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Hilbert schemes of 8 points

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The Hilbert scheme H_n^d of n points in $\mathbb{A}^d$ contains an irreducible component R_n^d which generically represents n distinct points in $\mathbb{A}^d$. We show that when n is at most 8, the Hilbert scheme H_n^d is reducible if and only if $n = 8$ and $d \geq 4$. In the simplest case of reducibility, the component $R_8^4 \subset H_8^4$ is defined by a single explicit equation, which serves as a criterion for deciding whether a given ideal is a limit of distinct points.

To understand the components of the Hilbert scheme, we study the closed subschemes of H_n^d which parametrize those ideals which are homogeneous and have a fixed Hilbert function. These subschemes are a special case of multigraded Hilbert schemes, and we describe their components when the colength is at most 8. In particular, we show that the scheme corresponding to the Hilbert function $(1, 3, 2, 1)$ is the minimal reducible example.

1. Introduction

The Hilbert scheme H_n^d of n points in affine d-space parametrizes 0-dimensional, degree n subschemes of $\mathbb{A}^d$. Equivalently, the k-valued points of H_n^d parametrize ideals $I \subset S = k[x_1, \dots, x_d]$ such that S/I is an n-dimensional vector space over k. The smoothable component $R_n^d \subset H_n^d$ is the closure of the set of ideals of distinct points. The motivating problem of this paper is characterizing the ideals which lie in the smoothable component, that is, the 0-dimensional subschemes which are limits of distinct points. We determine the components of the schemes H_n^d for $n \leq 8$, and find explicit equations defining $R_8^4 \subset H_8^4$.

We assume that k is a field of characteristic not 2 or 3.

Theorem 1.1. *Suppose n is at most* 8 *and d is any positive integer. Then the Hilbert scheme H_n^d is reducible if and only if $n = 8$ and $d \geq 4$, in which case it consists of exactly two irreducible components: the smoothable component, of dimension $8d$, and a component denoted G_8^d, of dimension $8d - 7$, which consists of local algebras isomorphic to homogeneous algebras with Hilbert function* $(1, 4, 3)$.

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It is known that for d at least 3 and n sufficiently large the Hilbert scheme of points is always reducible [Iarrobino 1972]. The fact that the Hilbert scheme H_8^4 has at least two components appears in [Iarrobino and Emsalem 1978]. In contrast, for the Hilbert scheme of points in the plane ($d = 2$), the smoothable component is the only component [Fogarty 1968].

To show that the Hilbert scheme of n points is irreducible, it suffices to show that each isomorphism type of local algebras of rank at most n is smoothable, and for n at most 6 there are finitely many isomorphism types of local algebras. In contrast, there are infinitely many nonisomorphic local algebras of degree 7. Relying on a classification of the finitely many isomorphism types in degree 6, Mazzola [1980] proves the irreducibility of H_n^d for $n = 7$.

In our approach, a coarser geometric decomposition replaces most of the need for classification. We divide the local algebras in H_n^d into sets $H_{\boldsymbol{h}}^d$ by their Hilbert function $\boldsymbol{h}$, and we determine which components of these sets are smoothable. The main advantage to this approach is that there are fewer Hilbert functions than isomorphism classes, and this enables us to extend the smoothability results of [Mazzola 1980] up to degree 8.

In order to determine the components of $H_{\boldsymbol{h}}^d$, we first determine the components of the standard graded Hilbert scheme $\mathcal{H}_{\boldsymbol{h}}^d$, which parametrizes homogeneous ideals with Hilbert function $\boldsymbol{h}$. By considering the map $\pi_{\boldsymbol{h}} : H_{\boldsymbol{h}}^d \to \mathcal{H}_{\boldsymbol{h}}^d$ which sends a local algebra to its associated graded ring, we relate the components of $\mathcal{H}_{\boldsymbol{h}}^d$ to those of $H_{\boldsymbol{h}}^d$. The study of standard graded Hilbert schemes leads to the following analogue of Theorem 1.1:

Theorem 1.2. *Let $\mathcal{H}_{\boldsymbol{h}}^d$ be the standard graded Hilbert scheme for Hilbert function $\boldsymbol{h}$, where $\sum h_i \leq 8$. Then $\mathcal{H}_{\boldsymbol{h}}^d$ is reducible if and only if*

$$\boldsymbol{h} = (1, 3, 2, 1) \quad \textit{or} \quad \boldsymbol{h} = (1, 4, 2, 1),$$

in which case it has exactly two irreducible components. In particular, $\mathcal{H}^3_{(1,3,2,1)}$ is the minimal example of a reducible standard graded Hilbert scheme.

As in the ungraded case, all standard graded Hilbert schemes in the plane are smooth and irreducible [Evain 2004].

In the case when $d = 4$ and $n = 8$, we describe the intersection of the two components of H_8^4 explicitly. Let $S = k[x, y, z, w]$ and S_1 be the vector space of linear forms in S. Let S_2^* denote the space of symmetric bilinear forms on S_1. Then, the component G_8^4 is isomorphic to $\mathbb{A}^4 \times \mathrm{Gr}(3, S_2^*)$, where $\mathrm{Gr}(3, S_2^*)$ denotes the Grassmannian of 3-dimensional subspaces of S_2^*.

Theorem 1.3. *The intersection $R_8^4 \cap G_8^4$ is a prime divisor on G_8^4. We have the following equivalent descriptions of $R_8^4 \cap G_8^4 \subset G_8^4$:*

Set-theoretic: *For a point $I \in G_8^4 \cong \mathbb{A}^4 \times \mathrm{Gr}(3, S_2^*) \cong \mathbb{A}^4 \times \mathrm{Gr}(7, S_2)$ let V be the corresponding 7-dimensional subspace of S_2. Then $I \in G_8^4$ belongs to the intersection if and only if the skew-symmetric bilinear form*

$$\langle\ ,\ \rangle_I : (S_1 \otimes S_2/V)^{\otimes 2} \to \bigwedge^3 (S_2/V) \cong k$$

given by

$$\langle l_1 \otimes q_1, l_2 \otimes q_2 \rangle_I = (l_1 l_2) \wedge q_1 \wedge q_2$$

is degenerate.

Local equations: *Around any $I \in G_8^4$, choose an open neighborhood $U_I \subset G_8^4$ such that the universal Grassmannian bundle over the U_I is generated by three sections. Since these sections are bilinear forms we may represent them as symmetric 4×4 matrices A_1, A_2, and A_3 with entries in $\Gamma(U_I, \mathcal{O}_{G_8^4})$. The local equation for $R_8^4 \cap U_I$ is then the Pfaffian of the 12×12 matrix*:

$$\begin{pmatrix} 0 & A_1 & -A_2 \\ -A_1 & 0 & A_3 \\ A_2 & -A_3 & 0 \end{pmatrix}.$$

Note that specializing this equation to I gives the Pfaffian of $\langle\ ,\ \rangle_I$.

The local equation from the previous theorem gives an effective criterion for deciding whether an algebra of colength 8 belongs to the smoothable component. Moreover, it can be lifted to equations which cut out $R_8^4 \subset H_8^4$. Recall that H_8^4 can be covered by open affines corresponding to monomial ideals in $k[x, y, z, w]$ of colength 8.

Theorem 1.4. *On these monomial coordinate charts, $R_8^4 \subset H_8^4$ is cut out set-theoretically by*

(1) *the zero ideal on charts corresponding to monomial ideals with Hilbert functions other than* $(1, 4, 3)$, *and*

(2) *the pullback of the equations in Theorem 1.3 along the projection to homogeneous ideals in charts corresponding to monomial ideals with Hilbert function* $(1, 4, 3)$.

Remark 1.5. It is not known whether H_8^4 is reduced. If it is, then the equations in Theorem 1.4 cut out the smoothable component scheme-theoretically.

Remark 1.6. Every point of the Hilbert scheme of n points in $\mathbb{P}^d$ has an open neighborhood isomorphic to H_n^d. Therefore, analogues of Theorems 1.1, 1.3, and 1.4 hold for the Hilbert scheme of n points in $\mathbb{P}^d$. However, the most natural setting for our methods is the affine case and the language of multigraded Hilbert schemes.

The material in this paper is organized as follows: Section 2 contains background and definitions. Section 3 describes the geometry of standard graded Hilbert schemes of degree at most 8. Section 4 contains proofs of the smoothability of families of algebras and its main steps are collected in Table 4.1. Section 5 is devoted to the study of the components of H_8^4 and their intersection. Section 6 ties together these results to give proofs of all theorems mentioned above. Finally, Section 7 proposes some open questions.

2. Background

In this section, let k be a field and $S = k[x_1, \dots, x_d]$.

Multigraded Hilbert schemes. A grading of S by an abelian group A is a semigroup homomorphism $\deg : \mathbb{N}^d \to A$ which assigns to each monomial in S a degree in A. Let $h : A \to \mathbb{N}$ be an arbitrary function, which we will think of as a vector $\boldsymbol{h}$, with values h_a indexed by a in A. We say that a homogeneous ideal I in S has Hilbert function $\boldsymbol{h}$ if S_a/I_a has k-dimension h_a for all $a \in A$. Multigraded Hilbert schemes, introduced in [Haiman and Sturmfels 2004], parametrize homogeneous ideals with a fixed Hilbert function. More precisely these are quasiprojective schemes over k which represent the following functors [Haiman and Sturmfels 2004, Theorem 1.1]:

Definition 2.1. *For a fixed integer d, grading* deg, *and Hilbert function $\boldsymbol{h}$, the* multigraded Hilbert functor $\mathcal{H}_{\boldsymbol{h}} : \underline{k\text{-Alg}} \to \underline{\text{Set}}$ *assigns to each k-algebra T, the set of homogeneous ideals J in $S \otimes T$ such that the graded component $(S \otimes T/J)_a$ is a locally free T-module of rank h_a for all a in A. The* multigraded Hilbert scheme *is the scheme which represents the multigraded Hilbert functor.*

In particular, we will be interested in the following two special kinds of multigraded Hilbert scheme:

- *Let* $\deg : \mathbb{N}^d \to 0$ *be the constant function to the trivial group and define $h_0 = n$. In this case the multigraded Hilbert scheme is the* Hilbert scheme of n points in $\mathbb{A}^d$ *and will be denoted H_n^d.*
- *Let* $\deg : \mathbb{N}^d \to \mathbb{Z}$ *be the summation function, which induces the standard grading* $\deg(x_i) = 1$. *We call the corresponding multigraded Hilbert scheme the* standard graded Hilbert scheme for Hilbert function $\boldsymbol{h}$ *and denote it with $\mathcal{H}_{\boldsymbol{h}}^d$.*

If $n = \sum_{j \in \mathbb{N}} h_j$ there is a closed immersion $\mathcal{H}_{\boldsymbol{h}}^d \to H_n^d$ by [Haiman and Sturmfels 2004, Proposition 1.5].

Coordinates for the Hilbert scheme of points. In this section we briefly discuss some coordinate systems on H_n^d. The reader should refer to [Miller and Sturmfels

2005, Chapter 18] for an extended treatment. For a monomial ideal M of colength n with standard monomials λ, let $U_\lambda \subset H_n^d$ be the set of ideals I such that the monomials in λ are a basis for S/I. Note that the U_λ form an open cover of H_n^d. An ideal $I \in U_\lambda$ has generators of the form $m - \sum_{m' \in \lambda} c_{m'}^m m'$. The $c_{m'}^m$ are local coordinates for U_λ which define a closed immersion into affine space.

Suppose $V(I)$ consists of n distinct points $q^{(1)}, \ldots, q^{(n)}$ with coordinates $q_i^{(j)}$ for $1 \le i \le d$. Fix an order $\lambda = (m_1, \ldots, m_n)$ on the set of monomials λ and define $\Delta_\lambda = \det\big([m_i(q^{(j)})]_{i,j}\big)$. For example, if $\lambda = (1, x_1, \ldots, x_1^{n-1})$, then Δ_λ is the determinant of the Vandermonde matrix on the $q_1^{(j)}$. If $I \in U_\lambda$, we can express the $c_{m'}^m$ in terms of the $q_i^{(j)}$, using Cramer's rule, as

$$c_{m'}^m = \frac{\Delta_{\lambda - m' + m}}{\Delta_\lambda},$$

where $\lambda - m' + m$ is the ordered set of monomials obtained from λ by replacing m' with m. Note that the right-hand side of this equality is only defined for ideals of distinct points. The quotient or product of two Δ_λ's is S_n-invariant. Thus the formula does not depend on the order of λ. Gluing over the various U_λ, these quotients determine a birational map

$$(\mathbb{A}^d)^n /\!\!/ S_n \dashrightarrow R_n^d$$

that is regular when the points $q^{(j)}$ are all distinct. The rational functions $\Delta_{\eta_1}/\Delta_{\eta_2}$ are elements of the quotient field of either $(\mathbb{A}^d)^n /\!\!/ S_n$ or R_n^d. The expressions Δ_λ and their relationship to the local equations $c_{m'}^m$ were introduced in [Haiman 1998, Proposition 2.6].

