}_{\mathcal{M}}$ for a Mackey functor $\mathcal{M} : \mathfrak{D}(G) \to \mathcal{A}b$, where $\mathfrak{D}(G)$ denotes the category of finite G-sets, and $\mathcal{A}b$ the category of abelian groups. This Green ring $\mathcal{A}_{\mathcal{M}}$ is the smallest quotient of the Burnside ring which is a Green ring, and still acts on the Mackey functor. As defined, it has many convenient naturality properties, and generation results for $\mathcal{A}_{\mathcal{M}}$ will lead as usual to computation results for $\mathcal{M}$. We define the concept of a *Dress generating set* X for a Green ring in Definition 3.5. The main result (see Theorem 3.6) is:

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Theorem A. *A finite G-set X is a Dress generating set for a Green ring $\mathcal{G}$ if and only if it is a Dress generating set for the Burnside quotient Green ring $\mathcal{A}_{\mathcal{G}}$.*

The naturality of the Burnside quotient Green rings can now be used to obtain computability results for sub- or quotient Mackey functors (see Theorem 3.8). We also point out a useful refinement of Dress induction in Theorem 3.10. We use the Burnside quotient Green ring in Section 5 to study additive functors out of the categories RG-Morita defined in our paper of 1990, henceforth abbreviated [HTW 1990]; here R is a commutative ring with unit. The main examples of such functors include K-theory, Hochschild homology and cyclic homology [HTW 1990, 1.A.12]. We define a bifunctor $d : \mathcal{D}(G) \to RG$-Morita in 5.4 and prove the following computability result (see Theorem 5.5):

Theorem B. *Any additive functor $F : RG$-Morita $\to \mathcal{A}b$ gives a Mackey functor on $\mathcal{D}(G)$ by composition with $d : \mathcal{D}(G) \to RG$-Morita. Any such Mackey functor is hyperelementary computable.*

This is a refinement of [Oliver 1988, 11.2], and Theorem A provides the key new ingredient in the proof. The extra generality is useful for studying functors such as the Dade group and the units in the Burnside ring [Bouc 2007; Bouc 2008].

The Burnside quotient Green ring has been applied to study the permutation representations of finite groups in [Hambleton and Taylor 1999], free actions of finite groups on products of spheres in [Hambleton 2006], and to the computation of Bass nilgroups in [Hambleton and Lück 2007]. This theory was surveyed and used in [Hambleton and Taylor 2000]. Our results also apply to the computation of K and L-theory for infinite groups, based on an idea of Farrell and Hsiang [1981].

We introduce Mackey *pre*functors and *pseudo*-Mackey functors in Section 6. A Mackey prefunctor is a just prebifunctor $\mathcal{D}(G) \to \mathcal{A}b$, and a pseudo-Mackey functor is a Mackey prefunctor which admits a finite filtration by Mackey functors. Such structures have been observed in a number of different contexts: the main examples include the higher Whitehead groups $\mathrm{Wh}_n(\mathbf{Z}G)$, and the structure set of a compact manifold in surgery theory [Wall 1999, Chapter 9].

It turns out that the general scheme of Dress induction theory can be extended to pseudo-Mackey functors as well. In Section 7, we combine this idea with the Burnside quotient Green ring to study additive functors out of the category (RG, ω)-Morita [HTW 1990, 1C]. We have the corresponding computability result (see Theorem 7.2):

Theorem C. *Let $F : (RG, \omega)$-Morita $\to \mathcal{A}b$ be an additive functor. Then the composite $\mathcal{M} = F \circ d : \mathcal{D}(G) \to \mathcal{A}b$ is a Mackey prefunctor. Moreover*:

(i) *The 2-adic completion of $\mathcal{M}$ is 2-hyperelementary computable.*

(ii) *If $\mathcal{M}$ is a Mackey functor, then $\mathcal{M}$ is hyperelementary computable.*

As an application, we conclude from part (i) that the surgery obstruction groups $L_n(\mathbf{Z}G, \omega)$, with arbitrary orientation character $\omega : G \to \{\pm 1\}$, are 2-hyperelementary computable after 2-adic completion (see Example 7.3 for the meaning of computability in this setting). This computability result was certainly expected to be true, but the argument presented here seems to be the first actual proof in the nonoriented case. In the oriented case, where ω is trivial, part (ii) applies to L-theory and the computability is just [Dress 1975, Theorem 1]. For nontrivial ω, the surgery obstruction group $L_n(\mathbf{Z}G, \omega)$ is a Mackey functor if and only if it has exponent two (see [Taylor 1973], and see [Wall 1976, 5.2.5] for an example where the L-groups do not have exponent two). In Lemma 7.1 we give a general necessary and sufficient condition on F for part (ii) to apply to $\mathcal{M}$.

2. Dress induction

We will first recall some definitions Dress used in his formulation of induction theory [Dress 1975, page 301].

2A. ***Mackey functors.*** Let G be a finite group, and let $\mathfrak{D}(G)$ denote the category whose objects are finite, left G-sets and whose morphisms are G-maps. A *Mackey functor* is a bifunctor $\mathcal{M} = (\mathcal{M}_*, \mathcal{M}^*) : \mathfrak{D}(G) \to \mathcal{A}b$, where $\mathcal{A}b$ denotes the category of abelian groups and groups homomorphisms, such that $\mathcal{M}_*(S) = \mathcal{M}^*(S)$ for each object $S \in \mathfrak{D}(G)$, and the following two properties hold:

(M1) For any pullback diagram of finite G-sets

$$\begin{tikzcd} S \arrow[r, "\Psi"] \arrow[d, "\Phi"'] & S_1 \arrow[d, "\varphi"] \\ S_2 \arrow[r, "\psi"] & T \end{tikzcd}$$

the induced maps give an commutative diagram

$$\begin{tikzcd} \mathcal{M}(S) \arrow[r, "\Psi_{\mathcal{M}}"] & \mathcal{M}(S_1) \\ \mathcal{M}(S_2) \arrow[u, "\Phi^{\mathcal{M}}"] \arrow[r, "\psi_{\mathcal{M}}"] & \mathcal{M}(T) \arrow[u, "\varphi^{\mathcal{M}}"'] \end{tikzcd}$$

Here we denote the covariant maps by $\psi_{\mathcal{M}}$ and the contravariant maps by $\varphi^{\mathcal{M}}$.

(M2) The embeddings of S_1 and S_2 into the disjoint union $S_1 \sqcup S_2$ define an isomorphism $\mathcal{M}^*(S_1 \sqcup S_2) \to \mathcal{M}^*(S_1) \oplus \mathcal{M}^*(S_2)$. Let $\mathcal{M}(\varnothing) = 0$.

The property (M1) is the usual double coset formula, and (M2) gives additivity.

We remark that for any bifunctor satisfying (M1), the composition

$$\begin{aligned} \mathcal{M}_*(S_1) \oplus \mathcal{M}_*(S_2) \;\to\; \mathcal{M}_*(S_1 \sqcup S_2) &= \mathcal{M}^*(S_1 \sqcup S_2) \\ &\to\; \mathcal{M}^*(S_1) \oplus \mathcal{M}^*(M_2) = \mathcal{M}_*(S_1) \oplus \mathcal{M}_*(S_2) \end{aligned}$$

is just the identity matrix. It follows that any subbifunctor of a Mackey functor is Mackey.

Definition 2.1. If $\mathcal{M}$ and $\mathcal{N}$ are Mackey functors, then a *homomorphism* $\mathcal{M} \to \mathcal{N}$ of Mackey functors is a natural transformation of bifunctors $\Theta : \mathcal{M} \to \mathcal{N}$ such that for each object $S \in \mathcal{D}(G)$ the function $\Theta_S : \mathcal{M}(S) \to \mathcal{N}(S)$ is a homomorphism of abelian groups. It is easy to check that the kernel, $\ker \Theta$, the image, $\operatorname{Im} \Theta$, and the cokernel of Θ are all sub- or quotient Mackey functors of $\mathcal{M}$ or $\mathcal{N}$.

2B. *Pairings and Green functors.* If $\mathcal{M}$, $\mathcal{N}$, and $\mathcal{L}$ are Mackey functors, then a *pairing* is a family of bilinear maps

$$\mathcal{M}(S) \times \mathcal{N}(S) \to \mathcal{L}(S)$$

indexed by the objects of $\mathcal{D}(G)$, such that for any G-map $\varphi : S \to T$ the following formulas hold:

$$\begin{aligned} \varphi^{\mathcal{L}}(x \cdot y) &= \varphi^{\mathcal{M}}(x) \cdot \varphi^{\mathcal{N}}(y) && \text{for } x \in \mathcal{M}(T),\ y \in \mathcal{N}(T), \\ x \cdot \varphi_{\mathcal{N}}(y) &= \varphi_{\mathcal{L}}(\varphi^{\mathcal{M}}(x) \cdot y) && \text{for } x \in \mathcal{M}(T),\ y \in \mathcal{N}(S), \\ \varphi_{\mathcal{M}}(x) \cdot y &= \varphi_{\mathcal{L}}(x \cdot \varphi^{N}(y)) && \text{for } x \in \mathcal{M}(S),\ y \in \mathcal{N}(T). \end{aligned}$$

A *Green ring* is a Mackey functor $\mathcal{G}$ together with a pairing $\mathcal{G} \times \mathcal{G} \to \mathcal{G}$, and a collection of elements $\{1_S \in \mathcal{G}(S)\}$ such that the pairing defines an associative ring structure on each $\mathcal{G}(S)$ with unit 1_S, and $\varphi^{\mathcal{G}}(1_T) = 1_S$ for every G-map $\varphi : S \to T$.

A *homomorphism* of Green rings $\Theta : \mathcal{G} \to \mathcal{H}$ is a homomorphism of Mackey functors such that for each object $S \in \mathcal{D}(G)$ the function $\Theta_S : \mathcal{G}(S) \to \mathcal{H}(S)$ is a unital ring homomorphism. If Θ_S is injective for each object $S \in \mathcal{D}(G)$, we say that $\mathcal{G}$ is a *sub*-Green ring of $\mathcal{H}$. If Θ_S is surjective for each object $S \in \mathcal{D}(G)$, then we say that $\mathcal{H}$ is a *quotient* Green ring of $\mathcal{G}$. Similarly, we define subquotient Green rings.

If $\mathcal{M}$ is a Mackey functor, then $\mathcal{M}$ is a *Green module* over a Green ring $\mathcal{G}$ if there exists a pairing $\mathcal{G} \times \mathcal{M} \to \mathcal{M}$ such that $\mathcal{M}(S)$ becomes a left $\mathcal{G}(S)$-module from the pairing, and $1_S \cdot x = x$ for all $x \in \mathcal{M}(S)$.

Example 2.2. If $\mathcal{G} \to \mathcal{H}$ is a homomorphism of Green rings, then $\mathcal{H}$ is a Green module over $\mathcal{G}$ under the pairing $\mathcal{G} \times \mathcal{H} \to \mathcal{H}$ induced by the homomorphism.

2C. ***The Burnside ring.*** For any left G-set S, we let $\mathcal{D}_S(G)$ denote the category with objects (X, f), where X is a left G-set and $f : X \to S$ is a G-map. The morphisms $F : (X_1, f_1) \to (X_2, f_2)$ are G-maps $F : X_1 \to X_2$ such that $f_2 \circ F = f_1$. We define a bifunctor

$$\mathcal{A} : \mathcal{D}(G) \to \mathcal{A}b$$

by setting $\mathcal{A}(S) = K_0(\mathcal{D}_S(G))$. If $\varphi : S \to T$ is a G-map, then $\varphi_{\mathcal{A}} : \mathcal{A}(S) \to \mathcal{A}(T)$ is the map induced on K_0 by the composition $(X, f) \mapsto (X, \varphi \circ f)$. The contravariant map $\varphi^{\mathcal{A}} : \mathcal{A}(T) \to \mathcal{A}(S)$ is induced by the pullback construction applied to

$$S \xrightarrow{\varphi} T \xleftarrow{f} Y,$$

where (Y, f) is an object in $\mathcal{D}_T(G)$. Conditions (M1) and (M2) are easy to check, and $\mathcal{A}$ is a Mackey functor. There is also a pairing $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$ defined by pullback: let (X_1, f_1) and (X_2, f_2) represent elements of $\mathcal{A}(S)$, and form the pullback

$$X_1 \xrightarrow{f_1} S \xleftarrow{f_2} X_2$$

considered as a G-set over S. This object in $\mathcal{D}_S(G)$ represents the product, and each $\mathcal{A}(S)$ becomes an associative ring with unit element represented by $\mathrm{id} : S \to S$. The resulting Green ring is called the *Burnside ring*. Dress also remarks that the Burnside ring is the "universal" Green ring, since it acts on any Mackey functor $\mathcal{M}$. The required pairing $\mathcal{A} \times \mathcal{M} \to \mathcal{M}$ is defined by pairing an element of $\mathcal{A}(S)$ represented by a G-set (X, f) over S, and an element $x \in \mathcal{M}(S)$, to get $f_{\mathcal{M}}(f^{\mathcal{M}}(x)) \in \mathcal{M}(S)$. It is not hard to check that $\mathcal{M}(S)$ is a unital $\mathcal{A}(S)$-module under this bilinear pairing, so $\mathcal{M}$ is a Green module over $\mathcal{A}$.

We remark that a homomorphism $\mathcal{M} \to \mathcal{N}$ of Mackey functors is compatible with the $\mathcal{A}$-module action, so gives a map of $\mathcal{A}$-Green modules.

If $\mathcal{G}$ is a Green ring, the same checks show that $\mathcal{G}$ is an $\mathcal{A}$-algebra, implying in particular that $a \cdot (x \cdot y) = (a \cdot x) \cdot y$ for all $a \in \mathcal{A}(S)$ and all $x, y \in \mathcal{G}(S)$. It follows that the map $\iota : \mathcal{A} \to \mathcal{G}$ defined by $a \mapsto a \cdot 1_S$, for all $a \in \mathcal{A}(S)$, is a (unital) ring homomorphism. Indeed

$$(a \cdot 1_S) \cdot (b \cdot 1_S) = a \cdot (1_S \cdot (b \cdot 1_S)) = a \cdot (b \cdot 1_S) = (a \cdot b) \cdot 1_S$$

for all $a, b \in \mathcal{A}(S)$, since $\mathcal{G}(S)$ is a $\mathcal{A}(S)$-algebra. It is easy to check from the pairing formulas that $\iota : \mathcal{A} \to \mathcal{G}$ is also a homomorphism of Green rings.

2D. ***Ideals and quotient Green rings.*** There is a natural notion of a (left) *Green ideal* in a Green ring $\mathcal{G}$, namely a subbifunctor $I \subset \mathcal{G}$ such that $I(S) \subset \mathcal{G}(S)$ is a left ideal in the ring $\mathcal{G}(S)$. Similarly, we have right ideals and two-sided ideals. If $I \subset \mathcal{G}$ is a two-sided Green ideal, then the quotient functor $\mathcal{G}/I$, defined by $S \mapsto \mathcal{G}(S)/I(S)$, is a Green ring under the quotient pairing inherited from $\mathcal{G}$.

If $\mathcal{G} \times \mathcal{M} \to \mathcal{M}$ is a Green module structure on a Mackey functor $\mathcal{M}$, then we define the *Green ideal* $I_{\mathcal{M}} \subset \mathcal{G}$ as the subbifunctor of $\mathcal{G}$ with

$$I_{\mathcal{M}}(S) = \left\{a \in \mathcal{G}(S) \mid \varphi_{\mathcal{G}}(a) \cdot y = 0,\, \psi^{\mathcal{G}}(a) \cdot z = 0\right\}$$

for all $\varphi : S \to T$, $\psi : U \to S$, and all $y \in \mathcal{M}(T)$, $z \in \mathcal{M}(U)$. Note that elements of $I_{\mathcal{M}}(S)$ satisfy additional conditions (both "up" and "down") beyond just acting trivially on $\mathcal{M}(S)$.

The pairing formulas show directly that $I_{\mathcal{M}}(S)$ is a two-sided ideal in the ring $\mathcal{G}(S)$, for every finite G-set S. We will check that $I_{\mathcal{M}}$ is a subbifunctor of $\mathcal{G}$ by looking at the operations induced by G-maps $\mu : V \to S$ and $\lambda : S \to W$ on an arbitrary element $a \in I_{\mathcal{M}}(S)$.

First we consider $\lambda_{\mathcal{G}}(a) \in \mathcal{G}(W)$. Let $\varphi : W \to T$ and $\psi : U \to W$ be any G-maps. We have

$$\varphi_{\mathcal{G}}(\lambda_{\mathcal{G}}(a)) \cdot y = (\varphi \circ \lambda)_{\mathcal{G}}(a) \cdot y = 0$$

by definition of $I_{\mathcal{M}}(S)$. Let

$$\begin{tikzcd} X \arrow[r, "\tilde{\lambda}"] \arrow[d, "\tilde{\psi}"'] & U \arrow[d, "\psi"] \\ S \arrow[r, "\lambda"] & W \end{tikzcd}$$

be the pullback square, and from (M1) we get

$$\psi^{\mathcal{G}}(\lambda_{\mathcal{G}}(a)) \cdot z = \tilde{\lambda}_{\mathcal{G}}(\tilde{\psi}^{\mathcal{G}}(a)) \cdot z = \tilde{\lambda}_{\mathcal{G}}(\tilde{\psi}^{\mathcal{G}}(a) \cdot \tilde{\lambda}^{\mathcal{M}}(z)) = 0$$

so $\lambda_{\mathcal{G}}(a) \in I_{\mathcal{M}}(W)$.

Similarly, we must check that $\mu^{\mathcal{G}}(a) \in I_{\mathcal{M}}(V)$. Let $\varphi : V \to T$ and $\psi : U \to V$ be G-maps, and note that

$$\varphi_{\mathcal{G}}(\mu^{\mathcal{G}}(a)) \cdot y = \varphi_{\mathcal{M}}(\mu^{\mathcal{G}}(a) \cdot \varphi^{\mathcal{M}}(y)) = 0$$

and $\psi^{\mathcal{G}}(\mu^{\mathcal{G}}(a)) \cdot z = (\mu \circ \psi)^{\mathcal{G}}(a) \cdot z = 0$.

We have now checked that $I_{\mathcal{M}} \subset \mathcal{G}$ is a subbifunctor, and therefore $I_{\mathcal{M}}$ is a Mackey functor and a two-sided Green ideal in $\mathcal{G}$. We define the *quotient Green ring* $\mathcal{G}_{\mathcal{M}} = \mathcal{G}/I_{\mathcal{M}}$ to be the bifunctor whose value on objects is given by the quotient rings $\mathcal{G}_{\mathcal{M}}(S) = \mathcal{G}(S)/I_{\mathcal{M}}(S)$. It is straightforward to check that $\mathcal{G}_{\mathcal{M}}$ is a Green ring, since the formulas above show that the pairing $\mathcal{G} \times \mathcal{G} \to \mathcal{G}$ restricts to pairings $I_{\mathcal{M}} \times \mathcal{G} \to I_{\mathcal{M}}$ and $\mathcal{G} \times I_{\mathcal{M}} \to I_{\mathcal{M}}$ of Mackey functors. By construction, $\mathcal{M}$ is also a Green module over $\mathcal{G}_{\mathcal{M}}$.

Definition 2.3. Let $\mathcal{M}$ be a Mackey functor. The *Burnside quotient Green ring* of $\mathcal{M}$ is the Green ring $\mathcal{A}_{\mathcal{M}} := \mathcal{A}/I_{\mathcal{M}}$. Let $\iota_{\mathcal{M}} : \mathcal{A} \to \mathcal{A}_{\mathcal{M}}$ denote the epimorphism of Green rings given by the natural quotient map.

Remark 2.4. For $\mathscr{G}$ a Green ring, the map $\iota : \mathscr{A} \to \mathscr{G}$ defined above by $a \mapsto a \cdot 1_S$ factors through $\iota_{\mathscr{G}} : \mathscr{A} \to \mathscr{A}_{\mathscr{G}}$, and we obtain a *canonical* induced homomorphism of Green rings $\mathscr{A}_{\mathscr{G}} \to \mathscr{G}$. The next result shows that in fact $\mathscr{A}_{\mathscr{G}} = \operatorname{Im} \iota$, which gives a quick alternate definition of $\mathscr{A}_{\mathscr{G}}$. (For this observation, compare [Dress 1973, page 207; Oliver 1988, page 253; Hambleton and Taylor 2000, page 711; Bak 1995, page 236].)

Lemma 2.5. *Let $\mathscr{G}$ be a Green ring. Then the canonical homomorphism of Green rings, $\mathscr{A}_{\mathscr{G}} \to \mathscr{G}$ is injective.*

Proof. For each G-set S, the natural transformation of bifunctors in the statement maps $\mathscr{A}_{\mathscr{G}}(S) \to \mathscr{G}(S)$ by the ring homomorphism $a \mapsto a \cdot 1_S$, where $a \in \mathscr{A}(S)$ and $1_S \in \mathscr{G}(S)$ is the unit. If $a \cdot 1_S = 0$, and $\varphi : S \to T$, $\psi : U \to S$ are G-maps, then it follows as above that $\varphi_{\mathscr{A}}(a) \cdot 1_T = 0$ and $\psi^{\mathscr{A}}(a) \cdot 1_U = 0$, Therefore $\{a \in \mathscr{A}(S) \mid a \cdot 1_S = 0\} \subset I_{\mathscr{G}}(S)$, and the ring homomorphism $\mathscr{A}_{\mathscr{G}}(S) \to \mathscr{G}(S)$ is injective. □

We will explore Definition 2.3 by considering the Burnside quotient Green rings for filtrations of Mackey functors.

Definition 2.6. If $\mathscr{M}$ and $\mathscr{N}$ are Mackey functors, we say that $\mathscr{M}$ is a *subfunctor* of $\mathscr{N}$ (respectively $\mathscr{N}$ is a *quotient functor* of $\mathscr{M}$) if there is a natural transformation $\Theta : \mathscr{M} \to \mathscr{N}$ such that for each object $S \in \mathscr{D}(G)$ the function $\Theta_S : \mathscr{M}(S) \to \mathscr{N}(S)$ is an injective (respectively, surjective) homomorphism of abelian groups. We say that $\mathscr{M}$ is a *subquotient* of $\mathscr{N}$ if there is a finite sequence of Mackey functors $\mathscr{M} = \mathscr{L}_0, \mathscr{L}_1, \dots, \mathscr{L}_r = \mathscr{N}$ such that each $\mathscr{L}_i$ is either a subfunctor or a quotient functor of $\mathscr{L}_{i+1}$, for $i = 0, \dots, r-1$. Note that the relation "$\mathscr{M}$ is a subquotient of $\mathscr{N}$" is a transitive relation.

Example 2.7. If $\Theta : \mathscr{G} \to \mathscr{H}$ is a homomorphism of Green rings, then we can regard $\mathscr{H}$ as a Green module over $\mathscr{G}$. Furthermore, $\ker \Theta = I_{\mathscr{H}} \subset \mathscr{G}$, and there is an induced homomorphism $\mathscr{G}_{\mathscr{H}} \to \mathscr{H}$ of Green rings. If $\mathscr{H}$ is a quotient Green ring of $\mathscr{G}$, then $\mathscr{H} = \mathscr{G}_{\mathscr{H}} = \mathscr{G}/I_{\mathscr{H}}$.

Lemma 2.8. *Let $\mathscr{G}$ be a Green ring and $\mathscr{M}$ a Green module over $\mathscr{G}$. Then the Burnside quotient Green ring $\mathscr{A}_M$ is a quotient of $\mathscr{A}_{\mathscr{G}}$, and isomorphic to a subquotient of $\mathscr{G}$.*

Proof. Since $\mathscr{A}_{\mathscr{G}}$ is a sub-Green ring of $\mathscr{G}$, we just need to check that $\mathscr{A}_{\mathscr{M}}$ is a quotient Green ring of $\mathscr{A}_{\mathscr{G}}$ under the natural projection from $\mathscr{A}$. This is equivalent to the statement that $I_{\mathscr{G}} \subset I_{\mathscr{M}}$. Let $a \in I_{\mathscr{G}}(S)$, and consider G-maps $\varphi : S \to T$ and $\psi : U \to S$. For any $y \in \mathscr{M}(T)$,

$$\varphi_{\mathscr{A}}(a) \cdot y = \varphi_{\mathscr{A}}(a) \cdot (1_T \cdot y) = (\varphi_{\mathscr{A}}(a) \cdot 1_T) \cdot y = 0,$$

since $1_T \in \mathcal{G}(T)$. Similarly, for any $z \in \mathcal{M}(U)$,

$$\psi^{\mathcal{A}}(a) \cdot z = \psi^{\mathcal{A}}(a) \cdot 1_U \cdot z = 0,$$

and we see that $a \in I_{\mathcal{M}}(S)$. □

Lemma 2.9. *Let $\mathcal{M}$ and $\mathcal{N}$ be Mackey functors, with $\mathcal{M}$ a subquotient of $\mathcal{N}$. Then there is a surjective homomorphism of Green rings $\mathfrak{f}\colon \mathcal{A}_{\mathcal{N}} \to \mathcal{A}_{\mathcal{M}}$ such that $\mathfrak{f} \circ \iota_{\mathcal{N}} = \iota_{\mathcal{M}}$.*

Proof. We will establish this result for subfunctors and quotient functors, and note that the general subquotient case follows by an inductive argument on the length of the chain joining $\mathcal{M}$ to $\mathcal{N}$.

Suppose first that $\Theta\colon \mathcal{M} \to \mathcal{N}$ is a natural transformation, with $\Theta_S\colon \mathcal{M}(S) \to \mathcal{N}(S)$ injective for all finite G-sets S. Let $a \in I_{\mathcal{N}}(S)$ and let $\varphi : S \to T$, $\psi : U \to S$ be G-maps. Then for any $y \in \mathcal{M}(T)$, $\Theta_T(\varphi_{\mathcal{A}}(a) \cdot y) = \varphi_{\mathcal{A}}(a) \cdot (\Theta_T(y)) = 0$ since Θ is a $\mathcal{A}$-Green module map. Similarly, for any $z \in \mathcal{M}(U)$, $\Theta_U(\psi^{\mathcal{A}}(a) \cdot z) = \psi^{\mathcal{A}}(a) \cdot (\Theta_U(z)) = 0$. Since Θ_T and Θ_U are injective, it follows that $a \in I_{\mathcal{M}}(S)$, and $I_{\mathcal{N}} \subset I_{\mathcal{M}}$ so that $\mathcal{A}_{\mathcal{N}}$ maps onto $\mathcal{A}_{\mathcal{M}}$.

Next suppose that $\Theta\colon \mathcal{N} \to \mathcal{M}$ is a natural transformation, with $\Theta_S\colon \mathcal{N}(S) \to \mathcal{M}(S)$ surjective for all finite G-sets S. If $a \in I_{\mathcal{N}}(S)$, we check that $\varphi_{\mathcal{A}}(a) \cdot y = 0$ and $\psi^{\mathcal{A}}(a) \cdot z = 0$, for all $y \in \mathcal{M}(T)$ and all $z \in \mathcal{M}(U)$, by using the surjectivity of Θ_T and Θ_U, and the compatibility of Θ with the $\mathcal{A}$-module structures on $\mathcal{M}$ and $\mathcal{N}$. Therefore $I_{\mathcal{N}} \subset I_{\mathcal{M}}$. □

In general, if $\mathcal{M}$ is a sub-Mackey functor of $\mathcal{N}$ it is not true that $I_{\mathcal{M}} \subset I_{\mathcal{N}}$, so there is no natural map in the other direction from $\mathcal{A}_{\mathcal{M}}$ onto $\mathcal{A}_{\mathcal{N}}$, but here is one more situation that works.

We say that $\mathcal{M}$ is a *full lattice* in $\mathcal{N}$ if there is a natural transformation $\Theta : \mathcal{M} \to \mathcal{N}$ such that the induced maps $\Theta_S^* : \mathrm{Hom}(\mathcal{N}(S), \mathcal{N}(S)) \to \mathrm{Hom}(\mathcal{M}(S), \mathcal{N}(S))$ are injective for all finite G-sets S. Note that $\mathcal{M}$ need not be a sub-Mackey functor of $\mathcal{N}$ for this condition to hold.

Lemma 2.10. *Let $\mathcal{M}$ and $\mathcal{N}$ be Mackey functors, with $\mathcal{M}$ be a full lattice in $\mathcal{N}$. Then there exists a surjective homomorphism of Green rings $\mathfrak{g} : \mathcal{A}_{\mathcal{M}} \to \mathcal{A}_{\mathcal{N}}$ such that $\mathfrak{g} \circ \iota_M = \iota_N$. If $\mathcal{M}$ is also a subfunctor of $\mathcal{N}$, then $\mathfrak{g}$ is an isomorphism and the inverse to the $\mathfrak{f}\colon \mathcal{A}_{\mathcal{N}} \to \mathcal{A}_{\mathcal{M}}$ described previously.*

Proof. Let $\varphi : S \to T$ be a G-map. For each $a \in \mathcal{A}(S)$ we can consider the action map $y \mapsto \varphi_{\mathcal{A}}(a) \cdot y$ as an element of $\mathrm{Hom}(\mathcal{N}(T), \mathcal{N}(T))$. However if $a \in I_{\mathcal{M}}(a)$, this homomorphism is zero on the image of Θ_T, and therefore it vanishes identically. Similarly, we check that $\psi^{\mathcal{A}}(a) \cdot z = 0$ for all $z \in \mathcal{N}(U)$ and any G-map $\psi : U \to S$. Therefore $I_{\mathcal{M}} \subset I_{\mathcal{N}}$. □

2E. *Amitsur complexes.* Dress proves computation results for Mackey functors via the contractibility of certain chain complexes. Let X, Y be finite G-sets, and define a semisimplicial set $Am(X, Y)$ inductively. Let $Am_0(X, Y) = Y$ and $Am_r(X, Y) = X \times Am_{r-1}(X, Y)$ for $r \geq 1$. There are G-maps

$$d_i^r : Am_r(X, Y) \to Am_{r-1}(X, Y)$$

for $0 \leq i < r$, defined by setting d_0^r as the projection

$$X \times Am_{r-1}(X, Y) \to Am_{r-1}(X, Y),$$

and for $i > 0$ by $d_i^r = 1_X \times d_{i-1}^{r-1}$.

Definition 2.11. Let $\mathcal{M}$ be a Mackey functor. For given finite G-sets X, Y, the *Amitsur complex* $\mathcal{M}\big(Am(X, Y)\big)$ is the chain bicomplex whose chain group in dimension r is $\mathcal{M}\big(Am_r(X, Y)\big)$, with boundary operators $\partial_r = \sum(-1)^i [d_i^r]_{\mathcal{M}}$ and $\delta^r = \sum(-1)^i [d_i^r]^{\mathcal{M}}$ for $r \geq 0$. and zero otherwise.

This construction has certain naturality properties.

Lemma 2.12. *Let $\mathcal{M}$ be a Mackey functor. The Amitsur complex gives a bifunctor*

$$\mathcal{M}(Am(\_\,,\_)) : \mathfrak{D}(G) \times \mathfrak{D}(G) \to Chain(\mathcal{A}b)$$

where $Chain(\mathcal{A}b)$ denotes the category of chain complexes of abelian groups.

For any Mackey functor $\mathcal{M}$, and any finite G-set S, let $\mathcal{M}_S$ denote the Mackey functor defined by $\mathcal{M}_S(T) = \mathcal{M}(S \times T)$, for any finite G-set T. There are natural transformations and $\Theta_S^{\mathcal{M}} : \mathcal{M} \to \mathcal{M}_S$ and $\Theta_{\mathcal{M}}^S : \mathcal{M}_S \to \mathcal{M}$ of Mackey functors induced by the projection maps $S \times T \to T$. Dress says that $\mathcal{M}$ is *S-injective* (respectively *S-projective*) if $\Theta_S^{\mathcal{M}}$ is split-injective (respectively $\Theta_{\mathcal{M}}^S$ is split surjective).

Lemma 2.13 [Dress 1975, Proposition 1.1′]. *A Mackey functor $\mathcal{M}$ is S-injective if and only if it is S-projective.*

Proof. Suppose that $\mathcal{M}$ is S-projective, so that $\Theta_{\mathcal{M}}^S$ is split-injective. Let $\Phi : \mathcal{M} \to \mathcal{M}_S$ be a natural transformation such that $\Theta_{\mathcal{M}}^S \circ \Phi = Id_{\mathcal{M}}$ (the identity natural transformation on $\mathcal{M}$). If $\Delta : S \to S \times S$ denotes the diagonal map and $p : S \times T \to T$ the second factor projection, we notice that

$$S \times T \xrightarrow{\Delta \times 1} S \times S \times T \xrightarrow{1 \times p} S \times T$$

is just the identity map on $S \times T$. It follows that

$$\Theta_{\mathcal{M}(T)}^S \circ (\Delta \times 1)^{\mathcal{M}} \circ \Phi_{S \times T} \circ \Theta_S^{\mathcal{M}(T)} = Id_{\mathcal{M}(T)}$$

for any finite G-set T. One can check that the formula

$$\widetilde{\Phi}(T) := \Theta_{\mathcal{M}(T)}^S \circ (\Delta \times 1)^{\mathcal{M}} \circ \Phi_{S \times T}$$

defines a natural transformation of bifunctors splitting $\Theta_S^{\mathcal{M}}$ and hence $\mathcal{M}$ is S-injective. The converse is similar. $\square$

Dress now proves that, for any finite G-set Y and whenever $\mathcal{M}$ is S-injective or S-projective, both Amitsur complexes $(\mathcal{M}_*(Am(S,Y),\partial)$ and $(\mathcal{M}^*(Am(S,Y),\delta)$ are contractible (we say $\mathcal{M}$ is *S-computable*). In particular, for $Y=\bullet$ there are exact sequences

$$\cdots \xrightarrow{\partial_3} \mathcal{M}(S\times S) \xrightarrow{\partial_2} \mathcal{M}(S) \xrightarrow{\partial_1} \mathcal{M}(\bullet) \longrightarrow 0$$

$$0 \longrightarrow \mathcal{M}(\bullet) \xrightarrow{\delta_1} \mathcal{M}(S) \xrightarrow{\delta_2} \mathcal{M}(S\times S) \xrightarrow{\delta_3} \cdots$$

which exhibit $\mathcal{M}(\bullet)$ as a limit of induction or restriction maps respectively.

Here is the main theorem of Dress induction theory:

Proposition 2.14 [Dress 1975, Proposition 1.2]. *Let $\mathcal{G}$ be a Green ring and S be a finite G-set. Then the following conditions are equivalent*:

(1) *The map $\varphi_{\mathcal{G}}:\mathcal{G}(S)\to\mathcal{G}(\bullet)$ associated to the projection $\varphi: S\to\bullet$ is surjective.*

(2) *$\mathcal{G}$ is S-injective.*

(3) *All $\mathcal{G}$-modules are S-injective.*

This result focuses attention on the task of finding a suitable Green ring which acts on $\mathcal{M}$, and then checking property (i). We remark that the Burnside ring $\mathcal{A}$ acts on any Mackey functor, but $\mathcal{A}$ is S-injective only if $\bullet\subset S$. Hence the Burnside ring itself has no useful induction properties.

