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A rooted-trees q-series lifting a one-parameter family of Lie idempotents

Frédéric Chapoton

We define and study a series indexed by rooted trees and with coefficients in $\mathbb{Q}(q)$. We show that it is related to a family of Lie idempotents. We prove that this series is a q-deformation of a more classical series and that some of its coefficients are Carlitz q-Bernoulli numbers.

1. Introduction

The aim of this article is to introduce and study a series Ω_q indexed by rooted trees, with coefficients that are rational functions of the indeterminate q.

The series Ω_q is in fact an element of the group G_{PL} of formal power series indexed by rooted trees, which is associated to the pre-Lie operad by a general functorial construction of a group from an operad. As there is an injective morphism of operads from the pre-Lie operad to the dendriform operad, there is an injection of groups from G_{PL} to the group $\mathsf{G}_{\mathrm{Dend}}$, which is a group of formal power series indexed by planar binary trees. This means that each series indexed by rooted trees can be mapped to a series indexed by planar binary trees, in a nontrivial way.

There is a conjectural description of the image of this injection of groups (see [Chapoton 2007, Corollary 5.4]). This can be stated roughly as the intersection in a bigger space (spanned by permutations) of the dendriform elements with the Lie elements. The inclusion of the image in the intersection is known, but the converse is not.

One starting point of this article was the existence of a one-parameter family of Lie idempotents belonging to the descent algebras of the symmetric groups [Duchamp et al. 1994; Krob et al. 1997]. As Lie idempotents, these are in particular Lie elements. As elements of the descent algebras, these are also dendriform elements. Therefore, according to the conjecture stated above, they should belong to the image of G_{PL} in $\mathsf{G}_{\mathrm{Dend}}$.

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Bypassing the conjecture, we prove this by exhibiting an element Ω_q of G_{PL} and then showing that its image is the expected sum of Lie idempotents.

We then obtain several results on Ω_q. First, we prove that the series Ω_q has only simple poles at nontrivial roots of unity and, in particular, can be evaluated at $q = 1$. Then we show that Ω_q is a q-deformation of a classical series Ω which is its value at $q = 1$. We also compute the value at $q = 0$ and the appropriate limit value when $q = \infty$.

We then consider the images of Ω_q in some other groups. There are two morphisms of groups from G_{PL} to usual groups of formal power series in one variable, both coming by functoriality from quotient operads of pre-Lie. Looking at linear trees only, that is, using the quotient map from pre-Lie to the associative operad, one gets a map from G_{PL} to the composition group of formal power series without constant term. The image of Ω_q is then a q-logarithm.

On the other hand, looking at corollas only, one gets a map from G_{PL} to the group of formal power series with constant term 1 for multiplication. The image of Ω_q is then the generating function of the q-Bernoulli numbers introduced by Carlitz. These numbers appear quite naturally here.

We recall in Appendix A the functorial definition of a group $\mathsf{G}_{\mathcal{P}}$ from an augmented operad $\mathcal{P}$. On this subject, the reader may also consult [Chapoton 2002a; 2007/08; van der Laan 2003; Chapoton and Livernet 2007].

In Appendix B, we give, for concreteness, the first few terms of the rooted-trees series that we consider.

Many useful computations and checks have been done using MuPAD.

2. General setting

We will work over the field $\mathbb{Q}$ of rational numbers and over the field $\mathbb{Q}(q)$ of fractions in the indeterminate q.

We have tried to avoid using operads as much as possible, but this language is necessary to define the ambient groups, and we will need it at some points in this article. The reader may consult [Loday 2001; Chapoton 2007/08] as references. The symbol $\circ_i$ will denote the (single) composition at position i in an operad and the symbols $\flat$ and $\natural$ will serve to note positions where composition is done.

Pre-Lie algebras. Recall (see for instance [Chapoton and Livernet 2001]) that a *pre-Lie algebra* is a vector space V endowed with a bilinear map $\curvearrowleft$ from $V \otimes V$ to V satisfying the axiom

$$(x \curvearrowleft y) \curvearrowleft z - x \curvearrowleft (y \curvearrowleft z) = (x \curvearrowleft z) \curvearrowleft y - x \curvearrowleft (z \curvearrowleft y). \tag{1}$$

This is sometimes called a *right pre-Lie algebra*.

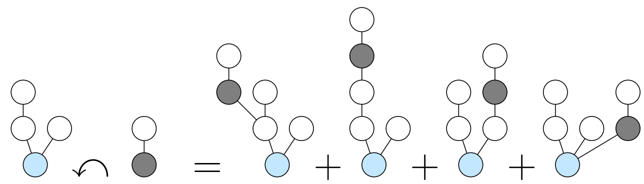


Figure 1. Example of operation in the free pre-Lie algebra PL.

The pre-Lie product $\curvearrowleft$ defines a Lie bracket on V as:

$$[x, y] = x \curvearrowleft y - y \curvearrowleft x. \tag{2}$$

One can easily check that the pre-Lie axiom (1) implies the Jacobi identity for the antisymmetric bracket $[\ ,\ ]$. The Lie algebra $(V, [\ ,\ ])$ will be called V_{Lie}.

The pre-Lie product $\curvearrowleft$ can also be considered as a right action $\curvearrowleft$ of the associated Lie algebra V_{Lie} on the vector space V. Indeed, one has

$$(x \curvearrowleft y) \curvearrowleft z - (x \curvearrowleft z) \curvearrowleft y = x \curvearrowleft [y, z]. \tag{3}$$

This should not be confused with the adjoint action of a Lie algebra on itself.

Free pre-Lie algebras and rooted trees. The free pre-Lie algebras have a simple description using rooted trees. Let us recall briefly this description and other properties. Details and proofs can be found in [Chapoton and Livernet 2001].

A *rooted tree* is a finite, connected and simply connected graph, together with a distinguished vertex called the *root*. We will picture rooted trees with their root at the bottom and orient (implicitly) the edges towards the root. There are two distinguished kinds of rooted trees: corollas, where every vertex other than the root is linked to the root by an edge; and linear trees, where at every vertex, there is at most one incoming edge. See Figure 4 on page 620 for examples. A *forest of rooted trees* is a finite graph whose connected components are rooted trees.

The free pre-Lie algebra $\mathsf{PL}(S)$ on a set S has a basis indexed by rooted trees decorated by S, that is, rooted trees together with a map from their set of vertices to S.

The pre-Lie product $T \curvearrowleft T'$ of a tree T' on another one T is given by the sum of all possible trees obtained from the disjoint union of T and T' by adding an edge from the root of T' to one of the vertices of T (the root of the resulting tree is the root of T). An example is depicted in Figure 1.

In particular, we will denote by PL the free pre-Lie algebra on one generator. This is the graded vector space $\mathsf{PL} = \oplus_{n \geq 1} \mathsf{PL}_n$ spanned by unlabeled rooted trees, where the degree of a tree T is the number $\#T$ of its vertices. The pre-Lie product obviously preserves this grading.

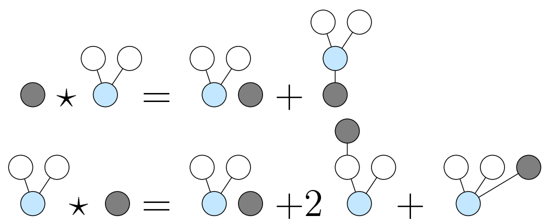


Figure 2. Products of trees in the universal enveloping algebra $U(\mathsf{PL})$.

Universal enveloping algebras of free pre-Lie algebras. The Lie algebras $\mathsf{PL}(S)_{\mathrm{Lie}}$ have the curious property that their universal enveloping algebras come naturally equipped with a basis, which depends on no choice and has nothing to do with a Poincaré–Birkhoff–Witt basis. Let us explain this in the case of $\mathsf{PL}_{\mathrm{Lie}}$.

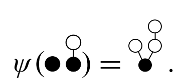
Let $U(\mathsf{PL})$ be the universal enveloping algebra of the Lie algebra $\mathsf{PL}_{\mathrm{Lie}}$. We will denote by $\star$ the associative product in $U(\mathsf{PL})$. We will freely identify right $\mathsf{PL}_{\mathrm{Lie}}$-modules with right $U(\mathsf{PL})$-modules. The crucial point is the following result [Chapoton and Livernet 2001, Theorem 3.3].

Theorem 2.1. *There exists a unique isomorphism ψ of graded right $\mathsf{PL}_{\mathrm{Lie}}$-modules between the free right $U(\mathsf{PL})$-module on one generator g of degree 1 and the $\mathsf{PL}_{\mathrm{Lie}}$-module $(\mathsf{PL}, \curvearrowleft)$ such that ψ maps the generator g to $\bullet$, the unique rooted tree with one vertex.*

This means that there is a commutative diagram as follows:

$$\begin{tikzcd} \mathbb{Q}g \otimes U(\mathsf{PL}) \otimes \mathsf{PL}_{\mathrm{Lie}} \arrow[r, "\psi\otimes\mathrm{Id}"] \arrow[d, "\mathrm{Id}\otimes\star"] & \mathsf{PL} \otimes \mathsf{PL}_{\mathrm{Lie}} \arrow[d, "\curvearrowleft"] \\ \mathbb{Q}g \otimes U(\mathsf{PL}) \arrow[r, "\psi"] & \mathsf{PL} \end{tikzcd} \tag{4}$$

As $\mathbb{Q}g$ has dimension 1, the map ψ can be considered as an isomorphism of vector spaces between $U(\mathsf{PL})$ and PL. One can therefore use ψ and the canonical basis of PL (indexed by rooted trees) to get a canonical basis of the enveloping algebra $U(\mathsf{PL})$. It is more convenient to index this basis by forests of rooted trees as follows. The inverse image $\psi^{-1}(T)$ in $U(\mathsf{PL})$ of a tree T in PL can be seen as an element of $U(\mathsf{PL})$. This element is defined to be the basis element corresponding to the forest F obtained from T by removal of its root. For example, one has



By using diagram (4) for the unit element 1 in $U(\mathsf{PL})$ (that is, the empty forest), one can show that the map from $\mathsf{PL}_{\mathrm{Lie}}$ to $U(\mathsf{PL})$ given by the universal property of the enveloping algebra corresponds to the inclusion of the set of rooted trees (as a

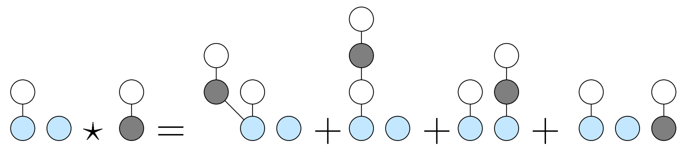


Figure 3. Product of a forest and a tree in $U(\mathsf{PL})$.

basis of PL) in the set of forests (as a basis of $U(\mathsf{PL})$). Indeed, for a rooted tree T, the top horizontal arrow maps $g \otimes 1 \otimes T$ to $\bullet \otimes T$. The right vertical arrow maps this to $\bullet \curvearrowleft T$. Then the inverse of ψ removes the root, which just gives T (seen as a forest with just one connected component). The image of the natural inclusion of $\mathsf{PL}_{\mathrm{Lie}}$ in $U(\mathsf{PL})$ is therefore the subspace spanned by rooted trees.

Note that one can use diagram (4) to compute the $\star$ product of a forest and a rooted tree in $U(\mathsf{PL})$, as a sum of forests (see Figure 3 for an example, compare with Figure 1). The usual spanning set of the universal enveloping algebra $U(\mathsf{PL})$ is the set of noncommutative monomials in rooted trees (for the $\star$ product). By induction on the length of the monomial, one can therefore map each such noncommutative monomial to a sum of forests. Examples are given in Figures 2 and 3.

In the basis of $U(\mathsf{PL})$ indexed by forests, there is a nice combinatorial description of the associative product $\star$. Let F and F' be forests in $U(\mathsf{PL})$. The product $F \star F'$ is the sum of all possible forests, obtained from the disjoint union of F and F' by the addition of some edges (possibly none), each of these new edges going from some root of F' to some vertex of F. Indeed, one can easily check that this operation is associative and coincide with $\star$ on rooted trees, hence the result.

There is a canonical projection π from $U(\mathsf{PL})$ to PL, defined using the canonical basis of $U(\mathsf{PL})$ by projection on the subspace spanned by rooted trees, annihilating the empty forest and all forests that are not trees.

Lemma 2.2. *Let F be a forest in $U(\mathsf{PL})$ and T be a rooted tree in PL. Then one has $\pi(F \star T) = \pi(F) \curvearrowleft T$.*

Proof. If F is not a tree, then each term of $F \star T$ is not a tree, therefore both sides vanish. If $F = \pi(F)$ is a tree, then $F \star T$ is the sum of $\pi(F) \curvearrowleft T$ with the disjoint union of F and T. Therefore $\pi(F \star T) = \pi(F) \curvearrowleft T$. $\square$

The reader can check this statement on the examples of Figures 2 and 3.

Lemma 2.3. *For all $n \geq 1$, the maps $T \mapsto \bullet \curvearrowleft T$ and $T \mapsto T \curvearrowleft \bullet$ are injective from PL_n to PL_{n+1}.*

Proof. This is obvious for the first map, which is even an injection on the set of rooted trees. For the second map, this follows from the fact that enveloping algebras are domains, by restriction of the commutative diagram (4). $\square$

Group associated to the pre-Lie operad. In the sequel, we will always work in the completed vector space $\widehat{\mathsf{PL}} = \prod_{n\geq 1} \mathsf{PL}_n$ and with its completed enveloping algebra $\widehat{U}(\mathsf{PL})$. All the results above are still true in this setting.

There is a group associated to each operad; see Appendix A and [Chapoton 2002a; 2007/08; van der Laan 2003; Chapoton and Livernet 2007]. We will need the group G_{PL} associated to the pre-Lie operad. Its elements are the elements of $\widehat{\mathsf{PL}}$ whose homogeneous component of degree 1 is $\bullet$. The product in G_{PL} is defined using the composition of the pre-Lie operad and $\bullet$ is the unit in G_{PL}. This group is contained in the bigger monoid $\widehat{\mathsf{PL}}$, on which it therefore acts on the right and on the left. The right action respects all the operations on $\widehat{\mathsf{PL}}$ induced by the product $\curvearrowleft$, including the product and the action of $\widehat{U}(\mathsf{PL})$.

Let us now introduce a special element of G_{PL} for later use. Let $\exp^* \in G_{\mathsf{PL}}$ be

$$\exp^* = \bullet \curvearrowleft ((\exp(\bullet) - 1)/\bullet), \tag{5}$$

where the rightmost factor is an element of $\widehat{U}(\mathsf{PL})$ defined using the formal power series $(\exp x - 1)/x$ and the $\star$ product.

The series $\exp^*$ is very classical, related to the flow of vector fields, and its coefficients are known as the Connes–Moscovici coefficients [Chapoton 2002a].

Consider the left action of $\exp^*$ on $\widehat{\mathsf{PL}}$. Let T be an element of $\widehat{\mathsf{PL}}$. Then $\exp^*(T)$ in $\widehat{\mathsf{PL}}$ is defined by

$$\exp^*(T) = \sum_{n\geq 1} \frac{1}{n!}((T \curvearrowleft T) \curvearrowleft \dots) \curvearrowleft T, \tag{6}$$

where there are n copies of T in the n-th term. As $\exp^*$ belongs to the group G_{PL}, the map $\exp^*$ defines a bijection from $\widehat{\mathsf{PL}}$ to itself.

Let us now relate the usual exponential map exp to the map $\exp^*$.

Let T be an element of $\widehat{\mathsf{PL}}$. Let $\exp T$ be the exponential of T in $\widehat{U}(\mathsf{PL})$ (which is defined by the usual series and using the $\star$ product). The map exp defines a bijection from $\widehat{\mathsf{PL}}$ to the set of group-like elements of $\widehat{U}(\mathsf{PL})$.

Therefore, the composite map $\exp^* \circ \exp^{-1}$ is a bijection from the set of group-like elements in $\widehat{U}(\mathsf{PL})$ to $\widehat{\mathsf{PL}}$. Let us show that this composite map is just a restriction of the canonical projection π.

Proposition 2.4. *Let T be an element of $\widehat{\mathsf{PL}}$. One has $\pi(\exp T) = \exp^*(T)$.*

Proof. Let F be in $\widehat{U}(\mathsf{PL})$. From Lemma 2.2 above, one knows that $\pi(F \star T)$ is exactly $\pi(F) \curvearrowleft T$. This implies that

$$\pi(T^{\star n}) = ((T \curvearrowleft T)\dots) \curvearrowleft T, \tag{7}$$

for all $n \geq 1$, hence the result. ☐

3. The classical case

Let us start by recalling the definition of a classical element Ω of $\widehat{\mathsf{PL}}$ with rational coefficients. It seems to have first appeared in [Agračev and Gamkrelidze 1980], was later considered under the name of $\log^*$ in [Chapoton 2002a] and has been since studied in [Murua 2006; Wright and Zhao 2003; Ebrahimi-Fard and Manchon 2009; Calaque et al. 2008].

Proposition 3.1. *There is a unique solution Ω in $\widehat{\mathsf{PL}}_{\mathbb{Q}}$ to the equation*

$$\bullet \curvearrowleft \left(\frac{\Omega}{\exp \Omega - 1}\right) = \Omega, \tag{8}$$

where $\Omega/(\exp \Omega - 1)$ is in the completed enveloping algebra $\widehat{U}(\mathsf{PL})$.

Proof. Let us write $\Omega = \sum_{n \geq 1} \Omega_n$ where each Ω_n is homogeneous of degree n.

Recall the Taylor expansion

$$\frac{x}{\exp x - 1} = \sum_{k \geq 0} \frac{B_k}{k!} x^k, \tag{9}$$

where the B_k are the Bernoulli numbers.

Then the homogeneous component of degree n of Equation (8) is

$$\Omega_n = \sum_{k \geq 0} \frac{B_k}{k!} \sum_{\substack{m_1 \geq 1, \ldots, m_k \geq 1 \\ m_1 + \cdots + m_k = n-1}} ((\bullet \curvearrowleft \Omega_{m_k}) \ldots) \curvearrowleft \Omega_{m_1}. \tag{10}$$

This gives a recursive definition of Ω_n, which implies the existence and uniqueness of Ω. $\square$

Remark. One can use Equation (10) to compute Ω up to order n in a $O(n^3)$ number of pre-Lie operations.

As the element $\Omega/(\exp \Omega - 1)$ is invertible in the completed enveloping algebra, Equation (8) is also equivalent to the following equation:

$$\Omega \curvearrowleft \left(\frac{\exp \Omega - 1}{\Omega}\right) = \bullet. \tag{11}$$

One can interpret Equation (11) as follows.

Proposition 3.2. *The series Ω is the inverse of* $\exp^*$ *in the group* G_{PL}.

Proof. By right action by the inverse Ω^{-1} of Ω in G_{PL} on (11), one shows that Ω^{-1} satisfies the same Equation (5) as $\exp^*$. $\square$

There is another equation for Ω.

Proposition 3.3. *The series* Ω *is the unique nonzero solution in* $\widehat{\mathsf{PL}}_{\mathbb{Q}}$ *to the equation*

$$\Omega \curvearrowleft (\exp \Omega - 1) = \bullet \curvearrowleft \Omega, \tag{12}$$

where $\exp \Omega - 1$ *is in the completed enveloping algebra* $\widehat{U}(\mathsf{PL})$.

Proof. First, by right action on (11) by Ω, one can see that the unique solution Ω of (8) is indeed a solution of (12).

Let us now prove uniqueness of a nonzero solution. Let Ω be any solution of (12). Let us write $\Omega = \sum_{n \geq 1} \Omega_n$ where each Ω_n is homogeneous of degree n.

Then the homogeneous component of degree n of Equation (12) is

$$\bullet \curvearrowleft \Omega_{n-1} = \sum_{k \geq 1} \frac{1}{k!} \sum_{\substack{m_1 \geq 1, \dots, m_k \geq 1, \ell \geq 1 \\ m_1 + \dots + m_k + \ell = n}} ((\Omega_\ell \curvearrowleft \Omega_{m_k}) \dots) \curvearrowleft \Omega_{m_1}. \tag{13}$$

If $n = 2$, this implies that Ω_1 is either 0 or $\bullet$.

Assume now that Ω is not zero. Let d be the degree of the first nonzero homogeneous component Ω_d of Ω. Assume that $d > 1$. Then Equation (13) in degree $d + 1$, together with Lemma 2.3, gives that $\Omega_d = 0$, a contradiction. Therefore necessarily, one has $d = 1$ and $\Omega_1 = \bullet$.

Let us look at the homogeneous component (13) in degree $n + 1 \geq 2$. The only terms involving Ω_n are $\bullet \curvearrowleft \Omega_n$ in the left-hand side and $\Omega_1 \curvearrowleft \Omega_n$, $\Omega_n \curvearrowleft \Omega_1$ in the right hand-side. As $\Omega_1 = \bullet$, two of them cancel out and one gets a recursive expression of $\Omega_n \curvearrowleft \bullet$ in terms of some Ω_j for $j < n$.

Using Lemma 2.3, this provides a recursive description of Ω (that may or may not possess a solution) and proves its uniqueness. □

The exponential of Ω has a simple shape.

Proposition 3.4. *In the enveloping algebra* $\widehat{U}(\mathsf{PL})$, *one has*

$$\exp \Omega = \sum_{n \geq 0} \frac{1}{n!} \bullet \bullet \dots \bullet, \tag{14}$$

where, in the n-th term, the forest has n nodes.

Proof. This is an equation for the exponential $\exp \Omega$ of the element Ω in the Lie algebra $\widehat{\mathsf{PL}}$. By Proposition 2.4, it is enough to prove that

$$\exp^*(\Omega) = \bullet, \tag{15}$$

because the image by π of the right side of (14) is $\bullet$.

But this amounts to saying that $\exp^*$ is the inverse of Ω in the group G_{PL}. This is none other than Proposition 3.2. □

It follows that

$$\Omega \curvearrowleft (\exp\Omega - 1) = \sum_{n\geq 2} \frac{1}{(n-1)!}\mathtt{Crl}_n^{\natural} \circ_{\natural} \Omega, \tag{16}$$

where $\mathtt{Crl}_n^{\natural}$ is the corolla with $n-1$ leaves and root labeled by $\natural$; see Figure 4.

Proposition 3.5. *The series Ω is the unique nonzero solution in $\widehat{\mathsf{PL}}_{\mathbb{Q}}$ to the equation*

$$\sum_{n\geq 2} \frac{1}{(n-1)!}\mathtt{Crl}_n^{\natural} \circ_{\natural} \Omega = \bullet \curvearrowleft \Omega, \tag{17}$$

where $\sum_{n\geq 2} \frac{1}{(n-1)!}\mathtt{Crl}_n^{\natural}$ is in $\widehat{\mathsf{PL}}$.

Proof. This follows from (16) and Proposition 3.3. □

4. The quantum case

We will introduce now an element Ω_q in $\widehat{\mathsf{PL}}$ with coefficients in $\mathbb{Q}(q)$. We will show later that this is a q-deformation of Ω.

If $A = \sum_{n\geq 1} A_n$ is an element of $\widehat{\mathsf{PL}}$, let $A[q]$ be the q-shift of A defined by

$$A[q] = \sum_{n\geq 1} q^n A_n. \tag{18}$$

Proposition 4.1. *There exists a unique solution Ω_q in $\mathsf{PL}_{\mathbb{Q}(q)}$ to the equation*

$$\Omega_q[q] \curvearrowleft (\exp\Omega - 1) + \Omega_q[q] - \Omega_q = \bullet \curvearrowleft \Omega_q + (q-1)\bullet. \tag{19}$$

Moreover, the series Ω_q has coefficients in the ring of fractions with poles only at roots of unity.

Proof. Write $\Omega_q = \sum_{n\geq 1} \Omega_{q,n}$ where each $\Omega_{q,n}$ is homogeneous of degree n. The homogeneous component of degree 1 of (19) implies that $\Omega_{q,1} = \bullet$.

Then for $n \geq 2$, the homogeneous component of degree n of Equation (19) is

$$\begin{aligned}(q^n - 1)&\Omega_{q,n} \\ &= \bullet \curvearrowleft \Omega_{q,n-1} - \sum_{k\geq 1}\frac{1}{k!} \sum_{\substack{m_1\geq 1,\dots,m_k\geq 1,\ell\geq 1\\ m_1+\dots+m_k+\ell=n}} q^{\ell}((\Omega_{q,\ell} \curvearrowleft \Omega_{m_k})\dots) \curvearrowleft \Omega_{m_1}.\end{aligned} \tag{20}$$

This provides an explicit recursion for $\Omega_{q,n}$ in terms of $\Omega_{q,j}$ and Ω_j for $j < n$. This gives existence and uniqueness and also implies that Ω_q has coefficients with poles only at roots of unity. □

On can reformulate the equation for Ω_q.

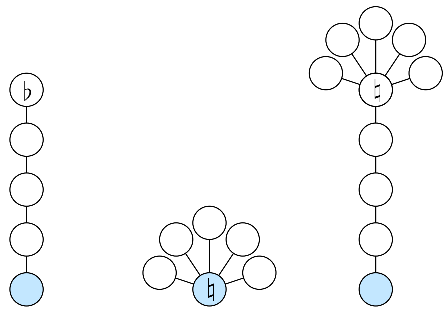


Figure 4. Rooted trees: $\mathtt{Lnr}^{\flat}_5$, $\mathtt{Crl}^{\natural}_6$ and $\mathtt{Frk}^{\natural}_{4,5} = \mathtt{Lnr}^{\flat}_5 \circ_\flat \mathtt{Crl}^{\natural}_6$.

Proposition 4.2. *The series Ω_q is the unique solution in $\widehat{\mathsf{PL}}_{\mathbb{Q}(q)}$ to the equation*

$$\sum_{n\geq 1} \frac{1}{(n-1)!} \mathtt{Crl}^{\natural}_n \circ_\natural \Omega_q[q] - \Omega_q = \bullet \curvearrowleft \Omega_q + (q-1)\bullet. \tag{21}$$

Proof. This follows from (16) and Proposition 4.1. □

Let $\mathtt{Frk}^{\natural}_{\ell,n}$ be the rooted tree with a linear trunk of ℓ vertices, a vertex $\natural$ on top of this trunk and a corolla with n leaves on top of the vertex $\natural$, see Figure 4. We will call this a *fork*. One has $\mathtt{Frk}_{\ell,n} = \mathtt{Lnr}^{\flat}_{\ell+1} \circ_\flat \mathtt{Crl}^{\natural}_{n+1}$.

Proposition 4.3. *The series Ω_q is the unique solution in $\widehat{\mathsf{PL}}_{\mathbb{Q}(q)}$ to the equation*

$$\Omega_q = \sum_{\ell\geq 0}\sum_{n\geq 0} \frac{(-1)^{\ell}}{n!} \mathtt{Frk}^{\natural}_{\ell,n} \circ_\natural \Omega_q[q] + (1-q)\sum_{\ell\geq 1}(-1)^{\ell-1}\mathtt{Lnr}_\ell. \tag{22}$$

Proof. Let us compute the right-hand side of (22), using (21) for Ω_q, written as

$$\Omega_q + \bullet \curvearrowleft \Omega_q + (q-1)\bullet = \sum_{n\geq 1} \frac{1}{(n-1)!} \mathtt{Crl}^{\natural}_n \circ_\natural \Omega_q[q]. \tag{23}$$

One gets

$$\sum_{\ell\geq 1}(-1)^{\ell-1}\mathtt{Lnr}^{\flat}_\ell \circ_\flat (\Omega_q + \bullet \curvearrowleft \Omega_q + (q-1)\bullet) + (1-q)\sum_{\ell\geq 1}(-1)^{\ell-1}\mathtt{Lnr}_\ell. \tag{24}$$

As $\mathtt{Lnr}^{\flat}_\ell \circ_\flat \bullet = \mathtt{Lnr}_\ell$ and $\mathtt{Lnr}^{\flat}_\ell \circ_\flat (\bullet \curvearrowleft \Omega_q) = \mathtt{Lnr}^{\flat}_{\ell+1} \circ_\flat \Omega_q$, the two rightmost terms cancels, and the sum simplifies to

$$\sum_{\ell\geq 1}(-1)^{\ell-1}\mathtt{Lnr}^{\flat}_\ell \circ_\flat \Omega_q - \sum_{\ell\geq 2}(-1)^{\ell-1}\mathtt{Lnr}^{\flat}_\ell \circ_\flat \Omega_q, \tag{25}$$

which is just Ω_q. This proves that Ω_q does satisfy (22).

It is then easy to see that (22) has only one solution in $\widehat{\mathsf{PL}}_{\mathbb{Q}(q)}$ by rewriting it as a recursion for the homogeneous components $\Omega_{q,n}$. □

5. Image in the free dendriform algebra

We describe in this section the image of Ω_q by the usual morphism from the free pre-Lie algebra to the free dendriform algebra. We show that this image is related to a family of Lie idempotents in the descent algebras of the symmetric groups. One deduces from that a nice explicit formula, that will be used later to get arithmetic information on Ω_q.

Dendriform algebra. Recall that a *dendriform algebra* [Loday 2001] is a vector space V endowed with two bilinear maps $\succ$ and $\prec$ from $V \otimes V$ to V satisfying the following axioms:

$$x \prec (y \prec z) + x \prec (y \succ z) = (x \prec y) \prec z, \tag{26}$$

$$x \succ (y \prec z) = (x \succ y) \prec z, \tag{27}$$

$$x \succ (y \succ z) = (x \succ y) \succ z + (x \prec y) \succ z. \tag{28}$$

Any dendriform algebra has the structure of a pre-Lie algebra given by

$$x \curvearrowleft y = y \succ x - x \prec y. \tag{29}$$

Any dendriform algebra has the structure of an associative algebra given by

$$x * y = x \succ y + x \prec y. \tag{30}$$

Remark. Equation (27) means that one can safely forget some parentheses. Equations (26) and (28) can be rewritten as

$$x \prec (y * z) = (x \prec y) \prec z, \tag{31}$$

$$x \succ (y \succ z) = (x * y) \succ z. \tag{32}$$

Let $\mathrm{Dend}(S)$ be the free dendriform algebra over a set S. This has an explicit basis indexed by *planar binary trees* with vertices decorated by S. For an example of a planar binary tree, see Figure 5. In particular, the free dendriform algebra on one generator, denoted by Dend, has a basis indexed by planar binary trees. This is a graded vector space, the degree $\#t$ of a planar binary tree t being the number of its inner vertices.

There is a unique morphism φ of pre-Lie algebras from PL to Dend that maps the rooted tree $\bullet$ to the planar binary tree $\vee$. This extends uniquely to a continuous morphism φ from $\widehat{\mathsf{PL}}$ to the completion $\widehat{\mathrm{Dend}}$ of Dend.

Remark. With some care, one can add a unit 1 to the free dendriform algebra Dend. Then $1 * x = 1 \succ x = x = x \prec 1 = x * 1$, but one has to pay attention to never write either $1 \prec x$ or $x \succ 1$. We will use this convention in the sequel.

There are two kinds of special planar binary trees: the left combs and the right combs. They can be defined as follows. Let $L = \sum_{n\geq 1} L_n$ be the unique solution in $\widehat{\mathsf{Dend}}$ to the equation

$$L = \curlyvee + L \succ \curlyvee = (1+L) \succ \curlyvee, \tag{33}$$

and let $R = \sum_{n\geq 1} R_n$ be the unique solution in $\widehat{\mathsf{Dend}}$ to

$$R = \curlyvee + \curlyvee \prec R = \curlyvee \prec (1+R). \tag{34}$$

Then L_n is called the left comb with n vertices and R_n be the right comb with n vertices.

If $A = \sum_{n\geq 1} A_n$ is an element of $\widehat{\mathsf{PL}}$ or $\widehat{\mathsf{Dend}}$, the *suspension* of A is $\widetilde{A} = \sum_{n\geq 1} (-1)^{n-1} A_n$.

Proposition 5.1. *The inverse of* $1+R$ *with respect to the* $*$ *product is* $1-\widetilde{L}$.

Proof. One has $\widetilde{L} = \curlyvee - \widetilde{L} \succ \curlyvee$. We compute

$$(1-\widetilde{L}) * (1+R) = 1 + R - \widetilde{L} * (1+R). \tag{35}$$

By the definition of $*$ and the convention on the unit 1, this is

$$1 + R - \widetilde{L} \prec (1+R) - \widetilde{L} \succ R. \tag{36}$$

By (33), this becomes

$$1 + R - \curlyvee \prec (1+R) + \widetilde{L} \succ \curlyvee \prec (1+R) - \widetilde{L} \succ R. \tag{37}$$

The last two terms cancel by (34) and one gets

$$1 + R - \curlyvee \prec (1+R), \tag{38}$$

which is just 1, again by (34). □

Equation for the dendriform image of $\boldsymbol{\Omega_q}$**.** Define a series $E = \sum_{n\geq 1} n L_n$ in $\widehat{\mathsf{Dend}}$. One can easily show that

$$E = L + E \succ \curlyvee. \tag{39}$$

Lemma 5.2. *The series* $B^\flat = \varphi(\sum_{n\geq 1} \mathtt{Lnr}_n^\flat)$ *satisfies*

$$B^\flat = \curlyvee^\flat + B^\flat \succ \curlyvee - \curlyvee \prec B^\flat. \tag{40}$$

Proof. This comes from a similar equation in PL. Let $\mathtt{Lnr}^\flat = \sum_{n\geq 1} \mathtt{Lnr}_n^\flat$. Then

$$\mathtt{Lnr}^\flat = \bullet^\flat + \bullet \curvearrowleft \mathtt{Lnr}^\flat, \tag{41}$$

as one can easily check. □

These relations can be taken as definitions of the elements E and $B^\flat$ of $\widehat{\mathsf{Dend}}$. One can forget the marking $\flat$ in $B^\flat$ to define a series B.

Proposition 5.3. *The series $B = \varphi(\sum_{n\geq 1} \mathtt{Lnr}_n)$ satisfies*

$$E = (1+L) * B. \tag{42}$$

Proof. One has to show that $E = (1+L) * B$. It is enough to prove that $(1+L) * B$ does satisfy the defining relation (39) of E.

One computes, using (40) for B,

$$(1+L) * B = (1+L) * (\curlyvee + B \succ \curlyvee - \curlyvee \prec B). \tag{43}$$

Expanding the $*$ product, this is

$$(1+L) \succ \curlyvee + L \prec \curlyvee + (1+L) \succ (B \succ \curlyvee - \curlyvee \prec B) + L \prec (B \succ \curlyvee - \curlyvee \prec B). \tag{44}$$

Using (33) and the dendriform axioms, this becomes

$$L + L \prec \curlyvee + ((1+L) * B) \succ \curlyvee - (1+L) \succ \curlyvee \prec B + L \prec (B \succ \curlyvee - \curlyvee \prec B). \tag{45}$$

Using (33) again, one gets

$$L + ((1+L) * B) \succ \curlyvee + L \prec (\curlyvee - B + B \succ \curlyvee - \curlyvee \prec B). \tag{46}$$

This simplifies, by (40) for B, to

$$L + ((1+L) * B) \succ \curlyvee, \tag{47}$$

as expected. □

Lemma 5.4. *The image of $\displaystyle\sum_{n\geq 1} \frac{1}{(n-1)!} \mathtt{Crl}_n^\natural$ by φ is*

$$(1+R) \succ \curlyvee^\natural \prec (1 - \widetilde{L}). \tag{48}$$

Proof. This was proved in [Ronco 2000; 2001; Chapoton 2002b]. □

Proposition 5.5. *The image of $\displaystyle\sum_{\ell\geq 0} \sum_{n\geq 0} \frac{(-1)^\ell}{n!} \mathtt{Frk}_{\ell,n}^\natural$ by φ is*

$$(1+R) * \curlyvee^\natural * (1 - \widetilde{L}), \tag{49}$$

where $\curlyvee^\natural$ is the planar binary tree $\curlyvee$ with vertex labeled by $\natural$.

Proof. Let $D = (1+R) * \curlyvee^\natural * (1 - \widetilde{L})$. Let us first show that

$$D = (1+R) \succ \curlyvee^\natural \prec (1 - \widetilde{L}) + \curlyvee \prec D - D \succ \curlyvee. \tag{50}$$

Expanding the $*$ product, one computes

$$D = ((1+R) * \curlyvee^\natural) \prec (1 - \widetilde{L}) - ((1+R) * \curlyvee^\natural) \succ \widetilde{L}. \tag{51}$$

Then one gets, by expanding again,

$$(1+R) \succ \curlyvee^\natural \prec (1 - \widetilde{L}) + (R \prec \curlyvee^\natural) \prec (1 - \widetilde{L}) - ((1+R) * \curlyvee^\natural) \succ \widetilde{L}. \tag{52}$$

Using the dendriform axioms, this is

$$(1+R) \succ \curlyvee^{\natural} \prec (1-\widetilde{L}) + R \prec (\curlyvee^{\natural} * (1-\widetilde{L})) - ((1+R) * \curlyvee^{\natural}) \succ \widetilde{L}. \tag{53}$$

Then by (33) and (34), this can be rewritten

$$\begin{aligned}(1+R) \succ \curlyvee^{\natural} \prec (1-\widetilde{L}) + (\curlyvee \prec (1+R)) \prec (\curlyvee^{\natural} * (1-\widetilde{L}))& \\ - ((1+R) * \curlyvee^{\natural}) \succ ((1-\widetilde{L}) \succ \curlyvee).&\end{aligned} \tag{54}$$

One gets, using the dendriform axioms,

$$\begin{aligned}(1+R) \succ \curlyvee^{\natural} \prec (1-\widetilde{L}) + \curlyvee \prec ((1+R) * \curlyvee^{\natural} * (1-\widetilde{L}))& \\ - ((1+R) * \curlyvee^{\natural} * (1-\widetilde{L})) \succ \curlyvee.&\end{aligned} \tag{55}$$

This proves Equation (50) for D.

Let us show now that

$$D' = \left(\sum_{\ell \geq 1} (-1)^{\ell-1} \varphi(\mathtt{Lnr}^{\flat}_{\ell})\right) \circ_{\flat} \left(\sum_{n \geq 1} \frac{1}{(n-1)!} \varphi(\mathtt{Crl}^{\natural}_{n})\right)$$

does satisfy the same equation as D.

By Lemma 5.4, one has

$$D' = \widetilde{B}^{\flat} \circ \flat((1+R) \succ \curlyvee^{\natural} \prec (1-\widetilde{L})). \tag{56}$$

By Lemma 5.2, one has $\widetilde{B}^{\flat} = \curlyvee^{\flat} - \widetilde{B}^{\flat} \succ \curlyvee + \curlyvee \prec \widetilde{B}^{\flat}$, hence

$$D' = (1+R) \succ \curlyvee^{\natural} \prec (1-\widetilde{L}) + \curlyvee \prec D' - D' \succ \curlyvee. \tag{57}$$

By uniqueness of the solution D of (50), one has $D = D'$, that is,

$$(1+R) * \curlyvee^{\natural} * (1-\widetilde{L}) = \left(\sum_{\ell \geq 1} (-1)^{\ell-1} \varphi(\mathtt{Lnr}^{\flat}_{\ell})\right) \circ_{\flat} \left(\sum_{n \geq 1} \frac{1}{(n-1)!} \varphi(\mathtt{Crl}^{\natural}_{n})\right). \tag{58}$$

Therefore

$$(1+R) * \curlyvee^{\natural} * (1-\widetilde{L}) = \varphi\left(\left(\sum_{\ell \geq 1} (-1)^{\ell-1} \mathtt{Lnr}^{\flat}_{\ell}\right) \circ_{\flat} \left(\sum_{n \geq 1} \frac{1}{(n-1)!} \mathtt{Crl}^{\natural}_{n}\right)\right), \tag{59}$$

which is exactly the expected image by φ of a sum over forks. □

One can now deduce a useful functional equation for the image of Ω_q by φ, using only the associative product $*$ of Dend.

Proposition 5.6. *The series $\varphi(\Omega_q)$ is the unique solution in $\widehat{\mathrm{Dend}}$ of*

$$\varphi(\Omega_q) = (1-\widetilde{L})^{-1} * \varphi(\Omega_q)[q] * (1-\widetilde{L}) + (1-q) \sum_{\ell \geq 1} (-1)^{\ell-1} \varphi(\mathtt{Lnr}_{\ell}). \tag{60}$$

Proof. Let us start from (22). By Proposition 5.5, we know the image by φ of the sum over forks. One gets

$$\varphi(\Omega_q) = ((1+R) * \curlyvee^{\natural} * (1-\widetilde{L})) \circ_{\natural} \varphi(\Omega_q)[q] + (1-q) \sum_{\ell \geq 1} (-1)^{\ell-1} \varphi(\mathtt{Lnr}_\ell). \quad (61)$$

Then one can use Proposition 5.1 to replace $1+R$ by the inverse of $1-\widetilde{L}$. $\square$

Explicit formula. We will prove in this section that the image of Ω_q by φ coincides (in some sense) with a known family of Lie idempotents, and has an explicit description using q-binomial coefficients, descents and major indices of planar binary trees. To obtain this description, we use a result on noncommutative symmetric functions. We refer to [Gelfand et al. 1995; Duchamp et al. 1997; Krob et al. 1997] for background on this subject. We will use the notation of this last article.

The algebra **Sym** of noncommutative symmetric function is the free unital associative algebra on generators $S_1, S_2, \ldots$. It is a graded algebra (with S_i of degree i), with a basis $(S_I)_I$ indexed by compositions. There is another basis $(R_I)_I$ obtained from the basis $(S_I)_I$ by Möbius inversion on compositions ordered by refinement. By convention, S_0 is the unit of **Sym**.

As **Sym** is free, there is a unique morphism of associative algebras θ from **Sym** to Dend which maps S_i to the left comb L_i for each $i \geq 0$, with the convention that L_0 is the unit of Dend.

One can check that θ is the usual morphism from **Sym** to Dend, considered for instance in [Hivert et al. 2005, Section 4.8] and [Loday and Ronco 1998].

In **Sym**, there are elements Ψ_i for $i \geq 1$, uniquely defined by the conditions

$$n S_n = \sum_{i=0}^{n-1} S_i * \Psi_{n-i}, \quad (62)$$

for all $n \geq 1$.

Proposition 5.7. *The image of Ψ_i by θ is $\varphi(\mathtt{Lnr}_i)$.*

Proof. This is a corollary of Proposition 5.3. Indeed, one has

$$1 + L = \sum_{n \geq 0} \theta(S_n) \quad \text{and} \quad E = \sum_{n \geq 1} n\, \theta(S_n). \quad (63)$$

Therefore

$$B = \sum_{n \geq 1} \theta(\Psi_n). \quad \square$$

The leaves of a planar binary tree with n vertices are labeled from 0 to n from left to right. The leaves with labels different from 0 and n are called *inner leaves*. A *descent* in a planar binary tree t is the label of an \-oriented inner leaf. The descent set $D(t)$ of t is the set of its descents.

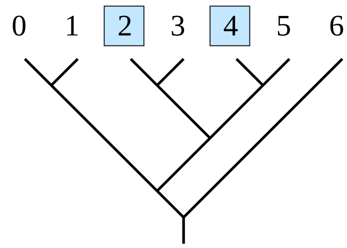


Figure 5. A planar binary tree t with descents at 2 and 4.

The number of descents of a planar binary tree t will be denoted $d(t)$. It satisfies $0 \leq d(t) \leq n-1$ for a tree t of degree n.

The *major index* $\mathrm{maj}(t)$ of t is the sum of its descents. For example, Figure 5 displays a planar binary tree with descent set $\{2,4\}$ and major index $2+4=6$.

Let us recall that the descent set $D(I)$ corresponding to a composition $I = (i_1, \dots, i_k)$ of n is the set $\{i_1, i_1+i_2, \dots, i_1+\cdots+i_k\}$.

Proposition 5.8. *The image by θ of R_I is the sum*

$$\sum_{\substack{\#t=n \\ D(t)=D(I)}} t \tag{64}$$

of all planar binary trees with n vertices and descent set $D(I)$.

Proof. This is a well-known property of the injection of **Sym** in Dend. □

In [Krob et al. 1997], elements $\Psi_n(A/(1-q))$, for $n \geq 1$, are defined by some "change of alphabet" applied to the elements Ψ_n. According to the proof of [Krob et al. 1997, Theorem 6.11], they are characterized by

$$\sum_{n\geq 1} \Psi_n\Big(\frac{A}{1-q}\Big) = \Big(\sum_{n\geq 0} S_n\Big)^{-1} \Big(\sum_{n\geq 1} q^n \Psi_n\Big(\frac{A}{1-q}\Big)\Big)\Big(\sum_{n\geq 0} S_n\Big) + \sum_{n\geq 1} \Psi_n. \tag{65}$$

There is a classical isomorphism of vector spaces α from **Sym** to the direct sum of all descent algebras of symmetric groups. By this morphism, each $\Psi_n(A/(1-q))$ is mapped, up to a multiplicative constant, to a Lie idempotent with coefficients in $\mathbb{Q}(q)$ in the descent algebra of the n-th symmetric group.

We can now state the precise relation between Ω_q and these Lie idempotents, or rather with the elements $\Psi_n(A/(1-q))$.

Proposition 5.9. *The image of $(1-q)\Psi(A/(1-q))$ by θ is $\varphi(\widetilde{\Omega_q})$.*

Proof. Indeed, by Proposition 5.6, one has

$$\sum_{n\geq 1} \varphi(\widetilde{\Omega_q}) = (1+L)^{-1} * (\varphi(\widetilde{\Omega_q})[q]) * (1+L) + (1-q) \sum_{n\geq 1} \varphi(\mathtt{Lnr}_n). \tag{66}$$

Then using Proposition 5.7 and (65), one gets that $\theta\big((1-q)\Psi(A/(1-q))\big)$ and $\varphi(\widetilde{\Omega_q})$ satisfy the same equation, hence they are equal. □

Let $\Omega_{q,n}$ be the homogeneous component of degree n of Ω_q.

