

Algebra & Number Theory

Volume 3

2009

No. 8

Log minimal models according to Shokurov

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mathematical sciences publishers

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Following Shokurov's ideas, we give a short proof of the following klt version of his result: termination of terminal log flips in dimension d implies that any klt pair of dimension d has a log minimal model or a Mori fibre space. Thus, in particular, any klt pair of dimension 4 has a log minimal model or a Mori fibre space.

1. Introduction

All the varieties in this paper are assumed to be over an algebraically closed field k of characteristic zero. We refer the reader to [Section 2](#) for notation and terminology.

The following conjecture is perhaps the most important problem in birational geometry.

Conjecture 1.1 (Minimal model). Let $(X/Z, B)$ be a Kawamata log terminal (klt) pair. Then it has a log minimal model or a Mori fibre space.

The 2-dimensional case of this conjecture is considered to be classical. The 3-dimensional case was settled in the 80's and 90's by the efforts of many mathematicians, in particular Mori, Shokurov and Kawamata. The higher-dimensional case has seen considerable progress in recent years, thanks primarily to Shokurov's work on the existence of log flips, which paved the way for further progress. The conjecture is also settled for pairs of general type [[Birkar et al. 2006](#)], and inductive arguments have been proposed for pairs of nonnegative Kodaira dimension [[Birkar 2007](#)]. For a more detailed account of the known cases of this conjecture, see the introduction to [[Birkar 2007](#)].

Shokurov [[2006](#)] proved that the log minimal model program (LMMP) in dimension $d - 1$ and termination of terminal log flips in dimension d imply [Conjecture 1.1](#) in dimension d even for log canonical (lc) pairs. (In this paper, by termination of *terminal log flips in dimension d* , we will mean termination of any sequence $X_i \dashrightarrow X_{i+1}/Z_i$ of log flips/ Z starting with a d -dimensional klt pair $(X/Z, B)$ which is terminal in codimension ≥ 2 ; see [Section 2](#) for a more precise formulation.)

MSC2000: 14E30.

Keywords: minimal models, Mori fibre spaces.

Following Shokurov's method and using results of [Birkar et al. 2006], we give a short proof of:

Theorem 1.2. *Termination of terminal log flips in dimension d implies Conjecture 1.1 in dimension d ; more precisely, for a klt pair $(X/Z, B)$ of dimension d one constructs a log minimal model or a Mori fibre space by running the LMMP/ Z on $K_X + B$ with scaling of a suitable big/ Z \mathbb{R} -divisor and proving that it terminates.*

As in [Shokurov 2006], one immediately derives the following:

Corollary 1.3. *Conjecture 1.1 holds in dimension 4.*

Note that when $(X/Z, B)$ is effective (for example if it is of nonnegative Kodaira dimension), log minimal models are constructed in [Birkar 2007], using different methods, in dimension ≤ 5 .

2. Basics

Let k be an algebraically closed field of characteristic zero. For an \mathbb{R} -divisor D on a variety X over k , we use D^\sim to denote the birational transform of D on a specified birational model of X .

Definition 2.1. A pair $(X/Z, B)$ consists of normal quasiprojective varieties X, Z over k , an \mathbb{R} -divisor B on X with coefficients in $[0, 1]$ such that $K_X + B$ is \mathbb{R} -Cartier, and a projective morphism $X \rightarrow Z$. $(X/Z, B)$ is called *log smooth* if X is smooth and $\text{Supp } B$ has simple normal crossing singularities.

For a prime divisor D on some birational model of X with a nonempty centre on X , $a(D, X, B)$ denotes the log discrepancy. $(X/Z, B)$ is *terminal in codimension ≥ 2* if $a(D, X, B) > 1$ whenever D is exceptional/ X . Log flips preserve this condition but divisorial contractions may not.

