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Every finite-dimensional representation of an algebraic group G gives a trace symmetric bilinear form on the Lie algebra of G. We give criteria in terms of root system data for the existence of a representation such that this form is nonzero or nondegenerate. As a corollary, we show that a Lie algebra of type E_8 over a field of characteristic 5 does not have a "quotient trace form", answering a question posed in the 1960s.

Let G be an algebraic group over a field F, acting on a finite-dimensional vector space V via a homomorphism $\rho : G \to \mathrm{GL}(V)$. The differential $\mathrm{d}\rho$ of ρ maps the Lie algebra $\mathrm{Lie}(G)$ of G into $\mathfrak{gl}(V)$, and we put Tr_ρ for the symmetric bilinear form

$$\mathrm{Tr}_\rho(x, y) := \mathrm{trace}(\mathrm{d}\rho(x)\,\mathrm{d}\rho(y)) \quad \text{for } x, y \in \mathrm{Lie}(G).$$

We call Tr_ρ a *trace form of* G. Such forms appear, for example, in the hypotheses for the Jacobson–Morozov Theorem [Carter 1985, 5.3.1], in Richardson's proof that there are finitely many conjugacy classes of nilpotent elements in the Lie algebra of a semisimple algebraic group as in [Jantzen 2004, §2] or [Humphreys 1995, §§3.8, 3.9], and in the "explicit" construction of a Springer isomorphism in [Bardsley and Richardson 1985, §9.3]. We prove:

Theorem A. *Let G be a split and almost simple linear algebraic group over a field F.*

(1) *There is a representation ρ of G with Tr_ρ nondegenerate if and only if the characteristic of F is very good for G.*

(2) *There is a representation ρ of G with Tr_ρ nonzero if and only if the characteristic of F is as indicated in Table I.*

A weaker version (up to isogeny) of the "if" direction of part (1) is standard; see for example [Springer and Steinberg 1970, I.5.3] or [Carter 1985, 1.16]. (After this paper was released as a preprint, I learned that Alexander Premet had previously

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G	Every ρ has Tr_ρ degenerate if char F	Every ρ has Tr_ρ zero if char F
$\mathrm{SL}_n/\boldsymbol{\mu}_m$ with m odd	divides n	divides $\gcd(m, n/m)$
$\mathrm{SL}_n/\boldsymbol{\mu}_m$ with m even	divides n	divides $2\gcd(m, n/m)$
Sp_{2n}	equals 2	[does not occur]
SO_n, Spin_n, PSO_n with $n \geq 7$, HSpin_{4n} for $n \geq 3$, PSp_{2n}	equals 2	equals 2
E_6 adjoint, G_2	equals 2 or 3	equals 2
E_6 simply connected, E_7, F_4	equals 2 or 3	equals 2 or 3
E_8	equals 2, 3, or 5	equals 2, 3, or 5

Table I. Primes where Tr_ρ is degenerate or zero for every ρ. The middle column lists the primes that are not very good for G. For simply connected G, the right column lists the torsion primes for G as defined in, for example, [Steinberg 1975, 1.13].

proved the "only if" direction of part (1) for groups not of type A_n, but his proof has not been published [Premet 1995, p. 80].)

Both directions of part (1) are deduced from part (2). The crux of the proof of part (2) is a formula for the trace form Tr_ρ, given in Proposition 4.1 below.

We remark that the characteristics in part (1) of the theorem depend only on the isogeny class of G, whereas the characteristics in part (2) of the theorem are more sensitive. For example, Sp_{2n} has a representation with nonzero trace form over every field, whereas its quotient PSp_{2n} has such only in characteristic different from 2; this is no surprise because PSp_{2n} has fewer representations than Sp_{2n}. But the opposite phenomenon also occurs: a simply connected group of type E_6 has a representation with nonzero trace form only in characteristics $\neq 2, 3$, whereas its quotient the adjoint E_6 has one over every field of characteristic $\neq 2$. This opposite phenomenon is related to the number $E(G)$ defined in Section 3 below.

For G of type E_8, we can strengthen Theorem A. Given a representation ψ of the Lie algebra $\mathrm{Lie}(G)$ of G, one can define a trace form Tr_ψ on $\mathrm{Lie}(G)$ by setting $(x, y) \mapsto \mathrm{trace}(\psi(x)\,\psi(y))$. We prove

Theorem B. *If F has characteristic* 2, 3, *or* 5 *and G is of type E_8, then the trace form of every representation of* $\mathrm{Lie}(G)$ *is zero.*

This is a strengthening of Theorem A because many representations ψ of $\mathrm{Lie}(G)$ are not differentials of representations of G. The proof of Theorem B is given in 11.3; it amounts to a combination of Theorem A and a result generously provided by Premet, presented in the Appendix. The converse of Theorem B is of course true; in characteristic $\neq 2, 3, 5$, the Killing form is nondegenerate.

In characteristic 5, Theorem B easily gives an apparently stronger statement, namely that $\mathrm{Lie}(G)$ has no quotient trace form, see Corollary 11.4. This answers a question posed in the early 1960s, see for example [Block 1962, p. 554], [Block and Zassenhaus 1964, p. 543], or [Seligman 1967, p. 48].

From the point of view of Lie algebras, this paper addresses the existence of restricted representations with nonzero or nondegenerate trace forms on Lie algebras of almost simple algebraic groups. These algebras are approximately the simple Lie algebras of classical type. For fields of characteristic ≥ 5 and simple Lie algebras of other types (necessarily Cartan or Melikian by Block–Premet–Strade–Wilson, see [Strade 2004] or [Mathieu 2000], every representation has zero trace form by Block [1962, Corollary 3.1].

Notation. All algebraic groups discussed here are linear. Such a group G over a field F is *almost simple* if it is semisimple and has no proper connected, closed, normal subgroups defined over F. In case F is separably closed, the almost simple algebraic groups are the semisimple groups whose Dynkin diagrams are connected.

PSO_n denotes the adjoint group of the (split) special orthogonal group SO_n; when n is odd it is the same as SO_n. Similarly, PSp_{2n} is the adjoint group of type C_n; it can be viewed as $\mathrm{Sp}_{2n}/\boldsymbol{\mu}_2$. The groups SO_n, Spin_n, and PSO_n for $n = 3, 5$, and 6 are isogenous to SL_2, Sp_4, or SL_4 and appear in Table I in that alternative form. For $n \geq 3$, we write HSpin_{4n} for the nontrivial quotient of Spin_{4n} that is neither SO_{4n} nor adjoint.

1. The number $N(G)$ and the Dynkin index

1.1. Fix a simple root system R. We write P for its weight lattice and $\langle\ ,\ \rangle$ for the canonical pairing between P and its dual. Fix a long root $\alpha \in R$ and write $\alpha^\vee$ for the associated coroot. For each subset X of P that is invariant under the Weyl group, we put

$$N(X) := \tfrac{1}{2} \sum_{x \in X} \langle x, \alpha^\vee \rangle^2 \in \mathbb{Z}\big[\tfrac{1}{2}\big].$$

The number $N(X)$ *does not depend on the choice of* α because the long roots are conjugate under the Weyl group.

Furthermore, $N(X)$ *is an integer.* To see this, note that the reflection s in the hyperplane orthogonal to α satisfies $\langle sx, \alpha^\vee \rangle = \langle x, s\alpha^\vee \rangle = -\langle x, \alpha^\vee \rangle$, so in the definition of $N(X)$, the sum can be taken to run over those x satisfying $x \neq sx$. For such x, we have $\langle x, \alpha^\vee \rangle^2 + \langle sx, \alpha^\vee \rangle^2 = 2\langle x, \alpha^\vee \rangle^2$, proving the claim.

Example 1.2. The computations in [Springer and Steinberg 1970, pp. 180, 181] show that $N(R) = 2h^\vee$, where $h^\vee$ denotes the *dual Coxeter number* of R, which is defined as follows. Fix a set of simple roots Δ of R. Write $\tilde{\alpha}$ for the highest root;

the corresponding coroot $\tilde{\alpha}^\vee$ is

$$\tilde{\alpha}^\vee = \sum_{\delta \in \Delta} m_\delta^\vee \delta^\vee$$

for some natural numbers $m_\delta^\vee$. The dual Coxeter number $h^\vee$ is defined by

$$h^\vee := 1 + \sum m_\delta^\vee.$$

In case all the roots of R have the same length, it is the (usual) Coxeter number h and is given in the tables in [Bourbaki 2002].

