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In 1956, Brauer showed that there is a partitioning of the p-regular conjugacy classes of a group according to the p-blocks of its irreducible characters with close connections to the block theoretical invariants. In a previous paper, the first explicit block splitting of regular classes for a family of groups was given for the 2-regular classes of the symmetric groups. Based on this work, the corresponding splitting problem is investigated here for the 2-regular classes of the alternating groups. As an application, an easy combinatorial formula for the elementary divisors of the Cartan matrix of the alternating groups at $p = 2$ is deduced.

1. Introduction

Richard Brauer [1956] introduced the idea of not only distributing characters into p-blocks but also of associating p-regular conjugacy classes to p-blocks. He showed that it is possible to distribute the p-regular classes into blocks in a way that fits with the blocks of irreducible Brauer characters (and suitable subsets of ordinary irreducible characters in the blocks); this is to say that the determinant of the corresponding block part of the Brauer character table (or a suitable part of the ordinary character table) is not congruent to 0 modulo $\mathfrak{p}$ (a prime ideal over p). Given such a splitting of p-regular classes into blocks, Brauer showed that the elementary divisors of the Cartan matrix of a block are then exactly the p-parts in the orders of the centralizers of elements in the classes corresponding to the block. He also observed that in general there may be several such block splittings, and there did not seem to be any natural choice for a given finite group.

But while it is well known how to determine the p-blocks of irreducible characters, for the p-regular classes only the existence of such a block splitting is known by Brauer's work — concrete examples for providing such a distribution for families of groups were not known for a long time. Only recently, such an explicit block splitting in the sense of Brauer was exhibited for the conjugacy classes of odd order elements and the 2-blocks of the symmetric groups [Bessenrodt 2007];

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in fact, in this case the 2-block splitting of the 2-regular classes is unique. The proof exploited detailed information on the double covers of the symmetric groups, in particular results on the 2-powers in the spin character values of these groups [Bessenrodt and Olsson 2000] as well as on the 2-block distribution of the spin characters [Bessenrodt and Olsson 1997] turned out to be important ingredients.

Based on these results, the present paper investigates the corresponding problem of constructing a 2-block splitting of the 2-regular classes for the alternating groups. We provide a basic set of characters for the alternating groups, and find a natural choice for a block splitting of the classes. As an application, we deduce an easy combinatorial description of the invariants of the Cartan matrices for the 2-blocks of the alternating groups.

Here is a brief outline of the sections. In Section 2, we recall Brauer's results on block splittings for finite groups which motivated the present work. Then, in Section 3, some combinatorial notations needed in the representation theory of the symmetric groups is introduced, and we state some results from [Bessenrodt 2007] on the block splitting of 2-regular classes for the symmetric groups that are the basis for the new results on alternating groups. In particular, the class labels for the 2-block splitting of S_n are recalled. In Section 4, we first collect the necessary information on characters of the alternating groups, and prove some preliminary results towards the construction of a class splitting for the alternating groups. In the main Theorem 4.7 properties of the determinants of the corresponding block character tables are proved which imply that the construction gives indeed a block splitting of the classes. By Brauer's Theorem, our result then implies an easy combinatorial description of the Cartan invariants for the 2-blocks of the alternating groups (Corollary 4.9).

2. Brauer's block splitting

Let G be a finite group, p a prime, $(\mathfrak{K}, R, F)$ a p-modular splitting system for G, and $\mathfrak{p}$ a maximal ideal of R lying over p. Let $\ell(G)$ be the cardinality of the set $\mathrm{Cl}_{p'}(G)$ of p-regular conjugacy classes in G. For each $K \in \mathrm{Cl}_{p'}(G)$ we let x_K denote an element in K. A defect group of K is a Sylow p-subgroup of $C_G(x)$ for some $x \in K$; if this has order p^d, then d is called the p-defect of K. We let $\mathrm{IBr}(G)$ denote the set of modular irreducible characters of G; then

$$\Phi_G = (\varphi(x_K))_{\substack{\varphi\in \mathrm{IBr}(G)\\ K\in \mathrm{Cl}_{p'}(G)}}$$

is the Brauer character table of G. It is well known by Brauer's work that the Brauer character table is nonsingular modulo p, that is,

$$\det \Phi_G \not\equiv 0 \pmod{\mathfrak{p}}\ .$$

Furthermore, we let $D=(d_{\chi\varphi})_{\chi\in\mathrm{Irr}(G),\,\varphi\in\mathrm{IBr}(G)}$ denote the p-decomposition matrix for G, and we let $C=D^tD$ denote its Cartan matrix. Let $\mathrm{Bl}_p(G)$ be the set of p-blocks of G. For $B\in\mathrm{Bl}_p(G)$, $\mathrm{Irr}(B)$ is the set of ordinary irreducible characters in B, $\mathrm{IBr}(B)$ is the set of modular irreducible characters in B, $\ell(B)=|\,\mathrm{IBr}(B)|$, $D(B)=(d_{\chi\varphi})_{\chi\in\mathrm{Irr}(B),\,\varphi\in\mathrm{IBr}(B)}$ denotes the p-decomposition matrix for B and $C(B)$ is the Cartan matrix for B.