Duality. First let us suppose that k has characteristic 0, and let S^* be the ring $k[y_1, \ldots, y_d]$, with the structure of an S-module via formal partial differentiation $x_i \cdot f = \partial f / \partial y_i$. If we look at homogeneous polynomials of a fixed degree j in each of the two rings, we have a pairing $S_j \times S_j^* \to S_0^* = k$. Any vector subspace of S_j has an orthogonal subspace in S_j^* of complementary dimension. In particular, if I is a homogeneous ideal in S, we have subspaces $I_j^\perp \subset S_j^*$ and we set $I^\perp = \oplus I_j^\perp$, which is often called the Macaulay inverse system of I. The subspace $I^\perp$ is closed under differentiation, that is, $\frac{\partial}{\partial y_i} I_j^\perp \subset I_{j-1}^\perp$ for all i and j. Conversely, any graded vector subspace $I^\perp \subset S^*$ which is closed under differentiation determines an orthogonal ideal $I \subset S$ with Hilbert function $h_j = \dim_k I_j^\perp$. Also, note that any linear change of variables in S induces a linear change of variables in S^*.

If k has positive characteristic p, then the same theory works for sufficiently small degree. Formal partial differentiation gives a perfect pairing $S_j \times S_j^* \to k$ if and only if j is less than p. Thus, we can associate orthogonal subspaces $I_j^\perp$ to a homogeneous ideal I so long as I_p contains all of S_p. In this case, we define

$I_j^\perp = 0$ for all degrees j at least p, and $I = \oplus I_j$ as before. Conversely, for a graded vector subspace $I^\perp \subset S_*$ which is closed under differentiation and with $I_j^\perp = 0$ for j at least p, the orthogonal space is a homogeneous ideal $I \subset S$ with Hilbert function $h_j = \dim_k I_j^\perp$.

3. Components of the standard graded Hilbert schemes

In this section, k will denote an algebraically closed field of characteristic not 2 or 3.

We will study the components of the standard graded Hilbert schemes $\mathcal{H}_{\boldsymbol{h}}^d$ with Hilbert function $\boldsymbol{h}$ where $\sum h_i \le 8$. These results will be important for the proofs of smoothability in the following section. From [Evain 2004, Theorem 1], we have that for $d = 2$, the standard graded Hilbert schemes are irreducible. Thus, we will only work with d at least 3. For the purposes of classifying irreducible components of $\mathcal{H}_{\boldsymbol{h}}^d$, it is convenient to work with homogeneous ideals which contain no linear forms, and thus we assume that $h_1 = d$. The following lemma allows us to restrict our attention to this case:

Proposition 3.1. *The standard graded Hilbert scheme $\mathcal{H}_{\boldsymbol{h}}^d$ with $d \ge h_1$ is a $\mathcal{H}_{\boldsymbol{h}}^{h_1}$-bundle over $\mathrm{Gr}(d - h_1, S_1)$. In particular, if $\mathcal{H}_{\boldsymbol{h}}^{h_1}$ is irreducible of dimension D then $\mathcal{H}_{\boldsymbol{h}}^d$ is irreducible of dimension $D + (d - h_1)d$.*

Proof. The degree 1 summand of the universal ideal sheaf of $\mathbb{O}_{\mathcal{H}_{\boldsymbol{h}}^d}[x_1, \dots, x_d]$ is locally free of rank $d - h_1$ and thus defines a morphism $\phi : \mathcal{H}_{\boldsymbol{h}}^d \to \mathrm{Gr}(d - h_1, S_1)$. Over an open affine $U \cong \mathbb{A}^{(d-h_1)h_1}$ in $\mathrm{Gr}(d - h_1, S_1)$, we have an isomorphism $\phi^{-1}(U) \cong U \times H_{\boldsymbol{h}}^{h_1}$ by taking a change of variables in $\mathbb{O}_U[x_1, \dots, x_d]$. □

Lemma 3.2. *Let be m a positive integer such that $m!$ is not divisible by the characteristic of k. Let $f(y_1, \dots, y_d)$ be a homogeneous polynomial in S_m^* whose partial derivatives form an r-dimensional vector subspace of S_{m-1}^*. Then f can be written as a polynomial in terms of some r-dimensional subspace of S_1^*.*

Proof. There exists a linear map from $S_1 \to S_{m-1}^*$ which sends $x_i \mapsto \frac{\partial f}{\partial y_i}$. After a change of variables, we can assume that $x_{r+1}, \dots, x_d$ annihilate f. Thus, any term of f contains only the variables $y_1, \dots, y_r$. □

Throughout this section, N will denote $\dim_k S_2 = \binom{d+1}{2}$, the dimension of the vector space of quadrics.

Proposition 3.3. *Let $0 \le e \le N$. The standard graded Hilbert scheme for Hilbert function $(1, d, e)$ is isomorphic to the Grassmannian $\mathrm{Gr}(N - e, S_2)$, and it is thus irreducible of dimension $(N - e)N$.*

Proof. We build the isomorphism via the functors of points of these schemes. For a k-algebra T let $\phi(T) : \mathcal{H}_h(T) \to \mathrm{Gr}(N - e, S_2)(T)$ be the morphism of sets which maps a homogeneous ideal $I \subset T \otimes_k S$ to I_2. Let $\psi(T) : \mathrm{Gr}(N-e, S_2)(T) \to \mathcal{H}_h(T)$ be the map which sends a k-submodule L of $T \otimes S_2$ to $L \oplus \bigoplus_{j \geq 3}(T \otimes_k S_j)$, which is an ideal of $T \otimes S$. The natural transformations ϕ and ψ are inverses of one another and the isomorphism follows from Yoneda's Lemma. $\square$

Proposition 3.4. *Let $\boldsymbol{h} = (1, d, 1, \dots, 1)$ and let $m \geq 3$ be the largest index such that h_m is nonzero. Then the standard graded Hilbert scheme for $\boldsymbol{h}$ is irreducible of dimension $d - 1$.*

Proof. We claim that the scheme $\mathcal{H}^d_{\boldsymbol{h}}$ is parametrized by $\mathrm{Gr}(1, S_1^*)$ by sending a vector space generated by $\ell \in S_1^*$ to the ideal generated by the quadrics orthogonal to ℓ^2 and all degree $m+1$ polynomials. This ideal has the right Hilbert function and the parametrization is clearly surjective. $\square$

Theorem 3.5. *If d is at least 3, the standard graded Hilbert scheme for Hilbert function $(1, d, 2, 1)$ is reducible and consists of the following two components:*

(1) *The homogeneous ideals orthogonal to ℓ^3, ℓ^2, and q where ℓ is a linear form and q is a quadric linearly independent of ℓ^2. We denote this component by $\mathcal{Q}_d$, and $\dim(\mathcal{Q}_d) = (d^2 + 3d - 6)/2$.*

(2) *The closure of the homogeneous ideals orthogonal to a cubic c and its partial derivatives, where the degree 1 derivatives of c have rank 2. We denote this component by $\mathcal{P}_d$, and $\dim(\mathcal{P}_d) = 2d - 1$.*

Proof. We compute the dimension of the first component. It is parametrized by the 1-dimensional subspace of S_1^* generated by ℓ and a 2-dimensional subspace of S_2^* which contains ℓ^2. These have dimensions $d - 1$ and $N - 1 - 1$ respectively, for a total of $(d^2 + 3d - 6)/2$.

Note also that an open subset of the second component, $\mathcal{P}_d$, is parametrized by a 2-dimensional subspace V of S_1^* and a cubic $c \in \mathrm{Sym}_3(V)$ which is not a perfect cube. The parametrization is by taking the ideal whose components of degrees 3 and 2 are orthogonal to c and to its derivatives respectively. The space of derivatives is 2-dimensional by our construction of c. The dimension of $\mathcal{P}_d$ is $3 + 2(d - 2) = 2d - 1$.

We claim that any homogeneous ideal with Hilbert function $(1, d, 2, 1)$ lies in one of these two components. Any such ideal is orthogonal to a cubic c, and the derivatives of c are at most 2-dimensional. If the derivatives are 1-dimensional, then c must be a perfect cube, so the ideal is in $\mathcal{Q}_d$. Otherwise, the ideal is in $\mathcal{P}_d$.

Finally, we will show that $\mathcal{P}_d$ has a point that does not lie on $\mathcal{Q}_d$. Let I be the ideal orthogonal to $y_1 y_2^2$, and its partial derivatives, $2y_1 y_2$, y_2^2. Then I is generated by x_2^3 and all degree 2 monomials other than x_2^2 and $x_1 x_2$. We will study the homomorphisms $\phi : I \to S/I$ of degree 0, as these correspond to the tangent space of $\mathcal{H}^d_{\boldsymbol{h}}$ at I. For any quadric generator q, we can write $\phi(q) = a_q x_2^2 + b_q x_1 x_2$. Note that $x_1\phi(q) = a_q x_1 x_2^2$ and $x_2\phi(q) = b_q x_1 x_2^2$. For any $i, j > 2$, ϕ must satisfy the conditions

$$x_1\phi(x_ix_j) = x_j\phi(x_ix_1) = 0, \qquad x_2\phi(x_ix_j) = x_j\phi(x_ix_2) = 0,$$
$$x_1\phi(x_1x_i) = x_i\phi(x_1^2) = 0, \qquad x_2\phi(x_1x_i) = x_1\phi(x_2x_i).$$

In matrix form, we see that ϕ must be in the form

$$\begin{array}{c c} & \begin{array}{ccccc} x_1^2 & x_1x_i & x_2x_i & x_ix_j & x_2^3 \end{array} \\ \begin{array}{c} x_2^2 \\ x_1x_2 \\ x_1x_2^2 \end{array} & \left(\begin{array}{ccccc} * & c_i & * & 0 & 0 \\ * & 0 & c_i & 0 & 0 \\ 0 & 0 & 0 & 0 & * \end{array}\right), \end{array}$$

where i and j range over all integers greater than 2. Thus there are at most $2(d-2)+3 = 2d-1$ tangent directions, but since $\mathcal{P}_d$ has dimension $2d-1$, there are exactly $2d-1$ tangent directions. On the other hand, $\mathcal{Q}_d$ has dimension $(d^2+3d-6)/2$, which is greater than $2d-1$ for d at most 3, so I cannot belong to $\mathcal{Q}_d$ and thus $\mathcal{P}_d$ is a component. □

Proposition 3.6. *The standard graded Hilbert scheme for Hilbert function* $\boldsymbol{h} = (1, d, 2, 2)$ *is irreducible of dimension* $2d-2$.

Proof. The Hilbert scheme is parametrized by a 2-dimensional subspace L of S_1^* and a subspace of S_2 of dimension $N-2$, and containing the $(N-3)$-dimensional subspace orthogonal to the square of L. The parametrization is by sending the subspace of S_2 to the ideal generated by that subspace, together with all degree 4 polynomials. The dimension of this parametrization is $2(d-2)+2 = 2d-2$. □

Proposition 3.7. *The standard graded Hilbert scheme for Hilbert function* $\boldsymbol{h} = (1, 3, 3, 1)$ *is irreducible of dimension* 9.

Proof. This Hilbert scheme embeds as a closed subscheme of the smooth 18-dimensional variety $\mathrm{Gr}(3, S_2) \times \mathrm{Gr}(9, S_3)$ by mapping an ideal to its degree 2 and 3 graded components. Furthermore, $\mathcal{H}_{\boldsymbol{h}}^3$ is defined by $9 = 3 \cdot 3$ equations, corresponding to the restrictions that the products of each of the 3 variables and each of the 3 quadrics in I_2 are in I_3. In particular, the dimension of each irreducible component is at least 9.

Now we will look at the projection of $\mathcal{H}_{\boldsymbol{h}}^3$ onto the Grassmannian $\mathrm{Gr}(9, S_3)$, which is isomorphic to $\mathrm{Gr}(1, S_3^*)$. The orthogonal cubic in S_3^* can be classified according to the vector space dimension of its derivatives. For a generic cubic, its three derivatives will be linearly independent and therefore the cubic will completely determine the orthogonal space. Thus, the projection from $\mathcal{H}_{\boldsymbol{h}}^3$ is a bijection over this open set, so the preimage is 9-dimensional. In the case where the derivatives of the cubic are 2-dimensional, we have that, after a change of coordinates, the cubic is written in terms of two variables. Thus, the parameter space of the cubic consists of a 2-dimensional choice of a subspace of S_1 and then a 3-dimensional choice of a cubic written in terms of this subspace. The fiber over any fixed cubic is isomorphic to the Grassmannian of 3-dimensional subspaces of the 4-dimensional subspace of S_2 orthogonal to the derivatives of the cubic. The dimension of the locus in $\mathscr{H}^3_{\boldsymbol{h}}$ is therefore $2+3+3=8$. By a similar logic, the locus where the cubic has a 1-dimensional space of derivatives is $2+2\cdot 3=8$. Therefore, $\mathscr{H}^3_{\boldsymbol{h}}$ is the disjoint union of three irreducible sets, of dimensions 9, 8, and 8. We conclude that $\mathscr{H}^3_{\boldsymbol{h}}$ is an irreducible complete intersection of dimension 9. □

Proposition 3.8. *Let $1 \leq e \leq N$. The standard graded Hilbert scheme for Hilbert function $(1, d, e, 1, 1)$ is irreducible of dimension $d-1+(N-e)(e-1)$.*

Proof. This Hilbert scheme is parametrized by a 1-dimensional subspace L of S_1^*, together with an e-dimensional subspace V of S_2^* which contains $\mathrm{Sym}_2(L)$. The parametrization is by mapping (L, V) to the ideal whose summands of degrees 2, 3, and 4 are orthogonal to V, L^3, and L^4, respectively. Note that this has the desired dimension $(d-1)+\big((N-1)-(e-1)\big)(e-1)$. □

Theorem 3.9. *With the exception of Hilbert functions $(1, 3, 2, 1)$ and $(1, 4, 2, 1)$, the standard graded Hilbert schemes with $\sum h_i \leq 8$ are irreducible.*

Proof. The cases when $d=2$ follow from [Evain 2004, Theorem 1]. The cases when d is at least 3 are summarized in Table 4.1. □

4. Smoothable 0-schemes of degree at most 8

In this section k will denote an algebraically closed field of characteristic not 2 or 3.