3. Dress generating sets

In the classical Mackey setting of G-functors given by Green [1971], computation is expressed in terms of families. A *family of subgroups* $\mathcal{F}$ of G is a collection of subgroups closed under conjugation and taking subgroups. For any finite G-set X let $\mathcal{F}(X)$ denote the family generated by the isotropy subgroups of X. For example, the family $\mathcal{F}(\bullet)=\{\text{All}\}$. Conversely, given a family $\mathcal{F}$ of subgroups, we can form the disjoint union $X(\mathcal{F})$ of G-sets G/H, one for each conjugacy class of maximal elements in $\mathcal{F}$, under the partial ordering from subgroup inclusion. For example, $X(\{\text{All}\})=\bullet$. We say that a family of subgroups $\mathcal{F}$ *contracts* a Mackey functor $\mathcal{M}$ if and only if $\mathcal{M}$ is $X(\mathcal{F})$-projective or $X(\mathcal{F})$-injective.

We have seen that a good strategy for computing a Mackey functor $\mathcal{M}$ is to study the Green rings acting on $\mathcal{M}$. We will apply this strategy to the Burnside quotient Green ring $\mathcal{A}_{\mathcal{M}}$ of $\mathcal{M}$.

Definition 3.1. Let $\mathcal{G}$ be a Green ring. A finite G-set X is a *generating set* for $\mathcal{G}$ if the natural map $\mathcal{G}(X)\to\mathcal{G}(\bullet)$ is surjective (equivalently, if $1_\bullet\in\operatorname{Im}\{\mathcal{G}(X)\to\mathcal{G}(\bullet)\}$).

By [Dress 1975, Proposition 1.2], X is a generating set for $\mathcal{G}$ if and only if $\mathcal{G}$ is X-injective or X-projective. It is not true in general that a generating set for a Green ring $\mathcal{G}$ is also a generating set for the sub-Green ring $\mathcal{A}_{\mathcal{G}}$. To obtain generation for $\mathcal{A}_{\mathcal{G}}$ it is usually necessary to enlarge the generating set.

For H a finite group and p a prime, let

$$O^p(H) = \bigcap \{H_0 \lhd H \mid H/H_0 \text{ is a } p\text{-group}\}$$

Notice that $O^p(H)$ is a characteristic subgroup of p-power index in H, and

$$O^p\big((O^p(H)\big) = O^p(H).$$

Definition 3.2. Let $\mathcal{F}$ be a family of subgroups of G and p a prime. Then *hyper$_p$-$\mathcal{F}$* is the family consisting of all subgroups H in G such that $O^p(H) \in \mathcal{F}$. If S is a G-set, then *hyper$_p$-S* is the corresponding G-set to *hyper$_p$-$\mathcal{F}(S)$*. This construction is due to Dress [1975, page 307].

It is easy to check that *hyper$_p$-$\mathcal{F}$* is closed under taking subgroups and conjugation, so we obtain a family of subgroups. By construction, there is a G-map $X \to$ *hyper$_p$-X* for any X and *hyper$_p$-hyper$_p$-X* $=$ *hyper$_p$-X*. One of Dress's main results is the following:

Theorem 3.3 [Dress 1973, page 207]. *Let $\mathcal{M}$ be a Mackey functor. For any prime p and for any finite G-set Y, let $\mathcal{K}(Y) = \ker\big(\mathcal{M}(\bullet) \otimes \mathbf{Z}_{(p)} \to \mathcal{M}(Y) \otimes \mathbf{Z}_{(p)}\big)$ and $\mathcal{I}(Y) = \operatorname{Im}\big(\mathcal{M}(hyper_p\text{-}Y) \otimes \mathbf{Z}_{(p)} \to \mathcal{M}(\bullet) \otimes \mathbf{Z}_{(p)}\big)$. Then $\mathcal{M}(\bullet) \otimes \mathbf{Z}_{(p)} = \mathcal{K}(Y) + \mathcal{I}(Y)$.*

If Y is a finite G-set, we will use the notation $\langle Y \rangle$ for the equivalence class of Y in the category $\mathcal{D}(G)$. One useful consequence is:

Lemma 3.4. *Let $\mathcal{G}_0$ be a sub-Green ring of $\mathcal{G}_1$. For any prime p, and any finite G-set Y with $\langle Y \rangle = \langle hyper_p\text{-}Y \rangle$, the natural map $\mathcal{G}_0(Y) \otimes \mathbf{Z}_{(p)} \to \mathcal{G}_0(\bullet) \otimes \mathbf{Z}_{(p)}$ is surjective if and only if $\mathcal{G}_1(Y) \otimes \mathbf{Z}_{(p)} \to \mathcal{G}_1(\bullet) \otimes \mathbf{Z}_{(p)}$ is surjective.*

Proof. For any Green ring $\mathcal{G}$ and any finite G-set Y, the image of $\mathcal{G}(Y) \otimes \mathbf{Z}_{(p)}$ in $\mathcal{G}(\bullet) \otimes \mathbf{Z}_{(p)}$ is an ideal. Hence either map is onto if and only if $1_{\mathcal{G}_i(\bullet)}$ is in the image. Since $1_{\mathcal{G}_0(\bullet)}$ goes to $1_{\mathcal{G}_1(\bullet)}$, this proves the first implication.

For the converse, the surjectivity of $\mathcal{G}_1(Y) \otimes \mathbf{Z}_{(p)} \to \mathcal{G}_1(\bullet) \otimes \mathbf{Z}_{(p)}$ implies that the Amitsur complex is contractible for the restriction maps induced by the transformation $Y \to \bullet$. In particular, $\mathcal{G}_1(\bullet) \otimes \mathbf{Z}_{(p)} \to \mathcal{G}_1(Y) \otimes \mathbf{Z}_{(p)}$ is injective. Therefore $\mathcal{G}_0(\bullet) \otimes \mathbf{Z}_{(p)} \to \mathcal{G}_0(Y) \otimes \mathbf{Z}_{(p)}$ is injective, and from Theorem 3.3 we conclude that $\mathcal{G}_0(hyper_p\text{-}Y) \otimes \mathbf{Z}_{(p)} \to \mathcal{G}_0(\bullet) \otimes \mathbf{Z}_{(p)}$ is surjective. $\square$

Suppose that $\mathcal{G}$ is a Green ring which acts on a Mackey functor $\mathcal{M}$. For many applications of induction theory, the "best" Green ring for $\mathcal{M}$ is the *Burnside quotient Green ring* $\mathcal{A}_{\mathcal{G}}$. This is a Green ring which acts on $\mathcal{M}$, and by construction $\mathcal{A}_{\mathcal{G}}$ is a sub-Green ring of $\mathcal{G}$. In particular, the natural map $\mathcal{A}_{\mathcal{G}} \to \mathcal{G}$ is an injection.

Definition 3.5. A finite G-set X is a *Dress generating set* for a Green ring $\mathcal{G}$, provided that $\mathcal{G}(hyper_p\text{-}X) \otimes \mathbf{Z}_{(p)} \to \mathcal{G}(\bullet) \otimes \mathbf{Z}_{(p)}$ is surjective for each prime p.

By Theorem 3.3, any finite G-set X such that the natural map $\mathcal{G}(\bullet) \to \mathcal{G}(X)$ is *injective* is a Dress generating set for $\mathcal{G}$. Notice that a Dress generating set for $\mathcal{G}$ is also a Dress generating set for any quotient Green ring of $\mathcal{G}$. The following result (Theorem A) is the main step in handling sub-Green rings.

Theorem 3.6. *A finite G-set X is a Dress generating set for a Green ring $\mathcal{G}$ if and only if it is a Dress generating set for the Burnside quotient Green ring $\mathcal{A}_{\mathcal{G}}$.*

Proof. We apply the result above to $Y = hyper_p\text{-}X$, for each prime p, and note that $\mathcal{A}_{\mathcal{G}}$ is a sub-Green ring of $\mathcal{G}$. □

The Burnside quotient Green ring can be used to compute Mackey functors obtained by subquotients.

Definition 3.7. A finite G-set X is a *Dress generating set* for a Mackey functor $\mathcal{M}$, provided that X is a Dress generating set for the Burnside quotient Green ring $\mathcal{A}_{\mathcal{M}}$ of $\mathcal{M}$.

This is consistent with our previous Definition 3.5 for a Green ring.

Theorem 3.8. *Let $\mathcal{G}$ be a Green ring and $\mathcal{M}$, $\mathcal{N}$ Mackey functors.*

(i) *If $\mathcal{M}$ is a $\mathcal{G}$-module and X is a Dress generating set for $\mathcal{G}$, then X is a Dress generating set for $\mathcal{M}$.*

(ii) *If $\mathcal{N}$ is a subquotient of $\mathcal{M}$ and X is a Dress generating set for $\mathcal{M}$, then X is a Dress generating set for $\mathcal{N}$.*

(iii) *If $\mathcal{M}$ is a full lattice in $\mathcal{N}$ and X is a Dress generating set for $\mathcal{M}$, then X is a Dress generating set for $\mathcal{N}$.*

Proof. Under the first assumption, $\mathcal{A}_{\mathcal{M}}$ is a subquotient of $\mathcal{G}$. In the other parts, $\mathcal{A}_{\mathcal{N}}$ is a quotient of $\mathcal{A}_{\mathcal{M}}$. □

We can translate this into a computability statement as follows:

Corollary 3.9. *Let p be a prime and $\mathcal{G}$ be a Green ring. Suppose that $\mathcal{F}$ is a $hyper_p$-closed family of subgroups of G. Then $\mathcal{G} \otimes \mathbf{Z}_{(p)}$ is $\mathcal{F}$-computable if and only if $\mathcal{A}_{\mathcal{G}} \otimes \mathbf{Z}_{(p)}$ is $\mathcal{F}$-computable.*

The advantage of $\mathcal{A}_{\mathcal{G}}$ over $\mathcal{G}$ is that $\mathcal{A}_{\mathcal{G}}$ acts on Mackey functors which are subfunctors or quotient functors of $\mathcal{M}$ but $\mathcal{G}$ does not in general. For example, $\mathcal{G}$ never acts on $\mathcal{A}_{\mathcal{G}}$ unless they are equal. We next point out another good feature of the Burnside quotient Green ring.

Theorem 3.10 [Hambleton 2006, Theorem 1.8]. *Suppose that $\mathcal{G}$ is a Green ring which acts on a Mackey functor $\mathcal{M}$, and $\mathcal{F}$ is a hyper$_p$-closed family of subgroups of G. If $\mathcal{G}\otimes\mathbf{Z}_{(p)}$ is $\mathcal{F}$-computable, then every $x\in\mathcal{M}(G)\otimes\mathbf{Z}_{(p)}$ can be written as*

$$x=\sum_{H\in\mathcal{F}} a_H \operatorname{Ind}_H^G\bigl(\operatorname{Res}_G^H(x)\bigr)$$

for some coefficients $a_H\in\mathbf{Z}_{(p)}$, where the a_H are the same for all x.

Proof. Since $\mathcal{G}\otimes\mathbf{Z}_{(p)}$ is $\mathcal{F}$-computable, we know that $\mathcal{A}_{\mathcal{G}}\otimes\mathbf{Z}_{(p)}$ is also $\mathcal{F}$-computable. Therefore, we can write $1=\sum_{K\in\mathcal{F}} b_K \operatorname{Ind}_K^G(y_K)$ for some $y_K\in\mathcal{A}_{\mathcal{G}}(K)\otimes\mathbf{Z}_{(p)}$ and $b_K\in\mathbf{Z}_{(p)}$. For any $x\in\mathcal{M}(G)\otimes\mathbf{Z}_{(p)}$ we now have the formula

$$x=1\cdot x=\sum_{K\in\mathcal{F}} b_K \operatorname{Ind}_K^G\bigl(y_K\cdot\operatorname{Res}_G^K(x)\bigr).$$

But each $y_K\in\mathcal{A}_{\mathcal{G}}(K)\otimes\mathbf{Z}_{(p)}$ can be represented by a sum $\sum c_{KH}[K/H]$, with $c_{KH}\in\mathbf{Z}_{(p)}$, under the surjection $\mathcal{A}(K)\to\mathcal{A}_{\mathcal{G}}(K)$. Therefore

$$\begin{aligned} x&=\textstyle\sum_{K\in\mathcal{F}} b_K \sum_{H\subseteq K} c_{KH} \operatorname{Ind}_K^G\bigl([K/H]\cdot\operatorname{Res}_G^K(x)\bigr)\\ &=\textstyle\sum_{K\in\mathcal{F}} b_K \sum_{H\subseteq K} c_{KH} \operatorname{Ind}_K^G\bigl(\operatorname{Ind}_H^K\bigl(\operatorname{Res}_K^H\bigl(\operatorname{Res}_G^K(x)\bigr)\bigr)\bigr)\\ &=\textstyle\sum_{K\in\mathcal{F}} b_K \sum_{H\subseteq K} c_{KH} \operatorname{Ind}_H^G\bigl(\operatorname{Res}_G^H(x)\bigr)\end{aligned}$$

We now define $a_H=\sum_{K\in\mathcal{F}} b_K c_{KH}$, and the formula becomes

$$x=\sum_{H\in\mathcal{F}} a_H \operatorname{Ind}_H^G\bigl(\operatorname{Res}_G^H(x)\bigr). \qquad \square$$

Example 3.11 (Representation theory). Recall that a p-(hyper)elementary group is a (semi)direct product $C\rtimes P$, where P is a p-group and C is cyclic of order prime to p. A Dress generating set for a Green ring $\mathcal{G}$ need not be a generating set for $\mathcal{G}$. For example, let E denote the finite G-set, $E=\coprod G/H$, where we have one H for each p-elementary subgroup of G. It is known that E is a generating set for the complex representation ring $R_{\mathbf{C}}(G)$, but not in general for the rational representation ring $R_{\mathbf{Q}}(G)$. On the other hand, complex representations are detected by characters, so any G-set with isotropy containing the cyclic family is a Dress generating set for $R_{\mathbf{C}}(G)$, or for the sub-Green ring $R_{\mathbf{Q}}(G)$ by Theorem 3.8 (ii). It follows that the hyperelementary family $\mathcal{H}$ gives a generating set $X_{\mathcal{H}}$ for $R_{\mathbf{Q}}(G)$. This implies the Brauer–Berman–Witt induction theorem for rational representations.

Example 3.12 (The Swan ring). The Swan ring is one of the main examples of Green rings in the classical setting of induction theory [Swan 1970]. For any finite group, let $SW(G,\mathbf{Z})$ denote the Grothendieck group of isomorphism classes of finitely-generated left $\mathbf{Z}G$-modules, with $[L] = [L'] + [L'']$ whenever there is a short exact sequence

$$0 \to L' \to L \to L'' \to 0$$

of such $\mathbf{Z}G$-modules. The operation $L \otimes_{\mathbf{Z}} L'$ gives a ring structure on this Grothendieck group, so we obtain a commutative ring. The usual induction and restriction operations for such modules give the Swan ring the structure of a Mackey functor. We let

$$SW_G : \mathfrak{D}(G) \to \mathcal{A}b$$

denote the Green ring (in the sense of Dress) defined by $SW_G(G/H) := SW(H, \mathbf{Z})$, and extended to $\mathfrak{D}(G)$ by additivity. Since $SW(G, \mathbf{Z})$ is hyperelementary computable by Swan's induction theorem [Dress 1973, page 211], we see that any Mackey functor on which this Green ring acts is hyperelementary computable.

It follows that the Burnside quotient Green ring of the Swan ring, denoted $\mathcal{A}_{SW}$, also has the hyperelementary set $X_{\mathcal{H}}$ as a Dress generating set (or more precisely, any G-set whose isotropy contains the cyclic family is a Dress generating set). In this case, $\mathcal{A}_{SW}(G/H) \subset SW_G(G/H)$ is the subring $P(H, \mathbf{Z}) \subset SW(H, \mathbf{Z})$ generated by the permutation modules $\mathbf{Z}[H/K]$, for all subgroups $K \subseteq H$.

4. Computation techniques

Dress generating sets can also be used to compute exact sequences of Mackey functors or filtrations of Mackey functors by subfunctors. We say that

$$\mathcal{M}_0 \xrightarrow{a} \mathcal{M}_1 \xrightarrow{b} \mathcal{M}_2$$

is an *exact sequence* of Mackey functors if a and b are homomorphisms of Mackey functors, such that the sequence $\mathcal{M}_0(S) \to \mathcal{M}_1(S) \to \mathcal{M}_2(S)$ is exact for each finite G-set S. We define long exact sequences in a similar way.

Proposition 4.1. *Suppose that $\mathcal{M}_0 \to \mathcal{M}_1 \to \mathcal{M}_2$ is an exact sequence of Mackey functors. Then X is a Dress generating set for $\mathcal{M}_1$ whenever X is a Dress generating set for $\mathcal{M}_0$ and $\mathcal{M}_2$.*

Proof. We may assume that $\mathcal{M}_0 \to \mathcal{M}_1$ is injective, and that $\mathcal{M}_1 \to \mathcal{M}_2$ is surjective, and the projections from $\mathcal{A}$ induce a natural transformation $\theta : \mathcal{A}_{\mathcal{M}_1} \to \mathcal{A}_{\mathcal{M}_0} \oplus \mathcal{A}_{\mathcal{M}_2}$ of Green rings. By exactness, $I_\theta(S) := \ker \theta_S$ is a nilpotent ideal (of nilpotence index 2). Let $\bar{\mathcal{A}}_{\mathcal{M}_1} = \mathcal{A}_{\mathcal{M}_1}/I_\theta$ denote the quotient Mackey functor. Since this Mackey functor is mapped injectively by θ into $\mathcal{A}_{\mathcal{M}_0} \oplus \mathcal{A}_{\mathcal{M}_2}$, X is a Dress generating set for $\bar{\mathcal{A}}_{\mathcal{M}_1}$. It follows that $\bar{\mathcal{A}}_{\mathcal{M}_1}(hyper_p\text{-}X) \otimes \mathbf{Z}_{(p)} \to \bar{\mathcal{A}}_{\mathcal{M}_1}(\bullet) \otimes \mathbf{Z}_{(p)}$ is surjective for every prime p. But an element in $\mathcal{A}_{\mathcal{M}_1}(\bullet) \otimes \mathbf{Z}_{(p)}$ hitting $1_\bullet \in \bar{\mathcal{A}}(\bullet) \otimes \mathbf{Z}_{(p)}$ has the form $1+u$, where $u \in I_\theta(\bullet) \otimes \mathbf{Z}_{(p)}$. Since $u^2 = 0$, $1+u$ is invertible and $(1+u)^{-1} = 1-u$. If $p_X : X \to \bullet$ denotes the projection map, and $(p_X)_*(a) = 1+u$, then we have

$(p_X)_*\big((p_X)^*(1_\bullet - u)\cdot a\big) = (1_\bullet - u)\cdot (p_X)_*(a) = 1_\bullet$ and hence X is a Dress generating set for $\mathscr{A}_{\mathcal{M}_1}$. □

Remark 4.2. In the proof of Proposition 4.1, we have shown for each prime p, there exists an element $a \in \mathscr{A}(hyper_p\text{-}X)\otimes \mathbf{Z}_{(p)}$ such that $a \mapsto 1_\bullet$ in each of the Burnside quotient Green rings $\mathscr{A}_{\mathcal{M}_i}(\bullet)\otimes \mathbf{Z}_{(p)}$, for $i = 0, 1, 2$. The same argument extends by induction to finite filtrations of a Mackey functor by sub-Mackey functors.

Corollary 4.3. *Let $0 = \mathcal{N}_0 \subset \cdots \subset \mathcal{N}_r = \mathcal{M}$ be a filtration of a Mackey functor by sub-Mackey functors. Then X is a Dress generating set for $\mathcal{M}$ if and only if X is a Dress generating set for each quotient $\mathcal{N}_i/\mathcal{N}_{i-1}$, for $1 \le i \le r$.*

A finite length *chain complex* of Mackey functors is a sequence $(\mathcal{N}_i, \partial_i)$ of Mackey functors $\mathcal{N}_i$, $0 \le i \le r$, and natural transformations $\partial_i : \mathcal{N}_i \to \mathcal{N}_{i-1}$, $1 \le i \le r$, such that $(\mathcal{N}(S), \partial_*)$ is a chain complexes of abelian groups for each finite G-set S. A chain complex $\mathcal{N}$ of Mackey functors has homology groups $H_i(\mathcal{N})$, $0 \le i \le r$, which are subquotient Mackey functors of $\mathcal{N}_i$.

Corollary 4.4. *Suppose that $\mathcal{N}$ is a finite length chain complex of Mackey functors. If X is a Dress generating set for each $\mathcal{N}_i$, $0 \le i \le r$, then X is a Dress generating set for each of the homology Mackey functors $H_i(\mathcal{N})$, $0 \le i \le r$.*

Another useful construction is completion.

Theorem 4.5. *Let $\mathcal{M}$ be a Mackey functor, and let $\mathcal{F}$ denote a (possibly infinite) filtration*

$$\mathcal{M} = F_0 \supseteq F_1 \supseteq \cdots \supseteq F_r \supseteq \cdots$$

of $\mathcal{M}$ by sub-Mackey functors. A finite G-set X is a Dress generating set for $\widehat{\mathcal{M}}_{\mathcal{F}} = \varprojlim \mathcal{M}/F_r$ if and only if X is a Dress generating set for each quotient Mackey functor F_{r-1}/F_r, $r \ge 1$.

Proof. Since each F_{r-1}/F_r is a subquotient of $\widehat{\mathcal{M}}_{\mathcal{F}}$, the necessity follows from the results above. For sufficiency, we first note by Corollary 4.3 that X is Dress generating set for each quotient $\mathcal{M}/F_r$. It is enough to prove that X generates the inverse limit $\varprojlim \mathscr{A}_{\mathcal{M}/F_r}$ of the Burnside quotient Green rings for the sequence $\{\mathcal{M}/F_r\}$. Suppose that X is a Dress generating set for each $\mathscr{A}_{\mathcal{M}/F_r}$, $r \ge 1$, and set $Y = hyper_p\text{-}X$. If $\{a_r\}$ is a sequence of elements in $\mathscr{A}_{\mathcal{M}/F_r}(Y)$ hitting $1^r_\bullet$, we can use the contractibility of the Y-Amitsur complex for $\mathscr{A}_{\mathcal{M}/F_{r+1}}$ inductively, to adjust each a_{r+1} by an element of $\mathscr{A}_{\mathcal{M}/F_{r+1}}(Y\times Y)$, so that $a_{r+1} \mapsto a_r$. This gives us an element in the inverse limit $\varprojlim \mathscr{A}_{\mathcal{M}/F_r}(Y)$ hitting $1_\bullet \in \varprojlim \mathscr{A}_{\mathcal{M}/F_r}(\bullet) := \mathscr{G}(\bullet)$, and hence the Green ring $\mathscr{G}$ acts on $\widehat{\mathcal{M}}_{\mathcal{F}}$ with X as a Dress generating set. Since $\mathscr{A}_{\mathscr{G}} \twoheadrightarrow \mathscr{A}_{\widehat{\mathcal{M}}_{\mathcal{F}}}$ is surjective, it follows that $\widehat{\mathcal{M}}_{\mathcal{F}}$ has X as a Dress generating set. □

Example 4.6. Here is an important special case. Let $\mathcal{G}$ be a Green ring acting on a Mackey functor $\mathcal{M}$. If $\mathcal{I} \subset \mathcal{G}$ is a Green ideal, we may filter $\mathcal{M}$ by the subbifunctors $F_r = \mathcal{I}^r \mathcal{M}$ and then $\widehat{\mathcal{M}}_{\mathcal{F}}$ is the $\mathcal{I}$-adic completion of $\mathcal{M}$.

In particular, for a given Mackey functor $\mathcal{M}$ we could take $\mathcal{I} = \langle \mathcal{I}_{\mathcal{M}}, 2\rangle$, and then $\widehat{\mathcal{M}}_{\mathcal{I}}$ is just the 2-adic completion of the Mackey functor $\mathcal{M}$. Note that if $\mathcal{M}(X)$ is finitely generated, then $\widehat{\mathcal{M}}_{\mathcal{I}}(X) \cong \mathcal{M}(X) \otimes \widehat{\mathbf{Z}}_2$.

5. Mackey functors and RG-Morita

To prove Theorem B we need to define the bifunctor $d : \mathcal{D}(G) \to RG$-Morita used in its statement. This involves some definitions and elementary properties of categories with bisets as morphisms, which are well-known to the experts. We include this material for the reader's convenience.

In [HTW 1990, 1.A.4] we introduced the category RG-Morita whose basic objects are finite groups H isomorphic to some subquotient of G, and whose morphisms were defined by a Grothendieck group construction on the isomorphism classes of finite H_2-H_1 bisets X, for which the order of the left stabilizer

$$_{H_2}I(x) = \{h \in H_2 \mid hx = x\}$$

is a unit in R, for all $x \in X$. Here R is a commutative ring with unit. We set $X \sim X'$ if RX is isomorphic to RX' as RH_2-RH_1 bimodules. The balanced product $X \times_{H_2} Y$ of an H_3-H_2 biset X and an H_2-H_1 biset Y is a H_3-H_1 biset. This defines the composition for morphisms. The Add-construction [MacLane 1971, page 194] is then applied to complete the definition. Many functors arising in algebraic K-theory and topology are actually functors out of RG-Morita, so it is of interest to recognize when these are Mackey functors.

To relate Mackey functors and RG-Morita, we will need the G-Burnside category, $\mathbf{A}(G)$, whose objects are subgroups $H \subset G$, and where $\mathrm{Hom}_{\mathbf{A}(G)}(H_1, H_2)$ is the Grothendieck construction applied to the isomorphism classes of finite bifree H_2-H_1 bisets (meaning both left and right actions are free). Because of the Grothendieck group construction, $\mathbf{A}(G)$ is an Ab–category, the morphism sets are abelian groups and the compositions are bilinear [MacLane 1971, I.8, page 28]. Let

$$u : \mathbf{A}(G) \to \mathbf{A}_\bullet(G)$$

denote the associated universal free additive category, and the universal inclusion [MacLane 1971, VII.2, problem 6, page 194].

The morphisms in $\mathbf{A}(G)$ are defined by the Grothendieck group construction with addition operation the disjoint union of bisets. By convention, the empty biset $\varnothing$ represents the zero element. Composition comes from the balanced product:

$$_{H_3}X_{H_2} \circ {}_{H_2}X_{H_1} = ({}_{H_3}X_{H_2}) \times_{H_2} ({}_{H_2}X_{H_1}).$$

The reader should check that this is well–defined on isomorphism classes of bisets and "bilinear" in that

$$\big({}_{H_3}X_{H_2} \sqcup {}_{H_3}Y_{H_2}\big) \circ {}_{H_2}X_{H_1} \cong \big({}_{H_3}X_{H_2} \circ {}_{H_2}X_{H_1}\big) \sqcup \big({}_{H_3}Y_{H_2} \circ {}_{H_2}X_{H_1}\big),$$

with a similar formula for disjoint union on the right. The morphisms in $\mathbf{A}_\bullet(G)$ are matrices of morphisms in $\mathbf{A}(G)$.

Definition 5.1. We define a contravariant involution $\tau : \mathbf{A}(G) \to \mathbf{A}(G)$, by the identity on objects, and on morphisms it is the map induced on the Grothendieck construction by the function which takes the finite bifree H_2-H_1 biset ${}_{H_2}X_{H_1}$ to the finite bifree H_1-H_2 biset ${}_{H_1}X_{H_2}$ which is X as a set and $h_1 \cdot x \cdot h_2$ is defined to be $h_2^{-1}xh_1^{-1}$.

The reader needs to check that isomorphic bisets are isomorphic after reversing the order, and should also check that the transpose conjugate of a disjoint union is isomorphic to the disjoint union of the conjugate transposes of the pieces. This means that τ is a functor which induces a homomorphism of Hom–sets. It is clearly an involution, not just up to natural equivalence. Since τ is a homomorphism on Hom-sets, it induces an additive contravariant involution $\tau^\bullet : \mathbf{A}_\bullet(G) \to \mathbf{A}_\bullet(G)$, called *conjugate transpose*, which commutes with the functor u. By definition, $\tau^\bullet$ acts on a matrix of morphisms by applying τ to each entry, and then transposing the matrix. There is a functor

$$a : \mathbf{A}_\bullet(G) \to RG\text{-Morita}$$

given by the inclusion on objects and morphisms (but the equivalence relation on morphisms is different in RG-Morita).

There is a functor $\mathbf{A}_\bullet(G) \to R$-Morita, called the *$R$-group ring functor*, where R-Morita has objects R-algebras and morphisms defined by stable isomorphism classes of bimodules [HTW 1990, 1.A.1]. This functor factors through RG-Morita: it sends $H \mapsto RH$ on objects, and $X \mapsto RX$ on morphisms.

We will define the following diagram of categories and functors:

$$\begin{tikzcd} & \mathbf{A}_\bullet(G) \arrow[d, "a"] \arrow[dr, "R\text{-group ring}"] & \\ \mathscr{D}(G) \arrow[ur, "j"] \arrow[r, "d"] & RG\text{-Morita} \arrow[r] & R\text{-Morita} \end{tikzcd} \tag{5.2}$$

To complete the definition of the functors in this diagram, we need to introduce another category. Let $\mathscr{D}_*(G)$ denote the category whose objects are pairs $(X, \mathbf{b})$, consisting of a finite G-space X and an ordered collection $\mathbf{b} = (b_1, \ldots, b_n)$ of base-points, one for each G-orbit of X. The morphisms are the G-maps (not necessarily base-point preserving). There is a functor

$$\mu : \mathscr{D}_*(G) \to \mathscr{D}(G)$$

defined by forgetting the base-points. Since every object of $\mathscr{D}(G)$ is isomorphic to the image $\mu(X, \mathbf{b})$ of an object of $\mathscr{D}_*(G)$, and μ induces a bijection on morphism sets, it follows that μ gives an equivalence between the categories $\mathscr{D}_*(G)$ and $\mathscr{D}(G)$, with inverse functor μ' [MacLane 1971, IV.4, Theorem 1, page 91].

We can now define two functors

$$(j_\bullet, j^\bullet) : \mathscr{D}_*(G) \to \mathbf{A}_\bullet(G).$$

The covariant functor $j_\bullet$ is the additive extension of the functor which sends an object $(G/H, eH)$ to the isotropy subgroup H, and sends the G map $f : G/H \to G/K$ to the biset ${}_K K_{g^{-1}Hg}$, where $f(eH) = gK$. If we change the coset representative and write $f(eH) = g_1 K$, then the map

$$\psi : {}_K K_{g^{-1}Hg} \to {}_K K_{g_1^{-1}Hg_1} \tag{5.3}$$

defined by $\psi(k) = k(g^{-1}g_1)$ gives an bijection of K-K bisets.
Note that $1_{G/H} : G/H \to G/H$ goes to ${}_H H_H$, which is the identity. Check that if $f_1 : G/H_1 \to G/H_2$ and $f_2 : G/H_2 \to G/H_3$ are G-maps and if $f_1(eH_1) = g_1 H_2$ and $f_2(eH_2) = g_2 H_3$ then $f_2 \circ f_1(eH_1) = (g_1 g_2) H_3$ and

$${}_{H_3}(H_3)_{g_2^{-1}H_2 g_2} \times_{H_2} {}_{H_2}(H_2)_{g_1^{-1}H_1 g_1}$$

is isomorphic to ${}_{H_3}(H_3)_{(g_1g_2)^{-1}H_1(g_1g_2)}$ by the map $(h_3, h_2) \mapsto h_3 g_2^{-1} h_2 g_2$.

The contravariant functor $j^\bullet$ agrees with $j_\bullet$ on objects, but sends the G map $f : G/H \to G/K$ to the biset ${}_{g^{-1}Hg} K_K$, where $f(eH) = gK$. Rather than checking the identity and composition directly, just note that ${}_{g^{-1}Hg} K_K$ is isomorphic to $\tau({}_K K_{g^{-1}Hg})$ by the function which sends k to k^{-1}, so $j^\bullet = \tau \circ j_\bullet$ and hence $j^\bullet$ is a contravariant functor.

Definition 5.4. We define the bifunctor

$$j : \mathscr{D}(G) \to \mathbf{A}_\bullet(G)$$

as the composition $j = (j^\bullet, j_\bullet) \circ \mu'$. Let

$$d = a \circ j : \mathscr{D}(G) \to RG\text{-Morita}$$

denote the composition in diagram (5.2). □

For any additive functor $F : \mathbf{A}_\bullet(G) \to \mathscr{A}b$, the composition $F \circ j : \mathscr{D}(G) \to \mathscr{A}b$ is a Mackey functor [HTW 2008]. Our main application is the following:

Theorem 5.5. *Any additive functor $F : RG$-Morita $\to \mathscr{A}b$ gives a Mackey functor on $\mathfrak{D}(G)$ by composition with $d : \mathfrak{D}(G) \to RG$-Morita. Any such Mackey functor is hyperelementary computable.*

Proof. The functor $d : \mathfrak{D}(G) \to RG$-Morita factors through $\mathbf{A}_\bullet(G)$, so we obtain Mackey functors by composition. We will show that any such Mackey functor, $\mathscr{M}$, is a Green module over the Burnside quotient Green ring $\mathscr{A}_{SW}$ of the Swan ring, and then apply Example 3.12. Let $L = \mathbf{Z}[H/K]$ denote a permutation module, for some subgroups $K \subset H$ of G, and let X denote an H-H-biset, which is free as a left H-set. Then $H/K \times X$ is again an H-H-biset by the formula $h_1(hK, x)h_2 = (h_1hK, h_1xh_2)$, for all $h, h_1, h_2 \in H$ and all $x \in X$. Note that $R[H/K \times X] = L \otimes_{\mathbf{Z}} RX$ as RH-RH bimodules, so this construction applied to $X = {}_H H_H$, sending $\mathbf{Z}[H/K] \mapsto H/K \times H$, gives a well-defined homomorphism

$$P(H, \mathbf{Z}) \to \operatorname{Hom}_{RG\text{-Morita}}(H, H)$$

from the Grothendieck group of permutation modules, for each subgroup $H \subset G$. The adjoints of these homomorphisms give a pairing $\mathscr{A}_{SW} \times \mathscr{M} \to \mathscr{M}$, and the Green module properties follow easily from bimodule identities (compare [Oliver 1988, 11.2]). Since $\mathscr{A}_{SW}$ is hyperelementary computable, we conclude that any Mackey functor out of R-Morita is hyperelementary computable. $\square$

Remark 5.6. As mentioned in the Introduction, this is a refinement of an earlier result of Oliver [1988, 11.2]. Oliver establishes hyperelementary computability for functors of the form $X(R[G])$, where X is an additive functor from the category of R-orders in semisimple K-algebras with bimodule morphisms to the category of abelian groups. Here R is a Dedekind domain with quotient field K of characteristic zero.