Suppose that there are two different root lengths in R; we write L for the set of long roots and S for the set of short roots. The arguments in [Springer and Steinberg 1970] are easily adapted to show that

$$N(L) = 2\left(1 + \sum_{\delta \in \Delta \cap L} m_\delta^\vee\right) \quad \text{and} \quad N(S) = 2 \sum_{\delta \in \Delta \cap S} m_\delta^\vee.$$

We obtain the following numbers:

type of R	h	$h^\vee$	$N(L)$	$N(S)$
B_n $(n \geq 2)$	$2n$	$2n-1$	$4(n-1)$	2
C_n $(n \geq 2)$	$2n$	$n+1$	4	$2(n-1)$
G_2	6	4	6	2
F_4	12	9	12	6

Definition 1.3. Fix a split almost simple linear algebraic group G over F. Fix also a pinning of G with respect to some maximal torus T; this includes a root system R and a set of simple roots Δ of G with respect to T. For a representation ρ of G over F, one defines

$$N(\rho) := \sum_{\substack{\text{dominant} \\ \text{weights } \lambda}} \binom{\text{multiplicity of } \lambda}{\text{as a weight of } \rho} \cdot N(W\lambda) \in \mathbb{Z}.$$

For example, the adjoint representation Ad has $N(\mathrm{Ad}) = 2h^\vee$ by Example 1.2. The number $N(\rho)$ is the *Dynkin index* of the representation ρ defined in [Dynkin 1952, p. 130] and studied in [Merkurjev 2003]. The Dynkin index of the fundamental irreducible representations of G (over $\mathbb{C}$) are listed in [Laszlo and Sorger 1997, Proposition 2.6] or [McKay et al. 1990, pp. 36–44], correcting some small errors in Dynkin's calculations.

We put

$$N(G) := \gcd N(\rho), \tag{1.4}$$

where the gcd runs over the representations of G defined over F. Because the map $\rho \mapsto N(\rho)$ depends only on the weights of ρ with multiplicity, it is compatible with short exact sequences

$$0 \longrightarrow \rho' \longrightarrow \rho \longrightarrow \rho/\rho' \longrightarrow 0 \tag{1.5}$$

in the sense that

$$N(\rho) = N(\rho') + N(\rho/\rho').$$

Writing RG for the representation ring of G, we obtain a homomorphism of abelian groups $N\colon RG \to \mathbb{Z}$ with image $N(G)\cdot\mathbb{Z}$.

In the definition of $N(G)$, it suffices to let the gcd run over generators of RG, for example, the irreducible representations of G. For an irreducible representation ρ, the highest weight λ has multiplicity 1 and all the other weights of ρ are lower in the partial ordering. Inducting on the partial ordering, we find

$$N(G) = \gcd\bigl\{N(W\lambda) \mid \lambda \in T^*\bigr\}.$$

In particular, $N(G)$ depends only on the root system R and the lattice T^*, and not on the field F.

Example 1.6. When G is simply connected, the number $N(G)$ is known as the *Dynkin index of* G and its value is listed in [Laszlo and Sorger 1997], for example. Examining the list of values, one finds that the primes dividing $N(G)$ (for G simply connected) are the torsion primes of G.

Example 1.7. Write Spin_n and SO_n for the spin and special orthogonal groups of an n-dimensional nondegenerate quadratic form of maximal Witt index. For $n \geq 7$, these groups are split and almost simple of type B_l (with $l \geq 3$) or D_l (with $l \geq 4$). The Dynkin index $N(\mathrm{Spin}_n)$ is 2; it obviously divides $N(\mathrm{SO}_n)$. On the other hand, the natural n-dimensional representation ρ of SO_n has $N(\rho) = 2$, so $N(\mathrm{SO}_n) = 2$.

Example 1.8. We claim that

$$N(\mathrm{PSp}_{2n}) = \begin{cases} 2 & \text{if } n \text{ is even,} \\ 4 & \text{if } n \text{ is odd,} \end{cases}$$

for $n \geq 2$. The number $N(\mathrm{PSp}_{2n})$ divides 4 and $2(n-1)$ by Example 1.2. Further, $N(\mathrm{PSp}_{2n})$ is even by [Merkurjev 2003, 14.2]. This shows that $N(\mathrm{PSp}_{2n})$ is 2 or 4, and is 2 in case n is even.

Suppose that n is odd. We must show that $N(W\lambda)$ is divisible by 4 for every element λ of the root lattice of PSp_{2n}. We use the same notation as [Merkurjev 2003, §14] for the weights of PSp_{2n}: they are a sum $\sum_{i=1}^{n} x_i e_i$ such that $\sum x_i$ is even. The Weyl group W is a semidirect product of $(\mathbb{Z}/2\mathbb{Z})^n$ (acting by flipping

the signs of the e_i) and the symmetric group on n letters (acting by permuting the e_i). Taking X for the $(\mathbb{Z}/2\mathbb{Z})^n$-orbit of $\sum x_i e_i$, we have

$$\tfrac{1}{2}\sum_{x\in X}\left\langle \sum\nolimits_i x_i e_i, (2e_n)^\vee\right\rangle^2 = 2^{r-1}x_n^2 \tag{1.9}$$

where r denotes the number of nonzero x_i's; see [Merkurjev 2003, proof of Lemma 14.2]. If $r = 1$, then the unique nonzero x_i is even, and we find that for $r \neq 2$, the sum — hence also $N(W\sum x_i e_i)$ — is divisible by 4. Suppose that x_1, x_2 are the only nonzero x_i; then by (1.9) we have

$$N(W(x_1e_1 + x_2e_2)) = \begin{cases} 2(n-1)(x_1^2 + x_2^2) & \text{if } x_1 \neq \pm x_2, \\ 2(n-1)x_1^2 & \text{if } x_1 = \pm x_2. \end{cases}$$

As n is odd, $N(W(x_1e_1 + x_2e_2))$ is divisible by 4, which completes the proof of the claim.

Example 1.10. *For G adjoint of type E_7, we have $N(G) = 12$.* To see this, we note that $N(G)$ is divisible by $N(\widetilde{G})$, where $\widetilde{G}$ is the universal covering of G, that is, 12 divides $N(G)$. Also, $N(G)$ divides $2h^\vee = 36$ by Example 1.2. For the minuscule representation ρ of $\widetilde{G}$, we have $\dim\rho = 56$ and $N(\rho) = 12$. The representation $\rho^{\otimes 2}$ of $\widetilde{G}$ factors through G and by the derivation formula

$$N(\rho_1 \otimes \rho_2) = (\dim\rho_1)\cdot N(\rho_2) + (\dim\rho_2)\cdot N(\rho_1)$$

(see for example [Merkurjev 2003, p. 122]) we have

$$N(\rho^{\otimes 2}) = 2(\dim\rho)N(\rho) = 2^6\cdot 3\cdot 7.$$

It follows that $N(G)$ equals 12, as claimed.

2. The Lie algebra of G

2.1. Let G be a split almost simple algebraic group over F; we fix a pinning for it. If $G_\mathbb{Z}$ is a split group over $\mathbb{Z}$ with the same root datum as G, the pinning identifies G with the group obtained from $G_\mathbb{Z}$ by the base change $\mathbb{Z} \to F$ and the maximal torus T in G (from the pinning) with the base change of a maximal torus $T_\mathbb{Z}$ in $G_\mathbb{Z}$. We have a root space decomposition of the Lie algebra of $G_\mathbb{Z}$

$$\mathrm{Lie}(G_\mathbb{Z}) = \mathrm{Lie}(T_\mathbb{Z}) \oplus \bigoplus_{\alpha\in R}\mathbb{Z}x_\alpha \tag{2.2}$$

and

$$\mathrm{Lie}(T_\mathbb{Z}) = \{h \in \mathrm{Lie}(T_\mathbb{C}) \mid \mu(h) \in \mathbb{Z} \text{ for all } \mu \in T^*\}, \tag{2.3}$$

see [Steinberg 1968, p. 64]. Because $\mathrm{Lie}(G_{\mathbb{Z}})$ is a free $\mathbb{Z}$-module, the Lie algebra $\mathrm{Lie}(G)$ of G is naturally identified with $\mathrm{Lie}(G_{\mathbb{Z}}) \otimes_{\mathbb{Z}} F$, and similarly for $\mathrm{Lie}(T)$; see [Demazure and Gabriel 1970, II.4.4.8].

2.4. Write $\widetilde{G}$ for the universal covering of G; we use the obvious analogues of the notations in 2.1 for $\widetilde{G}$. The group G acts on $\widetilde{G}$ by conjugation, hence also on $\mathrm{Lie}(\widetilde{G})$. If the kernel of the map $\widetilde{G} \to G$ is étale, then the representation $\mathrm{Lie}(\widetilde{G})$ is equivalent to the adjoint representation on $\mathrm{Lie}(G)$. In any case, the natural map $\mathrm{Lie}(\widetilde{G}) \to \mathrm{Lie}(G)$ is an isomorphism on the F-span of the x_α.

2.5. We claim that $\mathrm{Lie}(\widetilde{G})$ *is a Weyl module for* G in the sense of [Jantzen 2003, p. 183], that is, its character is given by Weyl's formula and it is generated as a G-module by a highest weight vector. The first condition holds by (2.2), so it suffices to check the second.

To check that the submodule $Gx_{\tilde{\alpha}}$ generated by the highest weight vector $x_{\tilde{\alpha}}$ is all of $\mathrm{Lie}(\widetilde{G})$, one quickly reduces to checking that $Gx_{\tilde{\alpha}}$ contains $\mathrm{Lie}(\widetilde{T}_{\mathbb{Z}})$. Equation (2.3) gives a natural isomorphism $\mathbb{Z}[R^\vee] \xrightarrow{\sim} \mathrm{Lie}(\widetilde{T}_{\mathbb{Z}})$ where $\widetilde{T}_{\mathbb{Z}}$ is the maximal torus in $\widetilde{G}_{\mathbb{Z}}$ mapping onto $T_{\mathbb{Z}}$. We write (as is usual) h_α for the image of $\alpha^\vee$ under this map. As $[x_\alpha, x_{-\alpha}] = h_\alpha$, the claim is proved.