Recall that a point I in H^d_n is smoothable if I belongs to the smoothable component R^d_n. In this section, we first reduce the question of smoothability to ideals I in H^d_n where S/I is a local k-algebra and I has embedding dimension d. Then we define the schemes $H^d_{\boldsymbol{h}}$ which parametrize local algebras, and we use these to show that each 0-dimensional algebra of degree at most 8 is either smoothable or is isomorphic to a homogeneous local algebra with Hilbert function $(1, 4, 3)$.

We use two different methods to show that a subscheme $H^d_{\boldsymbol{h}}$ belongs to the smoothable component.

(1) For each irreducible component of $H^d_{\boldsymbol{h}}$, consider a generic ideal I from that component. Apply suitable isomorphisms to put I into a nice form. Then show I is smoothable. Since the set of ideals isomorphic to I are dense in the component and smoothable, the entire component of $H^d_{\boldsymbol{h}}$ containing I must belong to R^d_n.

(2) Within each irreducible component of $H^d_{\boldsymbol{h}}$, find an ideal I such that I is a smooth point in H^d_n and I belongs to R^d_n. Then the whole component of $H^d_{\boldsymbol{h}}$ containing I must belong to R^d_n.

In each method we need to show that a particular ideal I is smoothable. We do this by showing $I = \operatorname{in} J$ with respect to some nonnegative weight vector for a smoothable ideal J. The corresponding Gröbner deformation induces a morphism $\mathbb{A}^1 \to R^d_n$ which maps 0 to I.

For $d = 2$, the Hilbert scheme H^2_n is smooth and irreducible [Fogarty 1968, Theorem 2.4]. Thus, we will limit our analysis to algebras with embedding dimension at least 3.

For a finite rank k-algebra A_0, we say that A_0 is smoothable if there exists a flat family $k[[t]] \to A$ such that the special fiber of A is isomorphic to A_0 and such that the generic fiber of A is smooth. This terminology is justified by the following result:

Lemma 4.1. *Let $I \subseteq S$ an ideal of colength n. Then I is smoothable if and only if S/I is smoothable* (*as a k-algebra*).

Proof. Let $I \subseteq S$ be a smoothable ideal, and let $U \subseteq R^d_n$ be the open set parametrizing smooth 0-schemes. Since $I \in R^d_n$, there exists a smooth curve C and a map $f : C \to R^d_n$ such that $f(P) = I$ for some point $P \in C$ and such that $f(C)$ meets U. By considering the completion of C at the point P, we obtain an induced map $\widehat{f} : \operatorname{Spec}(k[[t]]) \to R^d_n$ which sends the closed point to I and the generic point into U. The flat family over $\operatorname{Spec}(k[[t]])$ which corresponds to the map $\widehat{f}$ induces an abstract smoothing of S/I.

Conversely, let's assume that $A_0 := S/I$ admits an abstract smoothing $k[[t]] \to A$. Let $\phi : k[[t]][x_1, \dots, x_d] \to A$ be defined by sending each x_i to any lift of x_i from $S/I = A_0$. Since A is a finitely generated $k[[t]]$-module, the cokernel of ϕ is finitely generated, and so we can apply Nakayama's Lemma to show that ϕ is surjective. The map ϕ thus induces a morphism of schemes $f : \operatorname{Spec}(k[[t]]) \to H^d_n$ which sends the closed point to I and the generic point into U. It follows that $I \in \overline{U} = R^d_n$. □

If I is supported at multiple points, then S/I is the product of local Artin algebras and a smoothing of each factor over $k[[t]]$ yields a smoothing of S/I. Because of this observation and Lemma 4.1, we will now only consider ideals I in H^d_n which define local algebras with embedding dimension d.

The schemes $H^d_{\boldsymbol{h}}$. If $(A, \mathfrak{m})$ is a local algebra, its Hilbert function is defined by $h_i = \dim_k \mathfrak{m}^i/\mathfrak{m}^{i+1}$, which is equivalently the Hilbert function of the associated graded ring of A. When A is both local and graded, the two notions of Hilbert function coincide. We now define the schemes $H^d_{\boldsymbol{h}}$ and explore their irreducible components for each Hilbert function $\boldsymbol{h}$ with $\sum h_i \le 8$.

For each $\boldsymbol{h}$ such that $\sum h_i = n$, the subscheme $H^d_{\boldsymbol{h}} \subset H^d_n$ consists set-theoretically of the ideals I defining a local algebra S/I with maximal ideal $(x_1, \dots, x_d)$ whose Hilbert function equals $\boldsymbol{h}$. More precisely, let $\mathcal{A} = \mathbb{O}_{H^d_n}[x_1, \dots, x_d]/\mathcal{I}$ be the universal sheaf of algebras on H^d_n and let $\mathcal{M}$ be the ideal $(x_1, \dots, x_d)\mathcal{A}$. The fiber at an ideal I of the quotient sheaf $\mathcal{A}/\mathcal{M}^i$ is isomorphic to $S/(I + (x_1, \dots, x_d)^i)$. For any fixed $\boldsymbol{h}$, there is a locally closed subset of H^d_n consisting of those points such that the fiber of $\mathcal{A}/\mathcal{M}^i$ has dimension $h_0 + \cdots + h_{i-1}$ for all $i \geq 0$. Let $H^d_{\boldsymbol{h}}$ be the reduced subscheme on this subset, and then the restriction of each $\mathcal{A}/\mathcal{M}^i$ to $H^d_{\boldsymbol{h}}$ is locally free. Define $\mathcal{B}$ to be the sheaf of graded algebras on $H^d_{\boldsymbol{h}}$ whose i-th component is

$$\ker\bigl((\mathcal{A}/\mathcal{M}^{i+1})|_{H^d_{\boldsymbol{h}}} \to (\mathcal{A}/\mathcal{M}^i)|_{H^d_{\boldsymbol{h}}}\bigr),$$

which is locally free of rank h_i because it is the kernel of a surjection of locally free sheaves. Note that the fiber of $\mathcal{B}$ at I is the associated graded ring of S/I. There is a canonical surjection of graded algebras $O_{H^d_{\boldsymbol{h}}}[x_1, \dots, x_d] \to \mathcal{B}$ which defines a morphism $\pi_{\boldsymbol{h}} : H^d_{\boldsymbol{h}} \to \mathcal{H}^d_{\boldsymbol{h}}$ to the standard graded Hilbert scheme. The ideal I gets mapped to its initial ideal with respect to the weight vector $(-1, \dots, -1)$.

With the exception $\boldsymbol{h} = (1, 3, 2, 1, 1)$, we will show that the irreducible components of $H^d_{\boldsymbol{h}}$ and $\mathcal{H}^d_{\boldsymbol{h}}$ are in bijection via the map $\pi_{\boldsymbol{h}}$.

Proposition 4.2. *Each subscheme $H^d_{(1,d,e)}$ is irreducible.*

Proof. Since $H^d_{(1,d,e)} \cong \mathcal{H}^d_{(1,d,e)}$, this follows from Proposition 3.3. □

Proposition 4.3. *Fix $\boldsymbol{h} = (1, d, e, f)$. Let $m = (d+1)d/2 - e = \dim_k S_2/I_2$. Then every fiber of $\pi_{\boldsymbol{h}}$ is irreducible of dimension mf. In particular, the irreducible components of $H^d_{\boldsymbol{h}}$ are exactly the preimages of the irreducible components of $\mathcal{H}^d_{\boldsymbol{h}}$.*

Proof. Fix a point in $\mathcal{H}^d_{\boldsymbol{h}}$, which corresponds to a homogeneous ideal I. Let $q_1, \dots, q_m$ be quadratic generators of I, and let $c_1, \dots, c_f$ be cubics which form a vector space basis for S_3/I_3. Define a map $\phi : \mathbb{A}^{mf} \to H^d_n$ via the ideal

$$\Bigl\langle q_i - \textstyle\sum_{j=1}^{f} t_{ij} c_j \mid 1 \leq i \leq m \Bigr\rangle + I_{\geq 3},$$

where the t_{ij} are the coordinate functions of $\mathbb{A}^{mf}$. Because a product of any variable x_ℓ with any of these generators is in I, this ideal has the right Hilbert function and maps to the fiber of $\pi_{\boldsymbol{h}}$ over I. Furthermore, ϕ is bijective on field-valued points, so the fiber is irreducible of dimension mf.

For the last statement, we have that for any irreducible component of $\mathcal{H}^d_{\boldsymbol{h}}$, the restriction of $\pi_{\boldsymbol{h}}$ has irreducible equidimensional fibers over an irreducible base, so the preimage is irreducible. These closed sets cover $H^d_{\boldsymbol{h}}$ and because each lies over a distinct component of $\mathcal{H}^d_{\boldsymbol{h}}$, they are distinct irreducible components. □

Combining Theorem 3.5 with the above proposition, we see $H^d_{(1,d,2,1)}$ has exactly two components: $P_d := \pi^{-1}(\mathcal{P}_d)$ and $Q_d := \pi^{-1}(\mathcal{Q}_d)$. In addition, by Propositions 3.6, 3.7, and 4.3, $H^d_{(1,d,2,2)}$ and $H^3_{(1,3,3,1)}$ are irreducible.

Proposition 4.4. *Let $\boldsymbol{h} = (1, d, 1, \dots, 1)$ and let $m \geq 3$ be the largest index such that h_m is nonzero. Then $H^d_{\boldsymbol{h}}$ is irreducible of dimension $(d + 2m - 2)(d - 1)/2$. At a generic point, after a change of coordinates, we can take the ideal to be*

$$\langle x_d^{m+1}, x_i^2 - x_d^m, x_j x_k \mid 1 \leq i < d, 1 \leq j < k \leq d\rangle.$$

Proof. Fix an ideal $I \in \mathcal{H}^d_{\boldsymbol{h}}$, and after a change of coordinates, we can assume

$$I = \langle x_d^{m+1}, x_i x_j \mid 1 \leq i \leq d-1, 1 \leq j \leq d\rangle.$$

Let J be an ideal in the fiber above I. By assumption J contains an element of the form

$$x_i x_d - b_{i3} x_d^3 - \cdots - b_{im} x_d^m,$$

for $1 \leq i < d$. Let J' be the image of J after the change of coordinates

$$x_i \mapsto x_i + b_{i3} x_d^2 + \cdots b_{im} x_d^{m-1}, \tag{4-1}$$

and note that J' contains $x_i x_d$ for $1 \leq i < d$ and also lies over I. Thus, for each $1 \leq i \leq j < d$, J' contains an element of the form $f = x_i x_j - a_{ij} x_d^k - \cdots$ for some k. However, J' must also contain $x_j(x_i x_d) - x_d f = a_{ij} x_d^{k+1} + \cdots$, so in order to have I as the initial ideal, k must equal m. Therefore, J' is of the form

$$J' = \langle x_d^{m+1}, x_i x_j - a_{ij} x_d^m, x_k x_d \mid 1 \leq i \leq j \leq d, 1 \leq k < d\rangle.$$

Conversely, for any choice of a_{ij} and b_{ij}, applying the change of variables in (4-1) to the ideal J' gives a unique ideal J with I as an initial ideal. Thus, the fiber is irreducible of dimension $(m-2)(d-1) + (d-1)d/2 = (d-1)(d+2m-4)/2$, which, together with Proposition 3.4 proves the first statement.

For the second statement, note that the coefficients a_{ij} define a symmetric bilinear form. By taking the form to be generic and choosing a change of variables, we get the desired presentation of the quotient algebra. $\square$

The above propositions cover all Hilbert functions of length at most 8 except for $\boldsymbol{h} = (1, 3, 2, 1, 1)$. In this case the fibers of $\pi_{(1,3,2,1,1)}$ are not equidimensional. The dimension of the fiber depends on whether or not the homogeneous ideal requires a cubic generator.

Lemma 4.5. *No ideal in $\mathcal{H}^3_{(1,3,2,1,1)}$ requires a quartic generator.*

Proof. If I were such an ideal, then leaving out the quartic generator would yield an ideal with Hilbert function $(1, 3, 2, 1, 2)$. No such ideal exists, because no such monomial ideal exists. $\square$

Lemma 4.6. *There exists a* 4*-dimensional irreducible closed subset* $\mathcal{Z}$ *of* $\mathcal{H} = \mathcal{H}^3_{(1,3,2,1,1)}$ *where the corresponding homogeneous ideal requires a single cubic generator. On* $\mathcal{U} = \mathcal{H} \setminus \mathcal{Z}$*, the ideal does not require any cubic generators.*

Proof. Let $\mathcal{S}_j$ denote the j-th graded component of $\mathcal{O}_{\mathcal{H}}[x, y, z]$ and $\mathcal{I}_j \subset \mathcal{S}_j$ the j-th graded component of the universal family of ideals on $\mathcal{H}$. Consider the cokernel $\mathcal{Q}$ of the multiplication map on the coherent sheaves $\mathcal{I}_2 \otimes_{\mathcal{O}_{\mathcal{H}}} \mathcal{S}_1 \to \mathcal{I}_3$ on $\mathcal{H}$. The dimension of $\mathcal{Q}$ is upper semicontinuous. Furthermore, since it is not possible to have an algebra with Hilbert function $(1, 3, 2, 3)$, the dimension is at most 1. The set $\mathcal{Z}$ is exactly the support of $\mathcal{Q}$.

