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Centers of graded fusion categories

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Let $\mathscr{C}$ be a fusion category faithfully graded by a finite group G and let $\mathscr{D}$ be the trivial component of this grading. The center $\mathscr{Z}(\mathscr{C})$ of $\mathscr{C}$ is shown to be canonically equivalent to a G-equivariantization of the relative center $\mathscr{Z}_{\mathscr{D}}(\mathscr{C})$. We use this result to obtain a criterion for $\mathscr{C}$ to be group-theoretical and apply it to Tambara–Yamagami fusion categories. We also find several new series of modular categories by analyzing the centers of Tambara–Yamagami categories. Finally, we prove a general result about the existence of zeroes in S-matrices of weakly integral modular categories.

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1. Introduction

Throughout this paper we work over an algebraically closed field k of characteristic 0. All categories considered in this paper are finite, abelian, semisimple, and k-linear. We freely use the language and basic theory of fusion categories, module categories over them, braided categories, and Frobenius–Perron dimensions [Bakalov and Kirillov 2001; Ostrik 2003; Etingof et al. 2005].

Let G be a finite group. A fusion category $\mathscr{C}$ is *G-graded* if there is a decomposition

$$\mathscr{C} = \bigoplus_{g\in G} \mathscr{C}_g$$

of $\mathscr{C}$ into a direct sum of full abelian subcategories such that the tensor product of $\mathscr{C}$ maps $\mathscr{C}_g \times \mathscr{C}_h$ to $\mathscr{C}_{gh}$, for all $g, h \in G$. A *G-extension* of a fusion category $\mathscr{D}$ is a G-graded fusion category $\mathscr{C}$ whose trivial component $\mathscr{C}_e$, where e is the identity of G, is equivalent to $\mathscr{D}$.

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Gradings and extensions play an important role in the study and classification of fusion categories. For example, *nilpotent* fusion categories (that is, those categories that can be obtained from the trivial category by a sequence of group extensions) were studied in [Gelaki and Nikshych 2008]. It was proved in [Etingof et al. 2005] that every fusion category of prime power dimension is nilpotent. Group-theoretical properties of such categories were studied in [Drinfeld et al. 2007]. Recently, fusion categories of dimension $p^n q^m$, where p, q are primes, were shown to be Morita equivalent to nilpotent categories [Etingof et al. 2009].

The main goal of this paper is to describe the center $\mathcal{Z}(\mathcal{C})$ of a G-graded fusion category $\mathcal{C}$ in terms of its trivial component $\mathcal{D}$ (Theorem 3.5) and apply this description to the study of structural properties of $\mathcal{C}$ and the construction of new examples of modular categories.

The organization of the paper is as follows. In Section 2 we recall some basic notions, results, and examples of fusion categories, notably the notions of the relative center of a bimodule category [Majid 1991], group action on a fusion category and crossed product [Tambara 2001], equivariantization and de-equivariantization theory [Arkhipov and Gaitsgory 2003; Bruguières 2000; Gaitsgory 2005; Kirillov 2002; Müger 2000; Drinfeld et al. 2009], and braided G-crossed fusion categories [Turaev 2000; 2008].

In Section 3 we study the center $\mathcal{Z}(\mathcal{C})$ of a G-graded fusion category $\mathcal{C}$. We show that if $\mathcal{D}$ is the trivial component of $\mathcal{C}$, then the relative center $\mathcal{Z}_{\mathcal{D}}(\mathcal{C})$ has a canonical structure of a braided G-crossed category and there is an equivalence of braided fusion categories $\mathcal{Z}_{\mathcal{D}}(\mathcal{C})^G \cong \mathcal{Z}(\mathcal{C})$ (Theorem 3.5). Thus, the structure of $\mathcal{Z}(\mathcal{C})$ can be understood in terms of a smaller and more transparent category $\mathcal{Z}_{\mathcal{D}}(\mathcal{C})$. In particular, there is a canonical braided action (studied in detail in [Etingof et al. 2009]) of G on $\mathcal{Z}(\mathcal{D})$. In Corollary 3.10 we use this action to prove that $\mathcal{C}$ is group-theoretical if and only if $\mathcal{Z}(\mathcal{D})$ contains a G-stable Lagrangian subcategory. As an illustration, we describe the center of a crossed product fusion category $\mathcal{C} = \mathcal{D} \rtimes G$.

We apply the results from Section 4 to the study of Tambara–Yamagami categories [Tambara and Yamagami 1998]. We obtain a convenient description of the centers of such categories as equivariantizations and compute their modular data, that is, S- and T-matrices. This computation was previously done in [Izumi 2001] using different techniques. We establish a criterion for a Tambara–Yamagami category to be group-theoretical (Theorem 4.6). We also extend the construction of non-group-theoretical semisimple Hopf algebras from Tambara–Yamagami categories given in [Nikshych 2008].

In Section 5 we construct a series of new modular categories as factors of the centers of Tambara–Yamagami categories. One associates a pair of such categories $\mathcal{E}(q, \pm)$ with any nondegenerate quadratic form q on an abelian group A of odd order. The categories $\mathcal{E}(q, \pm)$ have dimension $4|A|$. They are group-theoretical if and only if A contains a Lagrangian subgroup with respect to q. We compute the S- and T-matrices of $\mathcal{E}(q, \pm)$ and write down several small examples explicitly.

Section 6 is independent from the rest of the paper and contains a general result about existence of zeroes in S-matrices of weakly integral modular categories (Theorem 6.1). This is a categorical analogue of a classical result of Burnside in character theory.

2. Preliminaries

2A. ***Dual fusion categories and Morita equivalence.*** Let $\mathcal{C}$ be a fusion category and let $\mathcal{M}$ be an indecomposable right $\mathcal{C}$-module category $\mathcal{M}$. The category $\mathcal{C}^*_{\mathcal{M}}$ of $\mathcal{C}$-module endofunctors of $\mathcal{M}$ is a fusion category, called the dual of $\mathcal{C}$ with respect to $\mathcal{M}$ [Etingof et al. 2005; Ostrik 2003].

Following [Müger 2003a], we say that two fusion categories $\mathcal{C}$ and $\mathcal{D}$ are *Morita equivalent* if $\mathcal{D}$ is equivalent to $\mathcal{C}^*_{\mathcal{M}}$, for some indecomposable right $\mathcal{C}$-module category $\mathcal{M}$. A fusion category is said to be *pointed* if all its simple objects are invertible (any such category is equivalent to the category Vec_G^{ω} of vector spaces graded by a finite group G with the associativity constraint given by a 3-cocycle $\omega \in Z^3(G, k^\times)$). A fusion category is called *group-theoretical* if it is Morita equivalent to a pointed fusion category. See [Ostrik 2003; Etingof et al. 2005; Nikshych 2008] for details of the theory of group-theoretical categories.

2B. ***The center of a bimodule category and the relative center of a fusion category.*** Let $\mathcal{C}$ be a fusion category with unit object $\mathbf{1}$ and associativity constraint $\alpha_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z)$ and let $\mathcal{M}$ be a $\mathcal{C}$-bimodule category.

Definition 2.1. The *center* of $\mathcal{M}$ is the category $\mathcal{Z}_{\mathcal{C}}(\mathcal{M})$ of $\mathcal{C}$-bimodule functors from $\mathcal{C}$ to $\mathcal{M}$.

Explicitly, the objects of $\mathcal{Z}_{\mathcal{C}}(\mathcal{M})$ are pairs (M, γ), where M is an object of $\mathcal{M}$ and

$$\gamma = \{\gamma_X : X \otimes M \xrightarrow{\sim} M \otimes X\}_{X \in \mathcal{C}} \tag{1}$$

is a natural family of isomorphisms making the diagram

$$\begin{array}{ccccc} & X \otimes (M \otimes Y) & \xrightarrow{\alpha^{-1}_{X,M,Y}} & (X \otimes M) \otimes Y & \\ {\scriptstyle \gamma_Y} \nearrow & & & & \searrow {\scriptstyle \gamma_X} \\ X \otimes (Y \otimes M) & & & & (M \otimes X) \otimes Y, \\ {\scriptstyle \alpha^{-1}_{X,Y,M}} \searrow & & & & \nearrow {\scriptstyle \alpha^{-1}_{M,X,Y}} \\ & (X \otimes Y) \otimes M & \xrightarrow[\gamma_{X \otimes Y}]{} & M \otimes (X \otimes Y) & \end{array} \tag{2}$$

commutative, where the α's denote the associativity constraints in $\mathcal{M}$.

Indeed, a $\mathscr{C}$-bimodule functor $F : \mathscr{C} \to \mathscr{M}$ is completely determined by the pair $(F(\mathbf{1}),\ \{\gamma_X\}_{X\in\mathscr{C}})$, where $\gamma = \{\gamma_X\}_{X\in\mathscr{C}}$ is the collection of isomorphisms

$$\gamma_X : X \otimes F(\mathbf{1}) \xrightarrow{\sim} F(X) \xrightarrow{\sim} F(\mathbf{1}) \otimes X,$$

coming from the $\mathscr{C}$-bimodule structure on F.

We will call the natural family of isomorphisms (1) the *central structure* of an object $X \in \mathscr{Z}_{\mathscr{C}}(\mathscr{M})$.

Remark 2.2. (i) The definition of the center of a bimodule category is parallel to that of the center of a bimodule over a ring.

(ii) We will often suppress the central structure while working with objects of $\mathscr{Z}_{\mathscr{C}}(\mathscr{M})$ and refer to (M, γ) simply as M.

(iii) $\mathscr{Z}_{\mathscr{C}}(\mathscr{M})$ is a semisimple abelian category. It has the obvious canonical structure of a $\mathscr{Z}(\mathscr{C})$-module category, where $\mathscr{Z}(\mathscr{C})$ is the center of $\mathscr{C}$ (see, for example, [Kassel 1995, Section XIII.4] for the definition of $\mathscr{Z}(\mathscr{C})$).

Here is an important special case of this construction. Let $\mathscr{C}$ be a fusion category and let $\mathscr{D} \subset \mathscr{C}$ be a fusion subcategory. Then $\mathscr{C}$ is a $\mathscr{D}$-bimodule category. We will call $\mathscr{Z}_{\mathscr{D}}(\mathscr{C})$ the *relative center* of $\mathscr{C}$.

Remark 2.3. The aforementioned construction of the relative center is a special case of a more general construction considered in [Majid 1991, Definition 3.2 and Theorem 3.3].

It is easy to see that $\mathscr{Z}_{\mathscr{D}}(\mathscr{C})$ is a tensor category with tensor product defined as follows. If (X, γ) and (X', γ') are objects in $\mathscr{Z}_{\mathscr{D}}(\mathscr{C})$ then

$$(X, \gamma) \otimes (X', \gamma') := (X \otimes X', \tilde{\gamma}),$$

where $\tilde{\gamma}_V : V \otimes (X \otimes X') \xrightarrow{\sim} (X \otimes X') \otimes V,\ V \in \mathscr{D}$, is defined by the diagram

$$\begin{array}{ccccc}
V \otimes (X \otimes X') & \xrightarrow{\alpha^{-1}_{V,X,X'}} & (V \otimes X) \otimes X' & \xrightarrow{\gamma_V} & (X \otimes V) \otimes X' \\
\downarrow{\scriptstyle \tilde{\gamma}_V} & & & & \downarrow{\scriptstyle \alpha_{X,V,X'}} \\
(X \otimes X') \otimes V & \xleftarrow{\alpha^{-1}_{X,X',V}} & X \otimes (X' \otimes V) & \xleftarrow{\gamma'_V} & X \otimes (V \otimes X').
\end{array} \tag{3}$$

The unit object of $\mathscr{Z}_{\mathscr{D}}(\mathscr{C})$ is $(\mathbf{1},\ \mathrm{id})$. The dual of (X, γ) is $(X^*,\ \overline{\gamma})$, where $\overline{\gamma}_V := (\gamma_{{}^*V})^*$.

Remark 2.4. Let $\mathscr{C}$ and $\mathscr{D}$ be as above.

(i) $\mathscr{Z}_{\mathscr{D}}(\mathscr{C})$ is dual to the fusion category $\mathscr{D} \boxtimes \mathscr{C}^{\mathrm{rev}}$ (where $\mathscr{C}^{\mathrm{rev}}$ is the fusion category obtained from $\mathscr{C}$ by reversing the tensor product and $\boxtimes$ is Deligne's tensor product of fusion categories) with respect to its module category $\mathscr{C}$, where $\mathcal{D}$ and $\mathcal{C}^{\text{rev}}$ act on $\mathcal{C}$ via the right and left multiplication respectively. In particular, $\mathcal{Z}_{\mathcal{D}}(\mathcal{C})$ is a fusion category.

(ii) $\operatorname{FPdim}(\mathcal{Z}_{\mathcal{D}}(\mathcal{C})) = \operatorname{FPdim}(\mathcal{C}) \operatorname{FPdim}(\mathcal{D})$, where FPdim denotes the Frobenius–Perron dimension of a category.

(iii) $\mathcal{Z}_{\mathcal{C}}(\mathcal{C})$ coincides with the center $\mathcal{Z}(\mathcal{C})$ of $\mathcal{C}$. This category has a canonical braiding given by

$$c_{(X,\gamma),\,(X',\gamma')} = \gamma_{X'} : (X, \gamma) \otimes (X', \gamma') \xrightarrow{\sim} (X', \gamma') \otimes (X, \gamma). \tag{4}$$

(iv) There is an obvious forgetful tensor functor:

$$\mathcal{Z}(\mathcal{C}) \mapsto \mathcal{Z}_{\mathcal{D}}(\mathcal{C}) : (X, \gamma) \mapsto (X, \gamma\,|_{\mathcal{D}}). \tag{5}$$

2C. *Centralizers in braided fusion categories.* Let $\mathcal{C}$ be a braided fusion category with braiding c. Two objects X and Y of $\mathcal{C}$ are said to *centralize* each other [Müger 2003b] if $c_{Y,X} c_{X,Y} = \mathrm{id}_{X\otimes Y}$.