There are two points of comparison: it should first be noted that Oliver [1988, page 246] is dealing with Mackey functors defined on the category of finite groups and monomorphisms, so the statement that any such functor $X(R[G])$ is a Mackey functor is straight-forward. In our case, relating RG-Morita to Mackey functors defined on finite G-sets in the sense of Dress [1975, page 301] involves some work (for example, in constructing the bifunctor d). The translation between the two versions of Mackey functors is also well-known to the experts [Dress 1975, Section 1], but in this paper we preferred to work only with the Dress G-set theory.

The second point of comparison is that Oliver's proof uses an action of the Swan ring on the Mackey functors $X(R[G])$, but the Swan ring does not act on our functors in any obvious way. The key new ingredient in our proof is the Burnside quotient Green ring of the Swan ring. Apart from this additional input, the argument is essentially the same. However, the extra generality can be useful since there are functors out of RG-Morita which do not appear to extend to the setting of Oliver [1988, 11.2]. □

Example 5.7 (Controlled topology). The bounded categories $\mathscr{C}_{M,G}(R)$ of [Hambleton and Pedersen 1991, §4], and the continuously controlled categories $\mathscr{B}_G(X \times [0, 1); R)$ of [Hambleton and Pedersen 2004, §6] are functors out of $\mathbf{A}_\bullet(G)$, for any finite group G, and hence any additive functor from these categories to abelian groups gives a Mackey functor on $\mathscr{D}(G)$.

Example 5.8 (Farrell–Hsiang induction). There is a useful extension of induction theory to (possibly) infinite groups, due to Farrell and Hsiang [1977]. Given any representation $\mathrm{pr} : \Gamma \to G$, with G finite, we get a new R-group ring functor $\mathscr{A}(G) \to R$-Morita by sending $G/H \mapsto R[\Gamma_H]$, where $\Gamma_H = \mathrm{pr}^{-1}(H)$ is the preimage of H in Γ. We have a generating set for the morphisms $\mathrm{Hom}_{\mathbf{A}(G)}(H_1, H_2)$ consisting of the bisets $H_2 \times_K H_1$, where $K \subset H_2 \times H_1$ is a subgroup [HTW 1990, 1.A.9]. We send the biset $H_2 \times_K H_1$ to the bimodule $R[\Gamma_{H_2}] \otimes_{R[\Gamma_K]} R[\Gamma_{H_1}]$. By composition with any additive functor $F : R\text{-Morita} \to \mathscr{A}b$, we again obtain Mackey functors. Since the Swan ring acts on R-Morita (by tensor product as above), any such Mackey functor is a Green module over the Swan ring, and we obtain hyperelementary computation as before. The main examples are listed in [HTW 1990, 1.A.12], including Quillen K-theory $K_n(R[\Gamma])$.

Remark 5.9. An alternate (and slightly sharper) formulation of this example could be given by defining $R\Gamma$-Morita for any discrete group Γ: the objects are finite groups H isomorphic to some subquotient $H \cong \Gamma_1/\Gamma_0$ of Γ, where $\Gamma_0 \lhd \Gamma_1$ and Γ_1 is finite index in Γ. The morphisms are H_2-H_1 bisets as before. Then from any representation $\mathrm{pr} : \Gamma \to G$, where G is finite, we get a functor $d : \mathscr{D}(G) \to R\Gamma$-Morita and Theorem 5.5 holds in this new setting.

Example 5.10 (Cohomotopy). Example 2.1 and Remark 2.4 (pages 108–109) of [Lam 1968] show that (ordinary or Tate) cohomology with twisted coefficients $H^i(?; M)$ is a Mackey functor on $\mathscr{D}(G)$ where M is a fixed G-module. Since the cohomotopy Green ring $H \mapsto \pi^0(BH)$ acts on this Mackey functor, it is Sylow computable. If $\mathrm{pr} : \Gamma \to G$ is a homomorphism and M is a Γ-module, then $H^i(\mathrm{pr}^{-1}(?); M)$ is also a Mackey functor on $\mathscr{D}(G)$ with the cohomotopy Green ring acting. An interesting example of this situation is Galois cohomology.

6. Pseudo-Mackey functors and pseudo-complexes

We wish to apply the computation strategy described above to a more general situation, namely to study functors which have induction and restriction but are not known to be Mackey. The main examples of interest are the higher Whitehead groups $\mathrm{Wh}_n(\mathbf{Z}G)$ and the nonoriented surgery obstruction groups $L_n(\mathbf{Z}G, \omega)$.

Definition 6.1. A covariant *prefunctor* $f : \mathcal{D} \to \mathcal{E}$ between two categories is just a function $S \mapsto ob(f)(S)$ on objects, and a function

$$\hom(f) : \mathrm{Hom}_{\mathcal{D}}(S_1, S_2) \to \mathrm{Hom}_{\mathcal{E}}(ob(f)(S_1), ob(f)(S_2))$$

on Hom-sets. A functor is a prefunctor which preserves identities and compositions. Similarly, we define a contravariant prefunctor, and a *prebifunctor* then consists of a pair (f_*, f^*) of prefunctors, where f_* is covariant, f^* is contravariant, and $ob(f_*) = ob(f^*)$. We call these *Mackey prefunctors* if $\mathcal{D} = \mathcal{D}(G)$ and $\mathcal{E} = \mathcal{A}b$.

A *prenatural transformation* $T : f_1 \to f_2$ is a function

$$S \mapsto T(S) \in \mathrm{Hom}_{\mathcal{E}}(ob(f_1)(S), ob(f_2)(S)) \ .$$

A *natural transformation* of (covariant) prefunctors is a prenatural transformation $T : f_1 \to f_2$ such that the diagram

$$\begin{array}{ccc} ob(f_1)(S_1) & \xrightarrow{\hom(f_1)(\phi)} & ob(f_1)(S_2) \\ {\scriptstyle T(S_1)}\downarrow & & \downarrow{\scriptstyle T(S_2)} \\ ob(f_2)(S_1) & \xrightarrow{\hom(f_2)(\phi)} & ob(f_2)(S_2) \end{array}$$

commutes for all pairs of objects $S_1, S_2 \in \mathcal{D}$ and all $\phi \in \mathrm{Hom}_{\mathcal{D}}(S_1, S_2)$. There is a similar definition for (pre)natural transformations of contravariant prefunctors, and a natural transformation of prebifunctors is a single function which is natural transformation for both the covariant and contravariant parts of the bifunctor. A *prepairing* between three Mackey prefunctors $\mathcal{M}$, $\mathcal{N}$ and $\mathcal{L}$ is a collection of functions $\mu(S) : \mathcal{M}(S) \times \mathcal{N}(S) \to \mathcal{L}(S)$. Finally, if $\mathcal{M} \to \mathcal{N}$ is an injective natural transformation of Mackey prefunctors, then we say that $\mathcal{M}$ is a sub-Mackey prefunctor of $\mathcal{N}$.

Note that if $\mathcal{M} : \mathcal{D}(G) \to \mathcal{A}b$ is a Mackey prefunctor, we can apply $\mathcal{M}$ to any of the Amitsur complexes $Am(X, Y)$, and obtain ∂_r and δ^r maps as usual, but we can not be sure that $\partial_r \circ \partial_{r+1} = 0$ or $\delta^{r+1} \circ \delta^r = 0$. We call $\mathcal{M}(Am(X, Y))$ a *pre-Amitsur* complex. This construction gives a prefunctor $\mathcal{D}(G) \times \mathcal{D}(G) \to Chain(\mathcal{A}b)$.

Definition 6.2. A Mackey prefunctor $\mathcal{M}$ is called a *pseudo-Mackey functor* provided that there exists a finite collection of Mackey prefunctors $0 = \mathcal{N}_0 \subset \mathcal{N}_1 \subset \cdots \subset \mathcal{N}_r = \mathcal{M}$ such that the quotient prebifunctors $\mathcal{N}_i/\mathcal{N}_{i-1}$ are actually Mackey functors, for $1 \le i \le r$. The collection $\{\mathcal{N}_i/\mathcal{N}_{i-1} \mid 1 \le i \le r\}$ will be called the *associated graded* Mackey functor to $\mathcal{M}$.

A natural transformation $\mathcal{M} \to \mathcal{N}$ of pseudo-Mackey functors is a natural transformation of Mackey prefunctors which preserves the filtrations. Notice that the Burnside ring $\mathcal{A}$ acts on a Mackey prefunctor via the usual formula (which gives a prepairing). The action of $\mathcal{A}$ on a pseudo-Mackey functor $\mathcal{M}$ preserves the filtration, and the induced action on the subquotients $\mathcal{N}_i/\mathcal{N}_{i-1}$ is the usual action.

We say that a finite G-set X is a Dress generating set for a pseudo-Mackey functor $\mathcal{M}$ provided X is a Dress generating set for each of the Mackey functors $\mathcal{N}_i/\mathcal{N}_{i-1}$ in its associated filtration. This agrees with our previous definitions if $\mathcal{M}$ is a Mackey functor filtered by Mackey subfunctors. Notice that the image of the natural map of Green rings $\mathcal{A} \to \bigoplus_{i=1}^{r} \mathcal{A}_{\mathcal{N}_i/\mathcal{N}_{i-1}}$ is a Green ring with X as a Dress generating set. It follows that there exists an element $a \in \mathcal{A}(hyper_p\text{-}X) \otimes \mathbf{Z}_{(p)}$, for each prime p, whose image in $\mathcal{A}(\bullet) \otimes \mathbf{Z}_{(p)}$ acts as $1_\bullet$ on each subquotient $\mathcal{N}_i/\mathcal{N}_{i-1}(\bullet) \otimes \mathbf{Z}_{(p)}$, $1 \le i \le r$.

Lemma 6.3. *Suppose that $\mathcal{M}_0 \to \mathcal{M}_1$ and $\mathcal{M}_1 \to \mathcal{M}_2$ are natural transformations of Mackey prefunctors, such that $\mathcal{M}_0(Y) \to \mathcal{M}_1(Y) \to \mathcal{M}_2(Y)$ is exact for every finite G-set Y. If $\mathcal{M}_0$ and $\mathcal{M}_2$ are pseudo-Mackey functors, then $\mathcal{M}_1$ is a pseudo-Mackey functor. Moreover, if X is a Dress generating set for $\mathcal{M}_0$ and $\mathcal{M}_2$, then X is a Dress generating set for $\mathcal{M}_1$.*

Proof. The preimage of the associated filtration for $\mathcal{M}_2$ gives a filtration $\mathcal{N}_0 \subset \mathcal{N}_1 \subset \cdots \subset \mathcal{N}_r = \mathcal{M}_1$, with $\mathcal{M}_0 \subset \mathcal{N}_i$ for $0 \le i \le r$. Since a subbifunctor of a Mackey functor is Mackey, we see that the quotient prefunctors $\mathcal{N}_i/\mathcal{N}_{i-1}$ are actually Mackey functors (and they all have Dress generating set X by Theorem 3.8). Now we extend this filtration by adjoining the associated filtration for $\mathcal{M}_0$. Since each of the subquotients in this extended filtration have Dress generating set X, the result follows. □

We also get a computational result for pseudo-Mackey functors. The Amitsur precomplex $(\mathcal{M}_*(Am(X,Y), \partial_*)$ is now a pseudo-complex, meaning that the boundary maps ∂_* are filtration-preserving (and the associated graded is an actual complex). It will be called *pseudo-contractible* if it is equipped with degree +1 filtration-preserving natural transformations

$$s_r : \mathcal{M}(Am_r(X,Y)) \to \mathcal{M}(Am_{r+1}(X,Y))$$

of prefunctors, for $r \ge 0$, which contract the Amitsur complexes for the associated graded Mackey functors to $\mathcal{M}$. The collection $s_* = \{s_r\}$ is called a *pseudo-contraction*. We make a similar definition for the cochain Amitsur complex and the degree -1 cochain pseudo-contractions σ^r.

We can construct pseudo-contractions by using any element $a \in \mathcal{A}(X)$ such that a acts as $1_\bullet \in$ on each subquotient $\mathcal{N}_i/\mathcal{N}_{i-1}(\bullet) \otimes \mathbf{Z}_{(p)}$, $1 \le i \le r$, to build chain homotopies $s_r(a)$ and cochain homotopies $\sigma^r(a)$. These are pseudo-contractions in the above sense.

Proposition 6.4. *Let $\mathcal{M}$ be a pseudo-Mackey functor, and X, Y finite G-sets. If $(\mathcal{M}_*(Am(X, Y), \partial_*)$ is pseudo-contractible with pseudo-contraction s_*, then there are canonical filtration-preserving natural transformations (∂'_*, s'_*) for which s'_* is a chain contraction and $(\mathcal{M}_*(Am(X, Y), \partial'_*)$ is a chain complex. If the pseudo-complex was already a complex, $\partial'_* = \partial_*$, and if in addition s_* was already a contraction, then $s'_* = s_*$.*

Proof. Let (C_i, ∂_i, s_i) be our data, where ∂_i and s_i are natural transformations. We assume that for $i < r$, $\partial_i \circ \partial_{i+1} = 0$, $\partial_{i+1} = \partial_{i+1} \circ s_i \circ \partial_{i+1}$, and $s_{i-1} \circ \partial_i + \partial_{i+1} \circ s_i = 1_{C_i}$. For $r \leq 0$ these identities clearly hold. We proceed to show how these conditions may be achieved for $i = r$ by modifying ∂_{r+1} and s_r (if necessary). Throughout the inductive construction, we do not change the maps induced by (∂_*, s_*) on the Amitsur complex for the associated graded Mackey functor to $\mathcal{M}$. We also note that the process does not change the given $\partial_1 : C_1 \to C_0$, but may change s_0 in the first step.

First, let $\partial'_{r+1} = \partial_{r+1} - s_{r-1} \circ \partial_r \circ \partial_{r+1}$. Then $\partial_r \circ \partial'_{r+1} = 0$ and if $\partial_r \circ \partial_{r+1} = 0$ we have $\partial'_{r+1} = \partial_{r+1}$. Note that both ∂'_{r+1} and ∂_{r+1} preserve the induced filtration from $\mathcal{M}$, and induce the same map on the Amitsur complexes for the associated graded Mackey functor to $\mathcal{M}$.

Next, we modify s_r. Let $\psi_r = s_{r-1} \circ \partial_r + \partial'_{r+1} \circ s_r$. By construction, ψ_r preserves the filtration and induces the identity on the associated graded. Hence, $\psi_r = 1_{C_r} + u$, where u is nilpotent, and ψ_r is invertible. Since $\partial_r \circ \psi_r = \partial_r$, we can set $s'_r = s_r \circ \psi_r^{-1}$ and obtain $s_{r-1} \circ \partial_r + \partial'_{r+1} \circ s'_r = 1_{C_r}$ by precomposing with ψ_r. Notice that if s_r was already part of a chain contraction, then we do not alter it. It follows that $\partial'_{r+1} = \partial'_{r+1} \circ s'_r \circ \partial'_{r+1}$ and the induction step is complete. The naturality of ∂'_r and ψ_r follow inductively from the explicit formulas. The naturality of ψ_r implies the naturality of s'_r for use at the next step of the induction. Since no choices were involved in the construction of (∂'_*, s'_*), the new maps are canonically determined by the original data (∂_*, s_*). $\square$

Remark 6.5. After this process, the new contractible complex gives an expression for $\mathcal{M}(Y)$ as a direct summand of $\mathcal{M}(X \times Y)$, with respect to the original induction map $\partial_1 : \mathcal{M}(X \times Y) \to \mathcal{M}(Y)$, and the new restriction map $s'_0 : \mathcal{M}(Y) \to : \mathcal{M}(X \times Y)$, since $\partial_1 \circ s'_0 = \mathrm{id}$. In this situation, we say that $\mathcal{M}(Y)$ is *computed* from the family $\mathcal{F}(X)$. If $\mathcal{M}$ was actually a Mackey functor, computability is this sense would agree with the notion previously defined. Similar remarks apply to the contravariant version $\mathcal{M}^*(Am(X, Y), \delta^*)$.

We will also need a slight extension of this result. A *filtered precomplex* (C, ∂) is a precomplex of abelian groups equipped with a filtration

$$C = F_0 C \supset F_1 C \supset F_2 C \supset \cdots$$

where each F_iC is a presubcomplex of (C, ∂), meaning that $\partial_r(F_iC_r) \subseteq F_iC_{r-1}$, for all i, r. We say that (C, ∂) is a *pseudo-complex* if the additional relation $\partial_r \circ \partial_{r+1} = 0$ holds, for all r, on each subquotient $F_iC/F_{i+1}C$. We say that a pseudo-complex has a *pseudo-contraction* $s_* = (s_r)$ provided that $s_r(F_iC_r) \subseteq F_iC_{r+1}$, and s_* induces an actual contraction on each subquotient complex $F_iC/F_{i+1}C$.

A pseudo-complex (C, ∂) has a natural completion

$$(C, \partial) \longrightarrow \varprojlim C/F_iC := (\widehat{C}, \widehat{\partial})$$

given by the inverse limit precomplex with respect to the natural projections $C \to C/F_iC$, $i \geq 0$. A pseudo-contraction s_* of (C, ∂) induces a precontraction $\hat{s}_*$ of $(\widehat{C}, \widehat{\partial})$.

Proposition 6.6. *Let (C, ∂) be a pseudo-complex with filtration $\{F_iC \mid i \geq 0\}$. If (C, ∂) admits a filtered pseudo-contraction s_*, then there exists canonical data for which $(\widehat{C}, \partial'_*, s'_*)$ is a contracted chain complex. If the pseudo-complex was already a complex, $\partial'_* = \hat{\partial}_*$, and if in addition s_* was already a contraction, then $s'_* = \hat{s}_*$.*

Proof. The proof follows the same outline as for Proposition 6.4, but we notice that the map $\psi_r = 1_{C_r} + u$ has the additional property that $u^{i+1} = 0$ on the quotient C_r/F_iC_r. This follows by induction from the exact sequences

$$0 \to F_{i+1}C/F_iC \to F_0C/F_{i+1}C \to F_0C/F_iC \to 0$$

of pseudo-contractible complexes. Then ψ_r induces an invertible map on C_r/F_iC_r, for each $i \geq 0$. We define $s'_r = s_r \circ \psi_r^{-1}$ on C_r/F_iC_r as before. By induction, we have constructed contraction data $(C/F_iC, \partial', s')$, for each $i \geq 0$. In addition, this contraction data is compatible with the projections $C/F_{i+1}C \to C/F_i$, and hence induce contraction data $(\widehat{C}, \partial', s')$ for the inverse limit complex. □

Remark 6.7. Once again, this process doesn't change ∂_1, so the new contractible complex gives an expression for $\widehat{C}_0$ as a direct summand of $\widehat{C}_1$, with respect to completion of the original boundary map $\partial_1 : C_1 \to C_0$.

Example 6.8 (Whitehead groups). Define the Whitehead groups, $Wh_n(\mathbf{Z}G)$, as the homotopy groups of the spectrum which is the cofibre of the Loday assembly map

$$BG^+ \wedge K(\mathbf{Z}) \to K(\mathbf{Z}G).$$

The Loday assembly map is a map of bifunctors [Nicas 1987, Main Theorem, page 223], and the Whitehead groups are bifunctors. Furthermore, the Wh_n, $n \leq 3$, are Mackey functors, but it is not obvious from this description that the other higher Whitehead groups are actually Mackey functors. However, from the long exact sequence in homotopy theory we see that they are pseudo-Mackey functors. From Example 3.12, Example 5.10 and Proposition 6.4, we see that the $Wh_n(\mathbf{Z}G)$ are computed by the hyperelementary family. Similarly, the $Wh_n(\mathbf{Z}G) \otimes \mathbf{Z}_{(p)}$ are computed by the p-hyperelementary family. See [Swan 1970; Lam 1968; Nicas 1987] for partial results in this direction).

Example 6.9 (Tate cohomology). The Tate cohomology of Wh_n or Quillen's K_n are bifunctors which are subquotients of Wh_n or K_n, and hence are computed by the hyperelementary family. The localization maps $Wh_n \to Wh_n \otimes \mathbf{Z}_{(2)}$ and $K_n \to K_n \otimes \mathbf{Z}_{(2)}$ induce isomorphisms on Tate cohomology. Hence the Tate cohomology is computed by the 2-hyperelementary family. Given any pseudo-Mackey subfunctor of Wh_n or K_n which is invariant under the involution, we can form the Tate cohomology and this Tate cohomology functor is computed by any family which contains the 2-hyperelementary family. □

7. Surgery obstruction groups

Dress [1975, Theorem 1] claims computability results for "any of the L-functors defined by C. T. C. Wall" (in [Wall 1976]). However, the nonoriented L-groups $L_n(\mathbf{Z}G, \omega)$ are not always Mackey functors, and so the techniques described in [Dress 1975] do not appear to be adequate to prove the result in this generality. The point is that an inner automorphism by an element $g \in G$ with $\omega(g) = -1$ induces multiplication by -1 (which may not be the identity) on $L_n(\mathbf{Z}G, \omega)$ [Taylor 1973]. One of the main applications of our more general techniques is to supply a proof that nonoriented L-theory is hyperelementary computable, in the sense that $L_n(\mathbf{Z}G, \omega)$ is the limit of restrictions or inductions involving hyperelementary subgroups of G.

Fix a finite group G, and the geometric antistructure for which $\theta = \mathrm{id}$ and $b = e \in G$ [HTW 1990, 1.B.3]. Let $\omega : G \to \{\pm 1\}$ be a fixed orientation homomorphism, and for each subgroup $H \subset G$ let $\omega_H = \omega|_H$. We define the following categories:

(1) $\mathbf{A}(G, \omega)$, with objects finite groups H isomorphic to some subgroup of G, and morphisms given by a Grothendieck group construction on finite biset forms (X, ω_X) (see [HTW 1990, page 256] for the definition). We construct $\mathbf{A}_\bullet(G, \omega)$ by taking the additive completion.

(2) $(R, -)$-Morita, with objects and morphisms as defined in [HTW 1990, 1.B.2], and the quotient category $(R, -)$-Witt from [HTW 1990, 1.C.2], for any commutative ring R with unit.

(3) (RG, ω)-Morita, with objects H isomorphic to some subquotient K/N of G, with $N \subset \ker \omega$, and morphisms given by the Grothendieck group construction on finite biset forms (X, ω_X), modulo an equivalence relation, as defined in [HTW 1990, 1.B.3]. We can define the analogous quotient category (RG, ω)-Witt by setting metabolic forms to zero in the morphisms [HTW 1990, page 254].

Notice that by forgetting the orientation map ω we get functors into the categories discussed in Section 5. The construction of Definition 5.4 gives a prebifunctor

$$j : \mathscr{D}(G) \to \mathbf{A}_\bullet(G, \omega)$$

extending the prefunctor $\mathbf{Or}(G) \to \mathbf{A}_\bullet(G, \omega)$ out of the orbit category, defined on objects by $G/H \mapsto H$ and on morphisms by sending the G-map $f : G/H \to G/K$, given by $f(eH) = gK$, to the biset form $({}_K K_{g^{-1}Kg}, \omega_K)$. This definition depends on the choice of coset representative g for the morphism f in $\mathbf{Or}(G)$, since this time, if $x \in K$ and $\omega(x) = -1$, the two morphisms $eH \mapsto gK$ and $eH \mapsto gxK$ are sent to different biset forms.

Lemma 7.1. *In $\mathbf{A}_\bullet(G, \omega)$, the morphism $\left[{}_H H_{x^{-1}Hx}, \omega_H\right] = \omega(x) \cdot \mathrm{id}$ for all $x \in H$. If $F : \mathbf{A}_\bullet(G, \omega) \to \mathscr{A}b$ is an additive functor, then*

$$F \circ j : \mathbf{A}_\bullet(G, \omega) \to \mathscr{A}b$$

is a Mackey prefunctor, which is a Mackey functor if and only if all the inner automorphism morphisms $F({}_H H_{x^{-1}Hx}, \omega_H) = \mathrm{id}$, for all $x \in H$.

Proof. The identity morphism in $\mathbf{A}_\bullet(G, \omega)$ is represented by the biset form

$$({}_H H_H, \omega_H).$$

The map $\psi : {}_H H_H \to {}_H H_{xHx^{-1}}$ of biset forms defined by $\psi(h) = hx^{-1}$, see (5.3), induces an isometry of biset forms $({}_H H_{x^{-1}Hx}, \omega_H) \cong ({}_H H_H, \omega(x) \cdot \omega_H)$ and hence

$$\left[{}_H H_{x^{-1}Hx}, \omega_H\right] = \omega(x) \cdot \mathrm{id}$$

in the Grothendieck group of morphisms of $\mathbf{A}_\bullet(G, \omega)$.

The property (M1) depends on conjugations acting trivially, or in other words, should induce $F(\psi) = \mathrm{id}$ for all $x \in H$ (including those with $\omega(x) = -1$). □

The R-group ring functor of [HTW 1990, 1.B.4] induces a functor from $\mathbf{A}_\bullet(G, \omega)$ to (RG, ω)-Morita or further into $(R, -)$-Witt. The required formulas are in section 1.B of [HTW 1990], including the remark that since our morphisms are formed via a Grothendieck construction, we are entitled to equate metabolics on isomorphic modules. There is a functor $a : \mathbf{A}_\bullet(G, \omega) \to (RG, \omega)$-Morita as before, and we let

$$d : \mathscr{D}(G) \to (RG, \omega)\text{-Morita}$$

be the prebifunctor $d = a \circ j$. There is a homomorphism from the Dress ring

$$GU(H, \mathbf{Z}) \to \mathrm{Hom}_{(R,-)\text{-Morita}}(H, H)$$

given by tensor product (see [Dress 1975] where it is asserted that $GU(G, \mathbf{Z})$ acts on L-theory, or [Hambleton, Ranicki and Taylor 1987, page 143] for explicit formulas). Dress [1975] showed that the hyperelementary family contracts the

Dress ring. We observe that the same formulas give an action of the Burnside quotient Green ring $\mathcal{A}_{GU}$ on (RG, ω)-Morita.

Theorem 7.2. *Let $F : (RG, \omega)$-Morita $\to \mathcal{A}b$ be an additive functor. Then*

$$F \circ d : \mathfrak{D}(G) \to \mathcal{A}b$$

is a Mackey prefunctor, and the 2-adic completion of any such Mackey prefunctor is 2-hyperelementary computable. If $\mathcal{M} = F \circ d$ is a Mackey functor, then $\mathcal{M}$ is hyperelementary computable.

Proof. In the oriented case ($\omega \equiv 1$) the prefunctor $\mathcal{M}$ is actually a Mackey functor, by Lemma 7.1. More generally, whenever $\mathcal{M} = F \circ d$ is a Mackey functor the result follows as in Theorem 5.5, since $\mathcal{M}$ is a Green module over $\mathcal{A}_{GU}$. By [Dress 1975, Theorem 3], and Theorem 3.6, the Burnside quotient Green ring of the Dress ring is hyperelementary computable.

In the nonoriented case, we define a filtration $F_i = 2^i F$, $i \geq 0$, with $F_0 = F$, and note that the subquotients $(F_i/F_{i+1}) \circ d$ are Mackey functors. Now we let (C, ∂) denote the filtered Amitsur pseudo-complex for $F_* \circ d$ with respect to 2-hyperelementary induction, and the result follows from Proposition 6.6. Notice that the passage from a pseudo-contractible pseudo-complex to a contractible complex does not change the first boundary map, so $F \circ d$ is 2-adically detected (generated) by the given restriction (induction) maps to the 2-hyperelementary subgroups. □

Example 7.3 (Nonoriented L-theory). The main example for us is the surgery obstruction group $L_n(\mathbf{Z}G, \omega)$. It is a foundational result of Wall [1974] that the surgery obstruction groups for finite groups are finitely-generated, with 2-primary torsion exponent. Theorem 7.2 computes $L_n(\mathbf{Z}G, \omega) \otimes \widehat{\mathbf{Z}}_2$ as a limit (and as a colimit) over the 2-hyperelementary subgroups $H \subset G$, $H \in \mathcal{H}$. These limits use the standard induction or restriction maps, for example, for induction we have the surjective map

$$\partial_1 : \bigoplus_{H \in \mathcal{H}} L_n(\mathbf{Z}H, \omega) \otimes \widehat{\mathbf{Z}}_2 \to L_n(\mathbf{Z}G, \omega) \otimes \widehat{\mathbf{Z}}_2$$

and our contraction data gives the relation subgroup $\ker \partial_1 = \operatorname{Im} \partial_2'$.

We conclude that $L_n(\mathbf{Z}G, \omega)$ is also effectively 2-hyperelementary computable: the torsion subgroup is isomorphic to that of $L_n(\mathbf{Z}G, \omega) \otimes \widehat{\mathbf{Z}}_2$, and the divisibility of the signatures is computable since the kernel and cokernel of the natural transformation

$$L_n(\mathbf{Z}G, \omega) \to L_n(\mathbb{R}G, \omega)$$

of pseudo-Mackey functors are both 2-primary torsion groups [Wall 1974, 7.3, 7.4]. The groups $L_n(\mathbb{R}G, \omega)$ were computed explicitly in [Wall 1976, 2.2.1] in terms of the irreducible characters of G. The proof of computability given here applies in the oriented case ($\omega \equiv 1$), but in that case the L-group is a Mackey functor and the argument is essentially the same as the one given by Dress. Other important examples were listed in [HTW 1990, 1.B.8].

Example 7.4 (L-theory with decorations). Let R be a commutative ring with unit, and consider any L-group $L_n^B(RG, \omega)$ for RG with antiinvolution given by $\omega : G \to \{\pm 1\}$ with decoration in any involution invariant subbifunctor, B, of $K_i(\mathbf{Z}G)$ or $Wh_i(\mathbf{Z}G)$, $i \leq 1$; see [Hambleton, Ranicki and Taylor 1987] for a summary of the definitions. It was checked in Theorem 5.3, Corollary 5.5 and Example 5.14 of that same reference that the corresponding round L-theories are functors out of $(\mathbf{Z}G, \omega)$-Morita. Hence these L-theories are pseudo-Mackey functors and are contracted by the hyperelementary family. It was also checked in Proposition 5.6, Corollary 5.7 and Example 5.14 of the same paper that the corresponding ordinary L-theories are functors out of $(\mathbf{Z}G, \omega)$-Morita, so the same computation result holds.

Example 7.5 (Localization). Dress [1975] shows that the Dress ring GU is contracted by any family containing the 2-hyperelementary and p-elementary families. More precisely, he showed that the 2-localization of the Dress ring is contracted by the 2-hyperelementary family, and the p-localization, p odd, is contracted by the p-elementary family.

Proposition 6.4 and a standard mixing argument shows that this smaller family suffices to contract the L^B functors described above. For subbifunctors B closed under the action of the Dress ring, this was proved by Dress [1975] and Wall [1976]. A similar argument shows that the odd-dimensional L^B-groups are contracted by the 2-hyperelementary family alone.

Example 7.6 (Symmetric, hyper-quadratic and lower L-theory). The Ranicki symmetric and hyper-quadratic L-theories [Ranicki 1992] are also functors out of $(\mathbf{Z}G, \omega)$-Morita and hence are contracted by the hyperelementary family. The hyper-quadratic theory is a 2-torsion group with an exponent so it is contracted by the 2-hyperelementary family (as above, we note that the 2-localization map induces an isomorphism on this functor and use the 2-local contraction of functors out of $(\mathbf{Z}, -)$-Morita by the 2-hyperelementary family). The lower L-theories for a ring with antistructure can be defined in terms of the L-theory of the ring with some Laurent variables adjoined [Ranicki 1992] and hence are functors out of $(\mathbf{Z}G, \omega)$-Morita. Therefore L-theories with decorations in sub-Mackey functors of K_i for $i < 0$ are contracted by the hyperelementary family. The higher L-theories of Weiss and Williams [1989] should also be amenable to these techniques.

Example 7.7 (Farrell–Hsiang induction). The technique of Farrell and Hsiang [1977, §§1–2] was originally introduced to apply induction theory to the L-groups of an infinite group Γ. Let $\mathrm{pr} : \Gamma \to G$ be a homomorphism to a finite group

G, and define an orientation character for Γ by the composition $\omega \circ \mathrm{pr}$, where $\omega : G \to \{\pm 1\}$ is an orientation character for G. Then $L_*^B(R\Gamma, \omega)$ is an additive functor (RG, ω)-Witt $\to \mathcal{A}b$, which defines a pseudo-Mackey functor as above. To check this, note that we again have a generating set for the morphisms consisting of the bisets $X = H_2 \times_K H_1$, where $K \subset H_2 \times H_1$ is a subgroup. To produce the needed biform on X, we adapt the formulas in [HTW 1990, 1.B] with $\theta_X = \mathrm{id}$. If $\omega \equiv 1$, it follows that these L-groups can be computed in terms of the L-theory of the various subgroups $\Gamma_H = \mathrm{pr}^{-1}(H)$, $H \subset G$. In particular, it is enough to use the hyperelementary subgroups H of G.

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Vanishing of trace forms in low characteristics

Skip Garibaldi
Appendix by Alexander Premet

Every finite-dimensional representation of an algebraic group G gives a trace symmetric bilinear form on the Lie algebra of G. We give criteria in terms of root system data for the existence of a representation such that this form is nonzero or nondegenerate. As a corollary, we show that a Lie algebra of type E_8 over a field of characteristic 5 does not have a "quotient trace form", answering a question posed in the 1960s.