2.6. See [Hiss 1984], [Hogeweij 1982, especially Corollary 2.7a], or [Steinberg 1961, §2] for descriptions of the composition series of $\mathrm{Lie}(\widetilde{G})$. They immediately give: *If the characteristic of* F *is very good for* G*, then* $\mathrm{Lie}(G)$ *is a simple Lie algebra. If additionally* F *is infinite then* $\mathrm{Lie}(G)$ *is an irreducible* $G(F)$*-module.*

3. The number $E(G)$

Definition 3.1. Maintain the notation of the preceding section. The Killing form on $\mathrm{Lie}(\widetilde{G}_{\mathbb{Z}})$ is divisible by $2h^\vee$ [Gross and Nebe 2004] and dividing by $2h^\vee$ gives an indivisible even symmetric bilinear form $\widetilde{b}$ on $\mathrm{Lie}(\widetilde{G}_{\mathbb{Z}})$ such that

$$\widetilde{b}(h_\alpha, h_\alpha) = 2 \quad \text{and} \quad \widetilde{b}(x_\alpha, x_{-\alpha}) = 1 \tag{3.2}$$

for long roots α; see [Springer and Steinberg 1970, p. 181] or [Bourbaki 2002, Lemma VIII.2.4.3]. For a short root β, we have: $\widetilde{b}(h_\beta, h_\beta) = 2c$ and $\widetilde{b}(x_\beta, x_{-\beta}) = c$, where c is the square-length ratio of α to β. For example, $G = \mathrm{SL}_n$ has Lie algebra the trace zero n-by-n matrices, and the form $\widetilde{b}$ is the usual trace bilinear form $(x, y) \mapsto \mathrm{trace}(xy)$; see [Bourbaki 2002, Exercise VIII.13.12].

The natural map $\mathrm{Lie}(\widetilde{G}_{\mathbb{Z}}) \to \mathrm{Lie}(G_{\mathbb{Z}})$ is an inclusion and extending scalars to $\mathbb{Q}$ gives an isomorphism. Therefore, $\widetilde{b}$ gives a rational-valued symmetric bilinear form on $\mathrm{Lie}(G_{\mathbb{Z}})$. We define $E(G)$ to be the smallest positive rational number such that $E(G) \cdot \widetilde{b}$ is integer-valued on $\mathrm{Lie}(G_{\mathbb{Z}})$; we write b for this form. Note that $E(G)$ is an integer by (3.2).

Clearly, $E(G)$ depends only on the root system of G and the character lattice T^* viewed as a sublattice of the weight lattice, and not on the field F.

3.3. Write $\bar{G}$ for the adjoint group of G; we use the obvious analogues of the notations in 2.1 for $\bar{G}$. We have a commutative diagram

$$\begin{array}{ccc} Q^\vee & \xrightarrow{\sim} & \mathrm{Lie}(\widetilde{T}_{\mathbb{Z}}) \\ \downarrow & & \downarrow \\ P^\vee & \xrightarrow{\sim} & \mathrm{Lie}(\bar{T}_{\mathbb{Z}}) \end{array}$$

where $Q^\vee$ and $P^\vee$ are the root and weight lattices of the dual root system. The form $\widetilde{b}$ restricts to be an inner product on $Q^\vee$ such that the square-length of a short coroot $\alpha^\vee$ is 2. This inner product extends to a rational-valued inner product on $P^\vee$, and $E(\bar{G})$ is the smallest positive integer such that $E(\bar{G}) \cdot \widetilde{b}$ is integer-valued on $P^\vee$.

Example 3.4. Consider the case where G is PSp_{2n} for some $n \geq 2$, that is, adjoint of type C_n. In the notation of the tables in [Bourbaki 2002], the form $\widetilde{b}$ is twice the usual scalar product, $\widetilde{b}(e_i, e_j) = 2\delta_{ij}$ (Kronecker delta). The fundamental weight ω_n has $\widetilde{b}(\omega_n, \omega_n) = n/2$. Checking $\widetilde{b}(\omega_i, \omega_j)$ for all i, j, shows that $E(\bar{G})$ is 1 if n is even and 2 if n is odd.

Example 3.5. Suppose that all the roots of G have the same length, so that we may identify the root system R with its dual and normalize lengths so that $\langle\,,\rangle$ is symmetric and equals $\widetilde{b}$ on $Q^\vee$.

(1) $E(\bar{G})$ *is the exponent of* P/Q, the weight lattice modulo the root lattice. Indeed, the isomorphism between P and $\mathrm{Lie}(\bar{T}_{\mathbb{Z}})$ shows that $E(\bar{G})$ is the smallest natural number such that $E(\bar{G}) \cdot \langle\,,\rangle$ is integer-valued on $P \times P$, equivalently, the smallest natural number e such that eP is contained in Q; this is the exponent of P/Q.

(2) The bilinear form

$$\widetilde{b}\colon \mathrm{Lie}(\widetilde{G}_{\mathbb{Z}}) \times \mathrm{Lie}(\bar{G}_{\mathbb{Z}}) \to \mathbb{Q}$$

has image $\mathbb{Z}$ and identifies $\mathrm{Lie}(\bar{G}_{\mathbb{Z}})$ with $\mathrm{Hom}_{\mathbb{Z}}(\mathrm{Lie}(\widetilde{G}_{\mathbb{Z}}), \mathbb{Z})$. (On the span of the x_α, this is clear from (3.2). On the Cartan subalgebras, it amounts to the statement that $\langle\,,\rangle$ identifies P with $\mathrm{Hom}(Q, \mathbb{Z})$.) It follows that $\mathrm{Lie}(\bar{G})$, as a G-module, is the dual of $\mathrm{Lie}(\widetilde{G})$, that is, $\mathrm{Lie}(\bar{G})$ is the module denoted by $H^0(\tilde{\alpha})$ in [Jantzen 2003].

Example 3.6. For $n = 3$ or $n \geq 5$, we claim that $E(\mathrm{SO}_n) = 1$.

For n odd, SO_n is adjoint of type B_l for $l = (n-1)/2$, and we compute as in 3.3 and Example 3.4. The dual root system is of type C_l, and the form $\widetilde{b}$ is

the usual scalar product, that is, $\widetilde{b}(e_i, e_j) = \delta_{ij}$. The fundamental weight ω_i is $e_1 + e_2 + \cdots + e_i$, so $E(\mathrm{SO}_{2l+1}) = 1$.

For n even, SO_n has type D_l for $l = n/2$. The character group T^* of a maximal torus in SO_n consists of the weights whose restriction to the center of Spin_n is 0 or agrees with the vector representation, that is, the weights $\sum c_i\omega_i$ such that $c_{l-1}+c_l$ is even. It follows that the cocharacter lattice T_* is generated by the (co)root lattice and

$$\omega_1 = \alpha_1 + \alpha_2 + \cdots + \alpha_{l-2} + \tfrac{1}{2}(\alpha_{l-1} + \alpha_l).$$

We have

$$\widetilde{b}(\omega_1, \omega_1) = \langle \omega_1, \omega_1 \rangle = 1,$$

so the form $\widetilde{b}$ is integer-valued on T_* and $E(\mathrm{SO}_{2l}) = 1$.

Example 3.7. Let HSpin_{4n} denote the image of Spin_{4n} under the irreducible representation with highest weight ω_l for $l := 2n$; it is a half-spin group. The character lattice T^* consists of weights $\sum c_i\omega_i$ such that $c_1 + c_{l-1}$ is even. The lattice generated by Q and

$$\omega_1 + \omega_{l-1} = \tfrac{1}{2}(3\alpha_1 + 4\alpha_2 + \cdots + l\alpha_{l-2}) + \tfrac{1}{4}(l+2)\alpha_{l-1} + \tfrac{1}{4}l\alpha_l$$

contains Q with index 2 and is contained in T_*, hence equals T_*. Since

$$\widetilde{b}(\omega_1 + \omega_{l-1}, \omega_1 + \omega_{l-1}) = \langle \omega_1 + \omega_{l-1}, \omega_1 + \omega_{l-1} \rangle = \tfrac{3}{2} + \tfrac{1}{4}(l+2) = \tfrac{1}{2}n + 2,$$

we conclude that

$$E(\mathrm{HSpin}_{4n}) = \begin{cases} 1 & \text{if } n \text{ is even,} \\ 2 & \text{if } n \text{ is odd.} \end{cases}$$

4. Formula for the trace

The integer-valued symmetric bilinear form b on $\mathrm{Lie}(G_{\mathbb{Z}})$ defined in 3.1 gives by scalar extension a symmetric bilinear form on $\mathrm{Lie}(G)$ which we denote by b_F.