Let $(X/Z, B)$ be a klt pair. By a *log flip/ Z* we mean the flip of a $K_X + B$ -negative extremal flipping contraction/ Z . A *sequence of log flips/ Z starting with $(X/Z, B)$* is a sequence $X_i \dashrightarrow X_{i+1}/Z_i$ in which $X_i \rightarrow Z_i \leftarrow X_{i+1}$ is a $K_{X_i} + B_i$ -flip/ Z and B_i is the birational transform of B_1 on X_1 , and $(X_1/Z, B_1) = (X/Z, B)$. By termination of terminal log flips in dimension d we mean termination of such a sequence in which $(X_1/Z, B_1)$ is a d -dimensional klt pair which is terminal in codimension ≥ 2 . Now assume that G is an \mathbb{R} -Cartier divisor on X . A *sequence of G -flops/ Z with respect to $(X/Z, B)$* is a sequence $X_i \dashrightarrow X_{i+1}/Z_i$ in which $X_i \rightarrow Z_i \leftarrow X_{i+1}$ is a G_i -flip/ Z such that $K_{X_i} + B_i \equiv 0/Z_i$ where G_i is the birational transform of G on $X = X_1$.

Definition 2.2 ([Birkar 2007, §2]). Let $(X/Z, B)$ be a klt pair, $(Y/Z, B_Y)$ a \mathbb{Q} -factorial klt pair, $\phi: X \dashrightarrow Y/Z$ a birational map such that ϕ^{-1} does not contract divisors, and B_Y be the birational transform of B (Note that since $X \rightarrow Z$ and

$Y \rightarrow Z$ are both projective, by the definition of a pair, X and Y have the same image on Z). Moreover, assume that

$$a(D, X, B) \leq a(D, Y, B_Y)$$

for any prime divisor D on birational models of X and assume that the strict inequality holds for any prime divisor D on X which is exceptional/ Y .

We say that $(Y/Z, B_Y)$ is a *log minimal model* of $(X/Z, B)$ if $K_Y + B_Y$ is nef/ Z . On the other hand, we say that $(Y/Z, B_Y)$ is a *Mori fibre space* of $(X/Z, B)$ if there is a $K_Y + B_Y$ -negative extremal contraction $Y \rightarrow Y'/Z$ such that $\dim Y' < \dim Y$.

Typically, one obtains a log minimal model or a Mori fibre space by a finite sequence of divisorial contractions and log flips.

Remark 2.3. Let $(X/Z, B)$ be a klt pair and $W \rightarrow X$ a log resolution. Let $B_W = B^\sim + (1 - \epsilon) \sum E_i$ where $0 < \epsilon \ll 1$ and E_i are the exceptional/ X divisors on W . Remember that B^\sim is the birational transform of B . If $(Y/X, B_Y)$ is a log minimal model of $(W/X, B_W)$, which exists by [Birkar et al. 2006], then by the negativity lemma $Y \rightarrow X$ is a small \mathbb{Q} -factorialisation of X . To find a log minimal model or a Mori fibre space of $(X/Z, B)$, it is enough to find one for $(Y/Z, B_Y)$. So, one could assume that X is \mathbb{Q} -factorial by replacing it with Y .

We recall a variant of the LMMP with scaling which we use in this paper. Let $(X/Z, B+C)$ be a \mathbb{Q} -factorial klt pair such that $K_X + B + C$ is nef/ Z and $B, C \geq 0$. By [Birkar 2007, Lemma 2.7], either $K_X + B$ is nef/ Z or there is an extremal ray R/Z such that

$$(K_X + B) \cdot R < 0 \text{ and } (K_X + B + \lambda_1 C) \cdot R = 0,$$

where

$$\lambda_1 := \inf\{t \geq 0 \mid K_X + B + tC \text{ is nef}/Z\}$$

and $K_X + B + \lambda_1 C$ is nef/ Z . Now assume that R defines a divisorial contraction or a log flip $X \dashrightarrow X'$. We can consider $(X'/Z, B' + \lambda_1 C')$, where $B' + \lambda_1 C'$ is the birational transform of $B + \lambda_1 C$ and continue the argument. That is, either $K_{X'} + B'$ is nef/ Z or there is an extremal ray R'/Z such that $(K_{X'} + B') \cdot R' < 0$ and $(K_{X'} + B' + \lambda_2 C') \cdot R' = 0$, where

$$\lambda_2 := \inf\{t \geq 0 \mid K_{X'} + B' + tC' \text{ is nef}/Z\}$$

and $K_{X'} + B' + \lambda_2 C'$ is nef/ Z . By continuing this process, we obtain a special kind of LMMP on $K_X + B$ which we refer to as the *LMMP with scaling of C* . If it terminates, then we obviously get a log minimal model or a Mori fibre space for $(X/Z, B)$. Note that the required log flips exist by [Birkar et al. 2006].