Let G be an algebraic group over a field F, acting on a finite-dimensional vector space V via a homomorphism $\rho : G \to \mathrm{GL}(V)$. The differential $\mathrm{d}\rho$ of ρ maps the Lie algebra $\mathrm{Lie}(G)$ of G into $\mathfrak{gl}(V)$, and we put Tr_ρ for the symmetric bilinear form

$$\mathrm{Tr}_\rho(x, y) := \mathrm{trace}(\mathrm{d}\rho(x)\,\mathrm{d}\rho(y)) \quad \text{for } x, y \in \mathrm{Lie}(G).$$

We call Tr_ρ a *trace form of G*. Such forms appear, for example, in the hypotheses for the Jacobson–Morozov Theorem [Carter 1985, 5.3.1], in Richardson's proof that there are finitely many conjugacy classes of nilpotent elements in the Lie algebra of a semisimple algebraic group as in [Jantzen 2004, §2] or [Humphreys 1995, §§3.8, 3.9], and in the "explicit" construction of a Springer isomorphism in [Bardsley and Richardson 1985, §9.3]. We prove:

Theorem A. *Let G be a split and almost simple linear algebraic group over a field F.*

(1) *There is a representation ρ of G with Tr_ρ nondegenerate if and only if the characteristic of F is very good for G.*
(2) *There is a representation ρ of G with Tr_ρ nonzero if and only if the characteristic of F is as indicated in Table I.*

A weaker version (up to isogeny) of the "if" direction of part (1) is standard; see for example [Springer and Steinberg 1970, I.5.3] or [Carter 1985, 1.16]. (After this paper was released as a preprint, I learned that Alexander Premet had previously

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G	Every ρ has Tr_ρ degenerate if char$F$	Every ρ has Tr_ρ zero if char$F$
$\mathrm{SL}_n/\boldsymbol{\mu}_m$ with m odd	divides n	divides $\gcd(m, n/m)$
$\mathrm{SL}_n/\boldsymbol{\mu}_m$ with m even	divides n	divides $2\gcd(m, n/m)$
Sp_{2n}	equals 2	[does not occur]
SO_n, Spin_n, PSO_n with $n \geq 7$, HSpin_{4n} for $n \geq 3$, PSp_{2n}	equals 2	equals 2
E_6 adjoint, G_2	equals 2 or 3	equals 2
E_6 simply connected, E_7, F_4	equals 2 or 3	equals 2 or 3
E_8	equals 2, 3, or 5	equals 2, 3, or 5

Table I. Primes where Tr_ρ is degenerate or zero for every ρ. The middle column lists the primes that are not very good for G. For simply connected G, the right column lists the torsion primes for G as defined in, for example, [Steinberg 1975, 1.13].

proved the "only if" direction of part (1) for groups not of type A_n, but his proof has not been published [Premet 1995, p. 80].)

Both directions of part (1) are deduced from part (2). The crux of the proof of part (2) is a formula for the trace form Tr_ρ, given in Proposition 4.1 below.

We remark that the characteristics in part (1) of the theorem depend only on the isogeny class of G, whereas the characteristics in part (2) of the theorem are more sensitive. For example, Sp_{2n} has a representation with nonzero trace form over every field, whereas its quotient PSp_{2n} has such only in characteristic different from 2; this is no surprise because PSp_{2n} has fewer representations than Sp_{2n}. But the opposite phenomenon also occurs: a simply connected group of type E_6 has a representation with nonzero trace form only in characteristics $\neq 2, 3$, whereas its quotient the adjoint E_6 has one over every field of characteristic $\neq 2$. This opposite phenomenon is related to the number $E(G)$ defined in Section 3 below.

For G of type E_8, we can strengthen Theorem A. Given a representation ψ of the Lie algebra $\mathrm{Lie}(G)$ of G, one can define a trace form Tr_ψ on $\mathrm{Lie}(G)$ by setting $(x, y) \mapsto \mathrm{trace}(\psi(x)\,\psi(y))$. We prove

Theorem B. *If F has characteristic* 2, 3, *or* 5 *and G is of type E_8, then the trace form of every representation of* $\mathrm{Lie}(G)$ *is zero.*

This is a strengthening of Theorem A because many representations ψ of $\mathrm{Lie}(G)$ are not differentials of representations of G. The proof of Theorem B is given in 11.3; it amounts to a combination of Theorem A and a result generously provided by Premet, presented in the Appendix. The converse of Theorem B is of course true; in characteristic $\neq 2, 3, 5$, the Killing form is nondegenerate.

In characteristic 5, Theorem B easily gives an apparently stronger statement, namely that $\mathrm{Lie}(G)$ has no quotient trace form, see Corollary 11.4. This answers a question posed in the early 1960s, see for example [Block 1962, p. 554], [Block and Zassenhaus 1964, p. 543], or [Seligman 1967, p. 48].

From the point of view of Lie algebras, this paper addresses the existence of restricted representations with nonzero or nondegenerate trace forms on Lie algebras of almost simple algebraic groups. These algebras are approximately the simple Lie algebras of classical type. For fields of characteristic ≥ 5 and simple Lie algebras of other types (necessarily Cartan or Melikian by Block–Premet–Strade–Wilson, see [Strade 2004] or [Mathieu 2000], every representation has zero trace form by Block [1962, Corollary 3.1].

Notation. All algebraic groups discussed here are linear. Such a group G over a field F is *almost simple* if it is semisimple and has no proper connected, closed, normal subgroups defined over F. In case F is separably closed, the almost simple algebraic groups are the semisimple groups whose Dynkin diagrams are connected.

PSO_n denotes the adjoint group of the (split) special orthogonal group SO_n; when n is odd it is the same as SO_n. Similarly, PSp_{2n} is the adjoint group of type C_n; it can be viewed as $\mathrm{Sp}_{2n}/\boldsymbol{\mu}_2$. The groups SO_n, Spin_n, and PSO_n for $n = 3, 5$, and 6 are isogenous to SL_2, Sp_4, or SL_4 and appear in Table I in that alternative form. For $n \geq 3$, we write HSpin_{4n} for the nontrivial quotient of Spin_{4n} that is neither SO_{4n} nor adjoint.

1. The number $N(G)$ and the Dynkin index

1.1. Fix a simple root system R. We write P for its weight lattice and $\langle\ ,\ \rangle$ for the canonical pairing between P and its dual. Fix a long root $\alpha \in R$ and write $\alpha^\vee$ for the associated coroot. For each subset X of P that is invariant under the Weyl group, we put

$$N(X) := \tfrac{1}{2} \sum_{x \in X} \langle x, \alpha^\vee \rangle^2 \in \mathbb{Z}\big[\tfrac{1}{2}\big].$$

The number $N(X)$ *does not depend on the choice of* α because the long roots are conjugate under the Weyl group.

Furthermore, $N(X)$ *is an integer.* To see this, note that the reflection s in the hyperplane orthogonal to α satisfies $\langle sx, \alpha^\vee \rangle = \langle x, s\alpha^\vee \rangle = -\langle x, \alpha^\vee \rangle$, so in the definition of $N(X)$, the sum can be taken to run over those x satisfying $x \neq sx$. For such x, we have $\langle x, \alpha^\vee \rangle^2 + \langle sx, \alpha^\vee \rangle^2 = 2\langle x, \alpha^\vee \rangle^2$, proving the claim.

Example 1.2. The computations in [Springer and Steinberg 1970, pp. 180, 181] show that $N(R) = 2h^\vee$, where $h^\vee$ denotes the *dual Coxeter number* of R, which is defined as follows. Fix a set of simple roots Δ of R. Write $\tilde{\alpha}$ for the highest root;

the corresponding coroot $\tilde{\alpha}^\vee$ is

$$\tilde{\alpha}^\vee = \sum_{\delta \in \Delta} m_\delta^\vee \delta^\vee$$

for some natural numbers $m_\delta^\vee$. The dual Coxeter number $h^\vee$ is defined by

$$h^\vee := 1 + \sum m_\delta^\vee.$$

In case all the roots of R have the same length, it is the (usual) Coxeter number h and is given in the tables in [Bourbaki 2002].

Suppose that there are two different root lengths in R; we write L for the set of long roots and S for the set of short roots. The arguments in [Springer and Steinberg 1970] are easily adapted to show that

$$N(L) = 2\Bigg(1 + \sum_{\delta \in \Delta \cap L} m_\delta^\vee\Bigg) \quad \text{and} \quad N(S) = 2 \sum_{\delta \in \Delta \cap S} m_\delta^\vee.$$

We obtain the following numbers:

type of R	h	$h^\vee$	$N(L)$	$N(S)$
B_n $(n \geq 2)$	$2n$	$2n-1$	$4(n-1)$	2
C_n $(n \geq 2)$	$2n$	$n+1$	4	$2(n-1)$
G_2	6	4	6	2
F_4	12	9	12	6

Definition 1.3. Fix a split almost simple linear algebraic group G over F. Fix also a pinning of G with respect to some maximal torus T; this includes a root system R and a set of simple roots Δ of G with respect to T. For a representation ρ of G over F, one defines

$$N(\rho) := \sum_{\substack{\text{dominant} \\ \text{weights } \lambda}} \begin{pmatrix} \text{multiplicity of } \lambda \\ \text{as a weight of } \rho \end{pmatrix} \cdot N(W\lambda) \in \mathbb{Z}.$$

For example, the adjoint representation Ad has $N(\mathrm{Ad}) = 2h^\vee$ by Example 1.2. The number $N(\rho)$ is the *Dynkin index* of the representation ρ defined in [Dynkin 1952, p. 130] and studied in [Merkurjev 2003]. The Dynkin index of the fundamental irreducible representations of G (over $\mathbb{C}$) are listed in [Laszlo and Sorger 1997, Proposition 2.6] or [McKay et al. 1990, pp. 36–44], correcting some small errors in Dynkin's calculations.

We put

$$N(G) := \gcd N(\rho), \tag{1.4}$$

where the gcd runs over the representations of G defined over F. Because the map $\rho \mapsto N(\rho)$ depends only on the weights of ρ with multiplicity, it is compatible with short exact sequences

$$0 \longrightarrow \rho' \longrightarrow \rho \longrightarrow \rho/\rho' \longrightarrow 0 \tag{1.5}$$

in the sense that

$$N(\rho) = N(\rho') + N(\rho/\rho').$$

Writing RG for the representation ring of G, we obtain a homomorphism of abelian groups $N\colon RG \to \mathbb{Z}$ with image $N(G)\cdot\mathbb{Z}$.

In the definition of $N(G)$, it suffices to let the gcd run over generators of RG, for example, the irreducible representations of G. For an irreducible representation ρ, the highest weight λ has multiplicity 1 and all the other weights of ρ are lower in the partial ordering. Inducting on the partial ordering, we find

$$N(G) = \gcd\bigl\{N(W\lambda) \mid \lambda \in T^*\bigr\}.$$

In particular, $N(G)$ depends only on the root system R and the lattice T^*, and not on the field F.

Example 1.6. When G is simply connected, the number $N(G)$ is known as the *Dynkin index of G* and its value is listed in [Laszlo and Sorger 1997], for example. Examining the list of values, one finds that the primes dividing $N(G)$ (for G simply connected) are the torsion primes of G.

Example 1.7. Write Spin_n and SO_n for the spin and special orthogonal groups of an n-dimensional nondegenerate quadratic form of maximal Witt index. For $n \ge 7$, these groups are split and almost simple of type B_l (with $l \ge 3$) or D_l (with $l \ge 4$). The Dynkin index $N(\mathrm{Spin}_n)$ is 2; it obviously divides $N(\mathrm{SO}_n)$. On the other hand, the natural n-dimensional representation ρ of SO_n has $N(\rho) = 2$, so $N(\mathrm{SO}_n) = 2$.

Example 1.8. We claim that

$$N(\mathrm{PSp}_{2n}) = \begin{cases} 2 & \text{if } n \text{ is even,} \\ 4 & \text{if } n \text{ is odd,} \end{cases}$$

for $n \ge 2$. The number $N(\mathrm{PSp}_{2n})$ divides 4 and $2(n-1)$ by Example 1.2. Further, $N(\mathrm{PSp}_{2n})$ is even by [Merkurjev 2003, 14.2]. This shows that $N(\mathrm{PSp}_{2n})$ is 2 or 4, and is 2 in case n is even.

Suppose that n is odd. We must show that $N(W\lambda)$ is divisible by 4 for every element λ of the root lattice of PSp_{2n}. We use the same notation as [Merkurjev 2003, §14] for the weights of PSp_{2n}: they are a sum $\sum_{i=1}^{n} x_i e_i$ such that $\sum x_i$ is even. The Weyl group W is a semidirect product of $(\mathbb{Z}/2\mathbb{Z})^n$ (acting by flipping the signs of the e_i) and the symmetric group on n letters (acting by permuting the e_i). Taking X for the $(\mathbb{Z}/2\mathbb{Z})^n$-orbit of $\sum x_i e_i$, we have

$$\frac{1}{2}\sum_{x\in X}\Big\langle \sum\nolimits_i x_i e_i,\, (2e_n)^\vee \Big\rangle^2 = 2^{r-1}x_n^2 \tag{1.9}$$

where r denotes the number of nonzero x_i's; see [Merkurjev 2003, proof of Lemma 14.2]. If $r=1$, then the unique nonzero x_i is even, and we find that for $r\neq 2$, the sum — hence also $N(W\sum x_i e_i)$ — is divisible by 4. Suppose that x_1, x_2 are the only nonzero x_i; then by (1.9) we have

$$N(W(x_1e_1+x_2e_2)) = \begin{cases} 2(n-1)(x_1^2+x_2^2) & \text{if } x_1\neq \pm x_2,\\ 2(n-1)x_1^2 & \text{if } x_1 = \pm x_2.\end{cases}$$

As n is odd, $N(W(x_1e_1+x_2e_2))$ is divisible by 4, which completes the proof of the claim.

Example 1.10. *For G adjoint of type E_7, we have $N(G)=12$.* To see this, we note that $N(G)$ is divisible by $N(\widetilde{G})$, where $\widetilde{G}$ is the universal covering of G, that is, 12 divides $N(G)$. Also, $N(G)$ divides $2h^\vee = 36$ by Example 1.2. For the minuscule representation ρ of $\widetilde{G}$, we have $\dim\rho = 56$ and $N(\rho)=12$. The representation $\rho^{\otimes 2}$ of $\widetilde{G}$ factors through G and by the derivation formula

$$N(\rho_1\otimes\rho_2) = (\dim\rho_1)\cdot N(\rho_2) + (\dim\rho_2)\cdot N(\rho_1)$$

(see for example [Merkurjev 2003, p. 122]) we have

$$N(\rho^{\otimes 2}) = 2(\dim\rho)N(\rho) = 2^6\cdot 3\cdot 7.$$

It follows that $N(G)$ equals 12, as claimed.

2. The Lie algebra of G

2.1. Let G be a split almost simple algebraic group over F; we fix a pinning for it. If $G_{\mathbb{Z}}$ is a split group over $\mathbb{Z}$ with the same root datum as G, the pinning identifies G with the group obtained from $G_{\mathbb{Z}}$ by the base change $\mathbb{Z}\to F$ and the maximal torus T in G (from the pinning) with the base change of a maximal torus $T_{\mathbb{Z}}$ in $G_{\mathbb{Z}}$. We have a root space decomposition of the Lie algebra of $G_{\mathbb{Z}}$

$$\operatorname{Lie}(G_{\mathbb{Z}}) = \operatorname{Lie}(T_{\mathbb{Z}}) \oplus \bigoplus_{\alpha\in R}\mathbb{Z}x_\alpha \tag{2.2}$$

and

$$\operatorname{Lie}(T_{\mathbb{Z}}) = \{h\in \operatorname{Lie}(T_{\mathbb{C}}) \mid \mu(h)\in\mathbb{Z} \text{ for all } \mu\in T^*\}, \tag{2.3}$$

see [Steinberg 1968, p. 64]. Because $\mathrm{Lie}(G_{\mathbb{Z}})$ is a free $\mathbb{Z}$-module, the Lie algebra $\mathrm{Lie}(G)$ of G is naturally identified with $\mathrm{Lie}(G_{\mathbb{Z}}) \otimes_{\mathbb{Z}} F$, and similarly for $\mathrm{Lie}(T)$; see [Demazure and Gabriel 1970, II.4.4.8].

2.4. Write $\widetilde{G}$ for the universal covering of G; we use the obvious analogues of the notations in 2.1 for $\widetilde{G}$. The group G acts on $\widetilde{G}$ by conjugation, hence also on $\mathrm{Lie}(\widetilde{G})$. If the kernel of the map $\widetilde{G} \to G$ is étale, then the representation $\mathrm{Lie}(\widetilde{G})$ is equivalent to the adjoint representation on $\mathrm{Lie}(G)$. In any case, the natural map $\mathrm{Lie}(\widetilde{G}) \to \mathrm{Lie}(G)$ is an isomorphism on the F-span of the x_α.

2.5. We claim that $\mathrm{Lie}(\widetilde{G})$ *is a Weyl module for* G in the sense of [Jantzen 2003, p. 183], that is, its character is given by Weyl's formula and it is generated as a G-module by a highest weight vector. The first condition holds by (2.2), so it suffices to check the second.

To check that the submodule $Gx_{\tilde{\alpha}}$ generated by the highest weight vector $x_{\tilde{\alpha}}$ is all of $\mathrm{Lie}(\widetilde{G})$, one quickly reduces to checking that $Gx_{\tilde{\alpha}}$ contains $\mathrm{Lie}(\widetilde{T}_{\mathbb{Z}})$. Equation (2.3) gives a natural isomorphism $\mathbb{Z}[R^\vee] \xrightarrow{\sim} \mathrm{Lie}(\widetilde{T}_{\mathbb{Z}})$ where $\widetilde{T}_{\mathbb{Z}}$ is the maximal torus in $\widetilde{G}_{\mathbb{Z}}$ mapping onto $T_{\mathbb{Z}}$. We write (as is usual) h_α for the image of $\alpha^\vee$ under this map. As $[x_\alpha, x_{-\alpha}] = h_\alpha$, the claim is proved.

2.6. See [Hiss 1984], [Hogeweij 1982, especially Corollary 2.7a], or [Steinberg 1961, §2] for descriptions of the composition series of $\mathrm{Lie}(\widetilde{G})$. They immediately give: *If the characteristic of F is very good for G, then* $\mathrm{Lie}(G)$ *is a simple Lie algebra. If additionally F is infinite then* $\mathrm{Lie}(G)$ *is an irreducible $G(F)$-module.*

3. The number $E(G)$

Definition 3.1. Maintain the notation of the preceding section. The Killing form on $\mathrm{Lie}(\widetilde{G}_{\mathbb{Z}})$ is divisible by $2h^\vee$ [Gross and Nebe 2004] and dividing by $2h^\vee$ gives an indivisible even symmetric bilinear form $\widetilde{b}$ on $\mathrm{Lie}(\widetilde{G}_{\mathbb{Z}})$ such that

$$\widetilde{b}(h_\alpha, h_\alpha) = 2 \quad \text{and} \quad \widetilde{b}(x_\alpha, x_{-\alpha}) = 1 \tag{3.2}$$

for long roots α; see [Springer and Steinberg 1970, p. 181] or [Bourbaki 2002, Lemma VIII.2.4.3]. For a short root β, we have: $\widetilde{b}(h_\beta, h_\beta) = 2c$ and $\widetilde{b}(x_\beta, x_{-\beta}) = c$, where c is the square-length ratio of α to β. For example, $G = \mathrm{SL}_n$ has Lie algebra the trace zero n-by-n matrices, and the form $\widetilde{b}$ is the usual trace bilinear form $(x, y) \mapsto \mathrm{trace}(xy)$; see [Bourbaki 2002, Exercise VIII.13.12].

The natural map $\mathrm{Lie}(\widetilde{G}_{\mathbb{Z}}) \to \mathrm{Lie}(G_{\mathbb{Z}})$ is an inclusion and extending scalars to $\mathbb{Q}$ gives an isomorphism. Therefore, $\widetilde{b}$ gives a rational-valued symmetric bilinear form on $\mathrm{Lie}(G_{\mathbb{Z}})$. We define $E(G)$ to be the smallest positive rational number such that $E(G) \cdot \widetilde{b}$ is integer-valued on $\mathrm{Lie}(G_{\mathbb{Z}})$; we write b for this form. Note that $E(G)$ is an integer by (3.2).

Clearly, $E(G)$ depends only on the root system of G and the character lattice T^* viewed as a sublattice of the weight lattice, and not on the field F.

3.3. Write $\bar{G}$ for the adjoint group of G; we use the obvious analogues of the notations in 2.1 for $\bar{G}$. We have a commutative diagram

$$\begin{array}{ccc} Q^\vee & \xrightarrow{\sim} & \mathrm{Lie}(\widetilde{T}_{\mathbb{Z}}) \\ \big\downarrow & & \big\downarrow \\ P^\vee & \xrightarrow{\sim} & \mathrm{Lie}(\bar{T}_{\mathbb{Z}}) \end{array}$$

where $Q^\vee$ and $P^\vee$ are the root and weight lattices of the dual root system. The form $\widetilde{b}$ restricts to be an inner product on $Q^\vee$ such that the square-length of a short coroot $\alpha^\vee$ is 2. This inner product extends to a rational-valued inner product on $P^\vee$, and $E(\bar{G})$ is the smallest positive integer such that $E(\bar{G}) \cdot \widetilde{b}$ is integer-valued on $P^\vee$.

Example 3.4. Consider the case where G is PSp_{2n} for some $n \geq 2$, that is, adjoint of type C_n. In the notation of the tables in [Bourbaki 2002], the form $\widetilde{b}$ is twice the usual scalar product, $\widetilde{b}(e_i, e_j) = 2\delta_{ij}$ (Kronecker delta). The fundamental weight ω_n has $\widetilde{b}(\omega_n, \omega_n) = n/2$. Checking $\widetilde{b}(\omega_i, \omega_j)$ for all i, j, shows that $E(\bar{G})$ is 1 if n is even and 2 if n is odd.

Example 3.5. Suppose that all the roots of G have the same length, so that we may identify the root system R with its dual and normalize lengths so that $\langle\, , \rangle$ is symmetric and equals $\widetilde{b}$ on $Q^\vee$.

(1) $E(\bar{G})$ *is the exponent of* P/Q, the weight lattice modulo the root lattice. Indeed, the isomorphism between P and $\mathrm{Lie}(\bar{T}_{\mathbb{Z}})$ shows that $E(\bar{G})$ is the smallest natural number such that $E(\bar{G}) \cdot \langle\, , \rangle$ is integer-valued on $P \times P$, equivalently, the smallest natural number e such that eP is contained in Q; this is the exponent of P/Q.

(2) The bilinear form

$$\widetilde{b} \colon \mathrm{Lie}(\widetilde{G}_{\mathbb{Z}}) \times \mathrm{Lie}(\bar{G}_{\mathbb{Z}}) \to \mathbb{Q}$$

has image $\mathbb{Z}$ and identifies $\mathrm{Lie}(\bar{G}_{\mathbb{Z}})$ with $\mathrm{Hom}_{\mathbb{Z}}(\mathrm{Lie}(\widetilde{G}_{\mathbb{Z}}), \mathbb{Z})$. (On the span of the x_α, this is clear from (3.2). On the Cartan subalgebras, it amounts to the statement that $\langle\, , \rangle$ identifies P with $\mathrm{Hom}(Q, \mathbb{Z})$.) It follows that $\mathrm{Lie}(\bar{G})$, as a G-module, is the dual of $\mathrm{Lie}(\widetilde{G})$, that is, $\mathrm{Lie}(\bar{G})$ is the module denoted by $H^0(\tilde{\alpha})$ in [Jantzen 2003].

Example 3.6. For $n = 3$ or $n \geq 5$, we claim that $E(\mathrm{SO}_n) = 1$.

For n odd, SO_n is adjoint of type B_l for $l = (n-1)/2$, and we compute as in 3.3 and Example 3.4. The dual root system is of type C_l, and the form $\widetilde{b}$ is the usual scalar product, that is, $\widetilde{b}(e_i, e_j) = \delta_{ij}$. The fundamental weight ω_i is $e_1 + e_2 + \cdots + e_i$, so $E(\mathrm{SO}_{2l+1}) = 1$.

For n even, SO_n has type D_l for $l = n/2$. The character group T^* of a maximal torus in SO_n consists of the weights whose restriction to the center of Spin_n is 0 or agrees with the vector representation, that is, the weights $\sum c_i\omega_i$ such that $c_{l-1}+c_l$ is even. It follows that the cocharacter lattice T_* is generated by the (co)root lattice and

$$\omega_1 = \alpha_1 + \alpha_2 + \cdots + \alpha_{l-2} + \tfrac{1}{2}(\alpha_{l-1} + \alpha_l).$$

We have

$$\widetilde{b}(\omega_1, \omega_1) = \langle \omega_1, \omega_1 \rangle = 1,$$

so the form $\widetilde{b}$ is integer-valued on T_* and $E(\mathrm{SO}_{2l}) = 1$.

Example 3.7. Let HSpin_{4n} denote the image of Spin_{4n} under the irreducible representation with highest weight ω_l for $l := 2n$; it is a half-spin group. The character lattice T^* consists of weights $\sum c_i\omega_i$ such that $c_1 + c_{l-1}$ is even. The lattice generated by Q and

$$\omega_1 + \omega_{l-1} = \tfrac{1}{2}(3\alpha_1 + 4\alpha_2 + \cdots + l\alpha_{l-2}) + \tfrac{1}{4}(l+2)\alpha_{l-1} + \tfrac{1}{4}l\alpha_l$$

contains Q with index 2 and is contained in T_*, hence equals T_*. Since

$$\widetilde{b}(\omega_1 + \omega_{l-1}, \omega_1 + \omega_{l-1}) = \langle \omega_1 + \omega_{l-1}, \omega_1 + \omega_{l-1} \rangle = \tfrac{3}{2} + \tfrac{1}{4}(l+2) = \tfrac{1}{2}n + 2,$$

we conclude that

$$E(\mathrm{HSpin}_{4n}) = \begin{cases} 1 & \text{if } n \text{ is even,} \\ 2 & \text{if } n \text{ is odd.} \end{cases}$$

4. Formula for the trace

The integer-valued symmetric bilinear form b on $\mathrm{Lie}(G_{\mathbb{Z}})$ defined in 3.1 gives by scalar extension a symmetric bilinear form on $\mathrm{Lie}(G)$ which we denote by b_F.

Proposition 4.1. *Let ρ be a representation of a split and almost simple algebraic group G over F.*

(1) *$E(G)$ divides $N(\rho)$.*

(2) $\mathrm{Tr}_\rho = \dfrac{N(\rho)}{E(G)} b_F$.

Proof. We first suppose that F is the complex numbers. Write $\pi : \widetilde{G} \to G$ for the universal covering of G as in Section 2. We compute $\mathrm{Tr}_{\rho\pi}$. If we decompose the representation ρ with respect to the action of $\widetilde{T}$ and write V_μ for the eigenspace relative to the weight μ, then h_α acts on V_μ by scalar multiplication by $\langle \mu, \alpha^\vee \rangle$, hence $\mathrm{Tr}_\rho(h_\alpha, h_\alpha) = \sum \dim(V_\mu)\langle \mu, \alpha^\vee\rangle^2$. By putting together the μ in an orbit $W\lambda$ (where λ is dominant) and taking α to be a long root, one gets

$$\mathrm{Tr}_{\rho\pi}(h_\alpha, h_\alpha) = 2\,N(\rho). \tag{4.2}$$

The representation $\mathrm{Lie}(\widetilde{G}_{\mathbb{Z}})\otimes\mathbb{C}$ is irreducible and has a nondegenerate $\widetilde{G}_{\mathbb{C}}$-invariant symmetric bilinear form, so by Schur's Lemma we have

$$\mathrm{Hom}_{\widetilde{G}_{\mathbb{C}}}\big(\mathrm{Lie}(\widetilde{G})\otimes\mathbb{C}, (\mathrm{Lie}(\widetilde{G})^*)\otimes\mathbb{C}\big) = \mathbb{C}.$$

In particular, $\mathrm{Tr}_{\rho\pi}$ equals $z\,\widetilde{b}$ for some complex number z and

$$2N(\rho) = \mathrm{Tr}_{\rho\pi}(h_\alpha, h_\alpha) = z\,\widetilde{b}(h_\alpha, h_\alpha) = 2z.$$

Hence $\mathrm{Tr}_{\rho\pi} = N(\rho)\,\widetilde{b}$. (This argument can be viewed as restating pages 130–131 of [Dynkin 1952].) The Lie algebra $\mathrm{Lie}(G_{\mathbb{Z}})\otimes\mathbb{C}$ is naturally identified with $\mathrm{Lie}(\widetilde{G}_{\mathbb{Z}})\otimes\mathbb{C}$, and (2) follows from the equation $E(G)\,\widetilde{b} = b$ in the case $F = \mathbb{C}$.

Now allow F to be arbitrary but suppose that ρ is a Weyl module. There is a $\mathbb{Z}$-form $\rho_{\mathbb{Z}}$ of ρ, and the form $\mathrm{Tr}_{\rho_{\mathbb{Z}}}$ is the restriction of $\mathrm{Tr}_{\rho_{\mathbb{C}}}$ on $\mathrm{Lie}(G_{\mathbb{C}})$ to $\mathrm{Lie}(G_{\mathbb{Z}})$. Because (2) holds over the complex numbers, it holds over the integers, and by scalar extension it holds over the field F as well. Clearly the form $\mathrm{Tr}_{\rho_{\mathbb{Z}}}$ is integer valued; as b is indivisible, it follows that $E(G)$ divides $N(\rho)$.

We now treat the case of an arbitrary representation ρ. The number $N(\rho)$ depends only on the class of ρ in the representation ring RG. As the Weyl modules generate RG as an abelian group and $E(G)$ divides $N(\psi)$ for every Weyl module ψ, (1) follows.

For (2), we note that the map $\rho \mapsto \mathrm{Tr}_\rho - \big(N(\rho)/E(G)\big)\,b_F$ is compatible with exact sequences like (1.5) in the sense that $\mathrm{Tr}_\rho = \mathrm{Tr}_{\rho'} + \mathrm{Tr}_{\rho/\rho'}$. We obtain a homomorphism of abelian groups

$$RG \to \boxed{\text{symmetric bilinear forms on } \mathrm{Lie}(G)}$$

that vanishes on the Weyl modules, hence is zero. □

4.3. Because b is indivisible (as a form over $\mathbb{Z}$), the form b_F is not zero. Part (2) of Proposition 4.1 immediately gives:

Fact 4.4. *Let ρ be a representation of a split and almost simple algebraic group G over F. Then Tr_ρ is zero if and only if the characteristic of F divides $N(\rho)/E(G)$.*

Furthermore, we defined $N(G)$ to be $\gcd N(\rho)$ as ρ varies over the representations of G. We have proved:

Fact 4.5. *Let G be split and almost simple. The trace Tr_ρ is zero for every representation ρ of G if and only if the characteristic of F divides the integer $N(G)/E(G)$.*

We have now finished half of the proof of Theorem A(2); it remains to check that the primes dividing $N(G)/E(G)$ are the primes in the rightmost column of Table I.

5. The ratio $N(G)/E(G)$ for $G = \mathrm{SL}_n/\boldsymbol{\mu}_m$

In this section, we fix natural numbers m and n with m dividing n, and we prove

Proposition 5.1. *For $G = \mathrm{SL}_n/\boldsymbol{\mu}_m$, the primes dividing $N(G)/E(G)$ are precisely the primes dividing*

$$\begin{cases} \gcd(m, n/m) & \text{if } m \text{ is odd,} \\ 2\gcd(m, n/m) & \text{if } m \text{ is even.} \end{cases}$$

Here $\boldsymbol{\mu}_m$ denotes the group scheme of m-th roots of unity, identified with the corresponding scalar matrices in SL_n.

In the important special cases where G is simply connected ($m = 1$), G is adjoint ($m = n$), or n is square-free, the gcd in the proposition is 1, and we have that $N(G)/E(G)$ *is* 1 *if m is odd and* 2 *if m is even.*

Lemma 5.2.

$$E(\mathrm{SL}_n/\boldsymbol{\mu}_m) = \frac{m}{\gcd(m, n/m)}\,.$$

Proof. Use the notation from [Bourbaki 2002] for the simple roots and fundamental weights of the root system A_{n-1} of SL_n. Let Λ denote the lattice generated by the root lattice Q and

$$\beta := \frac{n}{m}\,\omega_{n-1} = \frac{1}{m}\,(\alpha_1 + 2\alpha_2 + \cdots + (n-1)\alpha_{n-1})\,.$$

We claim that Λ is identified with the cocharacter lattice T_* for a pinning of $\mathrm{SL}_n/\boldsymbol{\mu}_m$. Certainly, Λ/Q is cyclic of order m, so it suffices to check that the set of inner products $\langle \Lambda, T^* \rangle$ consists of integers. But T^* is the collection of weights $\sum c_i\omega_i$ with $c_i \in \mathbb{Z}$ such that $\sum_{i=1}^{n-1} i c_i$ is divisible by m. We have

$$\Big\langle \beta, \sum c_i\omega_i \Big\rangle = \sum_i \frac{1}{m} i c_i\,,$$

which is an integer when $\sum c_i\omega_i$ is in T^*, so $T_* = \Lambda$ as claimed.

Finally, we compute

$$\langle \beta, \alpha_{n-1} \rangle = \frac{n}{m} \in \mathbb{Z} \quad \text{and} \quad \langle \beta, \beta \rangle = \Big\langle \frac{1}{m} \sum i\alpha_i, \frac{n}{m}\omega_{n-1} \Big\rangle = \frac{n(n-1)}{m^2}.$$

Since m divides n, it is relatively prime to $n-1$, so the minimum multiplier of $\langle\, , \rangle$ that takes integer values on T_* is $m/\gcd(m, n/m)$, as claimed. $\square$

5.3. Weights of representations of $\mathrm{SL}_n/\boldsymbol{\mu}_m$**.** Fix the usual pinning of SL_n, where the torus T consists of diagonal matrices and the dominant weights are the maps

$$\begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} \mapsto \prod_{i=1}^{n-1} t_i^{e_i},$$

where $e_1 \geq e_2 \geq \cdots \geq e_{n-1} \geq 0$. Such a weight restricts to $x \mapsto x^{\sum e_i}$ on the center of SL_n; in particular, m divides $\sum e_i$ for every dominant weight λ of a representation of $\mathrm{SL}_n/\boldsymbol{\mu}_m$. The proof of [Merkurjev 2003, Lemma 11.4] shows that m divides $N(W\lambda)$, hence *m divides $N(\mathrm{SL}_n/\boldsymbol{\mu}_m)$*.

5.4. We recall how to compute $N(W\lambda)$ from [Merkurjev 2003, p. 136]. Write $a_1 > a_2 > \cdots > a_{k-1} > a_k = 0$ for the distinct values of the exponents e_i in λ, where a_i appears r_i times, so that $n = \sum r_i$. We have

$$N(W\lambda) = \frac{(n-2)!}{r_1!\, r_2! \cdots r_k!}\left[n\Bigl(\sum_i r_i a_i^2\Bigr) - \Bigl(\sum_i r_i a_i\Bigr)^2 \right]. \tag{5.5}$$

The dominant weight λ with $e_1 = m$ and $e_i = 0$ for $i > 1$ vanishes on $\boldsymbol{\mu}_m$ and has $N(W\lambda) = m^2$ by (5.5), so *$N(\mathrm{SL}_n/\boldsymbol{\mu}_m)$ divides m^2*.

Example 5.6. Let λ be a dominant weight of G and let r_i, a_i be as in the preceding paragraph. Suppose that

$$\upsilon_2\Bigl(\sum r_i a_i\Bigr) \geq \upsilon_2(n) > 0,$$

where $\upsilon_2(x)$ is the 2-adic valuation of x, that is, the exponent of the largest power of 2 dividing x. We claim that

$$\upsilon_2(N(W\lambda)) > \upsilon_2(n). \tag{5.7}$$

Write $\sum r_i a_i = 2^\theta t$ and $n = 2^\nu u$ where $\theta = \upsilon_2(\sum r_i a_i)$ and $\nu = \upsilon_2(n)$. Our hypothesis is that $0 < \nu \leq \theta$. We rewrite (5.5) as

$$N(W\lambda) = \frac{(n-2)!}{r_1!\, r_2! \cdots r_k!}\left[u\Bigl(\sum_i r_i a_i^2\Bigr) - 2^{2\theta-\nu} t^2 \right] \cdot 2^\nu. \tag{5.8}$$

Write l for the minimum of $\upsilon_2(r_i)$, and fix an index j such that $\upsilon_2(r_j) = l$. Note that since $\sum r_i = n$, we have $l \leq \nu \leq 2\theta - \nu$.