Proposition 4.1. *Let ρ be a representation of a split and almost simple algebraic group G over F.*

(1) *$E(G)$ divides $N(\rho)$.*

(2) $\mathrm{Tr}_\rho = \dfrac{N(\rho)}{E(G)} b_F.$

Proof. We first suppose that F is the complex numbers. Write $\pi : \widetilde{G} \to G$ for the universal covering of G as in Section 2. We compute $\mathrm{Tr}_{\rho\pi}$. If we decompose the representation ρ with respect to the action of $\widetilde{T}$ and write V_μ for the eigenspace relative to the weight μ, then h_α acts on V_μ by scalar multiplication by $\langle \mu, \alpha^\vee \rangle$,

hence $\mathrm{Tr}_\rho(h_\alpha, h_\alpha) = \sum \dim(V_\mu)\langle \mu, \alpha^\vee\rangle^2$. By putting together the μ in an orbit $W\lambda$ (where λ is dominant) and taking α to be a long root, one gets

$$\mathrm{Tr}_{\rho\pi}(h_\alpha, h_\alpha) = 2\,N(\rho). \tag{4.2}$$

The representation $\mathrm{Lie}(\widetilde{G}_\mathbb{Z})\otimes\mathbb{C}$ is irreducible and has a nondegenerate $\widetilde{G}_\mathbb{C}$-invariant symmetric bilinear form, so by Schur's Lemma we have

$$\mathrm{Hom}_{\widetilde{G}_\mathbb{C}}\big(\mathrm{Lie}(\widetilde{G})\otimes\mathbb{C}, (\mathrm{Lie}(\widetilde{G})^*)\otimes\mathbb{C}\big) = \mathbb{C}.$$

In particular, $\mathrm{Tr}_{\rho\pi}$ equals $z\,\widetilde{b}$ for some complex number z and

$$2N(\rho) = \mathrm{Tr}_{\rho\pi}(h_\alpha, h_\alpha) = z\,\widetilde{b}(h_\alpha, h_\alpha) = 2z.$$

Hence $\mathrm{Tr}_{\rho\pi} = N(\rho)\,\widetilde{b}$. (This argument can be viewed as restating pages 130–131 of [Dynkin 1952].) The Lie algebra $\mathrm{Lie}(G_\mathbb{Z})\otimes\mathbb{C}$ is naturally identified with $\mathrm{Lie}(\widetilde{G}_\mathbb{Z})\otimes\mathbb{C}$, and (2) follows from the equation $E(G)\,\widetilde{b} = b$ in the case $F = \mathbb{C}$.

Now allow F to be arbitrary but suppose that ρ is a Weyl module. There is a $\mathbb{Z}$-form $\rho_\mathbb{Z}$ of ρ, and the form $\mathrm{Tr}_{\rho_\mathbb{Z}}$ is the restriction of $\mathrm{Tr}_{\rho_\mathbb{C}}$ on $\mathrm{Lie}(G_\mathbb{C})$ to $\mathrm{Lie}(G_\mathbb{Z})$. Because (2) holds over the complex numbers, it holds over the integers, and by scalar extension it holds over the field F as well. Clearly the form $\mathrm{Tr}_{\rho_\mathbb{Z}}$ is integer valued; as b is indivisible, it follows that $E(G)$ divides $N(\rho)$.

We now treat the case of an arbitrary representation ρ. The number $N(\rho)$ depends only on the class of ρ in the representation ring RG. As the Weyl modules generate RG as an abelian group and $E(G)$ divides $N(\psi)$ for every Weyl module ψ, (1) follows.

For (2), we note that the map $\rho \mapsto \mathrm{Tr}_\rho - \big(N(\rho)/E(G)\big)\,b_F$ is compatible with exact sequences like (1.5) in the sense that $\mathrm{Tr}_\rho = \mathrm{Tr}_{\rho'} + \mathrm{Tr}_{\rho/\rho'}$. We obtain a homomorphism of abelian groups

$$RG \to \boxed{\text{symmetric bilinear forms on } \mathrm{Lie}(G)}$$

that vanishes on the Weyl modules, hence is zero. □

4.3. Because b is indivisible (as a form over $\mathbb{Z}$), the form b_F is not zero. Part (2) of Proposition 4.1 immediately gives:

Fact 4.4. *Let ρ be a representation of a split and almost simple algebraic group G over F. Then Tr_ρ is zero if and only if the characteristic of F divides $N(\rho)/E(G)$.*

Furthermore, we defined $N(G)$ to be $\gcd N(\rho)$ as ρ varies over the representations of G. We have proved:

Fact 4.5. *Let G be split and almost simple. The trace Tr_ρ is zero for every representation ρ of G if and only if the characteristic of F divides the integer $N(G)/E(G)$.*

We have now finished half of the proof of Theorem A(2); it remains to check that the primes dividing $N(G)/E(G)$ are the primes in the rightmost column of Table I.

5. The ratio $N(G)/E(G)$ for $G = \mathrm{SL}_n/\boldsymbol{\mu}_m$

In this section, we fix natural numbers m and n with m dividing n, and we prove

Proposition 5.1. *For $G = \mathrm{SL}_n/\boldsymbol{\mu}_m$, the primes dividing $N(G)/E(G)$ are precisely the primes dividing*

$$\begin{cases} \gcd(m, n/m) & \text{if } m \text{ is odd,} \\ 2\gcd(m, n/m) & \text{if } m \text{ is even.} \end{cases}$$

Here $\boldsymbol{\mu}_m$ denotes the group scheme of m-th roots of unity, identified with the corresponding scalar matrices in SL_n.

In the important special cases where G is simply connected ($m = 1$), G is adjoint ($m = n$), or n is square-free, the gcd in the proposition is 1, and we have that $N(G)/E(G)$ *is* 1 *if m is odd and* 2 *if m is even.*

Lemma 5.2.

$$E(\mathrm{SL}_n/\boldsymbol{\mu}_m) = \frac{m}{\gcd(m, n/m)}\,.$$

Proof. Use the notation from [Bourbaki 2002] for the simple roots and fundamental weights of the root system A_{n-1} of SL_n. Let Λ denote the lattice generated by the root lattice Q and

$$\beta := \frac{n}{m}\,\omega_{n-1} = \frac{1}{m}\,(\alpha_1 + 2\alpha_2 + \cdots + (n-1)\alpha_{n-1})\,.$$

We claim that Λ is identified with the cocharacter lattice T_* for a pinning of $\mathrm{SL}_n/\boldsymbol{\mu}_m$. Certainly, Λ/Q is cyclic of order m, so it suffices to check that the set of inner products $\langle \Lambda, T^* \rangle$ consists of integers. But T^* is the collection of weights $\sum c_i\omega_i$ with $c_i \in \mathbb{Z}$ such that $\sum_{i=1}^{n-1} i c_i$ is divisible by m. We have

$$\Big\langle \beta, \sum c_i\omega_i \Big\rangle = \sum_i \frac{1}{m} i c_i,$$

which is an integer when $\sum c_i\omega_i$ is in T^*, so $T_* = \Lambda$ as claimed.

Finally, we compute

$$\langle \beta, \alpha_{n-1} \rangle = \frac{n}{m} \in \mathbb{Z} \quad \text{and} \quad \langle \beta, \beta \rangle = \Big\langle \frac{1}{m} \sum i\alpha_i, \frac{n}{m}\omega_{n-1} \Big\rangle = \frac{n(n-1)}{m^2}.$$

Since m divides n, it is relatively prime to $n-1$, so the minimum multiplier of $\langle\,,\,\rangle$ that takes integer values on T_* is $m/\gcd(m, n/m)$, as claimed. □

5.3. Weights of representations of $\mathrm{SL}_n/\boldsymbol{\mu}_m$. Fix the usual pinning of SL_n, where the torus T consists of diagonal matrices and the dominant weights are the maps

$$\begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} \mapsto \prod_{i=1}^{n-1} t_i^{e_i},$$

where $e_1 \geq e_2 \geq \cdots \geq e_{n-1} \geq 0$. Such a weight restricts to $x \mapsto x^{\sum e_i}$ on the center of SL_n; in particular, m divides $\sum e_i$ for every dominant weight λ of a representation of $\mathrm{SL}_n/\boldsymbol{\mu}_m$. The proof of [Merkurjev 2003, Lemma 11.4] shows that m divides $N(W\lambda)$, hence *m divides $N(\mathrm{SL}_n/\boldsymbol{\mu}_m)$*.

5.4. We recall how to compute $N(W\lambda)$ from [Merkurjev 2003, p. 136]. Write $a_1 > a_2 > \cdots > a_{k-1} > a_k = 0$ for the distinct values of the exponents e_i in λ, where a_i appears r_i times, so that $n = \sum r_i$. We have

$$N(W\lambda) = \frac{(n-2)!}{r_1!\, r_2! \cdots r_k!}\left[n\Big(\sum_i r_i a_i^2\Big) - \Big(\sum_i r_i a_i\Big)^2\right]. \tag{5.5}$$

The dominant weight λ with $e_1 = m$ and $e_i = 0$ for $i > 1$ vanishes on $\boldsymbol{\mu}_m$ and has $N(W\lambda) = m^2$ by (5.5), so *$N(\mathrm{SL}_n/\boldsymbol{\mu}_m)$ divides m^2*.