Proposition 5.10. *One has*

$$\varphi(\Omega_{q,n}) = \frac{(-1)^{n-1}}{[n]_q} \sum_{\#t=n} (-1)^{d(t)} {\begin{bmatrix} n-1 \\ d(t) \end{bmatrix}}_q^{-1} q^{\mathrm{maj}(t)-\binom{d(t)+1}{2}}\, t. \tag{67}$$

Proof. Theorem 6.11 of [Krob et al. 1997] says that the element $(1-q)\Psi_n\Big(\frac{A}{1-q}\Big)$ is

$$\frac{1}{[n]_q} \sum_{|I|=n} (-1)^{d(I)} {\begin{bmatrix} n-1 \\ d(I) \end{bmatrix}}_q^{-1} q^{\mathrm{maj}(I)-\binom{d(I)+1}{2}}\, R_I. \tag{68}$$

By Proposition 5.9, the image under θ of this formula is $(-1)^{n-1}\varphi(\Omega_{q,n})$. By Proposition 5.8, this becomes the expected formula. □

6. Arithmetic properties

In this section, we obtain some properties of the denominators in Ω_q and consider what happens when q is specialized to $1, 0$ and ∞.

The case $q = 1$. First note that the morphism φ from $\widehat{\mathsf{PL}}$ to the completed free dendriform algebra $\widehat{\mathrm{Dend}}$ is defined over $\mathbb{Q}$ and injective. Hence one can deduce results on Ω_q from results on its image by φ.

Proposition 6.1. *The series Ω_q is regular at $q = 1$ and $\Omega_{q=1} = \Omega$.*

Proof. By Proposition 5.10, the image $\varphi(\Omega_q)$ is regular at $q = 1$, as q-binomial coefficients become usual binomial coefficients when $q = 1$. Therefore Ω_q itself is regular at $q = 1$.

At $q = 1$, (19) becomes (12). By uniqueness in Proposition 3.3, the value of Ω_q at $q = 1$ is Ω. □

Remark. Knowing that $\Omega_{q=1} = \Omega$, one can use (20) to compute simultaneously Ω_q and Ω up to order n in a $O(n^3)$ number of pre-Lie operations.

There is a lot of cancellation in the coefficients of Ω_q, leading to a reduced complexity of the denominators. Note that the expected denominator of $\Omega_{q,n}$ (from recursion (20)) is the product $\prod_{d=2}^n (q^d - 1)$. Let Φ_d be the d-th cyclotomic polynomial.

Proposition 6.2. *The common denominator of the coefficients of the element $\Omega_{q,n}$ divides the product $\prod_{d=2}^n \Phi_d$.*

Proof. For the image of Ω_q by φ, this follows from 5.10 and a simple property of the q-binomial coefficients: their only roots are simple roots at roots of unity, see [Guo and Zeng 2006, Proposition 2.2]. This implies the same result for Ω_q. □

Remark. The true denominator of each individual coefficient in Ω_q is often smaller than the complete product $\prod_{d=2}^{n} \Phi_d$, see for instance the first few terms in Appendix B.

The case q = 0. Let us consider now what happens when $q = 0$. Then Ω_0 is well-defined, $\Omega_q[q]$ vanishes and (19) becomes

$$\Omega_0 = \bullet - \bullet \curvearrowleft \Omega_0. \tag{69}$$

It follows that Ω_0 is the alternating sum of linear trees.

The case q = ∞. Let us now consider what happens when $q = \infty$. Let $\omega_{q,T}$ be the coefficient of the rooted tree T in the expansion of Ω_q. We define the *valuation* at $q = \infty$ as the smallest exponent in the formal Laurent expansion in powers of q^{-1} of an element of $\mathbb{Q}(q)$.

Proposition 6.3. *The valuation of $\omega_{q,T}$ at $q = \infty$ is at least $\#T - 1$.*

Proof. This will follow from the recursion (20). This is true in degree $n = 1$. Let us assume that $n \geq 2$. Then the valuation of $\bullet \curvearrowleft \Omega_{q,n-1}$ is at least $n-2$ by induction and the valuation of each term of the rightmost sum in (20) is at least -1. Hence the valuation of $\Omega_{q,n}$ is at least $n - 1$. □

Hence there exists a limit Ω_∞ for $\Omega_q[q]/q$ when q goes to ∞ and the limit of Ω_q/q is zero.

Equation (19), divided by q, becomes, at $q = \infty$,

$$\Omega_\infty \curvearrowleft \exp \Omega = \bullet. \tag{70}$$

By right action by $\exp(-\Omega)$, this is equivalent to

$$\Omega_\infty = \bullet \curvearrowleft \exp(-\Omega). \tag{71}$$

The element $\exp(-\Omega)$ is the inverse of $\exp \Omega$ in $\widehat{U}(\mathsf{PL})$. This has been computed in [Chapoton and Livernet 2007, Section 6.4]. More precisely, the inverse of $\sum_{n\geq 1} \frac{1}{(n-1)!} \mathtt{Crl}_n$ in the group of characters of the Connes–Kreimer Hopf algebra was shown there to be

$$\sum_T \frac{(-1)^{\#T-1}}{\operatorname{aut}(T)} T, \tag{72}$$

where $\operatorname{aut}(T)$ is the cardinal of the automorphism group of the rooted tree T. But it is known [Chapoton and Livernet 2001] that this group of characters is isomorphic to the group of group-like elements in $\widehat{U}(\mathsf{PL})$. Going through the isomorphism, one gets the following result.

Proposition 6.4. *The series Ω_∞ is given by*

$$\Omega_\infty = \sum_T \frac{(-1)^{\#T-1}}{\mathrm{aut}(T)} T. \tag{73}$$

7. Morphisms and images

In this section, we consider several quotients of the free pre-Lie algebra PL and the images of Ω_q in some of these quotients. The first two quotients come from quotients operads of the pre-Lie operad.

As shown in [Chapoton and Livernet 2001], the pre-Lie operad can be described in terms of labeled rooted trees. We recall here briefly (see the article just cited for details) the definition of the composition of two labeled rooted trees T and S on the vertex sets I and J, respectively. Let $i \in I$; the composition of S at the vertex i of T is given by

$$T \circ_i S = \sum_f T \circ_i^f S, \tag{74}$$

where the sum runs over all maps f from the set of incoming edges of the vertex i of T to the set of vertices of S, and $T \circ_i^f S$ can be described as follows: replace the vertex i by the tree S, grafting back the subtrees of T previously attached to i, according to the map f.

Morphism to the free associative algebra.

Proposition 7.1. *The subspace of pre-Lie spanned by nonlinear labeled trees is an ideal. The quotient map κ coincides with the usual map from pre-Lie to the associative operad* As.

Proof. Using the description above of the composition map of the operad pre-Lie, it is clear that the composition of two labeled trees, at least one of which is nonlinear, is again nonlinear. The quotient operad, spanned by labeled linear trees, has dimension $n!$ in rank n. Its composition can be easily identified with the associative operad As. The quotient map is then checked on generators of pre-Lie to be the same as the usual map. □

Proposition 7.2. *The group G_{As} is isomorphic to the group of invertible formal power series in $x\mathbb{Q}[[x]]$ for the composition product.*

Proof. It is more convenient here to work at the monoid level with $\widehat{\mathrm{As}}$ and $x\mathbb{Q}[[x]]$. The vector space $\mathrm{As}(n)_{\mathfrak{S}_n}$ is one dimensional for all n, with a basis element θ_n. By left linearity of both monoids, it is sufficient to check the product rule for θ_m and $f = \sum_{n\geq 1} f_n \theta_n$. One finds that

$$\theta_m \times f = \sum_{n_1,\dots,n_m \geq 1} f_{n_1} \dots f_{n_m} \theta_{n_1+\dots+n_m}, \tag{75}$$

which proves that the linear map defined by $x^n \mapsto \theta_n$ is an isomorphism between the monoids $\widehat{\mathrm{As}}$ and $x\mathbb{Q}[\![x]\!]$. The proposition follows by taking invertible elements. $\square$

The operad morphism κ induces a morphism of pre-Lie algebras from PL to the free (nonunital) associative algebra $\mathbb{Q}[x]_+$ on one generator x, sending $\bullet$ to x. This extends to a morphism from $\widehat{\mathsf{PL}}$ to the algebra $\mathbb{Q}[\![x]\!]_+$ of formal power series in x without constant term. This last morphism restricts to a group morphism from $\mathsf{G}_{\mathrm{pre-Lie}}$ to G_{As}. We will denote by κ all these morphisms.

One can see that κ sends the linear trees $\mathtt{Lnr}_n = (\bullet \curvearrowleft \bullet) \dots \curvearrowleft \bullet$ to the monomials $(xx)\dots x = x^n$ and maps all other trees to 0.

From (8), one obtains that $\kappa(\Omega)$ satisfies

$$\exp(\kappa(\Omega)) - 1 = x. \tag{76}$$

Therefore $\kappa(\Omega)$ is the formal power series

$$\log(1+x) = \sum_{n\geq 1} \frac{(-1)^{n-1}}{n} x^n. \tag{77}$$

One then deduces from (19) that the image of Ω_q is the q-logarithm defined by

$$\log_q(x) = \sum_{n\geq 1} \frac{(-1)^{n-1}}{[n]_q} x^n, \tag{78}$$

which is the unique solution to the functional equation

$$x \log_q(qx) = x \log_q(x) + (q-1)\, x - \log_q(qx) + \log_q(x). \tag{79}$$

Morphism to corollas and point-wise multiplication. Let the *depth* of a rooted tree be the maximum number of vertices in a chain of adjacent vertices between the root and a leaf. Corollas are then the rooted trees of depth at most 2.

Proposition 7.3. *The subspace of pre-Lie spanned by labeled noncorollas is an ideal.*

Proof. Using the description above of the composition of pre-Lie, one shows that the depth of the composition of two labeled trees is greater or equal than the maximum of the depths of these labeled trees. Therefore, if one of the labeled trees has depth greater or equal to three, so does the composition. $\square$

Let us denote by Mu the quotient operad and by λ the quotient map from pre-Lie to Mu. One can give a simple description of Mu. It has dimension n in rank n with basis given by the image of the labeled corollas with n nodes. Let us call μ_i^n the image of the corolla with n nodes and with root labeled by i for $i = 1, \dots, n$.

Then μ_1^1 is the unit of Mu and the composition is given by

$$\begin{cases} \mu_i^n \circ_i \mu_j^\ell = \mu_{i+j-1}^{n+\ell-1}, & \\ \mu_i^n \circ_h \mu_j^\ell = 0 & \text{for } h \neq i \text{ and } \ell \geq 2. \end{cases} \tag{80}$$

Let G_1 be the group of formal power series of the form $1 + x\mathbb{Q}[[x]]$ for the pointwise multiplication product and G_2 be the multiplicative group $\mathbb{Q}^*$. There is an action of G_2 on G_1 by substitution: $\lambda \cdot f(x) = f(\lambda x)$.

From the description of Mu above, one deduces that

Proposition 7.4. *The group* G_{Mu} *is isomorphic to* $G_2 \ltimes G_1$.

Proof. The vector space $\mathrm{Mu}(n)_{\mathfrak{S}_n}$ is one-dimensional for all n, with basis given by the image of the corolla with n nodes. Let us denote this basis element by ν_{n-1}. Any element of G_{Mu} can be uniquely written as the product $\lambda\big(\sum_{m\geq 0} f_m \nu_m\big)$ of $\lambda \in \mathbb{Q}^*$ and $f = \sum_{m\geq 0} f_m \nu_m$ with $f_0 = 1$. Let us compute the product of $\lambda f = \lambda\big(\sum_{m\geq 0} f_m \nu_m\big)$ and $\theta g = \theta\big(\sum_{n\geq 0} g_n \nu_n\big)$ with the conventions $f_0 = 1$ and $g_0 = 1$. One finds that

$$\lambda f \times \theta g = \sum_{m\geq 0}\sum_{n\geq 0} \lambda f_m \theta^m (\theta g_n) \nu_{n+m} = \lambda\theta \sum_{m\geq 0}\sum_{n\geq 0} \theta^m f_m g_n \nu_{n+m}. \tag{81}$$

One defines a map from G_{Mu} to $G_2 \ltimes G_1$ by $\lambda\big(\sum_m f_m \nu_m\big) \mapsto (\lambda, f(x))$ with $f(x) = \sum_m f_m x^m$. The product in $G_2 \ltimes G_1$ is given by

$$(\lambda, f(x))(\theta, g(x)) = (\lambda\theta, f(\theta x) g(x)). \tag{82}$$

Hence the map is an isomorphism. $\square$

The quotient map λ induces a morphism of pre-Lie algebras from PL to the following pre-Lie algebra. Let us identify the image of the corolla $\mathtt{Crl}_{n+1}$ with n leaves to x^n for all $n \geq 0$. In particular, the tree $\bullet$ is mapped to 1. The underlying vector space is therefore identified with $\mathbb{Q}[x]$ and the quotient pre-Lie product is

$$x^p \curvearrowleft x^q = \begin{cases} x^{p+1} & \text{if } q = 0, \\ 0 & \text{else.} \end{cases} \tag{83}$$

One can show (using the description of the quotient $\curvearrowleft$ product given above) that the right action by $\lambda(\Omega)$ is given by the product by x. Therefore, the right action of the image of $\exp(\lambda(\Omega)) - 1$ is just given by the product by $\exp x - 1$. One sees as well that the right action by $\lambda(\Omega_q)$ is also given by the product by x.

Then one deduces from Proposition 3.3 that the image $\lambda(\Omega)$ is the generating function $x/(\exp x - 1)$ for the Bernoulli numbers.

Next, from (19), one gets that the image $F_q(x)$ of Ω_q satisfies the equation

$$(\exp x - 1)[q F_q(qx)] = x + q - 1 - q F_q(qx) + F_q(x). \tag{84}$$

This functional equation is known (see for instance [Satoh 1989]) to describe the generating function

$$F = \sum_{n \geq 0} \beta_n(q) \frac{x^n}{n!}, \tag{85}$$

where $\beta_q(n)$ are the q-Bernoulli numbers of [Carlitz 1948; 1954; 1958].

Therefore the coefficients of the corollas in Ω_q are the q-Bernoulli numbers of Carlitz. One may wonder whether it is possible to describe directly the coefficient of each rooted tree in Ω_q.

Morphism to a pre-Lie algebra of vector fields. There exists an interesting morphism from PL to a pre-Lie algebra of vector fields. We describe it here only as a side remark, as the image of Ω_q seems to have no special property.

Consider the vector space $V = \mathbb{Q}[x]_+$, endowed with the following pre-Lie product:

$$f \curvearrowleft g = xf\, \partial_x g. \tag{86}$$

Then there is a unique morphism from PL to V sending $\bullet$ to x.

This map has the following nice property: the coefficient of x^n in the image of a series A is the sum of the coefficients of the trees in the homogeneous component A_n of A. The proof is just a check that this sum-of-coefficients map defines a morphism of pre-Lie algebra from PL to V.

Appendix A. A group associated to an augmented operad

We briefly recall here the definition for each augmented operad of a group of formal power series. This can also be found in [Chapoton 2002a; 2007/08; van der Laan 2003; Chapoton and Livernet 2007]. We use in this section the definition of the notion operad by a multiple composition map γ, which is equivalent (using the unit) to the definition using the single compositions $\circ_i$ that we have used elsewhere in the article.

Let $\mathcal{P}$ be an operad in the category $\mathrm{Vect}_{\mathbb{Q}}$ of vector spaces over $\mathbb{Q}$ and assume that $\mathcal{P}(0) = \{0\}$ and that $\mathcal{P}(1) = \mathbb{Q}e$ where e is the unit of $\mathcal{P}$. Such an operad is called *augmented*.

Let $\mathsf{F}\mathcal{P} = \oplus_n \mathcal{P}(n)_{\mathfrak{S}_n}$ be the direct sum of the coinvariant spaces, which can be identified with the underlying vector space of the free $\mathcal{P}$-algebra on one generator, and $\widehat{\mathcal{P}} = \prod_n \mathcal{P}(n)_{\mathfrak{S}_n}$ be its completion.

Let $x = \sum_m x_m$, $y = \sum_n y_n$ be two elements of $\widehat{\mathcal{P}}$. Choose any representatives $\bar{x}_m$ of x_m (resp. $\bar{y}_n$ of y_n) in the operad $\mathcal{P}$. Then one can check that the following formula defines a product on $\widehat{\mathcal{P}}$:

$$x \times y = \sum_{m \geq 1} \sum_{n_1,\dots,n_m \geq 1} \langle \gamma\,(\bar{x}_m, \bar{y}_{n_1}, \dots, \bar{y}_{n_m})\rangle, \tag{A.1}$$

where $\langle\ \rangle$ is the quotient map to the coinvariants and γ is the (multiple) composition map of the operad $\mathcal{P}$.

Proposition A.1. *The product $\times$ defines the structure of an associative monoid on the vector space $\widehat{\mathcal{P}}$. Furthermore, this product is $\mathbb{Q}$-linear on its left argument.*

Proof. We first prove associativity. On the one hand,

$$
\begin{aligned}
(x \times y) \times z
&= \sum_{m} \sum_{p_1,\dots,p_m} \langle \gamma\, (\overline{(x \times y)}_m, \bar{z}_{p_1}, \dots, \bar{z}_{p_m}) \rangle \\
&= \sum_{m} \sum_{n_1,\dots,n_m} \sum_{p_1,\dots,p_{n_1+\dots+n_m}} \langle \gamma\, (\gamma\, (\bar{x}_m, \bar{y}_{n_1}, \dots, \bar{y}_{n_m}), \bar{z}_{p_1}, \dots, \bar{z}_{p_{n_1+\dots+n_m}}) \rangle. \qquad \text{(A.2)}
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
x \times (y \times z)
&= \sum_{m} \sum_{n_1,\dots,n_m} \big\langle \gamma\, (\bar{x}_m, \overline{(y \times z)}_{n_1}, \dots, \overline{(y \times z)}_{n_m}) \big\rangle \\
&= \sum_{m} \sum_{n_1,\dots,n_m} \sum_{(q_{i,j})} \big\langle \gamma\, (\bar{x}_m, \gamma\, (\bar{y}_{n_1}, \bar{z}_{q_{1,1}}, \dots, \bar{z}_{q_{1,n_1}}), \dots, \gamma\, (\bar{y}_{n_m}, \bar{z}_{q_{m,1}}, \dots, \bar{z}_{q_{m,n_m}})) \big\rangle. \qquad \text{(A.3)}
\end{aligned}
$$

Using then the "associativity" of the operad, one gets the associativity of $\times$. It is easy to check that the image of the unit e of the operad $\mathcal{P}$ is a two-sided unit for the $\times$ product. The left $\mathbb{Q}$-linearity is clear from the formula (A.1). □

Proposition A.2. *An element y of $\widehat{\mathcal{P}}$ is invertible for $\times$ if and only if the first component y_1 of y is nonzero.*

Proof. The direct implication is trivial. The reverse one is proved by a very standard recursive argument. □

Let us call $\mathsf{G}_{\mathcal{P}}$ the set of invertible elements of $\widehat{\mathcal{P}}$ for the $\times$ product.

Proposition A.3. G *is a functor from the category of augmented operads to the category of groups.*

Proof. The functoriality follows from inspection of the definitions of $\widehat{\mathcal{P}}$ and $\times$. □

In fact, one can see $\mathsf{G}_{\mathcal{P}}$ as the group of $\mathbb{Q}$-points of a proalgebraic group. The Lie algebra of this proalgebraic group is given by the usual linearization process on the tangent space $\widehat{\mathcal{P}}$, resulting in the formula

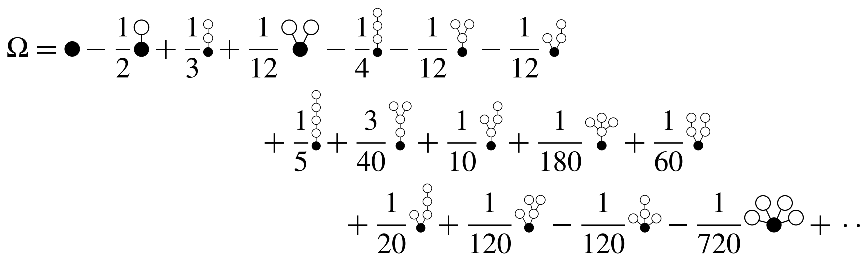
$$
[x, y] = \sum_{m \geq 1} \sum_{n \geq 1} \langle \bar{x}_m \circ \bar{y}_n - \bar{y}_n \circ \bar{x}_m \rangle, \qquad \text{(A.4)}
$$

where

$$\bar{x}_m \circ \bar{y}_n = \sum_{i=1}^{m} \gamma\,(\bar{x}_m, \underbrace{e, \dots, e}_{i-1 \text{ units}}, \bar{y}_n, e, \dots, e). \tag{A.5}$$

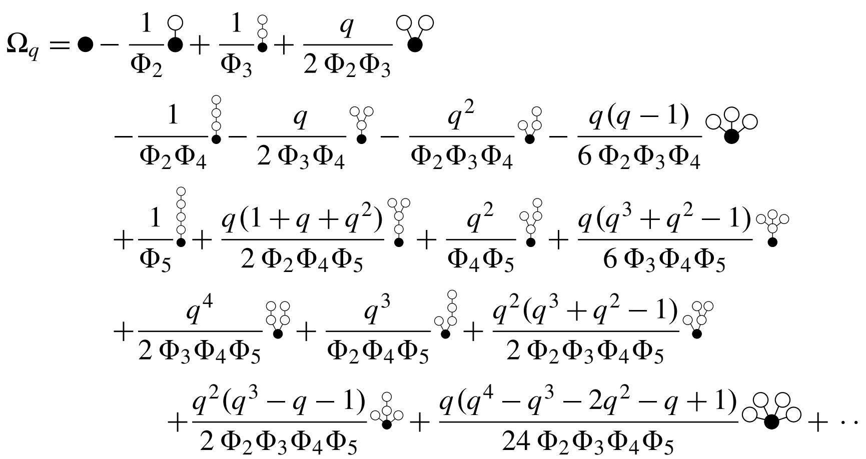
The graded Lie algebra structure on $\mathsf{F}\mathcal{P}$ defined by the same formulas has already appeared in the work of Kapranov and Manin on the category of right modules over an operad [Kapranov and Manin 2001, Theorem 1.7.3].

Appendix B. First terms of some expansions

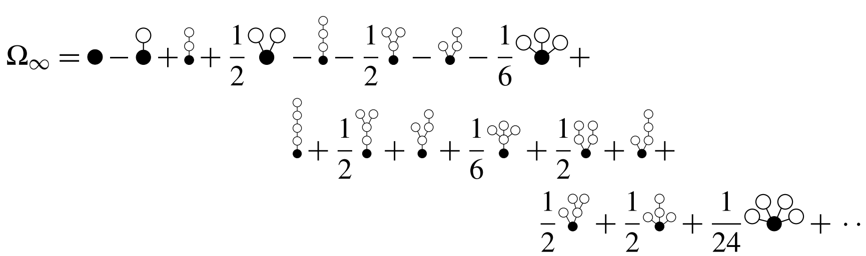


(B.1)

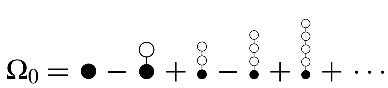
For $n \geq 1$, let Φ_n be the n-th cyclotomic polynomial.



(B.2)



(B.3)



(B.4)

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ALGEBRA AND NUMBER THEORY 3:6(2009)

A pencil of Enriques surfaces of index one with no section

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Monodromy arguments and deformation-and-specialization are used to prove existence of a pencil of Enriques surfaces with no section and index 1. The same technique "completes" the strategy from Graber et al. (2005) proving that the family of witness curves for dimension d depends on the integer d.

1. Introduction

This paper uses monodromy and deformation-and-specialization to answer some questions related to [Graber et al. 2005].

Theorem 1.1. *There exists a positive integer t (which the reader can readily compute) such that the following holds. Let k be any algebraically closed field with* $\mathrm{char}(k) \neq 2, 3$ *whose transcendence degree over the prime field is $\geq t$. This holds, for instance, if k is uncountable. There exists a flat, projective k-morphism $\pi : \mathscr{X} \to \mathbb{P}^1_k$ with the following properties.*

(i) *The geometric generic fiber of π is a smooth Enriques surface.*

(ii) *The invertible sheaf $\pi_*[\omega_\pi^{\otimes 2}]$ has degree* 6.

(iii) *For the function field K of $\mathbb{P}^1_k$ and the generic fiber X_K of π, the residue field of each closed point of X_K has degree ≥ 3 as an extension of K. Moreover, there exists at least one closed point of degree* 3 *and at least one closed point of degree* 4. *So the greatest common divisor of all degrees of closed points equals* 1.

Moreover every "very general" Enriques surface over k is a fiber of such a family.

What is the relevance of this result? First of all, in [Graber et al. 2005, Corollary 1.4] it was proved that there exists an Enriques surface X defined over the function field $K = \mathbb{C}(B)$ of some complex curve B such that X has no K-rational points.

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This answered a question which Serre asked [Colmez and Serre 2001, page 153]. Next, shortly after the results of [Graber et al. 2005] were proved, Lafon [2004] even gave an *explicit* pencil of Enriques surfaces defined over $(\mathbb{Z}[1/2])[t][1/f(t)]$ for an explicit monic polynomial $f(t)$ such that for every field k of characteristic $\neq 2$, the base change of the Enriques surface to $K = k(t)$ has no K-rational point.

Motivated by this, Hélène Esnault posed to Joe Harris the important question of whether or not these examples can be somehow explained "cohomologically". In particular, she posited that the Enriques surfaces from [Graber et al. 2005] have the stronger property that every 0-cycle has degree divisible by 4. Similarly, for Lafon's Enriques surface, every 0-cycle has even degree. This suggests that these examples, and by extension, perhaps all examples, can be understood in terms of obstructions which are defined as elements in Galois cohomology groups of the fraction field K of the base B. These obstructions are compatible with restriction and corestriction for finite, separable field extensions L/K. In concrete terms what this implies is that the order of the obstruction (as an element in the appropriate torsion Galois cohomology group) divides the degree over K of the residue field L of every closed point of X (for which L/K is separable). Thus, if one of these obstructions is nonzero, then the greatest common divisor of the degrees of all (separable) closed points is > 1. Therefore, since the greatest common divisor of all (separable) closed points is > 1 in the Enriques surfaces of [Graber et al. 2005] and [Lafon 2004], perhaps there is a cohomological explanation for those Enriques surfaces.

In Theorem 1.1, the degree 3 and degree 4 points are separable (since the characteristic is not 2, 3), and $\gcd(3, 4)$ equals 1. Therefore nonexistence of rational points is not due to a cohomological obstruction as above. Of course there may be a cohomological obstruction defined in a different way which does "explain" the nonexistence of rational points of these varieties.

For comparison, note that the greatest common divisor of degrees of closed points in Lafon's example equals 2. Although I am not certain, I expect that also in the examples from [Graber et al. 2005] (which are defined only in a very indirect manner), the greatest common divisor is again > 1.

Proposition 1.2. *Let k be an algebraically closed field having infinite transcendence degree over its prime subfield, for example, an uncountable algebraically closed field. Let B be an integral, normal, projective k-scheme of dimension ≥ 2. Let M be an integral quasiprojective k-variety. And let $\mathscr{C} \subset M \times_k B$ be a closed subscheme which is flat of relative dimension* 1 *over M with irreducible geometric fibers and which dominates B, that is, $(M, \mathscr{C})$ is a family $(\mathscr{C}_m)_{m \in M}$ of irreducible curves $\mathscr{C}_m$ in B such that $\cup_{m \in M} \mathscr{C}_m$ contains a dense open subset of B. For all integers n which are sufficiently positive, there exists a projective, dominant morphism of integral k-schemes $\pi : \mathscr{X} \to B$ having relative dimension n and whose* *restriction to each geometric fiber of $\mathscr{C} \to M$ has a section, but whose restriction to some smooth curve in B has no section. In the language of* [Graber et al. 2005], *this means that for every family of curves* $(M, \mathscr{C})$, *for all sufficiently positive integers* n, *the family is not a* witness family *for relative dimension* n.

This proposition completes the sketch given in [Graber et al. 2005, Section 7.3]. Just to repeat, the significance is that there does not exist a family of curves in B which is simultaneously for all integers n a witness family for relative dimension n.

In proving Theorem 1.1 and Proposition 1.2 it will be useful to recall the definition of the *index*.

Definition 1.3. Let X be a finite type scheme, algebraic space, algebraic stack, etc. over a field K. The *index* and the *minimal degree* are,

$$\begin{aligned} I(K, X) &= \gcd\{[L : K] | X(L) \neq \varnothing\}, \\ M(K, X) &= \min\{[L : K] | X(L) \neq \varnothing\}. \end{aligned}$$

The proofs of Theorem 1.1 and Proposition 1.2 both use the same technique; here is a brief description for Theorem 1.1. Over $\mathbb{P}^1$ a family of reducible surfaces is given whose monodromy group acts as the full group of symmetries of the dual graph of the geometric generic fiber. This dual graph is the 2-skeleton of a cube. There is an action of $\mathbb{Z}/2\mathbb{Z}$ acting fiberwise on the family. The quotient by this action is a pencil $\mathscr{X}/\mathbb{P}^1$ of *degenerate Enriques surfaces*, that is, the geometric generic fiber deforms to a smooth Enriques surface. The 8 vertices of the cube give a degree 4 multisection of the pencil. The 6 faces of the cube give a degree 3 multisection of the pencil. By monodromy considerations every multisection of X has degree ≥ 3. The pencil X together with the degree 3 and degree 4 multisections deforms to a pencil of surfaces whose geometric generic fiber is a smooth Enriques surface. For such a deformation which is sufficiently general, $M(K, X_K)$ equals 3 and $I(K, X_K)$ equals 1.

2. The construction for hypersurfaces

Let $n-1$ be a positive integer. The goal of this section is to construct the morphism π of relative dimension $n-1$ satisfying the condition from Proposition 1.2. The morphism is constructed as a family of hypersurfaces in $\mathbb{P}^n$ of degree $d > n$ parameterized by B, that is, $\mathscr{X}$ will be constructed as a closed subscheme of $B \times_k \mathbb{P}^n_k$ whose fibers over B, considered as closed subschemes of $\mathbb{P}^n$, are hypersurfaces of degree d. Of course this is equivalent to giving a rational transformation from B to the Hilbert scheme $\mathrm{Hilb}^{P_d(t)}_{\mathbb{P}^n_k/k}$ of degree d hypersurfaces in $\mathbb{P}^n$ (where $P_d(t)$ is the appropriate Hilbert polynomial). This Hilbert scheme is itself a projective space $\mathbb{P}^{N_{d,n}}_k$, where $N_{d,n}$ equals $\binom{n+d}{n} - 1$. The goal of this section is to construct

1-parameter families of degree d hypersurfaces in $\mathbb{P}^n$ with no section, and then explain why some smooth curves in B give such 1-parameter families.

The parameter space for degree d hypersurfaces. Let k be a field. For simplicity assume that k is algebraically closed. Let $d, n > 0$ be integers. Let V be a k-vector space of dimension $n+1$ so that $\mathbb{P}(V)$ is isomorphic to $\mathbb{P}^n_k$. For keeping track of the dense notation, it is simpler to use $\mathbb{P}(V)$ throughout rather than $\mathbb{P}^n_k$. There seem to be different conventions as to the meaning of $\mathbb{P}(V)$. Thus to make precise, $\mathbb{P}(V)$ is defined here to be

$$\mathbb{P}(V) := \mathrm{Proj} \bigoplus_{i \geq 0} \mathrm{Sym}^i_k(V^\vee).$$

In particular, there exists an invertible sheaf $\mathbb{O}_{\mathbb{P}(V)}(-1)$ and a monomorphism of coherent sheaves,

$$u : \mathbb{O}_{\mathbb{P}(V)}(-1) \to V \otimes_k \mathbb{O}_{\mathbb{P}(V)}$$

whose restriction over every sufficiently small Zariski open subset of $\mathbb{P}(V)$ is a split monomorphism. Moreover, the triple $(\mathbb{P}(V), \mathbb{O}_{\mathbb{P}(V)}(-1), u)$ is universal among all such triples of a k-scheme, an invertible sheaf, and a locally split monomorphism of coherent sheaves. This is the universal property of projective space used here. (Of course, by taking adjoint maps, it is equivalent to the property appearing elsewhere which uses locally split epimorphisms to an invertible sheaf.)

There is a minor positive characteristic issue that merits some discussion. The symmetric product $\mathrm{Sym}^d(V^\vee)$ is the maximal k-vector space quotient of $\bigotimes^d_k(V^\vee)$, which is invariant for the natural k-linear action of the symmetric group $\mathfrak{S}_d$, that is, $\mathrm{Sym}^d(V^\vee)$ is the space of $\mathfrak{S}_d$-coinvariants. In particular, $\bigoplus_{i \geq 0} \mathrm{Sym}^i(V^\vee)$ is the ring of polynomial functions from V to k. The adjoint of the coinvariant quotient map gives a canonical k-vector space isomorphism between the dual k-vector space of $\mathrm{Sym}^d(V^\vee)$ and the subspace $\mathrm{Syt}^d(V)$ of $\mathfrak{S}_d$-invariants in $\bigotimes^d_k(V^\vee)$, that is,

$$\mathrm{Hom}_k(\mathrm{Sym}^d(V^\vee), k) = \mathrm{Syt}^d(V).$$

Of course if $\mathrm{char}(k)$ is $> d$, then the induced map from invariants to coinvariants,

$$\mathrm{Syt}^d(V) \to \mathrm{Sym}^m(V),$$

is an isomorphism of k-vector spaces. But since Proposition 1.2 involves integers d which are arbitrarily large, it is important to distinguish between the invariants and coinvariants.

Degree d hypersurfaces in $\mathbb{P}(V)$ are parametrized by the projective space,

$$\mathbb{P}\,\mathrm{Sym}^d(V^\vee) = \mathrm{Proj} \bigoplus_{i \geq 0} \mathrm{Sym}^i(\mathrm{Syt}^d(V)).$$

To be precise, there is a closed subscheme $\mathcal{Y} \subset \mathbb{P}\,\mathrm{Sym}^d(V^\vee) \times \mathbb{P}(V)$ such that the projection to $\mathbb{P}\,\mathrm{Sym}^d(V^\vee)$ is flat with geometric fibers being degree d hypersurfaces in $\mathbb{P}(V)$. And the pair $(\mathbb{P}\,\mathrm{Sym}^d(V^\vee), \mathcal{Y})$ represents the Hilbert functor of $\mathbb{P}(V)$ for the appropriate Hilbert polynomial $P_{d,n}(t)$ of degree d hypersurfaces, that is, for

$$P_{d,n}(t) = \binom{n+t}{n} - \binom{n+t-d}{n}.$$

***Covers of* $\mathbb{P}^1$ *by* $\mathbb{P}^1$.** As an intermediate step, we will construct pencils of *unions of d hyperplanes*. To be a little more precise, for a parameter curve B (which we will assume to be $\mathbb{P}^1$ for simplicity), a pencil of unions of d hyperplanes over B is a closed subscheme $\mathcal{X} \subset B \times_k \mathbb{P}(V)$ such that the projection $\mathcal{X} \to B$ is flat and the geometric generic fiber is a union of d hyperplanes in $\mathbb{P}(V)$. Consider the normalization $(\mathcal{X})_{\mathrm{nor}}$ of $\mathcal{X}$. The Stein factorization of the associated morphism $(\mathcal{X})_{\mathrm{nor}} \to B$ is a finite morphism $C \to B$. To construct the closed subscheme $\mathcal{X} \subset B \times_k \mathbb{P}(V)$, we will first construct the morphism $C \to B$. For simplicity, we will construct it so that both B and C are smooth, proper, connected curves of genus 0. Thus, let B and C be given k-curves isomorphic to $\mathbb{P}^1_k$.

Lemma 2.1. *For every integer $d \geq 2$, there exists a degree d, separably-generated k-morphism $f : C \to B$ such that* $\mathrm{Gal}(k(C)/k(B))$ *is the full symmetric group $\mathfrak{S}_d$.*

Proof. One can prove this in many ways. The following proof is simple, but is only valid when $\mathrm{char}(k)$ is not 2. The proofs I know for characteristic 2 are considerably longer. One such proof constructs the cover f as a general deformation of a degree d, finite, flat, local complete intersection morphism $C \to B$ of nodal curves of genus 0 such that every component C_i of C maps to its image B_j in B as either an isomorphism or as a degree 2 cover,

$$\mathbb{P}^1 \to \mathbb{P}^1, \quad [X_0, X_1] \mapsto [X_0^2, X_1^2 - X_0 X_1],$$

where the ramification points, resp. branch points, are smooth points of C, resp. B, that is, the total map is not ramified at the nodes of C. In characteristic 2, notice that the map above is an Artin–Schreier map. By analyzing the combinatorics of the associated set map from the dual graph of C to the dual graph of B "decorated" by the information of the degree of $C_i \to B_j$, one can show that for a general such morphism, every sufficiently general deformation is a morphism as in the lemma. So the result reduces to a small amount of combinatorial analysis. On the other hand, the following argument involves no combinatorial analysis.

Choose an isomorphism of C with $\mathbb{P}^1$ and let $v_d : C \to \mathbb{P}^d_k$ be the associated d-uple Veronese morphism. Let B be a general line in the dual projective plane, $B \subset (\mathbb{P}^d_k)^\vee$. Let $\Gamma \subset C \times_k B$ be the corresponding pencil of degree d divisors in C parameterized by B. By [Deligne and Katz 1973, Théorème XVII.2.5], this is a

Lefschetz pencil. Thus the base locus is transverse to C. Since C is 1-dimensional, this means the base locus is empty. Therefore the projection $\Gamma \to C$ is an isomorphism. In other words, Γ equals the graph of a morphism $f : C \to B$, that is, $\Gamma = \Gamma_f$. Since Γ is a pencil of degree d divisors in C, the morphism f has degree d. Since this is a Lefschetz pencil, every ramification point c_i of f is an ordinary double point and the images $b_i = f(c_i)$ are all distinct points of B. Since char(k) is not 2, this means that f is a tamely ramified cover. By Riemann–Hurwitz for tamely ramified morphisms, there are $2d-2$ branch points $b_1, \dots, b_{2d-2}$.

Denote $U = B \setminus \{b_1, \dots, b_{2d-2}\}$. Denote by $\overline{u}$ a geometric point of U. The morphism $f : f^{-1}(U) \to U$ is a finite étale morphism of degree d. Therefore it defines a group homomorphism from the tamely ramified fundamental group

$$\phi : \pi_1^t(U, \overline{u}) \to \mathfrak{S}_d,$$

which is well-defined up to inner automorphism of $\mathfrak{S}_d$. By the description of the tamely ramified fundamental group in [Grothendieck 1962, Corollaire XIII.2.12], $\pi_1^t(U, \overline{u})$ is topologically generated by elements $\sigma_1, \dots, \sigma_{2d-2}$ such that the image of σ_i is a topological generator of the inertia group at b_i. Because f is simply ramified over b_i, the image $\phi(\sigma_i)$ is a transposition in $\mathfrak{S}_d$. Since $f^{-1}(U)$ is irreducible, Image(ϕ) is a transitive subgroup of $\mathfrak{S}_d$. Since Image(ϕ) is a transitive subgroup of $\mathfrak{S}_d$ which is generated by transpositions, Image(ϕ) equals all of $\mathfrak{S}_d$. Thus the Galois group of $k(C)/k(B)$ is the full symmetric group $\mathfrak{S}_d$. □

Norm sheaves and norm maps. Given a degree d cover $f : C \to B$ as in Lemma 2.1, and given a k-morphism from $g : C \to \mathbb{P}(V^\vee)$ from C to $\mathbb{P}(V^\vee)$, the parameter space of hyperplanes in $\mathbb{P}(V)$, there is an associated k-morphism from B to $\mathbb{P}(Sym^d(V^\vee))$ sending a geometric point b of B to the reducible hypersurface which is the union over the d points c of $f^{-1}(b)$ of the corresponding hyperplane. This is made precise using norm sheaves and norm maps.

For simplicity, let $g : C \to \mathbb{P}(V^\vee)$ be a closed immersion whose image is a rational normal curve of degree n, for example,

$$\mathbb{P}^1 \to \mathbb{P}^n, \quad [X_0, X_1] \mapsto [X_0^n, X_0^{n-1}X_1, \dots, X_0^{n-i}X_1^i, \dots, X_0X_1^{n-1}, X_1^n].$$

Consider the pullback under g of the tautological surjection, $V \otimes_k \mathcal{O}_C \to g^*\mathcal{O}(1)$. Of course $V \otimes_k \mathcal{O}_C$ is canonically isomorphic to $f^*(V \otimes_k \mathcal{O}_B)$. By adjointness of f^* and f_*, there is an associated morphism of $\mathcal{O}_B$-modules,

$$\beta : V \otimes_k \mathcal{O}_B \to f_*(g^*\mathcal{O}(1)).$$

Now for every locally free $\mathcal{O}_C$-module $\mathscr{E}$ there is the *associated norm sheaf* on B, defined as

$$\mathrm{Nm}_f(\mathscr{E}) = \mathrm{Hom}_{\mathcal{O}_B}\big(\textstyle\bigwedge^d(f_*\mathcal{O}_C), \bigwedge^d(f_*\mathscr{E})\big).$$

There is also the *associated norm map* $\alpha'_{\mathscr{E}} : \bigotimes^{d}(f_*\mathscr{E}) \to \mathrm{Nm}_f(\mathscr{E})$ of $\mathbb{O}_B$-modules, defined by

$$e_1 \otimes \cdots \otimes e_d \mapsto (c_1 \wedge \cdots \wedge c_d \mapsto (c_1 \cdot e_1) \wedge \cdots \wedge (c_d \cdot e_d)),$$

for $e_1 \otimes \cdots \otimes e_d \in \bigotimes^d (f_*\mathscr{E})$ and $c_1 \wedge \cdots \wedge c_d \in \bigwedge^d (f_*\mathbb{O}_B)$. In fact we will only need the restriction to the subsheaf of symmetric tensors, which we denote as follows

$$\alpha_{\mathscr{E}} : \mathrm{Syt}^d(f_*\mathscr{E}) \to \mathrm{Nm}_f(\mathscr{E}).$$

In particular, observe that $\mathrm{Nm}_f(\mathbb{O}_C)$ equals $\mathbb{O}_B$ and that for every local section b of $f_*\mathbb{O}_C$, the norm $\alpha_{\mathbb{O}_C}(b \otimes \cdots \otimes b) \in \mathbb{O}_B$ is the usual norm of b.

The family of unions of hyperplanes. Denote by γ the composition,

$$\mathrm{Syt}^d(V) \otimes_k \mathbb{O}_B \xrightarrow{\mathrm{Syt}^d(\beta)} \mathrm{Syt}^d(f_*g^*\mathbb{O}(1)) \xrightarrow{\alpha_{g^*\mathbb{O}(1)}} \mathrm{Nm}_f(g^*\mathbb{O}(1)).$$

Because β is surjective, also γ is surjective. Because f is finite, there exists a Zariski open covering $\{U_i\}$ of B such that $g^*\mathbb{O}(1)$ is trivial on each open $f^{-1}(U_i)$. Thus

$$\mathrm{Nm}_f(g^*\mathbb{O}(1))$$

is locally isomorphic to $\mathrm{Nm}_f(\mathbb{O}_C)$, that is, to $\mathbb{O}_B$. So $\mathrm{Nm}_f(g^*\mathbb{O}(1))$ is an invertible $\mathbb{O}_B$-module and γ is locally a split epimorphism. Therefore, by the universal property of projective space, there is an induced morphism $h : B \to \mathbb{P}\,\mathrm{Sym}^d(V^\vee)$. Let b be a geometric point of B whose fiber $f^{-1}(b)$ is a reduced set $\{c_1, \ldots, c_d\}$. For every $i = 1, \ldots, d$, the image $g(c_i)$ equals $[L_i]$ for a linear functional $L_i \in V^\vee$. And then $h(b)$ equals $[L_1 \times \cdots \times L_d]$. The degree of $\mathrm{Nm}_f(g^*\mathbb{O}(1))$ equals n. Thus $h^*\mathbb{O}(1)$ is an invertible $\mathbb{O}_B$-module of degree n.

Denote by $\mathscr{Y}_h \subset B \times \mathbb{P}(V)$ the preimage under (h, Id) of the universal hypersurface $\mathscr{Y}$ in $\mathbb{P}\,\mathrm{Sym}^d(V^\vee) \times \mathbb{P}(V)$. And denote by $\pi : \mathscr{Y}_h \to B$ the projection. Let $m = \min(d, n)$ and let $S_{d,n} \subset \mathbb{Z}_{\geq 0}$ denote the additive semigroup generated by $\binom{d}{i}$ for $i = 1, \ldots, m$. Denote $k(B)$ by K and denote by $\mathscr{Y}_{h,K}$ the generic fiber of π.

Proposition 2.2. *For every irreducible multisection of π, there exists an integer $i = 1, \ldots, m$ such that the degree of the multisection is divisible by $\binom{d}{i}$. The degree of every multisection is in the semigroup $S_{d,n}$. In particular, if d is greater than n then $M(K, \mathscr{Y}_{h,K})$ equals d and $I(K, \mathscr{Y}_{h,K})$ is divisible by* $\gcd\big(d, \binom{d}{2}, \ldots, \binom{d}{n}\big)$.

Proof. Denote by $U \subset B$ the largest open subset over which f is étale and define $W = f^{-1}(U)$. For each $i = 1, \ldots, m$, denote by W_i/U the relative Hilbert scheme $\mathrm{Hilb}^i_{W/U}$. Because W is étale over U, the fiber of f over a geometric point b of B is a set of d distinct points, $f^{-1}(b) = \{c_1, \ldots, c_d\}$, and the fiber of $\mathrm{Hilb}^i_{W/U}$ is the set of subsets of $f^{-1}(b)$ of size i. Every geometric fiber of $\mathscr{Y}_h \times_B U \to U$ is a union of d hyperplanes. Denote by

$$\mathcal{Y}_h \times_B U = \mathcal{Y}_h^1 \sqcup \mathcal{Y}_h^2 \sqcup \cdots \sqcup \mathcal{Y}_h^n$$

the locally closed stratification where $\mathcal{Y}_h^i$ is the set of points x in precisely i irreducible components of the geometric fiber $\mathcal{Y}_h \otimes_{\mathbb{O}_B} \overline{\kappa}(\pi(x))$.