3. Extremal rays

In this section, for convenience of the reader, we give the proofs of some results about extremal rays [Shokurov 2006, Corollary 9, Addendum 4]. The norm $\|G\|$ of an \mathbb{R} -divisor G denotes the maximum of the absolute value of its coefficients.

Let $X \rightarrow Z$ be a projective morphism of normal quasiprojective varieties. A curve Γ on X is called *extremal/Z* if it generates an extremal ray R/Z which defines a contraction $X \rightarrow S/Z$, and if for some ample/ Z divisor H we have $H \cdot \Gamma = \min\{H \cdot \Sigma\}$, where Σ ranges over curves generating R . If $(X/Z, B)$ is divisorial log terminal (dlt) and $(K_X + B) \cdot R < 0$, then by [Kawamata 1991] there is a curve Σ generating R such that $(K_X + B) \cdot \Sigma \geq -2 \dim X$. On the other hand, since Γ and Σ both generate R we have

$$\frac{(K_X + B) \cdot \Gamma}{H \cdot \Gamma} = \frac{(K_X + B) \cdot \Sigma}{H \cdot \Sigma},$$

hence

$$(K_X + B) \cdot \Gamma = (K_X + B) \cdot \Sigma \frac{H \cdot \Gamma}{H \cdot \Sigma} \geq -2 \dim X.$$

Remark 3.1. Let $(X/Z, B)$ be a \mathbb{Q} -factorial klt pair, F be a reduced divisor on X whose support contains that of B , and V be the \mathbb{R} -vector space of divisors generated by the components of F .

- (i) By [Shokurov 1992, Property 1.3.2; 1996, First Main Theorem 6.2 and Remark 6.4], the sets

$$\mathcal{L} = \{\Delta \in V \mid (X/Z, \Delta) \text{ is lc}\} \quad \text{and} \quad \mathcal{N} = \{\Delta \in \mathcal{L} \mid K_X + \Delta \text{ is nef}/Z\}$$

are rational polytopes in V . Since $B \in \mathcal{L}$, there are rational boundaries $B^1, \dots, B^r \in \mathcal{L}$ and nonnegative real numbers a_1, \dots, a_r such that $B = \sum a_j B^j$, $\sum a_j = 1$, and each $(X/Z, B^j)$ is klt. In particular, there is $m \in \mathbb{N}$ such that $m(K_X + B^j)$ are Cartier, and for any curve Γ on X the intersection number $(K_X + B) \cdot \Gamma$ can be written as $\sum a_j \frac{n_j}{m}$ for certain $n_1, \dots, n_r \in \mathbb{Z}$. Moreover, if Γ is extremal/ Z , then the n_j satisfy $n_j \geq -2m \dim X$.

- (ii) If $K_X + B$ is nef/ Z , then $B \in \mathcal{N}$ and so one can choose the B^j so that $K_X + B^j$ are nef/ Z .

Lemma 3.2. *Let $(X/Z, B)$ be a \mathbb{Q} -factorial klt pair. There is a real number $\alpha > 0$ such that:*

- (i) *If Γ is any extremal curve/ Z and if $(K_X + B) \cdot \Gamma > 0$, then $(K_X + B) \cdot \Gamma > \alpha$.*
- (ii) *If $K_X + B$ is nef/ Z , then for any \mathbb{R} -divisor G , any sequence $X_i \dashrightarrow X_{i+1}/Z_i$ of G -flops/ Z with respect to $(X/Z, B)$, and any extremal curve Γ on X_i , if $(K_{X_i} + B_i) \cdot \Gamma > 0$, then $(K_{X_i} + B_i) \cdot \Gamma > \alpha$ where B_i is the birational transform of B .*

Proof. (i) If B is a \mathbb{Q} -divisor, then the statement is trivially true. Let $B^1, \dots, B^r, a_1, \dots, a_r$, and m be as in Remark 3.1(i). Let Γ be an extremal curve/ Z . Then, $(K_X + B) \cdot \Gamma = \sum a_j (K_X + B^j) \cdot \Gamma$ and since for each j we have $(K_X + B^j) \cdot \Gamma \geq -2 \dim X$, the existence of α is clear for (i).