Example 5.6. Let λ be a dominant weight of G and let r_i, a_i be as in the preceding paragraph. Suppose that

$$v_2\Big(\sum r_i a_i\Big) \geq v_2(n) > 0,$$

where $v_2(x)$ is the 2-adic valuation of x, that is, the exponent of the largest power of 2 dividing x. We claim that

$$v_2(N(W\lambda)) > v_2(n). \tag{5.7}$$

Write $\sum r_i a_i = 2^\theta t$ and $n = 2^\nu u$ where $\theta = v_2(\sum r_i a_i)$ and $\nu = v_2(n)$. Our hypothesis is that $0 < \nu \leq \theta$. We rewrite (5.5) as

$$N(W\lambda) = \frac{(n-2)!}{r_1!\, r_2! \cdots r_k!}\left[u\Big(\sum_i r_i a_i^2\Big) - 2^{2\theta-\nu} t^2\right] \cdot 2^\nu. \tag{5.8}$$

Write l for the minimum of $v_2(r_i)$, and fix an index j such that $v_2(r_j) = l$. Note that since $\sum r_i = n$, we have $l \leq \nu \leq 2\theta - \nu$.

The first term on the right side of (5.8) has 2-adic value $\geq -l$ [Merkurjev 2003, p. 137]. The term in brackets has value $\geq l$. Therefore, to prove claim (5.7), it suffices to consider the case where $v_2\big(\sum r_i a_i^2\big) = l$ and the first term on the right side of (5.8) has value $-l$; this latter condition implies that

$$s_2(n-1) = s_2(r_1) + \cdots + s_2(r_{j-1}) + s_2(r_j - 1) + s_2(r_{j+1}) + \cdots + s_2(r_k), \tag{5.9}$$

where s_2 denotes the number of 1's appearing in the binary representation of the integer [Merkurjev 2003, p. 137]. That is, when adding up the numbers $r_1, \ldots, r_{j-1}$, $r_j - 1, r_{j+1}, \ldots, r_k$ in base 2 (to get $n-1$), there are no carries. We check that this is impossible.

Suppose first that $l < v$. Equation (5.9) implies that there are exactly two indices, say, j, j' with $v_2(r_j) = v_2(r_{j'}) = l$. As 2^{l+1} divides $\sum r_i a_i$, it also divides $r_j a_j + r_{j'} a_{j'}$, hence a_j and $a_{j'}$ have the same parity. It follows that 2^{l+1} divides $r_j a_j^2 + r_{j'} a_{j'}^2$, contradicting the hypothesis that $v_2\left(\sum r_i a_i^2\right) = l$.

We are left with the case where $l = v$. By (5.9), r_j is the unique r_i with 2-adic valuation l. As $v_2\left(\sum r_i a_i^2\right) = l$, the number a_j is odd and we have $l = v_2\left(\sum r_i a_i\right) = \theta$. Hence both $u \cdot \left(\sum r_i a_i^2\right)$ and $2^{2\theta - v} t$ have 2-adic valuation l. It follows that the term in brackets in (5.8) has 2-adic valuation strictly greater than l, and claim (5.7) is proved.

Proof of Proposition 5.1. We write G for $\mathrm{SL}_n/\boldsymbol{\mu}_m$. Paragraphs 5.3 and 5.4 give the bounds: m divides $N(G)$ divides m^2. Also, $N(G)$ divides $2n$ by Example 1.2. Applying Lemma 5.2 shows that $\gcd(m, n/m)$ divides $N(G)/E(G)$, which in turn divides $\gcd(m, n/m)\gcd(m, 2n/m)$. This completes the proof for m odd.

Clearly, an odd prime divides $N(G)/E(G)$ if and only if it divides $\gcd(m, n/m)$. So suppose that m is even and 2 does not divide $\gcd(m, n/m)$, that is, $v_2(m) = v_2(n)$. Then every dominant weight of a representation of G satisfies the hypotheses of Example 5.6, hence $v_2(N(G)) > v_2(n) = v_2(m)$. By Lemma 5.2, $v_2(E(G)) = v_2(m)$, so 2 divides $N(G)/E(G)$. This completes the proof. □

6. Conclusion of proof of Theorem A(2)

For a split and almost simple algebraic group G, we now verify that the primes dividing $N(G)/E(G)$ are those in the last column of Table I. Together with 4.5, this will prove Theorem A(2).

For G simply connected, $E(G)$ is 1 and $N(G)$ is divisible precisely by the torsion primes of G, see 1.6. We assume that G is not simply connected and write $\widetilde{G}$ for the universal covering of G; obviously $N(\widetilde{G})$ divides $N(G)$.

For $G = \mathrm{PSp}_{2n}$, SO_n, or adjoint of type E_7, one combines Examples 1.8 and 3.4, 1.7 and 3.6, or 1.10 and 3.5, respectively.

For G adjoint of type D_n, we have $E(G) = 2$ by Example 3.5. Also, 4 divides $N(G)$ by [Merkurjev 2003, 15.2]. On the other hand, the spinor representations of $\widetilde{G}$ have Dynkin index 2^{n-3} [Laszlo and Sorger 1997], and it is easy to use this as in Example 1.10 to construct a representation ρ of G with $N(\rho)$ a power of 2. This shows that $N(G)/E(G)$ is a power of 2 and is not 1.

Now let $G = \mathrm{HSpin}_{4n}$ for some $n \geq 3$. The dual of the center of Spin_{4n} is the Klein four-group, and we write χ for the unique nonzero element that vanishes

on the kernel of the map $\mathrm{Spin}_{4n} \to \mathrm{HSpin}_{4n}$. The gcd of $N(W\lambda)$ as λ varies over the weights that restrict to χ (respectively, 0) on the center of Spin_{4n} is 2^{2n-3} (respectively, divisible by 4) by [Merkurjev 2003, p. 146], hence $N(G)$ is a power of 2 and at least 4. On the other hand, $E(\mathrm{HSpin}_{4n})$ is 1 or 2. We conclude that $N(G)/E(G)$ is a power of 2 and is not 1.

For G adjoint of type E_6, the number $N(G)$ is divisible by $N(\widetilde{G}) = 6$ and divides $2h^\vee = 24$ by Example 1.2. By Example 3.5, $N(G)/E(G)$ is 2, 4, or 8. This completes the proof of Theorem A(2). □

Example 6.1. Suppose that the characteristic of F is an odd prime p, and let n be a natural number divisible by p^2. Every trace form of $\mathrm{SL}_n/\boldsymbol{\mu}_p$ is zero by Theorem A(2), even though the universal covering SL_n and adjoint group PGL_n have representations with nonzero trace forms.

7. Proof of Theorem A(1)

We now prove Theorem A(1); we show that the following three statements are equivalent:

(a) *The characteristic of F is very good for G.*

(b) $\mathrm{Lie}(G)$ *is a simple algebra and there is a representation ρ of G with* Tr_ρ *nonzero.*

(c) *There is a representation ρ of G with* Tr_ρ *nondegenerate.*

Suppose (a) holds. Then $\mathrm{Lie}(G)$ is simple, as in 2.6. The existence of a representation ρ with nonzero trace follows from Theorem A(2), so (b) holds. It is easy to check that for a representation ψ of $\mathrm{Lie}(G)$, $\mathrm{Tr}_\psi([x, y], z) = \mathrm{Tr}_\psi(x, [y, z])$ for all $x, y, z \in \mathrm{Lie}(G)$. So the radical of a trace form on $\mathrm{Lie}(G)$ is an ideal, and (b) implies (c).

Now suppose that (a) fails; we check that (c) also fails. By Theorem A(2), we only need to consider those cases where the characteristic of F appears in the middle column of Table I and not in the right column, namely the cases:

(i) G has type G_2 and $\mathrm{char}F = 3$, or G is Sp_{2n} and $\mathrm{char}F = 2$, or G is $\mathrm{SL}_n/\boldsymbol{\mu}_m$ and $\mathrm{char}F$ is odd and divides n/m but not m.

(ii) G is adjoint of type E_6 and $\mathrm{char}F = 3$, or G is $\mathrm{SL}_n/\boldsymbol{\mu}_m$ and $\mathrm{char}F$ divides m but not n/m.

We write $\pi : \widetilde{G} \to G$ for the universal covering of G. In case (i), the kernel of π is étale, so $\mathrm{Lie}(G)$ is a Weyl module by 2.5. For all three of the types listed, $\mathrm{Lie}(G)$ has a nontrivial submodule M, namely the subalgebra generated by the short roots (for G_2) or the center (in the other two cases). It follows that M is contained in the radical of Tr_ρ — see [Garibaldi 2008, 6.2], for example — hence (c) fails.