For any fusion subcategory $\mathcal{D} \subseteq \mathcal{C}$ its *centralizer* $\mathcal{D}'$ is the full fusion subcategory of $\mathcal{C}$ consisting of all objects $X \in \mathcal{C}$ centralizing every object in $\mathcal{D}$. The category $\mathcal{C}$ is said to be *nondegenerate* if $\mathcal{C}' = \mathrm{Vec}$. In this case one has $\mathcal{D}'' = \mathcal{D}$ [Müger 2003b]. If $\mathcal{C}$ is a premodular category, that is, has a spherical structure, then it is nondegenerate if and only if it is modular.

A braided fusion category $\mathcal{E}$ is called *Tannakian* if it is equivalent to the representation category $\mathrm{Rep}(G)$ of a finite group G as a braided fusion category. Here $\mathrm{Rep}(G)$ is considered with its standard symmetric braiding. The group G is defined by $\mathcal{E}$ up to an isomorphism [Deligne 1990].

A fusion subcategory $\mathcal{L}$ of a braided fusion category is called *Lagrangian* if it is Tannakian and $\mathcal{L} = \mathcal{L}'$.

Theorem 2.5 [Drinfeld et al. 2007]. *A fusion category $\mathcal{C}$ is group-theoretical if and only if $\mathcal{Z}(\mathcal{C})$ contains a Lagrangian subcategory.*

2D. *Group actions on fusion categories and equivariantization.* Let G be a finite group, and let $\underline{G}$ denote the monoidal category whose objects are elements of G, whose morphisms are identities, and whose tensor product is given by multiplication in G. Recall that an action of G on a fusion category $\mathcal{C}$ is a monoidal functor $\underline{G} \to \mathrm{Aut}_{\otimes}(\mathcal{C}) : g \mapsto T_g$. For any $g, h \in G$, let

$$\gamma_{g,h} = T_g \circ T_h \simeq T_{gh}$$

be the isomorphism defining the monoidal structure on the functor $\underline{G} \to \mathrm{Aut}_{\otimes}(\mathcal{C})$.

Definition 2.6. A *G-equivariant object* in $\mathcal{C}$ is a pair $(X, \{u_g\}_{g\in G})$ consisting of an object X of $\mathcal{C}$ together with a collection of isomorphisms $u_g : T_g(X) \simeq X,\ g \in G,$

such that the diagram

$$\begin{array}{ccc} T_g(T_h(X)) & \xrightarrow{T_g(u_h)} & T_g(X) \\ {\scriptstyle \gamma_{g,h}(X)}\downarrow & & \downarrow{\scriptstyle u_g} \\ T_{gh}(X) & \xrightarrow{u_{gh}} & X \end{array}$$

commutes for all $g, h \in G$. One defines morphisms of equivariant objects to be morphisms in $\mathcal{C}$ commuting with $u_g,\ g \in G$.

Equivariant objects in $\mathcal{C}$ form a fusion category, called the *equivariantization* of $\mathcal{C}$ and denoted by $\mathcal{C}^G$ [Tambara 2001; Arkhipov and Gaitsgory 2003; Gaitsgory 2005]. One has $\mathrm{FPdim}(\mathcal{C}^G) = |G|\,\mathrm{FPdim}(\mathcal{C})$.

There is another fusion category that comes from an action of G on $\mathcal{C}$. It is the *crossed product* category $\mathcal{C} \rtimes G$ defined as follows [Tambara 2001; Nikshych 2008]. As an abelian category, $\mathcal{C} \rtimes G := \mathcal{C} \boxtimes \mathrm{Vec}_G$, where Vec_G denotes the fusion category of G-graded vector spaces. The tensor product in $\mathcal{C} \rtimes G$ is given by

$$(X \boxtimes g) \otimes (Y \boxtimes h) := (X \otimes T_g(Y)) \boxtimes gh, \qquad X, Y \in \mathcal{C}, \quad g, h \in G. \tag{6}$$

The unit object is $\mathbf{1} \boxtimes e$ and the associativity and unit constraints come from those of $\mathcal{C}$. Clearly, $\mathcal{C} \rtimes G$ is faithfully G-graded with the trivial component $\mathcal{C}$.

As explained in [Nikshych 2008], $\mathcal{C}$ is a right $\mathcal{C} \rtimes G$-module category via

$$Y \otimes (X \boxtimes g) := T_{g^{-1}}(Y \otimes X),$$

and the corresponding dual category $(\mathcal{C} \rtimes G)^*_{\mathcal{C}}$ is equivalent to $\mathcal{C}^G$. It follows from [Müger 2003a] that there is an equivalence of braided fusion categories

$$\mathcal{Z}(\mathcal{C} \rtimes G) \cong \mathcal{Z}(\mathcal{C}^G).$$

Let G be a finite group. For any conjugacy class K of G fix a representative $a_K \in K$. Let G_K denote the centralizer of a_K in G.

Proposition 2.7. *Let $\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$ be a G-graded fusion category with an action $g \mapsto T_g$ of G on $\mathcal{C}$ such that T_g carries $\mathcal{C}_h$ to $\mathcal{C}_{ghg^{-1}}$. Let $H := \{g \in G \mid \mathcal{C}_g \neq 0\}$. There is a bijection between the set of isomorphism classes of simple objects of $\mathcal{C}^G$ and pairs $(K,\ X)$, where $K \subset H$ is a conjugacy class of G and X is a simple G_K-equivariant object of $\mathcal{C}_{a_K}$.*

Proof. A simple G-equivariant object of $\mathcal{C}$ must be supported on a single conjugacy class K. Let $Y = \oplus_{g \in K} Y_g$ be such an object. Then Y_{a_K} is a simple G_K-equivariant object.

Conversely, given a G_K-equivariant object X in $\mathcal{C}_{a_K}$ let

$$Y = \bigoplus_h T_h(X),$$

where the summation is taken over the set of representatives of cosets of G_K in G. It is easy to see that Y acquires the structure of a simple G-equivariant object.

Clearly, the two constructions are inverses of each other. □

Remark 2.8. The Frobenius–Perron dimension of the simple object corresponding to a pair (K, X) in Proposition 2.7 is $|K|\,\mathrm{FPdim}(X)$.

2E. ***De-equivariantization of fusion categories.*** Let $\mathscr{C}$ be a fusion category. Let $\mathscr{E}=\mathrm{Rep}(G)$ be a Tannakian category along with a braided tensor functor $\mathscr{E}\to\mathscr{Z}(\mathscr{C})$ such that the composition $\mathscr{E}\to\mathscr{Z}(\mathscr{C})\to\mathscr{C}$ (where the second arrow is the forgetful functor) is fully faithful. The following construction was introduced in [Bruguières 2000] and [Müger 2000]. Let $A:=\mathrm{Fun}(G)$ be the algebra of functions on G. It is a commutative algebra in $\mathscr{E}$ and thus its image is a commutative algebra in $\mathscr{Z}(\mathscr{C})$. This fact allows us to view the category $\mathscr{C}_G$ of A-modules in $\mathscr{C}$ as a fusion category, called *de-equivariantization* of $\mathscr{C}$. There is a canonical surjective tensor functor

$$F:\mathscr{C}\to\mathscr{C}_G : X\mapsto A\otimes X. \tag{7}$$

It was explained in [Müger 2000; Drinfeld et al. 2009] that the group G acts on $\mathscr{C}_G$ by tensor autoequivalences (this action comes from the action of G on A by right translations). Furthermore, there is a bijection between subcategories of $\mathscr{C}$ containing the image of $\mathscr{E}=\mathrm{Rep}(G)$ and G-stable subcategories of $\mathscr{C}_G$. This bijection preserves Tannakian subcategories.

The procedures of equivariantization and de-equivariantization are inverses of each other: that is, there are canonical equivalences $(\mathscr{C}_G)^G\cong\mathscr{C}$ and $(\mathscr{C}^G)_G\cong\mathscr{C}$.

In particular, the construction above applies when $\mathscr{C}$ is a braided fusion category containing a Tannakian subcategory $\mathscr{E}=\mathrm{Rep}(G)$. In this case the braiding of $\mathscr{C}$ gives rise to an additional structure on the de-equivariantization functor (7). Namely, there is natural family of isomorphisms

$$X\otimes F(Y)\xrightarrow{\sim}F(Y)\otimes X,\qquad X\in\mathscr{C}_G,\ Y\in\mathscr{C}, \tag{8}$$

satisfying obvious compatibility conditions. In other words, F can be factored through a braided functor $\mathscr{C}\to\mathscr{Z}(\mathscr{C}_G)$, that is, F is a *central* functor.

If $\mathscr{E}\subset\mathscr{C}'$ then $\mathscr{C}_G$ is a braided fusion category with the braiding inherited from that of $\mathscr{C}$. If $\mathscr{E}=\mathscr{C}'$, the category $\mathscr{C}_G$ is nondegenerate. (In the presence of a spherical structure this category is called the *modularization* of $\mathscr{C}$ by $\mathscr{E}$ [Bruguières 2000; Müger 2000].)

Remark 2.9. The category $\mathscr{C}_G$ is not braided in general. However it does have an additional structure — it is a *braided G-crossed fusion category*. See next section (2F) for details.

2F. *Braided G-crossed categories.* Let G be a finite group. Kirillov [2002] and Müger [2004] found a description of all braided fusion categories $\mathcal{D}$ containing $\mathrm{Rep}(G)$. Namely, they showed that the datum of a braided fusion category $\mathcal{D}$ containing $\mathrm{Rep}(G)$ is equivalent to the datum of a braided G-crossed category $\mathcal{C}$; see Theorem 2.12. The notion of a braided G-crossed category is due to Turaev [2000; 2008] and is recalled below.

Definition 2.10. A *braided G-crossed fusion category* is a fusion category $\mathcal{C}$ equipped with (i) a (not necessarily faithful) grading $\mathcal{C} = \bigoplus_{g\in G} \mathcal{C}_g$, (ii) an action $g \mapsto T_g$ of G on $\mathcal{C}$ such that $T_g(\mathcal{C}_h) \subset \mathcal{C}_{ghg^{-1}}$, and (iii) a natural collection of isomorphisms

$$c_{X,Y} : X \otimes Y \simeq T_g(Y) \otimes X, \qquad X \in \mathcal{C}_g,\ g \in G \text{ and } Y \in \mathcal{C}, \tag{9}$$

called the *G-braiding*. These structures are required to satisfy certain compatibility conditions, which we now state. Let $\gamma_{g,h} : T_g T_h \xrightarrow{\sim} T_{gh}$ denote the tensor structure of the functor $g \mapsto T_g$ and μ_g the tensor structure of T_g.

(a) The diagram

$$\begin{CD}
T_g(X)\otimes T_g(Y) @>{c_{T_g(X),T_g(Y)}}>> T_{ghg^{-1}}(T_g(Y))\otimes T_g(X)\\
@A{(\mu_g)^{-1}_{X,Y}}AA @VV{(\gamma_{ghg^{-1},g})_Y\otimes \mathrm{id}_{T_g(X)}}V\\
T_g(X\otimes Y) @. T_{gh}(Y)\otimes T_g(X)\\
@V{T_g(c_{X,Y})}VV @AA{(\gamma_{g,h})_Y\otimes\mathrm{id}_{T_g(X)}}A\\
T_g(T_h(Y)\otimes X) @>>{(\mu_g)^{-1}_{T_g(Y),X}}> T_g(T_h(Y))\otimes T_g(X)
\end{CD} \tag{10}$$

commutes for all $g, h \in G$ and objects $X \in \mathcal{C}_h,\ Y \in \mathcal{C}$.

(b) The diagram

$$\begin{array}{ccccc}
 & & (X\otimes Y)\otimes Z & & \\
 & \swarrow^{\alpha_{X,Y,Z}} & & \searrow^{c_{X,Y}\otimes \mathrm{id}_Z} & \\
X\otimes(Y\otimes Z) & & & & (T_g(Y)\otimes X)\otimes Z\\
\downarrow{\scriptstyle c_{X,Y\otimes Z}} & & & & \downarrow{\scriptstyle \alpha_{T_g(Y),X,Z}}\\
T_g(Y\otimes Z)\otimes X & & & & T_g(Y)\otimes(X\otimes Z)\\
\downarrow{\scriptstyle (\mu_g)^{-1}_{Y,Z}\otimes \mathrm{id}_X} & & & & \downarrow{\scriptstyle \mathrm{id}_{T_g(Y)}\otimes c_{X,Z}}\\
(T_g(Y)\otimes T_g(Z))\otimes X & & \xrightarrow{\alpha_{T_g(Y),T_g(Z),X}} & & T_g(Y)\otimes(T_g(Z)\otimes X)
\end{array} \tag{11}$$

commutes for all $g \in G$ and objects $X \in \mathcal{C}_g,\ Y, Z \in \mathcal{C}$.