We claim that $\mathcal{Z}$ is parametrized by the data of a complete flag $V_1 \subset V_2 \subset S_1^*$, and a 2-dimensional subspace Q of V_2^2 which contains V_1^2. The dimension of this parametrization is $2 + 1 + 1 = 4$. An ideal is formed by taking the ideal which is orthogonal to Q in degree 2 and to the powers V_1^3 and V_1^4 in degrees 3 and 4 respectively. After a change of variables, we can assume that the flag is orthogonal to $\langle x \rangle \subset \langle x, y \rangle \subset S_2$. Then the degree 2 generators of I are x^2, xy, xz and another quadric. It is easy to see that these only generate a codimension 2 subspace of S_3. Conversely, for any ideal with this property, the orthogonal cubic has a 1-dimensional space of derivatives. Furthermore, there exists a homogeneous ideal with Hilbert function $(1, 3, 2, 2, 1)$ contained in the original ideal. The cubics orthogonal to these have a 2-dimensional space of derivatives, so we can write them in terms of a 2-dimensional space of the dual variables. These two vector spaces determine the flag, and the parametrization is bijective on closed points. In particular, $\mathcal{Z}$ is irreducible of dimension 4. □

Lemma 4.7. *The preimage* $Z := \pi^{-1}(\mathcal{Z})$ *is irreducible of dimension* 11.

Proof. By Lemma 4.6, it suffices to prove that the fibers of π over $\mathcal{Z}$ are irreducible and 7-dimensional. Let I be a point in $\mathcal{Z}$. As in the proof of Lemma 4.6, we can assume that the ideal corresponding to a point of $\mathcal{Z}$ is generated by x^2, xy, xz, q, c, where q and c are a homogeneous quadric and cubic respectively. A point J in the fiber must be generated by $\mathfrak{m}^5$ and

$$g_1 := x^2 + a_1 z^3 + b_1 z^4, \qquad g_2 := xy + a_2 z^3 + b_2 z^4,$$
$$g_3 := xz + a_3 z^3 + b_3 z^4, \qquad g_4 := q + a_4 z^3 + b_4 z^4, \qquad g_5 := c + b_5 z^4.$$

The a_i, b_i are not necessarily free. We must impose additional conditions to ensure the initial ideal for the weight vector $(-1, -1, -1)$ is no larger than I. In particular, we must have

$$zg_1 - xg_3 = a_1 z^4 + b_1 z^5 - a_3 x z^3 - b_3 x z^4 \in J,$$
$$zg_2 - yg_3 = a_2 z^4 + b_2 z^5 - a_3 y z^3 - b_3 y z^4 \in J.$$

This implies $a_1 = a_2 = 0$, because the final three terms of each expression are already in J. By Buchberger's criterion, it is also sufficient for these conditions to be satisfied. Therefore the fibers are 7-dimensional. □

Lemma 4.8. *The preimage $U := \pi^{-1}(\mathcal{U})$ is irreducible of dimension* 12.

Proof. By Proposition 3.8, it suffices to show that the fibers of π over $\mathcal{U}$ are irreducible of dimension 6. Let I be an ideal in $\mathcal{U}$. Let V be the 1-dimensional subspace of S_1^* such that I_3 is orthogonal to $\mathrm{Sym}_3(V)$ and let $q_1, \dots, q_4$ be the degree 2 generators of I. Choose a basis x, y, z of S_1 such that x, y is a basis for $V^\perp$. Then any J in $\phi^{-1}(I)$ is of the form

$$\langle q_i + a_i z^3 + b_i z^4 \mid 1 \le i \le 4\rangle + \mathfrak{m}^5.$$

We claim that forcing $\mathrm{in}_{(-1,-1,-1)}(J) = I$ imposes two linear conditions on the a_is. Using the table of isomorphism classes of $(1, 3, 2)$ algebras in [Poonen 2008], one can check that for any 4-dimensional subspace $\langle q_1, q_2, q_3, q_4\rangle$ of $\mathrm{Sym}_2(V)$, the intersection of $\langle zq_i \mid 1 \le i \le 4\rangle_3$ and $\langle xq_j, yq_j \mid 1 \le j \le 4\rangle_3$ is 2-dimensional. After choosing a different basis for Q, we may assume $zq_1, zq_2 \in \langle xq_j, yq_j \mid 1 \le j \le 4\rangle$. By using a similar argument to the one in Lemma 4.7, we see $a_1 = a_2 = 0$. Since the only other linear syzygies among the q_i's have no z coefficients and xz^3, yz^3 and $\mathfrak{m}^5$ are in the ideal, these are the only conditions imposed. Therefore, the fiber is 6-dimensional. □

Therefore, it suffices to show the following irreducible sets are contained in the smoothable component.

$$H^d_{(1,d,1,\dots,1)},\ H^d_{(1,d,2)},\ P_d,\ Q_d,\ H^d_{(1,d,2,2)},\ H^3_{(1,3,4)},\ H^3_{(1,3,3)},\ H^3_{(1,3,3,1)},\ U,\ Z.$$

Smoothable algebras. In this section we prove that the irreducible sets

$$H^d_{(1,d,1,\dots,1)},\ H^d_{(1,d,2)},\ P_d,\ Q_d,\ H^d_{(1,d,2,2)},\ H^3_{(1,3,4)},$$

are in the smoothable component by showing that a generic algebra in each is smoothable. We then show that the remaining irreducible sets,

$$H^3_{(1,3,3)},\ H^3_{(1,3,3,1)},\ U,\ Z,$$

are in the smoothable component by finding a point in each which is smoothable and a smooth point on the Hilbert scheme.

Proposition 4.9. *All algebras in $H^d_{(1,d,1,\dots,1)}$ are smoothable.*

Proof. We prove this by induction on d. Note the $d = 1$ case is trivial. Let m be the greatest integer such that h_m is nonzero. Then, by Proposition 4.4 we can take a generic ideal to be

$$I = \langle x_1^2 - x_d^m, \dots, x_{d-1}^2 - x_d^m, x_d^{m+1}\rangle + \langle x_i x_j \mid i \ne j\rangle.$$

We define J to be

$$J = \langle x_1^2 + x_1 - x_d^m, \dots, x_{d-1}^2 - x_d^m, x_d^{m+1} \rangle + \langle x_i x_j \mid i \neq j \rangle.$$

Note that J admits a decomposition as $J = J_1 \cap J_2$ where $J_1 = \langle x_1 + 1, x_2, x_3, \dots x_d \rangle$ and

$$J_2 = \langle x_1 - x_d^m, x_2^2 - x_d^m, \dots, x_{d-1}^2 - x_d^m, x_d^{m+1} \rangle + \langle x_i x_j \mid i \neq j \rangle.$$

As the Hilbert function of J_2 equals $(1, d-1, 1, \dots, 1)$, the inductive hypothesis implies that J_2 is smoothable. Thus J itself is also smoothable. Next note that $I \subset \mathrm{in}_{(m,\dots,m,2)}(J)$. Since both I and J have the same colength, we obtain the equality $I = \mathrm{in}_{(m,\dots,m,2)}(J)$. The corresponding Gröbner degeneration induces a map $\mathbb{A}^1 \to R_n^d$ which sends 0 to I. Thus I is smoothable. $\square$

Proposition 4.10. *All algebras in $H^d_{(1,d,2)}$ are smoothable.*

Proof. The proof is by induction on d. The case $d = 2$ follows from Theorem 2.4 of [Fogarty 1968].

Assume d is at least 3. Note that $I_2^\perp$ defines a pencils of quadrics in d-variables. It then follows from [Harris 1992, Lemma 22.42] that, up to isomorphism, a generic ideal in $H^d_{(1,d,2)}$ is of the form

$$I = \langle x_i x_j \mid i \neq j \rangle + \langle x_i^2 - a_i x_{d-1}^2 - b_i x_d^2 \mid 1 \leq i \leq d-2 \rangle,$$

with a_i and b_i elements of k.

Define

$$J_1 = \langle x_i x_j \mid i \neq j \rangle + \langle x_1 + a_1 x_{n-1}^2 + b_1 x_d^2, x_i^2 - a_i x_{d-1}^2 - b_i x_d^2 \mid 2 \leq i \leq d-2 \rangle,$$
$$J_2 = \langle x_1 - 1, x_2, \dots, x_d \rangle.$$

Since J_1 has Hilbert function $(1, d-1, 2)$, it is thus smoothable by the induction hypothesis. One can check that $I = \mathrm{in}_{(1,\dots,1)}(J_1 \cap J_2)$. Therefore I is smoothable. $\square$

Proposition 4.11. *All algebras in P_d are smoothable.*

Proof. Let I be a generic ideal in P_d. After a change of variables we may assume

$$I = \langle x_i x_j, x_\ell^2 + x_1^3, x_1^3 - x_2^3 \mid 1 \leq i < j \leq d, 2 < \ell \leq d \rangle.$$

One can check

$$I = \mathrm{in}_{(2,2,3,\dots,3)} \big(\langle x_1 + 1, x_j \mid j > 1 \rangle \cap \langle x_i x_j, x_\ell^2 + x_1^2, x_1^2 - x_2^3 \mid 1 \leq i < j \leq d, 2 < \ell \leq d \rangle \big).$$

The second ideal in the intersection has Hilbert function $(1, d, 1, 1)$, hence is smoothable by Proposition 4.9. It follows that I is smoothable. $\square$

Proposition 4.12. *All algebras in Q_d are smoothable.*

Proof. Let I be a generic ideal in Q_d. After a change of variables, we may assume

$$I = \langle x_1 x_\ell, x_i x_j + b_{(i,j)} x_1^3, x_k^2 - x_{k+1}^2 + b_k x_1^3 \mid \ell \neq 1, 1 < i < j \leq d, 1 < k < d \rangle.$$

Define

$$J_1 = \langle x_1 x_\ell, x_i x_j + b_{(i,j)} x_1^2, x_k^2 - x_{k+1}^2 + b_k x_1^2 \mid \ell \neq 1, 1 < i < j \leq d, 1 < k < d \rangle,$$
$$J_2 = \langle x_1 + 1, x_2, \dots, x_d \rangle.$$

Then J_1 has Hilbert function $(1, d, 2)$ so is smoothable by Proposition 4.10. One can check that $I = \mathrm{in}_{(2,3,\dots,3)}(J_1 \cap J_2)$, and thus I is smoothable. □

Proposition 4.13. *All algebras in $H^d_{(1,d,2,2)}$ are smoothable.*

Proof. Let I be a generic ideal with Hilbert function $(1, d, 2, 2)$. After a change of variable, we may assume $(\pi(I))_2^\perp = \langle y_1^2, y_2^2 \rangle$. Thus I must be of the form

$$\langle x_\ell^2 - a_{\ell\ell} x_1^3 - b_{\ell\ell} x_2^3, x_i x_j - a_{ij} x_1^3 - b_{ij} x_2^3 \mid i < j, 2 < \ell \rangle + \mathfrak{m}^4.$$

Note I determines a symmetric bilinear map

$$\begin{array}{rcl} \phi : (\mathfrak{m} : \mathfrak{m}^3)/\mathfrak{m}^2 \times (\mathfrak{m} : \mathfrak{m}^3)/\mathfrak{m}^2 & \to & \mathfrak{m}^3 \cong k^2, \\ (x_i, x_j) & \mapsto & a_{ij} x_1^3 + b_{ij} x_2^3. \end{array}$$

By composing ϕ with projections onto the two coordinates, we get a pair of symmetric bilinear forms. For a generic ϕ, these are linearly independent and their span is invariant under a change of basis on $\mathfrak{m}^3$. By [Harris 1992, Lemma 22.42], there exists a basis for $(\mathfrak{m} : \mathfrak{m}^3)/\mathfrak{m}^2$ and $\mathfrak{m}^3$ such that these bilinear forms are represented by diagonal matrices. Thus I has the following form

$$\langle x_\ell^2 - a_\ell x_1^3 - b_\ell x_2^3, x_i x_j - a_{ij} x_1^3 - b_{ij} x_2^3 \mid i < j, 2 < \ell \rangle + \mathfrak{m}^4,$$

where $a_{ij} = b_{ij} = 0$ if i and j are both greater than 2 and a_ℓ, b_ℓ are nonzero for all $\ell > 2$. After suitable changes of variable, we may assume $b_{ij} = a_{ij} = 0$ for all i, j. This gives the ideal

$$I = \langle x_\ell^2 - a_\ell x_1^3 - b_\ell x_2^3, x_i x_j, x_1^4, x_2^4 \mid i < j, 2 < \ell \rangle.$$

Now consider the following ideals:

$$J_1 := \langle x_\ell^2 - a_\ell x_1^2 - b_\ell x_2^3, x_i x_j, x_1^3, x_2^4 \mid i < j, 2 < \ell \rangle,$$
$$J_2 := \langle x_1 + 1, x_2, \dots, x_d \rangle.$$

Note J_1 is a $(1, d, 2, 1)$ ideal and in fact lies in the component Q_d, and therefore is smoothable by Proposition 4.12. One can check that $I = \mathrm{in}_{(2,2,3,\dots,3)}(J_1 \cap J_2)$, and therefore I is smoothable. □

Proposition 4.14. *All algebras in $H^3_{(1,3,4)}$ are smoothable.*

Proof. Such algebras are given by a 2-dimensional subspace of the space of quadratic forms, with isomorphisms given by the action of GL_3. Arguing as in Proposition 4.10, we conclude that, up to isomorphism, a generic 2-dimensional space of quadrics is spanned by x^2+z^2 and y^2+z^2. Adding the necessary cubic generators, we conclude that $I = \langle y^2+z^2, x^2+z^2, z^3, yz^2, xz^2, xyz\rangle$ is a generic point of $H^3_{(1,3,4)}$.