(ii) By Remark 3.1(ii) we may also assume that $K_X + B^j$ are nef/ Z . Then, the sequence $X_i \dashrightarrow X_{i+1}/Z_i$ is also a sequence of flops with respect to each $(X/Z, B^j)$. In particular, $(X_i/Z, B_i^j)$ is klt and $m(K_{X_i} + B_i^j)$ is Cartier for any j, i where B_i^j is the birational transform of B^j . The rest is as in (i). \square

Proposition 3.3. *Let $(X/Z, B)$ be a \mathbb{Q} -factorial klt pair, F a reduced divisor on X whose support contains that of B , and \mathcal{L} as in Remark 3.1. There is a rational polytope $\mathcal{K} \subset \mathcal{L}$ of klt boundaries and of maximal dimension containing an open neighborhood of B in \mathcal{L} (with respect to the topology on \mathcal{L} induced by the norm $\|\cdot\|$) such that*

- (i) *if $\Delta \in \mathcal{K}$ and $(K_X + \Delta) \cdot R < 0$ for an extremal ray R/Z , then $(K_X + B) \cdot R \leq 0$, and*
- (ii) *if $K_X + B$ is nef/ Z , $\Delta \in \mathcal{K}$, we have a sequence $X_i \dashrightarrow X_{i+1}/Z_i$ of $K_X + \Delta$ -flops/ Z with respect to $(X/Z, B)$, and $(K_{X_i} + \Delta_i) \cdot R < 0$ for an extremal ray R/Z on some X_i , then $(K_{X_i} + B_i) \cdot R = 0$, where Δ_i, B_i are the birational transforms of Δ, B respectively.*

Proof. (i) Let $\mathcal{M} \subset \mathcal{L}$ be a rational polytope of klt boundaries and of maximal dimension containing an open neighborhood of B in \mathcal{L} . If the statement is not true then there is an infinite sequence of $\Delta_l \in \mathcal{M}$ and extremal rays R_l/Z such that for each l we have

$$(K_X + \Delta_l) \cdot R_l < 0, \quad (K_X + B) \cdot R_l > 0,$$

and $\|\Delta_l - B\|$ converges to 0. Let Ω_l be the point on the boundary of \mathcal{M} such that $\Omega_l - \Delta_l = b_l(\Delta_l - B)$ for some real number $b_l \geq 0$ and such that $\|\Omega_l - B\|$ is maximal. So, Ω_l is the most far away point in \mathcal{M} which is on the line determined by B and Δ_l , in the direction of Δ_l . Since $\|\Delta_l - B\|$ converges to 0, b_l converges to $+\infty$.

By assumptions, $(X/Z, \Omega_l)$ is klt and if Γ_l is an extremal curve/ Z generating R_l , then

$$(\Omega_l - \Delta_l) \cdot \Gamma_l = (K_X + \Omega_l) \cdot \Gamma_l - (K_X + \Delta_l) \cdot \Gamma_l \geq -2 \dim X.$$

This is not possible because by Lemma 3.2,

$$(K_X + \Delta_l) \cdot \Gamma_l + (B - \Delta_l) \cdot \Gamma_l = (K_X + B) \cdot \Gamma_l > \alpha,$$

and by the same arguments $(B - \Delta_l) \cdot \Gamma_l$ approaches 0.

By definition, the sequence $X_i \dashrightarrow X_{i+1}/Z_i$ is a sequence of $K_X + \Delta$ -flips which are numerically trivial with respect to $K_X + B$. Let \mathcal{H} be as in (i). Assume that R is an extremal ray/ Z on X_i such that $(K_{X_i} + \Delta_i) \cdot R < 0$ but $(K_{X_i} + B_i) \cdot R > 0$. Let Γ be an extremal curve/ Z generating R . Let Ω be the point on the boundary of \mathcal{H} which is chosen for Δ similarly as in (i). By assumptions, $(X_i/Z, \Delta_i)$ and $(X_i/Z, \Omega_i)$ are klt where Ω_i is the birational transform of Ω . So,

$$(\Omega_i - \Delta_i) \cdot \Gamma = (K_{X_i} + \Omega_i) \cdot \Gamma - (K_{X_i} + \Delta_i) \cdot \Gamma \geq -2 \dim X.$$