(c) The diagram

$$\begin{array}{ccc}
 & X\otimes(Y\otimes Z) & \\
\overset{\alpha_{X,Y,Z}}{\nearrow} & & \overset{\mathrm{id}_X\otimes c_{Y,Z}}{\searrow} \\
(X\otimes Y)\otimes Z & & X\otimes(T_h(Z)\otimes Y) \\
\uparrow c^{-1}_{X\otimes Y,Z} & & \downarrow \alpha^{-1}_{X,T_h(Z),Y} \\
T_{gh}(Z)\otimes(X\otimes Y) & & (X\otimes T_h(Z))\otimes Y \\
\uparrow (\gamma_{g,h})_Z\otimes \mathrm{id}_{X\otimes Y} & & \downarrow c_{X,T_h(Z)}\otimes \mathrm{id}_Y \\
T_gT_h(Z)\otimes(X\otimes Y) & \xrightarrow{\alpha^{-1}_{T_gT_h(Z),X,Y}} & (T_gT_h(Z)\otimes X)\otimes Y
\end{array} \tag{12}$$

commutes for all $g, h \in G$ and objects $X \in \mathcal{C}_g,\ Y \in \mathcal{C}_h,\ Z \in \mathcal{C}$.

Remark 2.11. The trivial component $\mathcal{C}_e$ of a braided G-crossed fusion category $\mathcal{C}$ is a braided fusion category with the action of G by braided autoequivalences. This can be seen by taking $X,\ Y \in \mathcal{C}_e$ in diagrams (10)–(12).

Theorem 2.12 ([Kirillov 2002; Müger 2004]). *The equivariantization and de-equivariantization constructions establish a bijection between the set of equivalence classes of G-crossed braided fusion categories and the set of equivalence classes of braided fusion categories containing* $\mathrm{Rep}(G)$ *as a symmetric fusion subcategory.*

We shall now sketch the proof of this theorem. An alternative approach is given in [Drinfeld et al. 2009].

Suppose $\mathcal{C}$ is a braided G-crossed fusion category. We define a braiding $\tilde{c}$ on its equivariantization $\mathcal{C}^G$ as follows.

Let $(X,\ \{u_g\}_{g\in G})$ and $(Y,\ \{v_g\}_{g\in G})$ be objects in $\mathcal{C}^G$. Let $X = \oplus_{g\in G} X_g$ be a decomposition of X with respect to the grading of $\mathcal{C}$. Define an isomorphism

$$\tilde{c}_{X,Y} : X\otimes Y = \bigoplus_{g\in G} X_g\otimes Y \xrightarrow{\oplus c_{X_g,Y}} \bigoplus_{g\in G} T_g(Y)\otimes X_g \xrightarrow{\oplus v_g\otimes \mathrm{id}_{X_g}} \bigoplus_{g\in G} Y\otimes X_g = Y\otimes X. \tag{13}$$

It follows from condition (a) of Definition 2.10 that $\tilde{c}_{X,Y}$ respects the equivariant structures, that is, it is an isomorphism in $\mathcal{C}^G$. Its naturality is clear. The fact that $\tilde{c}$ is a braiding on $\mathcal{C}^G$ (that is, the hexagon axioms) follows from the commutativity of diagrams (11) and (12). It is easy to check that $\tilde{c}$ restricts to the standard braiding on $\mathrm{Rep}(G) = \mathrm{Vec}^G \subset \mathcal{C}^G$. Hence, $\mathcal{C}^G$ contains a Tannakian subcategory $\mathrm{Rep}(G)$.

Conversely, let $\mathcal{C}$ be a braided fusion category with braiding c containing a Tannakian subcategory $\mathrm{Rep}(G)$. The restriction of the de-equivariantization functor F from (7) on $\mathrm{Rep}(G)$ is isomorphic to the fiber functor $\mathrm{Rep}(G) \to \mathrm{Vec}$. Hence for any object X in $\mathcal{C}_G$ and any object V in $\mathrm{Rep}(G)$ we have an automorphism of

$F(V)\otimes X$ defined as the composition

$$F(V)\otimes X \xrightarrow{\sim} X\otimes F(V) \xrightarrow{\sim} F(V)\otimes X, \tag{14}$$

where the first isomorphism comes from the fact that $F(V)\in \mathrm{Vec}$ and the second one is (8).

When X is simple we have an isomorphism $\mathrm{Aut}_{\mathcal{C}}(F(V)\otimes X)\cong \mathrm{Aut}_{\mathrm{Vec}}(F(V))$, hence we obtain a tensor automorphism i_X of $F|_{\mathrm{Rep}(G)}$. Since $\mathrm{Aut}_{\otimes}(F|_{\mathrm{Rep}(G)})\cong G$ we have an assignment $X\mapsto i_X\in G$. The hexagon axiom of braiding implies that this assignment is multiplicative, that is, that $i_Z=i_Xi_Y$ for any simple object Z contained in $X\otimes Y$. Thus, it defines a G-grading on $\mathcal{C}$:

$$\mathcal{C}=\bigoplus_{g\in G}\mathcal{C}_g, \qquad \text{where } \mathcal{O}(\mathcal{C}_g)=\{X\in \mathcal{O}(\mathcal{C})\mid i_X=g\}. \tag{15}$$

It is straightforward to check that $i_{T_g(X)}=ghg^{-1}$ whenever $i_X=h$.

Finally, to construct a G-crossed braiding on $\mathcal{C}$, observe that $\mathcal{C}$ and $\mathcal{C}^{\mathrm{rev}}$ are embedded into the crossed product category $\mathcal{C}\rtimes G=(\mathcal{C}^G)^*_{\mathcal{C}}$ as subcategories $\mathcal{C}_{\mathrm{left}}$ and $\mathcal{C}_{\mathrm{right}}$, consisting, respectively, of functors of left and right multiplications by objects of $\mathcal{C}$. Clearly, there is a natural family of isomorphisms

$$X\otimes Y \xrightarrow{\sim} Y\otimes X, \qquad \text{with } X\in \mathcal{C}_{\mathrm{left}} \text{ and } Y\in \mathcal{C}_{\mathrm{right}}, \tag{16}$$

satisfying obvious compatibility conditions. Note that $\mathcal{C}_{\mathrm{left}}$ is identified with the diagonal subcategory of $\mathcal{C}\rtimes G$ spanned by objects $X\boxtimes g$, $X\in\mathcal{C}_g$, $g\in G$, and $\mathcal{C}_{\mathrm{right}}$ is identified with the trivial component subcategory $\mathcal{C}\boxtimes e$. Using (6) we conclude that isomorphisms (16) give rise to a G-crossed braiding on $\mathcal{C}$.

One can check that the two constructions above (from braided fusion categories containing $\mathrm{Rep}(G)$ to braided G-crossed categories and vice versa) are inverses of each other; see [Kirillov 2002; Müger 2004; Drinfeld et al. 2009] for details.

Remark 2.13. Let $\mathcal{C}=\oplus_{g\in G}\mathcal{C}_g$ be a braided G-crossed fusion category. It was shown in [Drinfeld et al. 2009] that the braided category $\mathcal{C}^G$ is nondegenerate if and only if $\mathcal{C}_e$ is nondegenerate and the G-grading of $\mathcal{C}$ is faithful.

3. The center of a graded fusion category

Let G be a finite group and let $\mathcal{D}$ be a fusion category. Throughout this section $\mathcal{C}$ will denote a fusion category with a faithful G-grading, whose trivial component is $\mathcal{D}$; that is, $\mathcal{C}$ is a G-extension of $\mathcal{D}$:

$$\mathcal{C}=\bigoplus_{g\in G}\mathcal{C}_g, \qquad \mathcal{C}_e=\mathcal{D}. \tag{17}$$

In what follows we consider only *faithful* gradings: that is, those such that $\mathscr{C}_g \neq 0$ for all $g \in G$. An object of $\mathscr{C}$ contained in $\mathscr{C}_g$ will be called *homogeneous* of degree g.

Our goal is to describe the center $\mathscr{Z}(\mathscr{C})$ as an equivariantization of the relative center $\mathscr{Z}_{\mathscr{D}}(\mathscr{C})$ defined in Section 2B.

3A. ***The relative center $\mathscr{Z}_{\mathscr{D}}(\mathscr{C})$ as a braided G-crossed category.*** Let us define a canonical braided G-crossed category structure on $\mathscr{Z}_{\mathscr{D}}(\mathscr{C})$.

First of all, there is an obvious faithful G-grading on $\mathscr{Z}_{\mathscr{D}}(\mathscr{C})$:

$$\mathscr{Z}_{\mathscr{D}}(\mathscr{C}) = \bigoplus_{g\in G} \mathscr{Z}_{\mathscr{D}}(\mathscr{C}_g). \tag{18}$$

Indeed, it is clear that for every simple object X of $\mathscr{Z}_{\mathscr{D}}(\mathscr{C})$ the forgetful image of X in $\mathscr{C}$ must be homogeneous.

We now define the action of G on $\mathscr{Z}_{\mathscr{D}}(\mathscr{C})$. Take $g, h \in G$. Let $\mathrm{Fun}_{\mathscr{D}\boxtimes\mathscr{D}^{\mathrm{rev}}}(\mathscr{C}_g, \mathscr{C}_h)$ denote the category of $\mathscr{D}$-bimodule functors from $\mathscr{C}_g$ to $\mathscr{C}_h$. Clearly, it is a $\mathscr{Z}(\mathscr{D})$-bimodule category.

Proposition 3.1. *Let $g, h \in G$. The functors*

$$L_{g,h}: \mathscr{Z}_{\mathscr{D}}(\mathscr{C}_h) \xrightarrow{\sim} \mathrm{Fun}_{\mathscr{D}\boxtimes\mathscr{D}^{rev}}(\mathscr{C}_g, \mathscr{C}_{hg}) : Z \mapsto Z\otimes ?, \tag{19}$$

$$R_{g,h}: \mathscr{Z}_{\mathscr{D}}(\mathscr{C}_h) \xrightarrow{\sim} \mathrm{Fun}_{\mathscr{D}\boxtimes\mathscr{D}^{rev}}(\mathscr{C}_g, \mathscr{C}_{gh}) : Z \mapsto ?\otimes Z. \tag{20}$$

are equivalences of $\mathscr{Z}(\mathscr{D})$-bimodule categories.

Proof. We prove that (19) is an equivalence. Let $\mathrm{Fun}_{\mathscr{D}}(\mathscr{C}_g, \mathscr{C}_{hg})$ be the category of right $\mathscr{D}$-module functors from $\mathscr{C}_g$ to $\mathscr{C}_{hg}$. It suffices to prove that

$$M_{g,h} : \mathscr{C}_h \to \mathrm{Fun}_{\mathscr{D}}(\mathscr{C}_g, \mathscr{C}_{hg}) : X \mapsto X\otimes ? \tag{21}$$

is an equivalence. Indeed, $\mathscr{D}$-bimodule functor structures on $M_{g,h}(X)$ for $X \in \mathscr{C}_h$ are in bijection with central structures on X.

For every $g \in G$ choose a simple object $X_g \in \mathscr{C}_g$. Then $A_g := X_g \otimes X_g^*$ is an algebra in $\mathscr{D}$. It follows from [Ostrik 2003, Theorem 1] that the functor $Y \mapsto Y \otimes X_g^*$ is a left $\mathscr{C}$-module category equivalence between $\mathscr{C}$ and the category of right A_g-modules in $\mathscr{C}$. Since $Y \otimes X_g^*$ belongs to $\mathscr{D}$ if and only if Y is in $\mathscr{C}_g$ we see that the functor above restricts to a left $\mathscr{D}$-module category equivalence between $\mathscr{C}_g$ and the category of right A_g-modules in $\mathscr{D}$. There are also similar equivalences of right module categories.

It follows that for all $g, h \in G$ there is an equivalence

$$Y \mapsto X_g \otimes Y \otimes X_{hg}^* \tag{22}$$

between $\mathscr{C}$ and the category of $(A_g - A_{hg})$-bimodules in $\mathscr{C}$. The right-hand side of (22) belongs to $\mathscr{D}$ if and only if Y is in $\mathscr{C}_h$. Hence, (22) restricts to an equivalence between $\mathcal{C}_h$ and the category of $(A_g - A_{hg})$-bimodules in $\mathcal{D}$. The latter category is identified with the category of right $\mathcal{D}$-module functors between the categories of right A_g-modules and A_{hg}-modules in $\mathcal{D}$, that is, with $\mathrm{Fun}_{\mathcal{D}}(\mathcal{C}_g, \mathcal{C}_{hg})$. It is easy to see that upon this identification the restriction of equivalence (22) to $\mathcal{C}_h$ coincides with (21).

The proof of the equivalence (20) is completely similar. □

We define tensor functors

$$T_{g,h} := L^{-1}_{g,ghg^{-1}} R_{g,h} : \mathcal{Z}_{\mathcal{D}}(\mathcal{C}_h) \to \mathcal{Z}_{\mathcal{D}}(\mathcal{C}_{ghg^{-1}}), \qquad g, h \in G, \tag{23}$$

and set

$$T_g := \bigoplus_{h\in G} T_{g,h} : \mathcal{Z}_{\mathcal{D}}(\mathcal{C}) \to \mathcal{Z}_{\mathcal{D}}(\mathcal{C}). \tag{24}$$

The definition of T_g along with Proposition 3.1 give rise to the following natural isomorphism of $\mathcal{D}$-bimodule functors from $\mathcal{C}_g$ to $\mathcal{C}$:

$$c_{-,Y} : ? \otimes Y \xrightarrow{\sim} T_g(Y) \otimes ?. \tag{25}$$

It translates to a natural family of isomorphisms

$$c_{X,Y} : X \otimes Y \xrightarrow{\sim} T_g(Y) \otimes X, \qquad X \in \mathcal{C}_g,\ Y \in \mathcal{Z}_{\mathcal{D}}(\mathcal{C}),\ g \in G, \tag{26}$$

satisfying natural compatibility conditions corresponding to the $\mathcal{D}$-bimodule structure on (25). Since the grading (18) is faithful, we have $T_g(\mathcal{Z}_{\mathcal{D}}(\mathcal{C}_h)) \subset \mathcal{Z}_{\mathcal{D}}(\mathcal{C}_{ghg^{-1}})$.