Then C resp. D are the block direct sums of the matrices $C(B)$ resp. $D(B)$, $B\in\mathrm{Bl}_p(G)$.

The following result was proved by Brauer.

Theorem 2.1. [Brauer 1956, Section 5] *There exists a disjoint decomposition of* $\mathrm{Cl}_{p'}(G)$ *into blocks of p-regular conjugacy classes*

$$\mathrm{Cl}_{p'}(G)=\bigcup_{B\in\mathrm{Bl}_p(G)}\mathrm{Cl}_{p'}(B)$$

and a selection of characters $\mathrm{Irr}'(B)\subseteq\mathrm{Irr}(B)$ *for each p-block B of G such that the following conditions are fulfilled*:

(i) $|\,\mathrm{Cl}_{p'}(B)|=|\,\mathrm{Irr}'(B)|=\ell(B)$ *for all* $B\in\mathrm{Bl}_p(G)$;

(ii) *For* $X_B=(\chi(x_K))_{\substack{\chi\in\mathrm{Irr}'(B)\\ K\in\mathrm{Cl}_{p'}(B)}}$, *we have* $\det X_B\not\equiv 0 \pmod{\mathfrak{p}}$;

(iii) *For* $\Phi_B=(\varphi(x_K))_{\substack{\varphi\in\mathrm{IBr}(B)\\ K\in\mathrm{Cl}_{p'}(B)}}$, *we have* $\det\Phi_B\not\equiv 0 \pmod{\mathfrak{p}}$;

(iv) *For* $D_B=(d_{\chi\varphi})_{\substack{\chi\in\mathrm{Irr}'(B),\\ \varphi\in\mathrm{IBr}(B)}}$, *we have* $\det D_B\not\equiv 0 \pmod{\mathfrak{p}}$.

Furthermore, the elementary divisors of the Cartan matrix $C(B)$ are then exactly the orders of the p-defect groups of the conjugacy classes in $\mathrm{Cl}_{p'}(B)$, *for all B in* $\mathrm{Bl}_p(G)$.

The properties in (ii)–(iv) are not independent of each other, as $X_B=D_B\,\Phi_B$. In particular, if we have a suitable choice $\mathrm{Irr}'(B)$ of characters that satisfies (iv), and a suitable choice of classes that satisfies (iii), then these together are a suitable choice for (ii). If we have a basic set of irreducible characters, that is, a subset $\mathrm{Irr}'(G)\subseteq\mathrm{Irr}(G)$ giving a $\mathbb{Z}$-basis for the character restrictions to the p-regular classes, then the p-block decomposition of this set will give a suitable choice of sets $\mathrm{Irr}'(B)$ satisfying (iv).

3. The 2-block splitting for S_n

Let $n\in\mathbb{N}$. For the symmetric groups S_n, the corresponding combinatorial notions and their representation theory, we will follow mostly the usual notation in [James and Kerber 1981].

Let P be the set of partitions, $P(n)$ the partitions of n. For a partition λ of n, the number of its (nonzero) parts is called its length, and is denoted by $l(\lambda)$. The complex irreducible character of S_n corresponding to λ is denoted by $[\lambda]$. For any partition μ of n, we choose an element σ_μ in S_n of cycle type μ.

Let $\mu = (1^{m_1(\mu)}, 2^{m_2(\mu)}, \ldots)$ be written in exponential notation; then we set

$$a_\mu = \prod_{i\geq 1} i^{m_i(\mu)}, \quad b_\mu = \prod_{i\geq 1} m_i(\mu)!\,.$$

Let $z_\mu = |C_{S_n}(\sigma_\mu)|$; then $z_\mu = a_\mu b_\mu$.

Let p be a prime; we will soon fix this to $p = 2$. A partition is called p-regular if no part is repeated p or more times, and a partition is called p-class regular if no part is divisible by p.

Let $\mathcal{D}(n)$ be the set of partitions of n into distinct parts; this is thus the set of 2-regular partitions of n. Let $\mathcal{O}(n)$ be the set of partitions of n into odd parts only; this is the set of 2-class regular partitions of n. Let $\mathcal{O} = \bigcup_{n\in\mathbb{N}} \mathcal{O}(n)$ and let $\mathcal{D} = \bigcup_{n\in\mathbb{N}} \mathcal{D}(n)$. It is well known that

$$\mathrm{Irr}'(S_n) = \{[\lambda], \lambda \in \mathcal{D}(n)\}$$

forms a 2-basic set for S_n.

Then the 2-regular character table of the symmetric group S_n is defined to be

$$X_n = ([\lambda](\sigma_\alpha))_{\substack{\lambda\in\mathcal{D}(n)\\ \alpha\in\mathcal{O}(n)}}$$

where the partitions are ordered in a suitable way.