Because every finite subset of distinct closed points on a rational normal curve over an algebraically closed field is in linearly general position, $\mathcal{Y}_h^i$ is empty for all $i > m$. In particular every geometric fiber of $\mathcal{Y}_h \times_B U \to U$ is a simple normal crossings variety. For each $i = 1, \dots, m$ the morphism $\mathcal{Y}_h^i \to U$ factors as an $\mathbb{A}^{n-i}$-bundle over W_i over U.

For each irreducible multisection of π, there exists an integer $i \le m$ (depending on the multisection) such that the generic point of the multisection is contained in $\mathcal{Y}_h^i$. Because $\mathrm{Gal}(k(C)/k(B))$ is the full symmetric group $\mathfrak{S}_d$, in particular it is i-transitive. Thus W_i is irreducible. Therefore the degree of the multisection is divisible by $\deg(k(W_i)/k(U))$ which equals $\binom{d}{i}$. So the degree of every multisection, irreducible or not, is in $S_{d,n}$. In particular, the degree is $\ge d$, that is, $M(K, \mathcal{Y}_{h,K}) \ge d$. Conversely, the intersection of $\mathcal{X}_{h,K}$ with a general line in $\mathbb{P}(V \otimes_k K)$ is a degree d multisection. Therefore $M(K, \mathcal{Y}_{h,K})$ equals d. □

Denote by $\mathrm{Hom}(B, \mathbb{P}\,\mathrm{Sym}^d(V^\vee))$ the Hom scheme [Grothendieck 1961, 4.c, p. 19]. And denote by $H_n \subset \mathrm{Hom}(B, \mathbb{P}\,\mathrm{Sym}^d(V^\vee))$ the irreducible component parameterizing those morphisms of degree n. Denote the universal morphism by

$$\chi : H_n \times_k B \to \mathbb{P}\,\mathrm{Sym}^d(V^\vee).$$

Denote by $\mathcal{Y}_\chi \subset H_n \times_k B \times_k \mathbb{P}(V)$ the pullback under $\chi \times \mathrm{Id}_{\mathbb{P}(V)}$ of the universal degree d hypersurface $\mathcal{Y} \subset \mathbb{P}\,\mathrm{Sym}^d(V^\vee) \times_k \mathbb{P}(V)$. For every field k' and every $[j] \in H_n(k')$, denote by $\mathcal{Y}_j$ the restriction of $\mathcal{Y}_\chi$ to $\mathrm{Spec}\,(k') \times B$. Also denote by K' the function field $k'(B)$, and denote by $\mathcal{Y}_{j,K'}$ the fiber of $\mathcal{Y}_j$ over the generic point $\mathrm{Spec}\,(K')$ of $k'(B)$.

Corollary 2.3. *Assume $d > n$. In H_n there is a countable intersection of open dense subsets such that for every $[j]$ in this set, $M(K', \mathcal{Y}_{j,K'}) = d$ and $I(K', \mathcal{Y}_{j,K'})$ is divisible by $\gcd(d, \dots, \binom{d}{n})$. In particular this holds when $[j]$ equals the geometric generic point of H_n.*

Proof. The subset $H_n^{\mathrm{good}} \subset H_n$ where $\gcd\big(d, \dots, \binom{d}{n}\big)$ divides $I(K', \mathcal{X}_{h,K'})$ and where $M(K', \mathcal{X}_{K'})$ is at least d is a countable intersection of open subsets by standard Hilbert scheme arguments: the complement of this set is the union indexed by the countably many Hilbert polynomials $P(t)$ of multisections of degree which are either less than d or which are not divisible by $\gcd(d, \dots, \binom{d}{n})$ of the closed image in H_n of the relative Hilbert scheme $\mathrm{Hilb}^{P(t)}_{\mathcal{X}/H_n}$. By Proposition 2.2 H_n^{good} is nonempty. Therefore it is a countable intersection of open *dense* subsets. Of course the intersection of $\mathcal{Y}_{h,K'}$ with a general line in $\mathbb{P}(V \otimes_k K')$ gives a multisection of degree d. Therefore H_n^{good} is actually the set where $M(K', \mathcal{X}_{K'})$ equals d and where $\gcd(d, \dots, \binom{d}{n})$ divides $I(K', \mathcal{X}_{h,K'})$. $\square$

2.1. *Proof of Proposition 1.2.* Let k be an algebraically closed field having infinite transcendence degree over its prime subfield. The main case of Proposition 1.2 is actually the special case where B equals $\mathbb{P}^1_k \times_k \mathbb{P}^1_k$ and where the family of curves $(M, \mathcal{C})$ is the complete linear system $|\mathbb{O}(a, b)|$ for some integers $a, b \geq 0$.

Assume first that one of a or b equals 0, say $b = 0$. Let $f : Y \to \mathbb{P}^1_k$ be a finite, separably-generated morphism of irreducible curves of degree > 1, and let $\mathcal{X}$ be $Y \times \mathbb{P}^1$ with projection $\pi = (f, \mathrm{Id})$. Every divisor in $|\mathbb{O}(a, 0)|$ is a union of fibers of pr_1, so the restriction of π has a section. The restriction of π over every fiber of pr_2 is just f, and so has no rational section.

Thus assume next that both a and b are positive. Define $n = 4ab$ and define $d = n - 1$. Let V be a k-vector space of dimension $n+1$ as above. Let $C \subset \mathbb{P}^1 \times \mathbb{P}^1$ be a smooth curve in the linear system $|\mathbb{O}(1, 2b)|$. By Corollary 2.3, there exists a closed immersion of degree n, $j : C \to \mathbb{P}\,\mathrm{Sym}^d(V^\vee)$, such that $M(k(C), \mathcal{Y}_{j,k(C)})$ equals d, which is > 1. Of course j extends to a closed immersion

$$j : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}\,\mathrm{Sym}^d(V^\vee)$$

such that $j^*\mathbb{O}(1)$ is $\mathbb{O}(2a - 1, 2b)$. Indeed the restriction map

$$H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathbb{O}(2a - 1, 2b)) \to H^0(C, \mathbb{O}_C(n))$$

is surjective. Define $\pi : \mathcal{X} \to \mathbb{P}^1 \times \mathbb{P}^1$ to be the pullback under $j \times \mathrm{Id}_{\mathbb{P}(V)}$ of the universal hypersurface $\mathcal{Y} \subset \mathbb{P}\,\mathrm{Sym}^d(V^\vee) \times_k \mathbb{P}(V)$. By construction, the restriction over C has no section.

Every divisor in $|\mathbb{O}(a, b)|$ maps under j to a curve in $\mathbb{P}\,\mathrm{Sym}^d(V^\vee)$ whose degree with respect to $\mathbb{O}(1)$ is $\leq n - b$. Moreover, if the degree equals $n - b$, then j maps the divisor birationally to its image. Thus the arithmetic genus p_a of the image is at least the geometric genus of the divisor, that is, $p_a \geq (a - 1)(b - 1)$. A curve of arithmetic genus p_a and degree δ spans a linear space of (projective) dimension $\leq \delta - p_a$. Thus the span of the image of the divisor is either $\leq n - b - 1$ (if $\delta \leq n - b - 1$) or $\leq n - b - (a - 1)(b - 1)$ which is again $\leq n - b - 1$ (when δ equals $n - b$). The span is a linear system of hypersurfaces in $\mathbb{P}(V)$. Since $n - b - 1$ is $\leq n$, this linear system has a nonempty base locus. But every point in the base locus gives a section of the corresponding family of hypersurfaces. Thus it also gives a section of the restriction of $\mathcal{X}$ over the divisor. This proves the main case of Proposition 1.2, that is, the case when B equals $\mathbb{P}^1 \times \mathbb{P}^1$ and when M is a complete linear system $|\mathbb{O}(a, b)|$.

Next let B be an arbitrary normal, projective variety of dimension ≥ 2 and let M be an irreducible family of irreducible curves dominating B. There exists a smooth open subset $U \subset B$ whose complement has codimension ≥ 2, and there exists a dominant morphism $g : U \to \mathbb{P}^1 \times \mathbb{P}^1$. Intersecting U with general hyperplanes, there exists an irreducible closed subset $Z \subset U$ such that $g|_Z : Z \to \mathbb{P}^1 \times \mathbb{P}^1$ is generically finite of some degree $e > 0$. For the geometric generic point of M, the intersection of the corresponding curve with U is nonempty, and the closure of the image under f is a divisor in the linear system $|\mathcal{O}(a', b')|$ for some nonnegative integers a', b'. Let $a \geq a'$, and $b \geq b'$ be integers such that $4ab > e + 1$. There exists a projective, dominant morphism $\pi : \mathscr{X} \to \mathbb{P}^1 \times \mathbb{P}^1$ whose restriction over every divisor in $|\mathcal{O}(a, b)|$ has a section, but whose restriction over a general divisor in $|\mathcal{O}(1, 2b)|$ has minimal degree $4ab - 1$.

Define $\mathscr{X}_B \subset B \times \mathscr{X}$ to be the closure of $U \times_{\mathbb{P}^1 \times \mathbb{P}^1} \mathscr{X}$. Then $\pi_B : \mathscr{X}_B \to B$ is a projective dominant morphism. For the geometric generic point of M, the restriction of π_B to the curve has a section because the restriction of π to the image in $\mathbb{P}^1 \times \mathbb{P}^1$ has a section. Let $C_B \subset Z$ be the preimage of a general curve C in $|\mathcal{O}(1, 2b)|$. The morphism $C_B \to C$ has degree $e < 4ab - 1$. Because every multisection of π over C has degree $\geq 4ab - 1$, π_B has no section over C_B.

3. The construction for Enriques surfaces

Next let k be a field of characteristic $\neq 2, 3$ whose transcendence degree over the prime subfield is "sufficiently large". In fact, let us simply assume that k is uncountable. It is straightforward to trace through the following arguments to find an integer N such that everything remains valid if the transcendence degree over the prime subfield is $\geq N$. However, we think this extra bookkeeping would only distract from the proof, which is already burdened by heavy notation.

We will make use of one particular construction of Enriques surfaces over k. To that end, let V_+ and V_- be three-dimensional k-vector spaces. Denote $V = V_+ \oplus V_-$ and denote $V' = \mathrm{Sym}^2(V_+^\vee) \oplus \mathrm{Sym}^2(V_-^\vee)$. Denote by G the Grassmannian $\mathrm{Grass}(3, V')$ parametrizing three-dimensional *subspaces* of V'. This is a parameter space for Enriques surfaces, as we shall explain.

In fact there are two descriptions of the universal family, each useful. First, let $\pi_Z : Z \to \mathbb{P}(V_+) \times \mathbb{P}(V_-)$ be the projective bundle of the locally free sheaf $\mathrm{pr}_+^* \mathcal{O}_{\mathbb{P}(V_+)}(-2) \oplus \mathrm{pr}_-^* \mathcal{O}_{\mathbb{P}(V_-)}(-2)$. A general complete intersection of three divisors in $|\mathcal{O}_Z(1)|$ is an Enriques surface. Because $H^0(Z, \mathcal{O}_Z(1)) = V'$, the parameter space for these complete intersections is G. Second, G parametrizes complete intersections in $\mathbb{P}(V)$ of three quadric divisors that are invariant under the involution ι of $\mathbb{P}(V)$ whose (-1)-eigenspace is V_- and whose $(+1)$-eigenspace is V_+. A general such complete intersection is a K3 surface on which ι acts as a fixed-point-free involution. The quotient by ι is an Enriques surface.

These two descriptions are equivalent. The involution extends to an involution $\tilde{\imath}$ on the blowing up $\widetilde{\mathbb{P}(V)}$ of $\mathbb{P}(V)$ along $\mathbb{P}(V_+) \cup \mathbb{P}(V_-)$ and the quotient is Z. Denote by $\mathscr{X} \to G$ the universal family of Enriques surfaces, and denote by $\mathscr{Y} \to G$ the universal family of K3 covers.

As in the proof of Corollary 2.3, the one-parameter family of Enriques surfaces in Theorem 1.1 will be a general deformation of a one-parameter family of reducible surfaces. As in Proposition 2.2, the one-parameter family of reducible surfaces will be constructed using a particular cover of $\mathbb{P}^1$ by $\mathbb{P}^1$. Thus let B, C, D be k-curves isomorphic to $\mathbb{P}^1_k$. By a result similar to Lemma 2.1, there exists a degree 2, separably-generated morphism $g : D \to C$ and a degree 3, separably-generated morphism $f : C \to B$ such that $\mathrm{Gal}(k(D)/k(B))$ is the full wreath product $\mathfrak{W}_{3,2}$, that is, the semidirect product $(\mathfrak{S}_2)^3 \rtimes \mathfrak{S}_3$. In characteristic 0, this holds whenever g and f have simple branching and the branch points of g are in distinct, reduced fibers of f. There is an involution $\imath_D$ of D commuting with g.

Now we will construct the family of reducible surfaces using norms as in Section 2. Let $j : D \to \mathbb{P}(V^\vee)$ be a closed immersion equivariant for $\imath_D$ and $\imath$ whose image is a rational normal curve of degree 5. By the construction in Section 2, there is an associated morphism $i : C \to \mathbb{P}\,\mathrm{Sym}^2(V^\vee)$. Because j is equivariant, i factors through $\mathbb{P}(V')$. In the rest of this paragraph we will compute that

$$i^*\mathbb{O}(1) = \mathrm{Nm}_g(j^*\mathbb{O}(1)) \cong \mathbb{O}_C(5).$$

Indeed, $j^*\mathbb{O}(1)$ has degree 5 on D by construction. And g has degree 2. Thus $j^*\mathbb{O}(1) \otimes g^*\mathbb{O}_C(-3)$ has degree -1, hence has both $h^0 = 0$ and $h^1 = 0$. Since g is affine, g_* is exact on $\mathbb{O}_D$-modules and $R^i g_*$ is zero on $\mathbb{O}_D$-modules for $i > 0$. Thus, by a Leray spectral sequence, $g_* j^*\mathbb{O}(1) \otimes_{\mathbb{O}_C} \mathbb{O}_C(-3)$ is a rank 2 locally free sheaf with $h^0 = 0$ and $h^1 = 0$. By Grothendieck's lemma, this sheaf is a direct sum of two invertible sheaves. And the condition on h^0 and h^1 implies that $g_* j^*\mathbb{O}(1) \otimes_{\mathbb{O}_C} \mathbb{O}_C(-3)$ is isomorphic to $\mathbb{O}_c(-1)^{\oplus 2}$. Thus $g_* j^*\mathbb{O}(1)$ is isomorphic to $\mathbb{O}_C(2)^{\oplus 2}$, which has determinant $\mathbb{O}_C(4)$. Again by considering h^0 and h^1 and using Grothendieck's lemma, $g_*\mathbb{O}_D$ is isomorphic to $\mathbb{O}_C \oplus \mathbb{O}_C(-1)$, which has determinant $\mathbb{O}_C(-1)$. Thus $\mathrm{Nm}_g(j^*\mathbb{O}(1))$ is isomorphic to $\mathrm{Hom}_{\mathbb{O}_C}(\mathbb{O}_C(-1), \mathbb{O}_C(4))$, i.e., $\mathbb{O}_C(5)$.

The pushforward by f_* of the pullback by i^* of the tautological surjection is a surjection $(V')^\vee \otimes \mathbb{O}_B \to f_* i^*\mathbb{O}(1)$. The sheaf $f_* i^*\mathbb{O}(1)$ is locally free of rank 3. By the same type of analysis of h^0 and h^1 as in the previous paragraph, $f_*\mathbb{O}_C(5) \otimes_{\mathbb{O}_B} \mathbb{O}_B(-2)$ is isomorphic to $\mathbb{O}_B(-1)^{\oplus 3}$. Thus $f_* i^*\mathbb{O}(1)$, that is, $f_* i^*\mathbb{O}_C(5)$, is isomorphic to $\mathbb{O}_B(1)^3$. So there is an induced morphism $h : B \to G$. Denote by $\pi_h : \mathscr{X}_h \to B$ and $\rho_h : \mathscr{Y}_h \to B$ the base-change by h of $\mathscr{X}$ and $\mathscr{Y}$. Denote $K = k(B)$ and denote by $\mathscr{X}_{h,K}$ the generic fiber of π_h.

Proposition 3.1. *Every irreducible multisection of π_h has degree divisible by* 3 *or* 4. *In particular $M(K, \mathscr{X}_{h,K}) = 3$.*

Proof. This is a combinatorial analysis, of precisely the sort we avoided in the proof of Lemma 2.1. Unfortunately here it seems necessary. Denote by $U \subset B$ the open set over which $f \circ g$ is étale, and denote by $W \subset D$ the preimage of U. Denote by $c : \widetilde{W} \to U$ the Galois closure of W/U. Then $c^* f_* \mathcal{O}_C|_U \cong \mathcal{O}_{\widetilde{W}}\{\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3\}$ for idempotents $\boldsymbol{a}_p$, $p = 1, 2, 3$. And $c^* g_* f_* \mathcal{O}_D|_U \cong \mathcal{O}_{\widetilde{W}}\{\boldsymbol{b}_{1,1}, \boldsymbol{b}_{1,2}, \boldsymbol{b}_{2,1}, \boldsymbol{b}_{2,2}, \boldsymbol{b}_{3,1}, \boldsymbol{b}_{3,2}\}$ for idempotents $\boldsymbol{b}_{p,q}$, $p = 1, 2, 3$, $q = 1, 2$. Of course $\boldsymbol{a}_p \mapsto \boldsymbol{b}_{p,1} + \boldsymbol{b}_{p,2}$, $p = 1, 2, 3$. The action of the Galois group $\mathfrak{W}_{3,2}$ on $\boldsymbol{a}_p$ is by the symmetric group $\mathfrak{S}_3$, and the action on $\boldsymbol{b}_{p,q}$ is the standard representation of the wreath product. In the next paragraph, every index p corresponds to an index p of $\boldsymbol{a}_p$. As here, these induces are permuted by $\mathfrak{W}_{3,2}$ through its quotient $\mathfrak{S}_3$. And in the next paragraph, every index (p, q) corresponds to an index (p, q) of $\boldsymbol{b}_{p,q}$. The action of $\mathfrak{W}_{3,2}$ on these indices is the standard representation of the wreath product.

For each $p = 1, 2, 3$ and $q = 1, 2$, denote by $j_{p,q} : \widetilde{W} \to \mathbb{P}(V^\vee)$ the morphism obtained by composing the idempotent $\boldsymbol{b}_{p,q} : \widetilde{W} \to \widetilde{W} \times_U W$ with the basechange of j. In particular, $\imath \circ j_{p,1} = j_{p,2}$. Denote by $\Lambda_{p,q} \subset \widetilde{W} \times \mathbb{P}(V)$ the pullback by $(j_{p,q}, \mathrm{Id})$ of the universal hyperplane. Denote by $\mathcal{Y}_{\widetilde{W}}$ the base-change to $\widetilde{W}$ of $\mathcal{Y}_h$. Then

$$\mathcal{Y}_{\widetilde{W}} = \bigcup_{(q_1,q_2,q_3) \in \{1,2\}^3} (\Lambda_{1,q_1} \cap \Lambda_{2,q_2} \cap \Lambda_{3,q_3}).$$

There is a locally closed stratification

$$\mathcal{Y}_{\widetilde{W}} = \mathcal{Y}^3_{\widetilde{W}} \sqcup \mathcal{Y}^4_{\widetilde{W}} \sqcup \mathcal{Y}^5_{\widetilde{W}},$$

where $\mathcal{Y}^l_{\widetilde{W}}$ is the set of points lying in the intersection of precisely l of the $\Lambda_{p,q}$. The stratum $\mathcal{Y}^3_{\widetilde{W}}$ is the union of eight connected open subsets,

$$\Lambda_{(q_1,q_2,q_3)} \subset (\Lambda_{1,q_1} \cap \Lambda_{2,q_2} \cap \Lambda_{3,q_3}),$$

for $q_1, q_2, q_3 \in \{1, 2\}$. Each connected component is a dense open subset of a $\mathbb{P}^2$-bundle over $\widetilde{W}$. The stratum $\mathcal{Y}^4_{\widetilde{W}}$ is the union of 12 connected open subsets,

$$\Lambda_{(*,q_2,q_3)} \subset (\Lambda_{1,1} \cap \Lambda_{1,2}) \cap \Lambda_{2,q_2} \cap \Lambda_{3,q_3},$$
$$\Lambda_{(q_1,*,q_3)} \subset \Lambda_{1,q_1} \cap (\Lambda_{2,1} \cap \Lambda_{2,2}) \cap \Lambda_{3,q_3},$$
$$\Lambda_{(q_1,q_2,*)} \subset \Lambda_{1,q_1} \cap \Lambda_{2,q_2} \cap (\Lambda_{3,1} \cap \Lambda_{3,2}),$$

for $q_1, q_2, q_3 \in \{1, 2\}$. Each connected component is a dense open subset of a $\mathbb{P}^1$-bundle over $\widetilde{W}$. Finally $\mathcal{Y}^5_{\widetilde{W}}$ is the union of the six connected sets

$$\Lambda_{(*,*,q_3)} = (\Lambda_{1,1} \cap \Lambda_{1,2}) \cap (\Lambda_{2,1} \cap \Lambda_{2,2}) \cap \Lambda_{3,q_3},$$
$$\Lambda_{(*,q_2,*)} = (\Lambda_{1,1} \cap \Lambda_{1,2}) \cap \Lambda_{2,q_2} \cap (\Lambda_{3,1} \cap \Lambda_{3,2}),$$
$$\Lambda_{(q_1,*,*)} = \Lambda_{1,q_1} \cap (\Lambda_{2,1} \cap \Lambda_{2,2}) \cap (\Lambda_{3,1} \cap \Lambda_{3,2}),$$

for $q_1, q_2, q_3 \in \{1, 2\}$. Each connected component projects isomorphically to $\widetilde{W}$.

There is a bijection between multisections of $\mathcal{Y}_h$ over U and Galois invariant multisections of $\mathcal{Y}_{\widetilde{W}}$ over $\widetilde{W}$. An irreducible multisection of $\mathcal{Y}_h$ determines a multisection of $\mathcal{Y}_{\widetilde{W}}$ contained in a single stratum $\mathcal{Y}^l_{\widetilde{W}}$. The action of the Galois group $\mathfrak{W}_{3,2}$ on the connected components of $\mathcal{Y}^l_{\widetilde{W}}$ is the obvious one; in particular, it acts transitively on the set of connected components. So every Galois invariant multisection in $\mathcal{Y}^3_{\widetilde{W}}$ has degree divisible by 8, every Galois invariant multisection in $\mathcal{Y}^4_{\widetilde{W}}$ has degree divisible by 12, and every Galois invariant multisection in $\mathcal{Y}^5_{\widetilde{W}}$ has degree divisible by 6. Therefore every irreducible multisection of $\mathcal{Y}_h$ has degree divisible by 8 or 6. Because $\mathcal{Y}_h$ is a double-cover of $\mathcal{X}_h$, every irreducible multisection of $\mathcal{X}_h$ has degree divisible by 4 or 3. In particular, the minimal degree of a multisection of $\mathcal{X}_h$ is 3. □

Because $f_* i^* \mathbb{O}(1) \cong \mathbb{O}_B(1)^3$, the scheme $\mathcal{X}_h \subset B \times Z$ is a complete intersection of three divisors in the linear system $|\operatorname{pr}_B^* \mathbb{O}_B(1) \otimes \operatorname{pr}_Z^* \mathbb{O}_Z(1)|$. A general deformation of this complete intersection is a pencil of Enriques surfaces satisfying Theorem 1.1 (i) and (ii) with $M(K, X_K) \geq 3$, $I(K, X_K) \mid 4$ (this is valid so long as $\operatorname{char}(k) \neq 2$). For (iii), it is necessary to deform the pencil together with the degree 3 multisection. This requires the hypothesis that $\operatorname{char}(k) \neq 2, 3$. We explain the argument below.

The stratum $\mathcal{Y}^5_{\widetilde{W}}$ is Galois invariant and determines a degree 3 multisection of $\mathcal{X}_h$. As a $\mathfrak{W}_{3,2}$-equivariant morphism to $\widetilde{W}$, $\mathcal{Y}^5_{\widetilde{W}}$ is just the base-change of D, and the morphism $\mathcal{Y}^5_{\widetilde{W}} \to \mathbb{P}(V)$ is Galois invariant. By étale descent it is the base-change of a morphism $j' : D \to \mathbb{P}(V)$. Now j' induces a morphism to $\widetilde{\mathbb{P}(V)}$, the blowing up of $\mathbb{P}(V)$ along $\mathbb{P}(V_+) \cup \mathbb{P}(V_-)$. Because j' is equivariant for ι and ι_D, the quotient morphism $D \to Z$ factors through C, that is, there is an induced morphism $i' : C \to Z$. By a straightforward enumerative geometry computation, j' has degree 5 with respect to $\mathbb{O}_{\mathbb{P}(V)}(1)$. Therefore i' has degree 5 with respect to $\mathbb{O}_Z(1)$. The degree 3 multisection of $\mathcal{X}_h$ is the image of $(f, i') : C \to B \times Z$.

Lemma 3.2. *If f, g and j are general, then $(i')^* : H^0(Z, \mathbb{O}_Z(1)) \to H^0(C, \mathbb{O}_C(5))$ is surjective.*

Proof. The condition that $(i')^*$ is surjective is an open condition in families, hence it suffices to verify $(i')^*$ is surjective for a single choice of f, g and j, even one for which $\operatorname{Gal}(k(D)/k(B))$ is not $\mathfrak{W}_{3,2}$. Choose homogeneous coordinates $[S_0, S_1]$ on D, $[T_0, T_1]$ on C and $[U_0, U_1]$ on B. Define $g([S_0, S_1]) = [S_0^2, S_1^2]$ and $f([T_0, T_1]) = [T_0^3, T_1^3]$. Denote by μ_6 the group scheme of 6th roots of unity. There is an action of μ_6 on D by $\zeta \cdot [S_0, S_1] = [S_0, \zeta S_1]$. This identifies μ_6 with $\operatorname{Gal}(k(D)/k(B))$.

Let $\boldsymbol{e}_{+,0}$, $\boldsymbol{e}_{+,1}$, $\boldsymbol{e}_{+,2}$ and $\boldsymbol{e}_{-,0}$, $\boldsymbol{e}_{-,1}$, $\boldsymbol{e}_{-,2}$ be ordered bases of V_+ and V_- respectively, and let $X_{+,0}$, $X_{+,1}$, $X_{+,2}$ and $X_{-,0}$, $X_{-,1}$, $X_{-,2}$ be the dual ordered bases of $V_+^\vee$ and $V_-^\vee$ respectively. There is an action of μ_6 on V by

$$\zeta \cdot [X_{+,0}, X_{+,1}, X_{+,2}, X_{-,0}, X_{-,1}, X_{-,2}] \\ = [X_{+,0}, \zeta^2 X_{+,1}, \zeta^4 X_{+,2}, \zeta X_{-,0}, \zeta^3 X_{-,1}, \zeta^5 X_{-,2}]$$

and a dual action on $V^\vee$. Define $j : D \to \mathbb{P}(V)$ with respect to the ordered basis $\mathbf{e}_{+,0}, \dots, \mathbf{e}_{-,2}$ to be the μ_6-equivariant morphism

$$j([S_0, S_1]) = [S_0^5, S_0^3 S_1^2, S_0 S_1^3, S_0^4 S_1, S_0^2 S_1^3, S_1^5].$$

In this case $U = D_+(U_0 U_1) \subset B$ and $\widetilde{W} = W = D_+(S_0 S_1) \subset C$. It is straightforward to compute j' with respect to the dual ordered basis $X_{+,0}, \dots, X_{-,2}$:

$$j'([S_0, S_1]) = [S_1^5, S_0^2 S_1^3, S_0^4 S_1, S_0 S_1^4, S_0 S_1^4, S_0^3 S_1^2, S_0^5].$$

As a double-check, observe this is μ_6-equivariant. The induced map $(j')^*$ is

$$\begin{array}{lll} X_{+,0}X_{+,0} \mapsto T_1^5, & X_{+,0}X_{+,1} \mapsto T_0 T_1^4, & X_{+,0}X_{+,2} \mapsto T_0^2 T_1^3, \\ X_{+,1}X_{+,1} \mapsto T_0^2 T_1^3, & X_{+,1}X_{+,2} \mapsto T_0^3 T_1^2, & X_{+,2}X_{+,2} \mapsto T_0^4 T_1, \\ X_{-,0}X_{-,0} \mapsto T_0 T_1^4, & X_{-,0}X_{-,1} \mapsto T_0^2 T_1^3, & X_{-,0}X_{-,2} \mapsto T_0^3 T_1^2, \\ X_{-,1}X_{-,1} \mapsto T_0^3 T_1^2, & X_{-,1}X_{-,2} \mapsto T_0^4 T_1, & X_{-,2}X_{-,2} \mapsto T_0^5. \end{array}$$

This is surjective by inspection. □

Proof of Theorem 1.1. The subvariety $\mathscr{X}_h \subset B \times Z$ is a complete intersection of three divisors in the linear system $|\operatorname{pr}_B^* \mathbb{O}_B(1) \otimes \operatorname{pr}_Z^* \mathbb{O}_Z(1)|$, each containing $(f, i')(C)$. Denote by $\mathcal{I}$ the ideal sheaf of $(f, i')(C) \subset B \times Z$, and set

$$I = H^0(B \times Z, \mathcal{I} \otimes \operatorname{pr}_B^* \mathbb{O}_B(1) \otimes \operatorname{pr}_Z^* \mathbb{O}_Z(1)).$$

The projective space of I is the linear system of divisors on $B \times Z$ in the linear system $|\operatorname{pr}_B^* \mathbb{O}_B(1) \otimes \operatorname{pr}_Z^* \mathbb{O}_Z(1)|$ that contain $(f, i')(C)$. The Grassmannian $G' = \operatorname{Grass}(3, I)$ is the parameter space for deformations of $\mathscr{X}_h$ that contain $(f, i')(C)$. For the same reason as in Corollary 2.3, in G' there is a countable intersection of dense open subsets parametrizing subvarieties $\mathscr{X}' \subset B \times Z$ with $M(K, \mathscr{X}'_K) \geq 3$ and $I(K, \mathscr{X}'_K) \mid 4$. By construction, $\mathscr{X}'$ contains the degree 3 multisection $(f, i')(C)$. Therefore $M(K, \mathscr{X}'_K) = 3$ and $I(K, \mathscr{X}'_K) = 1$. It is straightforward to compute $\operatorname{pr}_B *[\omega_{\mathscr{X}'/B}^{\otimes 2}] \cong \mathbb{O}_B(6)$. So to prove the theorem, it suffices to prove every "very general" Enriques surface occurs as a fiber of some $\mathscr{X}'$, that is, for a general $[X] \in G$, X occurs as $\operatorname{pr}_Z(\mathscr{X}' \cap \pi_B^{-1}(b))$ for some choice of f, g, i and $b \in B$.

A general zero-dimensional, length 3 subscheme of Z occurs as $i'(f^{-1}(b))$ for some choice of f, g, i and $b \in B$. So for a general Enriques surface $[X] \in G$ and a general choice of zero-dimensional, length 3 subscheme of X, X is a complete intersection of three divisors in the linear system $|\mathbb{O}_Z(1)|$ containing $i'(f^{-1}(b))$ for some choice of f, g, i and b. To prove that a general $[X] \in G$ is the fiber over b of $\mathscr{X}'$ for some f, g, i and $[\mathscr{X}'] \in G'$, it suffices to prove every divisor in the linear system $|\mathbb{O}_Z(1)|$ containing $i'(f^{-1}(b))$ is the fiber over b of a divisor in the linear system $|\mathcal{I} \otimes \mathbb{O}_B(1) \otimes \mathbb{O}_Z(1)|$.

There is a short exact sequence

$$0 \to \mathcal{I} \otimes \mathrm{pr}_Z^* \mathbb{O}_Z(1) \to \mathrm{pr}_Z^* \mathbb{O}_Z(1) \to \mathrm{pr}_Z^* \mathbb{O}_Z(1)|_C \to 0,$$

giving a short exact sequence

$$0 \to \mathrm{pr}_{B,*}(\mathcal{I} \otimes \mathrm{pr}_Z^* \mathbb{O}_Z(1)) \to \mathrm{pr}_{B,*} \mathrm{pr}_Z^* \mathbb{O}_Z(1) \to \mathrm{pr}_{B,*}(\mathrm{pr}_Z^* \mathbb{O}_Z(1)|_C) \to 0.$$

Because $(i')^*$ is surjective, $\mathrm{pr}_{B,*}(\mathcal{I} \otimes \mathrm{pr}_Z^* \mathbb{O}_Z(1))$ is a locally free sheaf with $h^1 = 0$. So it is $\cong \mathbb{O}_B^6 \oplus \mathbb{O}_B(-1)^3$. Twisting by $\mathbb{O}_B(1)$, $\mathrm{pr}_{B,*}(\mathcal{I} \otimes \mathrm{pr}_B^* \mathbb{O}_B(1) \otimes \mathrm{pr}_Z^* \mathbb{O}_Z(1))$ is generated by global sections. Therefore every divisor on Z in the linear system $|\mathbb{O}_Z(1)|$ containing the scheme $i'(f^{-1}(b))$ is the fiber over b of a divisor on $B \times Z$ in the linear system $|\mathcal{I} \otimes \mathrm{pr}_B^* \mathbb{O}_B(1) \otimes \mathrm{pr}_Z^* \mathbb{O}_Z(1)|$. □

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Compactified moduli of projective bundles

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We present a method for compactifying stacks of PGL_n-torsors (Azumaya algebras) on algebraic spaces. In particular, when the ambient space is a smooth projective surface we use our methods to show that various moduli spaces are irreducible and carry natural virtual fundamental classes. We also prove a version of the Skolem–Noether theorem for certain algebra objects in the derived category, which allows us to give an explicit description of the boundary points in our compactified moduli problem.

1. Introduction

In this paper, we present a method for constructing compactified moduli of principal PGL_n-bundles on an algebraic space. As a demonstration of its usefulness, we will prove the following theorem.

Irreducibility Theorem (Theorem 6.3.1). *Let X be a smooth projective surface over an algebraically closed field k and n a positive integer which is invertible in k. For any cohomology class $\alpha \in \mathrm{H}^2(X, \boldsymbol{\mu}_n)$, the stack of stable* PGL_n*-torsors on X with cohomology class α and sufficiently large c_2 is of finite type and irreducible whenever it is nonempty, and it is nonempty infinitely often.*

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Here the number c_2 is meant to be the second Chern class of the adjoint vector bundle associated to a PGL_n-torsor. For the definition of stability of a PGL_n-torsor, we refer the reader to Definition 6.1.2 below; in characteristic 0 it is equivalent to slope-stability of the adjoint vector bundle, while in arbitrary characteristic one quantifies only over ideals in the adjoint (with respect to its natural Azumaya algebra structure).

The Irreducibility Theorem may be viewed as a zeroth-order algebraic version of results of Mrowka and Taubes (see [Taubes 1989], for instance) on the stable topology of the space of PGL_n-bundles; we show that π_0 is a singleton. Our proof arises out of a reduction of the moduli problem to another recently studied problem: moduli of twisted sheaves. Before making a few historical remarks, let us outline the contents of the paper.

In Section 3.4 we present a general theory of twisted objects in a stack, including the resulting deformation theory and the relationship between twisted and untwisted virtual fundamental classes. In Section 4 we apply this theory to PGL_n-torsors to show that the stack of twisted sheaves is naturally a cover of a compactification of the stack of PGL_n-torsors.

In Section 5, we give a reinterpretation of the results of Section 4 using certain algebra objects of the derived category (generalized Azumaya algebras), with the ultimate aim being an approach to virtual fundamental classes for spaces of stable PGL_n-torsors. The key result there is Theorem 5.1.5, a version of the Skolem–Noether theorem for these algebra objects, which we believe should be of independent interest.

In Section 6, we specialize the whole picture to study moduli of PGL_n-torsors on smooth projective surfaces. We develop the theory of stability in Section 6.1 and use the known structure theory of moduli spaces of twisted sheaves on a surface to prove the Irreducibility Theorem in Section 6.3. In Sections 6.4 and 6.5, we use the interpretation of the moduli problem in terms of generalized Azumaya algebras to produce virtual fundamental classes on moduli spaces of stable PGL_n-torsors on surfaces. In Section 6.6, we record a question due to de Jong regarding potentially new numerical invariants for division algebras over function fields of surfaces arising out the virtual fundamental classes constructed in Section 6.5.

Historical remarks. As has become clear in the history of algebraic geometry, a propitious choice of compactification of a moduli problem can lead to concrete results about the original (usually open) subproblem which is being compactified. Thus, Deligne and Mumford proved that $\mathcal{M}_g$ is irreducible by embedding it as an open substack of $\overline{\mathcal{M}}_g$ and connecting points by first degenerating them to the boundary. Similarly, O'Grady approached the moduli of semistable vector bundles on a surface by considering the larger space of semistable torsion free sheaves and showing that the boundary admits a stratification by spaces fibered over moduli of stable vector bundles with smaller c_2. Combining this inductive structure with delicate numerical estimates allowed him to prove that the spaces of semistable vector bundles with sufficiently large c_2 are irreducible. (This is very beautifully explained in Chapter 9 of [Huybrechts and Lehn 1997].) The similarity between the Irreducibility Theorem above and O'Grady's results for stable sheaves is traceable to the fact that our compactification is closely related to the space of twisted sheaves, so that we get a similar inductive structure on the moduli problem from the geometry of its boundary.

Various attempts have been made at constructing compactified moduli spaces of G-torsors for arbitrary (reductive) groups G (the reader can consult [Gómez and Sols 2005; Hyeon 2002; Langer 2005; Schmitt 2002 2004] for a sampling of moduli problems and techniques). Of course, when $G = \mathrm{GL}_n$, one can take torsion free sheaves, and when $G = \mathrm{SL}_n$, one can take torsion-free sheaves with a trivialized determinant. In the existing literature, most compactifications proceed (at least in the case where the center of G is trivial) essentially by encoding a degeneration of a principal G-bundle in a degeneration of its adjoint bundle to a torsion-free sheaf along with data which remember the principal G-bundle structure over the open subspace on which the degenerate sheaf is locally free.

We show how one can analyze the case $G = \mathrm{PGL}_n$ using more subtle methods, which roughly amount to allowing a principal bundle to degnerate by degenerating its associated adjoint bundle to an object of the derived category (rather than simply a torsion-free sheaf). By controlling the nature of these derived objects, we arrive at a compact moduli stack whose geometry is as tightly controlled as that of the stack of SL_n-bundles. This "tight control" is formalized precisely by the covering using twisted sheaves.

2. Notation

All stacks will be stacks in groupoids. Thus, given an algebraic structure such as a torsion-free sheaf, the stack of objects with that structure will be assumed to keep track only of isomorphisms.

Given a geometric morphism $f : X \to S$ of topoi and a stack $\mathscr{S}$ on X, $f_*\mathscr{S}$ will denote the stack on S whose sections over an object $T \in S$ are the sections of $\mathscr{S}$ over $\pi^{-1}T \in X$. We will often write $T \to S$ for the map to the final object of S and we will often use $X \times_S T$ to denote the object $f^{-1}(T)$. Most of the topoi we encounter will be the usual étale or fppf topos of an algebraic space or stack, but we do include a few which are slightly less conventional, such as the relative small étale topos of Section 3.2.

A stack over an algebraic space will be called *quasiproper* if it satisfies the existence part of the valuative criterion of properness over discrete valuation rings (allowing finite extensions, as is usually required for algebraic stacks). Given a Deligne–Mumford stack $\mathcal{M}$ with a coarse moduli space, we will let $\mathcal{M}^{\mathrm{mod}}$ denote the coarse space. Given a moduli space (stack) M of sheaves on a proper algebraic space X, we will let M^{lf} denote the open subspace parametrizing locally free sheaves.

The notation $(\mathbb{O}, \mathfrak{m}, \kappa)$ will mean that $\mathbb{O}$ is a local ring with maximal ideal $\mathfrak{m}$ and residue field κ.

3. Generalities

Throughout this section, we fix a geometric morphism of ringed topoi $f : X \to S$. (In various subsections, there will be additional hypotheses on the nature of X, S, or f, but the notation will remain unchanged.)

3.1. ***Stacks of sheaves.*** There are various types of sheaves which will be important for us. We recall important definitions and set notations in this section. Let Z be an algebraic space and $\mathcal{F}$ a quasicoherent sheaf of finite presentation on Z.

The sheaf $\mathcal{F}$ is

(1) *perfect* if its image in $\mathbf{D}(\mathcal{O}_Z)$ is a perfect complex;

(2) *pure* if for every geometric point $z \to Z$ the stalk $\mathcal{F}_z$ (which is a module over the local ring $\mathcal{O}^{\mathrm{hs}}_{z,Z}$) has no embedded primes;

(3) *totally supported* if the natural map $\mathcal{O}_Z \to \mathcal{E}nd(\mathcal{F})$ is injective;

(4) *totally pure* if it is pure and totally supported.

The key property of perfect sheaves for us will be the fact that one can form the determinant of any such sheaf. The reader is referred to [Knudsen and Mumford 1976] for the construction and basic facts.

It is clear that all of these properties are local in the étale topology on Z (in the sense that they hold on Z if and only if they hold on an étale cover). Thus, we can define various stacks on the small étale site of Z. We will write

(1) $\mathcal{T}_Z$ for the stack of totally supported sheaves;

(2) $\mathcal{T}_Z^{\mathrm{parf}}$ for the stack of perfect totally supported sheaves;

(3) $\mathcal{P}_Z$ for the stack of pure sheaves;

(4) $\mathcal{P}_Z^{\mathrm{parf}}$ for the stack of perfect pure sheaves;

(5) if $\mathcal{M}$ denotes any of the preceding stacks, we will use $\mathcal{M}(n)$ to denote the substack parametrizing sheaves with rank n at each maximal point of Z.

In particular, if $n > 0$ then $\mathcal{P}_Z^{\mathrm{parf}}(n)$ parametrizes perfect totally pure sheaves.

3.2. ***The relative small étale site.*** We recall a few pieces of pure nonsense that will help us apply the techniques of Section 3.4 below to study moduli problems. In this section, we assume that f is a morphism of algebraic spaces.

Definition 3.2.1. The *relative small étale site of X/S* is the site whose underlying category consists of pairs (U, T) with $T \to S$ a morphism and $U \to X \times_S T$ an étale morphism. A morphism $(U, T) \to (U', T')$ is an S-morphism $T \to T'$ and an T-morphism $U \to U' \times'_T T$. A covering is a collection of maps $\{(V_i, T)\} \to \{(U, T)\}$ such that $V_i \to U$ form a covering.

We will denote the topos of sheaves on the relative small étale site by $X_{\text{rét}}$. There is an obvious geometric morphism of topoi $X_{\text{rét}} \to T_{\text{ÉT}}$. (In fact, $X_{\text{rét}}$ is just the "total space" of a fibered topos over $T_{\text{ÉT}}$ whose fiber over $T \to S$ is just the small étale topos of $X \times_S T$.)

The relative small étale topos is naturally suited to studying moduli of T-flat sheaves on X (as pushforwards of X-stacks).

Proposition 3.2.2. *Pullback defines a natural equivalence of the category of quasicoherent sheaves on X with the category of quasicoherent sheaves on $X_{\text{rét}}$. Moreover,*

(1) *there is a stack $\mathscr{C}_{X/S} \to X_{\text{rét}}$ whose sections over (U, T) parametrize quasicoherent sheaves on U which are T-flat and which are locally of finite presentation*;

(2) *if $\mathscr{M}$ denotes any of the stacks from Section 3.1, there is a substack $\mathscr{M}_{X/S} \subset \mathscr{C}_{X/S}$ whose sections over (U, T) are T-flat quasicoherent sheaves $\mathscr{F}$ of finite presentation on U such that for each geometric point $t \to T$, the restriction $\mathscr{F}_t$ lies in $\mathscr{M}_{U_t}$*;

(3) *for each $\mathscr{M}$, there is a substack $\mathscr{M}^{\text{parf}}_{X/S} \subset \mathscr{M}_{X/S}$ parametrizing $\mathscr{F}$ such that each $\mathscr{F}_t$ is perfect*;

(4) *for any $\mathscr{M}^{\text{parf}}_{X/S}$ as in the previous item, there is a substack $\mathscr{M}^{\mathbb{O}}_{X/S}(n) \subset \mathscr{M}^{\text{parf}}_{X/S}$ parametrizing sheaves $\mathscr{F}$ such that each fiber $\mathscr{F}_t$ has rank n at each maximal point of U_t, along with a global trivialization* $\det \mathscr{F} \xrightarrow{\sim} \mathbb{O}_U$.

In the last item, we implicitly use the standard fact that a quasicoherent sheaf of finite presentation which is flat over the base and perfect on each geometric fiber is perfect. As an example, we have that $\mathscr{P}^{\mathbb{O}}_{X/S}(n)$ denotes the stack on $X_{\text{rét}}$ whose objects over (U, T) are pairs $(\mathscr{F}, \delta)$ with $\mathscr{F}$ perfect and T-flat, $\delta : \det \mathscr{F} \xrightarrow{\sim} \mathbb{O}_U$ an isomorphism, and such that for each geometric point $t \to T$, the sheaf $\mathscr{F}_t$ has rank n at each maximal point of U_t.

The proof of Proposition 3.2.2 is essentially a sequence of tautologies and is omitted. Note that we cannot make any claims about algebraicity of $\mathscr{C}$, $\mathscr{T}$, or

$\mathscr{P}$ because such a statement is meaningless for stacks on $X_{\text{rét}}$. However, when $f : X \to S$ is a proper morphism of finite presentation between algebraic spaces, it is of course standard that the pushforward of $\mathscr{C}_{X/S}$ to $S_{\text{ÉT}}$ is an algebraic stack (and similarly for any $\mathscr{M}_{X/S}$ or $\mathscr{M}^{\mathbb{O}}_{X/S}$, as the added conditions are open and the addition of a trivialization of the determinant is algebraic).