Take $X_1 \in \mathcal{C}_{g_1}$, $X_2 \in \mathcal{C}_{g_2}$ and set $X = X_1 \otimes X_2$ in (26). We obtain a natural isomorphism

$$T_{g_1} T_{g_2}(Y) \otimes X_1 \otimes X_2 \xrightarrow{\sim} T_{g_1 g_2}(Y) \otimes X_1 \otimes X_2. \tag{27}$$

Since every object $Z \in \mathcal{C}_{g_1 g_2}$ is contained in $X_1 \otimes X_2$ for some $X_1 \in \mathcal{C}_{g_1}$, $X_2 \in \mathcal{C}_{g_2}$, using naturality of (27) we obtain a natural isomorphism

$$T_{g_1} T_{g_2}(Y) \otimes Z \xrightarrow{\sim} T_{g_1 g_2}(Y) \otimes Z, \qquad Z \in \mathcal{C}_{g_1 g_2}, \tag{28}$$

of $\mathcal{D}$-bimodule functors $T_{g_1} T_{g_2}(Y) \otimes ?$ and $T_{g_1 g_2}(Y) \otimes ?$. By Proposition 3.1 this gives an isomorphism $T_{g_1} T_{g_2}(Y) \xrightarrow{\sim} T_{g_1 g_2}(Y)$, $Y \in \mathcal{Z}_{\mathcal{D}}(\mathcal{C})$, that is, an isomorphism of functors $T_{g_1} T_{g_2} \xrightarrow{\sim} T_{g_1 g_2}$. Thus, the assignment $g \mapsto T_g$ is an action of G on $\mathcal{Z}_{\mathcal{D}}(\mathcal{C})$ by tensor autoequivalences.

Suppose that X is an object in $\mathcal{Z}(\mathcal{C}_g)$. Then both sides of (26) have structure of objects in $\mathcal{Z}_{\mathcal{D}}(\mathcal{C})$ obtained by composing central structures of X and Y.

Lemma 3.2. *Isomorphisms (26) define a G-braiding on $\mathcal{Z}_{\mathcal{D}}(\mathcal{C})$.*

Proof. That isomorphisms (26) are indeed morphisms in $\mathcal{Z}_{\mathcal{D}}(\mathcal{C})$ follows from commutativity of the diagram

$$\begin{array}{ccccc} X\otimes Y\otimes V & \xrightarrow{\mathrm{id}_X\otimes\delta_V} & X\otimes V\otimes Y & \xrightarrow{\gamma_V\otimes\mathrm{id}_Y} & V\otimes X\otimes Y \\ {\scriptstyle c_{X,Y}\otimes\mathrm{id}_V}\downarrow & \swarrow{\scriptstyle c_{X\otimes V,Y}} & & \swarrow{\scriptstyle c_{V\otimes X,Y}} & \downarrow{\scriptstyle \mathrm{id}_V\otimes c_{X,Y}} \\ T_g(Y)\otimes X\otimes V & \xrightarrow[\mathrm{id}_{T_g(Y)}\otimes\gamma_V]{} & T_g(Y)\otimes V\otimes X & \xrightarrow[T_g(\delta)_V\otimes\mathrm{id}_X]{} & V\otimes T_g(Y)\otimes X, \end{array} \tag{29}$$

where $(X, \gamma) \in \mathcal{Z}_{\mathcal{D}}(\mathcal{C}_g)$, $(Y, \delta) \in \mathcal{Z}_{\mathcal{D}}(\mathcal{C})$, and $V \in \mathcal{D}$. Indeed, the parallelogram in the middle commutes by naturality of c, and the two triangular faces commute since the natural isomorphism (25) is an isomorphism of $\mathcal{D}$-bimodule functors.

It is straightforward to check that isomorphisms $c_{X,Y}$ satisfy the compatibility conditions of Definition 2.10. □

The constructions and arguments above prove the following theorem.

Theorem 3.3. *Let G be a finite group and let $\mathcal{C}$ be a fusion category with a faithful G-grading whose trivial component is $\mathcal{D}$. The relative center $\mathcal{Z}_{\mathcal{D}}(\mathcal{C})$ has a canonical structure of a braided G-crossed category.*

Remark 3.4. In particular, to every G-extension of a fusion category $\mathcal{D}$ we assigned an action of G by braided autoequivalences of $\mathcal{Z}(\mathcal{D})$. This assignment is studied in detail in [Etingof et al. 2009].

3B. *The center $\mathcal{Z}(\mathcal{C})$ as an equivariantization.* As before, let G be a finite group and let $\mathcal{C}$ be a fusion category with a faithful G-grading (17). Let $\mathcal{Z}_{\mathcal{D}}(\mathcal{C})$ be the braided G-crossed category constructed in Section 3A.

Theorem 3.5. *There is an equivalence of braided fusion categories*

$$\mathcal{Z}_{\mathcal{D}}(\mathcal{C})^G \xrightarrow{\sim} \mathcal{Z}(\mathcal{C}). \tag{30}$$

Proof. We see from (26) that a G-equivariant object in $\mathcal{Z}_{\mathcal{D}}(\mathcal{C})$ has a structure of a central object in $\mathcal{C}$ defined as in (13). It follows from definitions that the corresponding tensor functor $\mathcal{Z}_{\mathcal{D}}(\mathcal{C})^G \to \mathcal{Z}(\mathcal{C})$ is braided.

Conversely, given an object Y in $\mathcal{Z}(\mathcal{C})$, consider its forgetful image $\tilde{Y}$ in $\mathcal{Z}_{\mathcal{D}}(\mathcal{C})$. Combining the central structure of Y with isomorphism (26) we obtain a family of isomorphisms

$$\tilde{Y}\otimes X \xrightarrow{\sim} T_g(\tilde{Y})\otimes X, \qquad X\in\mathcal{C}_g,\ g\in G,$$

which gives rise to the isomorphism of $\mathcal{D}$-bimodule functors $\tilde{Y}\otimes ? \xrightarrow{\sim} T_g(\tilde{Y})\otimes ? : \mathcal{C}_g \to \mathcal{C}$. By Proposition 3.1 we obtain a natural isomorphism $\tilde{Y} \xrightarrow{\sim} T_g(\tilde{Y})$ and, hence, a G-equivariant structure on $\tilde{Y}$. Thus, we have a tensor functor $\mathcal{Z}(\mathcal{C}) \to \mathcal{Z}_{\mathcal{D}}(\mathcal{C})^G$. It is clear that the two functors are quasiinverses of each other. □

We describe the Tannakian subcategory $\mathcal{E} \cong \text{Rep}(G) \subset \mathcal{Z}(\mathcal{C})$ corresponding to equivalence (30). For any representation $\pi : G \to GL(V)$ of the grading group G, consider an object I_π in $\mathcal{Z}(\mathcal{C})$ where $I_\pi = V \otimes \mathbf{1}$ as an object of $\mathcal{C}$ with the permutation isomorphism

$$c_{I_\pi, X} := \pi(g) \otimes \text{id}_X : I_\pi \otimes X \cong X \otimes I_\pi, \qquad \text{when } X \in \mathcal{C}_g. \tag{31}$$

Then $\mathcal{E}$ is the subcategory of $\mathcal{Z}(\mathcal{C})$ consisting of objects I_π, where π runs through all finite-dimensional representations of G.

Remark 3.6. Here is another description of the subcategory $\mathcal{E}$: it consists of all objects in $\mathcal{Z}(\mathcal{C})$ sent to Vec by the forgetful functor $\mathcal{Z}(\mathcal{C}) \to \mathcal{Z}_{\mathcal{D}}(\mathcal{C})$.

Corollary 3.7. *Let $\mathcal{C}$ be a faithfully G-graded fusion category with the trivial component $\mathcal{D}$. Let $\mathcal{E} = \text{Rep}(G) \subset \mathcal{Z}(\mathcal{C})$ be the Tannakian subcategory constructed above. Then the de-equivariantization category $(\mathcal{E}')_G$ is braided tensor equivalent to $\mathcal{Z}(\mathcal{D})$.*

Proof. The statement follows from Theorem 3.5 since $(\mathcal{E}')_G$ is the trivial component of the grading of $\mathcal{Z}(\mathcal{C})_G = \mathcal{Z}_{\mathcal{D}}(\mathcal{C})$. □

Remark 3.8. The assignment above

$$\{G\text{-extensions of } \mathcal{D}\} \mapsto \{\text{braided } G\text{-crossed extensions of } \mathcal{Z}(\mathcal{D})\} \tag{32}$$

can be thought of as an analogue of the center construction for G-extensions.

Next, we describe simple objects of $\mathcal{Z}(\mathcal{C})$. For any conjugacy class K in G fix a representative $a_K \in K$. Let G_K denote the centralizer of a_K in G. Note that the action (24) of G on $\mathcal{Z}_{\mathcal{D}}(\mathcal{C})$ restricts to the action of G_K on $\mathcal{Z}_{\mathcal{D}}(\mathcal{C}_{a_K})$.

Proposition 3.9. *There is a bijection between the set of isomorphism classes of simple objects of $\mathcal{Z}(\mathcal{C})$ and pairs (K, X), where K is a conjugacy class of G and X is a simple G_K-equivariant object of $\mathcal{Z}_{\mathcal{D}}(\mathcal{C}_{a_K})$.*

Proof. By Theorem 3.5 we have $\mathcal{Z}(\mathcal{C}) \simeq \mathcal{Z}_{\mathcal{D}}(\mathcal{C})^G$, so the stated parameterization is immediate from the description of simple objects of the equivariantization category given in Proposition 2.7. □

3C. ***A criterion for a graded fusion category to be group-theoretical.*** We have seen in Corollary 3.7 that $\mathcal{Z}(\mathcal{C})$ contains a Tannakian subcategory $\mathcal{E} = \text{Rep}(G)$ such that the de-equivariantization $(\mathcal{E}')_G$ is braided equivalent to $\mathcal{Z}(\mathcal{D})$, where $\mathcal{D}$ is the trivial component of $\mathcal{C}$. Furthermore, by Remark 2.11, there is a canonical action of G on $\mathcal{Z}(\mathcal{D})$, by braided autoequivalences. By [Drinfeld et al. 2009], Tannakian subcategories of $\mathcal{Z}(\mathcal{C})$ containing $\mathcal{E}$ bijectively correspond to G-stable Tannakian subcategories of $(\mathcal{E}')_G \simeq \mathcal{Z}(\mathcal{D})$. Combining this observation with Theorem 2.5(ii) we obtain the following criterion.

Corollary 3.10. *A graded fusion category $\mathscr{C} = \bigoplus_{g \in G} \mathscr{C}_g$, $\mathscr{C}_e = \mathscr{D}$, is group-theoretical if and only if $\mathscr{Z}(\mathscr{D})$ contains a G-stable Lagrangian subcategory.*

Corollary 3.10 will be useful in Section 4D, where we characterize group-theoretical Tambara–Yamagami categories.

We can specialize Corollary 3.10 to equivariantization categories. Let G be a finite group acting on a fusion category $\mathscr{C}$. The equivariantization $\mathscr{C}^G$ is Morita equivalent to the crossed product category $\mathscr{C} \rtimes G$ (see Section 2D). Therefore, $\mathscr{Z}(\mathscr{C}^G) \cong \mathscr{Z}(\mathscr{C} \rtimes G)$. Clearly, the trivial component of $\mathscr{Z}(\mathscr{C} \rtimes G)_G$ is $\mathscr{Z}(\mathscr{C})$ and the canonical action of G on $\mathscr{Z}(\mathscr{C})$ is induced from the action of G on $\mathscr{C}$ in an obvious way.

Corollary 3.11. *The equivariantization $\mathscr{C}^G$ is group-theoretical if and only if there exists a G-stable Lagrangian subcategory of $\mathscr{Z}(\mathscr{C})$.*

Remark 3.12. Let G act on $\mathscr{C}$ as before. One can check (independently from the results of this section) that the G-set of Lagrangian subcategories of $\mathscr{Z}(\mathscr{C})$ is isomorphic to the G-set consisting of indecomposable $\mathscr{C}$-module categories $\mathscr{M}$ such that the dual category $\mathscr{C}^*_{\mathscr{M}}$ is pointed. This isomorphism is given by the map constructed in [Naidu and Nikshych 2008, Theorem 4.17]. Thus, the criterion in Corollary 3.11 is the same as [Nikshych 2008, Corollary 3.6].