The first term on the right side of (5.8) has 2-adic value $\geq -l$ [Merkurjev 2003, p. 137]. The term in brackets has value $\geq l$. Therefore, to prove claim (5.7), it suffices to consider the case where $\upsilon_2\bigl(\sum r_i a_i^2\bigr) = l$ and the first term on the right side of (5.8) has value $-l$; this latter condition implies that

$$s_2(n-1) = s_2(r_1) + \cdots + s_2(r_{j-1}) + s_2(r_j - 1) + s_2(r_{j+1}) + \cdots + s_2(r_k), \tag{5.9}$$

where s_2 denotes the number of 1's appearing in the binary representation of the integer [Merkurjev 2003, p. 137]. That is, when adding up the numbers $r_1, \dots, r_{j-1}$, $r_j - 1, r_{j+1}, \dots, r_k$ in base 2 (to get $n-1$), there are no carries. We check that this is impossible.

Suppose first that $l < \nu$. Equation (5.9) implies that there are exactly two indices, say, j, j' with $\upsilon_2(r_j) = \upsilon_2(r_{j'}) = l$. As 2^{l+1} divides $\sum r_i a_i$, it also divides $r_j a_j + r_{j'} a_{j'}$, hence a_j and $a_{j'}$ have the same parity. It follows that 2^{l+1} divides $r_j a_j^2 + r_{j'} a_{j'}^2$, contradicting the hypothesis that $\upsilon_2\left(\sum r_i a_i^2\right) = l$.

We are left with the case where $l = \nu$. By (5.9), r_j is the unique r_i with 2-adic valuation l. As $\upsilon_2\left(\sum r_i a_i^2\right) = l$, the number a_j is odd and we have $l = \upsilon_2\left(\sum r_i a_i\right) = \theta$. Hence both $u \cdot \left(\sum r_i a_i^2\right)$ and $2^{2\theta - \nu} t$ have 2-adic valuation l. It follows that the term in brackets in (5.8) has 2-adic valuation strictly greater than l, and claim (5.7) is proved.

Proof of Proposition 5.1. We write G for $\mathrm{SL}_n/\boldsymbol{\mu}_m$. Paragraphs 5.3 and 5.4 give the bounds: m divides $N(G)$ divides m^2. Also, $N(G)$ divides $2n$ by Example 1.2. Applying Lemma 5.2 shows that $\gcd(m, n/m)$ divides $N(G)/E(G)$, which in turn divides $\gcd(m, n/m)\gcd(m, 2n/m)$. This completes the proof for m odd.

Clearly, an odd prime divides $N(G)/E(G)$ if and only if it divides $\gcd(m, n/m)$. So suppose that m is even and 2 does not divide $\gcd(m, n/m)$, that is, $\upsilon_2(m) = \upsilon_2(n)$. Then every dominant weight of a representation of G satisfies the hypotheses of Example 5.6, hence $\upsilon_2(N(G)) > \upsilon_2(n) = \upsilon_2(m)$. By Lemma 5.2, $\upsilon_2(E(G)) = \upsilon_2(m)$, so 2 divides $N(G)/E(G)$. This completes the proof. $\square$

6. Conclusion of proof of Theorem A(2)

For a split and almost simple algebraic group G, we now verify that the primes dividing $N(G)/E(G)$ are those in the last column of Table I. Together with 4.5, this will prove Theorem A(2).

For G simply connected, $E(G)$ is 1 and $N(G)$ is divisible precisely by the torsion primes of G, see 1.6. We assume that G is not simply connected and write $\widetilde{G}$ for the universal covering of G; obviously $N(\widetilde{G})$ divides $N(G)$.

For $G = \mathrm{PSp}_{2n}$, SO_n, or adjoint of type E_7, one combines Examples 1.8 and 3.4, 1.7 and 3.6, or 1.10 and 3.5, respectively.

For G adjoint of type D_n, we have $E(G) = 2$ by Example 3.5. Also, 4 divides $N(G)$ by [Merkurjev 2003, 15.2]. On the other hand, the spinor representations of $\widetilde{G}$ have Dynkin index 2^{n-3} [Laszlo and Sorger 1997], and it is easy to use this as in Example 1.10 to construct a representation ρ of G with $N(\rho)$ a power of 2. This shows that $N(G)/E(G)$ is a power of 2 and is not 1.

Now let $G = \mathrm{HSpin}_{4n}$ for some $n \geq 3$. The dual of the center of Spin_{4n} is the Klein four-group, and we write χ for the unique nonzero element that vanishes on the kernel of the map $\mathrm{Spin}_{4n} \to \mathrm{HSpin}_{4n}$. The gcd of $N(W\lambda)$ as λ varies over the weights that restrict to χ (respectively, 0) on the center of Spin_{4n} is 2^{2n-3} (respectively, divisible by 4) by [Merkurjev 2003, p. 146], hence $N(G)$ is a power of 2 and at least 4. On the other hand, $E(\mathrm{HSpin}_{4n})$ is 1 or 2. We conclude that $N(G)/E(G)$ is a power of 2 and is not 1.

For G adjoint of type E_6, the number $N(G)$ is divisible by $N(\widetilde{G}) = 6$ and divides $2h^\vee = 24$ by Example 1.2. By Example 3.5, $N(G)/E(G)$ is 2, 4, or 8. This completes the proof of Theorem A(2). $\square$

Example 6.1. Suppose that the characteristic of F is an odd prime p, and let n be a natural number divisible by p^2. Every trace form of $\mathrm{SL}_n/\boldsymbol{\mu}_p$ is zero by Theorem A(2), even though the universal covering SL_n and adjoint group PGL_n have representations with nonzero trace forms.

7. Proof of Theorem A(1)

We now prove Theorem A(1); we show that the following three statements are equivalent:

(a) *The characteristic of F is very good for G.*

(b) $\mathrm{Lie}(G)$ *is a simple algebra and there is a representation ρ of G with* Tr_ρ *nonzero.*

(c) *There is a representation ρ of G with* Tr_ρ *nondegenerate.*

Suppose (a) holds. Then $\mathrm{Lie}(G)$ is simple, as in 2.6. The existence of a representation ρ with nonzero trace follows from Theorem A(2), so (b) holds. It is easy to check that for a representation ψ of $\mathrm{Lie}(G)$, $\mathrm{Tr}_\psi([x, y], z) = \mathrm{Tr}_\psi(x, [y, z])$ for all $x, y, z \in \mathrm{Lie}(G)$. So the radical of a trace form on $\mathrm{Lie}(G)$ is an ideal, and (b) implies (c).

Now suppose that (a) fails; we check that (c) also fails. By Theorem A(2), we only need to consider those cases where the characteristic of F appears in the middle column of Table I and not in the right column, namely the cases:

(i) G has type G_2 and $\mathrm{char} F = 3$, or G is Sp_{2n} and $\mathrm{char} F = 2$, or G is $\mathrm{SL}_n/\boldsymbol{\mu}_m$ and $\mathrm{char} F$ is odd and divides n/m but not m.

(ii) G is adjoint of type E_6 and $\mathrm{char} F = 3$, or G is $\mathrm{SL}_n/\boldsymbol{\mu}_m$ and $\mathrm{char} F$ divides m but not n/m.

We write $\pi : \widetilde{G} \to G$ for the universal covering of G. In case (i), the kernel of π is étale, so $\mathrm{Lie}(G)$ is a Weyl module by 2.5. For all three of the types listed, $\mathrm{Lie}(G)$ has a nontrivial submodule M, namely the subalgebra generated by the short roots (for G_2) or the center (in the other two cases). It follows that M is contained in the radical of Tr_ρ — see [Garibaldi 2008, 6.2], for example — hence (c) fails.

In case (ii), every representation ρ of G gives a representation $\rho\pi$ of $\widetilde{G}$ whose trace form $\mathrm{Tr}_{\rho\pi}$ vanishes on $\mathrm{Lie}(\widetilde{G})$ by Theorem A(2) (for E_6) or 5.3 (for SL_n). Hence the image of $\mathrm{d}\pi$ is a totally isotropic subspace for Tr_ρ. As

$$\dim(\operatorname{im} \mathrm{d}\pi) = \dim \widetilde{G} - \dim(\ker \mathrm{d}\pi) = \dim G - 1$$

is strictly greater than half the dimension of G, the form Tr_ρ is degenerate and (c) fails. This concludes the proof of Theorem A(1). □

8. Richardson's condition

In the literature, the weak version of the "if" direction of Theorem A(1) is used to deduce Richardson's condition from [Richardson 1967, p. 3]. Our slightly finer version of the "if" direction gives a slightly finer version of Richardson's condition; we state it here for the convenience of the reader. As in Theorem A, G is a split almost simple algebraic group over a field F.

Proposition 8.1. *If the characteristic of F is very good for G, then there is a representation $\rho\colon G \to \mathrm{GL}(V)$ such that $\mathrm{d}\rho$ is an injection $\mathrm{Lie}(G) \hookrightarrow \mathfrak{gl}(V)$ and there is a subspace M of V such that $\mathfrak{gl}(V) = \mathrm{d}\rho(\mathrm{Lie}(G)) \oplus M$, Id_V is in M, and $\mathrm{Ad}(\rho(G))M \subseteq M$.*

Proof. Theorem A(1) gives a representation ρ so that Tr_ρ is nondegenerate. In particular, the restriction of the symmetic bilinear form $(x, y) \mapsto \mathrm{trace}(xy)$ on $\mathfrak{gl}(V)$ to $\mathrm{d}\rho(\mathrm{Lie}(G))$ is nondegenerate. (And obviously $\mathrm{d}\rho$ must be injective.)

Take M to be the space of $x \in \mathfrak{gl}(V)$ such that $\mathrm{trace}(\mathrm{d}\rho(\mathrm{Lie}(G))x) = 0$. Trivially, M is invariant under $\mathrm{Ad}(\rho(G))$. Nondegeneracy of Tr_ρ shows that M meets $\mathrm{d}\rho(\mathrm{Lie}(G))$ only at 0, and dimension count shows that $V = \mathrm{d}\rho(\mathrm{Lie}(G)) \oplus M$. As G is semisimple, the image $\rho(G)$ is contained in $\mathrm{SL}(V)$, hence $\mathrm{d}\rho(\mathrm{Lie}(G))$ lies in $\mathfrak{sl}(V)$, that is, Id_V belongs to M. □

The proposition is essentially known, but the usual argument as in [Richardson 1967, §5], [Jantzen 2004, 2.6], [Humphreys 1995, p. 48], or [Springer and Steinberg 1970, p. 184] is different. For example, the usual approach to treating an adjoint group G of type C_n or D_n replaces G with its covering $G' = \mathrm{Sp}_{2n}$ or SO_{2n} and then gives a representation of G' with the desired properties.

9. Complements: Characteristic 2

In characteristic 2, one might prefer to consider, instead of the symmetric bilinear form Tr_ρ, the quadratic form

$$s_\rho : x \mapsto -\mathrm{trace}\big(\textstyle\bigwedge^2 \mathrm{d}\rho(x)\big),$$

which gives the negative of the degree 2 coefficient of the characteristic polynomial of $\mathrm{d}\rho(x)$. (Because $\mathrm{d}\rho(\mathrm{Lie}(G))$ consists of trace zero matrices, s_ρ is the map $x \mapsto \mathrm{trace}(\mathrm{d}\rho(x)^2)/2$; our definition has the advantage that it obviously makes sense in characteristic 2 as well.) The bilinear form derived from s_ρ, that is, $(x, y) \mapsto s_\rho(x+y) - s_\rho(x) - s_\rho(y)$, is Tr_ρ.

Theorem A(2) is easy to extend. In case G is simply connected, $\mathrm{Lie}(G)$ is a Weyl module by 2.5 and s_ρ is zero if and only if Tr_ρ is zero by [Garibaldi 2008, Proposition 6.4(1)]. That is, the conditions in Theorem A(2) are equivalent to: *For every representation ρ of G, the quadratic form s_ρ is zero.* (This is true in all characteristics but is only nontrivial in characteristic 2.)

Alternatively, one can proceed as follows. The bilinear form $\widetilde{b}$ on $\mathrm{Lie}(\widetilde{G}_{\mathbb{Z}})$ is even [Gross and Nebe 2004, Proposition 4], so it is the bilinear form derived from a unique quadratic form $\widetilde{q}$ on $\mathrm{Lie}(\widetilde{G}_{\mathbb{Z}})$. The form $\widetilde{q}$ extends to a rational-valued quadratic form on $\mathrm{Lie}(G_{\mathbb{Z}})$ and we write $E_q(G)$ for the smallest positive rational number such that $E_q(G)\,\widetilde{q}$ is integer-valued on $\mathrm{Lie}(G_{\mathbb{Z}})$. The number $E_q(G)$ is $E(G)$ or $2E(G)$, and both cases can occur. (For example, take $G = \widetilde{G}$ or SO_{2l}, respectively.) The statements and proofs of 4.4 and 4.5 go through if we replace Tr_ρ, $E(G)$, and b with s_ρ, $E_q(G)$, and $E_q(G)\,\widetilde{q}$ respectively.

10. Complements: Nonsplit groups

We can extend our results above to the case where G is not split, that is, we can replace the hypotheses "G is split and almost simple" with "G is absolutely almost simple". Indeed, suppose that G is absolutely almost simple over F, that is, there is a split and almost simple group G' over F and an isomorphism $f : G' \to G$ defined over a separable closure F_{sep} of F. Fix a pinning for G' and write b' for the indivisible bilinear form on $\mathrm{Lie}(G'_{\mathbb{Z}})$ defined in 3.1. Clearly, the automorphism group of G', which is an affine group scheme over $\mathbb{Z}$, leaves b' invariant, so it maps into the orthogonal group of b'. Galois descent (via f) gives a G-invariant symmetric bilinear form b_F on $\mathrm{Lie}(G)$ such that the differential $\mathrm{d}f$ identifies $b'_F \otimes F_{\mathrm{sep}}$ with $b_F \otimes F_{\mathrm{sep}}$.

Given a representation ρ of G over F, we get a representation ρf of G' over F_{sep} and an integer $N(\rho f)$ defined in 1.3; put $N(\rho) := N(\rho f)$. (In the special case where G is split over F, this agrees with our previous definition.) We define $N(G)$ as in (1.4); it is the gcd of $N(\rho)$ as ρ varies over the representations of G defined over F. Obviously, $N(G)$ is divisible by $N(G_{F_{\mathrm{sep}}})$, that is, $N(G')$, and it depends on the field F.

We put $E(G) := E(G')$. It does not depend on the field F.

With these definitions for $N(G)$ and $E(G)$, conclusions (1) and (2) of Proposition 4.1 hold for absolutely almost simple G. Indeed, it suffices to check them over F_{sep}, where they hold by the original version of the proposition. It follows immediately that the conclusions of 4.4 and 4.5 hold for every absolutely almost simple group G.

We now extend Theorem A. Recall that there is a natural action of the absolute Galois group $\mathrm{Gal}(F)$ of F on the Dynkin diagram Δ of G [Tits 1966, 2.3]. As in [Tits 1966, p. 54], we say, for example, that G *has type* ${}^3D_4$ if Δ has type D_4 and the image of the map $\mathrm{Gal}(F) \to \mathrm{Aut}(\Delta)$ has order 3. We say that the characteristic of F is not very good for G if and only if it is not very good for the corresponding split group G'; these primes are listed in the middle column of Table I.

Theorem A′. *Let G be an absolutely almost simple algebraic group over a field F.*

(1) *Every representation ρ of G over F has Tr_ρ degenerate if and only if the characteristic of F*

$$\begin{cases} \text{divides } 2n & \text{if } G \text{ has type } {}^2A_{n-1} \text{ for some odd } n \geq 3; \\ \text{is } 2 \text{ or } 3 & \text{if } G \text{ has type } {}^3D_4 \text{ or } {}^6D_4; \\ \text{is not very good for } G & \text{otherwise.} \end{cases}$$

(2) *Suppose G is not simply connected and not of type A. Every representation of G has Tr_ρ zero if and only if the characteristic of F is as in the table*:

type of G	$\mathrm{char}\, F$
B_n $(n \geq 3)$; C_n $(n \geq 2)$; ${}^1D_n$ *or* ${}^2D_n$ $(n \geq 4)$; *or* E_6	2
${}^3D_4$, ${}^6D_4$, E_7	2 *or* 3

Regarding the omitted cases in part (2), for G simply connected, the number $E(G)$ is 1, so every representation ρ of G has Tr_ρ zero if and only if the characteristic divides $N(G)$ by 4.5; this number (using that G is simply connected) is calculated in [Merkurjev 2003, §§11–16]. We leave the type A case of (2) as an exercise for the reader.

Proof of Theorem A′. To prove (2), by 4.5 it remains to show that the primes in the table are those dividing $N(G)/E(G)$. As $N(G')$ divides $N(G)$ and $E(G')$ equals $E(G)$, we have the trivial equation

$$\frac{N(G)}{E(G)} = \frac{N(G)}{N(G')} \frac{N(G')}{E(G')} \tag{10.1}$$

where all three terms are integers. The primes dividing $N(G')/E(G')$ are listed in Table I, so it suffices to check which primes divide $N(G)/N(G')$ and are not in that table.

For G adjoint of type E_6, the proof that $N(G)/E(G)$ is a power of 2 from the end of Section 6 goes through without change.

The proof of [Merkurjev 2003, 10.11] shows that every prime dividing $N(G)/N(G')$ divides the exponent of P/Q (the weight lattice modulo the root lattice) or the order of the image of $\mathrm{Gal}(F) \to \mathrm{Aut}(\Delta)$. For G of type B_n ($n \geq 3$), C_n ($n \geq 2$), ${}^1D_n$ or ${}^2D_n$ ($n \geq 4$), or E_7, the exponent of P/Q is 2 and the image of $\mathrm{Gal}(F) \to \mathrm{Aut}(\Delta)$ has order at most 2. As 2 divides $N(G')/E(G')$, part (2) is proved for these groups.

For G adjoint of type ${}^3D_4$ or ${}^6D_4$, write $\widetilde{G} \to G$ for the universal covering of G. The number $N(\widetilde{G})$ is 6 or 12 by [Merkurjev 2003, 16.5] and divides $N(G)$. As $E(G)$ is 2 by Example 3.5, $N(G)/E(G)$ is divisible by 3. Part (2) of the theorem is proved.

(We remark that applying the argument from the two previous paragraphs in the case where G has type A_{n-1} shows that every prime dividing $N(G)/N(G')$ divides $2n$. If n is odd and ≥ 3 and G has type ${}^2A_{n-1}$, then 2 divides $N(\widetilde{G})$ by [Merkurjev 2003, 12.6] hence also $N(G)$, yet $E(G)$ is odd by Lemma 5.2, so $N(G)/E(G)$ is even.)

We now prove part (1) by imitating Section 7. We replace (a) with the condition that the characteristic of F is not as in the statement of Theorem A$'$(1); we denote this condition by (a)$'$.

Suppose that (a)$'$ holds. The characteristic is very good for G and $\mathrm{Lie}(G) \otimes F_{\mathrm{sep}}$ is simple as in 2.6, hence $\mathrm{Lie}(G)$ is simple. If G is neither simply connected nor of type A, then there is a representation ρ of G with Tr_ρ nonzero by part (2) and (b) holds. If G is simply connected, then checking [Merkurjev 2003] verifies that $N(G)$ is not divisible by the characteristic and again (b) holds. In the remaining case where G has type A, the characteristic does not divide $N(G')$ by 5.4 nor does it divide $N(G)/N(G')$ by the discussion above; by (10.1), we find that (b) holds.

As in Section 7, (b) trivially implies (c).

Finally, suppose that (a)$'$ fails; we will show that (c) fails. We assume that the characteristic is very good, otherwise (c) fails because it does so over F_{sep}. That is, we are in one of the cases

(i) $\mathrm{char} F = 3$ and G has type ${}^3D_4$ or ${}^6D_4$; or

(ii) $\mathrm{char} F = 2$ and G has type ${}^2A_{n-1}$ for some odd n.

But in these cases the characteristic divides $N(G)/E(G)$ by the proof of part (2) above, and (c) fails. □

Example 10.2. Let F be a field of prime characteristic p with a central division F-algebra A of degree p. Take G to be the group $\mathrm{SL}(A)$ whose F-points are the elements of A with determinant 1. This group is simply connected, so $N(G)/E(G)$ is p by [Merkurjev 2003, 11.5]. That is, Tr_ρ is zero for every representation ρ of G over F. On the other hand, $N(G_{F_{\mathrm{sep}}})$ is 1, so there are representations of G defined over F_{sep} (for example, the natural representation of SL_n) that have a trace form that is not zero.

A similar statement holds for groups of type ${}^3D_4$ or ${}^6D_4$ over fields of characteristic 3.

11. Trace forms and Lie algebras

This section collects some results regarding Tr_ψ, where ψ is a representation of the Lie algebra of an algebraic group G and we do not assume that ψ is the differential of a representation of G.

11.1. Fix a positive integer n and assume that the characteristic of F is a prime dividing n and $\neq 2, 3$. The Lie algebra $\mathfrak{sl}_n$ of trace zero n-by-n matrices has center $\mathfrak{c}$ the scalar matrices and $\mathfrak{sl}_n/\mathfrak{c}$ is simple [Steinberg 1961, 2.6]. We give a new proof of:

Proposition 11.2 [Block 1962, Theorem 6.2]. *Under the hypotheses of 11.1, every representation of* $\mathfrak{sl}_n/\mathfrak{c}$ *has zero trace form.*

Proof. For sake of contradiction, suppose that there is an irreducible representation ψ of $\mathfrak{sl}_n/\mathfrak{c}$ with nonzero trace form. Then ψ is restricted by [Block 1962, Theorem 5.1], using that F has characteristic $\neq 2, 3$. The composition of ψ with $\mathfrak{sl}_n \to \mathfrak{sl}_n/\mathfrak{c}$ is a restricted irreducible representation of $\mathfrak{sl}_n$, which is the differential of a representation ρ of SL_n by [Curtis 1960; Steinberg 1963].

By construction Tr_ρ is not zero and $\mathrm{d}\rho$ vanishes on the scalar matrices. Identifying the center of SL_n with the (nonreduced) group scheme $\boldsymbol{\mu}_n$ identifies the restriction of ρ to $\boldsymbol{\mu}_n$ with a map $x \mapsto x^l$. Our hypothesis on $\mathrm{d}\rho$ says that l is divisible by the characteristic p of F, hence ρ factors through the natural map $\mathrm{SL}_n \to \mathrm{SL}_n/\boldsymbol{\mu}_p$. Paragraph 5.3 says that $N(\rho)$ is divisible by p, hence Tr_ρ vanishes by 4.4, a contradiction.

Since every irreducible representation has zero trace form, the same holds for every representation like at the end of the proof of Proposition 4.1. $\square$

11.3. *Proof of Theorem B.* Let G be a group of type E_8, and suppose that there is a representation ψ of $\mathrm{Lie}(G)$ such that Tr_ψ is not zero. We may assume that ψ is irreducible. Then the Block–Premet theorem on page 563 implies that ψ is restricted, hence is the differential of a representation of G. Theorem A(2) implies that the characteristic of F is $\neq 2, 3, 5$. $\square$

We close by proving that over a field of characteristic 5, the Lie algebra of a group of type E_8 has no quotient trace form. For a Lie algebra L over F and a representation ψ of L, write $\mathrm{rad}\,\psi$ for the radical of the trace bilinear form Tr_ψ; it is an ideal of L.

Corollary 11.4. *For every representation ψ of every Lie algebra L over a field of characteristic* 5, *the quotient $L/\operatorname{rad}\psi$ is* not *isomorphic to the Lie algebra of an algebraic group of type E_8.*

Proof. Suppose the corollary is false. That is, suppose that there is a group G of type E_8 and a Lie algebra L with a representation ψ and a surjection $\pi : L \to \mathrm{Lie}(G)$ with kernel the radical of Tr_ψ.

By [Block 1962, Lemma 2.1] — using that the characteristic is $\neq 2, 3$ — we may assume that the radical of Tr_ψ is contained in the center of L, that is, L is a central extension of $\mathrm{Lie}(G)$. It follows that there is a map $f : \mathrm{Lie}(G) \to L$ such that πf is the identity [Steinberg 1962, Theorem 6.1(c)]. Clearly, the representation ψf of $\mathrm{Lie}(G)$ has nonzero trace form. As in the proof of Proposition 11.2, we deduce that G has a representation ρ such that Tr_ρ is not zero, but this is impossible by Theorem A(2). $\square$

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Appendix: On trace forms of Lie algebras of type E_8

by Alexander Premet

All basic notions and results of modular Lie theory used in this appendix can be found in [Premet 1995] and references therein.

Let G be an algebraic group of type E_8 over an algebraically closed field of characteristic $p > 0$ and $\mathfrak{g} = \mathrm{Lie}(G)$. It is well known that $\mathfrak{g}$ is a simple Lie algebra carrying an $(\mathrm{Ad}\, G)$-equivariant $[p]$-th power map $x \mapsto x^{[p]}$. Since the universal enveloping algebra $U(\mathfrak{g})$ is a finite module over its central subalgebra generated by all $x^p - x^{[p]}$ with $x \in \mathfrak{g}$, all irreducible $\mathfrak{g}$-modules are finite dimensional. Furthermore, for every irreducible $\mathfrak{g}$-module M there is a linear function $\xi = \xi_M$ on $\mathfrak{g}$ such that $x^p - x^{[p]}$ acts on M as the scalar operator $\xi(x)^p \,\mathrm{Id}_M$. The function ξ_M is called the *p-character* of M. Denote by I_ξ the two-sided ideal of $U(\mathfrak{g})$ generated by all elements $x^p - x^{[p]} - \xi(x)^p$, where $x \in \mathfrak{g}$. The factor-algebra of $U(\mathfrak{g})/I_\xi$ is called the *reduced enveloping algebra* associated with ξ and denoted $U_\xi(\mathfrak{g})$. It has dimension $p^{\dim \mathfrak{g}}$. Clearly, M is a $U_\xi(\mathfrak{g})$-module. We say that M is *restricted* if $\xi_M = 0$.

For $p > 3$, the theorem below was first proved in [Block 1962]. The aim of this appendix is to give a proof valid in any positive characteristic.

Theorem. *If $\psi\colon \mathfrak{g} \to \mathfrak{gl}(V)$ is an irreducible representation with $\mathrm{Tr}_\psi \neq 0$, then V is a restricted $\mathfrak{g}$-module.*

Proof. Suppose ψ is not restricted and let χ be the p-character of V. Then χ is a nonzero linear function on $\mathfrak{g}$. We show that Tr_ψ is zero.

Let T be a maximal torus of G and $\mathfrak{t} := \mathrm{Lie}(T)$. As in sections 1 and 2, we write R for the root system of G relative to T and h_α for the image of the coroot $\alpha^\vee$ in $\mathfrak{t} := \mathrm{Lie}(T)$. (In our case, the group G is both adjoint and simply connected.) Since $\mathfrak{g}$ is an irreducible $(\mathrm{Ad}\, G)$-module, every nonzero adjoint G-orbit spans $\mathfrak{g}$. Thus, replacing $\mathfrak{t}$ by its G-conjugate if necessary, we may assume that $\chi(h_\beta) \neq 0$ for some $\beta \in R$.

There are root vectors $e_{\pm\beta} \in \mathfrak{g}^{\pm\beta}$ such that $\mathfrak{s} := Fe_{-\beta} \oplus Fh_\beta \oplus Fe_\beta$ is isomorphic to $\mathfrak{sl}_2$. Replacing $\mathfrak{t}$ by its conjugate $(\mathrm{Ad}\, x_{-\beta}(\lambda))(\mathfrak{t})$ for a suitable $x_{-\beta}(\lambda)$ in the unipotent root subgroup $U_{-\beta}$ of G, we may assume without loss of generality that $\chi|_{\mathfrak{s}} \neq 0$ and $\chi(e_\beta) = 0$. Then every $\mathfrak{s}$-composition factor M of V is a baby Verma module, that is, $M \cong Z_\xi(a)$, where $\xi = \chi|_{\mathfrak{s}}$ and $a \in F$ is a root of the equation $X^p - X = \xi(h_\alpha)^p$. Note that $\dim M = p$, the operator h_β acts semisimply on M, and the h_β-weights of M are $a, a-2, \ldots, a-2(p-1)$.

First suppose $p > 3$. Then $\mathrm{trace}_M(h_\beta^2) = \sum_{i=0}^{p-1} (a-2i)^2 = pa^2 - 2ap(p-1) + \frac{2}{3}(p-1)p(2p-1) = 0$. Since this holds for every $\mathfrak{s}$-composition factor M of V, we obtain $\mathrm{Tr}_\psi(h_\beta, h_\beta) = 0$. As $\mathfrak{g}$ is a simple Lie algebra and Tr_ψ is $\mathfrak{g}$-invariant, Tr_ψ is a multiple of the form b_F from Section 4. Hence Tr_ψ is zero.

Next suppose $p = 3$. Then the h_β-weights of M are a, $a+1$, $a-1$, hence $\mathrm{trace}_M(h_\beta^2) = a^2 + (a+1)^2 + (a-1)^2 = 2$. It follows that $\mathrm{trace}_M(h_\beta^2)$ is independent of M. Since all $\mathfrak{s}$-composition factors of V are three-dimensional, we deduce that $\mathrm{Tr}_\psi(h_\beta, h_\beta) = 2(\dim V)/3$. Note that $\mathfrak{s}$ can be included into a Levi subalgebra of type A_7; call it $\mathfrak{l}$. Since $\mathfrak{s} \subset \mathfrak{l}$, all $\mathfrak{l}$-composition factors of V have the same nonzero p-character. But then the Kac–Weisfeiler conjecture, which thanks to [Premet 1995, Theorem 3.10] holds for $\mathfrak{sl}_8$ in characteristic 3, implies that all such factors have dimension divisible by 9. Then 9 divides $\dim V$, forcing $\mathrm{Tr}_\psi(h_\beta, h_\beta) = 0$. As in the $p > 3$ case, Tr_ψ is zero.

Finally, suppose $p = 2$. Then the $\mathfrak{sl}_2$-algebra $\mathfrak{s} = Fe \oplus Fh \oplus Ff$ is nilpotent and h lies in the center of $\mathfrak{s}$. However, the reduced enveloping algebra $U_\xi(\mathfrak{s})$ is semisimple whenever $\xi(h) \neq 0$. Indeed, $U_\xi(\mathfrak{s})$ then possesses two nonequivalent two-dimensional irreducible modules, M and N, induced from one-dimensional modules over a Borel subalgebra of $\mathfrak{s}$. The central element h of $\mathfrak{s}$ acts on M and N by different scalars. There are exactly two choices here, namely, a and $a+1$, where a is a root of the equation $X^2 - X = \xi(h)^2$. As a consequence, $U_\xi(\mathfrak{s})$ maps onto a direct sum of two copies of $\mathrm{Mat}_2(F)$. Since $\dim U_\xi(\mathfrak{s}) = 2^3 = 8$, this map is an isomorphism. Thus, $U_\xi(\mathfrak{s})$ is semisimple with two isoclasses of simple modules, both of which are two-dimensional.

Suppose now that we have found two commuting $\mathfrak{sl}_2$-subalgebras $\mathfrak{s}_i = Fe_i \oplus Fh_i \oplus Ff_i$ in $\mathfrak{g}$, where $i = 1, 2$, such that

(a) the sum $\mathfrak{s}_1 + \mathfrak{s}_2$ is direct;

(b) $\chi(h_i) \neq 0$ for $i = 1, 2$;

(c) $e_1 \in \mathfrak{g}^\gamma$ and $f_1 \in \mathfrak{g}^{-\gamma}$ for some $\gamma \in R$.

Our preceding remark then would show that V is a semisimple module over the subalgebra $U_\chi(\mathfrak{s}_1 \oplus \mathfrak{s}_2) \cong U_\chi(\mathfrak{s}_1) \otimes U_\chi(\mathfrak{s}_2)$ of $U_\chi(\mathfrak{g})$ (to ease notation, we identify χ with its restriction to $\mathfrak{s}_i$, $i = 1, 2$). Let M and N be two irreducibles for $U_\chi(\mathfrak{s}_1)$ described earlier. Then V decomposes as a tensor product $V = (M \otimes P) \bigoplus (N \otimes Q)$ for some semisimple $U_\chi(\mathfrak{s}_2)$-modules P and Q. Therefore,

$$\mathrm{Tr}_\psi(e_1, f_1) = r \dim P + s \dim Q,$$

where $r = \mathrm{trace}_M(e_1 f_1)$ and $s = \mathrm{trace}_N(e_1 f_1)$. As both P and Q must have even dimension by our preceding remark, this would yield $\mathrm{Tr}_\psi(\mathfrak{g}^\gamma, \mathfrak{g}^{-\gamma}) = 0$. Hence $\mathrm{Tr}_\psi = 0$ by (3.2), using that γ is (trivially) a long root.

So it remains to find two commuting $\mathfrak{sl}_2$-triples as above. We adopt Bourbaki's numbering of simple roots; see [Bourbaki 2002]. Since $\chi \neq 0$ and the adjoint G-orbit of e_{α_7} spans $\mathfrak{g}$ by the simplicity of $\mathfrak{g}$, we may assume that $\chi(e_{\alpha_7}) \neq 0$. If $\chi(h_{\alpha_6}) \neq 0$ and $\chi(h_{\alpha_8}) \neq 0$, then we can take $\mathfrak{s}_1 = Fe_{\alpha_6} \oplus Fh_{\alpha_6} \oplus Fe_{-\alpha_6}$ and $\mathfrak{s}_2 = Fe_{\alpha_8} \oplus Fh_{\alpha_8} \oplus Fe_{-\alpha_8}$. If this is not the case, then we replace $\mathfrak{t}$ by $(\mathrm{Ad}\, x_{\alpha_7}(t_0))(\mathfrak{t})$ for a suitable $x_{\alpha_7}(t_0)$ in the unipotent root subgroup U_{α_7} of G.