The following lemma will be useful later.

Lemma 3.2.3. *Suppose $f : Y \to Z$ is a flat morphism of locally Noetherian schemes and $\mathscr{F}$ is a Z-flat coherent sheaf on Y. If the restriction of $\mathscr{F}$ to every fiber of f is totally supported, then $\mathscr{F}$ is totally supported on Y.*

Proof. We may assume that $X = \operatorname{Spec} B$ and $S = \operatorname{Spec} A$ are local schemes and that f is the map associated to a local homomorphism $\varphi : A \to B$. Write F for the stalk of $\mathscr{F}$ at the closed point of B. Choosing generators $x_1, \ldots, x_n$ for F, we find a surjection $B^n \twoheadrightarrow F$ which yields an injection $\operatorname{End}(F) \hookrightarrow F^n$. The composition of this injection with the natural inclusion of B sends $1 \in B$ to the n-tuple $(x_1, \ldots, x_n) \in F^n$. We will show that this map $\iota : B \to F^n$ is an injection. Note that ι respects base change in the sense that for any A-algebra C, $\iota \otimes_A C$ is the map corresponding to the composition $C \to \operatorname{End}_C(F \otimes_A C) \to (F \otimes_A C)^n$. As the right-hand map in that sequence is always an injection, we find that the left-hand map is an injection if and only if $\iota \otimes_A C$ is an injection.

We proceed by "infinitesimal induction" relative to A, i.e., we write A with the $\mathfrak{m}_A$-adic topology as an inverse limit of small extensions $\{A_m\}$ with $A_0 = k(A)$, the residue field of A. We will show that $\varprojlim \iota_m : \widehat{B} \to \widehat{F}^n$ is an injection. Krull's theorem and the obvious compatibility then show that ι itself is an injection.

By hypothesis ι_0 is an injection. Suppose by induction that ι_m is an injection. Let ε generate the kernel of $A_{m+1} \to A_m$. By flatness, there are identifications $\varepsilon B_{m+1} \cong (\varepsilon) \otimes_{A_{m+1}} B_{m+1} \cong B_0$ and $\varepsilon F^n_{m+1} \cong (\varepsilon) \otimes F^n_{m+1} \cong F^n_0$, and under these identifications, $\varepsilon \cdot \iota_{m+1}$ is identified with ι_0. Now consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & \varepsilon B_{m+1} \arrow[r] \arrow[d] & B_{m+1} \arrow[r] \arrow[d] & B_m \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \varepsilon F^n_{m+1} \arrow[r] & F^n_{m+1} \arrow[r] & F^n_m \arrow[r] & 0.
\end{tikzcd}$$

By the Snake Lemma and the inductive hypothesis, the kernel of the left-hand vertical map is identified with the kernel of the middle map (which is ι_{m+1}). But the left-hand map is identified with ι_0, hence is injective. $\square$

In particular, a section of $\mathscr{T}^{\mathbb{O}}_{X/S}$ over $T \to S$ lies in $\mathscr{T}^{\text{parf}}_{X_T}(X_T)$. This will be essential when we study relative generalized Azumaya algebras in Lemma 3.2.3.

3.3. ***Azumaya algebras.*** For the sake of completeness, we recall a few basic facts about Azumaya algebras, which can be thought of as coherent models for PGL_n-torsors. We suppose that $f : X \to S$ is a proper morphism of finite presentation between algebraic spaces. By abuse of notation, we will also write f for the induced geometric morphism $X_{\text{rét}} \to S_{\text{ÉT}}$.

Let $G \to S$ be a flat linear algebraic S-group of finite presentation. It follows from the definition that $f_* \mathsf{B}G_X$ is the stack of (étale) G-torsors on X, whose sections over an S-scheme T are G_T-torsors on X_T.

Lemma 3.3.1. *The stack $f_* \mathsf{B}G_X$ is an Artin stack locally of finite presentation over S.*

Sketch of proof. By the usual arguments, we may assume that S is the spectrum of an excellent Noetherian ring (even a finite type $\mathbb{Z}$-algebra if we so desire) and that there is a closed immersion $G \hookrightarrow \mathrm{GL}_{n,S}$ for some n. It is well known that the stack $f_* \mathsf{B}\mathrm{GL}_n$ is an Artin stack locally of finite presentation over S. (One can see [Laumon and Moret-Bailly 2000] for the case of X projective or apply the main theorem of [Artin 1974] — the standard deformation theory of [Grothendieck 1971] and the usual Grothendieck existence theorem [1961] — in the arbitrary proper case.) Furthermore, extension of structure group yields a 1-morphism

$$\varepsilon : f_* \mathsf{B}G \to f_* \mathsf{B}\mathrm{GL}_n;$$

it suffices to show that ε is representable by algebraic spaces locally of finite presentation. To see this, let $T \to f_* \mathsf{B}\mathrm{GL}_n$ be any morphism over S, corresponding to some GL_n-torsor V on X_T. The fiber product $f_* \mathsf{B}G \times_{f_* \mathsf{B}\mathrm{GL}_n} T$ is identified with the sheaf of reductions of structure group of V to G, which is simply V/G. Thus, we will be done if we show that $f_*(V/G)$ is an algebraic space locally of finite presentation over T.

By [Artin 1974, Corollary 6.3], the quotient sheaf V/G is representable by a separated algebraic space of finite presentation over X_T. The fact that $f_*(V/G)$ is an algebraic space may be seen in several ways. Here is one of them: we can identify it with the fiber of $\underline{\mathrm{Hom}}_T(X_T, V/G) \to \underline{\mathrm{Hom}}_T(X_T, X_T)$ over the section id_{X_T}. Thus, it suffices to show that $\underline{\mathrm{Hom}}_T(X_T, V/G)$ is an algebraic space locally of finite presentation over T; this is a standard result, as X_T is proper and V/G is separated. Its algebraicity follows from, for example, Artin's theorem or from the methods of [Lieblich 2006b]. $\square$

In the case of $G = \mathrm{PGL}_{n,S}$, there is a natural closed immersion $G \hookrightarrow \mathrm{GL}_{n^2}$ given by the action of PGL_n on $\mathrm{M}_n(\mathscr{O})$ by conjugation (the adjoint representation). In this case, in fact, $\mathrm{PGL}_n = \mathscr{A}ut_{\mathrm{alg}}(\mathrm{M}_n(\mathscr{O}))$. Thus, there this is a very concrete way to describe the bundles admitting a reduction of structure group to PGL_n, since these bundles are such that the associated locally free sheaves of rank n^2 carry the structure of an *Azumaya algebra*.

Definition 3.3.2. An *Azumaya algebra* $\mathscr{A}$ of degree n on a ringed topos T is a form of $\mathrm{M}_n(\mathbb{O}_T)$.

More precisely, to give a reduction of structure group on a GL_{n^2}-torsor is to give a multiplication on the associated locally free sheaf making it into an Azumaya algebra. The diagram

$$\begin{tikzcd}
1 \arrow[r] & \mathbf{G}_m \arrow[r] & \mathrm{GL}_n \arrow[r] & \mathrm{PGL}_n \arrow[r] & 1 \\
1 \arrow[r] & \boldsymbol{\mu}_n \arrow[r] \arrow[u] & \mathrm{SL}_n \arrow[r] \arrow[u] & \mathrm{PGL}_n \arrow[r] \arrow[u, equal] & 1
\end{tikzcd}$$

(where the horizontal sequences are exact in the fppf topology, with the bottom exact in the étale topology only if n is invertible on S) gives rise to a diagram of coboundary maps in non-abelian (flat) cohomology

$$\begin{tikzcd}
 & \mathrm{H}^2(T, \mathbf{G}_m) \\
\mathrm{H}^1(T, \mathrm{PGL}_n) \arrow[ur] \arrow[dr] & \\
 & \mathrm{H}^2(T, \boldsymbol{\mu}_n). \arrow[uu]
\end{tikzcd}$$

In Giraud's theory [1971, Section V.4.2], one can be more precise: given a PGL_n-torsor $P \to T$, the cohomology class $\mathrm{cl}(P) \in \mathrm{H}^2(T, \boldsymbol{\mu}_n)$ is precisely that given by the $\boldsymbol{\mu}_n$-gerbe of liftings (reductions of structure group) of T to an SL_n-torsor. In the language of Azumaya algebras, this is accomplished by looking at the gerbe of trivializations: a trivialization of $\mathscr{A}$ is given by a triple $(\mathscr{V}, \delta, \varphi)$ with $\mathscr{V}$ a locally free sheaf, $\delta : \det \mathscr{V} \xrightarrow{\sim} \mathbb{O}$ a trivialization of the determinant, and $\varphi : \mathscr{E}nd(\mathscr{V}) \xrightarrow{\sim} \mathscr{A}$ an isomorphism.

3.4. ***Twisted objects and rigidifications.*** In this section, we give a possible definition for a *twisted object* in a stack (relative to an abelian gerbe). We then review a basic stack-theoretic construction of Abramovich, Corti, and Vistoli [Abramovich et al. 2003] and show how pushing it forward naturally yields coverings by stacks of twisted objects.

3.4.1. *Twisted objects.* Let $\mathscr{S} \to X$ be a stack. Suppose (for the sake of simplicity) that A is an abelian sheaf on X admitting a central injection $\chi : A \to \mathscr{I}(\mathscr{S})$ into the inertia stack of $\mathscr{S}$. Let $\mathscr{X} \to X$ be an A-gerbe on X. (Since A is abelian, we may view this as an X-stack along with an identification of A with the inertia stack $\mathscr{I}(\mathscr{X})$.)

Definition 3.4.1.1. An *$\mathscr{X}$-twisted section of $\mathscr{S}$* over $T \to X$ is a 1-morphism $f : \mathscr{X} \times_X T \to \mathscr{S}$ such that the induced map $A \to \mathscr{I}(\mathscr{X} \times_X T) \to f^*\mathscr{I}(\mathscr{S})$ is identified with the pullback under f of the canonical inclusion $\chi : A \to \mathscr{I}(\mathscr{S})$.

The collection of $\mathscr{X}$-twisted sections of $\mathscr{S}$ forms a substack of the Hom-stack $\underline{\mathrm{Hom}}_X(\mathscr{X}, \mathscr{S})$, as the condition on the inertial morphism is local on the base of any family. We will write this substack as $\mathscr{S}^{\mathscr{X}}$. Note that there is a natural central injection $A \to \mathscr{I}(\mathscr{S}^{\mathscr{X}})$ given by acting on a map $\mathscr{X} \to \mathscr{S}$ by acting on sections of $\mathscr{S}$, or (what amounts to the same thing by the twisted condition) on the sections of $\mathscr{X}$.

The following transition results will prove useful.

Lemma 3.4.1.2. *Let $\mathscr{S}$ be an X-stack and $\sigma : X \to \mathscr{S}$ a section. There is an essentially unique* 1*-morphism $\mathsf{B}\mathscr{A}ut(\sigma) \to \mathscr{S}$ sending the section corresponding to the trivial torsor to σ.*

Proof. Let $\overline{\sigma} \subset \mathscr{S}$ be the stack-theoretic image of σ (so that σ factors as an epimorphism $X \to \overline{\sigma}$ followed by a monomorphism $\overline{\sigma} \to \mathscr{S}$). By definition, $\overline{\sigma}$ is the substack of $\mathscr{S}$ consisting of objects which are locally isomorphic to $\sigma(X)$. Given an object Y of $\overline{\sigma}$ over some X-space T, the sheaf $\mathrm{Isom}_T(Y, \sigma(T))$ is an $\mathscr{A}ut(\sigma)_T$-torsor; this defines a 1-morphism $\gamma : \overline{\sigma} \to \mathsf{B}\mathscr{A}ut(\sigma)$.

To check that this is a 1-isomorphism, we choose a cleavage for $\mathscr{S}$. It is enough to prove that γ is fully faithful on fiber categories, as it is clear that any torsor is locally in the image of γ. Let Y and Y' be two objects of $\overline{\sigma}_T$, and consider the induced map of sheaves $\mathrm{Isom}(Y, Y') \to \mathrm{Isom}(\mathrm{Isom}(X, Y), \mathrm{Isom}(X, Y'))$. Since Y and Y' are both locally isomorphic to X, this map of sheaves is trivially a surjection. Thus, we are done once we show that it is injective, for which it suffices (by the universality of the argument) to show that it is injective on global sections.

The map described in the statement is simply the 1-inverse of $\overline{\sigma} \to \mathsf{B}\mathscr{A}ut(\sigma)$. □

Proposition 3.4.1.3. *Given a section $\sigma : X \to \mathscr{X}$, the natural restriction map $\mathscr{S}^{\mathscr{X}} \to \underline{\mathrm{Hom}}_X(X, \mathscr{S}) = \mathscr{S}$ is a* 1*-isomorphism.*

Proof. Given a section of $\mathscr{S}$, the injection $A \to \mathscr{I}(\mathscr{S})$ combined with Lemma 3.4.1.2 yields an essentially unique induced map $\mathsf{B}A \to \mathscr{S}$ which respects the A-structures on the inertia stacks. This construction gives an isomorphism $\mathscr{S} \to \mathscr{S}^{\mathsf{B}A}$. Using σ to identify $\mathscr{X}$ with $\mathsf{B}A$, we have just described the inverse of the natural map given in the statement. □

Proposition 3.4.1.4. *Let $\mathscr{X}$ and $\mathscr{Y}$ be A-gerbes on X. There is a natural* 1*-isomorphism*

$$(\mathscr{S}^{\mathscr{X}})^{\mathscr{Y}} \xrightarrow{\sim} \mathscr{S}^{\mathscr{Y} \wedge \mathscr{X}}$$

of stacks of twisted objects.

Proof. Consider the diagram

$$\begin{tikzcd}
\underline{\mathrm{Hom}}(\mathscr{Y}, \underline{\mathrm{Hom}}(\mathscr{X}, \mathscr{S})) \arrow[r, equal] & \underline{\mathrm{Hom}}(\mathscr{Y} \times \mathscr{X}, \mathscr{S}) \\
 & \underline{\mathrm{Hom}}(\mathscr{Y} \wedge \mathscr{X}, \mathscr{S}) \arrow[u] \\
(\mathscr{S}^{\mathscr{X}})^{\mathscr{Y}} \arrow[uu] \arrow[r, dashed] & \mathscr{S}^{\mathscr{Y} \wedge \mathscr{X}} \arrow[u]
\end{tikzcd}$$

The top equality comes from the natural adjunction and the uppermost vertical right map comes from the natural map $m : \mathscr{Y} \times \mathscr{X} \to \mathscr{Y} \wedge \mathscr{X}$. The map $A \times A \to \mathscr{I}(\mathscr{Y} \times \mathscr{X}) \to m^*\mathscr{I}(\mathscr{Y} \wedge \mathscr{X}) = A$ is just the addition map, from which it follows that a unique (up to 2-isomorphism) dashed arrow exists filling in the diagram. The fact that every arrow is either an equality or an inclusion shows that the dashed arrow is a 1-isomorphism. □

3.4.1.5. There is a more ad hoc description of $\mathscr{X}$-twisted objects in terms of a cocycle representing the cohomology class of $\mathscr{X}$. This can be useful for its value in constructing quick (but perhaps not philosophically satisfying) proofs, but we will not use this formalism. In order to make the definition, we must fix a cleavage (pseudo-functor structure) on $\mathscr{S}$.

Given $\mathscr{X}$, we can choose a hypercovering $U_\bullet \to X$ which splits $\mathscr{X}$, in the following sense:

(1) there is a section σ of $\mathscr{X}$ over U_0, and

(2) the two pullbacks of σ to U_1 are isomorphic, say via φ.

Computing the coboundary of φ and using the fact that $\mathscr{X}$ is an A-gerbe yields a 2-cocycle $a \in \Gamma(U_2, A)$. It is a standard fact that this cocycle represents the same cohomology class as $\mathscr{X}$. Slightly more subtle is the fact that one can explicitly construct a gerbe from a cocycle on a hypercovering. (This gerbe is just the stack of "twisted A-torsors"; we will not describe it in detail here.)

Definition 3.4.1.6. Given $(U_\bullet, a)$ as above, a $(U_\bullet, a)$-twisted section of $\mathscr{S}$ over $T \to X$ is given by

(1) a 1-morphism $\varphi : U_0 \times_X T \to \mathscr{S}$, and

(2) a 2-morphism $\psi : (p_0^1)^*\varphi \xrightarrow{\sim} (p_1^1)^*\varphi$, where p_0^1 and p_1^1 are the two natural maps $U_1 \times_X T \to U_0 \times_X T$,

subject to the condition that the coboundary $\delta\psi \in \mathrm{Aut}((p_0^2)^*\varphi)$ is equal to the action of a (via the inclusion of A in $\mathscr{I}(\mathscr{S})$).

It is clear that $(U_\bullet, a)$-twisted objects of $\mathscr{S}$ form a stack on X.

Moreover, given an $\mathscr{X}$-twisted object of $\mathscr{S}$ over T, the construction of $(U_\bullet, a)$ induces a $(U_\bullet, a)$-twisted object of $\mathscr{S}$ over T.

Proposition 3.4.1.7. *There is a natural equivalence between the stack of $\mathscr{X}$-twisted objects of $\mathscr{S}$ and $(U_\bullet, a)$-twisted objects of $\mathscr{S}$.*

Sketch of proof. Let $f : \mathscr{X} \to \mathscr{S}$ be an $\mathscr{X}$-twisted object of $\mathscr{S}$. Let $\varphi : U_0 \to \mathscr{X} \to \mathscr{S}$ be the composition of f with the map coming from the chosen trivialization of $\mathscr{X}$ over U_0. Via the cleavage on $\mathscr{S}$, the two maps $U_1 \to U_0$ give an isomorphism $\psi : (p_0^1)^*\varphi \xrightarrow{\sim} (p_1^1)^*\varphi$ of the pullbacks. The condition that f be $\mathscr{X}$-twisted shows that the action of the coboundary is precisely multiplication by a, giving a $(U_\bullet, a)$-twisted object of $\mathscr{S}$.

By descent theory, the statement that this gives an equivalence boils down to the proposition that a morphism $\mathscr{X} \to \mathscr{S}$ is equivalent to a natural transformation between fibered categories. (This requires some careful justification, which can come from the realization that $\mathscr{X}$ is the stack of a-twisted torsors with respect to the pair $(U_\bullet, a)$. Since we will not use this formalism in this paper, we will not go into the rather unpleasant details.) □

3.4.2. *Pushing forward rigidifications.* Let $\mathscr{S} \to X$ be a stack on X with inertia stack $\mathscr{I}(\mathscr{S}) \to \mathscr{S}$. Suppose A is an abelian sheaf on X admitting a central injection $A_{\mathscr{S}} \hookrightarrow \mathscr{I}(\mathscr{S})$. Abramovich, Corti, and Vistoli constructed in [Abramovich et al. 2003] the *rigidification of $\mathscr{S}$ along A*, which we denote by $\mathscr{S} /\!\!/ A$, following [Romagny 2005, Section 5]. It is characterized by a universal property: there is a 1-morphism $\mathscr{S} \to \mathscr{S} /\!\!/ A$ which is 1-universal among morphisms $\varphi : \mathscr{S} \to \mathscr{T}$ for which $A_{\mathscr{S}}$ is in the kernel of the induced map $\mathscr{I}(\mathscr{S}) \to \varphi^*\mathscr{I}(\mathscr{T})$. We will freely use the standard fact that $\mathscr{S} \to \mathscr{S} /\!\!/ A$ is representable by A-gerbes.

Remark 3.4.2.1. While all existing references discuss rigidifications only for algebraic stacks on the category of S-schemes for some scheme S, the abstract nonsense works perfectly well for stacks on any site. We will implicitly use this in what follows.

In this section we study the morphism $f_*\mathscr{S} \to f_*(\mathscr{S} /\!\!/ A)$. Given an S-space $T \to S$ and a 1-morphism $\gamma : T \to f_*(\mathscr{S} /\!\!/ A)$, an A_T-gerbe results on $X \times_S T$, coming from the fact that $\mathscr{S} \to \mathscr{S} /\!\!/ A$ is represented by A-gerbes and the fact that $T \to f_*(\mathscr{S} /\!\!/ A)$ corresponds to a morphism $X \times_S T \to \mathscr{S} /\!\!/ A$.

Definition 3.4.2.2. With the above notation, the A-gerbe associated to γ will be denoted $\mathscr{X}_\gamma$ and called the *gerbe of γ*. The class of $\mathscr{X}_\gamma$ in $\mathrm{H}^2(X \times_S T, A)$ will be called the (*cohomology*) *class of γ*.

Proposition 3.4.2.3. *Let $\mathscr{X} \to X$ be an A-gerbe. There is a canonical isomorphism $\mathscr{S}^{\mathscr{X}} /\!\!/ A \cong \mathscr{S} /\!\!/ A$. Moreover, for any $T \to S$, a 1-morphism $\gamma : T \to f_*(\mathscr{S} /\!\!/ A)$ lifts to a 1-morphism $T \to f_*(\mathscr{S}^{\mathscr{X}_\gamma})$.*

Proof. Consider the diagram

$$\begin{tikzcd}
 & \mathscr{S}^{\mathscr{X}} \arrow[r] \arrow[d, dashed] & \underline{\mathrm{Hom}}_X(\mathscr{X}, \mathscr{S}) \arrow[d] \\
 & \underline{\mathrm{Hom}}_X(X, \mathscr{S} /\!\!/ A) \arrow[r] \arrow[d, equal] & \underline{\mathrm{Hom}}_X(\mathscr{X}, \mathscr{S} /\!\!/ A) \\
 & \mathscr{S} /\!\!/ A. &
\end{tikzcd} \tag{3.4.2.3.1}$$

The $\mathscr{X}$-twisted condition shows that the dashed arrow exists, and since every arrow in question is a monomorphism, the dashed arrow is unique up to 2-isomorphism. By the universal property of $/\!\!/$, there results a natural morphism $\nu : \mathscr{S}^{\mathscr{X}} /\!\!/ A \to \mathscr{S} /\!\!/ A$.

To show that ν is an equivalence, we may work locally on X and assume that $\mathscr{X}$ is trivial. In this case, Proposition 3.4.1.3 shows that the dashed arrow in (3.4.2.3.1) is the image of a 1-isomorphism $\mathscr{S} \xrightarrow{\sim} \mathscr{S}^{\mathscr{X}}$ which respects the A-structures. It follows that the map on rigidifications is an isomorphism, as desired. □

Definition 3.4.2.4. Given an A-gerbe $\mathscr{X} \to X$, let $f_*^{\mathscr{X}}(\mathscr{S} /\!\!/ A)$ denote the stack-theoretic image of $f_*(\mathscr{S}^{\mathscr{X}})$ under the natural map $f_*(\mathscr{S}^{\mathscr{X}}) \to f_*(\mathscr{S} /\!\!/ A)$. We will call $f_*^{\mathscr{X}}(\mathscr{S} /\!\!/ A)$ the *$\mathscr{X}$-twisted part of* $f_*(\mathscr{S} /\!\!/ A)$.

Lemma 3.4.2.5. *Given an A-gerbe $\mathscr{X} \to X$ and a 1-morphism $\varphi : T \to f_*^{\mathscr{X}}(\mathscr{S} /\!\!/ A)$, there is an étale surjection $U \to T$ and an isomorphism*

$$T \times_{f_*^{\mathscr{X}}(\mathscr{A} /\!\!/ A)} f_*(\mathscr{S}^{\mathscr{X}})|_U \cong f_* \mathsf{B}A|_U.$$

Proof. By construction $f_*(\mathscr{S}^{\mathscr{X}}) \to f_*^{\mathscr{X}}(\mathscr{S} /\!\!/ A)$ is an epimorphism of stacks, so there is some $U \to T$ such that $\varphi|_U$ lifts into $f_*(\mathscr{S}^{\mathscr{X}})$. Thus, it suffices to show that if φ lifts to $f_*(\mathscr{S}^{\mathscr{X}})$ then the fiber product is isomorphic to $f_* \mathsf{B}A$. In this case the gerbe $\mathscr{X}_\gamma \to X \times_S T$ is isomorphic to $\mathsf{B}A$. The result follows from the compatibility of the formation of fiber product with pushforward. □

3.4.3. *Deformation theory.* In this section we assume that f is a proper morphism of finite presentation between algebraic spaces and A a tame constructible abelian étale sheaf.

We assume throughout this section that f_*A and $\mathbf{R}^1 f_*A$ are finite étale over S. (It is known that they are both constructible; if f is smooth and A is the pullback of a finite étale group scheme then this hypothesis will be satisfied. This will be the case in applications of interest to us.)

Lemma 3.4.3.1. *There is a natural morphism* $f_* \mathsf{B}A \to \mathbf{R}^1 f_* A$ *which realizes* $f_* \mathsf{B}A$ *as a* f_*A*-gerbe over* $\mathbf{R}^1 f_* A$*.*

Proof. Recall that $\mathbf{R}^1 f_* A$ is defined as the sheafification of the functor $(T \to S) \mapsto \mathrm{H}^1(X \times_S T, A)$ on the big étale site of S. A section of $f_* \mathsf{B}A$ over $T \to S$ corresponds to an A-torsor $\mathscr{T}$ on $X \times_S T$. In fact, the set of isomorphism classes of objects of $f_* \mathsf{B}A$ over T is naturally isomorphic to $\mathrm{H}^1(X \times_S T, A)$. Thus, $\mathbf{R}^1 f_* A$ is the sheafification of the stack $f_* \mathsf{B}A$, from which it immediately follows that $f_* \mathsf{B}A \to \mathbf{R}^1 f_* A$ is a gerbe. It remains to identify the inertia stack $\mathscr{I}(f_* \mathsf{B}A)$. But the inertia stack of $\mathsf{B}A$ is naturally identified with A, from which it follows that there is a natural isomorphism $f_*A \xrightarrow{\sim} \mathscr{I}(f_* \mathsf{B}A)$. □

Corollary 3.4.3.2. *The cotangent complex of* $f_* \mathsf{B}A$ *over* S *is trivial.*

Proof. By the usual triangles and the fact that $\mathbf{R}^1 f_* A$ is étale over S, it suffices to show that if Γ is a finite étale group scheme and $\mathscr{Y} \to Y$ is a Γ-gerbe then the cotangent complex of $\mathscr{Y}$ over Y is trivial. But this follows immediately from the fact that $\mathscr{Y}$ there is a surjection $U \to \mathscr{Y}$ such that U is étale over both $\mathscr{Y}$ and Y. □

Corollary 3.4.3.3. *The natural map* $\chi : f_*(\mathscr{S}^{\mathscr{X}}) /\!\!/ f_* A \to f_*^{\mathscr{X}}(\mathscr{S} /\!\!/ A)$ *is representable by finite étale covers.*

Proof. By Lemma 3.4.2.5, the fiber of χ is locally $f_* \mathsf{B}A /\!\!/ f_* A$. Applying Lemma 3.4.3.1 shows that this is precisely $\mathbf{R}^1 f_* A$. □

Lemma 3.4.3.4. *If* $f : \mathscr{S} \to \mathscr{S}'$ *is a map of* S*-stacks which is representable by fppf morphisms of algebraic stacks then* $\mathscr{S}$ *is algebraic if and only if* $\mathscr{S}'$ *is.*

Proof. First, we show that the diagonal of $\mathscr{S}$ is separated, quasicompact, and representable by algebraic spaces if and only if the same is true for $\mathscr{S}'$. To this end, let $T' \to \mathscr{S}' \times \mathscr{S}'$ be a morphism with T' an affine scheme. Consider the diagram

$$\begin{tikzcd}
& \mathscr{S} \arrow[rr] \arrow[dd] & & \mathscr{S} \times \mathscr{S} \arrow[dd] \\
I \arrow[ur] \arrow[rr] \arrow[dd] & & T \arrow[ur] \arrow[dd] & \\
& \mathscr{S}' \arrow[rr] & & \mathscr{S}' \times \mathscr{S}' \\
I' \arrow[ur] \arrow[rr] & & T' \arrow[ur] &
\end{tikzcd}$$

whose terms we now explain. The sheaf I' is the pullback of T' along the diagonal. By assumption, the fiber product $\mathscr{S} \times \mathscr{S} \times_{\mathscr{S}' \times \mathscr{S}'} T'$ is an algebraic stack over T' with fppf structure morphism. Thus, we may let T be a scheme which gives a smooth cover, and then we let I be the pullback sheaf of T along the diagonal of $\mathcal{S}$. We see that $I \to I'$ is relatively representable by fppf morphisms of algebraic spaces. By a result of Artin [Laumon and Moret-Bailly 2000, 10.1], I is an algebraic space if and only if I' is.

It remains to show that $\mathcal{S}$ has a smooth cover by an algebraic space if and only if $\mathcal{S}'$ does. In fact, it suffices to replace the word "smooth" by "fppf," by Artin's theorem [*ibid.*]. But then the statement is clear. □

Proposition 3.4.3.5. *Given an X-stack $\mathcal{S}$ and an A-gerbe $\mathcal{X} \to X$, the stack $f_*(\mathcal{S}^{\mathcal{X}})$ is an Artin (resp. DM) stack if and only if the stack $f_*^{\mathcal{X}}(\mathcal{S} /\!\!/ A)$ is an Artin (resp. DM) stack.*

Proof. This follows immediately from Lemmas 3.4.2.5, 3.4.3.1, and 3.4.3.4. □

3.4.3.6. We apply the preceding considerations to give a relation between certain virtual fundamental classes on $f_*(\mathcal{S}^{\mathcal{X}})$ and $f_*^{\mathcal{X}}(\mathcal{S} /\!\!/ A)$.

Proposition 3.4.3.7. *Let $\xi : \mathfrak{X} \to \mathcal{W}$ be a map of S-stacks. Suppose there is a tame finite étale group scheme $G \to S$ and a central injection $G_{\mathfrak{X}} \hookrightarrow \mathcal{I}(\mathfrak{X})$ such that*

(1) *the map $G_{\mathfrak{X}} \to \mathcal{I}(\mathfrak{X}) \to \xi^*\mathcal{I}(\mathcal{W})$ is trivial;*

(2) *the induced 1-morphism $\mathfrak{X} /\!\!/ G \to \mathcal{W}$ is representable by finite étale morphisms of degree invertible on $\mathcal{W}$.*

Given a perfect complex $\mathcal{E} \in \mathbf{D}(\mathcal{W})$ and a map $\xi^*\mathcal{E} \to L_{\mathfrak{X}/S}$ which gives a perfect obstruction theory, there is a map $\mathcal{E} \to L_{\mathcal{W}/S}$ giving a perfect obstruction theory.

Proof. Write $\overline{\mathfrak{X}} := \mathfrak{X} /\!\!/ G$. We have a diagram $\mathfrak{X} \to \overline{\mathfrak{X}} \to \mathcal{W}$ with the property that the relative cotangent complex of any pair vanishes. We first claim that any map $\xi^*\mathcal{E} \to L_{\mathfrak{X}/S}$ is the pullback of a map $\mathcal{E}|_{\overline{\mathfrak{X}}} \to L_{\overline{\mathfrak{X}}/S}$. This follows from the fact that A acts trivially on the sheaves making up the complexes $\mathcal{E}$ and $L_{\mathfrak{X}/S}$ and the usual description of sheaves on gerbes in terms of the representation theory of A.

Thus, to prove the result, we are reduced to the case where $\mathfrak{X} \to \mathcal{W}$ is representable by finite étale maps with invertible degrees. In this case, there is a splitting trace map given by dividing the trace of the covering by the degree. Note that $L_{\mathfrak{X}/S} = \xi^* L_{\mathcal{W}/S}$, so that the perfect obstruction theory becomes a map $\alpha : \xi^*\mathcal{E} \to \xi^* L_{\mathcal{W}/S}$. Taking the splitting trace produces a map $\mathcal{E} \to L_{\mathcal{W}/S}$, which is a perfect obstruction theory because it is a summand of $\xi_*\alpha$. □

Corollary 3.4.3.8. *A complex in $\mathbf{D}(f_*^{\mathcal{X}}(\mathcal{S} /\!\!/ A))$ can be realized as a perfect obstruction theory if and only if its pullback to $f_*(\mathcal{S}^{\mathcal{X}})$ can be realized as a perfect obstruction theory.*

4. Compactified moduli of PGL_n-torsors: an abstract approach

In this section we compactify the moduli of PGL_n-torsors using the techniques of Section 3.4 and use the structure of our compactification to prove the Irreducibility Theorem. In Section 5 we will give a more concrete description of the abstract compactification we construct here and use it to describe the virtual fundamental class on the moduli stack.

Throughout this section, $f : X \to S$ will be a proper flat morphism of finite presentation between algebraic spaces, n will be an integer invertible on S, and $\pi : \mathscr{X} \to X$ will be a fixed $\boldsymbol{\mu}_n$-gerbe. We will abuse notation and let f stand for the geometric morphism $X_{\text{rét}} \to S_{\text{ÉT}}$ as well.

4.1. ***Twisted sheaves.*** We briefly describe how the theory developed in Section 3.4.1 works out in the case of twisted objects of the stack of coherent sheaves on X/S. We fix the $\boldsymbol{\mu}_n$-gerbe $\mathscr{X} \to X$ and do the twisting with respect to the natural inclusion of $\boldsymbol{\mu}_n$ into the inertia stack of $f_*\mathscr{C}_{X/S}$ (see Proposition 3.2.2).

Definition 4.1.1. An $\mathscr{X}$-twisted object of $f_*\mathscr{C}_{X/S}$ over $T \to S$ is called *a flat family of $\mathscr{X}$-twisted coherent sheaves parametrized by T.*

If the fibers of the family are torsion-free, we will speak of a flat family of torsion-free $\mathscr{X}$-twisted sheaves, etc. The reader is referred to [Lieblich 2007, paragraph 2.2.6.3] for a discussion of associated points, purity, and torsion-free sheaves on Artin stacks.

Concretely, an $\mathscr{X}$-twisted sheaf is a sheaf $\mathscr{F}$ on $\mathscr{X}$ such that the representation of $\boldsymbol{\mu}_n$ on each geometric fiber of $\mathscr{F}$ is given by scalar multiplication. These sheaves were originally introduced in [Giraud 1971] and have found various recent applications in mathematical physics and in algebra.

Notation 4.1.2. The stack of totally pure $\mathscr{X}$-twisted coherent sheaves with rank n and trivialized determinant will be denoted $\mathbf{Tw}_{\mathscr{X}/S}(n, \mathscr{O})$.

It is relatively straightforward to prove that $\mathbf{Tw}_{\mathscr{X}/S}(n, \mathscr{O})$ is an Artin stack locally of finite presentation over S. This is done in detail in section 2.3 of [Lieblich 2007]. Earlier work on twisted sheaves in the context of elliptic fibrations and $K3$ surfaces was carried out by Căldăraru [2002a; 2002b], and a study of their moduli for projective varieties, with a description of the moduli spaces associated to $K3$ and abelian surfaces, by [Yoshioka 2006]. Applications of Yoshioka's results to a conjecture of Căldăraru were discovered by Huybrechts and Stellari (described in the appendix of [Yoshioka 2006]). The abstract approach taken here has also proven useful in the study of certain arithmetic questions [Lieblich 2008b]. When $\mathscr{X}$ admits a locally free twisted sheaf $\mathscr{V}$ (that is, when its class in $\mathrm{H}^2(X, \mathbf{G}_m)$ lies in the Brauer group of X), $\mathscr{X}$-twisted sheaves are equivalent to $\pi_*\mathscr{E}nd(\mathscr{V})$-modules, where the problem of constructing moduli under various stability conditions was first studied in [Simpson 1994]. The case in which the $\pi_*\mathscr{E}nd(\mathscr{V})$-modules have rank 1 was studied in [Hoffmann and Stuhler 2005]; they also produced a symplectic structure on the moduli space when X is a $K3$ or abelian surface, giving results analogous to those of Yoshioka.

4.2. ***Compactification by rigidification.*** The natural map $\mathrm{SL}_n \to \mathrm{PGL}_n$ gives an "extension of structure group" morphism $\varepsilon : \mathsf{B}\mathrm{SL}_n \to \mathsf{B}\mathrm{PGL}_n$.

Lemma 4.2.1. *The map ε induces an isomorphism $\mathsf{B}\mathrm{SL}_n /\!\!/ \boldsymbol{\mu}_n \xrightarrow{\sim} \mathsf{B}\mathrm{PGL}_n$.*

Proof. Given a stack $\mathscr{S}$, there is a natural equivalence of categories between morphisms $\mathsf{B}\mathrm{SL}_n \to \mathscr{S}$ and SL_n-equivariant objects of $\mathscr{S}$. (The reader is referred to [Kovács and Lieblich 2006, Section 3.A] for a description of this equivalence.) On the other hand, there is clearly a natural equivalence between PGL_n-equivariant objects of $\mathscr{S}$ and SL_n-equivariant objects on which the $\boldsymbol{\mu}_n \subset \mathrm{SL}_n$ acts trivially. But these correspond precisely to morphisms $\mathsf{B}\mathrm{SL}_n \to \mathscr{S}$ such that the induced map on inertia annihilates $\boldsymbol{\mu}_n \subset \mathscr{I}(\mathsf{B}\mathrm{SL}_n)$. The lemma follows from the universal property of the rigidification. $\square$

Taking the associated locally free sheaf with trivialized determinant yields an inclusion $\mathsf{B}\mathrm{SL}_n \subset \mathscr{T}^{\mathbb{O}}_{X/S}(n)$. (Note that the natural target is not $\mathscr{P}^{\mathbb{O}}_{X/S}(n)$ unless the fibers of X/S are Cohen–Macaulay.) Moreover, there is a natural inclusion $\boldsymbol{\mu}_n \hookrightarrow \mathscr{I}(\mathscr{T}^{\mathbb{O}}_{X/S}(n))$ extending the inclusion over $\mathsf{B}\mathrm{SL}_n$. It follows that there is an inclusion $\mathsf{B}\mathrm{PGL}_n \hookrightarrow (\mathscr{T}^{\mathbb{O}}_{X/S}(n) /\!\!/ \boldsymbol{\mu}_n)$.

There is a natural morphism from $\chi : f_*\,\mathsf{B}\mathrm{PGL}_n \to \mathbf{R}^2 f_*\boldsymbol{\mu}_n$ which we may define as follows. (Note that since X is proper over S, the sheaf $\mathbf{R}^2 f_*\boldsymbol{\mu}_n$ on $S_{\text{ÉT}}$ is a quasifinite algebraic space of finite presentation by Artin's theorem. A proof in terms of algebraic spaces may be found in the last chapter of [Artin 1973].) Given an object of $f_*\,\mathsf{B}\mathrm{PGL}_n = f_*(\mathsf{B}\mathrm{SL}_n /\!\!/ \boldsymbol{\mu}_n)$ over some $T \to S$, there is an associated $\boldsymbol{\mu}_n$-gerbe on $X \times_S T$ (see Definition 3.4.2.2), and we simply take the image in $\mathrm{H}^0(T, \mathbf{R}^2 f_*\boldsymbol{\mu}_n)$.

Lemma 4.2.2. *The stack $f_*(\mathscr{T}^{\mathbb{O}}_{X/S}(n) /\!\!/ \boldsymbol{\mu}_n)$ is an Artin stack locally of finite presentation over S. If in addition f is smooth then the stack is quasiproper.*

Proof. Since $\mathbf{R}^2 f_*\boldsymbol{\mu}_n$ is an algebraic space, it suffices to show that χ makes $f_*(\mathscr{T}^{\mathbb{O}}_{X/S}(n))$ into an algebraic $(\mathbf{R}^2 f_*\boldsymbol{\mu}_n)$-stack of finite presentation. To prove this, it suffices to work locally on $\mathbf{R}^2 f_*\boldsymbol{\mu}_n$. Thus, as any section of $\mathbf{R}^2 f_*\boldsymbol{\mu}_n$ (and in particular, the "universal section" given by the identity map) is locally associated to the cohomology class of a $\boldsymbol{\mu}_n$-gerbe, we see that it suffices to prove that, given a $\boldsymbol{\mu}_n$-gerbe $\mathscr{X} \to X$, the stack $f^{\mathscr{X}}_*(\mathscr{T}^{\mathbb{O}}_{X/T}(n) /\!\!/ \boldsymbol{\mu}_n)$ is an Artin stack locally of finite presentation. Applying Proposition 3.4.3.5, we see that it suffices to prove that $f_*((\mathscr{T}^{\mathbb{O}}_{X/T}(n))^{\mathscr{X}})$ is an Artin stack locally of finite presentation. But this is an open substack of the stack of perfect $\mathscr{X}$-twisted sheaves with trivialized determinant, which is known to be Artin and locally of finite presentation. (For the proof that it is an Artin stack, the reader is referred to [Lieblich 2007, Section 2.3]. The condition that the fibers be perfect is clearly an open condition, and closed if X/S is smooth.)

Suppose f is smooth, and let $(R, (t), \kappa)$ be a discrete valuation ring over S with fraction field K. Suppose $\operatorname{Spec} K \to f_*(\mathscr{T}^{\mathbb{O}}_{X/S}(n) /\!\!/ \boldsymbol{\mu}_n)$ is a 1-morphism. We may suppose without loss of generality that $S = \operatorname{Spec} R$. Since f is proper and smooth, $\mathbf{R}^2 f_* \boldsymbol{\mu}_n$ is finite and étale over R. Making a finite base change (which is permitted in the stacky version of the valuative criterion), we may assume that $\mathbf{R}^2 f_* \boldsymbol{\mu}_n$ is a disjoint union of sections over $\operatorname{Spec} R$. It follows that to prove that $f_*(\mathscr{T}^{\mathbb{O}}(n) /\!\!/ \boldsymbol{\mu}_n)$ is quasiproper, it suffices to prove that $f_*((\mathscr{T}^{\mathbb{O}}_{X/S}(n))^{\mathscr{X}})$ is quasiproper, where $\mathscr{X} \to X$ is an arbitrary $\boldsymbol{\mu}_n$-gerbe. Since X is regular, the condition that the twisted sheaf be perfect is trivial, and the result comes down to showing that given a discrete valuation ring R and a torsion-free $\mathscr{X}$-twisted sheaf $\mathscr{F}$ of rank n with trivialized determinant over the generic point of R, there is an extension of $\mathscr{F}$ to a flat family over a finite flat extension of R such that the trivialization of the determinant extends.

Let K be the fraction field of R and κ its residue field. It is easy to see that any flat extension $\mathscr{G}$ of $\mathscr{F}$ will have trivial determinant (as all invertible sheaves on $\operatorname{Spec} R$ are trivial). Choose an isomorphism $\iota : \det \mathscr{G} \xrightarrow{\sim} \mathbb{O}$. Composing with the fixed generic isomorphism $\det \mathscr{F} \xrightarrow{\sim} \mathbb{O}_{X_K}$ yields an injection $\alpha : \det \mathscr{G} \to \mathbb{O}_{X_K}$ (the latter being viewed as a sheaf on X by pushforward from X_K). Since X is geometrically connected, the trivial invertible $\mathbb{O}_{X_R}$-subsheaves of $\mathbb{O}_{X_K}$ all have the form $t^s \mathbb{O}_{X_R}$ for some $s \in \mathbb{Z}$. Taking an n-th root of t if necessary (which may result in a finite extension of R), we may assume that $s = ns'$ for some integer s'. Replacing $\mathscr{G}$ by $\mathscr{G}(t^{-s'})$ yields $\det \mathscr{G}(t^{-s'}) = (\det \mathscr{G})(t^{-s})$. Thus, via ι and the given isomorphism $\det \mathscr{F} \xrightarrow{\sim} \mathbb{O}$, $\det \mathscr{G}(t^{-s'})$ gets identified with $t^{-s} t^s \mathbb{O}$, that is, ι yields a trivialization of $\det \mathscr{G}(t^{-s'})$ which extends that of $\det \mathscr{F}$, as desired. □

Lemma 4.2.3. *The natural map $f_* \mathsf{BPGL}_n \to f_*(\mathscr{T}^{\mathbb{O}}_{X/S}(n) /\!\!/ \boldsymbol{\mu}_n)$ is representable by open immersions.*

Proof. It again suffices to prove this for $f_* \mathsf{BSL}_n^{\mathscr{X}}$ and $f_*(\mathscr{T}^{\mathbb{O}}_{X/S}(n))^{\mathscr{X}}$, where we note that $f_* \mathsf{BSL}_n^{\mathscr{X}}$ parametrizes locally free $\mathscr{X}$-twisted sheaves of rank n and trivialized determinant and hence constitutes an open substack, as desired. □

4.2.4. When the fibers of X/S are Cohen–Macaulay, the entire discussion from the beginning of the section until the present paragraph also yields a compactification coming from the induced inclusion $\mathsf{BPGL}_n \hookrightarrow \mathscr{P}^{\mathbb{O}}_{X/S}$. We omit the details; the statements of the results are literally identical, with $\mathscr{P}$ replacing $\mathscr{T}$. Since $\mathscr{T}^{\mathbb{O}}_{X/S}$ is much larger than $\mathscr{P}^{\mathbb{O}}_{X/S}$, it is preferable to use the latter whenever possible. Thus, for example, if X/S is a smooth morphism, then there results an open immersion into a quasiproper Artin stack $f_* \mathsf{BPGL}_n \hookrightarrow f_*(\mathscr{P}^{\mathscr{O}}_{X/S}(n) /\!\!/ \boldsymbol{\mu}_n)$. This latter stack will play an important role in what follows. We endow it with the following notation.

Notation 4.2.5. Given a $\boldsymbol{\mu}_n$-gerbe $\mathscr{X}$, let $\mathscr{M}^{\mathscr{X}}_n := f^{\mathscr{X}}_*(\mathscr{P}^{\mathscr{O}}_{X/S} /\!\!/ \boldsymbol{\mu}_n)$.

There is a surjective map $\mathbf{Tw}_{\mathscr{X}/S}(n, \mathscr{O}) \to \mathscr{M}^{\mathscr{X}}_n$ which is universally closed and submersive.