Consider

$$J = \langle y^2+z^2, x+x^2+z^2, z^3, yz^2, xz^2, xyz\rangle.$$

Note that J is the intersection of an ideal of colength 3 and an ideal of colength 5:

$$J = \langle x+1, y^2, yz, z^2\rangle \cap \langle x+z^2, y^2+z^2, z^3, yz^2\rangle.$$

Since both ideals in the above intersection are smoothable, J itself is smoothable. One can check that $I = \mathrm{in}_{(1,1,1)}(J)$. Therefore I is smoothable. □

To prove that the remaining families are smoothable, we show that they contain a smooth point which is also smoothable. For this, we use the following result, which is well known (see for example [Miller and Sturmfels 2005, Lemma 18.10] in characteristic 0), but we give the proof in arbitrary characteristic for the reader's convenience:

Proposition 4.15. *All monomial ideals are smoothable.*

Proof. Suppose we have a monomial ideal of colength n, written in multiindex notation $I = \langle \boldsymbol{x}^{\alpha^{(1)}}, \ldots, \boldsymbol{x}^{\alpha^{(m)}}\rangle$. Since k is algebraically closed, we can pick an arbitrarily long sequence $a_1, a_2, \ldots$ consisting of distinct elements in k. Define

$$f_i = \prod_{j=1}^{d} \big((x_j - a_1)(x_j - a_2)\cdots(x_j - a_{\alpha_j^{(i)}})\big).$$

Note that $\mathrm{in}(f_i) = \boldsymbol{x}^{\alpha^{(i)}}$ with respect to any global term order. Let J be the ideal generated by the f_i for $1 \le i \le m$ and then $\mathrm{in}(J) \supset I$ and so J has colength at most n. However, for any standard monomial $\boldsymbol{x}^{\beta}$ in I, we have a distinct point $(a_{\beta_1}, \ldots, a_{\beta_d})$ in $\mathbb{A}^d$, and each f_i vanishes at this point. Therefore, J must be the radical ideal vanishing at exactly these points and have initial ideal I. Thus, I is smoothable. □

Furthermore, the tangent space of an ideal I in the Hilbert scheme is isomorphic to $\mathrm{Hom}_S(I, S/I)$. We use this fact to compute the dimension of the tangent space of a point I.

Proposition 4.16. *All algebras in $H^3_{(1,3,3)}$ are smoothable.*

Proof. This irreducible set includes the smoothable monomial ideal I generated by x^2, y^2, z^2, xyz. A direct computation shows I has a 21-dimensional tangent space, so I is a smooth point in H^3_7. Thus, any algebra in $H^3_{(1,3,3)}$ is smoothable. □

Proposition 4.17. *All algebras in $H^3_{(1,3,3,1)}$ are smoothable.*

Proof. The ideal $I = \langle x^2, y^2, z^2\rangle$ in this locus is smoothable by Proposition 4.15, and one can check that the Hilbert scheme is smooth at this point as well. Therefore $H^3_{(1,3,3,1)}$ is contained in the smoothable component of the Hilbert scheme. □

Proposition 4.18. *All algebras in Z are smoothable.*

Proof. Consider $I = \langle x^2, xy, xz, yz, z^3 - y^4\rangle \in Z$ and note that

$$I = \mathrm{in}_{(1,0,0)}\left(\langle x+1, y, z\rangle \cap \langle x, yz, z^3 - y^4\rangle\right).$$

The second ideal is smoothable [Fogarty 1968, Theorem 2.4], so I is smoothable. One can also check I is smooth in the Hilbert scheme by computing the dimension of $\mathrm{Hom}_S(I, S/I)$. Therefore Z is contained in the smoothable component of the Hilbert scheme. □

Proposition 4.19. *All algebras in $U \subset H^3_{(1,3,2,1,1)}$ are smoothable.*

Proof. Consider the ideal $I = \langle x^2, xy - z^4, y^2 - xz, yz\rangle \in U$. One can check that

$$I = \mathrm{in}_{(7,5,3)}\left(\langle x, y, z-1\rangle \cap \langle x^2, xy - z^3, y^2 - xz, yz\rangle\right).$$

The second ideal in the intersection is in Q_3 and therefore smoothable by Proposition 4.12. Therefore I is smoothable by the same argument in the proof of Proposition 4.14. One can also check I has a 24-dimensional tangent space in the Hilbert scheme and is thus smooth. Therefore U is contained in the smoothable component. □

Theorem 4.20. *With the exception of local algebras with Hilbert function* $(1, 4, 3)$, *every algebra with $n \leq 8$ is smoothable.*

Proof. The possible Hilbert functions are exactly the Hilbert functions of monomial ideals, and for d at least 3, one can check that there are no possibilities other those listed in Table 4.1. For d at most 2, smoothability follows from Theorem 2.4 of [Fogarty 1968] □

In particular, this implies that there are no components other than the ones listed in Theorem 1.1.

degree	Hilbert function $\boldsymbol{h}$	$\mathcal{H}_{\boldsymbol{h}}^d$ c.d.	reference	$H_{\boldsymbol{h}}^d$ c.d.	smoothability reference
4	1, 3	0	Prop. 3.3	0	Prop. 4.15
5	1, 3, 1	5	Prop. 3.3	5	Prop. 4.9
	1, 4	0	Prop. 3.3	0	Prop. 4.15
6	1, 3, 1, 1	2	Prop. 3.4	7	Prop. 4.10
	1, 4, 1	9	Prop. 3.3	9	Prop. 4.9
	1, 5	0	Prop. 3.3	0	Prop. 4.15
7	1, 3, 1, 1, 1	2	Prop. 3.4	9	Prop. 4.9
	1, 3, 2, 1	5, 6	Thm. 3.5	9, 10	Prop. 4.12, 4.11
	1, 3, 3	9	Prop. 3.3	9	Prop. 4.16
	1, 4, 1, 1	3	Prop. 3.4	12	Prop. 4.9
	1, 4, 2	16	Prop. 3.3	16	Prop. 4.10
	1, 5, 1	14	Prop. 3.3	14	Prop. 4.9
	1, 6	0	Prop. 3.3	0	Prop. 4.15
8	1, 3, 1, 1, 1, 1	2	Prop. 3.4	11	Prop. 4.9
	1, 3, 2, 1, 1	6	Prop. 3.8	11(?), 12	Prop. 4.19, 4.18
	1, 3, 2, 2	4	Prop. 3.6	12	Prop. 4.13
	1, 3, 3, 1	9	Prop. 3.7	12	Prop. 4.17
	1, 3, 4	8	Prop. 3.3	8	Prop. 4.14
	1, 4, 1, 1, 1	3	Prop. 3.4	15	Prop. 4.9
	1, 4, 2, 1	7, 11	Thm. 3.5	15, 19	Prop. 4.12, 4.11
	1, 4, 3	21	Prop. 3.3	21	*
	1, 5, 2	26	Prop. 3.3	26	Prop. 4.10
	1, 5, 1, 1	4	Prop. 3.4	18	Prop. 4.9
	1, 6, 1	20	Prop. 3.3	20	Prop. 4.9
	1, 7	0	Prop. 3.3	0	Prop. 4.15

Table 4.1. Summary of the decomposition of Hilbert schemes by Hilbert function of the local algebra with $h_1 \geq 3$. Here c.d. stands for "component dimensions". The component dimensions of $H_{\boldsymbol{h}}^d$ are computed using Propositions 4.3 and 4.4. In the case of $\boldsymbol{h} = (1, 3, 2, 1, 1)$, Lemmas 4.7 and 4.8 show that $H_{\boldsymbol{h}}^d$ is the union of two irreducible sets, but we don't know whether the smaller set is contained in the closure of the larger one.

5. Characterization of smoothable points of H_8^4

In this section k will denote a field of characteristic not 2 or 3, except for Section 5D where $k = \mathbb{C}$.

We show that besides the smoothable component, the Hilbert scheme H_8^4 contains a second component parametrizing the local algebras with $\boldsymbol{h} = (1, 4, 3)$. We prove that the intersection of the two components can be described as in Theorem 1.3, and as a result we determine exactly which algebras with Hilbert function $(1, 4, 3)$ are smoothable. We will use G_0 to denote the standard graded Hilbert scheme $\mathscr{H}^4_{(1,4,3)} \cong \mathrm{Gr}(7, S_2)$, which we will think of as a closed subscheme of H^4_8.

In Section 5B we introduce and investigate the Pfaffian which appears in Theorem 1.3, and we prove the crucial fact that it is the unique GL_4-invariant of minimal degree. In Section 5C, we give a first approximation of the intersection locus. We then use the uniqueness results from Section 5B to prove Theorem 1.3. We begin by proving reducibility,

Proposition 5.1. *For d at least 4, the Hilbert scheme H^d_8 is reducible.*

Proof. It is sufficient to find a single ideal whose tangent space dimension is less than $8d = \dim R^d_8$. Consider the ideal

$$J = \langle x_1^2, x_1x_2, x_2^2, x_3^2, x_3x_4, x_4^2, x_1x_4 + x_2x_3\rangle + \langle x_i \mid 4 < i \le d\rangle.$$

The tangent space of J in H^d_8 can be computed as $\dim_k \mathrm{Hom}_S(J, S/J)$. A direct computation shows that an arbitrary element of $\mathrm{Hom}_S(J, S/J)$ can be represented as a matrix

$$\begin{array}{c} \\ 1 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_1x_3 \\ x_1x_4 \\ x_2x_4 \end{array} \begin{array}{c} \begin{array}{cccccccc} x_1^2 & x_1x_2 & x_2^2 & x_3^2 & x_3x_4 & x_4^2 & x_1x_4{+}x_2x_3 & x_i \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & * \\ 2a_2 & a_1 & 0 & 0 & 0 & 0 & a_4 & * \\ 0 & a_2 & 2a_1 & 0 & 0 & 0 & a_3 & * \\ 0 & 0 & 0 & 2a_3 & a_4 & 0 & a_1 & * \\ 0 & 0 & 0 & 0 & a_3 & 2a_4 & a_2 & * \\ * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \end{pmatrix} \end{array},$$

where i again ranges over $4 < i \le d$, the a_i are any elements in k, and each $*$ represents an independent choice of an element of k. Thus, $\dim_k \mathrm{Hom}(J, S/J) = 4 + 21 + 8(d - 4) = 8d - 7$. The computation holds in all characteristics. Since $8d - 7 < 8d = \dim(R^d_8)$, we conclude that J is not smoothable and that H^d_8 is reducible. □

Remark 5.2. This proposition holds with the same proof even when $\mathrm{char}\, k = 2, 3$.

5A. *The irreducible components of H^4_8.* Consider H^4_8 with its universal ideal sheaf $\mathscr{I}$ and let $\mathscr{A} = \mathscr{O}_{H^4_8}[x_1, \dots, x_4]/\mathscr{I}$. On each open affine $U = \operatorname{Spec} B$ such that $\mathscr{A}|_U$ is free, define $f_i \in B$ to be $\frac{1}{8}\operatorname{tr}(X_i)$ where X_i is the operator on the free B-module $\mathscr{A}(U)$ defined by multiplication by x_i. We think of the f_i as being the "center of mass" functions for the subscheme of $\mathbb{A}^4_B$ defined by $\mathscr{I}|_U$. Note that the definitions of f_i commute with localization and thus they lift to define elements $f_i \in \Gamma(H_8^4, \mathcal{O}_{H_8^4})$, which determine a morphism $f : H_8^4 \to \mathbb{A}^4$.

Considered as an additive group, $\mathbb{A}^4$ acts on H_8^4 by translation. We define the "recentering" map r to be the composition

$$r : H_8^4 \xrightarrow{-f \times \mathrm{id}} \mathbb{A}^4 \times H_8^4 \longrightarrow H_8^4.$$

By forgetting about the grading of ideals, we have a closed immersion ι of $G_0 \cong \mathcal{H}^4_{(1,4,3)}$ into H_8^4 [Haiman and Sturmfels 2004, Proposition 1.5]. We define G to be the preimage of this closed subscheme via the "recentering" map, that is, the fiber product

$$\begin{tikzcd} G \arrow[r] \arrow[d] & H_8^4 \arrow[d, "r"] \\ G_0 \arrow[r, "\iota"] & H_8^4 \end{tikzcd}$$

We define intersections $W := G \cap R_8^4$ and $W_0 := G_0 \cap R_8^4$. We will focus on W_0, and the following lemma shows that this is sufficient for describing W.