As a special case of a result by Olsson [2003], we know that $|\det(X_n)| = \prod_{\mu\in\mathcal{O}(n)} a_\mu$, and thus in particular,

$$2 \nmid \det(X_n)\,.$$

The main result in [Bessenrodt 2007] provides a block version of this property, by distributing not only the characters but also the 2-regular conjugacy classes into blocks in such a way that the corresponding block parts of the character table have odd determinants. This block distribution of conjugacy classes provided a block splitting in the sense of Brauer as described in the previous section.

We recall this 2-block splitting for the symmetric groups below. The reader is referred to [Bessenrodt 2007] for the full results; these involve more detailed information on spin characters which we omit here since it would require recalling a lot of notation on double cover groups and their characters.

For the combinatorics of the p-modular representation theory for S_n, and in particular the p-block distribution of its characters, we refer to [James and Kerber 1981].

Let B be a 2-block of S_n, with associated 2-core $\kappa(B)$; this is then a staircase partition $\rho_k = (k, k-1, \ldots, 2, 1)$, $k \in \mathbb{N}_0$. For any partition λ, we denote by $\lambda_{(2)}$ the 2-core of λ. Then we define

$$\mathcal{D}_B = \{\lambda \in \mathcal{D}(n) \mid \lambda_{(2)} = \kappa(B)\}\,.$$

This is the set of labels of irreducible characters in B in the basic set mentioned above, and we set

$$\mathrm{Irr}'(B) = \{[\lambda] \mid \lambda \in \mathcal{D}_B\}\,.$$

To define the splitting of the classes we need a few more definitions.

For a partition $\lambda = (\lambda_1, \ldots, \lambda_m) \in \mathcal{D}(n)$ we set

$$\mathrm{dbl}(\lambda) = \left(\left\lceil\frac{\lambda_1+1}{2}\right\rceil, \left\lfloor\frac{\lambda_1}{2}\right\rfloor, \left\lceil\frac{\lambda_2+1}{2}\right\rceil, \left\lfloor\frac{\lambda_2}{2}\right\rfloor, \ldots, \left\lceil\frac{\lambda_m+1}{2}\right\rceil, \left\lfloor\frac{\lambda_m}{2}\right\rfloor\right),$$

the *doubling* of λ. For example, the staircase $\rho_k = (k, k-1, \ldots, 2, 1)$ is the doubling of the partition $\tau_k = (2k-1, 2k-5, \ldots)$.

The most natural way of defining the blocks of classes is based on the Glaisher map which we consider next.

J. W. L. Glaisher [1883] defined a bijection between partitions with parts not divisible by a given number k on the one hand and partitions where no part is repeated k times on the other hand; in particular for $k = 2$ this gives a bijection between $\mathcal{O}(n)$ and $\mathcal{D}(n)$. Here, Glaisher's map G is defined as follows. Let $\alpha = (1^{m_1}, 3^{m_3}, \cdots) \in \mathcal{O}(n)$. Write each multiplicity m_i as a sum of distinct powers of 2, that is, in its 2-adic decomposition: $m_i = \sum_j 2^{a_{ij}}$. Then $G(\alpha) \in \mathcal{D}(n)$ consists of the parts $(2^{a_{ij}} i)_{i,j}$, sorted in order to give a partition.

Let B be a 2-block of S_n, contained in a 2-block $\widetilde{B}$ of the double cover group $\widetilde{S}_n$ (see [Bessenrodt and Olsson 1997] for background and notation). Then we define the set

$$\mathcal{O}_B = \mathcal{O}_{\widetilde{B}} = \{\alpha \in \mathcal{O}(n) \mid \mathrm{dbl}(G(\alpha))_{(2)} = \kappa(B)\}\,.$$

Note that in [Bessenrodt 2007] we have used the language of $\bar{4}$-combinatorics for the description of the 2-block distribution of spin characters, and the set $\mathcal{O}_B$ is then the set of partitions of type $\mathcal{O}$ such that the Glaisher image has as $\bar{4}$-core the one associated to $\widetilde{B}$.

The sets $\mathcal{O}_B$ for the 2-blocks B of S_n give then a set partition $\mathcal{O}(n) = \bigcup_B \mathcal{O}_B$.

The set $\mathcal{O}_B$ is the set of labels of the 2-regular classes we want to associate to B, that is, we set

$$\mathrm{Cl}'(B) = \{\sigma_\alpha^{S_n} \mid \alpha \in O_B\}\,.$$

Defining

$$\mathcal{D}_{\widetilde{B}} = \{\lambda \in \mathcal{D}(n) \mid \mathrm{dbl}(\lambda)_{(2)} = \kappa(B)\}\,,$$

the Glaisher map then induces bijections $\mathcal{O}_B \to \mathcal{D}_{\widetilde{B}}$, for all 2-blocks B of S_n.