5. An explicit description of $\mathscr{M}^{\mathscr{X}}_n$: generalized Azumaya algebras

In this section, we use certain algebra objects of the derived category to give a concrete description of $\mathscr{M}^{\mathscr{X}}_n$. Using this description, we will show that when X is a smooth projective surface and $\mathscr{X} \to X$ has order n in $\mathrm{H}^2(X, \mathbf{G}_m)$ then $\mathscr{M}^{\mathscr{X}}_n$ has a virtual fundamental class.

5.1. ***Derived Skolem–Noether.*** In this section, we work primarily in the derived category of modules over a local commutative ring $(\mathscr{O}, \mathfrak{m}, k)$. For the sake of a smoother exposition, we assume that $\mathscr{O}$ is Noetherian, but this is unnecessary. On occasion, we will work in the category of chain complexes. However, we will use the word "complex" in both settings; it will be clear in context whether we mean an object of $\mathbf{D}(\mathscr{O})$ or an object of $\mathrm{K}(\mathscr{O})$. Similarly, "isomorphism" will be consistently used in place of "quasiisomorphism" and we will always assume that isomorphisms preserve whatever additional structures of objects are implicit. Given a scheme X, the symbol $\mathbf{D}(X)$ will denote a derived category of sheaves of $\mathscr{O}_X$-modules, with various conditions (boundedness, perfection, quasicoherence of cohomology) clear from context. In the end, it will suffice to work in the category denoted $\mathbf{D}_{\mathrm{fTd}}(X)$ in [Hartshorne 1966], so the hypotheses on $\mathbf{D}$ will not be a focus of attention.

Definition 5.1.1. Given a scheme X, an object $A \in \mathbf{D}(X)$ will be called a *weak $\mathscr{O}$-algebra* if there are maps $\mu : A \overset{\mathbf{L}}{\otimes} A \to A$ and $i : \mathscr{O} \to A$ in $\mathbf{D}(X)$ which satisfy the usual axioms for an associative unital algebra, the diagrams being required to commute in the derived category.

In other words, a weak algebra is an algebra object of the derived category. Note that the derived tensor product makes $\mathbf{D}(X)$ into a symmetric monoidal additive category (as the universal property of derived functors ensures that all different associations of an iterated tensor product are naturally isomorphic). Thus, it makes sense to speak of "associative" algebra structures.

Given an additive symmetric monoidal category, one can define most of the usual objects and maps of the theory of algebras: (unital) modules, bimodules, linear maps, derivations, inner derivations, maps of algebras, etc. We leave it to the reader to write down precise definitions of these terms, giving two examples:

Given a map of weak algebras $A \to B$, an *$\mathcal{O}$-linear derivation* from A to B is a map $\delta : A \to B$ in $\mathbf{D}(X)$ such that

$$\delta \circ \mu_A = \mu_B \circ (\mathrm{id} \overset{\mathbf{L}}{\otimes} \delta + \delta \overset{\mathbf{L}}{\otimes} \mathrm{id})$$

in $\mathbf{D}(X)$. A derivation from A to A is *inner* if $\delta = \mu \circ (\alpha \overset{\mathbf{L}}{\otimes} \mathrm{id}) - \mu \circ (\mathrm{id} \overset{\mathbf{L}}{\otimes} \alpha)$ for some $\alpha : \mathcal{O} \to A$.

Given a ring map $\mathcal{O} \to \mathcal{O}'$, the derived functor $(\cdot) \overset{\mathbf{L}}{\otimes}_{\mathcal{O}} \mathcal{O}' : \mathbf{D}(\mathcal{O}) \to \mathbf{D}(\mathcal{O}')$ respects the monoidal structure. There results a natural base change operation for weak algebras and modules. (This operation will be consistently written as a change of base on the right to avoid sign errors.)

Similarly, given a weak algebra A and a left A-module P, the functor $P \overset{\mathbf{L}}{\otimes} (\cdot)$ takes objects of $\mathbf{D}(\mathcal{O})$ to A-modules. This follows from the natural associativity of the derived tensor product.

The first nontrivial example of a weak algebra is given by

$$\mathbf{R}\mathrm{End}(K) := \mathbf{R}\mathrm{Hom}(K, K)$$

for a perfect complex K. Replacing K by a projective resolution, one easily deduces the weak algebra structure from the usual composition of functions: if we write K as a finite complex of free modules (which we will also call K), then $\mathbf{R}\mathrm{End}(K)$ has as its n-th module $\prod_p \mathrm{Hom}(K^p, K^{p+n})$, with differential

$$\partial^n(\alpha_p)_q = (-1)^{n+1}\alpha_{q+1} d + d\alpha_q.$$

Since K is perfect, the n-th module of $\mathbf{R}\mathrm{End}(K) \overset{\mathbf{L}}{\otimes} \mathbf{R}\mathrm{End}(K)$ is equal to

$$\prod_{a+b=n} \prod_{s,t} \mathrm{Hom}(K^s, K^{s+a}) \otimes \mathrm{Hom}(K^t, K^{t+b})$$

and the multiplication projects to the factors with $s = t + b$ and then composes functions as usual. Setting $K^\vee = \mathbf{R}\mathrm{Hom}(K, \mathcal{O})$ (the derived dual of K), we have the following basic lemma.

Lemma 5.1.2. *Let K be a perfect complex.*

(i) *There is a natural isomorphism $K \overset{\mathbf{L}}{\otimes} K^\vee \xrightarrow{\sim} \mathbf{R}\mathrm{End}(K)$.*

(ii) *There is a natural left action of $\mathbf{R}\mathrm{End}(K)$ on K.*

Tensoring the action $\mathbf{R}\mathrm{End}(K) \overset{\mathbf{L}}{\otimes} K \to K$ with $K^\vee$ on the right and using (i) yields the multiplication of $\mathbf{R}\mathrm{End}(K)$.

It is essential that the action be written on the left (when using the standard sign convention for forming the total complex of a double complex [Illusie 1971, I.1.2.1], [Matsumura 1989, Appendix]) and that $K^\vee$ be written on the right for the signs to work out. These kinds of sign sensitivities abound in the derived category and require vigilance.

An algebra of the form $\mathbf{R}\mathrm{End}(K)$ will be called a *derived endomorphism algebra*. Our goal is to reprove the classical results about matrix algebras for derived endomorphism algebras of perfect complexes.

Notation 5.1.3. The symbols P and Q will always be taken to mean perfect complexes with a chosen realization as a bounded complex of finite free modules. Thus, maps $P \to Q$ in the derived category will always come from maps of the "underlying complexes" (taken to mean the chosen realizations). Similarly, $\mathbf{R}\mathrm{End}(P)$ will have as chosen representative the complex constructed from the underlying complex of P as above: $\mathbf{R}\mathrm{End}(P)^n = \prod_t \mathrm{Hom}(P^t, P^{t+n})$ with differential $\partial(\alpha_t)_s = (-1)^{n+1}\alpha_{s+1}d + d\alpha_s$.

These conventions facilitate making certain basic arguments without speaking of replacing the object by a projective resolution, etc., but it is ultimately only important for this book-keeping reason; the reader may ignore it without fear (until it is explicitly invoked!).

Definition 5.1.4. Given $M \in \mathbf{D}(\mathscr{O})$, the *annihilator of* M is the kernel $\mathrm{Ann}(M)$ of the natural map from $\mathscr{O}$ to $\mathrm{End}_{\mathbf{D}(\mathscr{O})}(M)$. The quotient $\mathscr{O}/\mathrm{Ann}(M)$ will be denoted by $\mathscr{O}_M$.

Given an isomorphism $\psi : P \to Q(n)$, there is an isomorphism

$$\psi^* : \mathbf{R}\mathrm{End}(P) \to \mathbf{R}\mathrm{End}(Q)$$

given by functorial conjugation by ψ followed by the natural identification of $\mathbf{R}\mathrm{End}(Q(n))$ with $\mathbf{R}\mathrm{End}(Q)$. We will call this the *induced map*. The map ψ^* may also be described as follows: under the natural identification of $\mathbf{R}\mathrm{End}(P)$ with $P \overset{\mathbf{L}}{\otimes} P^\vee$, ψ^* is identified with $\psi \overset{\mathbf{L}}{\otimes} (\psi^\vee)^{-1}$.

Theorem 5.1.5. *Let P and Q be nonzero perfect complexes of $\mathscr{O}$-modules. If $\mathbf{R}\mathrm{End}(P) \cong \mathbf{R}\mathrm{End}(Q)$ as weak algebras, then there exists a unique n such that the map*

$$\mathrm{Isom}(P, Q(n)) \to \mathrm{Isom}(\mathbf{R}\mathrm{End}(P), \mathbf{R}\mathrm{End}(Q))$$

is surjective with each fiber a torsor under $\mathscr{O}_P^\times$. If $P = Q$, then $n = 0$ and the kernel is naturally a split torsor.

Corollary 5.1.6. *The sequence*

$$0 \to \mathscr{O}_P \to \mathrm{End}(P) \to \mathrm{Der}(\mathbf{R}\mathrm{End}(P)) \to 0$$

is exact. More generally, if P and $Q(n)$ are isomorphic, then the map

$$\mathrm{Hom}(P, Q(n)) \to \mathrm{Der}(\mathbf{R}\mathrm{End}(P), \mathbf{R}\mathrm{End}(Q))$$

is surjective with each fiber naturally a torsor under $\mathscr{O}_P$.

Proof. Apply Theorem 5.1.5 to $P[\varepsilon]$ (as a complex over $\mathbb{O}[\varepsilon]$) and look at automorphisms of the weak algebra $\mathbf{R}\mathrm{End}_{\mathbb{O}[\varepsilon]}(P[\varepsilon])$ reducing to the identity modulo ε. □

The proof of Theorem 5.1.5 will make use of the completion of $\mathbb{O}$ to lift the classical theorems on matrix algebras from the closed fiber by "infinitesimal induction."

Proposition 5.1.7. *If $\mathbb{O}$ is a field k then Theorem 5.1.5 and Corollary 5.1.6 hold.*

Proof. The bounded derived category of k is naturally identified with the category of $\mathbb{Z}$-graded finite k-modules by sending a complex to the direct sum of its cohomology spaces. Given perfect complexes P and Q, the algebras $\mathbf{R}\mathrm{End}(P)$ and $\mathbf{R}\mathrm{End}(Q)$ are then each identified with a matrix algebra carrying the induced grading from the grading of the vector spaces P and Q, respectively, and an isomorphism from $\mathbf{R}\mathrm{End}(P) \to \mathbf{R}\mathrm{End}(Q)$ is identified with an isomorphism of matrix algebras which respects the gradings. By the allowance of a shift, we may restrict our attention to graded spaces whose minimal nonzero graded piece is in degree 0; any reference in what follows to graded vector spaces will implicitly assume this hypothesis. (Of course the algebras involved will still carry nontrivial graded pieces with negative degrees.)

Let A be a graded matrix algebra of rank n^2 and V and W two graded n-dimensional vector spaces with nontrivial graded A-actions. From the Skolem–Noether theorem, we see that there is an A-linear isomorphism $\alpha : V \to W$. We claim that α is graded. To prove this, it suffices to show that given a nonzero vector $v \in V_0$, $\alpha(v)$ is in W_0 (because V and W are simple A-modules). Write $\alpha(v) = \sum w_i$. Since V is a simple A- module, $A_n \cdot v = V_n$; a similar statement holds for W (given a choice of nonzero weight 0 vector, which exists by the hypothesis on the gradings). Thus, the highest nontrivial grading N of A will equal the highest nontrivial grading of both V and W. Furthermore, given any i such that $w_i \neq 0$, the fact that $A_{-i} \cdot w_i = W_0$ means that $A_{-i} \neq 0$. Given $i > 0$ such that $w_i \neq 0$, we have for all $\tau \in A_{-i}$ that

$$0 = \alpha(0) = \alpha(\tau(v)) = \tau(\alpha(v)) = \tau\Big(\sum w_j\Big) = \tau(w_i) + \text{higher terms.}$$

Thus, $\tau(w_i) = 0$, which implies that $W_0 = 0$. This contradicts the assertion that W_0 is the minimal nontrivial graded piece. So $w_i = 0$ for all $i > 0$ and therefore $w \in W_0$. Translating this back into the derived language, we have proven that given an isomorphism $\varphi : \mathbf{R}\mathrm{End}(P) \to \mathbf{R}\mathrm{End}(Q)$, there is an isomorphism $P \to Q$ in $\mathbf{D}(k)$ which induces φ by functoriality. In fact, we have shown the rest of the proposition as well, because α is the unique choice for such an isomorphism up to scalars by the Skolem–Noether theorem.

To prove Corollary 5.1.6, let $V = \oplus V_i$ be a graded vector space and $T \in \mathrm{End}(V)$ a noncentral linear transformation. We wish to show that if the (nontrivial) inner derivation by T is homogeneous of degree 0 then T is homogeneous of degree 0. To do this, consider the restriction of T to the degree 0 part of $\mathrm{End}(V)$. Let T^n be a graded component of T (so that $T^n : V \to V$ shifts degrees by n). Let V^m be a graded component such that the induced transformation $T^n : V^m \to V^{m+n}$ is nonzero. Consider the graded linear transformation (of degree 0) $S : V \to V$ which acts as the identity on V^m and annihilates every other component. It is easy to see that the commutator $[T^n, S]$ is $T^n S$, which implies that $[T, S]$ has a nontrivial component of degree n. Since the derivation $[T, -]$ is homogeneous of degree 0, it follows that $n = 0$, and thus T is homogeneous of degree 0, as desired. ☐

Lemma 5.1.8. *Theorem 5.1.5 is true for $\mathbb{O}$ if it is true for $\widehat{\mathbb{O}}$.*

Proof. We proceed by reducing the problem to a question of linear algebra and then using the faithful flatness of completion.

Suppose given P and Q and an isomorphism $\varphi : \mathbf{R}\mathrm{End}(P) \to \mathbf{R}\mathrm{End}(Q)$; this defines an action of $A := \mathbf{R}\mathrm{End}(P)$ on Q. We claim that finding $u : P \to Q$ such that $\varphi = u^*$ is equivalent to finding an A-linear isomorphism from P to Q. Indeed, suppose $u : P \to Q$ is A-linear, so that the diagram

$$\begin{tikzcd}
\mathbf{R}\mathrm{End}(P) \overset{\mathbf{L}}{\otimes} P \arrow[r, "\varphi \overset{\mathbf{L}}{\otimes} u"] \arrow[d] & \mathbf{R}\mathrm{End}(Q) \overset{\mathbf{L}}{\otimes} Q \arrow[d] \\
P \arrow[r, "u"] & Q
\end{tikzcd}$$

commutes, where the vertical arrows are the actions. Tensoring the left side with $P^\vee$ and the right side with $Q^\vee$, we see that the resulting diagram

$$\begin{tikzcd}
\mathbf{R}\mathrm{End}(P) \overset{\mathbf{L}}{\otimes} P \overset{\mathbf{L}}{\otimes} P^\vee \arrow[r, "\varphi \overset{\mathbf{L}}{\otimes} u \overset{\mathbf{L}}{\otimes} (u^\vee)^{-1}"] \arrow[d] & \mathbf{R}\mathrm{End}(Q) \overset{\mathbf{L}}{\otimes} Q \overset{\mathbf{L}}{\otimes} Q^\vee \arrow[d] \\
P \overset{\mathbf{L}}{\otimes} P^\vee \arrow[r, "u \overset{\mathbf{L}}{\otimes} (u^\vee)^{-1}"] & Q \overset{\mathbf{L}}{\otimes} Q^\vee
\end{tikzcd}$$

also commutes. Applying Lemma 5.1.2 and writing B for $\mathbf{R}\mathrm{End}(Q)$, we find that the diagram

$$\begin{tikzcd}
A \overset{\mathbf{L}}{\otimes} A \arrow[r, "u^* \overset{\mathbf{L}}{\otimes} \varphi"] \arrow[d] & B \overset{\mathbf{L}}{\otimes} B \arrow[d] \\
A \arrow[r, "u^*"] & B
\end{tikzcd}$$

commutes, where the vertical arrows are the multiplication maps. Considering the units in the algebras, one readily concludes the proof of the claim. Note that to conclude that any such u as above is an isomorphism, it suffices for its reduction to the residue field to be an isomorphism (for example, because the complexes are bounded above).

It is easy to see (using the realization in terms of diagrams of finite flat $\mathbb{O}$-modules) that $\mathrm{Hom}_{\mathbf{D}(\mathbb{O})}$ is compatible with flat base change and completion when restricted to the category of perfect complexes: given a flat ring extension $\mathbb{O} \to \mathbb{O}'$, there is a natural isomorphism

$$\mathrm{Hom}_{\mathbf{D}(\mathbb{O}')}(M \overset{\mathbf{L}}{\otimes} \mathbb{O}', N \overset{\mathbf{L}}{\otimes} \mathbb{O}') \cong \mathrm{Hom}_{\mathbf{D}(\mathbb{O})}(M, N) \otimes \mathbb{O}'$$

for all perfect M and N in $\mathbf{D}(\mathbb{O})$. Furthermore, given a perfect weak algebra Ξ, the realization of the module of Ξ-linear maps as a kernel of maps of Hom-modules shows that the same statement is true for Hom_Ξ. Thus there is a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_\Xi(M, N) \arrow[rr] \arrow[d] & & \mathrm{Hom}_{\widehat{\Xi}}(\widehat{M}, \widehat{N}) \arrow[d] \\
\mathrm{Hom}_\Xi(M, N) \otimes_{\mathbb{O}} k \arrow[rr, equal] \arrow[dr] & & \mathrm{Hom}_{\widehat{\Xi}}(\widehat{M}, \widehat{N}) \otimes_{\widehat{\mathbb{O}}} k \arrow[dl] \\
& \mathrm{Hom}_{\Xi \overset{\mathbf{L}}{\otimes} k}(M \overset{\mathbf{L}}{\otimes} k, N \overset{\mathbf{L}}{\otimes} k). &
\end{tikzcd}$$

with surjective vertical arrows. This immediately applies to our situation to show that the map of Theorem 5.1.5 is surjective for $\mathbb{O}$ if it is for $\widehat{\mathbb{O}}$ (for a fixed n, which may be determined from the reduction to the residue field). Indeed, a $\widehat{\Xi}$-linear map $\widehat{M} \to \widehat{N}$ yields an element of $\mathrm{Hom}_{\widehat{\Xi}}(\widehat{M}, \widehat{N}) \otimes k$ whose image in the bottom module is an isomorphism. It follows from the diagram that there is a Ξ-linear map $u : M \to N$ whose (derived) reduction to k is an isomorphism, whence u is an isomorphism by Nakayama's lemma for perfect complexes (see, for example, [Lieblich 2006a, Lemma 2.1.3]).

Similarly, to verify that an isomorphism $\xi : P \xrightarrow{\sim} P$ in the kernel of the automorphism map is homotopic to a constant, it suffices to show that an element $\xi \in \mathrm{End}_{\mathbf{D}(\mathbb{O})}(P)$ is in $\mathbb{O}_P$ if and only if this is true after completing. But the module of maps homotopic to a constant is also clearly compatible with flat base change and completion is moreover *faithfully* flat (all modules involved are finite over $\mathbb{O}$ because the complexes involved are perfect), so ξ is in a submodule Z of $\mathrm{End}(P)$ if and only if its image in $\mathrm{End}(P) \otimes \widehat{\mathbb{O}}$ is contained in $Z \otimes \widehat{\mathbb{O}}$. □

From this point onward, *we assume that $\mathbb{O}$ is a complete local Noetherian ring*. Recall that a quotient of local rings $0 \to I \to \mathbb{O} \to \overline{\mathbb{O}} \to 0$ is *small* if I is generated by an element ε which is annihilated by the maximal ideal of $\mathbb{O}$ (so that, in particular, $\varepsilon^2 = 0$). We can choose a filtration $\mathbb{O} \supset \mathfrak{m} = I_0 \supset I_1 \supset I_2 \supset \cdots$ which is separated (i.e., so that $\cap_i I_i = 0$) and defines a topology equivalent to the $\mathfrak{m}$-adic topology such that for all $i \geq 0$, the quotient $0 \to I_i/I_{i+1} \to \mathbb{O}/I_{i+1} \to \mathbb{O}/I_i \to 0$ is a small extension, with I_i/I_{i+1} generated by ε_i. We fix such a filtration for remainder of this section, and we denote $\mathbb{O}/I_n$ by $\mathbb{O}_n$.

Lemma 5.1.9. *Let $0 \to I \to R \to \overline{R} \to 0$ be a surjection of rings. Let A be a weak R-algebra and P and Q two left A-modules. Let T denote the triangle in $\mathbf{D}(R)$ arising from the quotient map $R \to \overline{R}$.*

(i) *The maps in $P \overset{\mathbf{L}}{\otimes} T$ are A-linear (with the natural A-module structures).*

(ii) *Any A-linear map $\psi : P \to Q \overset{\mathbf{L}}{\otimes} \overline{R}$ factors through an A-linear map*

$$\overline{\psi} : P \overset{\mathbf{L}}{\otimes} \overline{R} \to Q \overset{\mathbf{L}}{\otimes} \overline{R}$$

which is the derived restriction of scalars of an $A \overset{\mathbf{L}}{\otimes} \overline{R}$-linear map from $P \overset{\mathbf{L}}{\otimes} \overline{R}$ to $Q \overset{\mathbf{L}}{\otimes} \overline{R}$.

(iii) *If $R \to \overline{R}$ is a small extension of local rings with residue field k, then the natural identification $P \overset{\mathbf{L}}{\otimes} I \xrightarrow{\sim} P_k$ induced by a choice of basis for I over k is A-linear.*

Proof. Note that basic results about homotopy colimits allow us to replace any object of $\mathbf{D}(R)$ by a complex of projectives, so there are no boundedness conditions on any of the complexes involved. Part (i) follows immediately from the fact that $P \overset{\mathbf{L}}{\otimes} (\cdot)$ is a functor from $\mathbf{D}(R)$ to A-modules. Part (ii) follows from writing P and A as complexes of projectives and representing the map $P \to Q \overset{\mathbf{L}}{\otimes} \overline{R}$ as a map on complexes. (Note that this factorization need not be unique as a map in $\mathbf{D}(R)$, but it is unique as the derived restriction of scalars from a map in $\mathbf{D}(\overline{R})$.) Part (iii) follows similarly from looking at explicit representatives of P and A. $\square$

Lemma 5.1.10. *Suppose $f, g : P \to Q$ are two maps of perfect complexes in $K(\mathbb{O})$. Let $P_n = P \otimes \mathbb{O}_n$, $Q_n = Q \otimes \mathbb{O}_n$, $f_n = f \otimes \mathbb{O}_n$, $g_n = g \otimes \mathbb{O}_n$. Suppose there are homotopies*

$$h(n) \in \prod_t \operatorname{Hom}(P^t, Q^{t-1} \otimes I_n)$$

such that for all n,

$$f_n - g_n = d\Big(\sum_{s<n} \bar{h}(s)\Big) + \Big(\sum_{s<n} \bar{h}(s)\Big)d$$

as maps of complexes, where $\bar{h}$ denotes the induced map. Then f is homotopic to g.

Proof. The element $h = \sum_{s=0}^{\infty} h(s)$ converges and defines the homotopy. $\square$

Lemma 5.1.11. *Let* $0 \to I \to R \to \overline{R} \to 0$ *be a small extension of local rings with residue field* k. *Let* P *and* Q *be perfect complexes of* R*-modules* (*with chosen realizations*) *and* $\varphi : \mathbf{R}\mathrm{End}(P) \to \mathbf{R}\mathrm{End}(Q)$ *an isomorphism of the derived endomorphism algebras*, *written as a map in that direction on the underlying complexes. If there exists an isomorphism of the underlying complexes* $\bar{u} : \overline{P} \xrightarrow{\sim} \overline{Q}$ *such that* $\overline{\varphi} = \bar{u}^*$ *as maps of complexes*, *then there is a lift* u *of* $\bar{u}$ *and a homotopy* h *between* φ *and* u^* *such that* $h(\mathbf{R}\mathrm{End}(P)) \subset \mathbf{R}\mathrm{End}(Q) \otimes I$. *In particular*, $\varphi = u^*$ *in* $\mathbf{D}(R)$.

Proof. Let $A = \mathbf{R}\mathrm{End}(P)$ and let A act on Q via φ. The identification of $\overline{\varphi}$ with $\bar{u}^*$ provides an $\overline{A}$-linear isomorphism $\overline{\gamma} : \overline{P} \to \overline{Q}$, and we wish to lift this to an A-linear isomorphism $P \to Q$. Consider the composition $P \to \overline{Q} \to Q \otimes I(1) \cong Q_k(1)$ in the derived category. By Lemma 5.1.9, this map is A-linear and factors through an A-linear map $\alpha : P_k \to Q_k(1)$ which comes by derived restriction of scalars from an A_k-linear map in $\mathbf{D}(k)$. By Proposition 5.1.7 (and the method of its proof), we see that α is either zero or an isomorphism. But $P_k \cong Q_k \not\cong 0$, which implies that $\alpha = 0$. This means that there is an *R-linear* lift γ of $\overline{\gamma}$. Now $(\gamma^*)^{-1} \circ \varphi - \mathrm{id}$ is identified with a map $\mathbf{R}\mathrm{End}_k(P_k) \to \mathbf{R}\mathrm{End}_k(P_k)$ in $\mathbf{D}(k)$ which is a derivation of the algebra, hence is homotopic to the inner derivation induced by a map $\omega_k : P_k \to P_k$ in $\mathbf{D}(k)$. Writing ω for the composition

$$P \longrightarrow P_k \xrightarrow{\omega_k} P_k \xrightarrow{\cong} P \overset{\mathbf{L}}{\otimes} I \longrightarrow P,$$

we see that there exists a homotopy between φ and $\gamma(1+\omega)^*$ having image in $\mathbf{R}\mathrm{End}(Q) \otimes I$, and that $\gamma(1+\omega)$ is a lift of $\overline{\gamma}$ as maps of complexes. □

Lemma 5.1.12. *Let* $0 \to I \to R \to \overline{R} \to 0$ *be a small extension of local rings with residue field* k. *Let* P *be a perfect complex of* R*-modules* (*with a chosen realization*) *and* $\psi : P \to P$ *an automorphism of the underlying complex such that* $\overline{\psi} = \overline{\alpha}$ *for some* $\overline{\alpha} \in \overline{R}_{\overline{P}}$ *as maps of the complex* $\overline{P}$ *and such that* ψ^* *is homotopic to the identity as a map of weak algebras. Then there is a unit* α *lifting* $\overline{\alpha}$ *and a homotopy* h *between* ψ *and* α *such that* $h(\mathbf{R}\mathrm{End}(P)) \subset \mathbf{R}\mathrm{End}(P) \otimes I$.

Proof. The proof is quite similar to the proof of Lemma 5.1.11, using the left half of the exact sequence of Corollary 5.1.6 rather than the right half. □

Proposition 5.1.13. *Theorem 5.1.5 holds for* $\mathbb{O}$ (*now assumed complete*).

Proof. Given an isomorphism $\varphi : \mathbf{R}\mathrm{End}(P) \xrightarrow{\sim} \mathbf{R}\mathrm{End}(Q)$, we may assume after adding zero complexes to P and Q, shifting Q, and applying a homotopy to φ, that there is an isomorphism $\psi_0 : P_0 \to Q_0$ such that $\varphi_0 = \psi_0^*$ as maps of complexes. We can now apply Lemma 5.1.11 to arrive at an isomorphism ψ_1 lifting ψ_0 and a homotopy $\bar{h}(0)$ with image in $\mathbf{R}\mathrm{End}_{\mathbb{O}_1}(Q_1) \otimes_{\mathbb{O}_1} I_0/I_1$ between φ_1 and ψ_1^*. Lift $\bar{h}(0)$ to a homotopy $h(0)$ with image in $\mathbf{R}\mathrm{End}(Q) \otimes I_0$. Then

$$(\varphi - (dh(0) + h(0)d))_1 = \psi_1^*$$

as maps of complexes, and we may find a homotopy $h(1)$, etc. By Lemma 5.1.10, we see that there is an isomorphism $\psi : P \to Q$ such that $\varphi = \psi^*$ in $\mathbf{D}(\mathscr{O})$. A similar argument shows that the kernel is $\mathscr{O}_P^*$. ☐

5.2. ***The construction of* GAz.** In this section, we define a stack which we will use to compactify the stack of Azumaya algebras. While the definition is rather technical in general, in the case of a relative surface it assumes a simpler and more intuitive form.

Let $(X, \mathscr{O})$ be a ringed topos.

Definition 5.2.1. A *pregeneralized Azumaya algebra* on X is a perfect algebra object $\mathscr{A}$ of the derived category $\mathbf{D}(X)$ of $\mathscr{O}_X$-modules such that there exists an object $U \in X$ covering the final object and a totally supported perfect sheaf $\mathscr{F}$ on U with $\mathscr{A}|_U \cong \mathbf{R}\mathscr{E}nd_U(\mathscr{F})$ as weak algebras. An isomorphism of pregeneralized Azumaya algebras is an isomorphism in the category of weak algebras.

5.2.1. *Stackification.* Consider the fibered category $\mathscr{P}\mathscr{R} \to Sch_{\text{ét}}$ of pregeneralized Azumaya algebras on the large étale topos over $\operatorname{Spec}\mathbb{Z}$. We will stackify this to yield the stack of generalized Azumaya algebras. This is slightly different from the construction given in [Laumon and Moret-Bailly 2000, 3.2], as we do not assume that the fibered category is a prestack.

Lemma 5.2.1.1. *Suppose T is a topos and $\mathscr{C} \to T$ is a category fibered in groupoids. There exists a stack $\mathscr{C}^s$, unique up to* 1*-isomorphism, and a* 1*-morphism $\mathscr{C} \to \mathscr{C}^s$ which is universal among* 1*-morphisms to stacks* (*up to* 2*-isomorphism*).

Proof. The proof is the usual type of argument. A reader interested in seeing a generalization to stacks in categories larger than groupoids should consult [Giraud 1971]. First, we may assume that in fact $\mathscr{C} \to T$ admits a splitting (after replacing $\mathscr{C}$ by a 1-isomorphic fibered category). Define a new fibered category $\mathscr{C}^p$ as follows: the objects will be the same, but the morphisms between two objects a and b over $t \in T$ will be the global sections of the sheafification of the presheaf $\operatorname{Hom}_t(a, b) : (s \xrightarrow{\varphi} t) \mapsto \operatorname{Hom}_s(\varphi^* a, \varphi^* b)$ on t. Clearly $\mathscr{C}^p$ is a prestack (that is, given any two sections a and b over t, the hom-presheaf just described is a sheaf) and the natural map $\mathscr{C} \to \mathscr{C}^p$ of fibered categories over T is universal up to 1-isomorphism for 1-morphisms of $\mathscr{C}$ into prestacks. Now we apply [Laumon and Moret-Bailly 2000, 3.2] to construct $\mathscr{C}^s$ as the stackification of $\mathscr{C}^p$. ☐

Definition 5.2.1.2. The stack of *generalized Azumaya algebras* on schemes is defined to be the stack in groupoids $\mathscr{P}\mathscr{R}^s \to Sch_{\text{ét}}$ associated to the fibered category of pregeneralized Azumaya algebras.

Remark 5.2.1.3. Explicitly, given a scheme X, to give a generalized Azumaya algebra on X is to give an étale 3-hypercovering $Y'' \,\substack{\longrightarrow\\[-1ex]\longrightarrow\\[-1ex]\longrightarrow}\, Y' \rightrightarrows Y \longrightarrow X$, a totally supported sheaf $\mathscr{F}$ on Y, and a gluing datum for $\mathbf{R}\mathscr{E}nd_Y(\mathscr{F})$ in the derived category $\mathbf{D}(Y')$ whose coboundary in $\mathbf{D}(Y'')$ is trivial. Two such objects $(Y_1, \mathscr{F}_1, \delta_1)$ and $(Y_2, \mathscr{F}_2, \delta_2)$ are isomorphic if and only if there is a common refinement Y_3 of the 3-hypercovers Y_1 and Y_2 and an isomorphism $\varphi : \mathscr{F}_1|_{Y_3} \xrightarrow{\sim} \mathscr{F}_2|_{Y_3}$ commuting with the restrictions of δ_1 and δ_2. Thus, a generalized Azumaya algebra is gotten by gluing "derived endomorphism algebras" together in the étale topology. When X is a quasiprojective smooth surface, or, more generally, a quasiprojective scheme smooth over an affine with fibers of dimension at most 2, then the sections of $\mathscr{P}\mathscr{R}$ over X are the same as the sections of $\mathscr{P}\mathscr{R}^s$ over X; see Section 6.4.

Example 5.2.1.4. Let $\pi : \mathscr{X} \to X$ be a $\boldsymbol{\mu}_n$-gerbe and $\mathscr{F}$ a totally supported perfect $\mathscr{X}$-twisted sheaf. The complex $\mathbf{R}\pi_*\mathbf{R}\mathscr{E}nd_{\mathscr{X}}(\mathscr{F}) \in \mathbf{D}(X)$ is a pregeneralized Azumaya algebra, hence has a naturally associated generalized Azumaya algebra. We will see below that the global sections of the stack $\mathscr{P}\mathscr{R}^s$ are precisely the weak algebras of this form.

Lemma 5.2.1.5. *Let $\mathscr{F}$ and $\mathscr{G}$ be totally supported perfect sheaves on a $\mathbf{G}_m$-gerbe $\mathscr{X} \to X$.*

(1) *The sheaf of isomorphisms between the generalized Azumaya algebras associated to the weak algebras $\mathbf{R}\pi_*\mathbf{R}\mathscr{E}nd(\mathscr{F})$ and $\mathbf{R}\pi_*\mathbf{R}\mathscr{E}nd(\mathscr{G})$ is naturally isomorphic to* $\operatorname{Isom}(\mathscr{F}, \mathscr{G})/\mathbf{G}_m$, *with $\mathbf{G}_m$ acting by scalar multiplication on $\mathscr{G}$.*

(2) *Any isomorphism of generalized Azumaya algebras*

$$\varphi : \mathbf{R}\pi_*\mathbf{R}\mathscr{E}nd(\mathscr{F}) \xrightarrow{\sim} \mathbf{R}\pi_*\mathbf{R}\mathscr{E}nd(\mathscr{G})$$

is the isomorphism associated to an isomorphism $\mathscr{F} \xrightarrow{\sim} \mathscr{L} \otimes \mathscr{G}$ for some invertible sheaf $\mathscr{L}$ on X.

Proof. Temporarily write $\mathscr{I}$ for the sheaf of isomorphisms of generalized Azumaya algebras from $\mathbf{R}\pi_*\mathbf{R}\mathscr{E}nd(\mathscr{F})$ to $\mathbf{R}\pi_*\mathbf{R}\mathscr{E}nd(\mathscr{G})$. There is clearly a map

$$\chi : \operatorname{Isom}(\mathscr{F}, \mathscr{G})/\mathbf{G}_m \to \mathscr{I}.$$

To verify that it is an isomorphism, it suffices to verify it étale-locally on X, whence we may assume that X is strictly local. Choosing an invertible $\mathscr{X}$-twisted sheaf and twisting down $\mathscr{F}$ and $\mathscr{G}$, we are reduced to showing the analogous statement for totally supported sheaves on X itself. Any local section of $\mathscr{I}$ comes from an isomorphism of weak algebras $\mathbf{R}\mathscr{E}nd(\mathscr{F}) \xrightarrow{\sim} \mathbf{R}\mathscr{E}nd(\mathscr{G})$, so Theorem 5.1.5 shows that χ is surjective. Similarly, any section of the kernel of χ must locally be trivial, whence χ is an isomorphism.

The second part is a formal consequence of the first: there is an étale covering $U \to X$ such that $\varphi|_U$ is associated to an isomorphism $\psi : \mathscr{F}|_U \xrightarrow{\sim} \mathscr{G}|_U$. The coboundary of ψ on the product $U \times_X U$ is multiplication by some scalar, which is a cocycle by a formal calculation. This gives rise to the invertible sheaf $\mathscr{L}$; tensoring with $\mathscr{L}$ makes the coboundary of ψ trivial, whence it descends to an isomorphism $\mathscr{F} \to \mathscr{L} \otimes \mathscr{G}$ inducing φ, as desired. □

Definition 5.2.1.6. Let $\mathscr{A}$ be a generalized Azumaya algebra on X. The *gerbe of trivializations* of $\mathscr{A}$, denoted $\mathscr{X}(\mathscr{A})$, is the stack on the small étale site $X_{\text{ét}}$ whose sections over $V \to X$ given by pairs $(\mathscr{F}, \varphi)$, where $\mathscr{F}$ is a totally supported sheaf on V and $\varphi : \mathbf{R}\mathscr{E}nd_V(\mathscr{F}) \xrightarrow{\sim} \mathscr{A}|_V$ is an isomorphism of generalized Azumaya algebras. The isomorphisms in the fiber categories are isomorphisms of the sheaves which respect the identifications with $\mathscr{A}$, as usual.

This is entirely analogous to the gerbe produced in [Giraud 1971, section V.4.2]. There is also an analogue of the $\boldsymbol{\mu}_n$-gerbe associated to an Azumaya algebra of degree n.

Definition 5.2.1.7. Given a generalized Azumaya algebra $\mathscr{A}$ of degree n on $\mathscr{X}$, the *gerbe of trivialized trivializations* of $\mathscr{A}$, denoted $\mathscr{X}_{\text{triv}}(\mathscr{A})$, is the stack on the small étale site of X whose sections over $U \to X$ consist of triples $(\mathscr{F}, \varphi, \delta)$ with $\varphi : \mathbf{R}\mathscr{E}nd_U(\mathscr{F}) \xrightarrow{\sim} \mathscr{A}_U$ an isomorphism of generalized Azumaya algebras and $\mathscr{O}_U \xrightarrow{\sim} \det(\mathscr{F})$ an isomorphism of invertible sheaves on U. The isomorphisms in the fiber categories are isomorphisms of the sheaves which preserve the identifications with $\mathscr{A}$ and the trivializations of the determinants.

Lemma 5.2.1.8. *The stack $\mathscr{X}(\mathscr{A})$ is a $\mathbf{G}_m$-gerbe. If $\mathscr{A}$ has degree n, then $\mathscr{X}_{triv}(\mathscr{A})$ is a $\boldsymbol{\mu}_n$-gerbe whose associated cohomology class maps to $[\mathscr{X}(\mathscr{A})]$ in $\mathrm{H}^2(X, \mathbf{G}_m)$.*

Proof. This follows immediately from the derived Skolem–Noether Theorem 5.1.5 and the fact that all of the sheaves $\mathscr{F}$ are totally supported. □

Corollary 5.2.1.9. *A generalized Azumaya algebra $\mathscr{A}$ has a class in $\mathrm{H}^2(X, \mathbf{G}_m)$. When the rank of $\mathscr{A}$ is n^2, $\mathscr{A}$ has a class in $\mathrm{H}^2(X, \boldsymbol{\mu}_n)$ (in the fppf topology).*

Definition 5.2.1.10. When $\operatorname{rk}\mathscr{A} = n^2$, we call the cohomology class in $\mathrm{H}^2(X, \boldsymbol{\mu}_n)$ the *class of* $\mathscr{A}$, and write $\mathrm{cl}(\mathscr{A})$.

Let $\pi : \mathscr{X}(\mathscr{A}) \to X$ denote the natural projection.

Lemma 5.2.1.11. *There is an $\mathscr{X}(\mathscr{A})$-twisted sheaf $\mathscr{F}$ and an isomorphism of generalized Azumaya algebras $\varphi : \mathbf{R}\pi_*\mathbf{R}\mathscr{E}nd_{\mathscr{X}(\mathscr{A})}(\mathscr{F}) \xrightarrow{\sim} \mathscr{A}$. The datum $(\mathscr{X}(\mathscr{A}), \mathscr{F}, \varphi)$ is functorial in $\mathscr{A}$.*

Proof. As usual, the construction of $\mathscr{X}(\mathscr{A})$ yields by first projection a twisted sheaf $\mathscr{F}$. Whenever $\mathscr{X}(\mathscr{A})$ has a section f over V, there is by construction an isomorphism $\mathbf{R}\mathscr{E}nd_V(f^*\mathscr{F}) \to \mathscr{A}|_V$, and this is natural in V and f. This is easily seen to imply the remaining statements of the lemma. □

Let $\mathscr{D} \to \mathit{Sch}_{\text{ét}}$ denote the fibered category of derived categories which to any scheme X associates the derived category $\mathbf{D}(X)$ of étale $\mathscr{O}_X$-modules.

Proposition 5.2.1.12. *There is a faithful morphism of fibered categories $\mathscr{P}\mathscr{R}^s \to \mathscr{D}$ which identifies $\mathscr{P}\mathscr{R}^s$ with the subcategory of $\mathscr{D}$ whose sections over X are weak algebras of the form $\mathbf{R}\pi_*\mathbf{R}\mathscr{E}nd_{\mathscr{X}}(\mathscr{F})$, where $\pi : \mathscr{X} \to X$ is a $\mathbf{G}_m$-gerbe, and whose isomorphisms $\mathbf{R}\pi_*\mathbf{R}\mathscr{E}nd_{\mathscr{X}}(\mathscr{F}) \xrightarrow{\sim} \mathbf{R}\pi'_*\mathbf{R}\mathscr{E}nd_{\mathscr{X}'}(\mathscr{F}')$ are naturally a pseudotorsor under $\mathscr{A}ut(\mathscr{F})/\mathbf{G}_m$.*

Proof. The morphism $\mathscr{P}\mathscr{R}^s \to \mathscr{D}$ comes from Lemma 5.2.1.11. Given $\mathscr{A}$ and $\mathscr{B}$, an isomorphism $\varphi : \mathscr{A} \to \mathscr{B}$ induces an isomorphism $\mathscr{X}(\mathscr{A}) \xrightarrow{\sim} \mathscr{X}(\mathscr{B})$. Thus, given $\mathscr{X}, \mathscr{F}, \mathscr{X}', \mathscr{F}'$, an isomorphism $\mathbf{R}\pi_*\mathbf{R}\mathscr{E}nd(\mathscr{F}) \xrightarrow{\sim} \mathbf{R}\pi'_*\mathbf{R}\mathscr{E}nd(\mathscr{F}')$ induces an isomorphism $\varepsilon : \mathscr{X} \to \mathscr{X}'$ and an isomorphism of generalized Azumaya algebras $\mathbf{R}\pi_*\mathbf{R}\mathscr{E}nd(\mathscr{F}) \xrightarrow{\sim} \mathbf{R}\pi_*\mathbf{R}\mathscr{E}nd(\varepsilon^*\mathscr{F}')$. (In particular, any isomorphism is identified with an isomorphism of the underlying weak algebras.) By Lemma 5.2.1.5, once there is an isomorphism the set of isomorphisms is a torsor under $\mathscr{A}ut(\mathscr{F})/\mathbf{G}_m$, as claimed. The faithfulness also results from Lemma 5.2.1.5. □

Remark 5.2.1.13. When $\mathscr{X} = \mathscr{X}'$ in Proposition 5.2.1.12, the sheaf of isomorphisms is simply identified with the quotient sheaf $\mathrm{Isom}(\mathscr{F}, \mathscr{F}')/\mathbf{G}_m$. This will be the case when we study the moduli of generalized Azumaya algebras on a surface, as the (geometric) Brauer class will be constant in families.

Thus, at the end of the complex process of stackification, one is left simply with the derived endomorphism algebras of twisted sheaves, with a subset of the quasiisomorphisms giving the isomorphisms.

5.2.2. *Identification with rigidifications.* Let $\mathscr{G}_X(n)$ be the stack of generalized Azumaya algebras on X of degree n.

Proposition 5.2.2.1. *The morphism $\rho : \mathscr{T}_X^{\mathrm{parf}} \to \mathscr{G}_X$ sending $\mathscr{F}$ to $\mathbf{R}\mathscr{E}nd(\mathscr{F})$ yields an isomorphism $\mathscr{T}_X /\!\!/ \mathbf{G}_m \xrightarrow{\sim} \mathscr{G}_X$.*

Proof. It follows from Theorem 5.1.5 that $\mathscr{G}_X$ is the stackification of the prestack given by taking totally supported sheaves and replacing $\mathrm{Isom}(\mathscr{F}, \mathscr{G})$ with $\mathrm{Isom}(\mathscr{F}, \mathscr{G})/\mathbf{G}_m$. But this is precisely how $\mathscr{T}_X^{\mathrm{parf}} /\!\!/ \mathbf{G}_m$ is constructed! □

Lemma 5.2.2.2. *If $\mathfrak{Q} \to \mathfrak{Q}'$ is a morphism of prestacks on a site which is fully faithful on fiber categories and an epimorphism* (*that is*, *any object of $\mathfrak{Q}'$ is locally in the image of $\mathfrak{Q}$*) *then the induced map of stackifications is an isomorphism.*

Proof. An object of the stackification is just an object of the prestack with a descent datum. Moreover, refining the descent datum yields a naturally isomorphic object of the stackification. Thus, we see that the map on stackifications $\widetilde{\mathfrak{Q}} \to \widetilde{\mathfrak{Q}'}$ is fully faithful and an epimorphism. (Indeed, after refining the descent datum on an object of $\mathfrak{Q}'$, we can assume the object and descent datum come from $\mathfrak{Q}$.) It follows that it must be an isomorphism. □

Proposition 5.2.2.3. *Assume n is invertible on X. The morphism*

$$\rho : \mathcal{T}_X^{\mathbb{O}}(n) \to \mathcal{G}_X(n)$$

sending $\mathcal{F}$ to $\mathbf{R}\mathcal{E}nd(\mathcal{F})$ yields an isomorphism $\mathcal{T}_X^{\mathbb{O}} /\!\!/ \boldsymbol{\mu}_n \xrightarrow{\sim} \mathcal{G}_X$.

