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Stable reduction of $X_0(p^3)$

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This paper is dedicated to Siegfried Bosch, whose foundational work in rigid analysis was invaluable in our development of the theory of semistable coverings.

We determine the stable models of the modular curves $X_0(p^3)$ for primes $p \geq 13$. An essential ingredient is the close relationship between the deformation theories of elliptic curves and formal groups, which was established in the Woods Hole notes of 1964. This enables us to apply results of Hopkins and Gross in our analysis of the supersingular locus.

1. Introduction

Let n be an integer and p a prime. It is known that if $n \geq 3$ and $p \geq 5$, or if $n \geq 1$ and $p \geq 11$, the modular curve $X_0(p^n)$ does not have a model with good reduction over the ring of integers of any complete subfield of $\mathbb{C}_p$. By a model for a scheme C over a complete local field K, we mean a scheme S over the ring of integers $\mathbb{O}_K$ of K such that $C \cong S \otimes_{\mathbb{O}_K} K$. When a curve C over K does not have a model with good reduction over $\mathbb{O}_K$, it may have the "next best thing", that is, a stable model. The stable model is unique up to isomorphism if it exists, which it does over the ring of integers in some finite extension of K, as long as the genus of the curve is at least 2. Moreover, if $\mathscr{C}$ is a stable model for C over $\mathbb{O}_K$, and $K \subseteq L \subseteq \mathbb{C}_p$, then $\mathscr{C} \otimes_{\mathbb{O}_K} \mathbb{O}_L$ is a stable model for $C \otimes_K L$ over $\mathbb{O}_L$. The special fiber of any stable model for C is called the stable reduction.

Here is a brief summary of prior results regarding the stable models of modular curves at prime power levels. Deligne and Rapoport [1973, §VI.6] found models for $X_0(p)$ and $X_1(p)$ over $\mathbb{Z}_p$ and $\mathbb{Z}_p[\mu_p]$ that become stable over the quadratic unramified extension. Edixhoven [1990, Theorem 2.1.2] found stable models for $X_0(p^2)$ over the ring of integers, R, in the Galois extension of $\mathbb{Q}_p^{\mathrm{unr}}$ of degree $(p^2-1)/2$. Bouw and Wewers [2004, Theorem 4.1 and Corollary 3.4] found stable models of $X_0(p)$ and $X(p)$ over $\mathbb{Z}_p$ and R by completely different means. Krir [1996, Théorème 1] proved that the Jacobian of $X_0(p^n)$ has a semistable

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model over the ring of integers of an explicit Galois extension L_n of $\mathbb{Q}_p^{\mathrm{unr}}$ of degree $p^{2(n-2)}(p^2-1)$ for $n \geq 2$, which implies that $X_0(p^n)$ has a stable model over the ring of integers of L_n by [Deligne and Mumford 1969, Theorem 2.4]. Also, stable models for $X_0(125)$ and $X_0(81)$ were computed explicitly in [McMurdy 2004, §2; 2008, §3], and [2008, §5] gave a conjectural stable reduction of $X_0(p^4)$. The main result of this paper is the construction of a stable model for $X_0(p^3)$, when $p \geq 13$, over the ring of integers of some finite extension of $\mathbb{Q}_p$ that is made explicit in [CM 2006].

We introduce the notion of a semistable covering of a smooth complete curve over a complete nonarchimedean field in Section 2C. We prove that any curve over a complete stable subfield of $\mathbb{C}_p$ has a semistable covering if and only if it has a semistable model, and moreover we can determine the corresponding reduction from the covering (see Theorem 2.36). Finding a semistable covering is often easier in practice than finding a semistable model directly, and this is what we do for $X_0(p^3)$ in Sections 6–9.

Overview. Our approach is rigid analytic, in that we construct a stable model of $X_0(p^3)$ by actually constructing a stable covering by wide opens (an equivalent rigid analytic notion which was introduced in [Coleman and McCallum 1988, §1]). A covering $\mathscr{C}^o$ of the ordinary locus can be obtained by extending the ordinary affinoids $\mathbf{X}_{ab}^{\pm}$ defined in [Coleman 2005, §1] to wide open neighborhoods $W_{ab}^{\pm}$. The supersingular locus essentially breaks up into the union of finitely many deformation spaces of height 2 formal groups with level structure [Lubin et al. 1964]. We use results from [Hopkins and Gross 1994] and [de Shalit 1994] to produce a covering $\mathscr{C}^s$ of this region. Finally, we show that the genus of the covering $\mathscr{C}^o \cup \mathscr{C}^s$ is at least the genus of $X_0(p^3)$, and therefore that the overall covering is stable. This argument is laid out as follows.