3D. ***Example: the relative center of a crossed product category.*** Let G be a finite group and let $g \mapsto T_g$, $g \in G$, be an action of G on a fusion category $\mathscr{D}$. Let $\mathscr{C} := \mathscr{D} \rtimes G$ be the crossed product category defined in Section 2D. It has a natural grading

$$\mathscr{C} = \bigoplus_{g \in G} \mathscr{C}_g, \qquad \text{where } \mathscr{C}_g = \{Y \boxtimes g \mid Y \in \mathscr{D}\}.$$

We describe the braided G-crossed fusion category structure on the relative center

$$\mathscr{Z}_{\mathscr{D}}(\mathscr{C}) = \bigoplus_{g \in G} \mathscr{Z}_{\mathscr{D}}(\mathscr{C}_g).$$

By definition, the objects of $\mathscr{Z}_{\mathscr{D}}(\mathscr{C}_g)$ are pairs $(Y \boxtimes g, \gamma)$, where $Y \in \mathscr{D}$ and

$$\gamma = \{\gamma_X : X \otimes Y \xrightarrow{\sim} Y \otimes T_g(X)\}_{X \in \mathscr{D}} \tag{33}$$

is a natural family of isomorphisms satisfying natural compatibility conditions. Thus, $\mathscr{Z}_{\mathscr{D}}(\mathscr{C}_g)$ can be viewed as a "deformation" of $\mathscr{Z}(\mathscr{D})$ by means of T_g.

The action of G on $\mathscr{D}$ induces an action $h \mapsto \tilde{T}_h$ on $\mathscr{Z}_{\mathscr{D}}(\mathscr{C})$ defined as follows. Applying T_h, $h \in G$, to $\gamma_{T_{h^{-1}}(X)}$ in (33), we obtain an isomorphism

$$\tilde{\gamma}_X : X \otimes T_h(Y) \xrightarrow{\sim} T_h(Y) \otimes T_{hgh^{-1}}(X). \tag{34}$$

Set $\tilde{T}_h(Y \boxtimes g, \gamma) := (T_h(Y) \boxtimes hgh^{-1}, \tilde{\gamma})$. Thus, $\tilde{T}_h$ maps $\mathscr{Z}_{\mathscr{D}}(\mathscr{C}_g)$ to $\mathscr{Z}_{\mathscr{D}}(\mathscr{C}_{hgh^{-1}})$.

Finally, the G-braiding between objects $(X\boxtimes h)\in\mathcal{Z}_{\mathcal{D}}(\mathcal{C}_h)$ and $(Y\boxtimes g)\in\mathcal{Z}_{\mathcal{D}}(\mathcal{C}_g)$ comes from the isomorphism

$$\begin{aligned}(X\boxtimes h)\otimes(Y\boxtimes g)=(X\otimes T_h(Y))\boxtimes hg \xrightarrow{\tilde{\gamma}} (T_h(Y)\otimes T_{hgh^{-1}}(X))\boxtimes hg\\ =(T_h(Y)\boxtimes hgh^{-1})\otimes(X\boxtimes h)\\ =\tilde{T}_h(Y\boxtimes g)\otimes(X\boxtimes h).\end{aligned}$$

By Theorem 3.5, the category $\mathcal{Z}(\mathcal{D}\rtimes G)\cong\mathcal{Z}(\mathcal{D}^G)$ is equivalent to the equivariantization of the braided G-crossed category above.

4. The centers of Tambara–Yamagami categories

Our goal in this section is to apply techniques developed in Section 3 to Tambara–Yamagami categories introduced in [Tambara and Yamagami 1998] (see Section 4A below for the definition). Namely, using the techniques in Section 3 we establish a criterion for a Tambara–Yamagami category to be group-theoretical. We then use this criterion together with Corollary 3.11 to produce a series of non-group-theoretical semisimple Hopf algebras. In this section we assume that our ground field k is the field of complex numbers $\mathbb{C}$. We begin by recalling the definition of a Tambara–Yamagami category.

4A. ***Definition of Tambara–Yamagami categories.*** Let $\mathbb{Z}_2=\langle\,\delta\mid\delta^2=1\,\rangle$ be the cyclic group of order 2.

Tambara and Yamagami [1998] completely classified all $\mathbb{Z}_2$-graded fusion categories in which all but one simple objects are invertible and the noninvertible simple object has nontrivial graded degree.

They showed that any such category $\mathcal{TY}(A,\chi,\tau)$ is determined, up to an equivalence, by a finite abelian group A, a nondegenerate symmetric bilinear form $\chi: A\times A\to k^\times$, and a square root $\tau\in k$ of $|A|^{-1}$. The category $\mathcal{TY}(A,\chi,\tau)$ is described as follows. It is a skeletal category (that is, such that any two isomorphic objects are equal) with simple objects $\{a\mid a\in A\}$ and m, and tensor product

$$a\otimes b=a+b,\quad a\otimes m=m,\quad m\otimes a=m,\quad m\otimes m=\bigoplus_{a\in A}a,$$

for all $a,b\in A$, and the unit object $0\in A$. The associativity constraints are given by

$$\alpha_{a,b,c}=\mathrm{id}_{a+b+c},\quad \alpha_{a,b,m}=\mathrm{id}_m,\quad \alpha_{a,m,b}=\chi(a,b)\,\mathrm{id}_m,\quad \alpha_{m,a,b}=\mathrm{id}_m,$$
$$\alpha_{a,m,m}=\bigoplus_{b\in A}\mathrm{id}_b,\quad \alpha_{m,a,m}=\bigoplus_{b\in A}\chi(a,b)\,\mathrm{id}_b,$$
$$\alpha_{m,m,a}=\bigoplus_{b\in A}\mathrm{id}_b,\quad \alpha_{m,m,m}=\bigoplus_{a,b\in A}\tau\chi(a,b)^{-1}\,\mathrm{id}_m\,.$$

The unit constraints are the identity maps. The category $\mathcal{TY}(A, \chi, \tau)$ is rigid with $a^* = -a$ and $m^* = m$ (with obvious evaluation and coevaluation maps).

Let $n := |A|$. The dimensions of simple objects of $\mathcal{TY}(A, \chi, \tau)$ are $\mathrm{FPdim}(a) = 1$, $a \in A$, and $\mathrm{FPdim}(m) = \sqrt{n}$. We have $\mathrm{FPdim}(\mathcal{TY}(A, \chi, \tau)) = 2n$.

The $\mathbb{Z}_2$-grading on $\mathcal{TY}(A, \chi, \tau)$ is

$$\mathcal{TY}(A, \chi, \tau) = \mathcal{TY}(A, \chi, \tau)_1 \oplus \mathcal{TY}(A, \chi, \tau)_\delta,$$

where $\mathcal{TY}(A, \chi, \tau)_1$ is the full fusion subcategory generated by the invertible objects $a \in A$ and $\mathcal{TY}(A, \chi, \tau)_\delta$ is the full abelian subcategory generated by the object m.

Let $\mathcal{C} := \mathcal{TY}(A, \chi, \tau)$ and $\mathcal{D} := \mathcal{TY}(A, \chi, \tau)_1$.

4B. *Braided $\mathbb{Z}_2$-crossed category* $\mathcal{Z}_{\mathcal{D}}(\mathcal{C})$. First, let us describe the simple objects of $\mathcal{Z}_{\mathcal{D}}(\mathcal{C}) = \mathcal{Z}(\mathcal{C}_1) \oplus \mathcal{Z}_{\mathcal{D}}(\mathcal{C}_\delta)$. Let $\widehat{A} := \mathrm{Hom}(A, k^\times)$. Clearly, $\mathcal{Z}(\mathcal{C}_1) = \mathcal{Z}(\mathrm{Vec}_A)$, so its simple objects are parameterized by $(a, \phi) \in A \times \widehat{A}$. The object $X_{(a,\phi)}$ corresponding to such a pair is equal to a as an object of $\mathcal{C}$ and its central structure is given by

$$\phi(x)\,\mathrm{id}_{a+x} : x \otimes X_{(a,\phi)} \xrightarrow{\sim} X_{(a,\phi)} \otimes x. \tag{35}$$

Using Definition 2.1 we see that simple objects of $\mathcal{Z}_{\mathcal{D}}(\mathcal{C}_\delta)$ are parameterized by functions $\rho : A \to k^\times$ satisfying

$$\rho(a+b) = \chi(a, b)^{-1} \rho(a)\rho(b), \quad a, b \in A \tag{36}$$

(clearly, such functions form a torsor over $\widehat{A}$). The corresponding object Z_ρ is equal to m as an object of $\mathcal{C}$ and has the relative central structure

$$\rho(x)\,\mathrm{id}_m : x \otimes Z_\rho \xrightarrow{\sim} Z_\rho \otimes x, \quad x \in A. \tag{37}$$

Let $A \to \widehat{A} : a \mapsto \widehat{a}$ be the homomorphism defined by $\widehat{a}(x) = \chi(x, a)$. Similarly, let $\widehat{A} \to A : \phi \mapsto \widehat{\phi}$ be the homomorphism defined by $\phi(x) = \chi(x, \widehat{\phi})$ (recall that χ is nondegenerate). Clearly, these two maps are inverses of each other.

The fusion rules of $\mathcal{Z}_{\mathcal{D}}(\mathcal{C})$ are computed using formula (3) :

$$\begin{aligned} X_{(a,\phi)} \otimes X_{(b,\psi)} &= X_{(a+b,\phi+\psi)}, \\ X_{(a,\phi)} \otimes Z_\rho &= Z_{\rho\phi(-\widehat{a})}, \\ Z_\rho \otimes X_{(a,\phi)} &= Z_{\rho\phi(-\widehat{a})}, \\ Z_{\rho'} \otimes Z_\rho &= \bigoplus_{a \in A} X_{(a, \widehat{a}\rho'/\overline{\rho})}. \end{aligned}$$

We have $X^*_{(a,\phi)} = X_{(-a,-\phi)}$ and $Z^*_\rho = Z_{\overline{\rho}}$, where $\overline{\rho}(x) = \rho(-x)$, $x \in A$.

Using the construction given in Section 3A we see that the action of $\mathbb{Z}_2$ on $\mathfrak{Z}_{\mathfrak{D}}(\mathscr{C})$ is given by

$$T_1 = \mathrm{id}_{\mathfrak{Z}_{\mathfrak{D}}(\mathscr{C})}; \qquad T_\delta(X_{(a,\phi)}) = X_{(-\widehat{\phi},\, -\widehat{a})}, \quad T_\delta(Z_\rho) = Z_\rho. \tag{38}$$

The monoidal functor structure on $\mathbb{Z}_2 \to \mathrm{Aut}_{\otimes}(\mathfrak{Z}_{\mathfrak{D}}(\mathscr{C}))$ is given by the natural isomorphism $\gamma := \gamma_{\delta,\delta} : T_\delta \circ T_\delta \xrightarrow{\sim} T_1$ defined by

$$\gamma_{X_{(a,\phi)}} = \phi(a)\, \mathrm{id}_{X_{(a,\phi)}}, \quad \gamma_{Z_\rho} = \left(\tau \sum_{x \in A} \rho(x)^{-1}\right) \mathrm{id}_{Z_\rho}.$$

The crossed braiding morphisms on $\mathfrak{Z}_{\mathfrak{D}}(\mathscr{C})$ are given by

$$\begin{aligned}
c_{X_{(a,\phi)},X_{(b,\psi)}} &= \psi(a)\, \mathrm{id}_{a+b} : X_{(a,\phi)} \otimes X_{(b,\psi)} \xrightarrow{\sim} X_{(b,\psi)} \otimes X_{(a,\phi)}, \\
c_{X_{(a,\phi)},Z_\rho} &= \rho(a)\, \mathrm{id}_m : X_{(a,\phi)} \otimes Z_\rho \xrightarrow{\sim} Z_\rho \otimes X_{(a,\phi)}, \\
c_{Z_\rho,X_{(a,\phi)}} &= \mathrm{id}_m : Z_\rho \otimes X_{(a,\phi)} \xrightarrow{\sim} X_{(-\widehat{\phi},\, -\widehat{a})} \otimes Z_\rho, \\
c_{Z_{\rho'},Z_\rho} &= \oplus_{a \in A}\, \rho(-a)^{-1}\, \mathrm{id}_a : Z_{\rho'} \otimes Z_\rho \xrightarrow{\sim} Z_\rho \otimes Z_{\rho'}.
\end{aligned}$$

4C. ***The equivariantization category*** $\mathfrak{Z}_{\mathfrak{D}}(\mathscr{C})^{\mathbb{Z}_2}$**.** A simple calculation of $\mathbb{Z}_2$-equivariant objects in $\mathfrak{Z}_{\mathfrak{D}}(\mathscr{C})$ establishes the following.