Proof. In light of Proposition 5.2.2.1, it is enough to prove that the natural map $\varphi : \mathcal{T}_X^{\mathbb{O}}(n) \to \mathcal{T}_X(n)$ yields an isomorphism of the appropriate rigidifications. But φ is clearly an epimorphism. Further, for any $\mathcal{F}$ and $\mathcal{G}$ with trivialized determinants, φ induces an isomorphism of sheaves $f : \mathrm{Isom}_{\det}(\mathcal{F}, \mathcal{G})/\boldsymbol{\mu}_n \xrightarrow{\sim} \mathrm{Isom}(\mathcal{F}, \mathcal{G})/\mathbf{G}_m$. To check this, it is enough to suppose $X = \operatorname{Spec} A$ is strictly Henselian. Since n is invertible on X, any unit of A has an n-th root, from which it follows that f is surjective. If $\gamma : \mathcal{F} \to \mathcal{G}$ and $\eta : \mathcal{F} \to \mathcal{G}$ are isomorphisms which preserve determinants and differ by multiplication by a scalar θ on $\mathcal{G}$ then θ must be an n-th root of unity, which shows that f is injective.

Forming prestacks by dividing out by the appropriate scalars, we thus find a morphism of prestacks $\mathfrak{Q} \to \mathfrak{Q}'$ which is fully faithful on fiber categories and is an étale epimorphism. Applying Lemma 5.2.2.2 completes the proof. □

5.2.3. *The relative case.* Now we push everything forward (with one important Warning 5.2.3.2 below) to define relative stacks of generalized Azumaya algebras.

Definition 5.2.3.1. Let $f : X \to S$ be a morphism. A *relative generalized Azumaya algebra* on X/S is a generalized Azumaya algebra on X whose local sheaves are S-flat and totally pure in each geometric fiber. This is equivalent to writing $\mathcal{A} \cong \mathbf{R}\pi_* \mathbf{R}\mathcal{E}nd_{\mathcal{X}}(\mathcal{F})$ with $\mathcal{X} \to X$ a $\mathbf{G}_m$-gerbe and $\mathcal{F}$ an S-flat $\mathcal{X}$-twisted sheaf which is totally pure in every geometric fiber.

Warning 5.2.3.2. Even though the absolute theory of generalized Azumaya algebras used totally supported sheaves, in the relative theory we will use totally pure sheaves. While this is not necessary for the abstract results to be true, it gives a better moduli theory when X/S is sufficiently nice (for example, a smooth projective surface).

As in Definition 5.2.1.6, one may define the class of such a generalized Azumaya algebra. Let $\mathcal{X} \to X$ be a fixed $\boldsymbol{\mu}_n$-gerbe, with $n \in \mathcal{O}_S(S)^\times$.

Notation 5.2.3.3. Let $\mathbf{GAz}_{\mathcal{X}/S}(n)$ denote the stack of generalized Azumaya algebras on X/S of rank n^2 in every geometric fiber whose class agrees with $[\mathcal{X}]$ étale locally around every point on the base. When we do not wish to specify the cohomology class, we will write $\mathbf{GAz}_{X/S}(n)$ for the stack of generalized Azumaya algebras of rank n^2 on each fiber.

When $X \to S$ is proper and n is invertible on S, the condition that the cohomology class agree with $[\mathcal{X}]$ étale-locally on S is equivalent to the condition that it agree with $[\mathcal{X}]$ in every geometric fiber.

5.2.4. *Identification of* $\mathbf{GAz}_{\mathscr{X}/S}(n)$ *with* $\mathscr{M}_n^{\mathscr{X}}$. Let $\mathscr{G}_{X/S}(n)$ be the stack on $X_{\text{rét}}$ parametrizing generalized Azumaya algebras that are locally isomorphic to $\mathbf{R}\mathscr{E}nd(\mathscr{F})$ with $\mathscr{F}$ an object of $\mathscr{P}^{\text{parf}}_{X/S}(n)$. An argument identical to Proposition 5.2.2.1 shows that the natural map $\mathscr{P}^{\mathbb{O}}_{X/S}(n) \to \mathscr{G}_{X/S}(n)$ yields an isomorphism

$$\mathscr{P}^{\mathbb{O}}_{X/S}(n) /\!\!/ \boldsymbol{\mu}_n \xrightarrow{\sim} \mathscr{G}_{X/S}(n).$$

On the other hand, it is easy to see that $\mathbf{GAz}_{X/S}(n) = f_*(\mathscr{G}_{X/S}(n))$ and that $\mathbf{GAz}_{\mathscr{X}/S}(n) = f_*^{\mathscr{X}}(\mathscr{G}_{X/S}(n))$. We conclude that $\mathbf{GAz}_{\mathscr{X}/S}(n) \cong \mathscr{M}_n^{\mathscr{X}}$, thus showing that generalized Azumaya algebras give a coherent model for $\mathscr{M}_n^{\mathscr{X}}$. Moreover, it is easy to see that $\mathscr{M}_n^{\mathscr{X}}(c)$ is identified with the stack of generalized Azumaya algebras of the form $\mathbf{R}\mathscr{H}om(\mathscr{F})$ with $\mathscr{F}$ an $\mathscr{X}$-twisted sheaf with trivialized determinant and $\deg c_2(\mathscr{F}) = c/2n$. This condition is equivalent to the condition that $\deg c_2(\mathbf{R}\mathscr{E}nd(\mathscr{F})) = c$, giving a coherent interpretation of $\mathscr{M}_n^{\mathscr{X}}(c)$.

For the reader uncomfortable with the stackification procedure (in spite of its concrete outcome), we will show in Section 6.4 that when X is a surface stackification is in fact unnecessary.

6. Moduli of stable PGL_n-torsors on surfaces

For the rest of this paper, we assume that $S = \operatorname{Spec} k$ with k a separably closed field and X/S a smooth projective surface with a fixed ample divisor H.

6.1. ***Stability of torsors.*** We first recall a basic definition.

Definition 6.1.1. Given a torsion-free sheaf $\mathscr{F}$, the *slope* of $\mathscr{F}$ is $\deg \mathscr{F} / \operatorname{rk} \mathscr{F}$.

To define stability for PGL_n-torsors, we use the adjoint sheaf. As described in Section 3.3, this adjoint sheaf naturally comes with an algebra structure, which we will use in our definition.

Definition 6.1.2. An Azumaya algebra $\mathscr{A}$ on X is *stable* if for all nonzero right ideals $\mathscr{I} \subset \mathscr{A}$ of rank strictly smaller than $\operatorname{rk} \mathscr{A}$ we have $\mu(\mathscr{I}) < 0$.

Remark 6.1.3. It is equivalent to quantify over left ideals. Thus, one could state the definition by omitting the word "right" and quantifying over arbitrary ideals, understood as right or left ideals. It is of course not sufficient to quantify over two-sided ideals.

Remark 6.1.4. This definition is meant to apply only to the classical notion of slope-stability, and not to the more refined notion due to Gieseker. While such notions of stability using normalized Hilbert polynomials are essential for the development of moduli theory using geometric invariant theory (GIT), they are somewhat artificial in the sense that they no longer correspond to the existence of a Hermite-Einstein connection. (However, recent work of Wang [2002; 2005] has clarified the analytic meaning of Gieseker-stability in terms of the existence of certain weak Hermite–Einstein connections.)

One way to understand the compactifications of the stack of slope-stable bundles — using slope-semistability or Gieseker-semistability, GIT-bound or purely stacky, etc. — is that each really only serves to impose the kind of inductive strucure on the moduli problem necessary to prove theorems about the actual part of interest: the open sublocus of slope-stable bundles. Working in a GIT-free manner (which is necessary in the context of twisted sheaves) frees us to ignore the subtleties (both algebraic and analytic) of Gieseker-stability. This is pursued in [Lieblich 2008a], where the asymptotic properties of moduli are proved entirely without GIT.

6.1.5. To relate Definition 6.1.2 to the cover of $f_*(\mathrm{BPGL}_n)$ by the stack of twisted sheaves, we recall some rudiments from the theory of Chern classes for twisted sheaves. A different development of the theory of Chern classes for twisted sheaves and the relationship to the theory described here is given in [Heinloth 2005].

Given a coherent $\mathscr{X}$-twisted sheaf $\mathscr{F}$, we can use the rational Chow theory of $\mathscr{X}$ to define Chern classes $c_i(\mathscr{F})$, $i = 1, 2$. (The first Chern class $c_1(\mathscr{F})$ is just the class in Chow theory associated to the invertible sheaf $\det \mathscr{F}$.) There is also a degree map from $d : A_0(\mathscr{X}) \to \mathbb{Q}$; this has the property that the 0-cycle supported over a closed point of X has degree $1/n$. We define a normalized degree function by $\deg = nd$. Using this degree, we have the following definition.

Definition 6.1.6. A torsion-free $\mathscr{X}$ twisted sheaf $\mathscr{V}$ is *stable* if for every subsheaf $\mathscr{W} \subset \mathscr{V}$ we have

$$\mu(\mathscr{W}) < \mu(\mathscr{V}).$$

As a special case of [Lieblich 2007, 2.2.7.22], it follows that if $\mathscr{F}$ is a flat family of coherent $\mathscr{X}$-twisted sheaves parametrized by T with trivialized determinant, then the function $t \mapsto \deg c_2(\mathscr{F}_t)$ is locally constant on T. Moreover, by [Lieblich 2008a, Proposition 4.3.1.2], we have that stability is an open condition in a flat family of torsionfree $\mathscr{X}$-twisted sheaves.

Notation 6.1.7. Let $\mathbf{Tw}_{\mathscr{X}/k}(n, \mathscr{O}, c) \subset \mathbf{Tw}_{\mathscr{X}/k}(n, \mathscr{O})$ denote the open and closed substack parametrizing families such that $\deg c_2(\mathscr{F}_t) = c$ in each geometric fiber. Let $\mathbf{Tw}^s_{\mathscr{X}/k}(n, \mathscr{O}, c) \subset \mathbf{Tw}_{\mathscr{X}/k}(n, \mathscr{O}, c)$ denote the open substack whose objects over T are families $\mathscr{F}$ such that the fiber $\mathscr{F}_t$ is stable for each geometric point $t \to T$.

Lemma 6.1.8. *A locally free $\mathscr{X}$-twisted sheaf $\mathscr{V}$ is stable if and only if the Azumaya algebra $\pi_*\mathscr{E}nd(\mathscr{V})$ is stable.*

Proof. Given a subsheaf $\mathscr{W} \subset \mathscr{V}$, a straightforward computation shows that

$$\mu(\mathscr{H}om(\mathscr{V}, \mathscr{W})) = \mu(\mathscr{W}) - \mu(\mathscr{V}).$$

On the other hand, any right ideal of $\mathscr{A}$ has the form $\mathscr{H}om(\mathscr{V}, \mathscr{W})$ for a subsheaf $\mathscr{W} \subset \mathscr{V}$. The result follows. $\square$

6.1.9. Suppose $B \to S$ is a k-scheme and $T \to X_B$ is a PGL_n-torsor. Let $\mathscr{A}$ be the locally free sheaf (of rank n^2) associated to the adjoint torsor (which is a GL_{n^2}-torsor). Using the Riemann-Roch theorem, the invariance of Euler characteristic in a flat family, and the fact that $\det \mathscr{A} \cong \mathscr{O}$, we see that the function $b \mapsto \deg c_2(\mathscr{A}_b)$ is locally constant on B. This provides a numerical invariant of a PGL_n-torsor which is constant in a family. Given a $\boldsymbol{\mu}_n$-gerbe $\mathscr{X} \to X$ and an integer c, let $(f_*^{\mathscr{X}} \mathsf{B}\mathrm{PGL}_n)(c)$ be the substack of $f_*^{\mathscr{X}} \mathsf{B}\mathrm{PGL}_n$ parametrizing families where the locally free sheaf associated to the adjoint bundle has $\deg c_2 = c$ in every fiber.

Thus, there is a decomposition

$$f_* \mathsf{B}\mathrm{PGL}_n = \sqcup_{\mathscr{X}} \sqcup_c (f_*^{\mathscr{X}} \mathsf{B}\mathrm{PGL}_n)(c),$$

where the first disjoint union is taken over a set of $\boldsymbol{\mu}_n$-gerbe representatives for $\mathrm{H}^2(X, \boldsymbol{\mu}_n)$ and the second is taken over $\mathbb{Z}$. Similarly, there is a decomposition

$$f_*(\mathsf{B}\mathrm{PGL}_n)^s = \sqcup_{\mathscr{X}} \sqcup_c (f_*^{\mathscr{X}}(\mathsf{B}\mathrm{PGL}_n)^s$$

of stable loci.

Lemma 6.1.10. *Given an integer c, the closed and open substack $\mathbf{Tw}_{\mathscr{X}/S}(n, \mathscr{O}, c/2n)$ is equal to the preimage of its image in $\mathscr{M}_n^{\mathscr{X}}$. Similarly, $\mathbf{Tw}^s_{\mathscr{X}/k}(n, \mathscr{O}, c/2n)$ is equal to the preimage of its image in $\mathscr{M}_n^{\mathscr{X}}$.*

Proof. Given a point p of $\mathscr{M}_n^{\mathscr{X}}$ which lifts into $[\mathscr{F}] \in \mathbf{Tw}_{\mathscr{X}/S}(n, \mathscr{O}, c/2n)$, it is easy to see that the full preimage of p in $\mathbf{Tw}_{\mathscr{X}/S}(n, \mathscr{O})$ is given by the twists $\mathscr{F} \otimes \mathscr{L}$ with $\mathscr{L} \in \mathscr{P}ic_{X/S}[n]$. But these have the same (rational) Chern classes as $\mathscr{F}$, as $\mathscr{L}$ is trivial, so they also lie in $\mathbf{Tw}_{\mathscr{X}/S}(n, \mathscr{O}, c/2n)$. The second statement follows from the fact that $\mathscr{F}$ is stable if and only if $\mathscr{F} \otimes \mathscr{L}$ is stable for an invertible sheaf $\mathscr{L}$. $\square$

Corollary 6.1.11. *There is an open substack $f_*(\mathsf{B}\mathrm{PGL}_n)^s \subset f_*(\mathsf{B}\mathrm{PGL}_n)$ parametrizing families $P \to X_T$ of PGL_n-torsors such that for all geometric points $t \to T$ the fiber $P_t \to X_t$ is a stable PGL_n-torsor.*

Proof. This follows from Lemma 6.1.10 together with the fact that $\mathbf{Tw}_{\mathscr{X}/k}(n, \mathscr{O})^{\mathrm{lf}} \to f_*(\mathsf{B}\mathrm{PGL}_n)$ is universally submersive. $\square$

Hence, since $\mathbf{Tw}_{\mathscr{X}/S}(n, \mathscr{O}, c/2n)$ is open and closed in $\mathbf{Tw}_{\mathscr{X}/S}(n, \mathscr{O})$, there is a well-defined open and closed substack $\mathscr{M}_n^{\mathscr{X}}(c)$ whose preimage is $\mathbf{Tw}_{\mathscr{X}/S}(n, \mathscr{O}, c/2n)$. There is an open substack $\mathscr{M}_n^{\mathscr{X}}(c)^s$ whose preimage is $\mathbf{Tw}^s_{\mathscr{X}/k}(n, \mathscr{O}, c/2n)$. Each $\mathscr{M}_n^{\mathscr{X}}(c)$ is quasiproper and there is an open immersion $f_*^{\mathscr{X}} \mathsf{B}\mathrm{PGL}_n(c) \hookrightarrow \mathscr{M}_n^{\mathscr{X}}(c)$ and an open immersion $f_*^{\mathscr{X}} \mathsf{B}\mathrm{PGL}_n(c)^s \hookrightarrow \mathscr{M}_n^{\mathscr{X}}(c)^s$. Moreover, each $\mathscr{M}_n^{\mathscr{X}}(c)$ is covered by $\mathbf{Tw}_{\mathscr{X}/S}(n, \mathscr{O}, c/2n)$ in such a way that the fibers are locally $\boldsymbol{\mu}_n$-gerbes over

$\mathrm{Pic}_{X/S}[n]$-torsors, and likewise for the open substacks parametrizing stable objects. This covering restricts to a covering of $(f_*^{\mathscr{X}}\,\mathrm{BPGL}_n)(c)^s$ by $\mathbf{Tw}^s_{\mathscr{X}/S}(n, \mathscr{O}, c/2n)^{\mathrm{lf}}$.

In particular, $\mathscr{M}_n^{\mathscr{X}}(c)^s$ is irreducible, separated, etc. if $\mathbf{Tw}^s_{\mathscr{X}/S}(n, \mathscr{O}, c/2n)$ has the same property; and $\mathscr{M}_n^{\mathscr{X}}(c)^s$ has a local property stable for the étale topology if and only if the same is true for $\mathbf{Tw}^s_{\mathscr{X}/S}(n, \mathscr{O}, c/2n)$ Moreover, there is a virtual fundamental class for $\mathscr{M}_n^{\mathscr{X}}(c)^s$ if and only if there is one for $\mathbf{Tw}^s_{\mathscr{X}/S}(n, \mathscr{O}, c/2n)$.

Notation 6.1.12. Let $\mathbf{GAz}_{\mathscr{X}/S}(n)^s$ denote the open substack parametrizing stable generalized Azumaya algebras via the isomorphism $\mathbf{GAz}_{\mathscr{X}/S}(n) \xrightarrow{\sim} (\mathscr{M}_n^{\mathscr{X}})^s$ of Section 5.2.4.

6.2. ***Structure of moduli of twisted sheaves.*** The following results show that infinitely many of the spaces $\mathbf{Tw}^s_{\mathscr{X}/k}(n, \mathscr{O}, \gamma)$ are nonempty. It is a geometric restatement of the fundamental result of [de Jong 2004] on the period-index problem for Brauer classes over functionf ields of algebraic surfaces.

Lemma 6.2.1. *There is a stable locally free $\mathscr{X}$-twisted sheaf of rank n.*

A proof of this result may be found in [Lieblich 2008b, Theorem 4.2.2.3] and [Lieblich 2008a, Proposition 5.1.2].

Lemma 6.2.2. *Suppose $\mathscr{F}$ is a coherent $\mathscr{X}$-twisted sheaf of rank n with* $\det \mathscr{F} \cong \mathscr{O}$ *and* $\deg c_2(\mathscr{F}) = \gamma$. *For each integer $\ell \geq 0$, there is a (noncanonical) subsheaf $\mathscr{F}_\ell \subset \mathscr{F}$ such that* $\dim \mathscr{F}/\mathscr{F}_\ell = 0$, $\det \mathscr{F}_\ell \cong \mathscr{O}$, *and* $\deg c_2(\mathscr{F}_\ell) = \gamma + \ell$. *If $\mathscr{F}$ is stable then so is $\mathscr{F}_\ell$.*

Proof. By induction, it suffices to construct $\mathscr{F}_1$. Choose a point $x \in X(k)$ around which $\mathscr{F}$ is locally free and let $\mathscr{F} \otimes \kappa(x) \twoheadrightarrow \mathscr{Q}$ be a quotient with geometric fiber of dimension 1. (In other words, given an algebraically closed extension field $L/\kappa(x)$ and a map $\operatorname{Spec} L \to \mathscr{X} \otimes \kappa(x)$, the pullback of $\mathscr{Q}$ to $\operatorname{Spec} L$ is the sheaf associated to a one-dimensional vector space.) We claim that $\deg(c_2(\mathscr{Q})) = -1$, from which the result follows by the multiplicativity of the total Chern polynomial. A proof of the claim uses the Grothendieck–Hirzebruch–Riemann–Roch theorem for representable morphisms of Deligne–Mumford stacks and can be found in the proof of [Lieblich 2007, Lemma 3.2.4.8] (where there is an unfortunate sign error in the statement, even though the proof is correct!).

To deduce stability of $\mathscr{F}_1$ from stability of $\mathscr{F}$, first note that the two sheaves agree in codimension 1. Since stability depends on a calculation of degree and this calculation depends only on a sheaf in codimension 1, we see that we need only quantify over saturated subsheaves [Huybrechts and Lehn 1997, Definition 1.1.5], which are determined by their values in codimension 1. Thus, the criterion determining stability of $\mathscr{F}$ and $\mathscr{F}_1$ quantifies over the same set of subsheaves with the same numerical calculations. ☐

Corollary 6.2.3. *If $\mathbf{Tw}^s_{\mathscr{X}/k}(n, \mathbb{O}, \gamma)$ is nonempty then so is $\mathbf{Tw}^s_{\mathscr{X}/k}(n, \mathbb{O}, \gamma + \ell)$ for all integers $\ell \geq 0$.*

Here is the fundamental structure theorem concerning these moduli spaces:

Theorem 6.2.4. *There exists a constant C such that for all $\gamma \geq C$,*

(1) *the open substack $\mathbf{Tw}^s_{\mathscr{X}/k}(n, \mathbb{O}, \gamma)^{lf} \subset \mathbf{Tw}^s_{\mathscr{X}/k}(n, \mathbb{O}, \gamma)$ is schematically dense*;

(2) *$\mathbf{Tw}^s_{\mathscr{X}/k}(n, \mathbb{O}, \gamma)$ is an irreducible proper normal lci tame Deligne–Mumford stack over k whenever it is nonempty*;

(3) *it is nonempty for infinitely many γ.*

Proof. The third statement follows immediately from Corollary 6.2.3. For the proof of the first and second, the reader is referred to paragraph 3.2.4.1 (and especially Theorem 3.2.4.11) of [Lieblich 2007]. □

6.2.5. Since every object of $\mathbf{Tw}^s_{\mathscr{X}/k}(n, \mathbb{O}, \gamma)$ is geometrically stable, it is simple (see, for example, Corollary 1.2.8 and Theorem 1.6.6 of [Huybrechts and Lehn 1997]); that is, its automorphisms are simply given by multiplication by scalars (in $\boldsymbol{\mu}_n$, since the determinant is trivialized). It follows that $\mathbf{Tw}^s_{\mathscr{X}/k}(n, \mathbb{O}, \gamma)$ is a $\boldsymbol{\mu}_n$-gerbe over its coarse moduli space $\mathrm{Tw}^s_{\mathscr{X}/k}(n, \mathbb{O}, \gamma)$.

6.3. *Consequences for $\mathscr{M}^{\mathscr{X}}_n$ and $f^{\mathscr{X}}_* \mathsf{B}\mathbf{PGL}_n$.*

Theorem 6.3.1. *There is a constant D such that for all $c \geq D$,*

(1) *the open substack $f^{\mathscr{X}}_* \mathsf{B}\mathrm{PGL}_n(c)^s$ is schematically dense in $\mathscr{M}^{\mathscr{X}}_n(c)^s$*;

(2) *$\mathscr{M}^{\mathscr{X}}_n(c)^s$ is an irreducible proper normal lci tame Deligne–Mumford stack over k whenever it is nonempty*;

(3) *it is nonempty for infinitely many c.*

In particular, the open substack $(f^{\mathscr{X}}_* \mathsf{B}\mathrm{PGL}_n)(c)^s$ is irreducible (and nonempty for infinitely many c).

Proof. The proof follows immediately by combining the covering described at the end of Section 4.2 with Lemma 6.2.1, Lemma 6.2.2, and Theorem 6.2.4. □

Recall that "lciq singularities" are by definition finite quotients of lci singularities.

Corollary 6.3.2. *For sufficiently large c, the coarse moduli space $(\mathscr{M}^{\mathscr{X}}_n(c)^s)^{\mathrm{mod}}$ is an irreducible proper normal algebraic space with lciq singularities.*

Proof. Lemma 3.4.2.5 gives rise to a finite morphism

$$\mathrm{Tw}^s_{\mathscr{X}/k}(n, \mathbb{O}, c/2n) \to (\mathscr{M}^{\mathscr{X}}_n(c)^s)^{\mathrm{mod}}$$

from the coarse space of Paragraph 6.2.5 which is invariant for the natural action of $\mathrm{Pic}_{X/k}[n]$ on $\mathrm{Tw}^s_{\mathscr{X}/k}(n, \mathscr{O}, c/2n)$ and such that the natural map

$$\chi : \mathrm{Tw}^s_{\mathscr{X}/k}(n, \mathscr{O}, c/2n)/\operatorname{Pic}_{X/k}[n] \to (\mathscr{M}^{\mathscr{X}}_n(c)^s)^{\mathrm{mod}}$$

is birational. Since the coarse space of a normal tame Deligne–Mumford stack is normal, it follows from Zariski's Main Theorem that χ is an isomorphism. Since $\mathbf{Tw}^s_{\mathscr{X}/k}(n, \mathscr{O}, c/2n)$ is lci, it follows that $(\mathscr{M}^{\mathscr{X}}_n(c)^s)^{\mathrm{mod}}$ is lciq. □

6.4. ***Stackification is unnecessary on a surface.*** Let $f : X \to S$ be a smooth projective relative surface. We will prove here that pregeneralized Azumaya algebras on X as in Section 5.2 form a stack on S.

Given a pregeneralized Azumaya algebra $\mathscr{A}$ on X, Lemma 5.2.1.11 produces a $\mathbf{G}_m$-gerbe $\mathscr{X}$, an $\mathscr{X}$-twisted sheaf $\mathscr{F}$, and an isomorphism of *generalized* Azumaya algebras $\mathscr{B} := \mathbf{R}\pi_*\mathbf{R}\mathscr{E}nd(\mathscr{F}) \xrightarrow{\sim} \mathscr{A}$. We will show that in fact $\mathscr{B}$ and $\mathscr{A}$ are isomorphic as *pregeneralized Azumaya algebras*. We will temporarily call $\mathscr{B}$ the *associated twisted derived endomorphism algebra* (or TDEA for short).

Proposition 6.4.1. *Suppose $f : X \to S$ is a smooth (possibly nonproper) relative surface over an affine scheme. Any pregeneralized Azumaya algebra $\mathscr{A}$ is isomorphic to the associated TDEA in $\mathscr{PR}$. Furthermore, the isomorphisms of two such weak algebras form a sheaf on S.*

Proof. By standard arguments (for example, [Beĭlinson et al. 1982, Theorem 3.2.4] or [Abramovich and Polishchuk 2006, Theorem 2.1.9]), it suffices to prove that $\mathrm{Ext}^{-i}(\mathscr{A}, \mathscr{A}) = 0$ for all $i > 0$ (as long as we allow $f : X \to S$ to be arbirary with the stated hypotheses). From the definition of pregeneralized Azumaya algebra, we know that $\mathscr{A}$ has cohomology only in degrees 0 and 1, that $\mathscr{H}^0(\mathscr{A})$ has totally pure fibers over S, and that $\mathscr{H}^1(\mathscr{A})$ has support with relative dimension 0. The natural triangle

$$\mathscr{H}^0(\mathscr{A}) \to \mathscr{A} \to \mathscr{H}^1(\mathscr{A})[-1] \xrightarrow{+}$$

gives rise to an exact sequence

$$\mathrm{Ext}^{-i}(\mathscr{H}^1(\mathscr{A}), \mathscr{A}[1]) \to \mathrm{Ext}^{-i}(\mathscr{A}, \mathscr{A}) \to \mathrm{Ext}^{-i}(\mathscr{H}^0(\mathscr{A}), \mathscr{A}).$$

The left-hand group fits into an exact sequence

$$\mathrm{Ext}^{-i}(\mathscr{H}^1(\mathscr{A}), \mathscr{H}^0(\mathscr{A})) \to \mathrm{Ext}^{-i}(\mathscr{H}^1(\mathscr{A}), \mathscr{A}[1]) \to \mathrm{Ext}^{-i}(\mathscr{H}^1(\mathscr{A}), \mathscr{H}^1(\mathscr{A}))$$

and the right-hand group fits into an exact sequence

$$\mathrm{Ext}^{-i}(\mathscr{H}^0(\mathscr{A}), \mathscr{H}^0(\mathscr{A})) \to \mathrm{Ext}^{-i}(\mathscr{H}^0(\mathscr{A}), \mathscr{A}) \to \mathrm{Ext}^{-i-1}(\mathscr{H}^0(\mathscr{A}), \mathscr{H}^1(\mathscr{A})).$$

(This is simply an explicit description of a certain spectral sequence, which is especially simple because $\mathscr{A}$ has so few cohomology sheaves.) We wish to show that the ends of the last two sequences vanish, for which it is enough to show (using the local-to-global Ext-spectral sequence) that the Ext-sheaves $\mathscr{E}xt^*(\mathscr{H}^*(\mathscr{A}), \mathscr{H}^*(\mathscr{A}))$ vanish for appropriate indices. But there are no negative Ext-groups for modules over a ring. This completes the proof. $\square$

Remark 6.4.2. When X/S is quasiprojective, one can also give an explicit proof of Proposition 6.4.1 (which does not rely on [Beĭlinson et al. 1982]) using resolutions by sums of powers of $\mathscr{O}(1)$.

6.5. ***Deformation theory and the virtual fundamental class.*** Let k be an algebraically closed field and X/k a smooth projective surface over k. Fix a $\boldsymbol{\mu}_n$-gerbe $\mathscr{X} \to X$ with n invertible in k.

6.5.1. *Perfect obstruction theory for twisted sheaves.* Let $\mathscr{F}$ be the universal $\mathscr{X}$-twisted sheaf on $\mathscr{X} \times \mathbf{Tw}^s_{\mathscr{X}/k}(n, \mathscr{O})$. Write p (resp. q) for the projection of $\mathscr{X} \times \mathbf{Tw}^s_{\mathscr{X}/k}(n, \mathscr{O})$ to $\mathscr{X}$ (resp. $\mathbf{Tw}^s_{\mathscr{X}/k}(n, \mathscr{O})$). Recall that there is a natural isomorphism

$$L_{\mathscr{X}\times\mathbf{Tw}^s_{\mathscr{X}/k}(n,\mathscr{O})} \cong \mathbf{L}p^*L_{\mathscr{X}} \oplus \mathbf{L}q^*L_{\mathbf{Tw}^s_{\mathscr{X}/k}(n,\mathscr{O})},$$

and that there is an isomorphism of functors (coming from Grothendieck duality for q)

$$\mathbf{L}q^* \cong \mathbf{L}q^! \overset{\mathbf{L}}{\otimes} \mathbf{L}p^*\omega_X^\vee[-2].$$

The Atiyah class

$$\mathscr{F} \to L_{\mathscr{X}\times\mathbf{Tw}^s_{\mathscr{X}/k}(n,\mathscr{O})} \overset{\mathbf{L}}{\otimes} \mathscr{F}[1]$$

yields by projection a map

$$\mathscr{F} \to \mathbf{L}q^*L_{\mathbf{Tw}^s_{\mathscr{X}/k}(n,\mathscr{O})}[1] \overset{\mathbf{L}}{\otimes} \mathscr{F}.$$

Since $\mathscr{F}$ is perfect, this is equivalent (by the cher à Cartan isomorphism) to a map

$$\mathbf{R}\mathscr{E}nd(\mathscr{F}) \to \mathbf{L}q^*L_{\mathbf{Tw}^s_{\mathscr{X}/k}(n,\mathscr{O})}[1],$$

yielding a map

$$\mathbf{R}\mathscr{E}nd(\mathscr{F}) \to \mathbf{L}q^!L_{\mathbf{Tw}^s_{\mathscr{X}/k}(n,\mathscr{O})}[-1] \overset{\mathbf{L}}{\otimes} \mathbf{L}p^*\omega_X^\vee.$$

Applying Grothendieck duality yields a morphism

$$\mathfrak{b} : \mathbf{R}q_*\mathbf{R}\mathscr{H}om(\mathscr{F}, \mathbf{L}p^*\omega_X \overset{\mathbf{L}}{\otimes} \mathscr{F}) \to L_{\mathbf{Tw}^s_{\mathscr{X}/k}(n,\mathscr{O})}[-1]$$

and restriction to the traceless part finally yields a morphism

$$\mathfrak{b}_0 : \mathbf{R}q_*\mathbf{R}\mathscr{H}om(\mathscr{F}, \mathbf{L}p^*\omega_X \overset{\mathbf{L}}{\otimes} \mathscr{F})_0 \to L_{\mathbf{Tw}^s_{\mathscr{X}/k}(n,\mathscr{O})}[-1]$$

Proposition 6.5.1.1. *The shifted map $\mathfrak{b}_0[1]$ gives a perfect obstruction theory for $\mathbf{Tw}^s_{\mathscr{X}/k}(n, \mathscr{O})$.*

Proof. There are three things to check: that the complex $\mathbf{R}q_*\mathbf{R}\mathcal{H}om(\mathcal{F}, \mathbf{L}p^*\omega_X \overset{\mathbf{L}}{\otimes} \mathcal{F})_0$ is perfect of amplitude $[0, 1]$, that $\mathfrak{b}_0$ induces an isomorphism on H^1 sheaves, and that $\mathfrak{b}_0$ induces a surjection on H^0 sheaves. The first assertion follows from the compatibility of the formation of the complex with base change on $\mathbf{Tw}^s_{\mathcal{X}/k}(n, \mathcal{O})$ (which shows that it is perfect, as its fibers are bounded complexes on regular schemes) and the fact that the fibers of $\mathbf{R}^2q_*\mathbf{R}\mathcal{H}om(\mathcal{F}, \mathbf{L}p^*\omega_X \overset{\mathbf{L}}{\otimes} \mathcal{F})_0$ over geometric points of $\mathbf{Tw}^s_{\mathcal{X}/k}(n, \mathcal{O})$ compute (by Serre duality) the traceless endomorphisms of stable sheaves, which must be trivial (so that the amplitude is as claimed). The other two assertions will follow from Illusie's theory. We already know (thanks to Illusie) that deformations and obstructions are governed by Atiyah classes; we will describe how this allows us to show that $\mathfrak{b}_0$ gives a perfect obstruction theory.

Since both the domain and codomain of $\mathfrak{b}_0$ have no cohomology above degree 1, to show that $\mathfrak{b}_0$ induces an isomorphism on H^1 sheaves, it suffices to show this after base change to geometric points of $\mathbf{Tw}^s_{\mathcal{X}/k}(n, \mathcal{O})$. Given a geometric point $x \to \mathbf{Tw}^s_{\mathcal{X}/k}(n, \mathcal{O})$ corresponding to a twisted sheaf on $\mathcal{X} \otimes \kappa(x)$, we know that $\mathrm{Hom}(L_{\mathbf{Tw}^s_{\mathcal{X}/k}(n,\mathcal{O})}, \kappa(x))$ is naturally identified with the space T of first order determinant-preserving deformations of F over $\kappa(x)[\varepsilon]$. Moreover, by the functoriality of our construction with respect to base change on $\mathbf{Tw}^s_{\mathcal{X}/k}(n, \mathcal{O})$, the map $\mathfrak{b}_0$ is the Serre dual of the Kodaira–Spencer map $T \to \mathrm{Ext}^1(F, F)_0$ (see, for example, [Huybrechts and Lehn 1997, Example 10.1.9]). This is well-known to give an isomorphism (for example, [Huybrechts and Lehn 1997, Section 10.2]).

To show that $\mathrm{H}^0(\mathfrak{b}_0)$ is surjective, we may proceed as follows. Recall from [Illusie 1971, IV.3.1.8] that if $A \to A_0$ is a small extension of B-algebras with kernel I in a topos and M is an A_0-module, then one can find the obstruction to deforming M to an A-module as the composition

$$M \to L_{A_0/B} \overset{\mathbf{L}}{\otimes} M[1] \to I \overset{\mathbf{L}}{\otimes} M[2] \to I \otimes M[2],$$

where the first map is the Atiyah class of M with respect to A_0/B, the second map comes from the morphism $L_{A_0/B} \to I$ parametrizing the class of the extension $A \to A_0$, and the third map is the natural augmentation onto the zeroth cohomology module. To apply this to our case, consider a situation

$$\mathrm{Spec}\, B \leftarrow \mathrm{Spec}\, B_0 \to \mathbf{Tw}^s_{\mathcal{X}/k}(n, \mathcal{O})$$

with $B \to B_0$ a small extension of strictly Henselian local rings and kernel annihilated by the maximal ideal of B. We let A be the structure sheaf of $\mathcal{X} \times \mathrm{Spec}\, B$ and A_0 that of $\mathcal{X} \times \mathrm{Spec}\, B_0 = \mathcal{X} \times \mathrm{Spec}\, B \times_{\mathrm{Spec}\, B} \mathrm{Spec}\, B_0$. Thus, we have that

$$L_{A_0/B} = \mathbf{L}p^* L_{\mathcal{X}} \oplus \mathbf{L}q^* L_{B_0/B}.$$

Moreover, it is clear that the morphism $L_{A_0/B} \to \mathcal{O}_{\mathscr{X}} \otimes I[1]$ parametrizing the extension $A \to A_0$ is given by the map

$$L_{A_0/B} \to \mathbf{L}q^* L_{B_0/B} \to \mathbf{L}q^* I[1].$$

By functoriality, composing the Atiyah class with the natural map

$$L_{\mathbf{Tw}^s_{\mathscr{X}/k}(n,\mathcal{O})}|_{B_0} \to L_{B_0/B}$$

gives rise to the map

$$\mathscr{F} \to \mathbf{L}q^* L_{B_0/B} \overset{\mathbf{L}}{\otimes} \mathscr{F}[1]$$

associated to the projection of the Atiyah class. Thus, we find that the obstruction to deforming $\mathscr{F}$ over B is the element corresponding by Serre duality to the composition

$$\mathbf{R}q_* \mathbf{R}\mathscr{H}om(\mathscr{F}, \mathbf{L}q^*\omega \overset{\mathbf{L}}{\otimes} \mathscr{F})|_{B_0} \to L_{\mathbf{Tw}_{\mathscr{X}/k}(n,\mathcal{O})}[-1]|_{B_0} \to L_{B_0/B}[-1] \to \kappa.$$

On the other hand, the last two arrows give precisely the obstruction to extending the map $\operatorname{Spec} B_0 \to \mathbf{Tw}_{\mathscr{X}/k}(n, \mathcal{O})$ to a map $\operatorname{Spec} B \to \mathbf{Tw}_{\mathscr{X}/k}(n, \mathcal{O})$. Thus, the entire composition is trivial if and only if the composition of the last two maps is trivial. Since any map $L_{\mathbf{Tw}_{\mathscr{X}/k}(n,\mathcal{O})} \to \kappa[1]$ factors through some deformation situation $B \to B_0$, this shows that $\mathrm{H}^0(\mathfrak{b})$ is surjective. Using the fact that n is invertible in k, and thus the existence of a splitting trace map, it is easy to see that the canonical obstruction given here actually lies in the traceless part of $\operatorname{Ext}^2(\mathscr{F}, \mathscr{F})$; this shows that in fact $\mathrm{H}^0(\mathfrak{b}_0)$ is surjective, as desired. $\square$

The deformation theory described here has a concrete form: given a generalized Azumaya algebra $\mathbf{R}\mathscr{E}nd(\mathscr{F})$ on X, the first-order infinitesimal deformations form a pseudo-torsor under the hypercohomology $\mathbf{H}^1(X, \mathbf{R}\mathscr{E}nd(\mathscr{F})_0)$, while there is naturally a class in $\mathbf{H}^2(X, \mathbf{R}\mathscr{E}nd(\mathscr{F})_0)$ giving the obstruction to deforming $\mathscr{F}$. When $\mathscr{F}$ is locally free, so that $\mathbf{R}\mathscr{E}nd(\mathscr{F}) \cong A$ is an Azumaya algebra, we recover the well-known fact that $\mathrm{H}^1(X, A_0)$ parametrizes deformations of A, while $\mathrm{H}^2(X, A_0)$ receives obstructions. If $A \cong \mathscr{H}om(\mathscr{V})$ is the sheaf of endomorphisms of a locally free sheaf on X with trivial(ized) determinant, our general machine simply says that the deformation and obstruction theory of the algebra A is the same as the deformation and obstruction theory of $\mathscr{V}$ as a locally free sheaf with trivialized determinant. (This similarly describes the deformation theory of a twisted sheaf with trivialized determinant.)

6.5.2. *The virtual fundamental class of* $\mathbf{GAz}_{\mathscr{X}/k}(n)^s$. By construction, $\mathbf{GAz}_{\mathscr{X}/k}(n)^s$ is a subfibered category of the fibered category of weak algebras on $X_{\text{rét}}$. As such, there is a universal generalized Azumaya algebra $\mathscr{A}$ on $X \times \mathbf{GAz}_{\mathscr{X}/k}(n)^s$ whose fibers over the moduli space have cohomology class $[\mathscr{X}]$. If

$$\pi : \mathscr{Y} \to X \times \mathbf{GAz}_{\mathscr{X}/k}(n)^s$$

is the gerbe of trivialized trivializations of $\mathscr{A}$, then $\mathscr{A} \cong \mathbf{R}\pi_* \mathbf{R}\mathscr{E}nd(\mathscr{F})$ for some $\mathscr{Y}$-twisted sheaf $\mathscr{F}$. Moreover, the covering $\mathbf{Tw}^s_{\mathscr{X}/k}(n, \mathscr{O}) \to \mathbf{GAz}_{\mathscr{X}/k}(n)^s$ gives rise to an isomorphism

$$\rho : \mathscr{Y} \times_{X \times \mathbf{GAz}_{\mathscr{X}/k}(n)^s} X \times \mathbf{Tw}^s_{\mathscr{X}/k}(n, \mathscr{O}) \xrightarrow{\sim} \mathscr{X} \times \mathbf{Tw}^s_{\mathscr{X}/k}$$

as well as an isomorphism $\mathscr{G} \to \rho^*\mathscr{F}$, where $\mathscr{G}$ is the universal twisted sheaf on $\mathscr{X} \times \mathbf{Tw}^s_{\mathscr{X}/k}(n, \mathscr{O})$. There results from this a natural isomorphism of weak algebras $\mathbf{R}\mathscr{E}nd(\mathscr{F}) \xrightarrow{\sim} \mathbf{L}\rho^*\mathscr{A}$.

Letting $\mathscr{A}_0 \subset \mathscr{A}$ be the traceless part, there is an induced isomorphism

$$\mathbf{R}\mathscr{E}nd(\mathscr{F})_0 \xrightarrow{\sim} \mathbf{L}\rho^*\mathscr{A}_0.$$

Applying Proposition 3.4.3.7 and Proposition 6.5.1.1, we conclude that there is a perfect obstruction theory $\mathscr{A}_0 \to L_{\mathbf{GAz}_{\mathscr{X}/k}(n)^s}$, giving rise to a virtual fundamental class on $\mathbf{GAz}_{\mathscr{X}/k}(n)^s$.

6.6. ***A potential application: numerical invariants of division algebras over function fields.*** Suppose that the cohomology class α of $\mathscr{X}$ in $\mathrm{H}^2(X, \mathbf{G}_m)$ has order n. If $\mathscr{A}$ is an Azumaya algebra of degree n with cohomology class α, then the generic fiber of $\mathscr{A}$ must be a finite dimensional central division algebra D over the function field $k(X)$. In this case, we have an especially nice description of the stable locus.

Lemma 6.6.1. *When $[\mathscr{X}]$ has order n in $\mathrm{H}^2(X, \mathbf{G}_m)$, any PGL_n-torsor T with class α is stable.*

Proof. Indeed, if $\mathscr{A}$ is the Azumaya algebra associated to T then any nonzero right ideal must have rank n^2, since the generic fiber of $\mathscr{A}$ is a division algebra. □

Thus, $\mathbf{GAz}_{\mathscr{X}/k}(n)^s$ is a proper Deligne–Mumford stack which carries a virtual fundamental class, as described in Section 6.5.2. The following question was asked by de Jong.

Question 6.6.2. Does the virtual class $[\mathbf{GAz}_{\mathscr{X}/k}(n)^s]^{\mathrm{vir}}$ lead to any new numerical invariants attached to D?

Via Proposition 3.4.3.7, any invariants coming from $[\mathbf{GAz}_{\mathscr{X}/k}(n)^s]^{\mathrm{vir}}$ will be closely related to similar numbers attached to $[\mathbf{Tw}^s_{\mathscr{X}/k}(n, \mathscr{O})^s]^{\mathrm{vir}}$. One might expect the latter invariants to be related to Donaldson invariants.

One interesting direct comparison might arise as follows: suppose given a family of surfaces $\mathscr{X} \to S$ and a class $\alpha \in \mathrm{H}^2(\mathscr{X}, \boldsymbol{\mu}_n)$ with n invertible on the base, such that there are two geometric points $0, 1 \to S$ such that $\alpha|_{\mathscr{X}_0}$ has order n in $\mathrm{H}^2(\mathscr{X}_0, \mathbf{G}_m)$ and $\alpha|_{\mathscr{X}_1}$ vanishes in $\mathrm{H}^2(\mathscr{X}_1, \mathbf{G}_m)$. (This happens whenever there is jumping in the rank of the Néron–Severi group in the family.) Assuming one could prove deformation invariance of whatever invariants one eventually defines, one would then be able to give a direct comparision between the division-algebra invariants attached to $\mathscr{X}_0$ and the classical invariants attached to $\mathscr{X}_1$.

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Effectiveness of the log Iitaka fibration for 3-folds and 4-folds

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We prove the effectiveness of the log Iitaka fibration in Kodaira codimension two for varieties of dimension ≤ 4. In particular, we finish the proof of effective log Iitaka fibration in dimension two. Also, we show that for the log Iitaka fibration, if the fiber is of dimension two, the denominator of the moduli part is bounded.

1. Introduction

One of the main problems in birational algebraic geometry is to understand the structure of pluricanonical maps. If X is a smooth complex projective variety of dimension n, then we define the pluricanonical maps

$$\phi_{rK_X} : X \dashrightarrow \mathbb{P}(H^0(X, \mathcal{O}_X(rK_X))),$$

determined by the linear system $|rK_X|$. If this linear system is nonempty for some natural number r, which is conjecturally equivalent to X being nonuniruled, then for r sufficiently divisible, the maps ϕ_{rK_X} become birational to a fixed algebraic fiber space $\phi : X' \longrightarrow Y'$ called the Iitaka fibration of X, and the Kodaira dimension $\kappa(X)$ is set to be $\dim Y'$. It is then natural to look for a uniform r (that is, an integer r that depends only on the dimension of X) for which we are inducing a map birational to the Iitaka fibration. After the monumental work [Birkar et al. 2006] of proving the finite generation of the pluricanonical rings, the effective Iitaka fibration problem is largely related to finding generators which are less than a uniform degree for a given dimension. If $\kappa(X) = \dim X$, then X is called of general type and in this case Hacon–M$^{\mathrm{c}}$Kernan [2006] and Takayama [2006], following ideas of Tsuji, have shown that for a smooth projective variety of general type and dimension n, there exists an integer r_n, that depends only on n, such that ϕ_{rK_X} is birational for $r \geq r_n$. The question remains widely open when the Kodaira dimension is not maximal and it is completely known only up to dimension three [Kawamata 1986; Fujino and Mori 2000; Viehweg and Zhang 2007; Ringler 2007].