In case (ii), every representation ρ of G gives a representation $\rho\pi$ of $\widetilde{G}$ whose trace form $\mathrm{Tr}_{\rho\pi}$ vanishes on $\mathrm{Lie}(\widetilde{G})$ by Theorem A(2) (for E_6) or 5.3 (for SL_n). Hence the image of $\mathrm{d}\pi$ is a totally isotropic subspace for Tr_ρ. As

$$\dim(\operatorname{im}\mathrm{d}\pi) = \dim\widetilde{G} - \dim(\ker\mathrm{d}\pi) = \dim G - 1$$

is strictly greater than half the dimension of G, the form Tr_ρ is degenerate and (c) fails. This concludes the proof of Theorem A(1). □

8. Richardson's condition

In the literature, the weak version of the "if" direction of Theorem A(1) is used to deduce Richardson's condition from [Richardson 1967, p. 3]. Our slightly finer version of the "if" direction gives a slightly finer version of Richardson's condition; we state it here for the convenience of the reader. As in Theorem A, G is a split almost simple algebraic group over a field F.

Proposition 8.1. *If the characteristic of F is very good for G, then there is a representation $\rho\colon G \to \mathrm{GL}(V)$ such that $\mathrm{d}\rho$ is an injection $\mathrm{Lie}(G) \hookrightarrow \mathfrak{gl}(V)$ and there is a subspace M of V such that $\mathfrak{gl}(V) = \mathrm{d}\rho(\mathrm{Lie}(G)) \oplus M$, Id_V is in M, and $\mathrm{Ad}(\rho(G))M \subseteq M$.*

Proof. Theorem A(1) gives a representation ρ so that Tr_ρ is nondegenerate. In particular, the restriction of the symmetic bilinear form $(x, y) \mapsto \mathrm{trace}(xy)$ on $\mathfrak{gl}(V)$ to $\mathrm{d}\rho(\mathrm{Lie}(G))$ is nondegenerate. (And obviously $\mathrm{d}\rho$ must be injective.)

Take M to be the space of $x \in \mathfrak{gl}(V)$ such that $\mathrm{trace}(\mathrm{d}\rho(\mathrm{Lie}(G))x) = 0$. Trivially, M is invariant under $\mathrm{Ad}(\rho(G))$. Nondegeneracy of Tr_ρ shows that M meets $\mathrm{d}\rho(\mathrm{Lie}(G))$ only at 0, and dimension count shows that $V = \mathrm{d}\rho(\mathrm{Lie}(G)) \oplus M$. As G is semisimple, the image $\rho(G)$ is contained in $\mathrm{SL}(V)$, hence $\mathrm{d}\rho(\mathrm{Lie}(G))$ lies in $\mathfrak{sl}(V)$, that is, Id_V belongs to M. □

The proposition is essentially known, but the usual argument as in [Richardson 1967, §5], [Jantzen 2004, 2.6], [Humphreys 1995, p. 48], or [Springer and Steinberg 1970, p. 184] is different. For example, the usual approach to treating an adjoint group G of type C_n or D_n replaces G with its covering $G' = \mathrm{Sp}_{2n}$ or SO_{2n} and then gives a representation of G' with the desired properties.

9. Complements: Characteristic 2

In characteristic 2, one might prefer to consider, instead of the symmetric bilinear form Tr_ρ, the quadratic form

$$s_\rho : x \mapsto -\operatorname{trace}\bigl(\textstyle\bigwedge^2 \mathrm{d}\rho(x)\bigr),$$

which gives the negative of the degree 2 coefficient of the characteristic polynomial of $\mathrm{d}\rho(x)$. (Because $\mathrm{d}\rho(\mathrm{Lie}(G))$ consists of trace zero matrices, s_ρ is the map $x \mapsto \mathrm{trace}(\mathrm{d}\rho(x)^2)/2$; our definition has the advantage that it obviously makes sense in characteristic 2 as well.) The bilinear form derived from s_ρ, that is, $(x, y) \mapsto s_\rho(x+y) - s_\rho(x) - s_\rho(y)$, is Tr_ρ.

Theorem A(2) is easy to extend. In case G is simply connected, $\mathrm{Lie}(G)$ is a Weyl module by 2.5 and s_ρ is zero if and only if Tr_ρ is zero by [Garibaldi 2008, Proposition 6.4(1)]. That is, the conditions in Theorem A(2) are equivalent to: *For every representation ρ of G, the quadratic form s_ρ is zero.* (This is true in all characteristics but is only nontrivial in characteristic 2.)

Alternatively, one can proceed as follows. The bilinear form $\widetilde{b}$ on $\mathrm{Lie}(\widetilde{G}_\mathbb{Z})$ is even [Gross and Nebe 2004, Proposition 4], so it is the bilinear form derived from a unique quadratic form $\widetilde{q}$ on $\mathrm{Lie}(\widetilde{G}_\mathbb{Z})$. The form $\widetilde{q}$ extends to a rational-valued quadratic form on $\mathrm{Lie}(G_\mathbb{Z})$ and we write $E_q(G)$ for the smallest positive rational number such that $E_q(G)\,\widetilde{q}$ is integer-valued on $\mathrm{Lie}(G_\mathbb{Z})$. The number $E_q(G)$ is $E(G)$ or $2E(G)$, and both cases can occur. (For example, take $G = \widetilde{G}$ or SO_{2l}, respectively.) The statements and proofs of 4.4 and 4.5 go through if we replace Tr_ρ, $E(G)$, and b with s_ρ, $E_q(G)$, and $E_q(G)\,\widetilde{q}$ respectively.

10. Complements: Nonsplit groups

We can extend our results above to the case where G is not split, that is, we can replace the hypotheses "G is split and almost simple" with "G is absolutely almost simple". Indeed, suppose that G is absolutely almost simple over F, that is, there is a split and almost simple group G' over F and an isomorphism $f : G' \to G$ defined over a separable closure F_{sep} of F. Fix a pinning for G' and write b' for the indivisible bilinear form on $\mathrm{Lie}(G'_\mathbb{Z})$ defined in 3.1. Clearly, the automorphism group of G', which is an affine group scheme over $\mathbb{Z}$, leaves b' invariant, so it maps into the orthogonal group of b'. Galois descent (via f) gives a G-invariant symmetric bilinear form b_F on $\mathrm{Lie}(G)$ such that the differential $\mathrm{d}f$ identifies $b'_F \otimes F_{\mathrm{sep}}$ with $b_F \otimes F_{\mathrm{sep}}$.

Given a representation ρ of G over F, we get a representation ρf of G' over F_{sep} and an integer $N(\rho f)$ defined in 1.3; put $N(\rho) := N(\rho f)$. (In the special case where G is split over F, this agrees with our previous definition.) We define $N(G)$ as in (1.4); it is the gcd of $N(\rho)$ as ρ varies over the representations of G defined over F. Obviously, $N(G)$ is divisible by $N(G_{F_{\mathrm{sep}}})$, that is, $N(G')$, and it depends on the field F.

We put $E(G) := E(G')$. It does not depend on the field F.

With these definitions for $N(G)$ and $E(G)$, conclusions (1) and (2) of Proposition 4.1 hold for absolutely almost simple G. Indeed, it suffices to check them

over F_{sep}, where they hold by the original version of the proposition. It follows immediately that the conclusions of 4.4 and 4.5 hold for every absolutely almost simple group G.

We now extend Theorem A. Recall that there is a natural action of the absolute Galois group $\mathrm{Gal}(F)$ of F on the Dynkin diagram Δ of G [Tits 1966, 2.3]. As in [Tits 1966, p. 54], we say, for example, that G *has type* ${}^3D_4$ if Δ has type D_4 and the image of the map $\mathrm{Gal}(F) \to \mathrm{Aut}(\Delta)$ has order 3. We say that the characteristic of F is not very good for G if and only if it is not very good for the corresponding split group G'; these primes are listed in the middle column of Table I.

Theorem A′. *Let G be an absolutely almost simple algebraic group over a field F.*

(1) *Every representation ρ of G over F has Tr_ρ degenerate if and only if the characteristic of F*

$$\begin{cases} \textit{divides } 2n & \textit{if } G \textit{ has type } {}^2A_{n-1} \textit{ for some odd } n \geq 3; \\ \textit{is } 2 \textit{ or } 3 & \textit{if } G \textit{ has type } {}^3D_4 \textit{ or } {}^6D_4; \\ \textit{is not very good for } G & \textit{otherwise.} \end{cases}$$

(2) *Suppose G is not simply connected and not of type A. Every representation of G has Tr_ρ zero if and only if the characteristic of F is as in the table*:

type of G	$\mathrm{char}F$
B_n $(n \geq 3)$; C_n $(n \geq 2)$; ${}^1D_n$ *or* ${}^2D_n$ $(n \geq 4)$; *or* E_6	2
${}^3D_4$, ${}^6D_4$, E_7	2 *or* 3

Regarding the omitted cases in part (2), for G simply connected, the number $E(G)$ is 1, so every representation ρ of G has Tr_ρ zero if and only if the characteristic divides $N(G)$ by 4.5; this number (using that G is simply connected) is calculated in [Merkurjev 2003, §§11–16]. We leave the type A case of (2) as an exercise for the reader.

Proof of Theorem A′. To prove (2), by 4.5 it remains to show that the primes in the table are those dividing $N(G)/E(G)$. As $N(G')$ divides $N(G)$ and $E(G')$ equals $E(G)$, we have the trivial equation

$$\frac{N(G)}{E(G)} = \frac{N(G)}{N(G')} \frac{N(G')}{E(G')} \tag{10.1}$$

where all three terms are integers. The primes dividing $N(G')/E(G')$ are listed in Table I, so it suffices to check which primes divide $N(G)/N(G')$ and are not in that table.