Proposition 4.1. *The following is a complete list of simple objects of* $\mathfrak{Z}_{\mathfrak{D}}(\mathscr{C})^{\mathbb{Z}_2} \cong \mathfrak{Z}(\mathscr{T}\mathscr{Y}(A, \chi, \tau))$ *up to an isomorphism*:

(1) $2n$ *invertible objects parameterized by pairs* (a, ϵ), *where* $a \in A$ *and* $\epsilon^2 = \chi(a, a)^{-1}$. *The corresponding object* $X_{a,\epsilon}$ *is equal to* $X_{(a,-\widehat{a})}$ *as an object of* $\mathfrak{Z}_{\mathfrak{D}}(\mathscr{C})$ *and has* $\mathbb{Z}_2$*-equivariant structure*

$$\epsilon\, \mathrm{id}_{X_{(a,-\widehat{a})}} : T_\delta(X_{(a,-\widehat{a})}) \xrightarrow{\sim} X_{(a,-\widehat{a})};$$

(2) $\frac{n(n-1)}{2}$ *two-dimensional objects parameterized by unordered pairs* (a, b) *of distinct objects in* A. *The corresponding object* $Y_{a,b}$ *is equal to* $X_{(a,-\widehat{b})} \oplus X_{(b,-\widehat{a})}$ *as an object of* $\mathfrak{Z}_{\mathfrak{D}}(\mathscr{C})$ *and has* $\mathbb{Z}_2$*-equivariant structure*

$$\left(\mathrm{id}_{X_{(a,-\widehat{b})}} \oplus \chi(a, b)^{-1}\, \mathrm{id}_{X_{(b,-\widehat{a})}}\right) : T_\delta(X_{(a,-\widehat{b})} \oplus X_{(b,-\widehat{a})}) \xrightarrow{\sim} X_{(a,-\widehat{b})} \oplus X_{(b,-\widehat{a})};$$

(3) $2n$ $\sqrt{n}$*-dimensional objects parameterized by pairs* (ρ, Δ), *where* $\rho : A \to k^\times$ *satisfies* (36) *and* $\Delta^2 = \tau \sum_{x \in A} \rho(x)^{-1}$. *The corresponding object* $Z_{\rho,\Delta}$ *is equal to* Z_ρ *as an object of* $\mathfrak{Z}_{\mathfrak{D}}(\mathscr{C})$ *and has* $\mathbb{Z}_2$*-equivariant structure*

$$\Delta\, \mathrm{id}_{Z_\rho} : T_\delta(Z_\rho) \xrightarrow{\sim} Z_\rho.$$

Recall from [Etingof et al. 2005] that in a braided fusion category of an integer Frobenius–Perron dimension there is a canonical choice of a twist θ such that the categorical dimensions of objects coincide with their Frobenius–Perron dimensions. Namely, for any simple object X the scalar θ_X is defined in such a way that the composition

$$\mathbf{1} \xrightarrow{\mathrm{coev}_X} X \otimes X^* \xrightarrow{\theta_X c_{X,X^*}} X^* \otimes X \xrightarrow{\mathrm{ev}_X} \mathbf{1} \tag{39}$$

is equal to $\mathrm{FPdim}(X)\,\mathrm{id}_X$.

Let θ be the canonical twist on $\mathcal{Z}(\mathcal{C})$. Using the previous observation, explicit formulas from Section 4B, and Section 2F, we immediately obtain the following.

$$\theta_{X_{a,\epsilon}} = \chi(a,a)^{-1}, \quad \theta_{Y_{a,b}} = \chi(a,b)^{-1}, \quad \theta_{Z_{\rho,\Delta}} = \Delta.$$

Using the fusion rules of $\mathcal{Z}(\mathcal{C})$ (which may be computed using the explicit formulas in Section 4B), values of the twists above, and the well known formula

$$S_{X,Y} = \theta_X^{-1}\theta_Y^{-1} \sum_Z N_{X,Y}^Z \theta_Z d_Z, \tag{40}$$

we obtain the S- and T-matrices of $\mathcal{Z}(\mathcal{C})$:

$$\begin{aligned}
&S_{X_{a,\epsilon},X_{a',\epsilon'}} = \chi(a,a')^2, \quad S_{X_{a,\epsilon},Y_{b,c}} = 2\chi(a,b+c),\\
&S_{X_{a,\epsilon},Z_{\rho,\Delta}} = \epsilon\sqrt{n}\rho(a), \quad S_{Y_{a,b},Y_{c,d}} = 2\left(\chi(a,d)\chi(b,c)+\chi(a,c)\chi(b,d)\right),\\
&S_{Y_{a,b},Z_{\rho,\Delta}} = 0, \qquad S_{Z_{\rho,\Delta},Z_{\rho',\Delta'}} = \frac{1}{\Delta\Delta'}\sum_{a\in A}\chi(a,a)^2\rho(a)\rho'(a);\\
&T_{X_{a,\epsilon}} = \chi(a,a)^{-1}, \quad T_{Y_{a,b}} = \chi(a,b)^{-1}, \quad T_{Z_{\rho,\Delta}} = \Delta.
\end{aligned}$$

Proposition 4.2. *The maximal pointed subcategory of $\mathcal{Z}(\mathcal{C})$ is nondegenerate if and only if $|A|$ is odd.*

Proof. Let $a \in A$ be an element of order 2. Then $X_{a,\epsilon}$ centralizes every invertible object of $\mathcal{Z}(\mathcal{C})$. □

Remark 4.3. We note that simple objects and the S- and T-matrices of $\mathcal{Z}(\mathcal{C})$ were described in [Izumi 2001] using very different methods.

4D. ***A criterion for a Tambara–Yamagami category to be group-theoretical.*** The group $A\times\widehat{A}$ is equipped with a canonical nondegenerate quadratic form $q : A\times\widehat{A} \to k^\times$ given by

$$q((a,\phi)) := \phi(a), \quad (a,\phi) \in A\times\widehat{A}.$$

We will call a subgroup $B \subset A\times\widehat{A}$ *Lagrangian* if $q|_B = 1$ and $B = B^\perp$ with respect to the bilinear form defined by q. Lagrangian subgroups of $A\times\widehat{A}$ correspond to Lagrangian subcategories of $\mathcal{Z}(\mathrm{Vec}_A) \cong \mathrm{Vec}_{A\times\widehat{A}}$.

The braided tensor autoequivalence T_δ of $\mathcal{Z}(\mathrm{Vec}_A)$ defined in Section 4B determines an order 2 automorphism of $A\times\widehat{A}$, which we denote simply by δ:

$$\delta((a,\phi)) = (-\widehat{\phi}, -\widehat{a}), \quad (a,\phi) \in A\times\widehat{A}. \tag{41}$$

Definition 4.4. We will say that a subgroup $L \subset A$ is *Lagrangian (with respect to χ)* if $L = L^{\perp}$ with respect to the inner product on A given by χ. Equivalently, $|L|^2 = |A|$ and $\chi|_L = 1$.

Lemma 4.5. *Let A be an abelian 2-group such that $|A| = 2^{2n}$ and let χ be a nondegenerate symmetric bilinear form on A. Then A contains a Lagrangian subgroup.*

Proof. It suffices to show that A contains an isotropic element, that is, an element $x \in A$, $x \neq 0$, such that $\chi(x, x) = 1$. Then one can pass from A to $\langle x\rangle^{\perp}/\langle x\rangle$ and use induction.

Suppose that A is cyclic with a generator a. Then $2^{2n}a = 0$ and $\chi(a, a)$ is a (2^{2n})th root of unity, hence $\chi(2^n a, 2^n a) = \chi(a, a)^{2^{2n}} = 1$.

If A is not cyclic then it contains a subgroup $A_0 = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. Let x_1, x_2 be distinct nonzero elements of A_0. Suppose $\chi(x_i, x_i) \neq 1$, $i = 1, 2$. Then $\chi(x_i, x_i) = -1$ and $\chi(x_1 + x_2, x_1 + x_2) = 1$, as desired. □

Theorem 4.6. *Let $\mathscr{C} = \mathscr{T}\mathscr{Y}(A, \chi, \tau)$ be a Tambara–Yamagami fusion category. Then $\mathscr{C}$ is group-theoretical if and only if A contains a Lagrangian subgroup* (*with respect to χ*).

Proof. By Corollary 3.10, $\mathscr{C}$ is group-theoretical if and only if $\mathscr{Z}(\mathscr{D})$ contains a T_δ-stable Lagrangian subcategory. Equivalently, $\mathscr{C}$ is group-theoretical if and only if $A \times \widehat{A}$ contains a Lagrangian subgroup B stable under the action

$$(a, \phi) \mapsto (\widehat{\phi}, \widehat{a}). \tag{42}$$

This condition on B is the same as being stable under the action of δ from (41).

Let L be a Lagrangian (with respect to χ) subgroup of A and let $\widehat{L} := \{\widehat{a} \mid a \in L\}$. Then $L \times \widehat{L}$ is a Lagrangian subgroup of $A \times \widehat{A}$ stable under (42). Hence $\mathscr{C}$ is group-theoretical.

Conversely, suppose that $\mathscr{C}$ is group-theoretical. Let us write $A = A_{\text{even}} \oplus A_{\text{odd}}$, where A_{even} is the Sylow 2-subgroup of A and A_{odd} is the maximal odd order subgroup of A. Since $|A|$ must be a square, we conclude that $|A_{\text{even}}|$ is a square, and so A_{even} contains a Lagrangian subgroup with respect to $\chi|_{A_{\text{even}}}$ by Lemma 4.5.

So it remains to show that A_{odd} contains a Lagrangian subgroup with respect to $\chi|_{A_{\text{odd}}}$. For this end we may assume that $|A|$ is odd. Let $B \subset A \times \widehat{A}$ be a Lagrangian subgroup stable under (42). Then $B = B_+ \oplus B_-$, where

$$B_{\pm} := \{(a, \pm\widehat{a}) \mid (a, \pm\widehat{a}) \in B\}.$$

Let $L_{\pm} = B_{\pm} \cap (A \times \{1\})$. Then $|L_+||L_-| = |A|$, and $\chi|_{L_{\pm}} = 1$. Hence, $L_{\pm}$ are Lagrangian subgroups of A. □

Remark 4.7. It was observed in [Etingof et al. 2005, Remark 8.48] that for an odd prime p and elliptic bicharacter χ on $A = (\mathbb{Z}/p\mathbb{Z})^2$, the category $\mathscr{T}\mathscr{Y}((\mathbb{Z}/p\mathbb{Z})^2, \chi, \tau)$ is not group-theoretical. The criterion from Theorem 4.6 extends this observation.

4E. ***A series of non-group-theoretical semisimple Hopf algebras obtained from Tambara–Yamagami categories.*** Here we apply Corollary 3.11 to produce a series of non-group-theoretical fusion categories admitting fiber functors (that is, representation categories of non-group-theoretical semisimple Hopf algebras), generalizing examples constructed in [Nikshych 2008]. We refer the reader to [Montgomery 1993] as a reference on Hopf algebra theory.

Let A be a finite abelian group with a nondegenerate bilinear form χ. Let $\mathrm{Aut}(A, \chi)$ denote the group of automorphisms of A preserving χ.

The following proposition was proved in [Nikshych 2008, Proposition 2.10].

Proposition 4.8. *There is an action of* $\mathrm{Aut}(A, \chi)$ *on* $\mathcal{TY}(A, \chi, \tau)$ *given by* $g \mapsto T_g$*, where*

$$T_g(A) = g(a), \quad T_g(m) = m, \qquad a \in A,\ g \in \mathrm{Aut}(A, \chi),$$

with the tensor structure of T_g *given by identity morphisms.*

Corollary 4.9. *Let* G *be a subgroup of* $\mathrm{Aut}(A, \chi)$. *Then the fusion category* $\mathcal{TY}(A, \chi, \tau)^G$ *is group-theoretical if and only if there is a Lagrangian subgroup of* (A, χ) *stable under the action of* G.

Proof. Combine Corollary 3.11 and Theorem 4.6. □

We will say that a nondegenerate symmetric bilinear form $\chi : A \times A \to k^\times$ is *hyperbolic* if there are Lagrangian subgroups $L, L' \subset A$ such that $A = L \oplus L'$. In this case L' is isomorphic to the group $\widehat{L} = \mathrm{Hom}(L, k^\times)$ of characters of L and χ is identified with the canonical bilinear form on $L \oplus \widehat{L}$.

It was demonstrated in Tambara [2000] that when $n = |A|$ is odd the category $\mathcal{TY}(A, \chi, \tau)$ admits a fiber functor (that is, $\mathcal{TY}(A, \chi, \tau)$ is equivalent to the representation category of a semisimple Hopf algebra) if and only if τ^{-1} is a positive integer and χ is hyperbolic.

Corollary 4.10. *Let* p *be an odd prime, let* $L = (\mathbb{Z}/p\mathbb{Z})^N$, $N \geq 1$, *let* $A = L \oplus \widehat{L}$, *and let* $\chi : A \times A \to k^\times$ *be the canonical bilinear form defined by*

$$\chi((a, \phi), (b, \psi)) = \psi(a)\phi(b), \quad a, b \in A,\ \phi, \psi \in \widehat{A}.$$

Suppose that G *is a subgroup of* $\mathrm{Aut}(A, \chi)$ *not contained in any conjugate of* $\mathrm{Aut}(L) \subset \mathrm{Aut}(A, \chi)$. *Then the equivariantization category* $\mathcal{TY}(A, \chi, p^{-N})^G$ *is a non-group-theoretical fusion category equivalent to the representation category of a semisimple Hopf algebra of dimension* $2p^{2N}|G|$.

Proof. Note that $\mathrm{Aut}(A, \chi)$ acts transitively on the set of Lagrangian subgroups of (A, χ) and the stabilizer of L is $\mathrm{Aut}(L)$. Apply Corollary 4.9. □

Remark 4.11. The series of fusion categories in Corollary 4.10 extends the one constructed in [Nikshych 2008], where the case of $N = 1$ and $G = \mathbb{Z}/2\mathbb{Z}$ was considered.

5. Examples of modular categories arising from quadratic forms

As before, let $\mathscr{C} := \mathscr{T}\mathscr{Y}(A, \chi, \tau)$ be a Tambara–Yamagami category and let $\mathscr{D} := \mathscr{T}\mathscr{Y}(A, \chi, \tau)_1$ be the trivial component of $\mathbb{Z}_2$-grading of $\mathscr{T}\mathscr{Y}(A, \chi, \tau)$. In this section we assume that our ground field k is the field of complex numbers $\mathbb{C}$.

Suppose that the symmetric bicharacter $\chi : A \times A \to k^\times$ comes from a quadratic form on A, that is, there is a function $q : A \to k^\times$ such that

$$q(a+b) = q(a)q(b)\chi(a,b), \quad a, b \in A \quad \text{and} \quad q(-a) = q(a).$$

From the description obtained in Section 4B we observe that $\mathscr{Z}_{\mathscr{D}}(\mathscr{C})$ contains a fusion subcategory spanned by the simple objects $X_{(a,\widehat{a})}$, $a \in A$, and $Z_{q^{-1}}$. It is clear from the Tambara–Yamagami classification in Section 4A that this category is equivalent to $\mathscr{C}$.