Lemma 5.3. *We have isomorphisms $G \cong G_0 \times \mathbb{A}^4$ and $W \cong W_0 \times \mathbb{A}^4$.*

Proof. We have a map $G \to G_0$, and a map $f : H_8^4 \to \mathbb{A}^4$. We claim that the induced map $\phi : G \to \mathbb{A}^4 \times G_0$ is an isomorphism.

Define $\psi : \mathbb{A}^4 \times G_0 \to H_8^4$ to be the closed immersion ι followed by translation. We work with an open affine $U \cong \operatorname{Spec} A \subset G_0$ such that the restriction $\iota|_U$ corresponds to an ideal $I \subset A[x_1, \dots, x_4]$ whose cokernel is a graded A-module with free components of ranks $(1, 4, 3)$. The map $\psi|_{\mathbb{A}^4 \times U}$ corresponds to the ideal $I' \subset A[t_1, \dots, t_4][x'_1, \dots, x'_4]$ where I' is the image of I under the homomorphism of A-algebras that sends x_i to $x'_i + t_i$. Then $A[t_1, \dots, t_4][x'_1, \dots, x'_4]/I' \cong (A[x_1, \dots, x_4]/I)[t_1, \dots, t_4]$ is a graded $A[t_1, \dots, t_4]$-algebra with $x'_i - t_i = x_i$ homogeneous of degree 1. The key point is that as operators on a free $A[t_1, \dots, t_4]$-module, the x_i have trace zero, so the trace of the x'_i is $8t_i$. Thus, $r \circ \psi : \mathbb{A}^4 \times G_0 \to H_8^4$ corresponds to an ideal $I'' \subset A[t_1, \dots, t_4][x''_1, \dots, x''_4]$ which is the image of I' under the homomorphism that takes x'_i to $x''_i - t_i$. This is of course the extension of $I \subset A[x_1, \dots, x_4]$ to $A[t_1, \dots, t_4][x''_1, \dots, x''_4]$ with $x''_i = x_i$, and so I'' has the required properties such that $r \circ \psi$ factors through the closed immersion ι. Thus, ψ maps to G. Furthermore, we see that $r \circ \psi$ is the projection onto the first coordinate of $\mathbb{A}^4 \times G_0$ and $f \circ \psi$ is projection onto the second coordinate. Thus, $\phi \circ \psi$ is the identity.

Second, we check that the composition $\psi \circ \phi$ is the identity on G. This is clear because $\psi \circ \phi$ amounts to translation by $-f$ followed by translation by f.

The isomorphism for W follows by the same argument. $\square$

Lemma 5.4. *W and W_0 are prime divisors in G and G_0 respectively.*

Proof. The point $I = \langle x_1^2, x_1x_2, x_2^2, x_3^2, x_3x_4, x_4^2, x_1x_4\rangle$ belongs to R_8^4 and to G and has a 33-dimensional tangent space in any characteristic. As a result, an open set U around I in the Hilbert scheme is a closed subscheme of a smooth 33-dimensional variety Y. By the subadditivity of codimension of intersections, as in [Harris 1992, Theorem 17.24], it follows that every component of W through I has codimension 1 in G.

To show integrality, fix a monomial ideal $M_\lambda \in G_0$, and let $U_\lambda \subset H_8^4$ be the corresponding open set, as in our discussion of coordinates for the Hilbert scheme of points (page 766). For any $I \in U_\lambda$ the initial ideal $\mathrm{in}_{(1,1,1,1)}(I) \in G_0 \cap U_\lambda$ and is generated by the $(1, 1, 1, 1)$-leading forms of the given generating set of I and all cubics. Thus we may define a projection morphism $\pi : U_\lambda \to G_0 \cap U_\lambda$ which corresponds to taking the $(1, 1, 1, 1)$-initial ideal. Since R_8^4 is integral, so is the image $\pi(R_8^4 \cap U_\lambda) = W_0 \cap U_\lambda$. Thus W_0 and $W_0 \times \mathbb{A}^4 \cong W$ are integral. □

5B. *A* GL$_4$*-invariant of a system of three quadrics.* In this section we study the Pfaffian which appears in the statement of Theorem 1.3. Recall that any $I \in G_0$ defines a 3-dimensional subspace $I_2^\perp \subset S_2^*$.

Let Q_1, Q_2, Q_3 be a basis of quadrics for $I_2^\perp$, let A_1, A_2, A_3 be the symmetric 4×4 matrices which represent the Q_i via $\vec{y}^t A_i \vec{y} = Q_i$ where $\vec{y}$ is the vector of formal variables (y_1, y_2, y_3, y_4).

Definition 5.5. *The* Salmon–Turnbull Pfaffian *is the Pfaffian* (*that is*, *the square root of the determinant*) *of the skew-symmetric* 12×12 *matrix*

$$M_I = \begin{pmatrix} 0 & A_1 & -A_2 \\ -A_1 & 0 & A_3 \\ A_2 & -A_3 & 0 \end{pmatrix}.$$

Lemma 5.6. *The Salmon–Turnbull Pfaffian of M_I coincides up to scaling with the Pfaffian of the skew-symmetric bilinear form $\langle\ ,\ \rangle_I : (S_1 \otimes S_2/I_2)^{\otimes 2} \to \bigwedge^3 S_2/I_2 \cong k$ defined by*

$$\langle l_1 \otimes q_1, l_2 \otimes q_2\rangle_I = (l_1 l_2) \wedge q_1 \wedge q_2.$$

In particular, *the vanishing of the Salmon–Turnbull Pfaffian is independent of the choice of basis of $I_2^\perp$ and invariant under the* GL$_4$ *action induced by linear change of coordinates on S.*

Proof. Let m_1, m_2, m_3 be any basis of S_2/I_2 and let x_1, x_2, x_3, x_4 be a basis for S_1. Let A_i be the matrix representation with respect to this basis of the symmetric bilinear form obtained by composing multiplication $S_1 \otimes S_1 \to S_2/I_2$ with projection onto m_i. Note that if m_1, m_2, m_3 form a basis dual to $\frac{1}{2}Q_1, \frac{1}{2}Q_2, \frac{1}{2}Q_3$ then this definition of A_i agrees with the definition of A_i above. Thus, $x_j x_{j'} =$

$\sum_i (A_i)_{jj'} m_i$ where $(A_i)_{jj'}$ is the (j, j') entry of A_i. Then we will use the basis $x_1 \otimes m_3, x_2 \otimes m_3, \dots, x_4 \otimes m_1$ for $S_1 \otimes S_2/I_2$. We compute the matrix representation of $\langle\, , \rangle_I$ in this basis:

$$\begin{aligned}\langle x_j \otimes m_i, x_{j'} \otimes m_{i'} \rangle_I &= (x_j x_{j'}) \wedge m_i \wedge m_{i'} \\ &= \Bigg(\sum_{1 \le \ell \le 3} (A_\ell)_{jj'} m_\ell \Bigg) \wedge m_i \wedge m_i'\end{aligned}$$

If $i = i'$, this quantity will be zero. Otherwise, let i'' be the index which is not i or i' and then we get

$$= (A_{i''})_{jj'} m_{i''} \wedge m_i \wedge m_{i'} = \pm (A_{i''})_{jj'} m_1 \wedge m_2 \wedge m_3,$$

where $\pm$ is the sign of the permutation which sends $1, 2, 3$ to i'', i, i' respectively. Thus, with $m_1 \wedge m_2 \wedge m_3$ as the basis for $\bigwedge^3 S_2/I_2$, $\langle\, , \rangle_I$ is represented as

$$\begin{pmatrix} 0 & A_1 & -A_2 \\ -A_1 & 0 & A_3 \\ A_2 & -A_3 & 0 \end{pmatrix}. \qquad \square$$

Since the vanishing of the Salmon–Turnbull Pfaffian depends only on the vector subspace $I_2^\perp \subset S_2^*$, it defines a function P on G_0 which is homogeneous of degree 2 in the Plücker coordinates. We next show that the Salmon–Turnbull Pfaffian is irreducible and that, over the complex numbers, it is uniquely determined by its degree and GL_4-invariance.

Lemma 5.7. *There are no polynomials of degree* 1 *in the Plücker coordinates of* G_0 *whose vanishing locus is invariant under the action of the algebraic group* GL_4. *Therefore, the Salmon–Turnbull Pfaffian is irreducible.*

Proof. We may prove this lemma by passing to the algebraic closure, and we thus assume that k is algebraically closed. Let $W = \bigwedge^3 S_2^*$ and consider the Plücker embedding of $\mathrm{Gr}(3, S_2^*)$ in $\mathbb{P}(W) = \mathrm{Proj}(R)$ where R is the polynomial ring $k[p_{ij\ell}]$ where $\{i, j, \ell\}$ runs over all unordered triplets of monomials in S_2^*. The Plücker coordinate ring A is the quotient of R by a homogeneous ideal J. In each degree e, we obtain a split exact sequence of GL_4-representations:

$$0 \to J_e \to \mathrm{Sym}_e(W) \to A_e \to 0.$$

Since $J_1 = 0$ we have $\mathrm{Sym}_1(W) = A_1$, and it suffices to show that this has no 1-dimensional subrepresentations. Given a monomial $i \in S_2^*$ let $\alpha_i \in \mathbb{N}^4$ be its multiindex. For $\theta = (\theta_1, \dots, \theta_4)$, let L be the diagonal matrix with $L_{mm} = \theta_m$. The action of L on the Plücker coordinate $p_{ij\ell}$ is to scale it by $\theta^{\alpha_i + \alpha_j + \alpha_\ell}$.

Suppose that there exists an invariant polynomial $F=\sum c_{ij\ell}p_{ij\ell}$ in $\mathrm{Sym}_1(W)$. Then $L\cdot F=\lambda F$ for some $\lambda\in k^\times$. But since $L\cdot F=\sum c_{ij\ell}\theta^{\alpha_i+\alpha_j+\alpha_\ell}p_{ij\ell}$ it follows that whenever $c_{ij\ell}$ and $c_{i'j'\ell'}$ are both nonzero, then $\alpha_i+\alpha_j+\alpha_\ell=\alpha_{i'}+\alpha_{j'}+\alpha_{\ell'}$. However there are no multiindices of total degree 6 which are also symmetric in $\theta_1,\theta_2,\theta_3,\theta_4$. Thus each $c_{ij\ell}=0$ and there are no nontrivial GL_4-invariant polynomials of degree 1. In particular no product of linear polynomials is GL_4-invariant, and thus the Salmon–Turnbull Pfaffian is irreducible. □

Lemma 5.8. *If $k=\mathbb{C}$, then there is a polynomial of degree 2 in the Plücker coordinates, unique up to scaling, whose vanishing locus is GL_4-invariant. This is the Salmon–Turnbull Pfaffian.*

Proof. We take the same notation as in the proof of Lemma 5.7 and recall that we have a split exact sequence of $\mathrm{GL}_4(\mathbb{C})$-representations:

$$0\to J_2\to \mathrm{Sym}_2(W)\to A_2\to 0.$$

We determine the irreducible subrepresentations of $\mathrm{Sym}_2(W)$ by computing the following Schur function decomposition of its character χ:

$$\begin{aligned}\chi=s_{(8,2,2)}+s_{(7,4,1)}+2s_{(7,3,1,1)}+s_{(7,2,1,1)}+s_{(6,6)}+3s_{(6,4,2)}+s_{(6,4,1,1)}\\ +2s_{(6,4,1,1)}+2s_{(6,3,2,1)}+s_{(6,2,2,2)}+2s_{(5,5,1,1)}+s_{(5,4,3)}+s_{(5,4,2,1)}+s_{(5,3,3,1)}\\ +s_{(4,4,4)}+s_{(4,4,3,1)}+2s_{(4,4,2,2)}+s_{(3,3,3,3)}.\end{aligned}$$

We conclude that $\mathrm{Sym}_2(W)$ contains a unique 1-dimensional subrepresentation with character $s_{(3,3,3,3)}$. It follows from this and Lemma 5.6 that, over $\mathbb{C}$, the Salmon–Turnbull Pfaffian is the only GL_4-invariant of degree 2 in the Plücker coordinates. □

Remark 5.9. Salmon gives a geometric description of the Salmon–Turnbull Pfaffian [Salmon 1874, pp. 242–244], where he shows that the Pfaffian vanishes whenever there exists a cubic form C and three linear differential operators d_1,d_2,d_3 such that $d_iC=Q_i$. Turnbull also describes this invariant in his study of ternary quadrics [Turnbull 1922].

5C. ***A first approximation to the intersection locus.*** Any point I in W_0 is a singular point in the Hilbert scheme. In Lemma 5.11, we construct an equation that cuts out the singular locus over an open set of G_0. The local equation defines a nonreduced divisor whose support contains W_0. The following subsets of G_0 will be used in this section:

$$\begin{aligned}G_0'&:=\{I\in G_0\mid \text{the ideal } I \text{ is generated in degree } 2\},\\ Z_1&:=G_0\setminus G_0',\\ Z_2&:=\{I\in G_0\mid \mathrm{Hom}(I,S/I)_{-2}\neq 0\}.\end{aligned}$$

Note that G'_0 is open in G_0 and that every ideal in G'_0 is generated by seven quadrics. The set Z_2 will be used in Lemma 5.16. If I is any ideal in G_0, the tangent space $\mathrm{Hom}_S(I, S/I)$ is graded. The following lemma shows that if we want to determine whether I is a singular point in the Hilbert scheme, then it suffices to compute only the degree -1 component of the tangent space.