There exists $b \in F^\times$ such that for every $t \in F$ we have

$$(\mathrm{Ad}\, x_{\alpha_7}(t))(h_{\alpha_i}) = h_{\alpha_i} + tb[e_{\alpha_7}, h_{\alpha_i}] = h_{\alpha_i} + tbe_{\alpha_7}, \quad i = 6, 8.$$

As $\chi(e_{\alpha_7}) \neq 0$, we can find $t_0 \in F$ such that

$$\chi(h_{\alpha_6} + t_0 be_{\alpha_7}) \neq 0 \quad \text{and} \quad \chi(h_{\alpha_8} + t_0 be_{\alpha_7}) \neq 0.$$

Hence we can take $(\mathrm{Ad}\, x_{\alpha_7}(t_0))(\mathfrak{s}_i)$, $i = 1, 2$, as our $\mathfrak{sl}_2$-triples. This completes the proof. □

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ALGEBRA AND NUMBER THEORY 3:5(2009)

Compatible associative products and trees

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We compute dimensions and characters of the components of the operad of two compatible associative products and give an explicit combinatorial construction of the corresponding free algebras in terms of planar rooted trees.

1. Introduction

Description of results. An algebra with two compatible associative products is a vector space V equipped with two binary operations such that each of them is an associative product and these two products are compatible (that is, any linear combination of these products is again an associative product). Such algebras were recently studied by Odesskii and Sokolov [2006], who classified simple finite-dimensional algebras of this type. In this paper, we study another extreme case: free algebras of this type. Namely, we compute dimensions of graded components of this algebra, and also give an interpretation of operations in terms of combinatorics of planar rooted trees.

Just as for an arbitrary algebraic structure, to get information about free algebras, one first computes the S_n-module structure (with respect to the action by permutations of the generators) on the "multilinear part" (that is, the space of elements in which each of the generators occurs exactly once) of the free algebra with n generators. Then this information is used in a rather straightforward way to compute the dimensions of all graded components.

As our computation shows the free 1-generated algebra with two compatible products has Catalan numbers as dimensions of its graded components. We provide a materialisation of this formula describing two compatible products on planar rooted trees and proving that the algebra of planar rooted trees is a free algebra with two compatible products. (We actually give a more general construction which is valid for any number of generators.) We use this construction to obtain yet another proof of the results on the Grossman–Larson algebra of planar rooted trees.

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Machinery. To compute dimensions and characters for spaces of multilinear elements, we use Koszul duality for operads and the theory of Koszul operads developed by V. Ginzburg and M. Kapranov. It turns out that the Koszul dual to the operad of two compatible products is much simpler than the original operad. For any Koszul operad, information on the dimensions of its components can be used to obtain similar information for the dual operad. Namely, the following assertion is true.

Proposition 1.1 [Ginzburg and Kapranov 1994]. *Let* $f_{\mathscr{O}}(x) := \sum_{n=1}^{\infty} \frac{\dim \mathscr{O}(n)}{n!} x^n$. *If* $\mathscr{O}$ *is a Koszul operad, then*

$$f_{\mathscr{O}}(-f_{\mathscr{O}^!}(-x)) = x.$$

A similar functional equation holds for the generating functions of characters of representations of the symmetric groups in the components of an operad.

Example 1.2. For the associative operad As, we have $\dim \mathrm{As}(n) = n!$, and thus $f_{\mathrm{As}}(x) = x/(1-x)$. This operad is Koszul and self-dual, which agrees with the functional equation

$$\frac{\frac{x}{1+x}}{1-\frac{x}{1+x}} = x.$$

Koszulness of the operad of two compatible products was proved in [Strohmayer 2008], and this result is crucial for us. In our study of free algebras over As^2, we use a simple but very elegant idea [Chapoton 2007; Fresse 1998] which applies in many cases when one wants to prove that some class of algebras consists of free algebras.

Outline of the paper. Throughout the paper, we assume that the reader is familiar with the main notions of operad theory. Still we briefly remind the reader of some of them when they appear in the text.

In Section 2, we recall some standard definitions of operad theory, define the operad of two compatible brackets, and list necessary facts about Koszul duality for operads. In Section 3, we compute the generating functions for the characters of our operads using functional equations on these generating functions and use them to identify the corresponding representations. In Section 4, we construct a monomial basis in the multilinear part of the free algebra with two compatible products. In Section 5, we prove that free algebras with two compatible products are free as associative algebras. In Section 6, we relate compatible associative products to combinatorics of trees. It turns out that the linear span of planar rooted trees has two compatible associative products and that the resulting algebra is a free algebra with two compatible products. We also give another proof of the result of [Grossman and Larson 1989] on the algebra of planar rooted trees. We conclude with some remarks and conjectural generalisations for other compatible structures.

All vector spaces and algebras throughout this paper are defined over an arbitrary field of zero characteristic.

2. Operads: summary

$\mathbb{S}$-modules and operads. An $\mathbb{S}$-module is a collection $\{\mathscr{V}(n), n \geq 1\}$ of vector spaces, where each $\mathscr{V}(k)$ is an S_k-module. Morphisms, direct sums, tensor products and duals of such objects are defined in the most straightforward way.

The module Det, where $\mathrm{Det}(n)$ is the sign representation of S_n, is an important example of an $\mathbb{S}$-module. We need the following version of the dual module: $\mathscr{V}^\vee = \mathscr{V}^* \otimes \mathrm{Det}$; this is the ordinary dual twisted by the sign representation. In some cases, we consider differential graded $\mathbb{S}$-modules; all preceding constructions are defined for them in a similar way. The graded analogue of Det is denoted by $\mathscr{E}$; the space $\mathscr{E}(n)_{1-n}$ is one-dimensional and is the sign representation of the symmetric group, while all other spaces $\mathscr{E}(n)_k$ are zero.

Each $\mathbb{S}$-module $\mathscr{V}$ gives rise to a functor from the category $\mathscr{F}in$ of finite sets (with bijections as morphisms) to the category of vector spaces. Namely, for a set I of cardinality n let

$$\mathscr{V}_I = \Bbbk\,\mathrm{Hom}_{\mathscr{F}in}([n], I) \otimes_{\Bbbk S_n} \mathscr{V}(n).$$

(Here $[n]$ stands for the "standard" set $\{1, 2, \ldots, n\}$.) This space is often denoted by $\mathscr{V}(I)$; we prefer to use a different notation to avoid confusion with free algebras later.

For $\mathbb{S}$-modules $\mathscr{V}$ and $\mathscr{W}$, define the composition $\mathscr{V} \circ \mathscr{W}$ as

$$(\mathscr{V} \circ \mathscr{W})(n) = \bigoplus_{m=1}^{n} \mathscr{V}(m) \otimes_{\Bbbk S_m} \Bigl(\bigoplus_{f\,:\,[n] \twoheadrightarrow [m]} \bigotimes_{l=1}^{m} \mathscr{W}_{f^{-1}(l)} \Bigr),$$

where the sum is taken over all surjections f. This operation equips the category of $\mathbb{S}$-modules with a structure of a monoidal category. An operad is a monoid in this category. See [Markl et al. 2002] for a more detailed definition. To simplify the definitions, we consider in this paper only operads $\mathscr{O}$ with $\mathscr{O}(1) = \Bbbk$.

Let V be a vector space. By definition, the operad End_V of linear mappings is the collection $\{\mathrm{End}_V(n) = \mathrm{Hom}(V^{\otimes n}, V), n \geq 1\}$ of all multilinear mappings of V into itself with the obvious composition maps.

Using the operad of linear mappings, we can define an algebra over an operad $\mathscr{O}$; a structure of such an algebra on a vector space is a morphism of the operad $\mathscr{O}$ into the corresponding operad of linear mappings. Thus an algebra over an operad $\mathscr{O}$ is a vector space W together with a collection $\mathscr{O}(n) \otimes_{\Bbbk S_n} W^{\otimes n} \to W$ of mappings with obvious compatibility conditions. The free algebra generated by a vector space X over an operad $\mathscr{O}$ is

$$\mathscr{O}(X) := \mathscr{O} \circ X = \bigoplus_{k=1}^{\infty} \mathscr{O}(n) \otimes_{\Bbbk S_n} X^{\otimes n}.$$

Operads defined by generators and relations. The free operad $\mathscr{F}_{\mathscr{G}}$ generated by an $\mathbb{S}$-module $\mathscr{G}$ (with $\mathscr{G}(1) = 0$) is defined as follows. A basis in this operad consists of some species of trees. These trees have a distinguished root (of degree one). A tree belonging to $\mathscr{F}_{\mathscr{G}}(n)$ has exactly n leaves, its internal vertices (neither leaves nor the root) labelled by basis elements of $\mathscr{G}$, any vertex with k siblings being labelled by an element of $\mathscr{G}(k)$. The unique tree whose set of internal vertices is empty generates the one-dimensional space $\mathscr{F}_{\mathscr{G}}(1)$. The composition of a tree t with l leaves and trees $t_1, \dots, t_l$ glues the roots of $t_1, \dots, t_l$ to the respective leaves of t. (In every case, two edges glued together become one edge, and the common vertex becomes an interior point of this edge.)

Free operads are used to define operads by generators and relations. Let $\mathscr{G}$ be an $\mathbb{S}$-module, and let $\mathscr{R}$ be an $\mathbb{S}$-submodule in $\mathscr{F}_{\mathscr{G}}$. An (operadic) ideal generated by $\mathscr{R}$ in $\mathscr{F}_{\mathscr{G}}$ is the linear span of all trees such that at least one internal vertex is labelled by an element of $\mathscr{R}$. An operad with generators $\mathscr{G}$ and relations $\mathscr{R}$ is the quotient of the free operad $\mathscr{F}_{\mathscr{G}}$ modulo this ideal.

Definition 2.1. The associative operad As is generated by one binary operation $\star\colon a, b \mapsto a \star b$. The relations in this operad are equivalent to the associativity condition for every algebra over this operad:

$$(a \star b) \star c = a \star (b \star c).$$

The operad of two compatible associative products As^2 is generated by two binary operations (products) $\star_1$ and $\star_2$. The relations in this operad are equivalent to the following identities in each algebra over this operad: the associativity conditions

$$(a \star_1 b) \star_1 c = a \star_1 (b \star_1 c) \quad \text{and} \quad (a \star_2 b) \star_2 c = a \star_2 (b \star_2 c)$$

for products and the four-term relation

$$(a \star_1 b) \star_2 c + (a \star_2 b) \star_1 c = a \star_1 (b \star_2 c) + a \star_2 (b \star_1 c) \tag{2-1}$$

between the products.

Koszul duality for operads. Let an operad $\mathscr{O}$ be defined by a set of binary operations $\mathscr{B}$ with quadratic relations $\mathscr{R}$ (that is, relations involving ternary operations obtained by compositions from the given binary operations). In this case, $\mathscr{O}$ is said to be quadratic. For quadratic operads, there is an analogue of Koszul duality for quadratic algebras. To a quadratic operad $\mathscr{O}$, this duality assigns the operad $\mathscr{O}^!$ with generators

$\mathscr{B}^\vee$ and with the annihilator of $\mathscr{R}$ under the natural pairing as the space of relations. Just as in the case of quadratic algebras, $(\mathscr{O}^!)^! \simeq \mathscr{O}$.

Example 2.2 [Ginzburg and Kapranov 1994]. The operad As is self-dual:

$$\mathrm{As}^! \simeq \mathrm{As}\,.$$

The cobar complex $\mathbf{C}(\mathscr{O})$ of an operad $\mathscr{O}$ is the free operad with generators $\{\mathscr{O}^*(n), n \geq 2\}$ equipped with a differential d with $d^2 = 0$ (see [Markl et al. 2002] for details). Once again we use twisting by the sign, now to get another version of the cobar complex, $\mathbf{D}(\mathscr{O}) = \mathbf{C}(\mathscr{O}) \otimes \mathscr{E}$. The zeroth cohomology of $\mathbf{D}(\mathscr{O})$ is isomorphic to the operad $\mathscr{O}^!$.

Definition 2.3. An operad $\mathscr{O}$ is said to be Koszul if $H^i(\mathbf{D}(\mathscr{O})) = 0$ for $i \neq 0$.

Proposition 2.4 [Strohmayer 2008]. *The operad* As^2 *is Koszul.*

Generating functions and characters. As we mentioned above, to each operad (and more generally, to each $\mathbb{S}$-module) $\mathscr{O}$ one can assign the formal power series (the exponential generating function of the dimensions)

$$f_{\mathscr{O}}(x) = \sum_{n=1}^{\infty} \frac{\dim \mathscr{O}(n)}{n!} x^n,$$

and if $\mathscr{O}$ is a Koszul operad, then $f_{\mathscr{O}}(-f_{\mathscr{O}^!}(-x)) = x$.

This functional equation is an immediate corollary of a functional equation relating more general generating functions that will be defined now.

The character of a representation M of the symmetric group S_n can be identified [Macdonald 1995] with a symmetric polynomial $F_M(x_1, x_2, \dots)$ of degree n in infinitely many variables. To each $\mathbb{S}$-module $\mathscr{V}$ we assign the element

$$F_{\mathscr{V}}(x_1, \dots, x_k, \dots) = \sum_{n\geq 1} F_{\mathscr{V}(n)}(x_1, \dots, x_k, \dots)$$

of the algebra Λ of symmetric functions. This algebra is the completion of the algebra of symmetric polynomials in infinitely many variables with respect to the valuation defined by the degree of a polynomial. It is isomorphic to the algebra of formal power series in Newton power sums $p_1, \dots, p_n, \dots$. The series $F_{\mathscr{V}}$ is a generating series for the characters of symmetric groups. Namely, by multiplying the coefficient of $p_1^{n_1} \dots p_k^{n_k}$ by $1^{n_1} n_1! \dots k^{n_k} n_k!$, we obtain the value of the character of $\mathscr{V}(n)$ on a permutation whose decomposition into disjoint cycles contains n_1 cycles of length $1, \dots, n_k$ cycles of length k. This definition can be generalised to the case of differential graded modules; for such a module $\mathscr{V} = \bigoplus_i \mathscr{V}_i$ we set $F_{\mathscr{V}} = \sum_i (-1)^i F_{\mathscr{V}_i}$ (the Euler characteristic of $\mathscr{V}$).

If $\mathscr{V}$ is equipped with an action of a group G commuting with the action of the symmetric groups, then for each n the group $S_n \times G$ acts on the space $\mathscr{V}(n)$. In this case, to $\mathscr{V}$ we assign an element of the algebra Λ_G of symmetric functions over the character ring of G (or, in other words, a character of G ranging over symmetric functions). We denote this element by $F_{\mathscr{V}}(x_1, \dots, x_n, \dots; g)$, where $g \in G$.

Remark 2.5. Further in this text we use the following properties of specialisations of our generating functions:

(1) $f_{\mathscr{V}}(x) = F_{\mathscr{V}}(p_1, \dots)|_{p_1=x,\ p_2=p_3=\dots=0}$.

(2) $F_{\mathscr{V}}(p_1, \dots) = F_{\mathscr{V}}(p_1, \dots; g)|_{g=e}$, where e is the identity element of the group G.

(3) for any finite-dimensional vector space V (considered as an $\mathbb{S}$-module concentrated in degree 1),

$$f_{\mathscr{V}(V)}(x) = F_{\mathscr{V}}(p_1, \dots)|_{p_1=x \dim V,\ p_2=x^2 \dim V,\ p_3=x^3 \dim V, \dots},$$

where $\mathscr{V}(V) = \mathscr{V} \circ V$ and $f_{\mathscr{V}(V)}(x)$ is the generating function for dimensions of the (graded) vector space $\mathscr{V}(V)$.

Functional equation for characters.

Definition 2.6. Fix $H(x_1, x_2, \dots; g) \in \Lambda_G$. The plethysm corresponding to H (the plethystic substitution of H) is the algebra homomorphism $F \mapsto F \circ H$ of Λ_G into itself that is linear over the character ring of G and is defined on symmetric functions by $p_n \circ H = H(x_1^n, x_2^n, \dots; g^n)$.

In particular,

$$p_n \circ (H(p_1, p_2, \dots, p_k, \dots; g)) = H(p_n, p_{2n}, \dots, p_{kn}, \dots; g^n).$$

Let ε be the involution of Λ_G linear over the character ring of G and taking p_n to $-p_n$.

Theorem 2.7 [Dotsenko and Khoroshkin 2007]. *Suppose that the operad $\mathscr{O}$ is Koszul. Then the following equation holds in Λ_G:*

$$\varepsilon(F_{\mathscr{O}}) \circ \varepsilon(F_{\mathscr{O}^!}) = p_1.$$

3. Calculation of dimensions and characters

Note that the components of As^2 and $(\mathrm{As}^2)^!$ are equipped with an action of SL_2 (arising from the action on the space of generators of the operad As^2), which commutes with the action of the symmetric groups. All information about these operads will follow from the functional equation on the characters and the explicit description of the representation $(\mathrm{As}^2)^!(n)$ of the group $\mathrm{SL}_2 \times S_n$.

The character ring of SL_2 is isomorphic to the ring of Laurent polynomials in one variable q — for example, the character of the n-dimensional irreducible representation $L(n-1)$ is equal to $(q^n - q^{-n})/(q - q^{-1})$. The element of Λ_{SL_2} corresponding to an $\mathbb{S}$-module $\mathscr{V}$ is denoted by $F_{\mathscr{V}}(p_1, \dots, p_n, \dots; q)$. This notation differs a little from the one introduced above, but we hope that this will not lead to confusion. In this case, the plethysm is defined by $p_n \circ q = q^n$.

Theorem 3.1. *For the operad of two compatible associative products*, *we have*

$$f_{\mathrm{As}^2}(x) = \sum_{n\geq 1} c_n x^n, \qquad F_{\mathrm{As}^2}(p_1, \dots) = \sum_{n\geq 1} c_n p_1^n,$$

$$F_{\mathrm{As}^2}(p_1, \dots; q) = \sum_{n\geq 1} p_1^n (q^{n-1} + N_{n,1} q^{n-3} + \dots + N_{n,k} q^{n-1-2k} + \dots),$$

where $c_n = \dfrac{1}{n+1}\dbinom{2n}{n}$ *and* $N_{n,k} = \dfrac{1}{n}\dbinom{n}{k}\dbinom{n}{k+1}$ *are the Catalan and Narayana numbers.*

Proof. Note that the substitution $p_1 = x$ transforms the second formula into the first one, and the substitution $q = 1$ transforms the third equation into the second (Narayana numbers refine Catalan numbers; see [Stanley 1999]). Thus, the third statement implies the other two, so we shall restrict ourselves to proving only the former. From the results from [Strohmayer 2008] (combined with results from [Dotsenko and Khoroshkin 2007] on SL_2-modules), it follows that as an $S_n \times \mathrm{SL}_2$-module,

$$(\mathrm{As}^2)^!(n) = \mathbb{Q}S_n \otimes L(n-1).$$

Thus, the $S_n \times \mathrm{SL}_2$-character of the Koszul dual operad is given by the formula

$$\sum_{n\geq 1} \frac{q^n - q^{-n}}{q - q^{-1}} p_1^n = \frac{p_1}{(1 - qp_1)(1 - q^{-1}p_1)}.$$

The functional equation for characters implies that the character

$$F_{\mathrm{As}^2}(p_1, \dots, p_n, \dots; q)$$

satisfies

$$\frac{F_{\mathrm{As}^2}}{(1 + qF_{\mathrm{As}^2})(1 + q^{-1}F_{\mathrm{As}^2})} = p_1. \tag{3-1}$$

From this equation it is clear that $F_{\mathrm{As}^2}(p_1, \dots; q)$ depends only on p_1 (and q).

On the other hand, it is well known (see, for example, [Stanley 1999], which should be adjusted to our parametrisation of Narayana numbers) that the generating function

$$N(t, x) = 1 + \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} N_{n,k} t^n x^k$$

of Narayana numbers satisfies the equation

$$txN^2(t,x) - txN(t,x) + tN(t,x) - N(t,x) + 1 = 0. \tag{3-2}$$

It is easy to see that the third statement of the theorem is equivalent to the following equation for generating functions:

$$N(p_1 q^{-1}, q^2) = 1 + q^{-1} F_{\mathrm{As}^2}(p_1, q).$$

Let

$$N(p_1 q^{-1}, q^2) = 1 + q^{-1} G(p_1, q).$$

From the functional (3-2) we deduce that G satisfies

$$p_1 q(1+q^{-1}G)^2 - p_1 q(1+q^{-1}G) + \frac{p_1}{q}(1+q^{-1}G) - (1+q^{-1}G) + 1 = 0,$$

which can be rewritten as

$$p_1(1 + (q+q^{-1})G + G^2) = G.$$

The latter equation coincides with (3-1), and determines G uniquely; thus $G = F_{\mathrm{As}^2}$. □

Corollary 3.2. (1) *As S_n-module*, $\mathrm{As}^2(n)$ *is free of rank c_n.*

(2) *As $S_n \times \mathrm{SL}_2$-module*,

$$\mathrm{As}^2(n) \simeq \mathbb{Q}S_n \otimes (L(n-1) + L(n-3)^{N_{n,1}-1} + L(n-5)^{N(n,2)-N(n,1)} + \cdots).$$

Proof. The SL_2-character of the module

$$L(n-1) + L(n-3)^{N_{n,1}-1} + L(n-5)^{N(n,2)-N(n,1)} + \cdots$$

is equal to

$$q^{n-1} + N_{n,1}q^{n-3} + N_{n,2}q^{n-5} + \cdots + N_{n,k}q^{n-1-2k} + \cdots + N_{n,n-1}q^{1-n},$$

so the second statement follows. The first statement is obtained from the second one, if we forget about the SL_2-action. □

Corollary 3.3. *The dimension of the k-th component of* $\mathrm{As}^2(V)$, *the free* As^2-*algebra generated by a vector space V, is equal to $c_k(\dim V)^k$. In particular, the dimension of the k-th graded component of the free* As^2-*algebra with one generator is equal to the k-th Catalan number.*

Proof. This follows immediately from our previous results: we just apply the third formula of Remark 2.5 to $\mathrm{As}^2(V)$. □

Remark 3.4. The relations in As^2 do not change the order of arguments of the operations. This means that this operad is a symmetrisation of a nonsymmetric operad, which justifies our observations above that

- $\mathrm{As}^2(n)$ is free as an S_n-module,
- $\dim \mathrm{As}^2(V)_k = \dfrac{\dim \mathrm{As}^2(k)}{k!}(\dim V)^k$, and
- $F_{\mathrm{As}^2}(p_1, \dots; q)$ depends only on p_1 (and q).

(No recourse to the functional equation is necessary, since these observations are reflections of a general fact on symmetrisations of nonsymmetric operads.)

4. A monomial basis for As^2

In this section we describe a monomial basis for the components of the operad As^2. One can compare the methods and structure of this paragraph to the same in [Dotsenko and Khoroshkin 2007] in the case of the operad of two compatible Lie brackets. In this section, we prefer to think of components of our operad in terms of the multilinear elements in free algebras.

Definition 4.1. Given a finite ordered set

$$A = \{a_1, a_2, \dots, a_n\},$$

with $a_1 < a_2 < \cdots < a_n$, define a family of monomials $\mathfrak{B}(A)$ in the free algebra $\mathrm{As}^2(A)$ recursively. Our recursive definition also assigns to a monomial m its "top level operation" $t(m) \in \{1, 2\}$, which is used to define further monomials.

- For $A = \{a_1\}$, let $\mathfrak{B}(A) = \{a_1\}$, and let $t(a_1) = 1$.
- For $n > 1$, a monomial b belongs to $\mathfrak{B}(A)$ if and only if it satisfies one of the two conditions:
  (1) $b = a_k \star_1 b'$, where $1 \le k \le n$ and $b' \in \mathfrak{B}(A \setminus \{a_k\})$; in this case we put $t(b) = 1$.
  (2) $b = b_1 \star_2 b_2$, where $b_1 \in \mathfrak{B}(A_1)$, $b_2 \in \mathfrak{B}(A_2)$ for some $A_1 \sqcup A_2 = A$, and $t(b_1) = 1$; in this case we put $t(b) = 2$.

Theorem 4.2. *The family of monomials $\mathfrak{B}(A)$ provides a basis for the multilinear part of the free algebra $\mathrm{As}^2(A)$.*

Proof. We shall prove that this family spans the multilinear part of $\mathrm{As}^2(A)$, and that its number of elements is equal to the dimension of this component — that is, the dimension of $\mathrm{As}^2(|A|)$. It will follow that it has to be a basis.

Lemma 4.3. *The family of monomials $\mathfrak{B}(A)$ spans the multilinear part of* $\mathrm{As}^2(A)$.

Proof. Consider a monomial m. It is a product of two monomials, and by induction we can assume that they both belong to families $\mathfrak{B}(A')$ for some sets $A' \subset A$. Using the associativity property for each of the products, we are left with only one case in which m does not belong to $\mathfrak{B}(A)$, namely

$$m = (m_1 \star_2 m_2) \star_1 m_3$$

for some m_1, m_2, m_3. In this case, we use the compatibility relation (2-1):

$$m = m_1 \star_2 (m_2 \star_1 m_3) + m_1 \star_1 (m_2 \star_2 m_3) - (m_1 \star_1 m_2) \star_2 m_3,$$

which shows that we can proceed by induction: in the first two summands, the degree of the first factor has decreased, and the last summand has fewer products of the second type in its first factor. □

Lemma 4.4. *The number of elements in* $\mathfrak{B}(A)$ *is equal to* $(2|A|)!/(|A|+1)!$.

Proof. Let $\beta_n = |\mathfrak{B}([n])|$. Moreover, for $i = 1, 2$ let $\beta_{i,n} = |\mathfrak{B}_i([n])|$, where $\mathfrak{B}_i([n])$ is the set of all monomials $b \in \mathfrak{B}([n])$ with $t(b) = i$. We use exponential generating functions again:

$$\beta(x) = \sum_{l \geq 1} \frac{\beta_l x^l}{l!}, \quad \beta_i(x) = \sum_{l \geq 1} \frac{\beta_{i,l} x^l}{l!},$$

The first condition implies that $\beta_{1,n+1} = (n+1)\beta_n$, which can be rewritten as

$$\beta_1(x) - x = x\beta(x). \tag{4-1}$$

The definition of $\mathfrak{B}([n])$ basically means that on the level of $\mathbb{S}$-modules,

$$\mathfrak{B}_2(n) = (\mathrm{As} \circ \mathfrak{B}_1)(n)$$

for $n \geq 2$, so

$$\beta_2(x) = (f_{\mathrm{As}}(x) - x) \circ \beta_1(x).$$

Let us rewrite this equation using the formulae

$$\beta_1(x) + \beta_2(x) = \beta(x) \quad \text{and} \quad f_{\mathrm{As}}(x) = \frac{x}{1-x}.$$

We get

$$\beta(x) = \frac{\beta_1(x)}{1 - \beta_1(x)}. \tag{4-2}$$

This can be rewritten as $\beta_1(x) = \dfrac{\beta(x)}{1+\beta(x)}$. Now we can substitute it into (4-1), and get the equation

$$\frac{\beta(x)}{1+\beta(x)} = x(1+\beta(x)),$$

which coincides with the functional equation for $f_{\mathrm{As}^2}(x)$ obtained from (3-1) by setting $q = 1$. This concludes the proof of the lemma and of Theorem 4.2. □

5. Free algebras over As^2

In this section, we prove that any free algebra with two compatible products is free as an associative algebra. Let us recall a theorem which is one of the main ingredients in our proof.

A criterion for free algebras. Let $\mathscr{P}$ be an operad. Assume that $\mathscr{P}(1) = \mathbb{Q}$ and let $\mathscr{P}^+$ be the $\mathbb{S}$-module such that $\mathscr{P} = \mathscr{P}(1) \oplus \mathscr{P}^+$. Let $\mathscr{A}$ be a $\mathscr{P}$-algebra in the category of $\mathbb{S}$-modules. The structure of a $\mathscr{P}$-algebra on $\mathscr{A}$ is given by a morphism $\mu\colon \mathscr{P} \circ \mathscr{A} \to \mathscr{A}$.

Let us define a decreasing $\mathscr{P}$-algebra filtration of $\mathscr{A}$: for each $k \geq 0$ we a define a subspace $\mathscr{A}_{\geq k}$ of $\mathscr{A}$. Let $\mathscr{A}_{\geq 0}$ be $\mathscr{A}$, and for $k > 0$ let $\mathscr{A}_{\geq k}$ be the image under μ of $\mathscr{P}^+ \circ \mathscr{A}_{\geq k-1}$.

We will assume that this filtration is separating, which is true, for instance, if $\mathscr{A}$ has a grading concentrated in positive degrees.

Let us define $H_0(\mathscr{A})$ to be the degree 0 component $\mathscr{A}_{\geq 0}/\mathscr{A}_{\geq 1}$ of the associated graded $\mathscr{P}$-algebra $\mathrm{gr}\,\mathscr{A}$.

Let us choose a section of $H_0(\mathscr{A})$ in $\mathscr{A}$. Consider $\mathscr{P}(H_0(\mathscr{A}))$, that is the free $\mathscr{P}$-algebra generated by $H_0(\mathscr{A})$. Then there exists a unique morphism θ of $\mathscr{P}$-algebras from $\mathscr{P}(H_0(\mathscr{A}))$ to $\mathscr{A}$ extending the chosen section.

Theorem 5.1 [Chapoton 2007; Fresse 1998]. *The morphism θ is surjective. Therefore, if dimensions (or graded characters) of $\mathscr{A}$ and $\mathscr{P}(H_0(\mathscr{A}))$ are equal, then θ is an isomorphism.*

Free algebras with two compatible products are free.

Theorem 5.2. *Free algebras with two compatible brackets are free as associative algebras.*

Proof. Let us first prove that there exists an $\mathbb{S}$-module $\mathscr{G}$ such that the $\mathbb{S}$-modules As^2 and $\mathrm{As} \circ \mathscr{G}$ are isomorphic. To do that, we apply the above criterion for free algebras in the case $\mathscr{P} = \mathrm{As}$ and $\mathscr{A} = \mathrm{As}^2$, where the $\mathscr{P}$-algebra structure is given by the second product. This means that we should put $\mathscr{G} := H_0(\mathscr{A})$, and in order to prove our theorem, we only need to prove that graded characters of As^2 and $\mathrm{As} \circ \mathscr{G}$ are equal. This is guaranteed by the next lemma.

Lemma 5.3. (1) *For each component of* As^2, *the part of its monomial basis consisting of elements b with $t(b) = 1$ can be taken as a lift $\theta\colon \mathscr{G} \to \mathrm{As}^2$;*

(2) $f_{\mathrm{As}^2}(x) = f_{\mathrm{As}} \circ f_{\mathscr{G}}(x)$.

Proof. From our proof of the spanning property, it follows that any monomial for which the top level operation is the second product belongs to the subspace spanned by all basis elements b with $t(b) = 2$, so the quotient by the space of all such monomials is identified with the complementary subspace.

Also, the (4-2) is

$$f_{\mathrm{As}^2}(x) = f_{\mathrm{As}} \circ \beta_1(x),$$

which is exactly what our second statement claims. □

Now we are ready to prove our theorem. For a vector space V,

$$\mathrm{As}^2(V) \simeq \mathrm{As}(\mathscr{G}(V)),$$

so the free As^2-algebra with generators V is isomorphic to the free associative algebra with generators $\mathscr{G}(V)$. □

6. Labelled rooted trees and compatible products

A planar rooted tree is an abstract rooted tree with a linear order on the set of children of every vertex. Alternatively, one can imagine a tree embedded into the plane in such a way that all children of any vertex v have their y-coordinates less than the y-coordinate of v (in this case, the linear order appears from reading the outgoing edges from left to right).

Proposition 6.1 [Stanley 1999]. *The number of planar rooted trees with $n+1$ vertices is equal to the Catalan number c_n.*

Thus, if we consider the planar rooted trees with $k+1$ vertices equipped with a labelling of all nonroot vertices by elements of some finite set S, the number of these objects is equal to $c_k(\#S)^k$, which is, by Corollary 3.3, equal to the dimension of the k-th component of the free As^2-algebra generated by S. In the remaining part of this section, we show that this fact is not a mere coincidence. Namely, we define two compatible associative products on the linear span of all planar rooted trees with S-labeled nonroot vertices, and show that this linear span is free as an As^2-algebra.

Denote by $\mathrm{RT}(S)$ the collection of all planar rooted trees whose nonroot vertices are labeled by elements of a finite set S (possibly with repeated labels). We start by defining several operations on the linear span $\mathbb{Q}\mathrm{RT}(S)$.

Definition 6.2. Let $T_1, T_2 \in \mathrm{RT}(S)$. Define the tree $T_1 \cdot T_2$ as the tree obtained by identifying the roots of T_1 and T_2; the linear ordering of the children of this vertex is uniquely defined by the condition that all children coming from T_1 precede all children coming from T_2. This operation is associative, and every $T \in \mathrm{RT}(S)$ whose root has k children can be uniquely decomposed as $T = T[1] \cdot T[2] \cdot \cdots \cdot T[k]$, where for each j the root of the tree $T[j]$ has only one child.

Let us denote by $\mathrm{Vertices}(T)$ the set of all vertices of a tree $T \in \mathrm{RT}(S)$ (including the root), and by $\mathrm{Internal}(T)$ the set of all internal vertices of T.

Definition 6.3. Let $T_1, T_2 \in \mathrm{RT}(S)$. Assume that the root of T_1 has k children (as in Definition 6.2).

(1) To every mapping $f\colon [k] \to \mathrm{Vertices}(T_2)$ we assign a new tree $T_1 \circ^f T_2$ which is obtained as follows. For each $v \in \mathrm{Vertices}(T_2)$ we let

$$f^{-1}(v) = \{i_1 < \cdots < i_s\},$$

form a tree $T_1[i_1] \cdot \cdots \cdot T_1[i_s]$, and identify the root of this tree with the vertex v (keeping the label of v) in a way that all children of this tree are placed left of all the children of v in T_2.

(2) To every mapping $g\colon [k] \to \mathrm{Internal}(T_2)$ we assign a new tree $T_1 \circ_g T_2$ which is obtained as follows. For each $v \in \mathrm{Internal}(T_2)$ we let

$$g^{-1}(v) = \{j_1 < \cdots < j_r\},$$

form a tree $T_1[j_1] \cdot \cdots \cdot T_1[j_r]$, and identify the root of this tree with the vertex v (keeping the label of v) in a way that all children of this tree are placed left of all the children of v in T_2.

We now define two products on $\mathbb{Q}\mathrm{RT}(S)$.