First, in Section 2, we recall or prove the general rigid analytic results that are necessary for a stable covering argument. These results are proved not only over complete subfields of $\mathbb{C}_p$, but over more general complete nonarchimedean-valued fields. For example, Proposition 2.34 is the aforementioned result that the genus of any stable covering must equal the genus of the curve. We also revise and extend results of Bosch, and of Bosch and Lütkebohmert, on the rigid geometry of algebraic curves. A rigid analytic version of the Riemann existence theorem is proved in Appendix A.

In Section 3, we recall and fix notation for some results specifically pertaining to $X_0(p^n)$ and its rigid subspaces. This is done from the moduli-theoretic point of view, which is that points of $X_0(p^n)$ correspond to pairs (E, C), where E is a generalized elliptic curve and C is a cyclic subgroup of order p^n. There is a detailed discussion in Section 3A of the theory of the canonical subgroup of an elliptic curve and its connection with the geometry of $X_0(p)$ [Buzzard 2003, §3]. Section 3B is where we define wide open neighborhoods, $W_{ab}^{\pm} \supseteq \mathbf{X}_{ab}^{\pm}$, of the irreducible affinoids that make up the ordinary locus of $X_0(p^n)$.

All of the necessary results regarding deformations of formal groups are given in Section 4. First we precisely state the relationship between deformations of elliptic curves and formal groups, which we call *Woods Hole theory* [Lubin et al. 1964, §6]. This is then used in Section 4A (along with the result of Howe in Appendix B) to prove that all of the connected components of the supersingular locus of $X_0(p^n)$ are (nearly) isomorphic. Because of this fact, we are able to focus on those regions $W_A(p^n)$ that correspond to a supersingular elliptic curve $A/\mathbb{F}_p$ for which $j(A) \neq 0, 1728$. Specifically, this enables us to directly apply results of de Shalit [1994, §3] for the forgetful map from $X_0(p)$ to the j-line. The other important consequence of Woods Hole theory is that it gives us a natural action of

$$\mathrm{Aut}(\hat{A}) \cong (\mathrm{End}(A) \otimes \mathbb{Z}_p)^*$$

on $W_A(p^n)$. In Section 4B we recall results from [Hopkins and Gross 1994] that describe this action in great detail, and we derive the specific consequences that we need for our analysis of $X_0(p^3)$.

Once the groundwork has been laid, the remaining sections are devoted to constructing stable coverings of $X_0(p^2)$ and $X_0(p^3)$. In Section 5 we construct a stable covering for $X_0(p^2)$ over an explicit Galois extension of $\mathbb{Q}_p$ of degree $12(p^2-1)$, essentially showing that the wide open subspaces defined in Section 3 are sufficient. To be more precise, the stable covering consists of

$$\{W_{20}, W_{11}^+, W_{11}^-, W_{02}\} \cup \{W_A(p^2) : A \text{ is supersingular}\}.$$

This reproves Edixhoven's [1990] result from the point of view of this paper. It also gives a moduli-theoretic interpretation to the wide opens and underlying affinoids in the stable covering.

As in the stable covering of $X_0(p^2)$, the ordinary region of $X_0(p^3)$ is covered by six wide opens: W_{30}, $W_{21}^{\pm}$, $W_{12}^{\pm}$, and W_{03}. Unlike $W_A(p^2)$, however, $W_A(p^3)$ must itself be covered by smaller wide opens, since its reduction contains multiple irreducible components. First of all, the reduction of $W_A(p^3)$ contains two isomorphic lifts of some supersingular component of $X_0(p^2)$, with each meeting exactly three of the ordinary components. These two "old" components are connected through a central genus-0 component that we call the *bridging component*. To complete the picture, the bridging component then meets (in distinct points) a certain number of isomorphic copies of the curve $y^2 = x^p - x$. A partial picture of the stable reduction of $X_0(p^3)$, including one complete supersingular region (corresponding to a fixed supersingular curve A) and the six ordinary components, is given in Figure 1. The

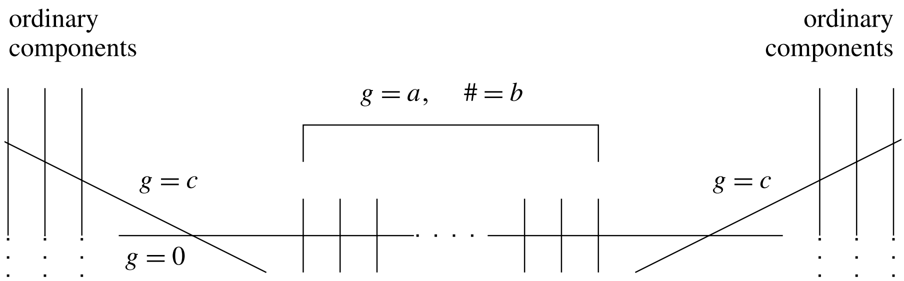

Figure 1. Partial picture of the stable reduction of $X_0(p^3)$.

number and genera of the components, as marked on the graph, are as follows:

$$(a, b, c) = \begin{cases} \big((p-1)/2, 2(p+1)/3, (p-5)/6\big) & \text{if } j(A) = 0, \\ \big((p-1)/2, p+1, (p-3)/4\big) & \text{if } j(A) = 1728, \\ \big((p-1)/2, 2(p+1), (p-1)/2\big) & \text{otherwise.} \end{cases}$$

Complete graphs with intersection multiplicities are given in Section 9A. As a consequence of these results, it follows that the new part of $J_0(p^3)$ has potential good reduction isogenous to the product of $(p^2-1)/6$ copies of the Jacobian of $y^2 = x^p - x$.