By [Bessenrodt and Olsson 1997], $|\mathcal{D}_B| = |\mathcal{D}_{\tilde{B}}| = p(w(B))$. Thus the following block parts of the character table are all square matrices:

$$X_B = ([\mu](\sigma_\alpha))_{\substack{\mu\in\mathcal{D}_B\\ \alpha\in\mathcal{O}_B}} .$$

Denoting the irreducible Brauer characters of S_n by φ^μ, $\mu \in \mathcal{D}(n)$, we also consider the corresponding block part of the Brauer character table:

$$\Phi_B = (\varphi^\mu(\sigma_\alpha))_{\substack{\mu\in\mathcal{D}_B\\ \alpha\in\mathcal{O}_B}} .$$

Theorem 3.1. [Bessenrodt 2007] *Let* $\mathrm{Irr}'(B)$ *and* $\mathrm{Cl}'(B)$ *for the* 2*-blocks* B *of* S_n *be defined as above. Then the determinants*

$$\det \Phi_B = \det X_B, \quad B \in \mathrm{Bl}(S_n)$$

of the associated block parts of the character table and the Brauer character table are all odd.

Thus the sets $\mathrm{Cl}'(B)$ *define a* 2*-block splitting of the* 2*-regular classes for* S_n.

Remarks 3.2. (i) More precisely, the determinant $\det X_B$ is (up to sign) the odd part of the determinant of the corresponding block part of the reduced spin character table for the 2-block $\tilde{B}$ of the double cover group $\tilde{S}_n$ containing B, that is,

$$Z_s(\tilde{B}) = (\langle\lambda\rangle(\tilde{\sigma}_\alpha))_{\substack{\lambda\in\mathcal{D}_{\tilde{B}}\\ \alpha\in\mathcal{O}_B}} .$$

See [Bessenrodt 2007] for the notation used here and details on this result.

(ii) By the 2-block splitting for S_n given above and Brauer's Theorem, the elementary divisors of the Cartan matrix C_B of a 2-block B of S_n are exactly the 2-powers

$$2^{k_\alpha} = |C_{S_n}(\sigma_\alpha)|_2, \ \alpha \in \mathcal{O}_B .$$

Here, the 2-defect of the class of type α in S_n may easily be computed as follows [Bessenrodt 2007]:

$$k_\alpha = l(\alpha) - l(G(\alpha)) .$$

This is a restatement of a formula from [Uno and Yamada 2006] which is based on [Bessenrodt and Olsson 1997]; a corrected version of an earlier formula from [Olsson 1986] already appeared in [Bessenrodt and Olsson 1997]. One should note, though, that this formula was used in the confirmation of the block splitting for S_n in [Bessenrodt 2007], so this does not give an independent proof for the elementary divisors of the Cartan matrix.

4. A 2-block splitting for alternating groups

We also have to introduce some notation for the alternating group A_n.

We let $P^+(n) = \{\lambda \in P(n) \mid (-1)^{n-l(\lambda)} = 1\}$ denote the set of even partitions in $P(n)$; these are the cycle types of elements in A_n.

The conjugacy classes in A_n are then of two types. The classes labeled by partitions $\mu \in P^+(n)\setminus(\mathscr{O}\cap\mathscr{D})(n)$ are the nonsplit classes, that is, those conjugacy classes of S_n which are also A_n-classes; we note that the corresponding A_n-centralizer is then of order $z'_\mu = z_\mu/2$. For the partitions $\mu \in (\mathscr{O}\cap\mathscr{D})(n)$, the S_n-class of σ_μ splits into two conjugacy classes in A_n, for which we denote representatives by σ_μ^+ and σ_μ^-; their centralizers are of order $z'_\mu = z_\mu$.

A set of representatives of the 2-regular classes of A_n is thus given by:

$$\mathscr{R}(n) = \{\sigma_\alpha \mid \alpha \in (\mathscr{O}\setminus\mathscr{O}\cap\mathscr{D})(n)\} \cup \{\sigma_\alpha^\pm \mid \alpha \in (\mathscr{O}\cap\mathscr{D})(n)\}.$$

Furthermore, we briefly have to recall some information on the irreducible A_n-characters [James and Kerber 1981, Section 2.5].

For a partition λ of n, let λ' denote the conjugate partition. Let

$$\mathscr{S}(n) = \{\lambda \in P(n) \mid \lambda = \lambda'\}$$

be the set of symmetric partitions of n.

If λ is nonsymmetric, then $[\lambda]\downarrow_{A_n} = [\lambda']\downarrow_{A_n}$ is irreducible. Let $\{\lambda\} = \{\lambda'\}$ denote this irreducible character of A_n.

If $\lambda = \lambda'$, then $[\lambda]\downarrow_{A_n} = \{\lambda\}_+ + \{\lambda\}_-$ is a sum of two distinct irreducible A_n-characters (which are conjugate in S_n).

This gives all the irreducible complex characters of A_n, that is,

$$\mathrm{Irr}(A_n) = \{\{\lambda\}_\pm \mid \lambda \in \mathscr{S}(n)\} \cup \{\{\lambda\} \mid \lambda \in (P\setminus\mathscr{S})(n)\}.$$

The characters $\{\lambda\}_\pm$, for $\lambda \in \mathscr{S}(n)$, are only distinguished by their values on the corresponding "critical" classes of cycle type $h(\lambda) = (h_1^\lambda, \ldots, h_d^\lambda)$, where $h_1^\lambda, \ldots, h_d^\lambda$ are the principal hook lengths in λ and $d = d(\lambda)$ is the diagonal length of λ. Note that $h(\lambda) \in (\mathscr{O}\cap\mathscr{D})(n)$, so the corresponding S_n-class splits.