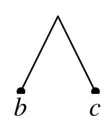
Definition 6.4. Let $T_1, T_2 \in \mathrm{RT}(S)$. Assume that the root of T_1 has k children. Define the products $T_1 \star_1 T_2$ and $T_1 \star_2 T_2$ by

$$T_1 \star_1 T_2 = \sum_{f\colon [k] \to \mathrm{Vertices}(T_2)} T_1 \circ^f T_2, \tag{6-1}$$

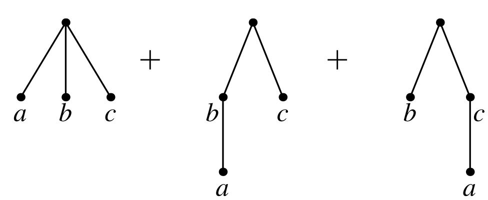
$$T_1 \star_2 T_2 = \sum_{g\colon [k] \to \mathrm{Internal}(T_2)} T_1 \circ_g T_2. \tag{6-2}$$

Example 6.5. For the trees

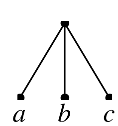
$T_1 =$ 

$T_2 =$ 

the product $T_1 \star_1 T_2$ is equal to



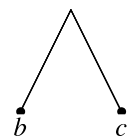
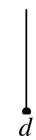
while the product $T_1 \star_2 T_2$ is equal to



Theorem 6.6. (1) *The products $\star_1$ and $\star_2$ are associative and compatible with each other.*

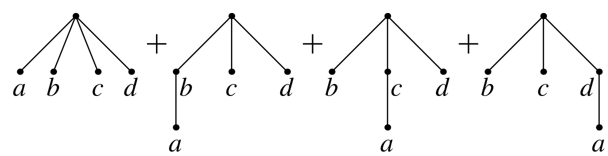
(2) *The* As^2*-algebra* $\mathbb{Q}\mathrm{RT}(S)$ *is isomorphic to the free* As^2*-algebra generated by S.*

Example 6.7. For the trees

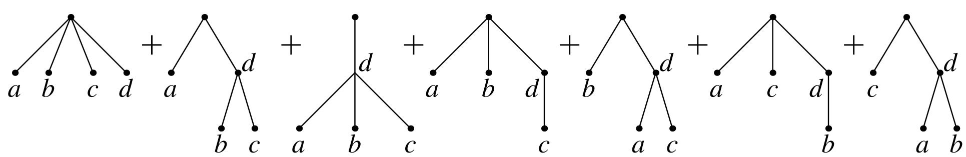
$T_1 =$  $T_2 =$  $T_3 =$ 

the four products that occur in the compatibility relation (2-1) are as follows.

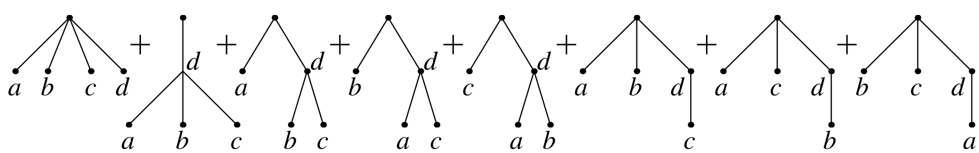
$T_1 \star_1 (T_2 \star_2 T_3)$:



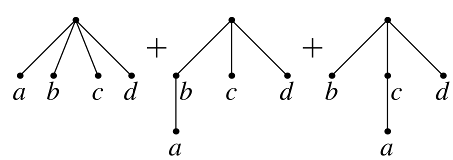
$T_1 \star_2 (T_2 \star_1 T_3)$:



$(T_1 \star_2 T_2) \star_1 T_3$:



$(T_1 \star_1 T_2) \star_2 T_3$:



Thus the compatibility condition is satisfied.

Proof. The associativity conditions for both products are pretty transparent; to show that all the terms in consecutive product $\Pi_1 = T_1 \star_1 (T_2 \star_1 T_3)$ appear in the product $\Pi_2 = (T_1 \star_1 T_2) \star_1 T_3$ one should just notice that to obtain the terms in Π_1 where subtrees of T_1 are attached directly to vertices of T_3 (all other terms appear in Π_2

for tautological reasons) we should just attach the corresponding subtrees to the root of T_2 when computing $T_1 \star_1 T_2$ for Π_2, then we can attach them as required when computing the final product. The same argument works for the second product.

We shall establish the compatibility condition rewritten in the form

$$(T_1 \star_2 T_2) \star_1 T_3 - T_1 \star_2 (T_2 \star_1 T_3) = T_1 \star_1 (T_2 \star_2 T_3) - (T_1 \star_1 T_2) \star_2 T_3.$$

The reason is that for our products both the left hand side and the right hand side are combinations of trees with nonnegative coefficients, and we can interpret the summands that appear there in a rather nice and simple way. Namely, the trees that appear on the left hand side are those for which there exist subtrees of T_1 that are attached to some leaves of T_3. Obviously, the left hand side has the same interpretation. Details are simple and we leave them to the reader.

Now we shall prove that the algebra $\mathbb{Q}\mathrm{RT}(S)$ is free as an As^2-algebra. Note that this algebra admits a natural grading by the number of nonroot vertices of a tree, and the dimensions of graded components are precisely the dimensions of the graded components of the free As^2-algebra generated by S. It remains to show that our algebra is generated as an As^2-algebra by elements of degree 1; it will follow that it is a quotient of the corresponding free algebra, and since it has the same dimensions of graded components, these two algebras should be isomorphic. Thus, it remains to prove the following lemma.

Lemma 6.8. *As an* As^2*-algebra*, $\mathbb{Q}\mathrm{RT}(S)$ *is generated by elements of degree* 1.

Proof. We use induction on degree. Assume that all trees of degree at most k are products of elements of degree 1. We show that the same holds for trees of degree $k+1$. For a tree T, let us call the children of the rightmost child of the root the *principal grandchildren* of the root; denote the number of principal grandchildren by $\mathrm{pg}(T)$. Let us prove the step of induction using the induction on $\mathrm{pg}(T)$.

For $\mathrm{pg}(T) = 0$, the rightmost child of the root is a leaf. Denote by s the label of that leaf, and by T' the tree obtained from T by deleting the rightmost child of the root. Then

$$T' \star_2 T(s) = T,$$

where $T(s)$ denotes the tree with two vertices whose nonroot vertex is labeled by s. The degree of T' is less, so our statement follows.

For $\mathrm{pg}(T) = k$, let us denote by T_1 the tree obtained from T by deleting all principal grandchildren of the root (and the trees they are the roots of), but keeping the rightmost child of the root (and its label). Also, denote by T_2 the complementary tree, that is, the subtree whose root is the rightmost child of the root of T (with its label deleted). Then the tree expansion of $T_2 \star_1 T_1$ consists of T and a combination of other trees $\widetilde{T}$ for which $\mathrm{pg}(\widetilde{T}) < k$, so we can proceed by induction. $\square$

Remark 6.9. Theorem 6.6 can be used to obtain an alternative proof of one of the main results in [Grossman and Larson 1989]. Namely, since the first product $T_1 \star_1 T_2$ is the Grossman–Larson product on $\mathbb{Q}\mathrm{RT}(S)$, it follows from Theorem 5.2 that the Grossman–Larson algebra of planar rooted trees is a free associative algebra; moreover, from our proofs it is easy to see that as a generating set of this algebra we can take all trees whose root has only one child. These are exactly the results of Grossman and Larson.

7. Remarks and open questions

Relation to the Grossman–Larson Hopf algebra structure. Recall that Grossman and Larson [1989] introduced an algebra of planar rooted trees as a Hopf algebra with the coproduct defined as follows.

Definition 7.1 [Grossman and Larson 1989]. Define the coproduct

$$\Delta : \mathbb{Q}\mathrm{RT}(S) \to \mathbb{Q}\mathrm{RT}(S) \otimes \mathbb{Q}\mathrm{RT}(S)$$

by the formula

$$\Delta(T) = \sum_{I \sqcup J = [k]} T[i_1] \cdot \cdots \cdot T[i_p] \otimes T[j_1] \cdot \cdots \cdot T[j_q],$$

where $T = T[1] \cdot \cdots \cdot T[k] \in \mathbb{Q}\mathrm{RT}(S)$, and the notation $I = \{i_1 < \cdots < i_p\}$, $J = \{j_1 < \cdots < j_q\}$ is used.

One could ask what is the relation between this coproduct and the second product that we introduced.

Proposition 7.2. *Consider $\mathbb{Q}\mathrm{RT}(S)$ as an associative algebra with respect to either of the products $\star_1, \star_2$ (and introduce the product on its tensor square accordingly). Then Δ is an algebra homomorphism.*

Proof. For the first product, this statement is proved in [Grossman and Larson 1989]. For the second product, one can use the same proof with some slight modifications (basically, what should be done is simply forgetting all summands where grafting to leaves occurs). □

Remark 7.3. It is worth mentioning that although the tensor product of two As^2-algebras can be turned into an As^2-algebra in many different ways, two products on the tensor square of the free algebra that we just described are not compatible; the family of products

$$(a_1 \otimes b_1) \star_{\lambda,\mu} (a_2 \otimes b_2) = (\lambda a_1 \star_1 a_2 + \mu a_1 \star_2 a_2) \otimes (\lambda b_1 \star_1 b_2 + \mu b_1 \star_2 b_2)$$

is a pencil of associative products, but it is not a linear pencil anymore (they rather resemble pencils of associative products from [Moerdijk 2001]). Thus, the relationship between Hopf algebra structure and the structure of an algebra with two compatible products is yet to be clarified.

Relation to other operads realised by planar trees. The following observation is due to Loday (private communication).

Remark 7.4. Consider the operad $\mathscr{O}_q$ generated by two binary operations $\circ$ and $\bullet$ which satisfy the relations

$$(x \circ y) \circ z = x \circ (y \circ z),$$
$$(x \bullet y) \circ z + q(x \circ y) \bullet z = x \bullet (y \circ z) + q\ x \circ (y \bullet z),$$
$$(x \bullet y) \bullet z = x \bullet (y \bullet z).$$

Then $\mathscr{O}_0$ is the operad Dup of duplicial algebras [Loday 2008], while $\mathscr{O}_1$ is the operad As_2.

Also, consider the operad $\mathscr{P}_t$ generated by two binary operations $\prec$ and $\succ$ which satisfy the relations

$$(x \prec y) \prec z = x \prec (y \prec z) + t\ x \prec (y \succ z),$$
$$(x \succ y) \prec z = x \succ (y \prec z),$$
$$(x \succ y) \succ z + t(x \prec y) \succ z = x \succ (y \succ z).$$

Then $\mathscr{P}_0$ is the operad Dup, while $\mathscr{P}_1$ is the operad Dend of dendriform algebras [Loday 2008].

It is known that free algebras over Dend and Dup can be realised by planar trees. It would be interesting to define in a pure combinatorial way a 2-parameter family of pairs of binary operations $\star_{1,q,t}$ and $\star_{2,q,t}$ on $\mathbb{Q}\mathrm{RT}(S)$ which have correct specialisations to $q = t = 0$ (duplicial case) $q = 1$, $t = 0$ (compatible associative products) and $q = 0$, $t = 1$ (dendriform case).

Other Hopf-algebraic families of trees. Some general phenomenon that we think is worth mentioning here is the existence of compatible associative products for many other well known algebras where the product is described via combinatorics of trees. The main idea is very simple. If the product in the linear span of rooted trees (planar or not) is defined for two trees $T_1 = T_1[1] \cdot T_1[2] \cdot \cdots \cdot T_1[k]$ and T_2 as the sum of all graftings of some type of trees $T_1[i]$ to vertices of the tree T_2, then another product over all graftings of the same type but only to internal vertices is compatible with the first product. For algebras of planar *binary* trees (which also often occur in literature) an analogous recipe holds: if a product is defined in terms of graftings, then graftings only to the "left-going" leaves produce a compatible product.

Example 7.5. The Connes–Kreimer Hopf algebra of renormalisation is a polynomial algebra on (abstract) rooted trees, or, in other words, an algebra on the linear span of forests of rooted trees [Connes and Kreimer 1998]. If we take its dual, and identify the dual of each forest with the rooted tree having all trees of the forest grafted at its root vertex, the coproduct of Connes and Kreimer yields a product on the linear span of rooted trees which is defined in terms of graftings as above. Thus, this algebra is naturally endowed with another product which is compatible with the original one.

Example 7.6. Similarly, consider the noncommutative Connes–Kreimer Hopf algebra NCK of Foissy [2002a; 2002b], which is a free associative algebra on planar rooted trees, or, in other words, an algebra on the linear span of (ordered) forests of planar rooted trees. If we take its dual, and identify the dual of each forest with the planar rooted tree having all trees of the forest grafted at its root vertex, the coproduct of Foissy leads to another product on $\mathbb{Q}\mathrm{RT}(S)$ which is again defined in terms of graftings. It follows that NCK has a natural structure of an algebra with two compatible products. Results of Foissy on isomorphisms of Hopf algebras also produce compatible products on some other algebras on trees, for example, the Brouder–Frabetti [2003] Hopf algebra of renormalisation in QED.

One can easily check that unlike the case of the Grossman–Larson product, the dual of the Foissy algebra is not a free algebra with two compatible products; for example, the $\star_2$-subalgebra of NCK generated by elements of degree 1 (that is, trees with one leaf) is commutative. We expect that this is in some sense the only obstruction to freeness.

Conjecture 7.7. As^2*-subalgebra of NCK generated by elements of degree* 1 *is a free algebra over the operad of two compatible associative products one of which is*, *in addition*, *commutative.*

The operad that shows up here does not seem to have many good properties. In particular, it is not Koszul, and not much is known about the growth of dimensions of its components.

It was pointed out by the referee that our results in Section 6 carry a certain resemblance with those of Patras and Schocker [2008], who studied the Hopf algebra structure on the linear span of set compositions (the twisted descent algebra). They prove that certain combinatorially defined algebras are free; labeled planar trees (with some restrictions on labels) do appear in their work, they also define two products on their trees for which the difference of two products is a combination of trees with nonnegative integer coefficients (just as in our case). However, their products are not compatible, and we do not know whether there is any relation of our results to those of Patras and Schocker.

General compatible structures. It is natural to ask which of our results have analogues for other operads of compatible structures (see [Strohmayer 2008] for the formal definition of compatible $\mathscr{O}$-structures for any operad $\mathscr{O}$). For the operad of compatible Lie brackets, we can prove an exact analogue of Theorem 5.2: free algebras over that operad are free as Lie algebras. The proof is similar to the proof for compatible associative structures and also makes use of an appropriate monomial basis. We expect that actually both of these statements are particular cases of a very general theorem.

The following conjecture consists of two parts. The first part generalises the main theorem of Strohmayer [2008], while the second one suggests that our theorem also holds in that generality.

Conjecture 7.8. (1) *Let $\mathscr{O}$ be a Koszul operad. Then the operad $\mathscr{O}^2$ of two compatible $\mathscr{O}$-structures is also Koszul.*

(2) *If the operad $\mathscr{O}$ is Koszul, then free $\mathscr{O}^2$-algebras are free as $\mathscr{O}$-algebras.*

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An algorithm for computing the integral closure

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We present an algorithm for computing the integral closure of a reduced ring that is finitely generated over a finite field.

Leonard and Pellikaan [2003] devised an algorithm for computing the integral closure of weighted rings that are finitely generated over finite fields. Previous algorithms proceed by building successively larger rings between the original ring and its integral closure [de Jong 1998; Seidenberg 1970; 1975; Stolzenberg 1968; Vasconcelos 1991; 2000]; the Leonard–Pellikaan algorithm instead starts with the first approximation being a finitely generated module that contains the integral closure, and successive steps produce submodules containing the integral closure. The weights in [Leonard and Pellikaan 2003] impose strong restrictions, and play a crucial role in various steps of their algorithm; see Remark 1.7. We present a modification of the Leonard–Pellikaan algorithm that works in much greater generality: it computes the integral closure of a reduced ring that is finitely generated over a finite field.

We discuss an implementation of the algorithm in Macaulay 2, and provide comparisons with de Jong's algorithm [1998].

1. The algorithm

Our main result is the following theorem; see Remark 1.5 for an algorithmic construction of an element D as below when R is a domain, and for techniques for dealing with the more general case of reduced rings.

Theorem 1.1. *Let R be a reduced ring that is finitely generated over a computable field of characteristic $p > 0$. Set $\overline{R}$ to be the integral closure of R in its total ring of fractions. Suppose D is a nonzerodivisor in the conductor ideal of R, that is, D is a nonzerodivisor with $D\overline{R} \subseteq R$.*

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(1) *Set* $V_0 = \frac{1}{D}R$, *and inductively define* $V_{e+1} = \{f \in V_e \mid f^p \in V_e\}$ *for* $e \geqslant 0$. *Then the modules* V_e *are algorithmically constructible.*

(2) *The descending chain* $V_0 \supseteq V_1 \supseteq V_2 \supseteq V_3 \supseteq \cdots$ *stabilizes. If* $V_e = V_{e+1}$, *then* V_e *equals* $\overline{R}$.

The prime characteristic enables us to use the Frobenius or pth power map; this is what makes the modules V_e algorithmically constructible.

Remark 1.2. For each integer $e \geqslant 0$, the module DV_e is an ideal of R; we set $U_e = DV_e$ and use this notation in the proof of Theorem 1.1 as well as in the Macaulay 2 code in the following section. The inductive definition of V_e translates to $U_0 = R$ and $U_{e+1} = \{r \in U_e \mid r^p \in D^{p-1}U_e\}$ for $e \geqslant 0$.

Proof of Theorem 1.1. (1) By Remark 1.2, it suffices to establish that the ideals U_e are algorithmically constructible. This follows inductively since

$$U_{e+1} = U_e \cap \ker\big(R \xrightarrow{\ F\ } R \xrightarrow{\ \pi\ } R/D^{p-1}U_e\big) \quad \text{for } e \geqslant 0,$$

where F is the Frobenius endomorphism of R, and π the canonical surjection.

(2) By construction, one has $V_{e+1} \subseteq V_e$ for each e. Moreover, it is a straightforward verification that

$$V_e = \{f \in V_0 \mid f^{p^i} \in V_0 \text{ for each } i \leqslant e\}.$$

Suppose $f \in \overline{R}$. Then $f^{p^i} \in \overline{R}$ for each $i \geqslant 0$, so $Df^{p^i} \in R$. It follows that $f \in V_e$ for each e.

If $V_{e+1} = V_e$ for some positive integer e, then it follows from the inductive definition that $V_{e+i} = V_e$ for each $i \geqslant 1$.

Let $v_1, \dots, v_s \colon R \longrightarrow \mathbb{Z} \cup \{\infty\}$ be the Rees valuations of the ideal DR, that is, v_i are valuations such that for each $n \in \mathbb{N}$, the integral closure of the ideal D^nR equals $\{r \in R \mid v_i(r) \geqslant nv_i(D) \text{ for each } i\}$. Let e be an integer such that $p^e > v_i(D)$ for each i. Suppose $r/D \in V_e$. Then $(r/D)^{p^e} \in V_0$, so $r^{p^e} \in D^{p^e-1}R$. It follows that $p^e v_i(r) \geqslant \big(p^e - 1\big)v_i(D)$ for each i, and hence that

$$v_i(r) \ \geqslant\ v_i(D) - v_i(D)/p^e \ >\ v_i(D) - 1$$

for each i. Since $v_i(r)$ is an integer, it follows that $v_i(r) \geqslant v_i(D)$ for each i, and therefore $r \in \overline{DR}$. But then r belongs to the integral closure of the ideal $D\overline{R}$ in $\overline{R}$. Since principal ideals are integrally closed in $\overline{R}$, it follows that $r \in D\overline{R}$, whence $r/D \in \overline{R}$. □

Remark 1.3. If R is an integral domain satisfying the Serre condition S_2, then each module V_e is S_2 as well:

Proceed by induction on e. Without loss of generality, assume R is local. Let x, y be part of a system of parameters for R. Suppose $yv \in xV_{e+1}$ for an element

$v \in V_{e+1}$. Then $yv/x \in V_{e+1}$, that is, $yv/x \in V_e$ and $y^p v^p / x^p \in V_e$, or equivalently, $yv \in xV_e$ and $y^p v^p \in x^p V_e$. Since V_e is S_2 by the inductive hypothesis, it follows that $v \in xV_e$ and $v^p \in x^p V_e$, hence $v \in xV_{e+1}$.

Remark 1.4. In the notation of Theorem 1.1, suppose e is an integer such that $V_e = V_{e+1}$. We claim that the integral closure of a principal ideal aR is

$$\{r \in R \mid Dr^{p^i} \in a^{p^i} R \text{ for each } i \leqslant e+1\}.$$

To see this, suppose r is an element of the ideal displayed above. Then $Dr^p = ga^p$ for some $g \in R$. Since

$$D(r/a)^{p^i} \in R \quad \text{for each } i \leqslant e+1,$$

it follows that

$$D(g/D)^{p^i} \in R \quad \text{for each } i \leqslant e.$$

But then $g/D \in V_e$, which implies that $g/D \in V_i$ for each i. Hence $D(r/a)^{p^i} \in R$ for each i, equivalently $r \in \overline{aR}$.

Remark 1.5. Let R be a reduced ring that is finitely generated over a perfect field K of prime characteristic p. We describe how to algorithmically obtain a nonzerodivisor D in the conductor ideal of R.

Case 1. Suppose R is an integral domain. Consider a presentation of R over K, say $R = K[x_1, \dots, x_n]/(f_1, \dots, f_m)$. Set $h = \operatorname{height}(f_1, \dots, f_m)$. Then the determinant of each $h \times h$ submatrix of the Jacobian matrix $(\partial f_i / \partial x_j)$ multiplies $\overline{R}$ into R; this may be concluded from the Lipman–Sathaye Theorem [1981] (also found as Theorem 12.3.10 in [Huneke and Swanson 2006]), as discussed in the following paragraph. At least one such determinant has nonzero image in R, and can be chosen as the element D in Theorem 1.1. Other approaches to obtaining an element D are via the proof of [Huneke and Swanson 2006, Theorem 3.1.3], or equivalently, via the results from [Stichtenoth 1993].

Let J be the ideal of R generated by the images of the $h \times h$ submatrices of $(\partial f_i / \partial x_j)$. We claim that J is contained in the conductor of R. By passing to the algebraic closure, assume K is algebraically closed. After a linear change of coordinates, assume that the x_i are in general position, specifically, that for any $n-h$ element subset Λ of $\{x_1, \dots, x_n\}$, the extension $K[\Lambda] \subseteq R$ is a finite integral extension, equivalently that $K[\Lambda]$ is a Noether normalization of R. By the Lipman–Sathaye Theorem, the relative Jacobian $J_{R/K[\Lambda]}$ is contained in the conductor ideal. The claim now follows since, as Λ varies, the relative Jacobian ideals $J_{R/K[\Lambda]}$ generate the ideal J.

Even when R is not necessarily an integral domain, the ideal J, as defined above, is not contained in any minimal prime of R; this follows from the Jacobian criterion, see, for example, [Huneke and Swanson 2006, Theorem 4.4.9].

Case 2. In the case where R is a reduced equidimensional ring, one may proceed as above and choose D to be the determinant of an $h \times h$ submatrix of $(\partial f_i/\partial x_j)$, and then test to see whether D is a nonzerodivisor. If it turns out that D is a nonzero zerodivisor, set $I_1 = (0 :_R D)$ and $I_2 = (0 :_R I_1)$. Then each of R/I_1 and R/I_2 is a reduced equidimensional ring, with fewer minimal primes than R, and $\overline{R} = \overline{R/I_1} \times \overline{R/I_2}$. Hence $\overline{R}$ may be computed by computing the integral closure of each R/I_i.

Case 3. If R is a reduced ring that is not necessarily equidimensional, one may compute the minimal primes $P_1, \dots, P_n$ of R using an algorithm for primary decomposition—admittedly an expensive step—and then compute $\overline{R}$ using Case 1 and the fact that $\overline{R} = \overline{R/P_1} \times \cdots \times \overline{R/P_n}$.

Remark 1.6. Theorem 1.1 may be extended as follows. Suppose a reduced ring R has an endomorphism φ with the property that for each valuation $v : R \longrightarrow \mathbb{Z} \cup \{\infty\}$, there exists an integer $k \geqslant 2$ such that

$$v(\varphi(r)) = kv(r) \quad \text{for each } r \in R. \tag{1.6.1}$$

Let D be a nonzerodivisor in the conductor of R. Set $V_0 = \frac{1}{D} R$ and

$$V_{e+1} = \{f \in V_e \mid \varphi(f) \in V_e\} \quad \text{for } e \geqslant 0.$$

Imitating the proof of Theorem 1.1, one sees that the descending chain

$$V_0 \supseteq V_1 \supseteq V_2 \supseteq V_3 \supseteq \cdots$$

stabilizes at $\overline{R}$. If colon ideals and kernels of endomorphisms are computable in R, then each V_e is algorithmically constructible.

As an example, consider a polynomial ring $A = \mathbb{F}[x_1, \dots, x_k]$ over a field $\mathbb{F}$. Let R be a subring of A that is generated, as an $\mathbb{F}$-algebra, by finitely many monomials. Fix an integer $k \geqslant 2$. The $\mathbb{F}$-algebra endomorphism of A with $x_i \mapsto x_i^k$ restricts to an endomorphism φ of R satisfying property (1.6.1). Thus, one obtains an algorithm for computing the integral closure of affine semigroup rings; see Bruns and Koch [2001] for another algorithm.

Remark 1.7. The Leonard–Pellikaan algorithm [2003] is based on earlier work of Leonard [2001]. These papers make use of the Frobenius endomorphism along with a *weighted total-degree monomial ordering*; this is a monomial ordering under which there are only finitely many elements preceding any given element, and this is an essential ingredient in proving the convergence of their algorithm. The affine domains considered in [Leonard and Pellikaan 2003] are constructed as towers in the following sense: R_0 is a finite field; if R_{j-1} is given with a weight function wt_{j-1}, then R_j is the integral closure of $R_{j-1}[x_j]/(\varphi_j(x_j))$ in $F_{j-1}[x_j]/(\varphi_j(x_j))$, as computed by their algorithm; here F_{j-1} is the field of fractions of R_{j-1}, and

$$\varphi_j(x_j) = x_j^{m_j} + u_j \prod_{i=1}^{j-1} x_i^{\alpha_{i,j}} + g_j(x_j, \dots, x_1)$$

is an element of $R_{j-1}[x_j]$ that is irreducible and monic in x_j, such that u_j is a nonzero element of R_0, and the weight function satisfies

$$\mathrm{wt}_j(g_j(x_j, \dots, x_1)) < \mathrm{wt}_j(x_j^{m_j}) = \mathrm{wt}_j\left(\prod_{i=1}^{j-1} x_i^{\alpha_{i,j}}\right),$$

where wt_j is a modification (not a simple extension) of wt_{j-1} that requires further technical assumptions on the m_j and $\alpha_{i,j}$. A complexity analysis of some aspects of the Leonard–Pellikaan algorithm is carried out in [Hu and Maharaj 2008].

2. Implementation and examples

Here is our code in Macaulay 2 [Grayson and Stillman], which uses this algorithm to compute the integral closure.

Input: An integral domain R that is finitely generated over a finite field, and, optionally, a nonzero element D of the conductor ideal of R.

Output: A set of generators for $\overline{R}$ as a module over R.

Macaulay 2 function:

```
icFracP = method(Options=>{conductorElement => null})
icFracP Ring := List => o -> (R) -> (
     P := ideal presentation R;
     c := codim P;
     S := ring P;
     if o.conductorElement === null then (
        J := promote(jacobian P,R);
        n := 1;
        det1 := ideal(0_R);
        while det1 == ideal(0_R) do (
           det1 = minors(c,J);
           n = n+1
        );
        D := det1_0;
     ) else D = o.conductorElement;
     p := char(R);
     K := ideal(1_R);
     U := ideal(0_R);
     F := apply(generators R, i-> i^p);
```

```
while (U != K) do (
   U = K;
   L := U*ideal(D^(p-1));
   f := map(R/L,R,F);
   K = intersect(kernel f, U);
);
U = mingens U;
if numColumns U == 0 then {1_R}
else apply(numColumns U, i-> U_(0,i)/D)
)
```

Since the Leonard–Pellikaan algorithm uses the Frobenius endomorphism, it is less efficient when the characteristic of the ring is a large prime. In the examples that follow, the computations are performed on a MacBook Pro computer with a 2 GHz Intel Core Duo processor; the time units are seconds. The comparisons are with de Jong's algorithm [1998] as implemented in the program `ICfractions` in Macaulay 2, version 1.1.

Example 2.1. Let $\mathbb{F}_2[x, y, t]$ be a polynomial ring over the field $\mathbb{F}_2$, and set $R = \mathbb{F}_2[x, y, x^2t, y^2t]$. Then R has a presentation $\mathbb{F}_2[x, y, u, v]/(x^2v - y^2u)$, which shows, in particular, that x^2 is an element of the conductor ideal. Setting $D = x^2$, the algorithm above computes that the integral closure of R is generated, as an R-module, by the elements 1 and xyt. Tracing the algorithm, one sees that V_0 is not equal to V_1, that V_1 is not equal to V_2, and that $V_2 = V_3$. Indeed, these R-modules are

$$V_0 = \frac{1}{x^2}R, \qquad V_1 = \frac{1}{x}R + ytR, \qquad V_e = R + ytR \text{ for } e \geqslant 2.$$

As is to be expected, the algorithm is less efficient as the characteristic of the ground field increases:

characteristic p	2	3	5	7	11	13	17	37	97
`icFracP`	0.04	0.03	0.04	0.04	0.04	0.05	0.05	0.13	0.59
`icFractions`	0.08	0.09	0.09	0.09	0.14	0.15	0.15	0.15	0.15

Integral closure of $\mathbb{F}_p[x, y, u, v]/(x^2v - y^2u)$.

We remark that R is an affine semigroup ring, so its integral closure may also be computed using the program `normaliz` of Bruns and Koch [2001].

Example 2.2. Consider the hypersurface

$$R = \mathbb{F}_p[u, v, x, y, z]/(u^2x^4 + uvy^4 + v^2z^4).$$

It is readily verified that R is a domain, and that $t = ux^4/v$ is integral over R. The ring $R[t]$ has a presentation $\mathbb{F}_p[u, v, x, y, z, t]/I$, where I is the ideal generated by the 2×2 minors of the matrix

$$\begin{pmatrix} u & t & -z^4 \\ v & x^4 & t+y^4 \end{pmatrix}.$$

Since the entries of the matrix form a regular sequence in $\mathbb{F}_p[u, v, x, y, z, t]$, the ring $R[t]$ is Cohen–Macaulay. Moreover, if $p \neq 2$, then the singular locus of $R[t]$ is $V(t, y, xz, vz, ux)$ which has codimension 2, so $R[t]$ is normal.

If $p = 2$ then the ring $R[t]$ is not normal; indeed, in this case, the integral closure of R is generated, as an R-module, by the elements

$$1, \qquad \sqrt{uv}, \qquad \frac{ux + z\sqrt{uv}}{y}, \qquad \frac{vz + x\sqrt{uv}}{y}, \qquad \frac{uxz + z^2\sqrt{uv}}{uy}.$$

For small values of p, these computations may be verified on Macaulay 2 using either algorithm; some computations times are recorded next. (Here and in the next table $*$ means that the computation did not terminate within six hours.)

characteristic p	2	3	5	7	11
`icFracP`	0.07	0.22	9.67	143	12543
`icFractions`	1.16	$*$	$*$	$*$	$*$

Integral closure of $\mathbb{F}_p[u, v, x, y, z]/(u^2x^4 + uvy^4 + v^2z^4)$.

Example 2.3. Consider the hypersurface

$$R = \mathbb{F}_p[u, v, x, y, z]/(u^2x^p + 2uvy^p + v^2z^p),$$

where p is an odd prime. We shall see that $\overline{R}$ has $p+1$ generators as an R-module, but first some comparisons:

characteristic p	3	5	7	11	13	17	19	23
`icFracP`	0.07	0.09	0.27	1.81	4.89	26	56	225
`icFractions`	1.49	75.00	4009	$*$	$*$	$*$	$*$	$*$

Integral closure of $\mathbb{F}_p[u, v, x, y, z]/(u^2x^p + 2uvy^p + v^2z^p)$.

We claim that $\overline{R}$ is generated, as an R-module, by the elements

$$1, \quad \sqrt{y^2 - xz}, \quad \text{and} \quad u^{i/p}v^{(p-i)/p} \ \text{ for } 1 \leqslant i \leqslant p-1. \tag{2.3.1}$$

It is immediate that these elements are integral over R; to see that they belong to the fraction field of R, note that

$$\sqrt{y^2 - xz} = \pm\frac{uy^p + vz^p}{u(y^2 - xz)^{(p-1)/2}}$$

and that, by the quadratic formula, one also has

$$\left(\frac{u}{v}\right)^{1/p} = \frac{-y \pm \sqrt{y^2 - xz}}{x}. \tag{2.3.2}$$

Moreover, using (2.3.2), it follows that

$$v^{1/p}\sqrt{y^2 - xz} = \pm(xu^{1/p} + yv^{1/p}),$$

and hence the R-module generated by the elements (2.3.1) is indeed an R-algebra. It remains to verify that the ring

$$A = R\left[\sqrt{y^2 - xz}, u^{i/p}v^{(p-i)/p} \mid 1 \leqslant i \leqslant p-1\right]$$

is normal. For this, it suffices to verify that

$$B = R\left[\sqrt{y^2 - xz}, u^{1/p}, v^{1/p}\right]$$

is normal, since A is a direct summand of B as an A-module: use the grading on B where $\deg x = \deg y = \deg z = 0$ and $\deg u^{1/p} = 1 = \deg v^{1/p}$, in which case A is the pth Veronese subring $\bigoplus_{i \in \mathbb{N}} B_{ip}$. The ring B has a presentation $\mathbb{F}_p[x, y, z, d, s, t]/I$, where I is generated by the 2×2 minors of the matrix

$$\begin{pmatrix} y+d & z & s \\ x & y-d & -t \end{pmatrix},$$

and $s \mapsto u^{1/p}$, $t \mapsto v^{1/p}$, $d \mapsto \sqrt{y^2 - xz}$. But then — after a change of variables — B is a determinantal ring, and hence normal.

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A general homological Kleiman–Bertini theorem

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Let G be a smooth algebraic group acting on a variety X. Let $\mathcal{F}$ and $\mathcal{E}$ be coherent sheaves on X. We show that if all the higher $\mathcal{T}or$ sheaves of $\mathcal{F}$ against G-orbits vanish, then for generic $g \in G$, the sheaf $\mathcal{T}or_j^X(g\mathcal{F}, \mathcal{E})$ vanishes for all $j \geq 1$. This generalizes a result of Miller and Speyer for transitive group actions and a result of Speiser, itself generalizing the classical Kleiman–Bertini theorem, on generic transversality, under a general group action, of smooth subvarieties over an algebraically closed field of characteristic 0.

1. Introduction

All schemes that we consider in this paper are of finite type over a fixed field k; we make no assumptions on the characteristic of k.