Proposition 5.1. *Suppose that the symmetric bicharacter χ comes from a quadratic form on A. Then $\mathscr{C}$ admits a $\mathbb{Z}_2$-crossed braided category structure. The equivariantization $\mathscr{C}^{\mathbb{Z}_2}$ is nondegenerate if and only if $|A|$ is odd.*

Proof. Clearly, $\mathscr{C}$ inherits the $\mathbb{Z}_2$-crossed braided category structure from $\mathscr{Z}_{\mathscr{D}}(\mathscr{C})$. The nondegeneracy claim follows from Proposition 4.2 and Remark 2.13. □

Let us assume that $n := |A|$ is odd. Then χ corresponds to a unique quadratic form q. Let $\mathscr{E}(q, \pm) := \mathscr{C}^{\mathbb{Z}_2}$ be the modular category constructed in Proposition 5.1 (the $\pm$ corresponding to $\tau = \pm\frac{1}{\sqrt{n}}$, respectively). In what follows we describe the fusion rules and S- and T-matrices of $\mathscr{E}(q, \pm)$.

5A. ***Fusion rules of*** $\mathscr{E}$**.** Clearly, $\mathscr{E}(q, \pm)$ is a fusion category of dimension $4n$. It has the following simple objects:

two invertible objects, $\mathbf{1} = X_+$ and X_-;

$\frac{n-1}{2}$ two-dimensional objects Y_a, $a \in A - \{0\}$ (with $Y_{-a} = Y_a$); and

two $\sqrt{n}$-dimensional objects Z_l, $l \in \mathbb{Z}/2\mathbb{Z}$.

Here we simplify the notation used in Section 4C and define

$$X_\pm := X_{0,\pm 1}, \quad Y_a := Y_{a,-a}, \quad Z_l := Z_{q^{-1},\Delta_l},$$

where Δ_l, $l \in \mathbb{Z}/2\mathbb{Z}$, are distinct square roots of $\pm\dfrac{1}{\sqrt{n}} \sum_{a\in A} q(a)$.

The fusion rules of $\mathscr{E}(q, \pm)$ are given by

$$X_- \otimes X_- = X_+, \qquad X_\pm \otimes Y_a = Y_a, \qquad X_+ \otimes Z_l = Z_l,$$
$$X_- \otimes Z_l = Z_{l+1}, \qquad Y_a \otimes Y_b = Y_{a+b} \oplus Y_{a-b}, \qquad Y_a \otimes Y_a = X_+ \oplus X_- \oplus Y_{2a},$$
$$Y_a \otimes Z_l = Z_0 \oplus Z_1, \quad Z_l \otimes Z_l = X_+ \oplus (\oplus Y_a), \quad Z_l \otimes Z_{l+1} = X_- \oplus (\oplus Y_a),$$

where $a,\ b \in A$ $(a \neq b)$ and $l \in \mathbb{Z}/2\mathbb{Z}$. All objects of $\mathscr{E}(q, \pm)$ are self-dual.

Remark 5.2. Note that the fusion rules of $\mathscr{E}(q, \pm)$ do not depend on the quadratic form q and the number τ. We show below that the S- and T-matrices of $\mathscr{E}(q, \pm)$ do depend on q and τ.

5B. *S- and T-matrices of* $\mathscr{E}$.

Lemma 5.3. *The Gauss sums corresponding to q and q^2 are equal up to a sign, that is,*

$$\frac{\sum_{a \in A} q(a)^2}{\sum_{a \in A} q(a)} \in \{\pm 1\}.$$

Proof. Consider the group $A \times A$ with a nondegenerate quadratic form $Q = q \times q$. The Gaussian sum for this form is

$$\tau(A \times A, Q) = \sum_{a,b \in A} q(a)q(b) = \tau(A, q)^2.$$

The restriction of Q on the diagonal subgroup $D := \{(a, a) \mid a \in A\}$ is nondegenerate since $|A|$ is odd. The restriction of Q on the orthogonal complement $D^\perp = \{(a, -a) \mid a \in A\}$ is nondegenerate as well. By the multiplicativity of Gaussian sums we have

$$\tau(A \times A, Q) = \tau(D, Q)\tau(D^\perp, Q) = (\sum_{a \in A} q(a)^2)^2,$$

which implies the result. □

Using the formulas for the S- and T- matrices of $\mathscr{Z}(\mathscr{C})$ given in Section 4C we can write down the S- and T- matrices of $\mathscr{E}(q, \pm)$:

$$S_{X_\pm, X_\pm} = 1, \qquad S_{X_\mp, X_\pm} = 1, \qquad S_{X_\pm, Y_a} = 2, \quad S_{Y_a, Z_l} = 0,$$
$$S_{X_+, Z_l} = \sqrt{n}, \quad S_{X_-, Z_l} = -\sqrt{n}, \quad S_{Y_a, Y_b} = 2\Big(\frac{q(a+b)^2}{q(a)^2 q(b)^2} + \frac{q(a)^2 q(b)^2}{q(a+b)^2}\Big),$$
$$S_{Z_l, Z_l} = \begin{cases} \pm\sqrt{n} & \text{if the Gauss sums of } q \text{ and } q^2 \text{ coincide,} \\ \mp\sqrt{n} & \text{otherwise,} \end{cases}$$
$$S_{Z_l, Z_{l+1}} = \begin{cases} \mp\sqrt{n} & \text{if the Gauss sums of } q \text{ and } q^2 \text{ coincide,} \\ \pm\sqrt{n} & \text{otherwise.} \end{cases}$$

$$T_{X_\pm} = 1, \quad T_{Y_a} = q(a)^2, \quad T_{Z_l} = \Delta_l.$$

(Recall that $\Delta_l, l \in \mathbb{Z}/2\mathbb{Z}$, are distinct square roots of $\pm\frac{1}{\sqrt{n}} \sum_{a\in A} q(a)$.)

5C. *Example with $A = \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$.* Let p be an odd prime and let $A := \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$. Let $\left(\frac{\cdot}{p}\right)$ denote the Legendre symbol modulo p, that is, $\left(\frac{a}{p}\right) = 1$ if $a \in (\mathbb{Z}/p\mathbb{Z})^\times$ is a square modulo p and -1 otherwise.

Let $a, b \in (\mathbb{Z}/p\mathbb{Z})^\times$ and $\xi := e^{2\pi i/p}$. Consider the following nondegenerate quadratic form q on A:

$$q(x_1, x_2) = \xi^{ax_1^2 - bx_2^2}.$$

It is hyperbolic if $\left(\frac{ab}{p}\right) = 1$ and elliptic if $\left(\frac{ab}{p}\right) = -1$.

Lemma 5.4. *For every $a, b \in A^\times$, we have*

$$\sum_{x\in\mathbb{Z}/p\mathbb{Z}} \xi^{ax^2} = \begin{cases} \left(\frac{a}{p}\right)\sqrt{p} & \text{if } p \equiv 1 \pmod 4, \\ \left(\frac{a}{p}\right) i\sqrt{p} & \text{if } p \equiv 3 \pmod 4, \end{cases}$$

and

$$\sum_{(x_1,x_2)\in\mathbb{Z}/p\mathbb{Z}\times\mathbb{Z}/p\mathbb{Z}} \xi^{ax_1^2 - bx_2^2} = \left(\frac{ab}{p}\right) p.$$

Proof. The first assertion is well known; see, for example, [Ireland and Rosen 1990]. The second assertion is an easy consequence of the first. □

Using Lemma 5.4 we can explicitly write the S-matrix of $\mathcal{E}(q, \pm)$:

$$\begin{aligned} S_{X_\pm,X_\pm} &= 1, S_{X_\mp,X_\pm} = 1, & S_{X_\pm,Y_{(x_1,x_2)}} &= 2, \\ S_{X_+,Z_l} &= p,\ S_{X_-,Z_l} = -p, & S_{Y_{(x_1,x_2)},Y_{(y_1,y_2)}} &= 4\,\mathrm{Re}(\xi^{4ax_1y_1 - 4bx_2y_2}), \\ S_{Y_{(x_1,x_2)},Z_l} &= 0, \quad S_{Z_l,Z_l} = \pm p, & S_{Z_l,Z_{l+1}} &= \mp p, \end{aligned}$$

and its T-matrix:

$$T_{X_\pm} = 1, \quad T_{Y_{(x_1,x_2)}} = \xi^{2ax_1^2 - 2bx_2^2}, \quad T_{Z_l} = \Delta_l,$$

where $\Delta_l, l \in \mathbb{Z}/2\mathbb{Z}$, are distinct square roots of $\pm\left(\frac{ab}{p}\right)$.

The central charge of the modular category $\mathcal{E}(q, \pm)$ is

$$\zeta(\mathcal{E}(q, \pm)) = \left(\frac{ab}{p}\right).$$

Below we give the S- and T-matrices of the modular category $\mathcal{E}(q, \pm)$ for $p = 3$. Order simple objects of $\mathcal{E}(q, \pm)$ as follows: $\mathbf{1}$, X_-, $Y_{(0,1)}$, $Y_{(1,0)}$, $Y_{(1,1)}$, $Y_{(1,2)}$, Z_+, Z_-. There are four modular categories $\mathcal{E}(q, \pm)$ of dimension 36 corresponding to the choices of hyperbolic/elliptic q and $\tau = \pm\frac{1}{3}$.

(a) When q is hyperbolic we have

$$S = \begin{pmatrix} 1 & 1 & 2 & 2 & 2 & 2 & 3 & 3 \\ 1 & 1 & 2 & 2 & 2 & 2 & -3 & 3 \\ 2 & 2 & -2 & 4 & -2 & -2 & 0 & 0 \\ 2 & 2 & 4 & -2 & -2 & -2 & 0 & 0 \\ 2 & 2 & -2 & -2 & 4 & -2 & 0 & 0 \\ 2 & 2 & -2 & -2 & -2 & 4 & 0 & 0 \\ 3 & -3 & 0 & 0 & 0 & 0 & \pm 3 & \mp 3 \\ 3 & -3 & 0 & 0 & 0 & 0 & \mp 3 & \pm 3 \end{pmatrix},$$

$$T = \operatorname{diag}\{1, 1, \xi^2, \xi, 1, 1, 1, -1\} \quad \text{when } \tau = \tfrac{1}{3},$$
$$T = \operatorname{diag}\{1, 1, \xi^2, \xi, 1, 1, i, -i\} \quad \text{when } \tau = -\tfrac{1}{3}.$$

Note that both the corresponding modular categories are group-theoretical with central charge 1; in fact the one with $\tau = \frac{1}{3}$ is equivalent to the representation category of the double $D(S_3)$ of the symmetric group S_3 and the one with $\tau = -\frac{1}{3}$ is equivalent to the twisted double of S_3.

(b) When q is elliptic we have

$$S = \begin{pmatrix} 1 & 1 & 2 & 2 & 2 & 2 & 3 & 3 \\ 1 & 1 & 2 & 2 & 2 & 2 & -3 & 3 \\ 2 & 2 & -2 & 4 & -2 & -2 & 0 & 0 \\ 2 & 2 & 4 & -2 & -2 & -2 & 0 & 0 \\ 2 & 2 & -2 & -2 & -2 & 4 & 0 & 0 \\ 2 & 2 & -2 & -2 & 4 & -2 & 0 & 0 \\ 3 & -3 & 0 & 0 & 0 & 0 & \pm 3 & \mp 3 \\ 3 & -3 & 0 & 0 & 0 & 0 & \mp 3 & \pm 3 \end{pmatrix},$$

$$T = \operatorname{diag}\{1, 1, \xi, \xi, \xi^2, \xi^2, i, -i\} \quad \text{when } \tau = \frac{1}{3},$$
$$T = \operatorname{diag}\{1, 1, \xi, \xi, \xi^2, \xi^2, 1, -1\} \quad \text{when } \tau = -\frac{1}{3}.$$

Both the corresponding modular categories are not group-theoretical. They both have central charge -1 and so are not equivalent to centers of fusion categories. In particular, they are not equivalent to representation categories of any twisted group doubles.

5D. *Example with $A = \mathbb{Z}/p\mathbb{Z}$.* Let p be an odd prime and let $A := \mathbb{Z}/p\mathbb{Z}$. Let $a \in (\mathbb{Z}/p\mathbb{Z})^\times$ and $\xi := e^{2\pi i/p}$. Up to isomorphism there are two nondegenerate quadratic forms q on A:

$$q(x) = \xi^{ax^2},$$

one corresponding to $\left(\frac{a}{p}\right)=1$ and another to $\left(\frac{a}{p}\right)=-1$.