Lemma 5.10. *For any $I \in G'_0$ we have* $\dim_k \mathrm{Hom}_S(I, S/I)_{-1} \geq 4$, *and I is singular in H_8^4 if and only if* $\dim_k \mathrm{Hom}_S(I, S/I)_{-1} \geq 5$.

Proof. Since S/I is concentrated in degrees 0, 1 and 2, and $I \in G'_0$ is minimally generated only in degree 2, we have that $\mathrm{Hom}_S(I, S/I)$ is concentrated in degrees $0, -1, -2$. Furthermore, since $I \in G'_0$ we have that $\dim_k \mathrm{Hom}_S(I, S/I)_0 = 21$, because any k-linear map $I_2 \to (S/I)_2$ will be S-linear. Next, note that the morphisms $t_i : I_2 \to (S/I)_1$ mapping q_j to the class of $\partial q_j / \partial x_i$ are S-linear morphisms, and thus we have $\mathrm{Hom}_S(I, S/I)_{-1}$ is at least 4-dimensional.

Since the dimension of G'_0 is 25, I must be singular if $\dim_k \mathrm{Hom}_S(I, S/I)_{-1} > 4$. Conversely, assume for contradiction that there exists an I such that I is singular and dimension of $\mathrm{Hom}_S(I, S/I)_{-1}$ is exactly 4. Since I is singular, we have that $\mathrm{Hom}_S(I, S/I)_{-2}$ is nontrivial. Let $\phi \in \mathrm{Hom}_S(I, S/I)_{-2}$ be a nonzero map. By changing the generators of I we may assume that $\phi(q_i) = 0$ for $i = 1, \ldots, 6$ and $\phi(q_7) = 1$. Now the vector space $\langle x_1\phi, x_2\phi, x_3\phi, x_4\phi \rangle$ is a 4-dimensional subspace of $\mathrm{Hom}(I, S/I)_{-1}$. Since we have assumed that $\dim_k \mathrm{Hom}(I, S/I)_{-1} = 4$ it must be the case that the space $\langle x_1\phi, x_2\phi, x_3\phi, x_4\phi \rangle$ equals the space $\langle t_1, \ldots, t_4 \rangle$. However, this would imply that all partial derivatives of q_1 are zero, which is impossible. $\square$

Now we will investigate those ideals which have extra tangent vectors in degree -1. If $\phi : I_2 \to (S/I)_1$ is a k-linear map then ϕ will be S-linear if and only if ϕ satisfies the syzygies of I modulo I. In other words, ϕ should belong to the kernel of

$$\mathrm{Hom}_k(I_2, (S/I)_1) \to \mathrm{Hom}_k(\mathrm{Syz}(I), (S/I)),$$
$$\phi \mapsto \big(\sigma \mapsto \overline{\sigma(\phi)}\big).$$

Since I contains $\mathfrak{m}^3$ and is generated by quadrics, it suffices to consider linear syzygies σ and we have an exact sequence

$$0 \to \mathrm{Hom}_S(I, S/I)_{-1} \longrightarrow \mathrm{Hom}_k(I_2, (S/I)_1) \xrightarrow{\psi} \mathrm{Hom}_k(\mathrm{Syz}(I)_1, (S/I)_2),$$

where $\mathrm{Syz}(I)_1$ is the vector space of linear syzygies. We see that the t_i from the previous lemma span a 4-dimensional subspace T of the kernel of ψ. We obtain

$$\mathrm{Hom}_k(I, S/I)_{-1}/T \xrightarrow{\overline{\psi}} \mathrm{Hom}_k(\mathrm{Syz}(I)_1, (S/I)_2),$$

and $I \in G'_0$ will be a singular point if and only if $\ker(\overline{\psi}) \neq 0$. Since I is generated by quadrics, it follows that $\mathrm{Syz}(I)_1$ has dimension $4 \cdot 7 - 20 = 8$. Therefore $\overline{\psi}$ is a map between 24-dimensional spaces. Thus $\det(\overline{\psi})$ vanishes if and only if $I \in G'_0$ is a singular point in H^4_8.

The global version of this determinant will give a divisor whose support contains W_0. On G'_0 we have the $\mathcal{O}_{G'_0}$-algebra $\mathcal{S} := \mathcal{O}_{G'_0}[x_1, x_2, x_3, x_4]$, which is graded in the standard way, $\mathcal{S} = \oplus_i \mathcal{S}_i$. We have a graded universal ideal sheaf $\mathcal{I} = \oplus \mathcal{I}_i$, and a universal sheaf of graded algebras $\mathcal{S}/\mathcal{I} = \oplus_i \mathcal{S}_i/\mathcal{I}_i$. For all i the sheaves $\mathcal{S}_i$, $\mathcal{I}_i$ and $\mathcal{S}_i/\mathcal{I}_i$ are coherent locally free $O_{G'_0}$-modules.

Let $\mu : \mathcal{I}_2 \otimes \mathcal{S}_1 \to \mathcal{I}_3$ be the multiplication map. Surjectivity of this map follows from the definition of G'_0. We define $\mathcal{K}_1$ to be the kernel of this map, so that we have the exact sequence

$$0 \to \mathcal{K}_1 \to \mathcal{I}_2 \otimes \mathcal{S}_1 \xrightarrow{\mu} \mathcal{I}_3 \to 0. \tag{5-1}$$

In other words, $\mathcal{K}_1$ is the sheaf of linear syzygies. Let U be an open subset of G'_0 such that $\mathcal{I}_2|_U$ is free. Denote generators of $\mathcal{I}_2(U)$ by $q_1, \dots, q_7$ and thus we have

$$\mathcal{K}_1(U) = \left\{ \left(\sum_{i=1}^{7} q_i \otimes l_i \right) \,\middle|\, l_i \in \mathcal{S}_1(U), \sum q_i l_i = 0 \in \mathcal{I}_3 \right\}.$$

To simplify notation in the following lemma we write $\mathcal{H}om$ to denote $\mathcal{H}om_{\mathcal{O}_{G'_0}}$.

Lemma 5.11. (1) *On G'_0 there is a morphism of locally free sheaves of ranks* 28 *and* 24 *respectively*:

$$h : \mathcal{H}om(\mathcal{I}_2, \mathcal{S}_1) \to \mathcal{H}om(\mathcal{K}_1, \mathcal{S}_2/\mathcal{I}_2),$$

such that for any $I \in G'_0$, we have $\ker(h \otimes k(I)) = \mathrm{Hom}(I, S/I)_{-1}$.

(2) *There is a locally free subsheaf of rank four $\mathcal{T} \subset \ker(h)$ inducing a morphism*:

$$\overline{h} : \mathcal{H}om(\mathcal{I}_2, \mathcal{S}_1)/\mathcal{T} \to \mathcal{H}om(\mathcal{K}_1, \mathcal{S}_2/\mathcal{I}_2),$$

such that $\ker(\overline{h} \otimes k(I)) \neq 0$ *if and only if* $\dim_k \mathrm{Hom}(I, S/I)_{-1} \geq 5$.

Proof. (1) We have a map of locally free $\mathcal{O}_{G'_0}$-modules: $\mathcal{K}_1 \to \mathcal{I}_2 \otimes \mathcal{S}_1$. This induces the map $\mathcal{K}_1 \otimes \check{\mathcal{S}}_1 \to \mathcal{I}_2$. Applying $\mathcal{H}om(-, \mathcal{S}_1)$ to both sides we get

$$\mathcal{H}om(\mathcal{I}_2, \mathcal{S}_1) \to \mathcal{H}om(\mathcal{K}_1 \otimes \check{\mathcal{S}}_1, \mathcal{S}_1) \cong \mathcal{H}om(\mathcal{K}_1, \mathcal{S}_1 \otimes \mathcal{S}_1).$$

For the isomorphism above, we are using identities about $\mathcal{H}om$, tensor product of sheaves, and sheaf duality from [Hartshorne 1977, p. 123]. The sequence $\mathcal{S}_1 \otimes \mathcal{S}_1 \to \mathcal{S}_2 \to \mathcal{S}_2/\mathcal{I}_2$ gives a map from $\mathcal{H}om(\mathcal{K}_1, \mathcal{S}_1 \otimes \mathcal{S}_1) \to \mathcal{H}om(\mathcal{K}_1, \mathcal{S}_2/\mathcal{I}_2)$. By composition we obtain the desired map:

$$h : \mathcal{H}om(\mathcal{I}_2, \mathcal{S}_1) \to \mathcal{H}om(\mathcal{K}_1, \mathcal{S}_2/\mathcal{I}_2).$$

Let us take a moment and consider h in concrete terms, since this will be used for proving part (2) of the lemma. Let $U \subset G_0'$ be an open subset such that all relevant locally free sheaves are in fact free. Let $q_1, \dots, q_7$ be the generators of $\mathcal{I}_2(U)$ and let $\sigma_j := \sum_{i=1}^{7} q_i \otimes l_{ij}$ for $1 \le j \le 8$ be the generators of $\mathcal{K}_1(U)$. Finally, let $\phi \in \operatorname{Hom}(\mathcal{I}_2, \mathcal{S}_1)$ be a map $(q_i \mapsto m_i)$. Then $h(\phi)$ is the map

$$\sigma_j \mapsto \sum \overline{m_i l_i},$$

where $\overline{m_i l_i}$ is the reduction of $m_i l_i$ modulo $\mathcal{I}_2$.

(2) Over any U where $\mathcal{I}_2$ is free, let $q_1, \dots, q_7$ the global generators. Then we define $t_1 : q_i \mapsto \frac{\partial}{\partial x_1} q_i$, and we define t_2, t_3, t_4 similarly. This defines a locally free subsheaf $\mathcal{T}(U) := \langle t_1, \dots, t_4 \rangle \subset \mathcal{H}om(\mathcal{I}_2, \mathcal{S}_1)$ of rank 4. By the proof of Lemma 5.10, the injection $\mathcal{T} \to \mathcal{H}om(\mathcal{I}_2, \mathcal{S}_1)$ remains exact under pullback to a point. It follows that the quotient $\mathcal{H}om(\mathcal{I}_2, \mathcal{S}_1)/\mathcal{T}$ is locally free of rank 24 [Hartshorne 1977, Ex II.5.8].

It remains to show that $\mathcal{T} \subset \ker(h)$ and that $\ker(\overline{h} \otimes k(I))$ is nontrivial if and only if $\dim_k \operatorname{Hom}(I, S/I)_1 \ge 5$. This is immediate from the discussion preceding this theorem. $\square$

By the previous lemma, $\overline{h}$ is a map between locally free sheaves of rank 24, and thus $\det(\overline{h})$ defines a divisor on G_0'. To ensure that this is the restriction of a unique divisor on G_0, we need to verify that Z_1 and Z_2 are not too large. For this, we construct the rational curve $\tau : \mathbb{P}^1 \to G_0$ defined for $t \neq \infty$ by

$$I_t = (x_1^2, x_2^2, x_3^2, x_4^2, x_1x_2, x_2x_3 + t x_3x_4, x_1x_4 + t x_3x_4). \tag{5-2}$$

Lemma 5.12. *$Z_1 \cup Z_2$ is a closed set of codimension at least 2 in G_0.*

Proof. Z_1 is closed because it is the support of the cokernel of the multiplication map $\mathcal{I}_2 \otimes \mathcal{S}_1 \to \mathcal{S}_3$. The intersection $Z_2 \cap G_0'$ is the degeneracy locus of

$$\mathcal{H}om(\mathcal{I}_2, \mathcal{S}_0) \to \mathcal{H}om(\mathcal{K}_1, \mathcal{S}_1) \oplus \mathcal{H}om(\mathcal{K}_2, \mathcal{S}_2/I_2),$$

which is the analogue of Lemma 5.11 (1) for computing $\operatorname{Hom}(I_2, S_0)$. Thus $Z_1 \cup Z_2$ is closed in G_0.

Because $\operatorname{Pic}(G_0) = \mathbb{Z}$ and G_0 is projective, checking that the 1-cycle τ does not intersect $Z_1 \cup Z_2$ will show that $Z_1 \cup Z_2$ has codimension at least 2. By passing to the algebraic closure, we can assume that k is algebraically closed. The group $k^\times$ acts on $\mathbb{A}^4$ by $\alpha \cdot (x_1, x_2, x_3, x_4) = (x_1, x_2, \alpha x_3, \alpha x_4)$, and taking $\alpha = t$ maps I_1 to I_t, for any t other than 0 or ∞. Thus, it suffices to check that the following three ideals do not intersect $Z_1 \cup Z_2$:

$$\begin{aligned} I_0 &= (x_1^2, x_2^2, x_3^2, x_4^2, x_1x_2, x_2x_3, x_1x_4), \\ I_1 &= (x_1^2, x_2^2, x_3^2, x_4^2, x_1x_2, x_2x_3 + x_3x_4, x_1x_4 + x_3x_4), \\ I_\infty &= (x_1^2, x_2^2, x_3^2, x_4^2, x_1x_2, x_2x_3 - x_1x_4, x_3x_4). \end{aligned}$$

It is obvious that these are generated in degree 2. A change of variables transforms I_∞ to the ideal J from Proposition 5.1, which is smooth, so $\operatorname{Hom}(I_t, S/I_t)_{-2} = 0$ for $t = 1, \infty$. One can also check that $\operatorname{Hom}(I_0, S/I_0)_{-2} = 0$, which holds in all characteristics because I_0 is a monomial ideal. Therefore, $Z_1 \cup Z_2$ has codimension at least 2. □

Lemma 5.13. *Let D be the divisor on G_0 defined locally by $\det(\bar{h})$. Then W_0 belongs to the support of D.*

Proof. The Hilbert scheme is singular on W_0, so $W_0 \cap G_0' \subset V(\det(\bar{h}))$. Since W_0 is a divisor, Lemma 5.12 tells us that W_0 intersects G_0', so the irreducibility of W_0 means that it is contained in D. □

5D. *An equation for W_0.* In this section, except for the last paragraph, we restrict to the case $k = \mathbb{C}$ in order to use the representation theory of $\mathrm{GL}_4(\mathbb{C})$.