Then we have $[\lambda](\sigma_{h(\lambda)}) = (-1)^{(n-d)/2} =: \varepsilon_\lambda$. We set $H_\lambda = \prod_{i=1}^d h_i^\lambda$. Then

$$\{\lambda\}_+(\sigma_{h(\lambda)}^\pm) = \tfrac{1}{2}\big(\varepsilon_\lambda \pm \sqrt{\varepsilon_\lambda H_\lambda}\,\big),$$
$$\{\lambda\}_-(\sigma_{h(\lambda)}^\pm) = \tfrac{1}{2}\big(\varepsilon_\lambda \mp \sqrt{\varepsilon_\lambda H_\lambda}\,\big).$$

For any other irreducible A_n-character the values on these two classes coincide.

We have the following easy and well known property:

Lemma 4.1. *The map $h : \mathscr{S}(n) \to (\mathscr{O}\cap\mathscr{D})(n)$ with $h(\lambda) = (h_1^\lambda, \ldots, h_{d(\lambda)}^\lambda)$, for $\lambda \in \mathscr{S}(n)$, is a bijection.*

Let $B \in \mathrm{Bl}(S_n)$ with 2-core $\rho_k = (k, k-1, \ldots, 2, 1) = \mathrm{dbl}(\tau_k)$, where, as before, $\tau_k = (2k-1, 2k-5, \ldots)$; let $\mathcal{D}_{\widetilde{B}}$ and $\mathcal{O}_B = \mathcal{O}_{\widetilde{B}}$ as before in Section 3 and $(\mathcal{O} \cap \mathcal{D})_{\widetilde{B}} = \mathcal{O}_{\widetilde{B}} \cap \mathcal{D}_{\widetilde{B}}$.

We set $\mathcal{S}_B = \{\lambda \in \mathcal{S}(n) \mid \lambda_{(2)} = \rho_k\}$.

In our context, we need the following refinement of Lemma 4.1:

Proposition 4.2. *The map h induces bijections $\mathcal{S}_B \to (\mathcal{O} \cap \mathcal{D})_{\widetilde{B}}$.*

Proof. We have to show that for any $\lambda \in \mathcal{S}(n)$, we have $\lambda_{(2)} = \mathrm{dbl}(h(\lambda))_{(2)}$. In the notation of $\bar{4}$-combinatorics an easy reduction argument shows that $\lambda_{(2)} = \mathrm{dbl}(h(\lambda)_{(\bar{4})})$; simultaneously removing 2-hooks from the diagram of λ that are symmetrically positioned in λ corresponds to removing 4-bars from $h(\lambda)$, namely, subtracting 4 from a part in $h(\lambda)$, and removing an inner 2×2 array corresponds to removing a pair 3, 1 (which is also a 4-bar). This ends at a staircase partition $\rho_k = \lambda_{(2)}$, and in parallel at the corresponding $\tau_k = h(\lambda)_{(\bar{4})}$.

By [Bessenrodt and Olsson 1997, Lemma 3.6], we obtain the equation on the 2-cores. □

Remark 4.3. It is not difficult to see [Olsson 1993, 12.5] that

$$|\mathcal{S}_B| = \begin{cases} 0 & \text{if } w(B) \text{ is odd,} \\ p(w/2) & \text{if } w(B) \text{ is even.} \end{cases}$$

For a character χ of G, let χ^o denote the restriction of χ to the 2-regular elements of G. The following useful proposition provides a good 2-basic set for the alternating groups. Note here that the set $(\mathcal{D} \cap \mathcal{S})(n)$ labeling the third subset of characters is nonempty only if n is a triangular number; in that case, if $n = \binom{k+1}{2}$, $(\mathcal{D} \cap \mathcal{S})(n) = \{\rho_k\}$, and thus both characters $\{\rho_k\}_\pm$ of defect 0 then belong to the basic set.

Proposition 4.4. *Set*

$$\mathcal{C}(n) = \{\{\lambda\} \mid \lambda \in (\mathcal{D} \setminus \mathcal{S})(n)\} \cup \{\{\lambda\}_+ \mid \lambda \in \mathcal{S}(n)\} \cup \{\{\lambda\}_- \mid \lambda \in (\mathcal{D} \cap \mathcal{S})(n)\}.$$

Then $\{\chi^o \mid \chi \in \mathcal{C}(n)\}$ is a basic set for A_n (at the prime $p = 2$).