Using Lemma 5.4 we can explicitly write the S-matrix of $\mathcal{E}(q,\pm)$:

$$\begin{aligned} &S_{X_\pm,X_\pm}=1, && S_{X_\mp,X_\pm}=1, && S_{X_\pm,Y_x}=2,\\ &S_{X_+,Z_l}=\sqrt{p}, && S_{X_-,Z_l}=-\sqrt{p}, && S_{Y_x,Y_y}=4\,\mathrm{Re}(\xi^{4axy}),\\ &S_{Y_a,Z_l}=0, && S_{Z_l,Z_l}=\pm\left(\frac{2}{p}\right)\sqrt{p}, && S_{Z_l,Z_{l+1}}=\mp\left(\frac{2}{p}\right)\sqrt{p}. \end{aligned}$$

Further, we have

$$T_{X_\pm}=1,\quad T_{Y_x}=\xi^{-2ax^2},\quad T_{Z_l}=\Delta_l,$$

where

$$\Delta_l,\ l\in\mathbb{Z}/2\mathbb{Z},\ \text{ are distinct square roots of }\begin{cases}\pm\left(\frac{a}{p}\right) & \text{if } p\equiv 1 \pmod 4,\\ \pm\left(\frac{a}{p}\right)i & \text{if } p\equiv 3 \pmod 4.\end{cases}$$

The central charge of the modular category $\mathcal{E}(q,\pm)$ is

$$\zeta(\mathcal{E}(q,\pm))=\begin{cases}\left(\frac{2a}{p}\right) & \text{if } p\equiv 1 \pmod 4,\\ -\left(\frac{2a}{p}\right)i & \text{if } p\equiv 3 \pmod 4.\end{cases}$$

Below we give the S- and T-matrices of the modular category $\mathcal{E}(q,\pm)$ for $p=3$ and 5. For $p=3$ we order the simple objects as $\mathbf{1}, X_-, Y_1, Z_0, Z_1$ and for $p=5$ we order them as $\mathbf{1}, X_-, Y_1, Y_2, Z_0, Z_1$. (In (c) and (d) below, $\xi=e^{2\pi i/5}$.)

(a) When $p=3$ and $a=1$ we have

$$S=\begin{pmatrix} 1 & 1 & 2 & \sqrt{3} & \sqrt{3}\\ 1 & 1 & 2 & -\sqrt{3} & -\sqrt{3}\\ 2 & 2 & -2 & 0 & 0\\ \sqrt{3} & -\sqrt{3} & 0 & \mp\sqrt{3} & \pm\sqrt{3}\\ \sqrt{3} & -\sqrt{3} & 0 & \pm\sqrt{3} & \mp\sqrt{3} \end{pmatrix},$$

$$T=\mathrm{diag}\left\{1,1,\frac{-1+i\sqrt{3}}{2},\frac{1+i}{\sqrt{2}},\frac{-1-i}{\sqrt{2}}\right\}\quad \text{when } \tau=\frac{1}{\sqrt{3}},$$

$$T=\mathrm{diag}\left\{1,1,\frac{-1+i\sqrt{3}}{2},\frac{1-i}{\sqrt{2}},\frac{-1+i}{\sqrt{2}}\right\}\quad \text{when } \tau=-\frac{1}{\sqrt{3}}.$$

The central charge of both the corresponding modular categories is i.

(b) When $p=3$ and $a=2$ we have

$$S = \text{ the } S\text{-matrix in (a)},$$

$$T = \operatorname{diag}\left\{1,\, 1,\, \frac{-1-i\sqrt{3}}{2},\, \frac{1-i}{\sqrt{2}},\, \frac{-1+i}{\sqrt{2}}\right\} \quad \text{when } \tau = \frac{1}{\sqrt{3}},$$

$$T = \operatorname{diag}\left\{1,\, 1,\, \frac{-1-i\sqrt{3}}{2},\, \frac{1+i}{\sqrt{2}},\, \frac{-1-i}{\sqrt{2}}\right\} \quad \text{when } \tau = \frac{1}{\sqrt{3}}.$$

The central charge of both the corresponding modular categories is $-i$.

(c) When $p=5$ and $a=1$ we have

$$S = \begin{pmatrix} 1 & 1 & 2 & 2 & \sqrt{5} & \sqrt{5} \\ 1 & 1 & 2 & 2 & -\sqrt{5} & -\sqrt{5} \\ 2 & 2 & \sqrt{5}-1 & -\sqrt{5}-1 & 0 & 0 \\ 2 & 2 & -\sqrt{5}-1 & \sqrt{5}-1 & 0 & 0 \\ \sqrt{5} & -\sqrt{5} & 0 & 0 & \mp\sqrt{5} & \pm\sqrt{5} \\ \sqrt{5} & -\sqrt{5} & 0 & 0 & \pm\sqrt{5} & \mp\sqrt{5} \end{pmatrix},$$

$$T = \operatorname{diag}\left\{1,\, 1,\, \xi^3,\, \xi^2,\, 1,\, -1\right\} \quad \text{when } \tau = \frac{1}{\sqrt{5}},$$

$$T = \operatorname{diag}\left\{1,\, 1,\, \xi^3,\, \xi^2,\, i,\, -i\right\} \quad \text{when } \tau = -\frac{1}{\sqrt{5}}.$$

The central charge of both the corresponding modular categories is -1.

(d) When $p=5$ and $a=2$ we have

$$S = \begin{pmatrix} 1 & 1 & 2 & 2 & \sqrt{5} & \sqrt{5} \\ 1 & 1 & 2 & 2 & -\sqrt{5} & -\sqrt{5} \\ 2 & 2 & -\sqrt{5}-1 & \sqrt{5}-1 & 0 & 0 \\ 2 & 2 & \sqrt{5}-1 & -\sqrt{5}-1 & 0 & 0 \\ \sqrt{5} & -\sqrt{5} & 0 & 0 & \mp\sqrt{5} & \pm\sqrt{5} \\ \sqrt{5} & -\sqrt{5} & 0 & 0 & \pm\sqrt{5} & \mp\sqrt{5} \end{pmatrix},$$

$$T = \operatorname{diag}\left\{1,\, 1,\, \xi,\, \xi^4,\, i,\, -i\right\} \quad \text{when } \tau = \frac{1}{\sqrt{5}},$$

$$T = \operatorname{diag}\left\{1,\, 1,\, \xi,\, \xi^4,\, 1,\, -1\right\} \quad \text{when } \tau = -\frac{1}{\sqrt{5}}.$$

The central charge of both the corresponding modular categories is 1.

6. Appendix: Zeroes in S-matrices

There is a classical result of Burnside in character theory saying that if χ is an irreducible character of a finite group G and $\chi(1) > 1$, then $\chi(g) = 0$ for some $g \in G$; see [Berkovich and Zhmud' 1999, Chapter 21].

In this appendix we establish a categorical analogue of this result for weakly integral modular categories. Recall from [Etingof et al. 2008] that a fusion category $\mathcal{C}$ is called *weakly integral* if its Frobenius–Perron dimension is an integer. In this case the Frobenius–Perron dimension of every simple object of $\mathcal{C}$ is the square root of an integer [Etingof et al. 2005].

Let $\mathcal{C}$ be a weakly integral modular category with the S-matrix S. Let $\mathbb{O}(\mathcal{C})$ denote the set of all (representatives of isomorphism classes of) simple objects of $\mathcal{C}$. Given $X \in \mathbb{O}(\mathcal{C})$ define the sets

$$T_X = \{Y \in \mathbb{O}(\mathcal{C}) \mid S_{X,Y} = 0\}, \quad D_X = \mathbb{O}(\mathcal{C}) - (T_X \cup \{\mathbf{1}\}).$$

Clearly, we have a partition $\mathbb{O}(\mathcal{C}) = T_X \cup D_X \cup \{\mathbf{1}\}$. Let $\mathcal{T}_X$ and $\mathcal{D}_X$ be full abelian subcategories of $\mathcal{C}$ generated by T_X and D_X, respectively.

Let K be the field extension of $\mathbb{Q}$ generated by the entries of S. It is known [de Boer and Goeree 1991; Coste and Gannon 1994] that there is a root of unity ξ such that $K \subset \mathbb{Q}(\xi)$. In particular, the operation of taking the square of an absolute value of an element of S is well defined. Let $G := \mathrm{Gal}(K/\mathbb{Q})$. Every element $\sigma \in G$ comes from a permutation σ of $\mathbb{O}(\mathcal{C})$ such that $\sigma(S_{X,Y}) = S_{X,\sigma(Y)}$ for all $X, Y \in \mathbb{O}(\mathcal{C})$.

Let $\mathcal{C}$ be a weakly integral modular category. It was shown in [Etingof et al. 2005] that there is a canonical spherical structure on $\mathcal{C}$ such that categorical dimensions in $\mathcal{C}$ coincide with Frobenius–Perron dimensions. Let us fix this structure for the remainder of this section. For any $X \in \mathbb{O}(\mathcal{C})$ let d_X denote the dimension of X. For any full abelian subcategory $\mathcal{A}$ of $\mathcal{C}$ let $\dim \mathcal{A}$ denote the sum of squares of dimensions of simple objects of $\mathcal{A}$.

Theorem 6.1. *Let $\mathcal{C}$ be a weakly integral modular category with the S-matrix S. Then T_X is not empty for every noninvertible simple object X of $\mathcal{C}$. That is, every row (column) of S corresponding to a noninvertible simple object contains at least one zero entry.*

Proof. Note that the statement of Proposition does not depend on the choice of spherical structure.

We have $\sum_{Y \in \mathbb{O}(\mathcal{C})} |S_{X,Y}|^2 = \dim \mathcal{C}$; hence,

$$1 = \frac{\dim \mathcal{C}}{d_X^2} - \sum_{Y \in D_X} \left| \frac{S_{X,Y}}{d_X} \right|^2 = \frac{1 + \dim \mathcal{T}_X}{d_X^2} - \left(\sum_{Y \in D_X} \left| \frac{S_{X,Y}}{d_X} \right|^2 - \frac{\dim \mathcal{D}_X}{d_X^2} \right), \quad (43)$$

where d_X denotes the dimension of X. It suffices to check that

$$\frac{1}{\dim \mathcal{D}_X} \sum_{Y \in D_X} \left| \frac{S_{X,Y}}{d_X} \right|^2 \geq \frac{1}{d_X^2}, \tag{44}$$

since then (43) implies that $1 \leq (1 + \dim \mathcal{T}_X)/d_X^2$, whence

$$\dim \mathcal{T}_X \geq d_X^2 - 1. \tag{45}$$

But X is noninvertible so $d_X > 1$ and $\mathcal{T}_X \neq 0$.

Rewriting the left hand side of (44) as the sum of $\dim \mathcal{D}_X$ terms and using the inequality of arithmetic and geometric means we obtain

$$\begin{aligned} \frac{1}{\dim \mathcal{D}_X} \sum_{Y \in D_X} \left| \frac{S_{X,Y}}{d_X} \right|^2 &= \frac{1}{\dim \mathcal{D}_X} \sum_{Y \in D_X} d_Y^2 \left| \frac{S_{X,Y}}{d_X d_Y} \right|^2 \\ &\geq \frac{1}{d_X^2} \left(\prod_{Y \in D_X} \left| \frac{S_{X,Y}}{d_Y} \right|^{2d_Y^2} \right)^{1/\dim \mathcal{D}_X}. \end{aligned}$$

The set D_X is clearly stable under all automorphisms in the Galois group, and hence so is the product $\prod_{Y \in D_X} \left| S_{X,Y}/d_Y \right|^{2d_Y^2}$. Therefore, this product belongs to $\mathbb{Q}$. Its factors are squares of absolute values of characters of $K_0(\mathcal{C})$ on X and hence are algebraic integers. Since all factors are positive, the product is ≥ 1, which implies (44). □

For $X \in \mathcal{O}(\mathcal{C})$ define

$$U_X = \{Y \in \mathcal{O}(\mathcal{C}) \mid |S_{X,Y}| = d_Y\}.$$

Let $\mathcal{U}_X$ be the full abelian subcategory of $\mathcal{C}$ generated by U_X.

Proposition 6.2. *Let $\mathcal{C}$ be a weakly integral modular category and let X be a simple noninvertible object in $\mathcal{C}$. Then*

$$3 \dim \mathcal{T}_X + \dim \mathcal{U}_X > \dim \mathcal{C}. \tag{46}$$

Proof. We may assume $d_X \geq \sqrt{2}$.

We will use the following theorem of Siegel [1945] from number theory. Let $K/\mathbb{Q}$ be a finite Galois extension with the Galois group $G = \mathrm{Gal}(K/\mathbb{Q})$. Let α be a totally positive algebraic integer in K, $\alpha \neq 1$. Then

$$\frac{1}{|G|} \sum_{\sigma \in G} \sigma(\alpha) \geq \frac{3}{2}.$$

We apply this to the situation when K is the extension of $\mathbb{Q}$ generated by entries of S. We compute

$$\begin{aligned}\dim \mathscr{C} &= \sum_{Y\in\mathscr{C}} |S_{X,Y}|^2 = d_X^2 + \sum_{Y\in U_X} d_Y^2 + \sum_{Y\in\mathbb{O}(\mathscr{C})-(T_X\cup U_X\cup\{\mathbf{1}\})} |S_{X,Y}|^2 \\ &= d_X^2 + \dim \mathscr{U}_X + \sum_{Y\in\mathbb{O}(\mathscr{C})-(T_X\cup U_X\cup\{\mathbf{1}\})} d_Y^2 \left(\frac{1}{|G|}\sum_{\sigma\in G}\sigma\left(\frac{|S_{X,Y}|^2}{d_Y^2}\right)\right) \\ &\geq 2 + \dim \mathscr{U}_X + \tfrac{3}{2}(\dim \mathscr{C} - \dim \mathscr{T}_X - \dim \mathscr{U}_X - 1);\end{aligned}$$

therefore $3\dim \mathscr{T}_X + \dim \mathscr{U}_X \geq \dim \mathscr{C} + 1 > \dim \mathscr{C}$, as required. □

Remark 6.3. Our proofs of Theorem 6.1 and Proposition 6.2 imitate the corresponding proofs for group characters given in [Berkovich and Zhmud' 1999].

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