We will use the result of Lemma 5.13 to give an upper bound on the degree of W_0 in terms of Plücker coordinates. This leads to a proof of Theorem 1.3 over $\mathbb{C}$. The restriction to $\mathbb{C}$ will be removed in the next section.

Let H be an effective divisor which generates $\operatorname{Pic}(G_0) = \mathbb{Z}$. First we compute the degree of D in Plücker coordinates, using the rational curve τ.

Lemma 5.14. *The curve τ has intersection multiplicities*

$$\tau \cdot H = 1 \quad \textit{and} \quad \tau \cdot D = 16.$$

Proof. For the first statement, let p_1 and p_2 be the Plücker coordinates corresponding to the $(x_1^2, x_2^2, x_3^2, x_4^2, x_1x_2, x_2x_3, x_3x_4)$- and $(x_1^2, x_2^2, x_3^2, x_4^2, x_1x_2, x_2x_3, x_1x_4)$-minors respectively. Then $L = V(p_1)$ does not meet I_t at infinity. For $t \neq \infty$, we see that $p_2(I_t) \neq 0$, so local equations for L valid at all common points of L and τ are given by $L = \frac{p_1}{p_2}$. Since this equation pulls back to t on $\mathbb{P}^1 - \infty$ the statement follows.

For the second statement, note from the proof of Lemma 5.12 that I_∞ is a smooth point and τ does not intersect Z_1 or Z_2. Therefore, it suffices to check the degree on the open affine defined by $t \neq \infty$.

For every $t \neq \infty$, I_t has the following 8 linear syzygies, where $q_1, \dots, q_7$ are the generators of I_t in the order in Equation (5-2).

$$\begin{array}{l|l}
\sigma_1 = x_2 q_1 - x_1 q_5 & \sigma_2 = x_4 q_1 - x_1 q_7 + t x_3 q_7 - t^2 x_4 q_3 \\
\sigma_3 = x_1 q_2 - x_2 q_5 & \sigma_4 = x_3 q_2 - x_2 q_6 + t x_4 q_6 - t^2 x_3 q_4 \\
\sigma_5 = x_2 q_3 - x_3 q_6 + t x_4 q_3 & \sigma_6 = x_1 q_4 - x_4 q_7 + t x_3 q_4 \\
\sigma_7 = x_3 q_5 - x_1 q_6 + t x_3 q_7 - t^2 x_4 q_3 & \sigma_8 = x_4 q_5 - x_2 q_7 + t x_4 q_6 - t^2 x_3 q_4
\end{array}$$

The intersection number $\tau \cdot D$ equals the degree of $\tau^*(\det(\overline{h}))$, which we compute by writing out $\tau^*(h)$ as a matrix. Let $\phi \in \operatorname{Hom}(I, S/I)_{-1}$ be written as $\phi(q_i) = c_{i,1}x_1 + c_{i,2}x_2 + c_{i,3}x_3 + c_{i,4}x_4$ and recall that, if $\sigma_j = \sum_i q_i \otimes l_{ij}$ then $h(\phi)(\sigma_j) = \sum \overline{\phi(q_i) l_{ij}}$ where the bar indicates that we are considering the image as an element of S_2/I_2. The monomials x_1x_3, x_2x_4 and x_1x_4 are a basis of S_2/I_2 for $t \neq \infty$, so we can explicitly express the $h(\phi)(\sigma_j)$ as follows:

$$\begin{array}{r|l}
h(\phi)(\sigma_1) & -c_{5,3}x_1x_3 + c_{1,4}x_2x_4 + (-tc_{1,3} + tc_{5,4})x_3x_4 \\
h(\phi)(\sigma_2) & (tc_{7,1} - c_{7,3})x_1x_3 + (c_{1,2} - t^2c_{3,2})x_2x_4 + \\
 & (-tc_{1,1} + c_{1,3} + t^3c_{3,1} - t^2c_{3,3} - t^2c_{7,2} + 2tc_{7,4})x_3x_4 \\
h(\phi)(\sigma_3) & c_{2,3}x_1x_3 - c_{5,4}x_2x_4 + (-tc_{2,4} + tc_{5,3})x_3x_4 \\
h(\phi)(\sigma_4) & (c_{2,1} - t^2c_{4,1})x_1x_3 + (tc_{6,2} - c_{6,4})x_2x_4 + \\
 & (-tc_{2,2} + c_{2,4} + t^3c_{4,2} - t^2c_{4,4} - t^2c_{6,1} + 2tc6z)x_3x_4 \\
h(\phi)(\sigma_5) & -c_{6,1}x_1x_3 + (tc_{3,2} + c_{3,4})x_2x_4 + (-t^2c_{3,1} + tc_{6,2} - c_{6,4})x_3x_4 \\
h(\phi)(\sigma_6) & (tc_{4,1} + c_{4,3})x_1x_3 - c_{7,2}x_2x_4 + (-t^2c_{4,2} + tc_{7,1} - c_{7,3})x_3x_4 \\
h(\phi)(\sigma_7) & (c_{5,1} - c_{6,3} + tc_{7,1})x_1x_3 - t^2c_{3,2}x_2x_4 + \\
 & (t^3c_{3,1} - t^2c_{3,3} - tc_{5,2} + c_{5,4} + tc_{6,4} - t^2c_{7,2} + tc_{7,4})x_3x_4 \\
h(\phi)(\sigma_8) & -t^2c_{4,1}x_1x_3 + (c_{5,2} + tc_{6,2} - c_{7,4})x_2x_4 + \\
 & (t^3c_{4,2} - t^2c_{4,4} - tc_{5,1} + c_{5,3} - t^2c_{6,1} + tc_{6,3} + tc_{7,3})x_3x_4
\end{array}$$

Each row of the above lines yields three linear equations so $\tau^*(h)$ is represented by a 24×28 matrix M as expected. Computation in Macaulay2 [Grayson and Stillman] shows that the ideal of 24×24 minors of M is (t^{16}) and the statement follows. □

Corollary 5.15. *The divisor D is linearly equivalent to* $16H$.

The following lemma allows us to determine the degree of W_0.

Lemma 5.16. *The divisor D vanishes with multiplicity at least* 8 *on* W_0.

Proof. By Lemma 5.13, we know that $|W_0| \subseteq |D|$. By Lemma 5.12, a general point of W_0 does not belong to $Z_1 \cup Z_2$. Let I be any such point. Since I is a singular point in R_8^4, I has tangent space dimension at least $\dim(R_8^4) + 1 = 33$, and so the null space of $\overline{h} \otimes k(I)$ must have dimension at least 8.

Choose 8 vectors from the null space as basis vectors, and any other 16 to complete a basis of the source of $\overline{h} \otimes k(I)$. This basis in the quotient ring lifts to a basis in the local ring $\mathcal{O}_{G'_0,I}$. When we represent the localization of the map

$(\overline{h})_I$ as a matrix with respect to this basis we see that the first 8 columns belong to the maximal ideal $\mathfrak{m}_I$ of $\mathcal{O}_{G_0',I}$. Thus $\det(\overline{h})$ belongs to $\mathfrak{m}_I^8$, and in turn D has multiplicity at least 8 at I. □

Lemma 5.17. *The ideal sheaf of D is (P^8), where P is the Salmon–Turnbull Pfaffian.*

Proof. Since D is a divisor on G_0 its defining ideal in the homogeneous coordinate ring of the Plücker embedding of G_0 is generated by a single element f of degree 16 in the Plücker coordinates. If g is the square-free part of f then Lemma 5.16 shows that g has degree at most 2. Since D is invariant under linear changes of variables, it follows from Lemmas 5.7 and 5.8 that $g = P$ and $f = P^8$. □

By combining Lemmas 5.4, 5.13, and 5.17 we have now proven our descriptions of W_0 and W:

Theorem 5.18. *The subscheme W_0 is defined by P.*

For the rest of the section, we return to the case that k is a field, not necessarily algebraically closed, of characteristic not 2 or 3.

Recall that if M_λ is any monomial ideal in G_0 then there are local coordinates $c_{m'}^m$ on $U_\lambda \cap H_8^4$. Moreover there is a surjection $\pi : R_8^4 \cap U_\lambda \to W_0 \cap U_\lambda$, and there is a rational map $\phi : (\mathbb{A}^4)^8 /\!/ S_8 \dashrightarrow R_8^4 \cap U_\lambda$ given by $c_{m'}^m = \frac{\Delta_{\lambda - m' + m}}{\Delta_\lambda}$ whose image is dense in $R_8^4 \cap U_\lambda$.

Lemma 5.19. *With U_λ as above, the function $P \circ \pi$ vanishes identically on $R_8^4 \cap U_\lambda$ over an arbitrary field k.*

Proof. The composition $P \circ \pi \circ \phi$ is a rational function with integer coefficients. Theorem 5.18 proves that $P \circ \pi \circ \phi = 0$ in $\mathbb{C}[q_i^{(j)}][\Delta_\lambda^{-1}]$. Therefore, $P \circ \pi \circ \phi = 0$ in $\mathbb{Z}[q_i^{(j)}][\Delta_\lambda^{-1}]$. □

Theorem 5.20. *The following irreducible subsets of G_0 coincide:*

(1) W_0;

(2) $V(P)$, *the vanishing of the pullback to G of the Salmon–Turnbull Pfaffian*;

(3) *the homogeneous ideals with Hilbert function* $(1, 4, 3)$ *which are flat limits of ideals of distinct points.*

As a consequence, if we let $\pi|_G : G \to G_0$ be the restriction of the projection from Lemma 5.4 then $W = V(P \circ \pi|_G)$.

Proof. For other fields, note that for the ideal J of Proposition 5.1 with $d = 4$, we have that $P \circ \pi(J) = P(J) = 1$ and thus $P \circ \pi$ does not vanish uniformly on G in any characteristic. By the previous lemma, $P \circ \pi$ vanishes uniformly on R_8^4 for any k. Thus $W \subseteq V(P \circ \pi|_G)$. As both W and $V(P \circ \pi|_G)$ are integral closed subschemes of codimension 1 in G, they are equal. □

6. Proofs of main results

In this section, k denotes a field of characteristic not 2 or 3. We have used our characteristic assumption in order to apply the theory of duality in Sections 3 and 4 and to define the trace map of Lemma 5.3.

Proof of Theorem 1.1. The irreducibility of H_n^d when d is at most 3 or n is at most 7 follows for an algebraically closed field from Theorem 4.20. For a nonalgebraically closed field, the Hilbert scheme is irreducible because it is irreducible after passing to the algebraic closure. Proposition 3.1 and the same argument as in Lemma 5.3 show that when d is at least 4, G_8^d is irreducible and $(8d-7)$-dimensional, and Proposition 5.1 shows that it is a separate component. Theorem 4.20 shows that there are no other components, again, by passing to the algebraic closure if necessary. □

Proof of Theorem 1.2. This follows from Theorem 3.9. □

Proof of Theorem 1.3. The statement that $R_8^4 \cap G_8^4$ is a prime divisor on G_8^4 is proved in Lemma 5.4. The equivalence of the set-theoretic description and the local equation description follows from Lemma 5.6. Theorem 5.20 proves that the Salmon–Turnbull Pfaffian is the correct local equation. □

Proof of Theorem 1.4. Let M_λ be some monomial ideal and consider the monomial chart U_λ. If M_λ does not have Hilbert function $(1,4,3)$ then $U_\lambda \cap G_8^4 = \varnothing$ so that the zero ideals will cut out R_8^4. If M_λ has Hilbert function $(1,4,3)$, then Lemma 5.19 and Theorem 5.20 show that the zero set of the pullback of the Pfaffian is precisely $R_8^4 \cap U_\lambda$. □

7. Open questions

The motivating goal behind this work is understanding the smoothable component of the Hilbert scheme as explicitly as possible, and not just as the closure of a certain set. This can be phrased more abstractly by asking what functor the smoothable component represents, or, more concretely, by describing those algebras which occur in the smoothable component. In this paper we have done the latter for n at most 8. The following are natural further questions to ask:

- For d greater than 4, which algebras with Hilbert function $(1,d,3)$ are smoothable? Generically, such algebras are not smoothable. Computer experiments lead us to conjecture that, for smoothable algebras, the analogue of the skew symmetric matrix in Theorem 1.3 has rank at most $2d+2$. However, a dimension count shows that this rank condition alone is not sufficient for such an algebra to be smoothable. What are the other conditions?

- What is the smallest n such that H^3_n is reducible? We have shown H^3_8 is irreducible and Iarrobino [1985, Example 3] has shown that H^3_{78} is reducible.
- Is H^d_n ever nonreduced? What is the smallest example? Does it ever have generically nonreduced components?

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