Proof. If $\chi = \{\mu\} = [\mu]\downarrow_{A_n}$ for $\mu \in (P \setminus \mathcal{S})(n)$, then, since the 2-regular partitions label a basic set for S_n,

$$\chi^o = [\mu]^o \downarrow_{A_n} = \sum_{\lambda \in \mathcal{D}(n)} c_{\mu\lambda} [\lambda]^o \downarrow_{A_n} = \sum_{\lambda \in (\mathcal{D} \setminus \mathcal{S})(n)} c_{\mu\lambda} \{\lambda\}^o + \sum_{\lambda \in (\mathcal{D} \cap \mathcal{S})(n)} c_{\mu\lambda} (\{\lambda\}^o_+ + \{\lambda\}^o_-),$$

with integer coefficients $c_{\mu\lambda}$; as explained just before the proposition, the second sum above has at most one partition $\lambda \in (\mathcal{D} \cap \mathcal{S})(n)$ giving a contribution, namely when n is a triangular number.

For $\chi = \{\mu\}_+$, $\mu \in \mathcal{S}(n)$, or $\chi = \{\mu\}_-$, $\mu \in (\mathcal{D} \cap \mathcal{S})(n)$, there is nothing to prove. If $\chi = \{\mu\}_-$, $\mu \in \mathcal{S} \setminus (\mathcal{D} \cap \mathcal{S})(n)$, then, again since the 2-regular partitions label a basic set for S_n,

$$\begin{aligned}\chi^o = \{\mu\}_-^o = [\mu]^o \downarrow_{A_n} - \{\mu\}_+^o &= \Big(\sum_{\lambda \in \mathcal{D}(n)} c_{\mu\,\lambda} [\lambda]^o \downarrow_{A_n} \Big) - \{\mu\}_+^o \\ &= \Big(\sum_{\lambda \in (\mathcal{D} \setminus \mathcal{S})(n)} c_{\mu\,\lambda} \{\lambda\}^o + \sum_{\lambda \in (\mathcal{D} \cap \mathcal{S})(n)} c_{\mu\,\lambda} (\{\lambda\}_+^o + \{\lambda\}_-^o) \Big) - \{\mu\}_+^o,\end{aligned}$$

an integral linear combination as desired.

Since $|\mathcal{C}(n)| = |\mathcal{D}(n)| + |\mathcal{S}(n)| = |\mathbb{O}(n)| + |(\mathbb{O} \cap \mathcal{D})(n)| = |\operatorname{Cl}'(A_n)|$, the set $\{\chi^o \mid \chi \in \mathcal{C}(n)\}$ is a $\mathbb{Z}$-basis of $\langle \chi^o \mid \chi \in \operatorname{Irr}(A_n) \rangle_{\mathbb{Z}}$, as claimed. □

For any set $Q \subseteq P(n)$, we set

$$a_Q = \prod_{\lambda \in Q} a_\lambda \,.$$

Using the basic set above, we define the 2-regular character table for the alternating group A_n to be

$$X_n^A = (\chi(\sigma))_{\substack{\chi \in \mathcal{C}(n) \\ \sigma \in \mathcal{R}(n)}} \,.$$

Using the properties of the irreducible characters of A_n stated above (see also [Bessenrodt and Olsson 2004]), as well as the formula for the 2-regular character table for S_n from [Olsson 2003] we deduce:

Corollary 4.5. $|\det X_n^A| = |\det X_n| \cdot \sqrt{a_{(\mathbb{O} \cap \mathcal{D})(n)}} = a_{\mathbb{O}(n)} \cdot \sqrt{a_{(\mathbb{O} \cap \mathcal{D})(n)}}\,.$

In the main theorem stated below, we will give a block refinement of the first equation above.

Remark 4.6. The 2-blocks of S_n and A_n are closely related [Olsson 1993] . Let $B \in \operatorname{Bl}(S_n)$. If $w(B) = 0$, then B covers two 2-blocks of A_n (of defect 0), say B_ε^A, $\varepsilon \in \{\pm\}$. This only occurs when n is a triangular number, say $n = \binom{k+1}{2}$, and $\kappa(B) = \rho_k = (k, k-1, \ldots, 2, 1)$; then $\operatorname{Irr}(B_\varepsilon^A) = \{\{\rho_k\}_\varepsilon\}$, $\varepsilon \in \{\pm\}$.

Note that there is then a suitable choice of signs $\bar{\varepsilon}$ for $\varepsilon \in \{\pm\}$ such that

$$\{\rho_k\}_\varepsilon(\sigma_{h(\rho_k),\bar{\varepsilon}}) \not\equiv 0 \mod \mathfrak{p}\,.$$

If $w(B) > 0$, then B covers only one 2-block B^A of A_n, and this block B^A is only covered by B. We then have

$$\operatorname{Irr}(B^A) = \{\{\lambda\}_{(\pm)} \mid \lambda \in P(n),\, \lambda_{(2)} = \kappa(B)\}\,.$$

Here, $\{\lambda\}_{(\pm)}$ means that we take the character $\{\lambda\}$ if λ is nonsymmetric, and both characters $\{\lambda\}_\pm$ if λ is symmetric.