On the other hand, $(K_{X_i} + B_i) \cdot \Gamma > \alpha$ where α is as in Lemma 3.2. By construction, there is some $b \geq 0$ such that $b(\Delta_i - B_i) = \Omega_i - \Delta_i$. Therefore,

$$(K_{X_i} + \Delta_i) \cdot \Gamma = (K_{X_i} + B_i) \cdot \Gamma + (\Delta_i - B_i) \cdot \Gamma > \alpha - \frac{2 \dim X}{b}.$$

which is not possible if $b\alpha > 2 \dim X$. In other words, if Δ is close enough to B then the statement of (ii) also holds, that is, we only need to shrink \mathcal{H} appropriately. □

4. Proof of the main results

Proof of Theorem 1.2. Let $(X/Z, B)$ be a klt pair of dimension d . By Remark 2.3, we can assume that X is \mathbb{Q} -factorial. Let $H \geq 0$ be an \mathbb{R} -divisor which is big/ Z so that $K_X + B + H$ is klt and nef/ Z . Run the LMMP/ Z on $K_X + B$ with scaling of H . If the LMMP terminates, then we get a log minimal model or a Mori fibre space. Suppose that we get an infinite sequence $X_i \dashrightarrow X_{i+1}/Z_i$ of log flips/ Z , where we may also assume that $(X_1/Z, B_1) = (X/Z, B)$.

Let λ_i be the threshold on X_i determined by the LMMP with scaling as explained in Section 2. So, $K_{X_i} + B_i + \lambda_i H_i$ is nef/ Z ,

$$(K_{X_i} + B_i) \cdot R_i < 0 \quad \text{and} \quad (K_{X_i} + B_i + \lambda_i H_i) \cdot R_i = 0,$$

where B_i and H_i are the birational transforms of B and H and, R_i is the extremal ray which defines the flipping contraction $X_i \rightarrow Z_i$. Obviously, $\lambda_i \geq \lambda_{i+1}$.

Put $\lambda = \lim_{i \rightarrow \infty} \lambda_i$. If the limit is attained, that is, $\lambda = \lambda_i$ for some i , then the sequence terminates by Corollary 1.4.2 of [Birkar et al. 2006]. So, we assume that the limit is not attained. Actually, if $\lambda > 0$, again [Birkar et al. 2006] implies that the sequence terminates. However, we do not need to use [Birkar et al. 2006] in this case. In fact, by replacing B_i with $B_i + \lambda H_i$, we can assume that $\lambda = 0$ hence $\lim_{i \rightarrow \infty} \lambda_i = 0$.

Put $\Lambda_i := B_i + \lambda_i H_i$. Since we are assuming that terminal log flips terminate, or, alternatively, by Corollary 1.4.3 of [Birkar et al. 2006], we can construct a terminal (in codimension ≥ 2) crepant model $(Y_i/Z, \Theta_i)$ of $(X_i/Z, \Lambda_i)$. A slight modification of the argument in Remark 2.3 would do this. Note that we can assume

that all the Y_i are isomorphic to Y_1 in codimension one, perhaps after truncating the sequence. Let $\Delta_1 = \lim_{i \rightarrow \infty} \Theta_i^\sim$ on Y_1 and let Δ_i be its birational transform on Y_i . The limit is obtained componentwise.

Since H_i is big/ Z and $K_{X_i} + \Lambda_i$ is klt and nef/ Z , $K_{X_i} + \Lambda_i$ and $K_{Y_i} + \Theta_i$ are semiample/ Z by the base point freeness theorem for \mathbb{R} -divisors. Thus, $K_{Y_i} + \Delta_i$ is a limit of movable/ Z divisors which in particular means that it is pseudo-effective/ Z . Note that if $K_{Y_i} + \Delta_i$ is not pseudo-effective/ Z , we get a contradiction by Corollary 1.3.2 of [Birkar et al. 2006].

Now run the LMMP/ Z on $K_{Y_1} + \Delta_1$. No divisor will be contracted again because $K_{Y_1} + \Delta_1$ is a limit of movable/ Z divisors. Since $K_{Y_1} + \Delta_1$ is terminal in codimension ≥ 2 , by assumptions, the LMMP terminates with a log minimal model $(W/Z, \Delta)$. By construction, Δ on W is the birational transform of Δ_1 on Y_1 , and $G_i := \Theta_i^\sim - \Delta$ on W satisfies $\lim_{i \rightarrow \infty} G_i = 0$.