Our starting point is this:

Theorem 1.1 [Miller and Speyer 2008]. *Let X be a variety with a transitive left action of a smooth algebraic group G. Let $\mathcal{F}$ and $\mathcal{E}$ be coherent sheaves on X, and for all k-points $g \in G$, let $g\mathcal{F}$ denote the push-forward of $\mathcal{F}$ along multiplication by g. Then there is a dense Zariski open subset U of G such that, for all k-rational points $g \in U$ and for all $j \geq 1$, the sheaf $\mathcal{T}or_j^X(g\mathcal{F}, \mathcal{E})$ is zero.*

As Miller and Speyer remark, their result is a homological generalization of the Kleiman–Bertini theorem: in characteristic 0, if $\mathcal{F} = \mathcal{O}_W$ and $\mathcal{E} = \mathcal{O}_Y$ are structure sheaves of smooth subvarieties of X and G acts transitively on X, then gW and Y meet transversally for generic g, implying that $\mathcal{O}_{gW} = g\mathcal{O}_W$ and $\mathcal{O}_Y$ have no higher $\mathcal{T}or$. Motivated by this, if $\mathcal{F}$ and $\mathcal{E}$ are quasicoherent sheaves on X with $\mathcal{T}or_j^X(\mathcal{F}, \mathcal{E}) = 0$ for $j \geq 1$, we will say that $\mathcal{F}$ and $\mathcal{E}$ are *homologically transverse*; if $\mathcal{E} = \mathcal{O}_Y$ for some closed subscheme Y of X, we will simply say that $\mathcal{F}$ and Y are homologically transverse.

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Homological transversality has a geometric meaning if $\mathcal{F} = \mathbb{O}_W$ and $\mathcal{E} = \mathbb{O}_Y$ are structure sheaves of closed subschemes of X. If P is a component of $Y \cap W$, then Serre's formula for the multiplicity of the intersection of Y and W at P [Hartshorne 1977, p. 427] is

$$i(Y, W; P) = \sum_{j \geq 0} (-1)^j \operatorname{len}_P(\mathcal{T}or_j^X(\mathcal{F}, \mathcal{E})),$$

where the length is taken over the local ring at P. Thus if Y and W are homologically transverse, their intersection multiplicity at P is simply the length of their scheme-theoretic intersection over the local ring at P.

It is natural to ask what conditions on the action of G are necessary to conclude that homological transversality is generic in the sense of Theorem 1.1. In particular, the restriction to transitive actions is unfortunately strong, as it excludes important situations such as the torus action on $\mathbb{P}^n$. On the other hand, suppose that $\mathcal{F}$ is the structure sheaf of the closure of a nondense orbit. Then for all k-points $g \in G$, we have $\mathcal{T}or_1^X(g\mathcal{F}, \mathcal{F}) = \mathcal{T}or_1^X(\mathcal{F}, \mathcal{F}) \neq 0$, and so the conclusion of Theorem 1.1 fails (as long as $G(k)$ is dense in G). Thus for nontransitive group actions some additional hypothesis is necessary.

The main result of this paper is that there is a simple condition for homological transversality to be generic:

Theorem 1.2. *Let X be a scheme with a left action of a smooth algebraic group G, and let $\mathcal{F}$ be a coherent sheaf on X. Let $\overline{k}$ be an algebraic closure of k. Consider the following conditions*:

(1) *For all closed points $x \in X \times \operatorname{Spec} \overline{k}$, the pull-back of $\mathcal{F}$ to $X \times \operatorname{Spec} \overline{k}$ is homologically transverse to the closure of the $G(\overline{k})$-orbit of x.*

(2) *For all coherent sheaves $\mathcal{E}$ on X, there is a Zariski open and dense subset U of G such that for all k-rational points $g \in U$, the sheaf $g\mathcal{F}$ is homologically transverse to $\mathcal{E}$.*

Then (1) *implies* (2). *If k is algebraically closed*, *then* (1) *and* (2) *are equivalent.*

If g is not k-rational, the sheaf $g\mathcal{F}$ can still be defined; in Section 2 we give this definition and a generalization of (2) that is equivalent to (1) in any setting (see Theorem 2.1).

If G acts transitively on X in the sense of [Miller and Speyer 2008], then the action is geometrically transitive, and so (1) is trivially satisfied. Thus Theorem 1.1 follows from Theorem 1.2. Since transversality of smooth subvarieties in characteristic 0 implies homological transversality, Theorem 1.2 also generalizes the following result of Robert Speiser:

Theorem 1.3 [Speiser 1988, Theorem 1.3]. *Suppose that k is algebraically closed of characteristic* 0*. Let X be a smooth variety, and let G be a* (*necessarily smooth*) *algebraic group acting on X. Let W be a smooth closed subvariety of X. If W is transverse to every G-orbit in X, then for any smooth closed subvariety $Y \subseteq X$, there is a dense open subset U of G such that if $g \in U$, then gW and Y are transverse.*

Speiser's result implies that the generic intersection $gW \cap Y$, for $g \in U$, is also smooth. We also give a more general homological version of this. For simplicity, we state it here for algebraically closed fields, although in the body of the paper (see Theorem 4.2) we remove this assumption.

Theorem 1.4. *Assume that $k = \overline{k}$. Let X be a scheme with a left action of a smooth algebraic group G, and let W be a Cohen–Macaulay* (*alternatively*, *Gorenstein*) *closed subscheme of X such that W is homologically transverse to the G-orbit closure of every closed point $x \in X$. Then for any Cohen–Macaulay* (*Gorenstein*) *closed subscheme Y of X, there is a dense open subset $U \subseteq G$ so that gW is homologically transverse to Y and $gW \cap Y$ is Cohen–Macaulay* (*Gorenstein*) *for all closed points $g \in U$.*

Theorem 1.2 was proved in the course of an investigation of certain rings, determined by geometric data, that arise in the study of noncommutative algebraic geometry. Given a variety X, an automorphism σ of X and an invertible sheaf $\mathcal{L}$ on X, Artin and Van den Bergh [1990] construct a *twisted homogeneous coordinate ring* $B = B(X, \mathcal{L}, \sigma)$. The graded ring B is defined via

$$B_n = H^0(X, \mathcal{L} \otimes_X \sigma^* \mathcal{L} \otimes_X \cdots \otimes_X (\sigma^{n-1})^* \mathcal{L})$$

with multiplication of sections given by the action of σ. A closed subscheme W of X determines a graded right ideal I of B, generated by sections vanishing on W. In [Sierra 2008], we study the *idealizer* of I; that is, the maximal subring R of B such that I is a two-sided ideal of R. It turns out that quite subtle properties of W and its motion under σ control many of the properties of R; in particular, for R to be left Noetherian one needs that for any closed subscheme Y, all but finitely many $\sigma^n W$ are homologically transverse to Y. (For details, we refer the reader to [Sierra 2008].) Thus we were naturally led to ask how often homological transversality can be considered "generic" behavior, and what conditions on W ensure this.

We make some remarks on notation. If x is any point of a scheme X, we denote the skyscraper sheaf at x by k_x. For schemes X and Y, we will write $X \times Y$ for the product $X \times_k Y$. Finally, if X is a scheme with a (left) action of an algebraic group G, we will always denote the multiplication map by

$$\mu : G \times X \to X.$$

2. Generalizations

We begin this section by defining homological transversality more generally. If W and Y are schemes over a scheme X, with (quasi)coherent sheaves $\mathcal{F}$ on W and $\mathcal{E}$ on Y respectively, then for all $j \geq 0$ there is a (quasi)coherent sheaf $\mathcal{T}or_j^X(\mathcal{F}, \mathcal{E})$ on $W \times_X Y$. This sheaf is defined locally. Suppose that $X = \operatorname{Spec} R$, $W = \operatorname{Spec} S$, and $Y = \operatorname{Spec} T$ are affine. Let $(\_\_)\tilde{}$ denote the functor that takes an R-module (or S- or T-module, respectively) to the associated quasicoherent sheaf on X (or W or Y). If F is an S-module and E is a T-module, we define $\mathcal{T}or_j^X(\widetilde{F}, \widetilde{E})$ to be $(\operatorname{Tor}_j^R(F, E))\tilde{}$. That these glue properly to give sheaves on $W \times_X Y$ for general W, Y, and X is the result in [Grothendieck 1963, 6.5.3]. As before, we will say that $\mathcal{F}$ and $\mathcal{E}$ are *homologically transverse* if the sheaf $\mathcal{T}or_j^X(\mathcal{F}, \mathcal{E})$ is zero for all $j \geq 1$.

We caution the reader that the maps from W and Y to X are implicit in the definition of $\mathcal{T}or_j^X(\mathcal{F}, \mathcal{E})$; at times we will write $\mathcal{T}or_j^{W \to X \leftarrow Y}(\mathcal{F}, \mathcal{E})$ to make this more obvious. We also remark that if $Y = X$, then $\mathcal{T}or_j^X(\mathcal{F}, \mathcal{E})$ is a sheaf on $W \times_X X = W$. As localization commutes with Tor, for any $w \in W$ lying over $x \in X$ we have in this case that $\mathcal{T}or_j^X(\mathcal{F}, \mathcal{E})_w = \operatorname{Tor}_j^{\mathcal{O}_{X,x}}(\mathcal{F}_w, \mathcal{E}_x)$.

Now suppose that $f : W \to X$ is a morphism of schemes and G is an algebraic group acting on X. Let $\mathcal{F}$ be a (quasi)coherent sheaf on W and let g be any point of G. We will denote the pull-back of $\mathcal{F}$ to $\{g\} \times W$ by $g\mathcal{F}$. There is a map

$$\{g\} \times W \longrightarrow G \times W \xrightarrow{1 \times f} G \times X \xrightarrow{\mu} X.$$

If Y is a scheme over X and $\mathcal{E}$ is a (quasi)coherent sheaf on Y, the (quasi)coherent sheaf $\mathcal{T}or_j^{\{g\} \times W \to X \leftarrow Y}(g\mathcal{F}, \mathcal{E})$ on $W \times_X Y \times k(g)$ will be written $\mathcal{T}or_j^X(g\mathcal{F}, \mathcal{E})$. Note that if $W = X$ and g is k-rational, then $g\mathcal{F}$ is simply the push-forward of $\mathcal{F}$ along multiplication by g.

In this context, we prove the following relative version of Theorem 1.2:

Theorem 2.1. *Let X be a scheme with a left action of a smooth algebraic group G, let $f : W \to X$ be a morphism of schemes, and let $\mathcal{F}$ be a coherent sheaf on W. We define maps*

$$\begin{array}{ccc} G \times W & \xrightarrow{\rho} & X \\ {\scriptstyle p}\big\downarrow & & \\ W, & & \end{array}$$

where ρ is the map $\rho(g, w) = gf(w)$ induced by the action of G and p is projection onto the second factor.

The following conditions are equivalent:

(1) *For all closed points $x \in X \times \operatorname{Spec} \overline{k}$, the pull-back of $\mathcal{F}$ to $W \times \operatorname{Spec} \overline{k}$ is homologically transverse to the closure of the $G(\overline{k})$-orbit of x.*

(2) *For all schemes $r : Y \to X$ and all coherent sheaves $\mathscr{E}$ on Y, there is a Zariski open and dense subset U of G such that for all closed points $g \in U$, the sheaf $g\mathscr{F}$ on $\{g\} \times W$ is homologically transverse to $\mathscr{E}$.*

(3) *The sheaf $p^*\mathscr{F}$ on $G \times W$ is ρ-flat over X.*

A related relative version of Theorem 1.3 is given in [Speiser 1988].

Our general approach to Theorem 2.1 mirrors that of [Speiser 1988], although the proof techniques are quite different. We first generalize Theorem 1.1 to apply to any flat map $f : W \to X$; this is a homological version of [Kleiman 1974, Lemma 1] and may be of independent interest.

Theorem 2.2. *Let X, Y, and W be schemes, let A be a generically reduced scheme, and suppose that there are morphisms*

$$\begin{tikzcd} & Y \arrow[d, "r"] \\ W \arrow[r, "f"] \arrow[d, "q"'] & X \\ A. & \end{tikzcd}$$

Let $\mathscr{F}$ be a coherent sheaf on W that is f-flat over X, and let $\mathscr{E}$ be a coherent sheaf on Y. For all $a \in A$, let W_a denote the fiber of W over a, and let $\mathscr{F}_a = \mathscr{F} \otimes_W \mathscr{O}_{W_a}$ be the fiber of $\mathscr{F}$ over a.

Then there is a dense open $U \subseteq A$ such that if $a \in U$, then $\mathscr{F}_a$ is homologically transverse to $\mathscr{E}$.

We note that we have not assumed that X, Y, W, or A are smooth.

3. Proofs

In this section we prove Theorems 1.2, 2.1, and 2.2.

Lemma 3.1. *Let*

$$X_1 \xrightarrow{\alpha} X_2 \xrightarrow{\gamma} X_3$$

be morphisms of schemes, and assume that γ is flat. Let $\mathscr{G}$ be a quasicoherent sheaf on X_1 that is flat over X_3. Let $\mathscr{H}$ be any quasicoherent sheaf on X_3. Then for all $j \geq 1$, we have $\mathscr{T}or_j^{X_2}(\mathscr{G}, \gamma^*\mathscr{H}) = 0$.

Proof. We may reduce to the local case. Thus let $x \in X_1$ and let $y = \alpha(x)$ and $z = \gamma(y)$. Let $S = \mathscr{O}_{X_2,y}$ and let $R = \mathscr{O}_{X_3,z}$. Then $(\gamma^*\mathscr{H})_y \cong S \otimes_R \mathscr{H}_z$. Since S is flat over R, we have

$$\mathrm{Tor}_j^R(\mathscr{G}_x, \mathscr{H}_z) \cong \mathrm{Tor}_j^S(\mathscr{G}_x, S \otimes_R \mathscr{H}_z) = \mathscr{T}or_j^{X_2}(\mathscr{G}, \gamma^*\mathscr{H})_x$$

by flat base change. The left-hand side is 0 for $j \geq 1$ since $\mathcal{G}$ is flat over X_3. Thus for $j \geq 1$ we have $\mathcal{T}or_j^{X_2}(\mathcal{G}, \gamma^*\mathcal{H}) = 0$. $\square$

To prove Theorem 2.2, we show that a suitable modification of the spectral sequences used by Miller and Speyer will work in our situation. Our key computation is the following lemma; compare to [Miller and Speyer 2008, Proposition 2].

Lemma 3.2. *Given the notation of Theorem 2.2, there is an open dense $U \subseteq A$ such that for all $a \in U$ and for all $j \geq 0$ we have*

$$\mathcal{T}or_j^W(\mathcal{F} \otimes_X \mathcal{E}, q^*k_a) \cong \mathcal{T}or_j^X(\mathcal{F}_a, \mathcal{E})$$

as sheaves on $W \times_X Y$.

Note that $\mathcal{F} \otimes_X \mathcal{E}$ is a sheaf on $W \times_X Y$ and thus $\mathcal{T}or_j^W(\mathcal{F} \otimes_X \mathcal{E}, q^*k_a)$ is a sheaf on $W \times_X Y \times_W W = W \times_X Y$ as required.

Proof. Since A is generically reduced, we may apply generic flatness to the morphism $q : W \to A$. Thus there is an open dense subset U of A such that both W and $\mathcal{F}$ are flat over U. Let $a \in U$. Away from $q^{-1}(U)$, both sides of the equality we seek to establish are zero, and so the result is trivial. Since $\mathcal{F}|_{q^{-1}(U)}$ is still flat over X, without loss of generality we may replace W by $q^{-1}(U)$; that is, we may assume that both W and $\mathcal{F}$ are flat over A.

The question is local, so assume that $X = \operatorname{Spec} R$, $Y = \operatorname{Spec} T$, and $W = \operatorname{Spec} S$ are affine. Let $E = \Gamma(Y, \mathcal{E})$ and let $F = \Gamma(W, \mathcal{F})$. Let $Q = \Gamma(W, q^*k_a)$; then $\Gamma(W, \mathcal{F}_a) = F \otimes_S Q$. We seek to show that

$$\operatorname{Tor}_j^S(F \otimes_R E, Q) \cong \operatorname{Tor}_j^R(F \otimes_S Q, E)$$

as $S \otimes_R T$-modules.

We will work on $W \times X$. For clarity, we lay out the various morphisms and corresponding ring maps in our situation. We have morphisms of schemes

$$\begin{tikzcd} W \times X \arrow[d, "p"'] & Y \arrow[d, "r"] \\ W \arrow[u, bend right, "\phi"'] \arrow[r, "f"'] & X, \end{tikzcd}$$

where p is projection onto the first factor and the morphism ϕ splitting p is given by the graph of f. Letting $B = S \otimes_k R$, we have corresponding maps of rings

$$\begin{tikzcd} B \arrow[d, bend left, "\phi^{\#}"] & T \\ S \arrow[u, "p^{\#}"] & R, \arrow[l, "f^{\#}"] \arrow[u, "r^{\#}"'] \end{tikzcd}$$

where $p^{\#}(s) = s \otimes 1$ and $\phi^{\#}(s \otimes r) = s \cdot f^{\#}(r)$. We make the trivial observation that

$$B \otimes_R E = (S \otimes_k R) \otimes_R E \cong S \otimes_k E.$$

Let $K_\bullet \to F$ be a projective resolution of F, considered as a B-module via the map $\phi^{\#} : B \to S$. As E is an R-module via the map $r^{\#} : R \to T$, there is a B-action on $S \otimes_k E$; let $L_\bullet \to S \otimes_k E$ be a projective resolution over B.

Let $P_{\bullet,\bullet}$ be the double complex $K_\bullet \otimes_B L_\bullet$. We claim the total complex of $P_{\bullet,\bullet}$ resolves $F \otimes_B (S \otimes_k E)$. To see this, note that the rows of $P_{\bullet,\bullet}$, which are of the form $K_\bullet \otimes_B L_j$, are acyclic, except in degree 0, where the homology is $F \otimes_B L_j$. The degree 0 horizontal homology forms a vertical complex whose homology computes $\operatorname{Tor}_j^B(F, S \otimes_k E)$. But $S \otimes_k E \cong B \otimes_R E$, and B is a flat R-module. Therefore $\operatorname{Tor}_j^B(F, S \otimes_k E) \cong \operatorname{Tor}_j^B(F, B \otimes_R E) \cong \operatorname{Tor}_j^R(F, E)$ by the formula for flat base change for Tor. Since F is flat over R, this is zero for all $j \geq 1$. Thus, via the spectral sequence

$$H_j^v(H_i^h P_{\bullet,\bullet}) \Rightarrow H_{i+j} \operatorname{Tot} P_{\bullet,\bullet}$$

we see that the total complex of $P_{\bullet,\bullet}$ is acyclic, except in degree 0, where the homology is $F \otimes_B S \otimes_k E \cong F \otimes_R E$.

Consider the double complex $P_{\bullet,\bullet} \otimes_S Q$. Since $\operatorname{Tot} P_{\bullet,\bullet}$ is a B-projective and therefore S-projective resolution of $F \otimes_R E$, the homology of the total complex of this double complex computes $\operatorname{Tor}_j^S(F \otimes_R E, Q)$.

Now consider the row $K_\bullet \otimes_B L_j \otimes_S Q$. As L_j is B-projective and therefore B-flat, the ith homology of this row is isomorphic to $\operatorname{Tor}_i^S(F, Q) \otimes_B L_j$. Since W and $\mathcal{F}$ are flat over A, by Lemma 3.1 we have $\operatorname{Tor}_i^S(F, Q) = 0$ for all $i \geq 1$. Thus this row is acyclic except in degree 0, where the homology is $F \otimes_B L_j \otimes_S Q$. The vertical differentials on the degree 0 homology give a complex whose jth homology is isomorphic to $\operatorname{Tor}_j^B(F \otimes_S Q, S \otimes_k E)$. As before, this is simply $\operatorname{Tor}_j^R(F \otimes_S Q, E)$.

Thus (via a spectral sequence) we see that the homology of the total complex of $P_{\bullet,\bullet} \otimes_S Q$ computes $\operatorname{Tor}_j^R(F \otimes_S Q, E)$. But we have already seen that the homology of this total complex is isomorphic to $\operatorname{Tor}_j^S(F \otimes_R E, Q)$. Thus the two are isomorphic. $\square$

Proof of Theorem 2.2. By generic flatness, there is no loss of generality in restricting to W flat over A. Since $\mathcal{F}$ and $\mathcal{E}$ are coherent sheaves on W and Y respectively, $\mathcal{F} \otimes_X \mathcal{E}$ is a coherent sheaf on $W \times_X Y$. Applying generic flatness to the composition $W \times_X Y \to W \to A$, we obtain a dense open $V \subseteq A$ such that $\mathcal{F} \otimes_X \mathcal{E}$ is flat over V. Therefore, by Lemma 3.1, if $a \in V$ and $j \geq 1$, we have $\mathcal{T}or_j^W(\mathcal{F} \otimes_X \mathcal{E}, q^* k_a) = 0$.

We apply Lemma 3.2 to choose a dense open $U \subseteq A$ such that for all $j \geq 1$, if $a \in U$, then $\mathcal{T}or_j^W(\mathcal{F} \otimes_X \mathcal{E}, q^* k_a) \cong \mathcal{T}or_j^X(\mathcal{F}_a, \mathcal{E})$. Thus if a is in the dense open set $U \cap V$, then for all $j \geq 1$ we have

$$\mathcal{T}or_j^X(\mathcal{F}_a, \mathcal{E}) \cong \mathcal{T}or_j^W(\mathcal{F} \otimes_X \mathcal{E}, q^* k_a) = 0,$$

as required. □

We now turn to the proof of Theorem 2.1; for the remainder of this paper, we will adopt the hypotheses and notation given there.

Lemma 3.3. *Let R, R', S, and T be commutative rings, and let*

$$\begin{tikzcd} R' \arrow[r] & T \\ R \arrow[u] \arrow[r] & S \arrow[u] \end{tikzcd}$$

be a commutative diagram of ring homomorphisms such that R'_R and T_S are flat. Let N be an R-module. Then for all $j \geq 0$, we have that

$$\operatorname{Tor}_j^{R'}(N \otimes_R R', T) \cong \operatorname{Tor}_j^R(N, S) \otimes_S T.$$

Proof. Let $P_\bullet \to N$ be a projective resolution of N. Consider the complex

$$P_\bullet \otimes_R R' \otimes_{R'} T \cong P_\bullet \otimes_R T \cong P_\bullet \otimes_R S \otimes_S T. \tag{3.4}$$

Since R'_R is flat, $P_\bullet \otimes_R R'$ is a projective resolution of $N \otimes_R R'$. Thus the jth homology of (3.4) computes $\operatorname{Tor}_j^{R'}(N \otimes_R R', T)$. Since T_S is flat, this homology is isomorphic to $H_j(P_\bullet \otimes_R S) \otimes_S T$. Thus $\operatorname{Tor}_j^{R'}(N \otimes_R R', T) \cong \operatorname{Tor}_j^R(N, S) \otimes_S T$. □

Lemma 3.5. *Let x be a closed point of X. Consider the multiplication map*

$$\mu_x : G \times \{x\} \to X.$$

Then for all $j \geq 0$ we have

$$\mathcal{T}or_j^X(\mathcal{F}, \mathcal{O}_{G \times \{x\}}) \cong \mathcal{T}or_j^{G \times X}(p^* \mathcal{F}, \mu^* k_x). \tag{3.6}$$

If k is algebraically closed, then we also have

$$\mathcal{T}or_j^{G \times X}(p^* \mathcal{F}, \mu^* k_x) \cong \mathcal{T}or_j^X(\mathcal{F}, \mathcal{O}_{\overline{Gx}}) \otimes_X \mathcal{O}_{G \times \{x\}}. \tag{3.7}$$

All isomorphisms are of sheaves on $G \times W$.

Proof. Note that μ_x maps $G \times \{x\}$ onto a locally closed subscheme of X, which we will denote Gx. Since all computations may be done locally, without loss of generality we may assume that Gx is in fact a closed subscheme of X.

Let $\nu : G \to G$ be the inverse map, and let $\psi = \nu \times \mu : G \times X \to G \times X$. Consider the commutative diagram

$$\begin{tikzcd}
G \times W \arrow[r, "1\times f"] \arrow[d, "p"'] & G \times X \arrow[d, "p"'] & G \times \{x\} \arrow[l, "\psi"'] \arrow[d, "\pi"] \arrow[dl, "\mu_x"'] \\
W \arrow[r, "f"'] & X & Gx, \arrow[l, "\supseteq"']
\end{tikzcd} \tag{3.8}$$

where π is the induced map and p is projection onto the second factor. Since $\psi^2 = \mathrm{Id}_{G\times X}$ and $\mu = p \circ \psi$, we obtain $\mu^* k_x \cong \psi^* p^* k_x \cong \psi_* \mathcal{O}_{G\times\{x\}}$, considered as sheaves on $G \times X$. Then the isomorphism (3.6) is a direct consequence of the flatness of p and Lemma 3.3. If k is algebraically closed, then π is also flat, and so the isomorphism (3.7) also follows from Lemma 3.3. $\square$

Proof of Theorem 2.1. (3) $\Rightarrow$ (2). Assume (3). Let $\mathcal{E}$ be a coherent sheaf on Y. Consider the maps

$$\begin{tikzcd}
 & Y \arrow[d, "r"] \\
G \times W \arrow[r, "\rho"] \arrow[d, "q"'] & X \\
G, &
\end{tikzcd}$$

where q is projection on the first factor.

Since G is smooth, it is generically reduced. Thus we may apply Theorem 2.2 to the ρ-flat sheaf $p^*\mathcal{F}$ to obtain a dense open $U \subseteq G$ such that if $g \in U$ is a closed point, then ρ makes $(p^*\mathcal{F})_g$ homologically transverse to $\mathcal{E}$. But $\rho|_{\{g\}\times W}$ is the map used to define $\mathcal{T}or_j^X(g\mathcal{F}, \mathcal{E})$; that is, $(p^*\mathcal{F})_g \cong g\mathcal{F}$ as sheaves over X. Thus (2) holds.

(2) $\Rightarrow$ (3). The morphism ρ factors as

$$G \times W \xrightarrow{1\times f} G \times X \xrightarrow{\mu} X.$$

Since the multiplication map μ is the composition of an automorphism of $G \times X$ and a projection, it is flat.

Therefore for any quasicoherent $\mathcal{N}$ on X and $\mathcal{M}$ on $G \times W$ and for any closed point $z \in G \times W$, we have

$$\mathcal{T}or_j^{G\times X}(\mathcal{M}, \mu^*\mathcal{N})_z \cong \mathcal{T}or_j^{\mathcal{O}_{X,\rho(z)}}(\mathcal{M}_z, \mathcal{N}_{\rho(z)}), \tag{3.9}$$

as in the proof of Lemma 3.1.

If $p^*\mathcal{F}$ fails to be flat over X, then flatness fails against the structure sheaf of some closed point $x \in X$, by the local criterion for flatness [Eisenbud 1995, Theorem 6.8]. Thus to check that $p^*\mathcal{F}$ is flat over X, it is equivalent to test flatness against structure sheaves of closed points of X. By (3.9), we see that $p^*\mathcal{F}$ is ρ-flat over X if and only if

$$\mathcal{T}or_j^{G\times X}(p^*\mathcal{F}, \mu^* k_x) = 0 \quad \text{for all closed points } x \in X \text{ and for all } j \geq 1. \tag{3.10}$$

Applying Lemma 3.5, we see that the flatness of $p^*\mathcal{F}$ is equivalent to the vanishing

$$\mathcal{T}or_j^X(\mathcal{F}, \mathcal{O}_{G\times\{x\}}) = 0 \quad \text{for all closed points } x \in X \text{ and for all } j \geq 1. \tag{3.11}$$

Assume (2). We will show that (3.11) holds for all $x \in X$. Fix a closed point $x \in X$ and consider the morphism $\mu_x : G \times \{x\} \to X$. By assumption, there is a closed point $g \in G$ such that $g\mathcal{F}$ is homologically transverse to $\mathcal{O}_{G\times\{x\}}$. Let $k' = k(g)$ and let g' be the canonical k'-point of $G \times \operatorname{Spec} k'$ lying over g. Let $G' = G \times \operatorname{Spec} k'$ and let $X' = X \times \operatorname{Spec} k'$. Let $\mathcal{F}'$ be the pull-back of $\mathcal{F}$ to $W' = W \times \operatorname{Spec} k'$. Consider the commutative diagram

$$\begin{tikzcd}
G \times \{x\} \times \operatorname{Spec} k' \arrow[r, "\mu_x \times 1"] \arrow[d] & X' \arrow[d] & \{g'\} \times_{k'} W' \arrow[l, "\rho"'] \arrow[d, "\cong"'] \\
G \times \{x\} \arrow[r, "\mu_x"] & X & \{g\} \times W. \arrow[l, "\rho"']
\end{tikzcd}$$

Since the vertical maps are faithfully flat and the left-hand square is a fiber square, by Lemma 3.3 $g'\mathcal{F}'$ is homologically transverse to

$$G \times \{x\} \times \operatorname{Spec} k' \cong G' \times \{x\}.$$

By $G(k')$-equivariance, $\mathcal{F}'$ is homologically transverse to $(g')^{-1}G' \times \{x\} = G' \times \{x\}$. Since

$$G' \times \{x\} \longrightarrow X' \xleftarrow{f} W'$$

is base-extended from

$$G \times \{x\} \longrightarrow X \xleftarrow{f} W,$$

we obtain that $\mathcal{F}$ is homologically transverse to $G \times \{x\}$. Thus (3.11) holds.

(1) $\Rightarrow$ (3). The ρ-flatness of $\mathcal{F}$ is not affected by base extension, so without loss of generality we may assume that k is algebraically closed. Then (3) follows directly from Lemma 3.5 and the criterion (3.10) for flatness.

(3) $\Rightarrow$ (1). As before, we may assume that k is algebraically closed. Let x be a closed point of X. We have seen that (3) and (2) are equivalent; by applying (2) to $\mathcal{E} = \mathcal{O}_{\overline{Gx}}$ there is a closed point $g \in G$ such that $g\mathcal{F}$ and $\overline{Gx}$ are homologically transverse. By $G(k)$-equivariance, $\mathcal{F}$ and $g^{-1}\overline{Gx} = \overline{Gx}$ are homologically transverse. $\square$

Proof of Theorem 1.2. If $\mathscr{F}$ is homologically transverse to orbit closures upon extension to $\overline{k}$, then, using Theorem 2.1(2), for any $\mathscr{E}$ there is a dense open $U \subseteq G$ such that, in particular, for any k-rational $g \in U$ the sheaves $g\mathscr{F}$ and $\mathscr{E}$ are homologically transverse.

The equivalence of (1) and (2) in the case that k is algebraically closed follows directly from Theorem 2.1. □

Theorem 1.2 is a statement about k-rational points in $U \subseteq G$. However, the proof shows that for any extension k' of k and any k'-rational $g \in U \times \operatorname{Spec} k'$, $g\mathscr{F}$ will be homologically transverse to $\mathscr{E}$ on $X \times \operatorname{Spec} k'$. Further, in many situations U will automatically contain a k-rational point of G. This holds, in particular, if k is infinite, G is connected and affine, and either k is perfect or G is reductive, by [Borel 1991, Corollary 18.3].

4. Singularities of generic intersections

We now specialize to consider generic intersections of two subschemes of X. That is, let X be a scheme with a left action of a smooth algebraic group G. Let Y and W be closed subschemes of X. By Theorem 1.3, if k is algebraically closed of characteristic 0, W is transverse to G-orbit closures, and X, Y, and W are smooth, then for generic $g \in G$ the subschemes gW and Y meet transversally, and so by definition $gW \cap Y$ is smooth. Here we remark that a homological version of this result holds more generally: if W is homologically transverse to G-orbit closures and Y and W are both Cohen–Macaulay or both Gorenstein, their generic intersection has the same property. We use the following result from commutative algebra:

Theorem 4.1. *Let $A \to B$ be a local homomorphism of Noetherian local rings, and let $\mathfrak{m}$ be the maximal ideal of A and $F = B/\mathfrak{m}B$. Assume that B is flat over A. Then B is Cohen–Macaulay* (*alternatively*, *Gorenstein*) *if and only if B and F are both Cohen–Macaulay* (*Gorenstein*).

Proof. See [Matsumura 1989, Corollary 23.3, Theorem 23.4]. □

Theorem 4.2. *Let X be a scheme with a left action of a smooth algebraic group G. Suppose that $f : W \to X$ and $r : Y \to X$ are morphisms of schemes and that $W \times \operatorname{Spec} \overline{k}$ is homologically transverse to the $G(\overline{k})$-orbit of x for all closed points $x \in X \times \operatorname{Spec} \overline{k}$. Further suppose that Y and W are Cohen–Macaulay* (*alternatively*, *Gorenstein*). *Then there is a dense open subset $U \subseteq G$ so that for all closed points $g \in U$, the scheme $\{g\} \times W$ is homologically transverse to Y and the fiber product $(\{g\} \times W) \times_X Y$ is Cohen–Macaulay* (*Gorenstein*).

Proof. Let $\rho : G \times W \to X$ be the map $\rho(g, w) = gf(w)$ induced by f and the action of G. Let $q : G \times W \to G$ be projection to the first factor. Thus there is a commutative diagram

$$\begin{tikzcd} G \times W \times_X Y \arrow[r, "\rho\times 1"] \arrow[d, "1\times r"'] & Y \arrow[d, "r"] \\ G \times W \arrow[r, "\rho"] \arrow[d, "q"'] & X \\ G. & \end{tikzcd}$$

By Theorem 2.1 applied to $\mathcal{F} = \mathbb{O}_W$, ρ is flat. Now, $G \times W$ is Cohen–Macaulay or Gorenstein, and so by Theorem 4.1, the fibers of ρ have the same property. Since Y is Cohen–Macaulay (Gorenstein) and $\rho \times 1$ is flat, applying Theorem 4.1 again, we see that $G \times W \times_X Y$ is also Cohen–Macaulay (Gorenstein). Now, by generic flatness and Theorem 2.1, there is a dense open $U \subset G$ such that $q \circ (1 \times r)$ is flat over U and $\{g\} \times W$ is homologically transverse to Y for all $g \in U$. For $g \in U$, the fiber $(\{g\} \times W) \times_X Y$ of $q \circ (1 \times r)$ is Cohen–Macaulay (Gorenstein), by Theorem 4.1 again. $\square$

We note that, although we did not assume that X is Cohen–Macaulay or Gorenstein, it follows from the flatness of ρ and from Theorem 4.1.

We also remark that if Y and W are homologically transverse local complete intersections in a smooth X, it is not hard to show directly that $Y \cap W$ is also a local complete intersection. We do not know if it is true in general that the homologically transverse intersection of two Cohen–Macaulay subschemes is Cohen–Macaulay, although it follows, for example, from [Fulton and Pragacz 1998, Lemma, p. 108] if X is smooth.

Theorem 1.4 follows directly from Theorem 4.2.

Thus we may refine Theorem 1.1 to obtain a result on transitive group actions that echoes the Kleiman–Bertini theorem even more closely.

Corollary 4.3. *Let X be a scheme with a geometrically transitive left action of a smooth algebraic group G. Let Y and W be Cohen–Macaulay* (*alternatively*, *Gorenstein*) *closed subschemes of X. Then there is a dense Zariski open subset U of G such that gW is homologically transverse to Y and $gW \cap Y$ is Cohen–Macaulay* (*Gorenstein*) *for all k-rational points $g \in U$.* $\square$

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