Theorem 4.7. *Let $B \in \mathrm{Bl}(S_n)$ with 2-core $\kappa(B) = \rho_k = \mathrm{dbl}(\tau_k)$.*
If $w(B) = 0$, then $\mathbb{O}_B = \{\tau_k\}$ and $\mathcal{D}_B = \{\rho_k\}$, and we set

$$\mathrm{Cl}'^A(B^A_\varepsilon) = \{\sigma^{A_n}_{\tau_k,\bar\varepsilon}\}\,, \quad \mathrm{Irr}'(B^A_\varepsilon) = \{\{\rho_k\}_\varepsilon\}\,, \qquad \textit{for } \varepsilon \in \{\pm\}\,.$$

If $w(B) > 0$, we set

$$\mathrm{Cl}'^A(B^A) = \{\sigma^{A_n}_{\alpha(,\pm)} \mid \alpha \in \mathbb{O}_B\}\,, \quad \mathrm{Irr}'(B^A) = \{\{\lambda\} \mid \lambda \in \mathcal{D}_B\} \cup \{\{\lambda\}_+ \mid \lambda \in \mathcal{S}_B\}\,.$$

Let $X_{B^A} = (\chi(x_K))_{\chi\in\mathrm{Irr}'(B^A),\, K\in\mathrm{Cl}'^A(B^A)}$. Then

$$|\det X_{B^A}| = |\det X_B| \cdot \sqrt{a_{(\mathbb{O}\cap\mathcal{D})_{\widetilde{B}}}}\,.$$

In particular, $|\det X_{B^A}| \not\equiv 0 \mod \mathfrak{p}$. Hence, the sets above, taken for all $B \in \mathrm{Bl}(S_n)$, define a 2-block splitting for A_n.

Proof. Using the notation above, let B^A be a 2-block of A_n. As seen above, the sets $\mathrm{Cl}'^A(B^A)$ and $\mathrm{Irr}'(B^A)$ are of the same cardinality. We have to show that all the block tables have a nonzero determinant modulo $\mathfrak{p}$.

For the case where $w(B) = 0$, we have already seen before that we can make a suitable choice (namely the one used in the statement of the Theorem) such that this holds for the two blocks of A_n covered by B.

Thus we may now assume that $w(B) > 0$; then we do not have an irreducible character labeled by a partition of type $\mathcal{D} \cap \mathcal{S}$ in B, that is, $\mathcal{D}_B \cap \mathcal{S}_B = \varnothing$. We consider the part of the character table of A_n corresponding to B^A, sorted such that among the character labels we first list the nonsymmetric ones and then the symmetric ones. The classes are ordered such that we first have the $\mathbb{O}_B$ classes, and among these classes the $(\mathbb{O}\cap\mathcal{D})_{\widetilde{B}}$ classes at the end, and here first the $(\mathbb{O}\cap\mathcal{D})^+_{\widetilde{B}}$ classes (where the "+" indicates that we take the representatives σ^+_α), followed by the corresponding $(\mathbb{O}\cap\mathcal{D})^-_{\widetilde{B}}$ classes. The classes of type $(\mathbb{O}\cap\mathcal{D})_{\widetilde{B}}$ are taken in some ordering, and the $\mathcal{S}_B$ characters are then taken in the corresponding order, that is, with the label $\mu \in \mathcal{S}_B$ corresponding to $h(\mu) \in (\mathbb{O}\cap\mathcal{D})_{\widetilde{B}}$.

Recall that for any $\lambda \in (P \setminus \mathcal{S})(n)$ and $\alpha \in (\mathbb{O}\cap\mathcal{D})(n)$, we have $\{\lambda\}(\sigma^+_\alpha) = \{\lambda\}(\sigma^-_\alpha) = [\lambda](\sigma_\alpha)$. Now take $\mu \in \mathcal{S}(n)$; then for $\alpha = h(\mu)$ we have

$$\{\mu\}_+(\sigma^\pm_{h(\mu)}) = \tfrac{1}{2}(\varepsilon_\mu \pm \sqrt{\varepsilon_\mu H_\mu}) =: y^\pm_\mu\,,$$

while for $\beta \in (\mathbb{O}\cap\mathcal{D})(n)$, $\beta \neq h(\mu)$, we have $\{\mu\}_+(\sigma^+_\beta) = \{\mu\}_+(\sigma^-_\beta)$. Set $c_\mu = \sqrt{\varepsilon_\mu H_\mu} = y^+_\mu - y^-_\mu$. Now for any $\alpha \in (\mathbb{O}\cap\mathcal{D})_{\widetilde{B}}$, subtract the column of the block character table X_{B^A} to the class of σ^+_α from the one to the class of σ^-_α. By the above, then the final columns to the $(\mathbb{O}\cap\mathcal{D})^-_{\widetilde{B}}$ classes are transformed into an upper zero part, corresponding to the characters labeled by nonsymmetric partitions, and below this a diagonal matrix with diagonal entries $-c_\mu$, $\mu \in \mathcal{S}_B$. The table

$$X_B^S = (\{\lambda\}(\sigma_\alpha^{(+)}))_{\substack{\lambda\in\mathcal{D}_B\\ \alpha\in\mathbb{O}_{\tilde{B}}}}$$

is the upper left hand block part of the table X_{B^A}. By the above and Section 3, this is exactly the block part of the 2-regular character table of the symmetric group corresponding to the block B of S_n, with the block splitting constructed for the symmetric groups, that is,

$$X_B^S = X_B = ([\lambda](\sigma_\alpha))_{\substack{\lambda\in\mathcal{D}_B\\ \alpha\in\mathbb{O}_B}}\,.$$