For G adjoint of type E_6, the proof that $N(G)/E(G)$ is a power of 2 from the end of Section 6 goes through without change.
The proof of [Merkurjev 2003, 10.11] shows that every prime dividing $N(G)/N(G')$ divides the exponent of P/Q (the weight lattice modulo the root lattice) or the order of the image of $\mathrm{Gal}(F) \to \mathrm{Aut}(\Delta)$. For G of type B_n ($n \geq 3$), C_n ($n \geq 2$), ${}^1D_n$ or ${}^2D_n$ ($n \geq 4$), or E_7, the exponent of P/Q is 2 and the image of $\mathrm{Gal}(F) \to \mathrm{Aut}(\Delta)$ has order at most 2. As 2 divides $N(G')/E(G')$, part (2) is proved for these groups.

For G adjoint of type ${}^3D_4$ or ${}^6D_4$, write $\widetilde{G} \to G$ for the universal covering of G. The number $N(\widetilde{G})$ is 6 or 12 by [Merkurjev 2003, 16.5] and divides $N(G)$. As $E(G)$ is 2 by Example 3.5, $N(G)/E(G)$ is divisible by 3. Part (2) of the theorem is proved.

(We remark that applying the argument from the two previous paragraphs in the case where G has type A_{n-1} shows that every prime dividing $N(G)/N(G')$ divides $2n$. If n is odd and ≥ 3 and G has type ${}^2A_{n-1}$, then 2 divides $N(\widetilde{G})$ by [Merkurjev 2003, 12.6] hence also $N(G)$, yet $E(G)$ is odd by Lemma 5.2, so $N(G)/E(G)$ is even.)

We now prove part (1) by imitating Section 7. We replace (a) with the condition that the characteristic of F is not as in the statement of Theorem A′(1); we denote this condition by (a)′.

Suppose that (a)′ holds. The characteristic is very good for G and $\mathrm{Lie}(G) \otimes F_{\mathrm{sep}}$ is simple as in 2.6, hence $\mathrm{Lie}(G)$ is simple. If G is neither simply connected nor of type A, then there is a representation ρ of G with Tr_ρ nonzero by part (2) and (b) holds. If G is simply connected, then checking [Merkurjev 2003] verifies that $N(G)$ is not divisible by the characteristic and again (b) holds. In the remaining case where G has type A, the characteristic does not divide $N(G')$ by 5.4 nor does it divide $N(G)/N(G')$ by the discussion above; by (10.1), we find that (b) holds.

As in Section 7, (b) trivially implies (c).

Finally, suppose that (a)′ fails; we will show that (c) fails. We assume that the characteristic is very good, otherwise (c) fails because it does so over F_{sep}. That is, we are in one of the cases

(i) $\mathrm{char} F = 3$ and G has type ${}^3D_4$ or ${}^6D_4$; or

(ii) $\mathrm{char} F = 2$ and G has type ${}^2A_{n-1}$ for some odd n.

But in these cases the characteristic divides $N(G)/E(G)$ by the proof of part (2) above, and (c) fails. □

Example 10.2. Let F be a field of prime characteristic p with a central division F-algebra A of degree p. Take G to be the group $\mathrm{SL}(A)$ whose F-points are the elements of A with determinant 1. This group is simply connected, so $N(G)/E(G)$ is p by [Merkurjev 2003, 11.5]. That is, Tr_ρ is zero for every representation ρ of G over F. On the other hand, $N(G_{F_{\mathrm{sep}}})$ is 1, so there are representations of G defined

over F_{sep} (for example, the natural representation of SL_n) that have a trace form that is not zero.

A similar statement holds for groups of type ${}^3D_4$ or ${}^6D_4$ over fields of characteristic 3.

11. Trace forms and Lie algebras

This section collects some results regarding Tr_ψ, where ψ is a representation of the Lie algebra of an algebraic group G and we do not assume that ψ is the differential of a representation of G.

11.1. Fix a positive integer n and assume that the characteristic of F is a prime dividing n and $\neq 2, 3$. The Lie algebra $\mathfrak{sl}_n$ of trace zero n-by-n matrices has center $\mathfrak{c}$ the scalar matrices and $\mathfrak{sl}_n/\mathfrak{c}$ is simple [Steinberg 1961, 2.6]. We give a new proof of:

Proposition 11.2 [Block 1962, Theorem 6.2]. *Under the hypotheses of 11.1, every representation of $\mathfrak{sl}_n/\mathfrak{c}$ has zero trace form.*

Proof. For sake of contradiction, suppose that there is an irreducible representation ψ of $\mathfrak{sl}_n/\mathfrak{c}$ with nonzero trace form. Then ψ is restricted by [Block 1962, Theorem 5.1], using that F has characteristic $\neq 2, 3$. The composition of ψ with $\mathfrak{sl}_n \to \mathfrak{sl}_n/\mathfrak{c}$ is a restricted irreducible representation of $\mathfrak{sl}_n$, which is the differential of a representation ρ of SL_n by [Curtis 1960; Steinberg 1963].

By construction Tr_ρ is not zero and $\mathrm{d}\rho$ vanishes on the scalar matrices. Identifying the center of SL_n with the (nonreduced) group scheme $\boldsymbol{\mu}_n$ identifies the restriction of ρ to $\boldsymbol{\mu}_n$ with a map $x \mapsto x^l$. Our hypothesis on $\mathrm{d}\rho$ says that l is divisible by the characteristic p of F, hence ρ factors through the natural map $\mathrm{SL}_n \to \mathrm{SL}_n/\boldsymbol{\mu}_p$. Paragraph 5.3 says that $N(\rho)$ is divisible by p, hence Tr_ρ vanishes by 4.4, a contradiction.

Since every irreducible representation has zero trace form, the same holds for every representation like at the end of the proof of Proposition 4.1. □

11.3. *Proof of Theorem B.* Let G be a group of type E_8, and suppose that there is a representation ψ of $\mathrm{Lie}(G)$ such that Tr_ψ is not zero. We may assume that ψ is irreducible. Then the Block–Premet theorem on page 563 implies that ψ is restricted, hence is the differential of a representation of G. Theorem A(2) implies that the characteristic of F is $\neq 2, 3, 5$. □

We close by proving that over a field of characteristic 5, the Lie algebra of a group of type E_8 has no quotient trace form. For a Lie algebra L over F and a representation ψ of L, write $\mathrm{rad}\,\psi$ for the radical of the trace bilinear form Tr_ψ; it is an ideal of L.

Corollary 11.4. *For every representation ψ of every Lie algebra L over a field of characteristic* 5, *the quotient $L/\operatorname{rad}\psi$ is* not *isomorphic to the Lie algebra of an algebraic group of type E_8.*

Proof. Suppose the corollary is false. That is, suppose that there is a group G of type E_8 and a Lie algebra L with a representation ψ and a surjection $\pi: L \to \operatorname{Lie}(G)$ with kernel the radical of Tr_ψ.

By [Block 1962, Lemma 2.1] — using that the characteristic is $\neq 2, 3$ — we may assume that the radical of Tr_ψ is contained in the center of L, that is, L is a central extension of $\operatorname{Lie}(G)$. It follows that there is a map $f: \operatorname{Lie}(G) \to L$ such that πf is the identity [Steinberg 1962, Theorem 6.1(c)]. Clearly, the representation ψf of $\operatorname{Lie}(G)$ has nonzero trace form. As in the proof of Proposition 11.2, we deduce that G has a representation ρ such that Tr_ρ is not zero, but this is impossible by Theorem A(2). □

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Appendix: On trace forms of Lie algebras of type E_8

by Alexander Premet

All basic notions and results of modular Lie theory used in this appendix can be found in [Premet 1995] and references therein.

Let G be an algebraic group of type E_8 over an algebraically closed field of characteristic $p > 0$ and $\mathfrak{g} = \operatorname{Lie}(G)$. It is well known that $\mathfrak{g}$ is a simple Lie algebra carrying an $(\operatorname{Ad} G)$-equivariant $[p]$-th power map $x \mapsto x^{[p]}$. Since the universal enveloping algebra $U(\mathfrak{g})$ is a finite module over its central subalgebra generated by all $x^p - x^{[p]}$ with $x \in \mathfrak{g}$, all irreducible $\mathfrak{g}$-modules are finite dimensional. Furthermore, for every irreducible $\mathfrak{g}$-module M there is a linear function $\xi = \xi_M$ on $\mathfrak{g}$ such that $x^p - x^{[p]}$ acts on M as the scalar operator $\xi(x)^p \operatorname{Id}_M$. The function ξ_M is called the *p-character* of M. Denote by I_ξ the two-sided ideal of $U(\mathfrak{g})$ generated by all elements $x^p - x^{[p]} - \xi(x)^p$, where $x \in \mathfrak{g}$. The factor-algebra of $U(\mathfrak{g})/I_\xi$ is called the *reduced enveloping algebra* associated with ξ and denoted $U_\xi(\mathfrak{g})$. It has dimension $p^{\dim \mathfrak{g}}$. Clearly, M is a $U_\xi(\mathfrak{g})$-module. We say that M is *restricted* if $\xi_M = 0$.