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The current approach for studying this problem is by using Kawamata's canonical bundle formula which identifies the pluricanonical ring of X with the pluricanonical ring of a pair $(Y, \Delta + L)$ of log general type [Fujino and Mori 2000; Prokhorov and Shokurov 2009], where (Y, Δ) is a KLT pair (KLT = Kawamata log canonical; see Definition 1.4) and L is a $\mathbb{Q}$-line bundle coming from variation of Hodge structure. This raises the natural question of a log analogue of the above statement. Another setting where boundary divisors naturally occur is in moduli problems. Because of the presence of a boundary divisor the best statement that we can hope for is the following: for (X, D) a KLT pair of Kodaira dimension $\kappa \geq 0$ there is a natural number m, that depends only on the coefficients of Δ and the dimension of X such that $\left|\lfloor m(K_X + \Delta)\rfloor\right|$ is birational to the Iitaka fibration. To be able to use the canonical bundle formula inductively we have to allow that the coefficients of Δ lie in a possibly infinite set of rational numbers. The condition that captures the property of this set is the *descending chain condition* (DCC). At first glance, it may seem to be quite technical and somewhat artificial. However, it turns out to be a very natural and useful condition arising in many questions. See [Shokurov 1992] and [Kollár 1994] for more detailed discussions of this. Thus the general conjecture is formulated as follows.

Conjecture 1.1 (Effective log Iitaka fibration conjecture). *Let (X, D) be a KLT (or ϵ-lc) pair such that $K_X + D$ is pseudo-effective and the coefficients in D are in a DCC set $\mathcal{A}$. Then there is a constant r depending only on the dimension X and on $\mathcal{A}$ such that $\left|\lfloor r(K_X + D)\rfloor\right|$ induces a map birational to the Iitaka fibration.*

For a pair (X, D), the log Kodaira dimension is defined as the Iitaka dimension $\kappa(K_X + D)$ [Lazarsfeld 2004, 2.1.3]. For some partial results toward Conjecture 1.1 see [Pacienza 2007]. In this note, we prove the following.

Theorem 1.2. *Let (X, Δ) be a KLT pair of dimension two, three or four and log Kodaira codimension two. Assume that the coefficients of Δ are in a DCC set of rational numbers $\mathcal{A} \subset [0, 1]$. Then there is an explicitly computable constant m depending only on the set $\mathcal{A}$ such that $\lfloor m(K_X + \Delta)\rfloor$ induces the Iitaka fibration.*

The proof of the above theorem contains three parts. First, we prove the dimension-two case of Theorem 1.2. This is done in Theorem 3.1 and the hardest case there is when X itself is a ruled surface over a curve of positive genus. This completes the effectiveness of log Iitaka fibration in dimension two (for the other cases, see [Todorov 2008]). For this case, it relies on results from [Alexeev 1994] and [Alexeev and Mori 2004].

Then for the higher dimensional case, the key tool is the canonical bundle formula [Kawamata 1998; Fujino and Mori 2000; Kollár 2007]. In our case it roughly says that if $f : X \longrightarrow Y$ is the Iitaka fibration for $K_X + \Delta$ and $K_X + \Delta = f^*D$ for some $\mathbb{Q}$-divisor D on Y, then we can define the *discriminant* or *divisorial part* on Y for $K_X + \Delta$ to be the $\mathbb{Q}$-Weil divisor $B_Y := \sum_P b_P P$, where $1 - b_P$ is the maximal real number t such that the log pair $(X, \Delta + tf^*(P))$ has log canonical singularities over the generic point of P. The sum runs over all codimension-one points of Y, but it has finite support. The *moduli part* or *J-part* is the $\mathbb{Q}$-line bundle M_Y on Y that satisfies

$$K_X + \Delta = f^*(K_Y + B_Y + M_Y).$$

If F is the general fiber of f and if $h^0(F, b(K_F + \Delta_{|F})) \neq 0$, then for every integer r divisible by b we have

$$H^0(X, \lfloor r(K_X + \Delta) \rfloor) = H^0(Y, \lfloor r(K_Y + B_Y + M_Y) \rfloor).$$

Using ideas of Mori and Fujino, and the two-dimensional case of Theorem 1.2, we can bound the denominators in relative dimension two.

Theorem 1.3. *When the relative dimension of f is two, there is a natural number m depending only on the coefficients of Δ such that mM is Cartier.*

Note that Theorem 1.3 holds for pairs of arbitrary dimension.

Finally, after these preparations, to finish the proof it suffices to observe that the results of [Fujino and Mori 2000] in Kodaira dimension-one case and of [Viehweg and Zhang 2007] in Kodaira dimension-two case can be indeed generalized to a log version. Namely, for an n-dimensional pair (X, Δ) of log Kodaira dimension one or two with Iitaka fibration $f : X \to Y$ with general fiber$F$, there is a constant m, that depends only on the number b satisfying $b(K_F+\Delta_{|F}) \sim 0$, the middle Betti number of the associated b-fold cyclic cover and the coefficients of Δ, such that $\lfloor m(K_X + \Delta) \rfloor$ induces the Iitaka fibration. In the three-dimensional case of Theorem 1.2, the base is a curve and the conclusions follow from Theorem 1.3 by the same arguments as in [Fujino and Mori 2000]. When the dimension is four, a key observation is that many results in [Alexeev and Mori 2004] can be improved by adding a nef divisor (Proposition 4.3). Similar statements were essentially proved in [Viehweg and Zhang 2007], but we will simplify the proof by putting it in the context of [Alexeev 1994; Alexeev and Mori 2004] and the recent mainstream of investigations on adjoint linear systems.

We remark that Conjecture 7.13 of [Prokhorov and Shokurov 2009] — in fact a list of conjectures — concerns the solution of the effective Iitaka fibration problem. We have shown item (2) of that list, in the relative dimension-two case. (However, we do not prove the semiampleness statements in items (1) and (3) of Prokhorov and Shorukov's Conjecture 7.13.)

The paper is structured as following. In Section 2, we show that the DCC assumption on the coefficients of the boundary indeed forces that the KLT surface pairs of log Kodaira dimension zero to be ϵ-log canonical, for some ϵ depending on the set of coefficients. In Section 3, we prove that under the same assumption there is a uniform b such that $b(K_S + B) \sim 0$. In Section 4, using ideas of Mori and Fujino, we deduce Theorem 1.3 and complete the proof of Theorem 1.2.

Notations and conventions. We will work over the field of complex numbers $\mathbb{C}$. A $\mathbb{Q}$-Cartier divisor D is nef if $D \cdot C \geq 0$ for any curve C on X. We call two $\mathbb{Q}$-divisors D_1, D_2 $\mathbb{Q}$-linearly equivalent $D_1 \sim_{\mathbb{Q}} D_2$ if there exists an integer $m > 0$ such that mD_i are integral and linearly equivalent. We call two $\mathbb{Q}$-Cartier divisors D_1, D_2 numerically equivalent $D_1 \equiv D_2$ if $(D_1 - D_2) \cdot C = 0$ for any curve C on X. A log pair (X, Δ) is a normal variety X and an effective $\mathbb{Q}$-Weil divisor Δ such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. A projective morphism $\mu : Y \longrightarrow X$ is a log resolution of the pair (X, Δ) if Y is smooth and $\mu^{-1}(\Delta) \cup \{\text{exceptional set of } \mu\}$ is a divisor with simple normal crossing support. For such μ we write $\mu^*(K_X + \Delta) = K_Y + \Gamma$, and $\Gamma = \Sigma a_i \Gamma_i$ where Γ_i are distinct integral divisors.

Definition 1.4. A pair is called KLT (resp. lc, ϵ-lc) if there is a log resolution $\mu : Y \longrightarrow X$ such that in the above notation we have $a_i < 1$ (resp. $a_i \leq 1$, $a_i \leq 1-\epsilon$). The number $1-a_i$ is called the *log discrepancy* of Γ_i with respect to the pair (X, Δ).

For conventions and results about DCC set we refer to [Alexeev and Mori 2004], in particular, 2.2–2.7 and 3.4–3.6. When we say some quantity is bounded, it always means there is a computable bound depending on the data we give. We will not keep track of the explicit bound, but it does not require much effort to do so, following the arguments in [Alexeev and Mori 2004]. (Most of the results about surfaces that we use appear already in [Alexeev 1994], but to make the constants explicitly computable, we refer to the joint paper, which is more recent.)

2. ϵ-log canonicity

The main result of this section is this:

Theorem 2.1. *Let $(S, B = \sum b_i B_i)$ be a KLT projective surface pair. If $K_S + B \equiv 0$ and the coefficients of B are in a DCC set $\mathcal{A}$, then there exists an $\epsilon = \epsilon(\mathcal{A}) > 0$ that only depends on $\mathcal{A}$ such that (S, B) is ϵ-log canonical.*

Proof. We can run a minimal model program for S,

$$S = S_0 \to S_1 \to \cdots \to S_n.$$

By pushing forward B to S_n, we see that it suffices to prove the statement for S_n. So we need only prove the case that S is a log del Pezzo surface of Picard number 1, S admits a Fano contraction to some curve with a general fiber $\mathbb{P}^1$, or S has $K_S \equiv 0$. This is done in Lemmas 2.4–2.6. $\square$

The main tool we use is the following theorem,

Theorem 2.2 [Alexeev and Mori 2004, Theorem 3.2]. *Let $\mathcal{A} \subset [0, 1]$ be an arbitrary DCC set. There exists a constant $\delta = \beta(\mathcal{A})$ depending only on $\mathcal{A}$ such that the following holds. Let X be a normal projective surface, B_j be divisors on X, and let b_j, x_j be positive real numbers. Assume that*

(1) *X is a singular $\mathbb{Q}$-factorial surface with $-K_X$ ample and $\rho(X) = 1$;*

(2) *$b_j > 0$ and $b_j \in \mathcal{A}$;*

(3) *$1 - \delta < x_j \leq 1$;*

(4) *at least one x_j is strictly less than 1;*

(5) *the pair $(X, \sum x_j b_j B_j)$ is lc.*

Then the divisor $K_X + \sum x_j b_j B_j$ is not numerical equivalent to 0.

Lemma 2.3 [Alexeev and Mori 2004, Lemma 3.6]. *Let $\mathcal{A}$ be a DCC set containing 1. Let $a, \delta > 0$ such that $\delta \leq mf_2(\mathcal{A}, a)$* (see [Alexeev and Mori 2004, 3.5] for the definition of the function $mf_2(\mathcal{A}, a)$). *Consider finitely many $b_j, x_j \in R$ such that $0 < b_j \in \mathcal{A}$, $1 - \delta < x_j \leq 1$ for all j, and $x_j < 1$ for some j. Then $\sum x_j b_j \neq a$.*

Lemma 2.4. *If S is a log del Pezzo surface of Picard number 1. $B = \sum b_j B_j$ is a $\mathbb{Q}$-divisor such that $K_S + B \equiv 0$, (S, B) is KLT and the coefficients of B are in a DCC set $\mathcal{A}$. Then S is $\epsilon = \epsilon(\mathcal{A})$-log canonical.*

Proof. We assume $1 \in \mathcal{A}$. If (S, B) is not ϵ-log canonical, then we can extract a divisor $f : S' \to S$ such that

$$f^*\bigl(K_S + \sum b_j B_j\bigr) \equiv K_{S'} + \sum b_j B_j + (1 - a)E,$$

with $a < \epsilon$. S' has Picard number 2, so it has two extremal rays, one of which is given by E. Contracting the other extremal ray we obtain a morphism $S' \to S_0$ to a surface S_0. In fact if S_0 were a curve, by taking the intersection of $K_{S'} + \sum b_j B_j + (1 - a)E$ with a general fiber, we have

$$\sum n_i b_j + n(1 - a) = 2,$$

where $n_i \geq 0$ and $n > 0$ are integers, which contradicts Lemma 2.3, provided we choose $\epsilon = mf_2(\mathcal{A}, 2)$. Then S_0 is a log del Pezzo of Picard number 1. Now g does not contract E, and

$$g_*\bigl(K_{S'} + \sum b_j B_j + (1 - a)E\bigr) = K_{S_0} + (1 - a)E + \sum b_j g_*(B_j) \equiv 0.$$

Since $K_S + B \equiv 0$ we still have that $(S_0, (1 - a)E + \sum b_j g_*(B_j))$ is a KLT pair. Then, applying Theorem 2.2 to S_0 with $b_E = 1$, $x_E = (1 - a)$, all other $x_j = 1$ and $\epsilon = \delta$, we conclude that $K_{S_0} + (1 - a)E + \sum b_j g_*(B_j)$ is not numerically equivalent to zero, which is a contradiction. $\square$

Lemma 2.5. *Let $(S, B) \to C$ be a morphism from a surface to a curve, such that the generic fiber is $\mathbb{P}^1$ and let (S, B) be KLT with the coefficients of B in a DCC set $\mathcal{A}$. Then S is $\epsilon = \epsilon(\mathcal{A})$-log canonical.*

Proof. Suppose that this is not the case. Then extract a divisor $f : S' \to S$ such that

$$f^*\left(K_S + \sum b_j B_j\right) \equiv K_{S'} + \sum b_j B_j + (1-a)E,$$

with $a < \epsilon$. Running a minimal model program for $(S', \sum b_j B_j)$ that does not contract E, we end up with a surface S'' that is either log del Pezzo or admits a Fano contraction to a curve. Furthermore the coefficients of E in the pair (S'', B'') is $(1-a)$ and $S'' + B'' \equiv 0$.

In the case S'' is log del Pezzo, we get a contradiction as in Lemma 2.4. If S'' admits a Fano contraction to a curve, by intersecting with a general fiber, we can apply the argument in Lemma 2.4 again. □

Finally we deal with the case in which K_S is numerically trivial.

Lemma 2.6. *There is $\epsilon > 0$ such that every KLT surface with $K_S \equiv 0$ is ϵ-log canonical.*

Proof. Set $\epsilon = \min\{\beta(\{1\}), mf_2(\{1\}', 2)\}$. Suppose that S is not ϵ-log canonical and extract a divisor E by $f : S' \to S$ such that $f^*(K_S) = K_{S'} + (1-a)E$ and $0 < a < \epsilon$. Then running a minimal model for S', which does not contract E, we can apply one of the above two cases. □

3. Bounding the index

Observe that the weighted projective space $\mathbb{P}(a, b, c)$ with three lines $(x_i = 0)$ forms an unbounded family of lc surface pairs when we vary a, b and c, and the coefficients are in the DCC set $\{1\}$. However by restricting to the KLT case we have the boundedness result:

Theorem 3.1. *If (S, B) is a KLT pair of dimension 2 such that $K_S + B \equiv 0$ and the coefficients of B are in a DCC set $\mathcal{A}$, there is a natural number $b = b(\mathcal{A})$ such that $b(K_S + B)$ is Cartier and $H^0(S, b(K_S + B)) = 1$.*

From Theorem 2.1 we know that there is an $\epsilon > 0$ such that (S, B) is ϵ-log canonical; we will assume $0 < \epsilon < 1/\sqrt{3}$.

Lemma 3.2 [Alexeev and Mori 2004, Lemma 1.2, Theorem 1.8]. *Let X be a nonsingular projective surface and $B = \sum b_j B_j$ be an $\mathbb{R}$-divisor on X with $0 \le b_j \le 1 - \epsilon < 1$. Assume $K_X + B \equiv 0$. Then*

(1) *if E is an irreducible curve on X and $E^2 < 0$, then $E \cong \mathbb{P}^1$ and $E^2 > -2/\epsilon$;*

(2) $\rho(X) \le 128/\epsilon^5$.

Proposition 3.3. *Let (S, B) be as in Theorem 3.1. There exists an integer $t = t(\epsilon)$ such that for any Weil divisor D on S, tD is Cartier.*

Proof. The argument is parallel to the one in [Alexeev and Mori 2004, Lemma 3.7], though there S is assumed to be a del Pezzo surface of Picard number 1.

Take the minimal resolution $f : \tilde{S} \to S$ of S and write

$$f^*(K_S) = K_{\tilde{S}} + \sum a_i E_i.$$

Applying Lemma 3.2, we conclude that the determinant t of the intersection matrix of the exceptional curves $(-F_i F_k)_{ik}$ is bounded by

$$t \leq [2/\epsilon]^{[128/\epsilon^5]}.$$

Since S has rational singularities, for any Weil divisor D on S, tD is a Cartier divisor. □

In the next proposition we repeatedly use the trivial property of DCC set stated in the following Lemma.

Lemma 3.4. *Let c be a positive real number and $\mathcal{A} \subset [0, 1]$ a DCC set. Then there are a finite number of ways to write c as the sum of $a_i \in \mathcal{A}$.*

Proposition 3.5. *Let (S, B) be as in Theorem 3.1. There exists an integer $t = t(\epsilon)$ such that $t(K_S + B)$ is Cartier.*

Proof. This follows from Proposition 3.3 once we make sure that there is a uniform multiple of $K_S + B$ that is a Weil divisor for all pairs $(S < B)$ satisfying our conditions.

Running a minimal model program for S, we end up with a KLT surface $f : S \to S'$. If $t(K'_S + B')$ is Cartier, since

$$f^*(t(K_{S'} + B')) \sim t(K_S + f_*^{-1} B' + a_i E_i)$$

and

$$K_S + f_*^{-1} B' + a_i E_i \equiv K_S + B \equiv 0,$$

we conclude that $K_S + f_*^{-1} B' + a_i E_i = K_S + B$, so it suffices to prove it for (S', B'). S' is either of Kodaira dimension 0, has a contraction to a curve with general fiber $\mathbb{P}^1$ or a log del Pezzo surface, so we need only prove the proposition in these cases.

<u>*Case 1:*</u> S has Kodaira dimension 0. Then $B = 0$ and it follows directly from Proposition 3.3.

<u>*Case 2:*</u> S admits a contraction to a curve $f : S \to C$ with the general fiber $\mathbb{P}^1$. By taking the intersection of (S, B) with a generic fiber, we have

$$2 = \sum_{f(B_j) \neq \mathrm{pt}} a_j, \quad \text{where } a_j \in \mathcal{A}'.$$

On the other hand, applying the canonical bundle formula in this simple case, we have

$$K_S+B\equiv f^*(K_C+B_C)\equiv 0,\qquad \text{where } B_C=\sum_{f(B_j)=P_j}\frac{a_j+n-1}{n}P_j.$$

From Lemma 3.4 we know that there are only finitely many choices of a_j, and we choose a natural number $N=N(\mathcal{A})$ that clears all the denominators of a_j.

<u>Case 3:</u> S is a log del Pezzo surface of Picard number one. For t as in Proposition 3.3, $K_S+\sum b_jB_j\equiv 0$ implies that

$$-tK_S^2=\sum tb_jB_j\cdot K_S.$$

Here tK_S^2 and $tB_j\cdot K_S$ are integers. From the proof of [Alexeev and Mori 2004, Lemma 3.7], we can bound tK_S^2 by

$$tK_S^2\le [2/\epsilon]^{[128/\epsilon^5]}([2/\epsilon]+2)^2.$$

Lemma 3.4 implies that there are only finitely many possible b_i appearing as the coefficients of the boundary divisor. So there is a positive number $N=N(\mathcal{A})$ such that NK_S and $N(b_jB_j)$ are all Cartier divisors. □

In view of this discussion, if we take the minimal resolution $\pi:\tilde{S}\to S$ of S and pull back K_S+B, then the denominators can be killed by a uniform multiple $t=t(\epsilon)$. Running a minimal model program for $\tilde{S}$, thus we only need to prove Theorem 3.1 in the case that S is a smooth minimal surface and that the coefficients of B all have the form r/t for some uniform t. When $K_S\equiv 0$, then $B=0$, then it is from the classification theory of smooth surfaces; when S is rational, a Cartier divisor is numerically trivial if and only if it is a trivial divisor. Thus the only tricky case is the following.

Lemma 3.6. *If S is a smooth minimal ruled surface over a curve of positive genus and there is a $\mathbb{Q}$-divisor B such that (S,B) is KLT and $K_S+B\equiv 0$, then $S=E\times\mathbb{P}^1$, where E is an elliptic curve. In particular, if there is some integer N such that NB is an integer divisor, then $\mathcal{O}(N(K_S+B))\cong\mathcal{O}_S$.*

Proof. We use conventions for ruled surfaces as in [Hartshorne 1977, V.2]. Let $\pi:S\to C$ be $\mathbb{P}(\mathscr{E})$ over a curve C of positive genus g and let the general fiber of π be F. We can assume $H^0(\mathscr{E})\ne 0$ and $H^0(\mathscr{E}\otimes L)=0$ for any line bundle L of degree -1. Then $e=-\deg(\mathscr{E})$ is an invariant of S. There is a section C_0 such that $C_0|_{C_0}=\bigwedge^2\mathscr{E}$, which we denote as D. Then $\deg(D)=-e$.

$$K_S\sim -2C_0+\pi^*(K_C+D).$$

Then we have

$$0 = (K_S + B) \cdot C_0 = e + 2g - 2 + tC_0^2 + \sum_{B_i \neq C_0} b_i B_i \cdot C_0 \geq 2g - 2 + (1-t)e,$$

where $t < 1$ is the coefficient of C_0 in B. So $e \leq 0$, and if $e = 0$, then $g = 1$. On the other hand, if $e < 0$, it follows from [Hartshorne 1977, V.2.21] that

$$B \cdot C_0 \geq (2C_0 + eF) \cdot C_0 \geq -e,$$

so again we conclude $g = 1$. And the equality holds only when for each irreducible component B_i of $\mathrm{Supp}(B)$, $g_i : B_i \to C$ is unramified.

Now we have $B_i \equiv a(2C_0 + eF)$ and $B_i^2 = 0$.

Claim. *For $g : B_i \to C$, there are line bundles M and N on B_i such that*

$$0 \to \mathcal{O} \to g^*(\mathcal{E}) \otimes M \to N \to 0 \quad \textit{and} \quad \deg N = 0.$$

Proof of the claim. It suffices to show that $h(B_i)^2 = 0$, where $h : B_i \to \mathbb{P}_{B_i}(g_i^*\mathcal{E})$ is the induced section. But for $B_i \subset \mathbb{P}(\mathcal{E})$,

$$0 \to I_{B_i}/I_{B_i}^2 \to \Omega^1_{\mathbb{P}(\mathcal{E})} \otimes B_i \to \Omega^1_{B_i} \to 0,$$

where $\deg(I_{B_i}/I_{B_i}^2) = 0$. Since $B_i \to C$ is unramified, the same exact sequence indeed also computes the conormal bundle of $h(B_i) \subset \mathbb{P}_{B_i}(g_i^*\mathcal{E})$; thus we conclude that $h(B_i)^2 = 0$. □

If $e > 0$, then $\mathcal{E}$ is indecomposable which also means it is stable. But the exact sequence of the claim shows that $g^*(E)$ is not semistable, which is a contradiction.

When $e = 0$, if $\mathcal{E}$ is indecomposable, then $\mathcal{E}$ is given by a nonsplit extension

$$0 \to \mathcal{O} \to \mathcal{E} \to \mathcal{O} \to 0.$$

For any isogeny $g : B \to C$, if there is a surjection $g^*(\mathcal{E}) \to G \to 0$ for a line bundle G of degree 0, then it is given by the pull-back of the above exact sequence. So $B = 2C_0$, which contradicts the assumption that (S, B) is KLT. Similarly, if $\mathcal{E} = \mathcal{O} \oplus L$ for some degree-0 bundle L, we have at most two candidates for B_i, which again contradicts (S, B) being KLT.

Finally, when $\mathcal{E} = \mathcal{O}^2$, all candidates of B_i are proportional to C_0. Thus the last statement of the lemma holds. □

4. Effectiveness of the Iitaka fibration

In this section we prove the effectiveness of the log Iitaka fibration for (X, Δ) a KLT pair of dimension three and four and log Kodaira codimension two.

Consider a log resolution $\pi : X' \longrightarrow X$ of (X, Δ) and write

$$\pi^*(K_X + \Delta) \equiv K_{X'} + (\pi^{-1})_* \Delta + \sum_i e_i E_i,$$

with E_i exceptional. There is a natural number n such that $e_i < 1 - \dfrac{1}{n}$ for every i. Define

$$\Delta' = (\pi^{-1})_* \Delta + \sum \Bigl(1 - \frac{1}{n}\Bigr) E_i.$$

Since all E_i are exceptional divisors, we have

$$H^0(X', \lfloor m(K_{X'} + \Delta') \rfloor) = H^0(X, \lfloor m(K_X + \Delta) \rfloor).$$

By replacing the $\mathscr{A}$ with the DCC set $\mathscr{A} \cup \{1 - \frac{1}{n} | n \in \mathbb{N}\}$, we can assume that X is smooth and Δ is simple normal crossing. Furthermore, we can assume that there is a morphism $f : X \longrightarrow Y$ giving the Iitaka fibration for $K_X + \Delta$ with Y a smooth projective variety of dimension m [Lazarsfeld 2004, 2.1.C]. For F the general fiber of f, we have that $\kappa(K_F + \Delta_F) = 0$.

From Kawamata's canonical bundle formula, we know that there is a $\mathbb{Q}$-line bundle $M = M^{SS}_{(M,\Delta)/Y}$, a $\mathbb{Q}$-divisor B on Y and a $\mathbb{Q}$-divisor R on X such that

$$K_X + \Delta \sim_{\mathbb{Q}} f^*(K_Y + B + M + B) + R,$$

where the terms in the formula satisfy the property that if $b(K_F + \Delta_F)$ is Cartier and $h^0(F, b(K_F + \Delta_F)) = 1$, then for any $r \in \mathbb{Z}_{\geq 0}$ divided by b,

$$H^0(X, \lfloor r(K_X + \Delta) \rfloor) = H^0(Y, \lfloor r(K_Y + M + B) \rfloor),$$

[Fujino and Mori 2000, Theorem 4.5]. M and B are usually referred as the moduli part and the boundary part of the algebraic fiber space.

Proof of Theorem 1.3. First we can assume that (F, Δ_F) is a minimal pair, that is, $K_F + \Delta_F \equiv 0$. In fact, from [Kollár 2007, Definition 8.4.6], we know the moduli part M only depends on the birational class of the morphism $X \to Y$, and we can choose one such that the generic fiber is minimal. Hence Theorem 2.1 implies that (F, Δ_F) is $\epsilon = \epsilon(\mathscr{A})$-lc.

Let d be the smallest positive integer such that $d(K_F + \Delta_F)$ is Cartier and $h^0(F, d(K_F + \Delta_F)) = 1$. Note that here we do not need the uniform integer from Theorem 1.3. Then we can construct the corresponding cyclic cover $g : E \to F$, and let $\phi : E' \to E$ be the minimal resolution of E. Then, as in [Fujino and Mori 2000, Theorem 3.1], it suffices to bound the second Betti number of E'. Write $g^*(K_F + \Delta_F) = K_E + \Delta_E$. From [Kollár and Mori 1998, 5.20.3], we know (E, Δ_E) is ϵ-log canonical, provided that so is (F, Δ_F), which is the case because of Theorem 2.1. Pulling back to E', we conclude that $(E', \Delta_{E'})$ is ϵ-log canonical.

Then [Alexeev and Mori 2004, Theorem 1.8] implies that $\rho(E') \le 128/\epsilon^5$. Thus the second Betti number of E' is bounded by a constant depending only on $\mathscr{A}$.

The remainder of the proof was indeed given in [Todorov 2008, Section 3]. We give a sketch here. First we assume that the generic fiber (F, Δ_F) is log smooth. Then, as in [Kollár 2007, 8.4.5(7)], we can define a local system $\mathbb{V}$ on an open set $U \subset X$. Furthermore, if the monodromy action on $R^2 f_*(\mathbb{V})$ is unipotent, the moduli part M is identified with the bottom Hodge filtration of $R^2 f_*(\mathbb{V})$, which is an integral divisor. In general, we take a log resolution $\tilde{F} \to F$ and write

$$\pi^*(K_F + \Delta_F) = K_{\tilde{F}} + \Delta_{\tilde{F}} - G,$$

where $\lfloor \Delta_{\tilde{F}} \lfloor = 0$, $G \ge 0$ is an integral divisor and $\operatorname{Supp}(\Delta_{\tilde{F}})$ has no common component with G. Near every codimension one point P, there is a Galois cover Y' of Y such that when we pull back everything to Y', the monodromy action becomes unipotent near P. Thus as in [Fujino and Mori 2000, 3.6], the remaining question is to bound the character of $G \to \mathbb{Y}^*$ corresponding to the representation of G on $H^0(\tilde{F}, \mathbb{O}(G))$. If we write $\tilde{E}$ as the degree-b cyclic cover corresponding to $b(K_{\tilde{F}} + \Delta_{\tilde{F}} - G) \sim 0$, then $H^0(\tilde{F}, \mathbb{O}(G))$ is a direct summand of the bottom piece of the Hodge filtration of $H^2(\tilde{E}, \mathbb{C})$, which is also the bottom piece of $H^2(E', \mathbb{C})$ since $\tilde{E}$ and E' are birational.

Then from [Fujino and Mori 2000, 3.8], we conclude that if the index of (F, Δ_F) is bounded by $b = b(\mathscr{A})$ and the second Betti number of the cyclic cover is bounded by $B = B(\mathscr{A})$, then the denominator of the moduli part is bounded by

$$a = b \cdot \operatorname{lcm}\{m \in \mathbb{Z}_{>0} | \varphi(m) \le B\},$$

where φ denotes the Euler φ-function. □

Theorem 4.1. *Let the notation be as above. Assume that the coefficients of* Δ *are in a DCC set of rational numbers* $\mathscr{A} \subset [0, 1]$. *Let the dimension of X be either three or four and the dimension of the general fiber be two. Then there is a constant N depending only on the set* $\mathscr{A}$ *such that* $\lfloor N(K_X + \Delta) \rfloor$ *induces the Iitaka fibration.*

We will prove this statement when the dimension of X is four, and leave the easier dimension-three case to the reader. In fact, this follows directly from the argument in [Fujino and Mori 2000, Section 6], and Theorem 1.3.

As we mentioned before, the main theorem of [Viehweg and Zhang 2007] says that for an arbitrary dimensional smooth variety X of Kodaira dimension 2, if we assume that the generic fiber of the Iitaka fibration is a smooth variety F, then there is a constant N depending on the middle Betti number b_m of F and the index of F, such that $|NK_X|$ gives the Iitaka fibration. The following discussion shows this is still true for the log case. Then because of Theorems 3.1 and 1.3, where we prove that the analogous bounds for the Betti number and the index, which only depend on the DCC set $\mathscr{A}$, exist even in the log case, provided the generic fiber is of dimension 2, we in fact get unconditional bounds (depending only on coefficients set) as in Theorem 1.2.

Theorem 4.2. *If (W, D) is a KLT surface, L is a nef $\mathbb{Q}$-divisor (not necessarily effective) such that and K_W+D+L is big. Assume that a is a positive integer such that aL is a Cartier divisor, the coefficients of D are in a DCC set $\mathscr{B}$, then we have a uniform $N=N(a,\mathscr{B})$ such that $\left|\lfloor N(K_W+D+L)\rfloor\right|$ gives a birational map.*

Proof of Theorem 4.1. We apply Theorem 4.2 to $W=C$, $D=B$, $L=M$. To check the assumptions, the coefficients of B are of the form

$$\frac{b+n-1}{n}, \qquad \text{for some } b\in\mathscr{A}\cup\{0\} \text{ and } n\in\mathbb{Z}_{>0},$$

[Kollár 2007, Theorem 8.3.7(2)], which forms a DCC set depending on $\mathscr{A}$. And for a, it can be chosen as in the last part of the proof of Theorem 1.3. □

Proof of Theorem 4.2. This is essentially proved in [Viehweg and Zhang 2007], so here we only give a sketch which streamlines the arguments by using tools in [Alexeev and Mori 2004]. As before, we can start by assuming that W is smooth. The main observation is that in fact many of the results of Alexeev and Mori can be strengthened in such a way that instead of assuming $\left(W, D=\sum b_jD_j\right)$ $(b_j\in\mathscr{B})$ is big, we can assume that $(W, D+L)$ is big, where L is a nef line bundle such that for any curve C, $L\cdot C$ is in another DCC set $\mathscr{C}$. Then we get the same conclusion by changing all our constants $c=c(\mathscr{B})$ to $c=c(\mathscr{B},\mathscr{C})$. In particular:

Proposition 4.3. *Let the notation be as in Theorem 4.2. Then there is a uniform $\beta=\beta(\mathscr{B},a)$ such that $K_W+\sum(1-x_j)b_jB_j+L$ is big, provided $x_j\le\beta$.*

Because of Proposition 4.3 and the fact that all positive b_j have a lower bound, we can indeed assume that all b_j are of the form n_j/m for some $m=m(\mathscr{B},a)$.

Then the usual argument of cutting log canonical centers [Todorov 2008] works as long as we can prove that the volume of $K_W+\sum(1-x_j)b_jB_j+L$ has a uniform lower bound.

Now we run the minimal model program, $f:(W,D)\to(W',D'=f_*(D))$, $L'=f_*L$. Because $f^*(L')\ge L$, we have

$$H^0(W,\lfloor n(K_W+D+L)\rfloor)=H^0(W',\lfloor n(K_{W'}+D'+L')\rfloor) \quad \text{for all } n.$$

<u>Case 1:</u> K_W+D is pseudo-effective. Then we end up with a minimal model (W',D') such that $K_{W'}+D'$ is nef. If $K_{W'}+D'$ is big, this is so from [Alexeev and Mori 2004]. Otherwise,

$$(K_{W'}+D'+L')^2=2(K_{W'}+D')\cdot L'+L'^2>0,$$

which has a uniform lower bound by our assumption.

Case 2: $K_W + D$ is not pseudo-effective. Then we define the *pseudo-effective threshold*, which is the smallest number e such that $K_W + D + eL$ is pseudo-effective. Note that since K_W+D+L is big we have that $e \leq 1$. Then we have two subcases, for which we give a sketch of the argument. For details see the proof of [Viehweg and Zhang 2007, Proposition 2.7].

Case 2a: We end up with a log del Pezzo surface (W', D') of Picard number 1. So

$$K_{W'} + D' + eL' \equiv 0.$$

The discussion above about the generalization of [Alexeev and Mori 2004] indeed implies that $1-e$ has a uniform lower bound. So $(K_{W'} + D' + L')^2 = (1-e)L'^2$.

Case 3b: We end up with a Fano contraction (W', D') to a curve. By the argument above, we can assume L' is not big. Then W' is of Picard number 2 and generated by the fiber of the Fano contraction and L'. Taking the intersection of $K'_W+D'+eL'$ with the fiber, we conclude that e is uniformly far from 1. Since the coefficients of D' have bounded denominators, the fact that $(K_{W'}+D')\cdot L'$ is positive actually implies it is uniformly away from 0. So the volume

$$(K_{W'} + D' + L')^2 = (1-e)(K_{W'} + D') \cdot L'$$

is bounded from below. This concludes all possible cases, and hence Theorem 4.2 is proved. □

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A formalism for equivariant Schubert calculus

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In previous work we have developed a general formalism for Schubert calculus. Here we show how this theory can be adapted to give a formalism for equivariant Schubert calculus consisting of a basis theorem, a Pieri formula and a Giambelli formula. Our theory specializes to a formalism for equivariant cohomology of grassmannians. We interpret the results in a ring that can be considered as the formal generalized analog of localized equivariant cohomology of infinite grassmannians.

Introduction

Schubert calculus, in the form of the cohomology, or Chow ring, of a Grassmann variety $\mathrm{Grass}^l(n)$ of $n-l$-planes in n-space, has a long and important history. Recently much of the story has been extended to the *equivariant cohomology ring*, which has a much richer structure, being an algebra over a polynomial ring with n generators. One knows generators and relations for this algebra, and formulas for Schubert classes, which form a basis.

In this article we give a different and more general perspective on these algebras. The idea comes from a previous article [Laksov and Thorup 2009] where a *generalized Schubert calculus* is considered as the ring of *symmetric polynomials* $A[T_1, \dots, T_l]^{\mathrm{sym}}$ in l variables over an arbitrary ring A acting on the *exterior product* $\bigwedge_A^l A[T]$ of the polynomial ring $A[T]$ in one variable (see also [Gatto 2005] and [Gatto and Santiago 2009]).

In the present article we show how this formalism, when expressed in terms of the basis of $A[T]$ consisting of *generalized factorial powers*

$$(T|y)^i = (T-y_1)\cdots(T-y_i)$$

for given elements $y_1, y_2, \dots$ in A, gives a *general equivariant Schubert calculus* consisting of a *basis theorem*, a *Pieri formula* and a *Giambelli formula*. The theory

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is further specialized for each n in [Laksov 2008] (which builds upon the results of the present article) to give the equivariant Schubert calculus for $\mathrm{Grass}^l(n)$, or more generally for the l-quotients $\mathrm{Grass}^l_S(\mathscr{E})$ of a locally free $\mathbb{O}_S$-module on any scheme S with a bivariant intersection theory. In Section 6 of [Laksov 2008] we gave a detailed account of the geometric interpretation of $A[T_1, \dots, T_l]^{\mathrm{sym}}$, and in Section 7 we showed how the theory of the present article can be used to recover the quantum and equivariant quantum cohomology of grassmannians.

The geometry whose cohomology we are generalizing can be realized as the union of the grassmannians $\mathrm{Grass}^l(n)$, as n goes to infinity, taken over the natural embedding of $\mathrm{Grass}^l(n)$ in $\mathrm{Grass}^l(m)$, for $n < m$, equivariant with respect to a natural embedding of $\mathrm{GL}(n)$ in $\mathrm{GL}(m)$. The surjections to equivariant cohomology of $\mathrm{Grass}^l(n)$ for each n can be constructed using the description of equivariant cohomology that comes from localization, so that an element of the equivariant cohomology is given by specifying a polynomial at each fixed point of the torus $(\mathbb{C}^*)^n$, subject to the conditions made explicit by M. Goresky, R. Kottwitz and R. MacPherson, and called GKM conditions.

In our generalization the connection between the above construction and the description of equivariant cohomology coming from *localization* is given, in the case when A is a polynomial ring $\mathbb{Z}[y_1, y_2, \dots]$ in independent variables $y_1, y_2, \dots$, via an $A[T_1, \dots, T_l]^{\mathrm{sym}}$-module isomorphism

$$\textstyle\bigwedge_A^l A[T] \to H(l),$$

where $H(l)$ is the A-algebra consisting of elements that are of bounded total degree in the variables $y_1, y_2, \dots$ and satisfy the GKM condition in the product $\prod_{\lambda \in \{{\infty \atop l}\}} A$ of A taken over all lists $\lambda : \lambda_1\lambda_2\dots$ of ones and zeros with exactly l zeros, with coordinatewise addition and multiplication. The ring $H(l)$ can be thought of as the graded limit of the equivariant cohomologies of the grassmannians $\mathrm{Grass}^l(n)$, as n goes to infinity.

The observation that equivariant cohomology could be interpreted within the framework of exterior powers was made in [Gatto and Santiago 2006] and [Santiago 2006]. In the latter reference it was proved that there exists an isomorphism between *Schubert calculus on exterior powers*, that is, Schubert calculus in a setting similar to the $\mathrm{Grass}^l(n)$ case mentioned above, and equivariant cohomology for Grassmann manifolds, and for simple examples (projective space $k = 1$, and the Knutson–Tao [2003] example with $k = 2$, $n = 4$) it was indicated what the isomorphism should look like; see [Gatto and Santiago 2006]. It was this work that inspired us to consider the equivariant cohomology of Grassmann schemes and to describe the explicit isomorphism in the general case.

We note that we obtain a generalization of the *full* equivariant Pieri formula, and not only the *Chevalley formula* for divisors (see, for example, [Knutson and Tao

2003; Mihalcea 2006; Lakshmibai et al. 2006; Kostant and Kumar 1986; Molev and Sagan 1999; Okun'kov and Ol'shanskiĭ 1997] for various forms of the latter formula). This general form was first given by T. Santiago and we essentially reproduce the calculations of [Santiago 2006] in our language. A full version is also obtained by S. Robinson [2002] by different methods.

Our adaption of the general Schubert calculus to the equivariant case is built upon *factorial Schur functions* used by L. C. Mihalcea [2006; 2008] to describe the *equivariant quantum cohomology ring* of *grassmannians* (see [Arabia 1989; Billey 1999; Lascoux 2003], for earlier related work on double Schubert polynomials and complete flag varieties). We develop the theory of factorial Schur functions in such a way that specialization of our formalism directly gives the Goresky–Kottwitz–MacPherson description of equivariant cohomology coming from localization (see also [Knutson and Tao 2003; Arabia 1989; Kostant and Kumar 1986]), and the theory of L. C. Mihalcea [2006; 2008]. It should be pointed out that our equivariant Giambelli formula is a generalization of an *unshifted* version of that of [Mihalcea 2008], that is, it uses only *generalized factorial powers* and not their shifted counterparts. A different, but similar, equivariant Giambelli formula is given in [Lakshmibai et al. 2006]. All versions specialize to the *classical* Giambelli formula [Fulton 1998; 1997]. Fulton [2007, Lecture 7] explains how the equivariant Giambelli formula for grassmannians amounts to a degeneracy formula [Kempf and Laksov 1974] in algebraic geometry.

1. Exterior powers and residues

In this section we interpret the main results of [Laksov and Thorup 2009] in terms of factorial Schur functions. Our general version of equivariant Schubert calculus is the general Schubert calculus interpreted using polynomials of the form

$$(T - y_1) \cdots (T - y_i)$$

for elements $y_1, y_2, \dots$ in A. These polynomials form the building blocks of the approach to equivariant quantum Schubert calculus by Mihalcea. To facilitate the understanding of the correspondence between our theory and Mihalcea's we use the notation of [Macdonald 1995, I §3 Example 20], which is also used in [Mihalcea 2008].

Notation 1.1. All rings in the following will be commutative with a unit. Let A be such a ring. All exterior powers and tensor products will be taken with respect to A. We denote by $A[T]$ and $A[T_1, \dots, T_l]$ the polynomial rings over A in 1, respectively l, independent variables. The symmetric functions in $A[T_1, \dots, T_l]$ we denote by $A[T_1, \dots, T_l]^{\mathrm{sym}}$. We identify the tensor product $\bigotimes_A^l A[T]$ with $A[T_1, \dots, T_l]$ and consider $\bigotimes_A^l A[T]$ as a module over $A[T_1, \dots, T_l]^{\mathrm{sym}}$ via this identification. The starting point of our interpretation of Schubert calculus is the easily verified observation that the $A[T_1,\dots,T_l]^{\mathrm{sym}}$-module structure on the tensor product $\bigotimes_A^l A[T]$ induces an $A[T_1,\dots,T_l]^{\mathrm{sym}}$-module structure on the exterior power $\bigwedge_A^l A[T]$ via the canonical surjection $\bigotimes_A^l A[T]\to\bigwedge_A^l A[T]$; [Laksov and Thorup 2009, Section 1].

Let $e_i=\cdots+b_{i,-l}T^{-l}+b_{i,-l+1}T^{-l+1}+\cdots+b_{i,-1}T^{-1}+\cdots$ for $i=1,\dots,l$ be a collection of *Laurent series* with coefficients $b_{i,j}$ in a ring. We write, as in [Laksov and Thorup 2009, 0.3],

$$\mathrm{Res}(e_1,\dots,e_l):=\det(b_{i,-j})=\begin{pmatrix} b_{1,-1} & b_{1,-2} & \cdots & b_{1,-l}\\ b_{2,-1} & b_{2,-2} & \cdots & b_{2,-l}\\ \vdots & \vdots & \ddots & \vdots\\ b_{l,-1} & b_{l,-2} & \cdots & b_{l,-l}\end{pmatrix}.$$

Let $y_1,y_2,\dots$ be elements in A and write

$$(T|y)^i=(T-y_1)\cdots(T-y_i)\quad\text{for}\quad i=0,1,\dots.$$

The polynomials $(T|y)^0,(T|y)^1,\dots$ are called *generalized factorial powers* and form a basis for the A-module $A[T]$. Let $Q(T)=(T-T_1)\cdots(T-T_l)$, and let $b:b_1\geq\cdots\geq b_l\geq 0$ be a *partition*. We write

$$s_b(T_1,\dots,T_l|y)=\mathrm{Res}((T|y)^{b_1+l-1}/Q(T),\dots,(T|y)^{b_l}/Q(T)).$$

The polynomials $s_b(T_1,\dots,T_l|y)$ we refer to as *factorial Schur functions*; [Macdonald 1995, I §3 Example 20].

We now rewrite the Main result 0.5 of [Laksov and Thorup 2009] in the basis $(T|y)^0,(T|y)^1,\dots$ of $A[T]$.

Theorem 1.2. *Let $Q(T)=(T-T_1)\cdots(T-T_l)$. For every collection $f_1,\dots,f_l$ of polynomials in $A[T]$ and for every partition $b:b_1\geq\cdots\geq b_l\geq 0$ we have*:

(1) (Poincaré duality) *The $A[T_1,\dots,T_l]^{\mathrm{sym}}$-module $\bigwedge_A^l A[T]$ is free of rank one with generator $(T|y)^{l-1}\wedge\cdots\wedge(T|y)^0$.*

(2) (The determinantal formula)

$$f_1\wedge\cdots\wedge f_l=\mathrm{Res}(f_1/Q,\dots,f_l/Q)((T|y)^{l-1}\wedge\cdots\wedge(T|y)^0).$$

(3) (The equivariant Giambelli–Gatto formula)

$$(T|y)^{b_1+l-1}\wedge\cdots\wedge(T|y)^{b_l}=s_b(T_1,\dots,T_l|y)((T|y)^{l-1}\wedge\cdots\wedge(T|y)^0).$$

Proof. Since we clearly have $(T|y)^{l-1} \wedge \cdots \wedge (T|y)^0 = T^{l-1} \wedge \cdots \wedge T^0$ the first and second assertion are equivalent to the first and second assertions of [Laksov and Thorup 2009, 0.5].