Thus we have $\det X_B^S \not\equiv 0 \mod \mathfrak{p}$. Hence

$$|\det X_{B^A}| = |\det X_B^S| \cdot \prod_{\mu\in\mathcal{S}_B} |c_\mu| = |\det X_B| \cdot \sqrt{a_{(\mathbb{O}\cap\mathcal{D})_{\tilde{B}}}} \not\equiv 0 \mod \mathfrak{p}\,,$$

and we have proved that our construction provides a 2-block splitting for A_n. □

Remark 4.8. In contrast to the case of symmetric groups, the block splitting of the 2-regular classes for the alternating groups as given above is not the only block splitting; already A_6 provides a counterexample. Indeed, instead of associating the classes to (1^6), $(1^3, 3)$, (3^2) to the principal 2-block of A_6, the choices (1^6), $(1^3, 3)$, $(1, 5)_+$ or (1^6), (3^2), $(1, 5)_+$ are also possible.

Recall that for $\alpha \in \mathbb{O}(n) \setminus (\mathbb{O}\cap\mathcal{D})(n)$, the corresponding conjugacy class of σ_α is nonsplit in A_n, so we then have $|C_{A_n}(\sigma_\alpha)|_2 = 2^{k_\alpha - 1}$, with $k_\alpha = l(\alpha) - l(G(\alpha))$ as before.

By Brauer's Theorem 2.1 we can now deduce from our 2-block splitting given in Theorem 4.7 the following result on the Cartan matrices of 2-blocks of alternating groups, providing a combinatorial formula for the elementary divisors which is easy to compute.

Corollary 4.9. *Let $B \in \mathrm{Bl}(S_n)$ of weight $w(B) > 0$, covering the block $B^A \in \mathrm{Bl}(A_n)$. Then the elementary divisors of the Cartan matrix C_{B^A} are*

$$|C_{A_n}(\sigma_\alpha)|_2 = 2^{k_\alpha - 1}, \quad \text{for } \alpha \in \mathbb{O}_B \setminus (\mathbb{O}\cap\mathcal{D})_{\tilde{B}}\,; \qquad 1^{2|(\mathbb{O}\cap\mathcal{D})_{\tilde{B}}|}\,.$$

In particular,

$$\det C_B = 2^{2\ell(B) - \ell(B^A)} \det C_{B^A}\,.$$

References

[Bessenrodt 2007] C. Bessenrodt, "The 2-block splitting in symmetric groups", *Algebra Number Theory* **1**:2 (2007), 223–238. MR 2009a:20021 Zbl 1160.20010

[Bessenrodt and Olsson 1997] C. Bessenrodt and J. B. Olsson, "The 2-blocks of the covering groups of the symmetric groups", *Adv. Math.* **129**:2 (1997), 261–300. MR 98i:20011 Zbl 0979.20014

[Bessenrodt and Olsson 2000] C. Bessenrodt and J. B. Olsson, "Spin representations and powers of 2", *Algebr. Represent. Theory* **3**:3 (2000), 289–300. MR 2001h:20009 Zbl 1037.20011

[Bessenrodt and Olsson 2004] C. Bessenrodt and J. B. Olsson, "On character tables related to the alternating groups", *Sém. Lothar. Combin.* **52** (2004), Art. B52c. MR 2006a:20021

[Brauer 1956] R. Brauer, "Zur Darstellungstheorie der Gruppen endlicher Ordnung", *Math. Z.* **63** (1956), 406–444. MR 17,824g Zbl 0093.03101

[Glaisher 1883] J. W. L. Glaisher, "A theorem in partitions", *Messenger of Math.* **12** (1883), 158–170.

[James and Kerber 1981] G. James and A. Kerber, *The representation theory of the symmetric group*, Encyclopedia of Mathematics and its Applications **16**, Addison-Wesley Publishing Co., Reading, MA, 1981. MR 83k:20003 Zbl 0491.20010

[Olsson 1986] J. B. Olsson, "Lower defect groups in symmetric groups", *J. Algebra* **104**:1 (1986), 37–56. MR 87m:20039 Zbl 0604.20016

[Olsson 1993] J. B. Olsson, *Combinatorics and representations of finite groups*, Vorlesungen aus dem Fachbereich Mathematik **20**, Universität Essen, 1993. MR 95b:20020 Zbl 0796.05095

[Olsson 2003] J. B. Olsson, "Regular character tables of symmetric groups", *Electron. J. Combin.* **10** (2003), Note 3. MR 2004c:20021

[Uno and Yamada 2006] K. Uno and H.-F. Yamada, "Elementary divisors of Cartan matrices for symmetric groups", *J. Math. Soc. Japan* **58**:4 (2006), 1031–1036. MR 2007i:20020 Zbl 1108.20009

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