By Proposition 3.3, for each G_i with $i \gg 0$, we can run the LMMP/ Z on $K_W + \Delta + G_i$ which will be a sequence of G_i -flops, that is, $K + \Delta$ would be numerically zero on all the extremal rays contracted in the process. No divisor will be contracted because $K_W + \Delta + G_i$ is movable/ Z . The LMMP ends up with a log minimal model $(W_i/Z, \Omega_i)$. Here, Ω_i is the birational transform of $\Delta + G_i$ and so of Θ_i . Let S_i be the lc model of $(W_i/Z, \Omega_i)$ which is the same as the lc model of $(Y_i/Z, \Theta_i)$ and that of $(X_i/Z, \Lambda_i)$ because $K_{W_i} + \Omega_i$ and $K_{Y_i} + \Theta_i$ are nef/ Z with W_i and Y_i being isomorphic in codimension one, and $K_{Y_i} + \Theta_i$ is the pullback of $K_{X_i} + \Lambda_i$. Also note that since $K_{X_i} + B_i$ is pseudoeffective/ Z , $K_{X_i} + \Lambda_i$ is big/ Z ; hence S_i is birational to X_i .

By construction, $K_{W_i} + \Delta^\sim$ is nef/ Z and it turns out that $K_{W_i} + \Delta^\sim \sim_{\mathbb{R}} 0/S_i$. Suppose that this is not the case. Then, $K_{W_i} + \Delta^\sim$ is not numerically zero/ S_i hence there is some curve C/S_i such that $(K_{W_i} + \Delta^\sim + G_i^\sim) \cdot C = 0$ but $(K_{W_i} + \Delta^\sim) \cdot C > 0$ which implies that $G_i^\sim \cdot C < 0$. Hence, there is a $K_{W_i} + \Delta^\sim + (1 + \tau)G_i^\sim$ -negative extremal ray R/S_i for any $\tau > 0$. This contradicts Proposition 3.3 because we must have

$$(K_{W_i} + \Delta^\sim + G_i^\sim) \cdot R = (K_{W_i} + \Delta^\sim) \cdot R = 0.$$

Therefore, $K_{W_i} + \Delta^\sim \sim_{\mathbb{R}} 0/S_i$. Now $K_{X_i} + \Lambda_i \sim_{\mathbb{R}} 0/Z_i$ implies that Z_i is over S_i and so $K_{Y_i} + \Delta_i \sim_{\mathbb{R}} 0/S_i$. On the other hand, $K_{X_i} + B_i$ is the pushdown of $K_{Y_i} + \Delta_i$; hence $K_{X_i} + B_i \sim_{\mathbb{R}} 0/S_i$. Thus, $K_{X_i} + B_i \sim_{\mathbb{R}} 0/Z_i$ and this contradicts the fact that $X_i \rightarrow Z_i$ is a $K_{X_i} + B_i$ -flipping contraction. So, the sequence of flips terminates and this completes the proof. \square

Proof of Corollary 1.3. Since terminal log flips terminate in dimension 4 by [Fujino 2004; Shokurov 2004] (see also [Alexeev et al. 2007]), the result follows from Theorem 1.2. \square

Acknowledgements

I thank V. V. Shokurov and Y. Kawamata for their comments and suggestions, which improved this paper.

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Communicated by Yujiro Kawamata

Received 2009-03-12

Revised 2009-09-07

Accepted 2009-10-06

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The subscription price for 2009 is US \$140/year for the electronic version, and \$200/year (+\$35 shipping outside the US) for print and electronic. Subscriptions, requests for back issues from the last three years and changes of subscribers address should be sent to Mathematical Sciences Publishers, Department of Mathematics, University of California, Berkeley, CA 94720-3840, USA.

Algebra & Number Theory (ISSN 1937-0652) at Mathematical Sciences Publishers, Department of Mathematics, University of California, Berkeley, CA 94720-3840 is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

ANT peer-review and production is managed by EditFLOW™ from Mathematical Sciences Publishers.

PUBLISHED BY

 **mathematical sciences publishers**
<http://www.mathscipub.org>

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Typeset in L^AT_EX

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Algebra & Number Theory

Volume 3 No. 8 2009

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