Assertion (3) is a particular case of assertion (2). $\square$

Notation 1.3. For every collection $f_1, \dots, f_l$ of elements in $A[T]$ we write

$$\big(f_i(T_j)\big) = \begin{pmatrix} f_1(T_1) & \cdots & f_1(T_l) \\ \vdots & & \vdots \\ f_l(T_1) & \cdots & f_l(T_l) \end{pmatrix},$$

and we let $\Delta = \prod_{1 \le i < j \le l}(T_i - T_j) = \det(T_j^{l-i})$. The polynomial $\det(f_i(T_j))$ is divisible by Δ because it is alternating in $T_1, \dots, T_l$.

Proposition 1.4. *Let $Q(T) = (T - T_1) \cdots (T - T_l)$. For every collection of polynomials $f_1, \dots, f_l$ in $A[T]$ we have*

$$\operatorname{Res}(f_1/Q, \dots, f_l/Q) = \det(f_i(T_j))/\Delta.$$

In particular, for every partition $b : b_1 \ge \cdots \ge b_l \ge 0$ we have

$$\begin{aligned} s_b(T_1, \dots, T_l|y) &= \operatorname{Res}((T|y)^{b_1+l-1}/Q, \dots, (T|y)^{b_l}/Q) \\ &= \det((T_j|y)^{b_i+l-i})/\det((T_j|y)^{l-i}). \end{aligned} \tag{1}$$

Proof. Both sides of the first equality of the proposition are multilinear and alternating in $f_1, \dots, f_l$. Hence it suffices to prove the equality when $f_i = T^{h_i+l-i}$ with $h_1 \ge \cdots \ge h_l$. An easy calculation (see [Laksov and Thorup 2009, 0.6]) shows that we then have an equality $\operatorname{Res}(f_1/Q, \dots, f_l/Q) = s_{h_1,\dots,h_l}(T_1, \dots, T_l)$, where the polynomials $s_{h_1,\dots,h_l}(T_1, \dots, T_l) = (s_{h_i-i+j})$, with s_i the i-th *complete symmetric function* in $T_1, \dots, T_l$, are the ordinary Schur functions (see [Macdonald 1995, I §3], for example). However, by the Jacobi–Trudi formula (see [Macdonald 1995, I §3 (3.4)], for example) we have $s_{h_1,\dots,h_l}(T_1, \dots, T_l) = \det(T_j^{h_i+l-i})/\det(T_j^{l-i})$. $\square$

The following result indicates a different approach to the determinantal formula from that presented in [Laksov and Thorup 2009].

Proposition 1.5. *There is an isomorphism of $A[T_1, \dots, T_l]^{\mathrm{sym}}$-modules*

$$\sigma^{\mathrm{sym}} : \textstyle\bigwedge_A^l A[T] \to A[T_1, \dots, T_l]^{\mathrm{sym}}$$

determined by mapping $f_1 \wedge \cdots \wedge f_l$ to $\det(f_i(T_j))/\Delta$.

Proof. The existence of the homomorphism follows since $\det(f_i(T_j))/\Delta$ is multilinear and alternating in $f_1, \dots, f_l$.

To prove that the homomorphism is $A[T_1, \dots, T_l]^{\mathrm{sym}}$-linear it suffices to prove that the homomorphism $\sigma : \bigotimes_A^l A[T] \to A[T_1, \dots, T_l]^{\mathrm{sym}}$ determined by $\sigma(f_1 \otimes$

$\cdots \otimes f_l) = \det(f_i(T_j))/\Delta$ is $A[T_1, \dots, T_l]^{\mathrm{sym}}$-linear. We first note that we have an equality $\sigma((T_1^{h_1} \cdots T_l^{h_l}) f_1(T_1) \cdots f_l(T_l)) = \det(T_j^{h_i} f_i(T_j))/\Delta$. Fix a permutation τ of $[1, l]$. The contribution to $\det(T_j^{h_i} f_i(T_j))$ from the elements in rows $1, \dots, l$ and the corresponding columns $\tau(1), \dots, \tau(l)$ is $T_{\tau(1)}^{h_1} \cdots T_{\tau(l)}^{h_l} f_1(T_{\tau(1)}) \cdots f_l(T_{\tau(l)})$. Let f be the sum of the elements $T_{\tau(1)}^{k_1} \cdots T_{\tau(l)}^{k_l}$ taken over all the different l-tuples $(k_1, \dots, k_l)$ that are permutations of $(h_1, \dots, h_l)$. Then the contribution to $\sigma(f(f_1(T_1) \otimes \cdots \otimes f_l(T_l)))$ from the elements in rows $1, \dots, l$ and corresponding columns $\tau(1), \dots, \tau(l)$ is equal to $f f_1(T_{\tau(1)}) \cdots f_l(T_{\tau(l)})$. Consequently we have $\sigma(f(f_1 \otimes \cdots \otimes f_l)) = f \det(f_i(T_j))/\Delta$. The homomorphism σ^{sym} is $A[T_1, \dots, T_l]^{\mathrm{sym}}$-linear since all symmetric functions are linear combinations of polynomials of the form f.

It follows from Theorem 1.2(1) that $\bigwedge_A^l A[T]$ is a free $A[T_1, \dots, T_l]^{\mathrm{sym}}$-module of rank one with generator $(T|y)^{l-1} \wedge \cdots \wedge (T|y)^0$. Since this generator maps to 1 by σ^{sym} and σ^{sym} is an $A[T_1, \dots, T_l]^{\mathrm{sym}}$-module homomorphism it follows that σ^{sym} is an isomorphism. $\square$

2. Strings, partitions and factorial Schur functions

In this section we give the main properties of factorial Schur functions. This will provide us with the natural foundation for the treatment of the generalization of the description of equivariant cohomology given by Goresky–Kottwitz–MacPherson (see also [Knutson and Tao 2003; Arabia 1989; Kostant and Kumar 1986]) and of the theory of Mihalcea, both mentioned above. To facilitate the understanding of the correspondence between the theories we have conformed to the notation of [Knutson and Tao 2003] as much as possible.

Notation 2.1. Denote by $\left\{ {\infty \atop l} \right\}$ all *strings* $\lambda : \lambda_1 \lambda_2 \dots$ consisting of zeros and ones, with exactly l zeros. We consider $\left\{ {\infty \atop l} \right\}$ as a *lattice* with inequality $\lambda' \geq \lambda$ if $\sum_{i=1}^{j} \lambda'_i \geq \sum_{i=1}^{j} \lambda_i$ for $j = 1, 2, \dots$.

An *inversion* in λ is a pair (i, j) with $i < j$ such that $1 = \lambda_i > \lambda_j = 0$. Denote by $\mathrm{inv}(\lambda)$ the inversions in λ and write $l(\lambda) = |\mathrm{inv}(\lambda)|$.

We introduce a similar terminology and notation for *partitions*. Let $\left\{ {\infty \atop l} \right\}_{\mathcal{P}}$ consist of all partitions $b : b_1 \geq \cdots \geq b_l \geq 0$. We consider $\left\{ {\infty \atop l} \right\}_{\mathcal{P}}$ as a *lattice* with inequality $b' \geq b$ when $b'_i \geq b_i$ for $i = 1, \dots, l$. To each partition $b : b_l \geq \cdots \geq b_l \geq 0$ we associate a strictly decreasing sequence $a_1 > \cdots > a_l > 0$, where $a_j = b_j + l - j + 1$ for $j = 1, \dots, l$.

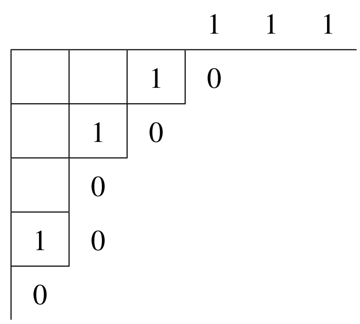
An *inversion* in b is a pair (i, a_j) such that $i < a_j$ and $i \notin \{a_{j+1}, \dots, a_l\}$. We denote the inversions in b by $\mathrm{inv}(b)$ and write $l(b) = |\mathrm{inv}(b)|$. Clearly $l(b) = \sum_{i=1}^{l} b_i$.

Let $0 < a(\lambda)_l < \cdots < a(\lambda)_1$ be the positions where the zeros appear in $\lambda : \lambda_1 \lambda_2 \dots$ for λ in $\left\{ {\infty \atop l} \right\}$, that is $\lambda_{a(\lambda)_i} = 0$ for $i = 1, \dots, l$. We obtain a *partition*

$b(\lambda) : b(\lambda)_1 \geq \cdots \geq b(\lambda)_l \geq 0$ in $\left\{ {\infty \atop l} \right\}_{\mathcal{P}}$, with $a(\lambda)_j = b(\lambda)_j + l - j + 1$ for $j = 1, \ldots, l$, and

$$b(\lambda)_i = \{\text{the number of ones to the left of zero number } l - i + 1 \text{ in the string } \lambda\}.$$

Example. Take $l = 5$. Consider the sequence $\lambda = 0\,1\,0\,0\,1\,0\,1\,0\,1\,1\,1\ldots.$ Then the sequence $a(\lambda)$ is equal to $1\,3\,4\,6\,8$, and the sequence $b(\lambda)$ to $3\,2\,1\,1\,0$. More geometrically, one can record the sequence λ by a planar path, read from southwest to northeast, starting from $(0, 0)$, and ending on the line $y = l$, by making zeros correspond to vertical steps and ones to horizontal steps. Then $b(\lambda)$ is the partition whose boundary is traced by λ.



It is clear that $\mathrm{inv}(\lambda) = \mathrm{inv}\, b(\lambda)$ and thus $l(\lambda) = \sum_{i=1}^{l} b(\lambda)_i = l(b(\lambda))$.

From a partition $b : b_1 \geq \cdots \geq b_l \geq 0$ we obtain, conversely, a string $\lambda(b) : \lambda(b)_1\lambda(b)_2\ldots$ in $\left\{ {\infty \atop l} \right\}$ defined by $\lambda(b)_{a_i} = 0$ for $i = 1, \ldots, l$, the remaining $\lambda(b)_i$ being one.

Lemma 2.2. *There is a length preserving bijection of lattices between partitions* $\left\{ {\infty \atop l} \right\}_{\mathcal{P}}$ *and* $\left\{ {\infty \atop l} \right\}$ *that maps b to $\lambda(b)$, and λ to $b(\lambda)$.*

Proof. It is obvious from the definitions that the map described in the lemma gives a bijection between $\left\{ {\infty \atop l} \right\}_{\mathcal{P}}$ and $\left\{ {\infty \atop l} \right\}$, and we observed above that the map preserves length.

That the map is a homomorphism of lattices follows since $b(\lambda)_i$ and $b(\lambda')_i$ are the number of ones to the left of zero number $l - i + 1$ in λ, respectively λ'. $\square$

Lemma 2.3. *Let λ, λ' be strings in* $\left\{ {\infty \atop l} \right\}$, *and let b, b' be the corresponding partitions in* $\left\{ {\infty \atop l} \right\}_{\mathcal{P}}$.

(1) *When λ and λ' differ only in the i-th and j-th positions and $(i, j) \in \mathrm{inv}(\lambda)$, then $j = a(\lambda)_p$ and $i = a(\lambda')_q$ for some p and q, and the remaining elements in the sequences $a(\lambda)_1 > \cdots > a(\lambda)_l$ and $a(\lambda')_1 > \cdots > a(\lambda')_l$ are the same.*

(2) *When b and b' are such that the sequences $a_l < \cdots < a_1$ and $a'_l < \cdots < a'_1$ differ only where $j = a_p$ and $i = a_q$ with $i < j$, then the strings $\lambda(b)$ and $\lambda(b')$ differ only in the positions i and j, and $(i, j) \in \mathrm{inv}(\lambda(b))$.*

Proof. Assertion (1) follows since the zeros in λ are in the same positions as the zeros in λ' except in positions i and j where $\lambda_j = 0$ and $\lambda'_i = 0$.

Similarly assertion (2) follows since the zeros in $\lambda(b)$ and $\lambda(b')$ are in the same positions except zero number j in $\lambda(b)$ and zero number i in $\lambda(b')$. $\square$

In the next two results we give the main properties of factorial Schur functions.

Theorem 2.4 (Vanishing Theorem; see also [Molev and Sagan 1999]). *For every partition $b : b_1 \geq \cdots \geq b_l \geq 0$ we have*:

(1) $s_b(y_{b_1+l}, \ldots, y_{b_l+1}|y) = \prod_{(i,j)\in \mathrm{inv}(b)}(y_j - y_i)$.

(2) *Let $h : h_1 \geq \cdots \geq h_l \geq 0$ be a partition that is not greater than or equal to b. Then $s_b(y_{h_1+l}, \ldots, y_{h_l+1}|y) = 0$.*

Proof. We may assume that A is the polynomial ring $\mathbb{Z}[y_1, y_2, \ldots]$ in the variables $y_1, y_2 \ldots$ because, once the theorem is proved in this case, we can, for general A, specialize the variables $y_1, y_2, \ldots$ to any sequence of elements in A.

(1) The (p, q)-th *entry* in $((T_j|y)^{b_i+l-i})$ is $(T_q|y)^{b_p+l-p} = \prod_{i=1}^{b_p+l-p}(T_q - y_i)$. If $p < q$ we have $b_q + l - q + 1 \leq b_p + l - p$, and thus $(T|y_{b_q+l-q+1})^{b_p+l-p} = 0$.

When $p = q$ we have $(T|y_{b_p+l-p+1})^{b_p+l-p} = \prod_{i=1}^{b_p+l-p}(y_{b_p+l-p+1} - y_i)$. Consequently the matrix $(T|y_{b_j+l-j+1})^{b_i+l-i}$ is lower triangular, and it follows from what we just saw and from Notation 2.1 that the product of the diagonal elements divided by $\prod_{1\leq i<j\leq l}(y_{b_i+l-i+1} - y_{b_j+l-j+1})$ is $\prod_{(i,j)\in \mathrm{inv}(\lambda(b))}(y_j - y_i)$.

(2) By assumption $h_p < b_p$ for some p and thus $h_p + l - p + 1 < b_p + l - p + 1$. Then, for $i \leq p$ and $p \leq j$ we have $h_j + l - j + 1 \leq h_p + l - p + 1 < b_p + l - p + 1 \leq b_i + l - i + 1$. Consequently $(T|y_{h_j+l-j+1})^{b_i+l-i} = 0$. Thus the $(p \times (l - p + 1))$-matrix in the upper right corner of $((T|y_{h_j+l-j+1})^{b_i+l-i})$ is zero, and thus $\det((T|y_{h_j+l-j+1})^{b_i+l-i}) = 0$. Since $\prod_{1\leq i<j\leq l}(y_{b_i+l-i+1} - y_{b_j+l-j+1})$ is not a zero divisor in $\mathbb{Z}[y_1, y_2, \ldots]$ we have $s_b(y_{h_1+l}, \ldots, y_{h_l+1}|y) = 0$. $\square$

The next result will be used in Section 3 to describe equivariant Schubert calculus as presented above. It will imply the main properties of Schubert classes generalizing the description Goresky–Kottwitz–MacPherson in the notation of [Knutson and Tao 2003]. We stress that the methods used to prove parts (2) and (3) of the next result are similar to those used by Knutson and Tao to prove corresponding results for Schubert classes (proof of Lemma 1 in Section 2.1 and of Proposition 1 in Section 2.4). We could have chosen the opposite approach and used the results of Knutson and Tao to obtain information on factorial Schur functions. It is however, more in the spirit of this work to focus on the properties of factorial Schur functions.

Theorem 2.5. *Let $A = \mathbb{Z}[y_1, y_2, \ldots]$ be the polynomial ring in the independent variables $y_1, y_2, \ldots$, and let $g \in A[T_1, \ldots, T_l]^{\mathrm{sym}}$. Moreover, let $\left\{ {\infty \atop l} \right\}_k$ be the subset of $\left\{ {\infty \atop l} \right\}_{\mathcal{P}}$ consisting of partitions $h : h_1 \geq \cdots \geq h_l \geq 0$ with $h_1 \leq k$.*

(1) *g satisfies the* GKM (Goresky–Kottwitz–MacPherson) *condition. That is*:

When b and b' are partitions in $\left\{ {\infty \atop l} \right\}_{\mathcal{P}}$ such that the sequences $b_l + 1 < \cdots < b_1 + l$ and $b'_l + 1 < \cdots < b'_1 + l$ differ only where the first sequence is equal to j and the second is equal to i with $i \neq j$, then $g(y_{b_l+1}, \ldots, y_{b_1+l}) - g(y_{b'_l+1}, \ldots, y_{b'_1+l})$ is divisible by $y_i - y_j$.

(2) *Let $g = \sum_{h \in \mathcal{I}} z_h s_h(T_1, \ldots, T_l | y)$ with $z_h \in A$ and with $\mathcal{I} \subseteq \left\{ {\infty \atop l} \right\}_k$, and assume that $g(y_{h_1+l}, \ldots, y_{h_l+1}) = 0$ for all partitions h in $\left\{ {\infty \atop l} \right\}_k$. Then $z_h = 0$ for all $h \in \mathcal{I}$.*

(3) *For a given partition b, conditions* (1) *and* (2) *of Theorem 2.4 characterize the homogeneous symmetric functions. More precisely and more generally*:

We have equality $g(y_{c_1+l}, \ldots, y_{c_l+1}) = s_b(y_{c_1+l}, \ldots, y_{c_l+1} | y)$ for all $c \in \left\{ {\infty \atop l} \right\}_k$ when the following three conditions are fulfilled:

(a) *For every partition $b \in \left\{ {\infty \atop l} \right\}_k$ we have an equality $g(y_{b_1+l}, \ldots, y_{b_l+1}) = \prod_{1 \leq i < j \leq \mathrm{inv}(b)} (y_j - y_i)$.*

(b) *$g(y_{h_1+l}, \ldots, y_{h_l+1}) = 0$ for all partitions $h \in \left\{ {\infty \atop l} \right\}_k$ that are not greater than or equal to b*

(c) *For all c in $\left\{ {\infty \atop l} \right\}_k$ we have that $g(y_{c_1+l}, \ldots, y_{c_l+1})$ is homogeneous in $y_1, y_2, \ldots$ of degree $l(b)$.*

Proof. Assertion (1) is clear since g is symmetric in $T_1, \ldots, T_l$.

To prove assertion (2) we assume that some z_h is non-zero and choose b minimal such that $z_b \neq 0$. If $z_h \neq 0$ we must then have $b_p < h_p$ for some p. It follows from Theorem 2.4(2) that $s_h(y_{b_1+l}, \ldots, y_{b_l+1} | y) = 0$. Consequently we have that $g(y_{b_1+l}, \ldots, y_{b_l+1})$ is equal to the sum $\sum_{h \in \mathcal{I}} z_h s_h(y_{b_1+l}, \ldots, y_{b_l+1} | y) = z_b s_b(y_{b_1+l}, \ldots, y_{b_l+1} | y)$ which is non-zero by Theorem 2.4(1). This contradicts the assumption of (2) and thus proves that $z_h = 0$ for all $h \in \mathcal{I}$.

We now prove assertion (3). Let g satisfy the conditions (i)–(iii) for some b. Assume that $g(y_{h_1+l}, \ldots, y_{h_l+1})$ is not equal to $s_b(y_{h_1+l}, \ldots, y_{h_l+1} | y)$ for some $h \in \left\{ {\infty \atop l} \right\}_k$ and let c be a minimal element in $\left\{ {\infty \atop l} \right\}_k$ with the property that

$$\big(g - s_b(T_1, \ldots, T_l | y)\big)(y_{c_1+l}, \ldots, y_{c_l+1}) \neq 0.$$

Then $c \geq b$; otherwise it follows from assumption (ii) and from Theorem 2.4(2) that $g(y_{c_1+l}, \ldots, y_{c_l+1}) = 0 = s_b(y_{c_1+l}, \ldots, y_{c_l+1} | y)$. Moreover c is strictly bigger than b, since $(g - s_b(T_1, \ldots, T_l | y))(y_{b_1+l}, \ldots, y_{b_l+1}) = 0$ by assumption (i) and by Theorem 2.4(1). In particular, $l(c) > l(b)$. But from the GKM condition 2.5(1) it follows that $\prod_{(i,j) \in \mathrm{inv}(c)} (y_j - y_i)$ divides $(g - s_b(T_1, \ldots, T_l | y))(y_{c_1+l}, \ldots, y_{c_l+1})$. This is impossible since $(g - s_b(T_1, \ldots, T_l | y))(y_{c_1+l}, \ldots, y_{c_l+1})$ is homogeneous of degree $l(b)$ in $y_1, y_2, \ldots$ by assumption (iii) of (3). We have thus a contradiction showing that there is an equality $g(y_{c_1+l}, \ldots, y_{c_l+1}) = s_b(y_{c_1+l}, \ldots, y_{c_l+1} | y)$ for all $c \in \left\{ {\infty \atop l} \right\}_k$, and we have proved assertion (3). $\square$

3. Factorial Schur functions and Schubert classes

In this section we present the generalization of the Goresky–Kottwitz–MacPherson description of equivariant cohomology alluded to several times above. We also give the precise correspondence between this formalism and the general Schubert calculus interpreted via factorial Schur functions.

Notation 3.1. Let $A=\mathbb{Z}[y_1, y_2, \dots]$ be the polynomial ring in the variables $y_1, y_2, \dots$ over $\mathbb{Z}$. We denote by $\prod_{\lambda\in\left\{ {\infty \atop l} \right\}} A$ all *lists* $\alpha=(\alpha|\lambda)$ of elements in A. This is a ring with componentwise multiplication, and the *unit* is the list with $1|\lambda=1_A$ for all λ. We consider $\prod_{\lambda\in\left\{ {\infty \atop l} \right\}} A$ as an A-algebra mapping $a\in A$ to the list α_a with $\alpha_a|\lambda=a$ for all λ, and we define an A-algebra homomorphism

$$\sigma^{\mathrm{equ}}: A[T_1,\dots,T_l]^{\mathrm{sym}}\to\prod_{\lambda\in\left\{ {\infty \atop l} \right\}} A$$

by $\sigma^{\mathrm{equ}}(f(T_1,\dots,T_l))=\alpha_f$ where $\alpha_f|\lambda=f(y_{b(\lambda)_1+l},\dots,y_{b(\lambda)_l+1})$.

For each partition $b: b_1\geq\cdots\geq b_l\geq 0$ in $\left\{ {\infty \atop l} \right\}_{\mathcal{P}}$ and for each string $\lambda:\lambda_1\lambda_2\dots$ in $\left\{ {\infty \atop l} \right\}$ we write

$$S_b=\sigma^{equ}(s_b(T_1,\dots,T_l|y))\quad\text{and}\quad S_\lambda=S_{b(\lambda)}.$$

Definition 3.2. An element $\alpha\in\prod_{\lambda\in\left\{ {\infty \atop l} \right\}} A$ is called a *class* if the polynomials $\alpha|\lambda$ for all $\lambda\in\left\{ {\infty \atop l} \right\}$ have bounded total degree in the variables $y_1, y_2, \dots$ and if it satisfies the GKM (Goresky–Kottwitz–MacPherson) condition, that is:

If λ,λ' in $\left\{ {\infty \atop l} \right\}$ differ in the positions i and j only, the element $\alpha|\lambda-\alpha|\lambda'$ is divisible by y_i-y_j.

It is clear that the classes in $\prod_{\lambda\in\left\{ {\infty \atop l} \right\}} A$ form an A-algebra with coordinatewise addition and multiplication. We denote this algebra by $H(l)$.

A class $\alpha\in\prod_{\lambda\in\left\{ {\infty \atop l} \right\}} A$ is a *Schubert class* corresponding to λ in $\left\{ {\infty \atop l} \right\}$ if it satisfies the following three conditions:

(1) $\alpha|\lambda=\prod_{(i,j)\in\mathrm{inv}(\lambda)}(y_j-y_i)$.

(2) If $\alpha|\lambda'\neq 0$ then $\lambda'\geq\lambda$.

(3) For all $\mu\in\left\{ {\infty \atop l} \right\}$ the element $\alpha|\mu$ is homogeneous of degree $l(\lambda)$ in $y_1, y_2, \dots$.

The ring $H(l)$ can be thought of as the formal generalized analog of the localized equivariant cohomology of the infinite grassmannian described in the introduction, with one torus fixed point for each partition. In the following result we give the exact connection between classes and symmetric polynomials. Observe that the proof of assertion (2) is modelled after the proof of Proposition 1 in [Knutson and Tao 2003].

Proposition 3.3. *Let $A=\mathbb{Z}[y_1, y_2, \dots]$ be the polynomial ring in the independent variables $y_1, y_2, \dots$ over $\mathbb{Z}$. Then*:

(1) *The image of the homomorphism* $\sigma^{\text{equ}} : A[T_1, \dots, T_l]^{\text{sym}} \to \prod_{\lambda \in \left\{ \substack{\infty \\ l} \right\}} A$ *consists of classes, that is,* σ^{equ} *induces an* A*-algebra homomorphism*

$$\sigma : A[T_1, \dots, T_l]^{\text{sym}} \to H(l).$$

Moreover, the images $S_b = \sigma(s_b(T_1, \dots, T_l|y))$ *of the factorial Schur functions are Schubert classes corresponding to* $\lambda(b)$, *for each partition* $b : b_1 \geq \dots \geq b_l \geq 0$ *in* $\left\{ \substack{\infty \\ l} \right\}_{\mathcal{P}}$.

(2) *Let* α *be a class. Then* α *is a linear combination, with coefficients in* A, *of the Schubert classes* $S_b = \sigma(s_b(T_1, \dots, T_l|y))$ *for partitions* $b : b_1 \geq \dots \geq b_l \geq 0$ *in* $\left\{ \substack{\infty \\ l} \right\}_{\mathcal{P}}$. *In particular* σ *is surjective.*

If all the elements $\alpha|\lambda$ *are in* $\mathbb{Z}[y_1, \dots, y_n]$ *for some* n, *the coefficients are in* $\mathbb{Z}[y_1, \dots, y_n]$, *and if* $\sigma^{\text{equ}}(s_b(T_1, \dots, T_l|y))$ *has a non-zero coefficient then* $b \in \left\{ \substack{\infty \\ l} \right\}_k$ *with* $k = n - l$.

Proof. It is clear that for $g \in A[T_1, \dots, T_l]^{\text{sym}}$ the elements $\sigma^{\text{equ}}(g)|\lambda$ are of total degree at most equal to the total degree of g plus the total degree of the coefficients. Moreover, it follows from Theorem 2.5(1) that $\sigma^{\text{equ}}(g)$ satisfies the GKM condition. Hence $\sigma^{\text{equ}}(g)$ is a class. That S_b is a Schubert class follows from Theorem 2.4. Thus we have proved assertion (1).

We next prove assertion (2). Let α be a non-zero class. Let μ be minimal in the support of α, that is, μ is minimal such that $\alpha|\mu \neq 0$. It follows from the GKM condition that $\alpha|\mu$ is a multiple β of $\prod_{(i,j)\in\text{inv}(\mu)}(y_j - y_i)$. Moreover, if $\alpha|\mu$ is in $\mathbb{Z}[y_1, \dots, y_n]$ we must have $b(\mu)_1 + l \leq n$, that is, $\mu \in \left\{ \substack{\infty \\ l} \right\}_k$ and $\beta \in \mathbb{Z}[y_1, \dots, y_n]$. Theorem 2.4(1) implies that $\alpha|\mu = \beta\sigma^{\text{equ}}(s_{b(\mu)}(T_1 \dots, T_l|y))|\mu$ and Theorem 2.4(2) that μ is not in the support of $\alpha - \beta\sigma^{\text{equ}}(s_{b(\mu)}(T_1, \dots, T_l|y))$ and that no element smaller than μ is in the support. Continuing this process we can successively reduce the support upwards. By the definition of a class the total degrees of $\alpha|\lambda$ for all $\lambda \in \left\{ \substack{\infty \\ l} \right\}$ are bounded. Thus the process must end. Moreover, in the process we have that if $\alpha|\lambda \in \mathbb{Z}[y_1, \dots, y_n]$ for all $\lambda \in \left\{ \substack{\infty \\ l} \right\}$ then the coefficients β are in $\mathbb{Z}[y_1, \dots, y_n]$ and the μ involved are in $\left\{ \substack{\infty \\ l} \right\}_k$. Hence assertion (2) holds. $\square$

The correspondence between the equivariant Schubert calculus of Section 1 and the generalization of the Goresky–Kottwitz–MacPherson description mentioned above (see also [Knutson and Tao 2003], [Arabia 1989] and [Kostant and Kumar 1986]) is given by the following results.

Theorem 3.4. (1) *The homomorphism* $\sigma : A[T_1, \dots, T_l]^{\text{sym}} \to H(l)$ *of Proposition 3.3 is an* A*-algebra isomorphism.*

(2) *The Schubert classes* S_λ *for* $\lambda \in \left\{ \substack{\infty \\ l} \right\}$ *form a basis for the* A*-module of classes.*

(3) S_λ *is the unique Schubert class belonging to* λ.

Proof. Let $g \in A[T_1, \dots, T_l]^{\text{sym}}$. The elements $(T|y)^{h_1+l-1} \wedge \dots \wedge (T|y)^{h_l}$ for all partitions $h_1 \geq \dots \geq h_l \geq 0$ form a basis for $\bigwedge_A^l A[T]$. In particular, it follows from Proposition 1.5 that g is the image by σ^{sym} of a unique element $\sum_{h \in \mathcal{I}} z_h((T|y)^{h_1+l-1} \wedge \dots \wedge (T|y)^{h_l})$ with $z_h \in A$, and where the sum is over a finite set of partitions in $\left\{ {\infty \atop l} \right\}$. Hence it follows from Theorem 1.2(1) and (3) that $g = \sum_{h \in \mathcal{I}} z_h s_h(T_1, \dots, T_l|y)$ and that the elements $s_b(T_1, \dots, T_l|y)$ for $b \in \left\{ {\infty \atop l} \right\}_{\mathcal{P}}$ form an A-basis for $A[T_1, \dots, T_l]^{\text{sym}}$. If $\sigma^{\text{equ}}(g) = 0$ it follows from Theorem 2.5(2) that $z_h = 0$ for all $h \in \mathcal{I}$. Thus σ^{equ} is injective. The A-algebra homomorphism σ^{equ} maps $A[T_1, \dots, T_l]^{\text{sym}}$ onto classes by Proposition 3.3(2). Thus we have proved assertion (1).

Since we just proved that the classes $s_b(T_1, \dots, T_l|y)$ for b in $\left\{ {\infty \atop l} \right\}_{\mathcal{P}}$ form a basis for $A[T_1, \dots, T_l]^{\text{sym}}$ assertion (2) follows from (1) and the definition of S_λ.

Assertion (3) follows easily from Theorem 2.5(3) and the definition of S_λ. $\square$

Corollary 3.5. *Consider $H(l)$ as an $A[T_1, \dots, T_l]^{\text{sym}}$-module via σ. Then the composite homomorphism*

$$\sigma\sigma^{\text{sym}} : \textstyle\bigwedge_A^l A[T] \to H(l)$$

determined by mapping

$$(T|y)^{b_1+l-1} \wedge \dots \wedge (T|y)^{b_l} = s_b(T_1, \dots, T_l|y)((T|y)^{l-1} \wedge \dots \wedge (T|y)^0) \quad \textit{to} \quad S_{\lambda(b)},$$

for each partition $b : b_1 \geq \dots \geq b_l \geq 0$, is an isomorphism of $A[T_1, \dots, T_l]^{\text{sym}}$-modules.

Proof. The proposition immediately follows from Proposition 1.5 and from assertion (1) of the theorem. $\square$

4. Pieri's formula

Let A be an arbitrary ring. From the action of $A[T_1, \dots, T_l]^{\text{sym}}$ on $\bigwedge_A^l A[T]$ we obtain that the product $s_h T^{h_1} \wedge \dots \wedge T^{h_l}$, where s_h is the h-th *complete symmetric function* in $T_1, \dots, T_l$ and $h_1 \geq \dots \geq h_l \geq 0$ is a partition, can be expressed as a linear combination of elements $T^{j_1} \wedge \dots \wedge T^{j_l}$ with coefficients in $A[T_1, \dots, T_l]^{\text{sym}}$, where $j_1 \geq \dots \geq j_l \geq 0$ is a partition such that $j_1 + \dots + j_l = h_1 + \dots + h_l + h$. After suitable cancellations the resulting formula is called Pieri's formula. In Sections 1 and 2 we have seen how the multiplication can be expressed in terms of *factorial Schur functions*. Here we shall give the explicit calculations of $s_{h0\dots0}(T_1, \dots, T_l|y)((T|y)^{h_1} \wedge \dots \wedge (T|y)^{h_l})$ in $\bigwedge_A^l A[T]$ and perform the necessary simplifications to obtain the *equivariant Pieri formula*.

Using the isomorphism in Corollary 3.5 the calculations and the resulting formulas can be translated into the algebra $H(l)$. There we obtain an explicit expression for the coordinatewise product $s_{h0\ldots0}s_h(T_1, \ldots, T_l|y)$ as linear combinations of elements $s_{i_1,\ldots,i_l}(T_1, \ldots, T_l|y)$ with coefficients in $A[T_1, \ldots, T_l]^{\mathrm{sym}}$ where $i_1 + \cdots + i_l = b_1 + \cdots + b_l + h$, the coefficients are those of [Knutson and Tao 2003]. After the appropriate cancellations of terms we obtain the Pieri formula in $H(l)$. The results are direct translations of those in $\bigwedge_A^l A[T]$ and we therefore do not repeat them.

Similar calculations to those performed in this section were first made by Santiago in [Santiago 2006] (see also [Gatto and Santiago 2009; Laksov and Thorup 2009]). The formula specializes to those of [Mihalcea 2006; 2008]. A different approach to Pieri's formula can be found in [Robinson 2002].

To make the expressions of the calculations more transparent we simplify the notation somewhat.

For independent variables $T_1, \ldots, T_l$ over A we denote by

$$s_h = s_h(T_1, \ldots, T_l)$$

the h-th *complete symmetric function* in $T_1, \ldots, T_l$, and by $s_h(y_1, \ldots, y_l)$ its value at $y_1, y_2, \ldots, y_l \in A$. Similarly, for elements $y_1, y_2, \ldots$ in A and for any h and m we denote by

$$c_h(y_1, \ldots, y_m)$$

the value at $y_1, \ldots, y_m$ of the h-th elementary symmetric function in m variables. We write

$$g_h = (T|y)^h = (T - y_1) \cdots (T - y_h)$$

and let

$$\begin{aligned} S_{h0\cdots0}(T|y) &= s_{h0\cdots0}(T_1, \ldots, T_l|y) \\ &= \mathrm{Res}((T|y)^{h+l-1}/Q(T), (T|y)^{l-2}/Q(T), \ldots, (T|y)^0/Q(T)) \end{aligned}$$

be the h-th *factorial Schur function* where $h \geq 0 \geq \cdots \geq 0$ is in $\left\{ {\infty \atop l} \right\}_{\mathcal{P}}$.

Lemma 4.1. *We have*

$$g_{h+l-1} \wedge g_{l-2} \wedge \cdots \wedge g_0 = \sum_{j=0}^{h} (-1)^j s_{h-j} c_j(y_1, \ldots, y_{h+l-1})(g_{l-1} \wedge \cdots \wedge g_0),$$

where $c_j(y_1, \ldots, y_{h+l-1})$ is the j-th elementary symmetric function in the variables $y_1, \ldots, y_{h+l-1}$. Moreover,

$$s_{h0\cdots0}(T|y) = \sum_{j=0}^{h} (-1)^j s_{h-j} c_j(y_1, \ldots, y_{h+l-1}).$$

Proof. We develop $g_{h+l-1}, g_{l-2}, \dots, g_0$ in powers of the variable T and obtain

$$g_{h+l-1} \wedge g_{l-2} \wedge \cdots \wedge g_0 = \Big(\sum_{j=0}^{h+l-1} (-1)^j c_j(y_1, \dots, y_{h+l-1}) T^{h+l-1-j} \Big) \wedge T^{l-2} \wedge \cdots \wedge T^0.$$

Theorem 1.2(3) implies that $T^{h+l-1-j} \wedge T^{l-2} \wedge \cdots \wedge T^0 = s_{h-j}(T^{l-1} \wedge \cdots \wedge T^0)$. Thus

$$g_{h+l-1} \wedge g_{l-2} \wedge \cdots \wedge g_0 = \sum_{j=0}^{h+l-1} (-1)^j s_{h-j} c_j(y_1, \dots, y_{h+l-1})(T^{l-1} \wedge \cdots \wedge T^0),$$

which gives the first part of the lemma since $s_{h-j} = 0$ for $j > k$, as required.

To obtain the last part of the lemma it suffices to compare the first equation of the lemma with the equation $g_{h+l-1} \wedge g_{l-2} \wedge \cdots \wedge g_0 = s_{h0\cdots0}(T|y)(g_{l-1} \wedge \cdots \wedge g_0)$ of Theorem 1.2(3). $\square$

Lemma 4.2. *We have*

$$T^h g_i = \sum_{j=0}^{h} s_{h-j}(y_{i+1}, \dots, y_{i+j+1}) g_{i+j}.$$

Proof. We prove the equation by induction on h. It holds trivially for $h = 0$. Assume it holds for $h > 0$. From $T g_i = g_{i+1} + y_{i+1} g_i$ and the induction hypothesis we get

$$\begin{aligned}
T^{h+1} g_i &= \sum_{j=0}^{h} s_{h-j}(y_{i+1}, \dots, y_{i+j+1})(g_{i+j+1} + y_{i+j+1} g_{i+j}) \\
&= \sum_{j=1}^{h+1} s_{h+1-j}(y_{i+1}, \dots, y_{i+j}) g_{i+j} + \sum_{j=0}^{h} s_{h-j}(y_{i+1}, \dots, y_{i+j+1}) y_{i+j+1} g_{i+j} \\
&= s_0(y_{i+1}, \dots, y_{i+h+1}) g_{i+h+1} \\
&\qquad + \sum_{j=1}^{h} (s_{h+1-j}(y_{i+1}, \dots, y_{i+j}) + s_{h-j}(y_{i+1}, \dots, y_{i+j+1}) y_{i+j}) g_{i+j} \\
&\qquad + s_h(y_{i+1}) y_{i+1} g_i \\
&= g_{i+h+1} + \sum_{j=1}^{h} s_{h+1-j}(y_{i+1}, \dots, y_{i+j+1}) g_{i+j} + s_{h+1}(y_{i+1}) g_i \\
&= \sum_{j=0}^{h+1} s_{h+1-j}(y_{i+1}, \dots, y_{i+j+1}) g_{i+j}. \qquad \square
\end{aligned}$$

Lemma 4.3. *Let $j_1, \dots, j_l, h_1, \dots, h_l, h$ be non-negative integers with $j_1 + \cdots + j_l \le h$. Then*

$$\sum_{i_1+\cdots+i_l=h} s_{i_1-j_1}(y_{h_1+1}, \dots, y_{h_1+j_1+1}) \cdots s_{i_l-j_l}(y_{h_l+1}, \dots, y_{h_l+j_l+1}) = s_{h-j_1-\cdots-j_l}(y_{h_1+1}, \dots, y_{h_1+j_1+1}, \dots, y_{h_l+1}, \dots, y_{h_l+j_l+1}).$$

Proof. It is clear that all the monomials on the left-hand side of the equation of the lemma appear in the right-hand side. Conversely, consider a monomial that appears on the right-hand side with a contribution of degree k_i from the variables $y_{h_i+1}, \dots, y_{h_i+j_i+1}$ for $i = 1, \dots, l$. Then $k_1 + \cdots + k_l = h - j_1 - \cdots - j_l$. Let $i_1 := h - j_2 - \cdots - j_l - k_2 - \cdots - k_l$. Then $i_1 - j_1 = k_1$. Correspondingly we define $i_2, \dots, i_l$ such that $i_p - j_p = k_p$ for $p = 1, \dots, l$. The monomial that we consider will then be the product of monomials in $y_{h_p+1}, \dots, y_{h_p+j_p+1}$ of degree $i_p - j_p = k_p$ for $p = 1, \dots, l$, and thus appear on the left-hand side of the equation in the lemma. □

Proposition 4.4. *Let $h_1, \dots, h_l, h$ be non-negative integers. Then*

$$s_h(g_{h_1} \wedge \cdots \wedge g_{h_l}) = \sum_{i=0}^{h} \sum_{j_1+\cdots+j_l=h-i} s_i(y_{h_1+1}, \dots, y_{h_1+j_1+1}, \dots, y_{h_l+1}, \dots, y_{h_l+j_l+1})(g_{h_1+j_1} \wedge \cdots \wedge g_{h_l+j_l}).$$

Proof. By definition $\omega := s_h g_{h_1} \wedge \cdots \wedge g_{h_l} = \sum_{i_1+\cdots+i_l=h} T^{i_1} g_{h_1} \wedge \cdots \wedge T^{i_l} g_{h_l}$. It follows from Lemma 4.2 that

$$\omega = \sum_{i_1+\cdots+i_l=h} \sum_{j_1=0}^{i_1} \cdots \sum_{j_l=0}^{i_l} s_{i_1-j_1}(y_{h_1+1}, \cdots, y_{h_1+j_1+1}) \cdots s_{i_l-j_l}(y_{h_l+1}, \dots y_{h_l+j_l+1})(g_{h_1+j_1} \wedge \cdots \wedge g_{h_l+j_l}).$$

Since $s_{i_p-j_p}(y_{h_p+1}, \dots, y_{h_p+j_p+1}) = 0$ when $j_p > i_p$ we obtain

$$\omega = \sum_{i_1+\cdots+i_l=h} \sum_{j_1+\cdots+j_l=0}^{h} s_{i_1-j_1}(y_{h_1+1}, \dots, y_{h_1+j_1+1}) \cdots s_{i_l-j_l}(y_{h_l+1}, \dots, y_{h_l+j_l+1})(g_{h_1+j_1} \wedge \cdots \wedge g_{h_l+j_l}).$$

Exchanging the order of summation we obtain by Lemma 4.3

$$\omega = \sum_{j_1+\cdots+j_l=0}^{h} s_{h-j_1-\cdots-j_l}(y_{h_1+1}, \dots, y_{h_1+j_1+1}, \dots, y_{h_l+1}, \dots, y_{h_l+j_l+1}) (g_{h_1+j_1} \wedge \cdots \wedge g_{h_l+j_l}),$$

that immediately gives the equation of the proposition. □

Theorem 4.5 (Pieri's formula). *Let $h_1 \geq \cdots \geq h_l \geq 0$ be a partition and let h be a non-negative integer. Then*

$$s_h(g_{h_1} \wedge \cdots \wedge g_{h_l}) = \sum_{i=0}^{h} \sum_{(j_1,\dots,j_l)\in\mathcal{J}_{h-i}} s_i(y_{h_1+1}, \dots, y_{h_1+j_1+1}, \dots, , y_{h_l+1}, \dots, y_{h_l+j_l+1})(g_{h_1+j_1} \wedge \cdots \wedge g_{h_l+j_l}),$$

where $\mathcal{J}_{h-i}$ is the collection of all l-tuples $(j_1, \dots, j_l)$ such that $j_1+\cdots+j_l = h-i$ and $j_1+h_1 \geq h_1 > j_2+h_2 \geq h_2 > \cdots > j_l+h_l \geq h_l$. We also have

$$
\begin{aligned}
&s_{h0\cdots 0}(T\,|\,y)(g_{h_1}\wedge\cdots\wedge g_{h_l})\\
&\quad=\sum_{j=0}^{h}\sum_{i=0}^{h}\sum_{(j_1,\dots,j_l)\in\mathcal{J}_{h-i}}(-1)^j c_j(y_1,\dots,y_{h+l-1})\\
&\qquad\qquad s_i(y_{h_1+1},\dots,y_{h_1+j_1+1},\dots,,y_{h_l+1},\dots,y_{h_l+j_l+1})(g_{h_1+j_1}\wedge\cdots\wedge g_{h_l+j_l}),
\end{aligned}
$$

Proof. Let $s_i(g_{h_1+j_1}\wedge\cdots\wedge g_{h_l+j_l})$ be a term on the right-hand side of the equation of Proposition 4.4. Assume that $h_p+j_p \geq h_{p-1}$ for some p. If $h_p+j_p = h_{p-1}+j_{p-1}$ the term is zero. Assume, on the other hand, that $j'_{p-1} := h_p+j_p-h_{p-1} \neq j_{p-1}$, and let $j'_p := h_{p-1}+j_{p-1}-h_p$. Then $j'_{p-1} \geq 0$ and $j'_p = h_{p-1}-h_p+j_{p-1} \geq 0$. Moreover we have $j'_{p-1}+j'_p = j_{p-1}+j_p$, and $j'_p \neq j_p$ because $j'_{p-1} \neq j_{p-1}$. On the right-hand side of the sum in Proposition 4.4 we thus have two terms

$$
\begin{aligned}
&s_i(\dots,y_{h_{p-1}+1},\dots,y_{h_{p-1}+j_{p-1}+1},y_{h_p+1},\dots,y_{h_p+j_p+1},\dots)\\
&\quad(\cdots\wedge g_{h_{p-1}+j_{p-1}}\wedge g_{h_p+j_p}\wedge\cdots)\\
&+s_i(\dots,y_{h_{p-1}+1},\dots,y_{h_{p-1}+j'_{p-1}+1},y_{h_p+1},\dots,y_{h_p+j'_p+1},\dots)\\
&\quad(\cdots\wedge g_{h_{p-1}+j'_{p-1}}\wedge g_{h_p+j'_p}\wedge\cdots).
\end{aligned}
$$

Since $h_{p-1}+j_{p-1} = h_p+j'_p$ and $h_p+j_p = h_{p-1}+j'_{p-1}$ and the s_i are symmetric in $y_{h_l+1},\dots,y_{h_1+j_1+l}$, these two terms cancel. In Proposition 4.4 there remain only the terms in Pieri's formula.

The last formula follows from the first and Lemma 4.1. □

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