It should be noted that the field of definition of our stable covering ultimately depends on the field of definition of certain elliptic curves that have "fake CM". In [CM 2006] we proved results about these fake CM curves that then made it possible for us to define the stable model over the ring of integers of an explicit finite extension of $\mathbb{Q}_p$ and compute the associated Weil group action, assuming the results of this paper. In [CM 2006] we also dealt with the $p \le 11$ cases explicitly and computed the stable reduction of $X_0(Np^3)$ for $(N, p) = 1$. We expect that our methods will extend to $X_0(Np^n)$, and will have applications to modular forms as in [CM 2006, Remark 6.10]. We understand that Wewers also has a different approach with applications to local Langlands.

2. Rigid analytic foundation

We fix some notation for the p-adic analysis and more general nonarchimedean analysis. Throughout this section, unless otherwise stated, we let K be a complete nonarchimedean valued field with absolute value $|\cdot|$. We denote the ring of integers of K by R_K, its maximal ideal by m_K, and the residue field by $\mathbb{F}_K$. Let p be the characteristic of $\mathbb{F}_K$ (which we allow to be 0). Let $\mathbf{C}$ be the completion of an algebraic closure of K, and denote its ring of integers, maximal ideal, and residue field by $\mathbf{R}$, $m_{\mathbf{R}}$, and $\overline{\mathbb{F}}$. Note that $\overline{\mathbb{F}}$ is then an algebraic closure of $\mathbb{F}_K$. Whenever

$\mathbb{F}_K$ is perfect and has positive characteristic, we let $W(\mathbb{F}) \subseteq \mathbf{R}$ denote the ring of Witt vectors for any field $\mathbb{F} \subseteq \overline{\mathbb{F}}$. The value group of K will be denoted $|K^*|$, and we let

$$\mathcal{R} := \mathcal{R}_K = \{x \in \mathbb{R} : x^n \in |K^*| \text{ for some } n \in \mathbb{N}\}$$

(equivalently, $\mathcal{R} := |\mathbf{C}^*|$). Then if $S \subseteq \mathbb{R}$, we let $\mathcal{R}S = \mathcal{R} \cap S$.

Occasionally, for technical reasons, we will need to assume that K is a stable field [Bosch et al. 1984, Definition 3.6.1/1]. By [1984, Proposition 3.6.2/6], this is the case if and only if $e(L/K)f(L/K) = [L : K]$ for all finite extensions L/K, where $e(L/K) = |L^*|/|K^*|$ and $f(L/K) = [\mathbb{F}_L : \mathbb{F}_K]$ are the ramification index and residue degree of L over K. There are also two special cases that we will consider for certain results. First, for a fixed prime p, let $\mathbb{C}_p$ be the completion of a fixed algebraic closure of $\mathbb{Q}_p$, let $\mathbb{R}_p \subseteq \mathbb{C}_p$ be its ring of integers, and let $m_{\mathbb{R}_p}$ be the maximal ideal of $\mathbb{R}_p$. Let v denote the unique valuation on $\mathbb{C}_p$ with $v(p) = 1$, and $|\cdot|$ the absolute value given by $|0| = 0$ and $|x| = p^{-v(x)}$ for $x \neq 0$. In this case $\mathcal{R} = |\mathbb{C}_p^*| = p^{\mathbb{Q}}$. Also $\mathbb{C}_p$ is stable, as is the completion of any tamely ramified extension of a finite extension of $\mathbb{Q}_p$. The second specific nonarchimedean valued field that will be considered is $\overline{\mathbb{F}}_p((T))$, for which the corresponding field $\mathbf{C}$ will be denoted $\mathbf{\Omega}_p$. Both $\overline{\mathbb{F}}_p((T))$ and $\mathbf{\Omega}_p$ are stable, and in this case we have $\mathcal{R} = |T|^{\mathbb{Q}}$.

Hypothesis T. The field $\mathbf{C}$ is isomorphic to either $\mathbb{C}_p$ or $\mathbf{\Omega}_p$.

In fact, for our purposes, this hypothesis can be relaxed to "$\mathbf{C}$ is an immediate extension[1] of $\mathbb{C}_p$ or $\mathbf{\Omega}_p$".