For $p > 3$, the theorem below was first proved in [Block 1962]. The aim of this appendix is to give a proof valid in any positive characteristic.

Theorem. *If $\psi : \mathfrak{g} \to \mathfrak{gl}(V)$ is an irreducible representation with $\mathrm{Tr}_\psi \neq 0$, then V is a restricted $\mathfrak{g}$-module.*

Proof. Suppose ψ is not restricted and let χ be the p-character of V. Then χ is a nonzero linear function on $\mathfrak{g}$. We show that Tr_ψ is zero.

Let T be a maximal torus of G and $\mathfrak{t} := \mathrm{Lie}(T)$. As in sections 1 and 2, we write R for the root system of G relative to T and h_α for the image of the coroot $\alpha^\vee$ in $\mathfrak{t} := \mathrm{Lie}(T)$. (In our case, the group G is both adjoint and simply connected.) Since $\mathfrak{g}$ is an irreducible $(\mathrm{Ad}\, G)$-module, every nonzero adjoint G-orbit spans $\mathfrak{g}$. Thus, replacing $\mathfrak{t}$ by its G-conjugate if necessary, we may assume that $\chi(h_\beta) \neq 0$ for some $\beta \in R$.

There are root vectors $e_{\pm\beta} \in \mathfrak{g}^{\pm\beta}$ such that $\mathfrak{s} := Fe_{-\beta} \oplus Fh_\beta \oplus Fe_\beta$ is isomorphic to $\mathfrak{sl}_2$. Replacing $\mathfrak{t}$ by its conjugate $(\mathrm{Ad}\, x_{-\beta}(\lambda))(\mathfrak{t})$ for a suitable $x_{-\beta}(\lambda)$ in the unipotent root subgroup $U_{-\beta}$ of G, we may assume without loss of generality that $\chi|_{\mathfrak{s}} \neq 0$ and $\chi(e_\beta) = 0$. Then every $\mathfrak{s}$-composition factor M of V is a baby Verma module, that is, $M \cong Z_\xi(a)$, where $\xi = \chi|_{\mathfrak{s}}$ and $a \in F$ is a root of the equation $X^p - X = \xi(h_\alpha)^p$. Note that $\dim M = p$, the operator h_β acts semisimply on M, and the h_β-weights of M are $a, a-2, \ldots, a-2(p-1)$.

First suppose $p > 3$. Then $\mathrm{trace}_M(h_\beta^2) = \sum_{i=0}^{p-1} (a-2i)^2 = pa^2 - 2ap(p-1) + \frac{2}{3}(p-1)p(2p-1) = 0$. Since this holds for every $\mathfrak{s}$-composition factor M of V, we obtain $\mathrm{Tr}_\psi(h_\beta, h_\beta) = 0$. As $\mathfrak{g}$ is a simple Lie algebra and Tr_ψ is $\mathfrak{g}$-invariant, Tr_ψ is a multiple of the form b_F from Section 4. Hence Tr_ψ is zero.

Next suppose $p = 3$. Then the h_β-weights of M are a, $a+1$, $a-1$, hence $\mathrm{trace}_M(h_\beta^2) = a^2 + (a+1)^2 + (a-1)^2 = 2$. It follows that $\mathrm{trace}_M(h_\beta^2)$ is independent of M. Since all $\mathfrak{s}$-composition factors of V are three-dimensional, we deduce that $\mathrm{Tr}_\psi(h_\beta, h_\beta) = 2(\dim V)/3$. Note that $\mathfrak{s}$ can be included into a Levi subalgebra of type A_7; call it $\mathfrak{l}$. Since $\mathfrak{s} \subset \mathfrak{l}$, all $\mathfrak{l}$-composition factors of V have the same nonzero p-character. But then the Kac–Weisfeiler conjecture, which thanks to [Premet 1995, Theorem 3.10] holds for $\mathfrak{sl}_8$ in characteristic 3, implies that all such factors have dimension divisible by 9. Then 9 divides $\dim V$, forcing $\mathrm{Tr}_\psi(h_\beta, h_\beta) = 0$. As in the $p > 3$ case, Tr_ψ is zero.

Finally, suppose $p = 2$. Then the $\mathfrak{sl}_2$-algebra $\mathfrak{s} = Fe \oplus Fh \oplus Ff$ is nilpotent and h lies in the center of $\mathfrak{s}$. However, the reduced enveloping algebra $U_\xi(\mathfrak{s})$ is semisimple whenever $\xi(h) \neq 0$. Indeed, $U_\xi(\mathfrak{s})$ then possesses two nonequivalent two-dimensional irreducible modules, M and N, induced from one-dimensional modules over a Borel subalgebra of $\mathfrak{s}$. The central element h of $\mathfrak{s}$ acts on M and N by different scalars. There are exactly two choices here, namely, a and $a+1$, where a is a root of the equation $X^2 - X = \xi(h)^2$. As a consequence, $U_\xi(\mathfrak{s})$ maps onto

a direct sum of two copies of $\mathrm{Mat}_2(F)$. Since $\dim U_\xi(\mathfrak{s}) = 2^3 = 8$, this map is an isomorphism. Thus, $U_\xi(\mathfrak{s})$ is semisimple with two isoclasses of simple modules, both of which are two-dimensional.

Suppose now that we have found two commuting $\mathfrak{sl}_2$-subalgebras $\mathfrak{s}_i = Fe_i \oplus Fh_i \oplus Ff_i$ in $\mathfrak{g}$, where $i = 1, 2$, such that

(a) the sum $\mathfrak{s}_1 + \mathfrak{s}_2$ is direct;

(b) $\chi(h_i) \neq 0$ for $i = 1, 2$;

(c) $e_1 \in \mathfrak{g}^\gamma$ and $f_1 \in \mathfrak{g}^{-\gamma}$ for some $\gamma \in R$.

Our preceding remark then would show that V is a semisimple module over the subalgebra $U_\chi(\mathfrak{s}_1 \oplus \mathfrak{s}_2) \cong U_\chi(\mathfrak{s}_1) \otimes U_\chi(\mathfrak{s}_2)$ of $U_\chi(\mathfrak{g})$ (to ease notation, we identify χ with its restriction to $\mathfrak{s}_i$, $i = 1, 2$). Let M and N be two irreducibles for $U_\chi(\mathfrak{s}_1)$ described earlier. Then V decomposes as a tensor product $V = (M \otimes P) \bigoplus (N \otimes Q)$ for some semisimple $U_\chi(\mathfrak{s}_2)$-modules P and Q. Therefore,

$$\mathrm{Tr}_\psi(e_1, f_1) = r \dim P + s \dim Q,$$

where $r = \mathrm{trace}_M(e_1 f_1)$ and $s = \mathrm{trace}_N(e_1 f_1)$. As both P and Q must have even dimension by our preceding remark, this would yield $\mathrm{Tr}_\psi(\mathfrak{g}^\gamma, \mathfrak{g}^{-\gamma}) = 0$. Hence $\mathrm{Tr}_\psi = 0$ by (3.2), using that γ is (trivially) a long root.

So it remains to find two commuting $\mathfrak{sl}_2$-triples as above. We adopt Bourbaki's numbering of simple roots; see [Bourbaki 2002]. Since $\chi \neq 0$ and the adjoint G-orbit of e_{α_7} spans $\mathfrak{g}$ by the simplicity of $\mathfrak{g}$, we may assume that $\chi(e_{\alpha_7}) \neq 0$. If $\chi(h_{\alpha_6}) \neq 0$ and $\chi(h_{\alpha_8}) \neq 0$, then we can take $\mathfrak{s}_1 = Fe_{\alpha_6} \oplus Fh_{\alpha_6} \oplus Fe_{-\alpha_6}$ and $\mathfrak{s}_2 = Fe_{\alpha_8} \oplus Fh_{\alpha_8} \oplus Fe_{-\alpha_8}$. If this is not the case, then we replace $\mathfrak{t}$ by $(\mathrm{Ad}\, x_{\alpha_7}(t_0))(\mathfrak{t})$ for a suitable $x_{\alpha_7}(t_0)$ in the unipotent root subgroup U_{α_7} of G.

There exists $b \in F^\times$ such that for every $t \in F$ we have

$$(\mathrm{Ad}\, x_{\alpha_7}(t))(h_{\alpha_i}) = h_{\alpha_i} + tb[e_{\alpha_7}, h_{\alpha_i}] = h_{\alpha_i} + tbe_{\alpha_7}, \quad i = 6, 8.$$

As $\chi(e_{\alpha_7}) \neq 0$, we can find $t_0 \in F$ such that

$$\chi(h_{\alpha_6} + t_0 be_{\alpha_7}) \neq 0 \quad \text{and} \quad \chi(h_{\alpha_8} + t_0 be_{\alpha_7}) \neq 0.$$

Hence we can take $(\mathrm{Ad}\, x_{\alpha_7}(t_0))(\mathfrak{s}_i)$, $i = 1, 2$, as our $\mathfrak{sl}_2$-triples. This completes the proof. □

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