Remark 2.1. Suppose K satisfies Hypothesis T. Then if A is an Abelian variety over K and $P \in A(\mathbf{C})$, then 0 is in the closure of $\{nP : n \in \mathbb{N}\}$; see the proof of Lemma 2.19.

Now, for $r \in \mathcal{R}$, we let $B_K^d[r]$ and $B_K^d(r)$ denote the closed and open d-dimensional polydisks over K of radius r around 0, that is, the rigid spaces over K whose $\mathbf{C}$-valued points are $\{(x_1, \ldots, x_d) \in \mathbf{C}^d : |x_i| \leq r\}$ and $\{(x_1, \ldots, x_d) \in \mathbf{C}^d : |x_i| < r\}$, respectively. In particular, let $B_K[r] := B_K^1[r]$ and $B_K(r) := B_K^1(r)$ denote the *closed disk* and *open disk* of radius r around 0. If $r, s \in \mathcal{R}$ and $r \leq s$, let $A_K[r, s]$ and $A_K(r, s)$ be the rigid spaces over K whose $\mathbf{C}$-valued points are $\{x \in \mathbf{C} : r \leq |x| \leq s\}$ and $\{x \in \mathbf{C} : r < |x| < s\}$, which we call *closed annuli* and *open annuli*. The *semiopen annuli*, $A_K[r, s)$ and $A_K(r, s]$, are similarly defined. The width of such an annulus is defined to be $\log_p(s/r)$ or $\ln(s/r)$ if $p = 0$. Note that all closed or open disks over K, and all closed or open annuli over K of the same width, are potentially isomorphic. Here and throughout the paper, we use the adverb "potentially" in various contexts to mean "after finite base extension." A closed

[1] In the classical theory, an extension of valued fields is said to be immediate if the corresponding value groups and residue fields are isomorphic. This notion was introduced by Krull.

annulus of width 0 will be called a *circle*, and we will also denote the circle, $A_K[s,s]$, by $C_K[s]$.

If X is a rigid space over K and $f \in A(X) := \mathcal{O}_X(X)$, let $|f|_{\sup}$ denote the sup of $|f(x)|$ over all $x \in X(\mathbf{C})$. Then set

$$\begin{aligned} A^o(X) &= \{f \in A(X) : |f|_{\sup} \le 1\}, \\ A^+(X) &= \mathrm{cl}\,\{f \in A(X) : |f|_{\sup} < 1\}, \text{ and} \\ \overline{A(X)} &= A^o(X)/A^+(X), \end{aligned}$$

where cl is the closure in $A^o(X)$. We define the reduction $\overline{X}$ of X to be the affine scheme Spec $\overline{A(X)}$. Suppose now that $X = \mathrm{Sp}\,(A)$ is an affinoid. Then $|f|_{\sup}$ is just the usual spectral seminorm of f, which we also denote by $\|f\|_X$ when X is reduced and $|\cdot|_{\sup}$ is a norm. There is a canonical reduction map $\mathrm{Red}: X(\mathbf{C}) \to \overline{X}(\overline{\mathbb{F}})$, which we denote by $x \mapsto \bar{x}$. If X is reduced and $\widetilde{Y}$ is any subscheme of $\overline{X}$, then $Y := \mathrm{Red}^{-1}\,\widetilde{Y}$ is the rigid space admissibly covered by affinoid subdomains Z of X such that $\overline{Z}$ maps into $\widetilde{Y}$. As a special case, when $\widetilde{Y} \subseteq \overline{X}$ is an open affine, Y is the unique subaffinoid of X such that $Y(\mathbf{C}) = \{x \in X(\mathbf{C}) : \bar{x} \in \tilde{Y}(\overline{\mathbb{F}})\}$, and we call Y a Zariski subaffinoid of X. When $\overline{X}$ is a reduced affine curve, we let $\overline{X}^c$ denote the unique complete curve that contains $\overline{X}$ as an affine open and is nonsingular at all other points (which we call the *points at infinity*).

If X is a rigid space over K, and $L \supseteq K$ is a complete subfield of $\mathbf{C}$, we write $P \in X(L)$ to mean that P is an L-valued point of X. An unspecified $P \in X$ should be read as $P \in X(\mathbf{C})$. We use the notations X_L and $X_{\mathbb{F}_L}$ for the extensions of X and $\overline{X}$ by scalars.

2A. ***Annuli.***

Proposition 2.2. *Let $f : \mathcal{A}_1 \to \mathcal{A}_2$ be a degree d unramified surjection of annuli over K (open or closed). Then the width of $\mathcal{A}_1$ is $1/d$ times the width of $\mathcal{A}_2$.*

Proof. Extend scalars to $\mathbf{C}$, and choose isomorphisms $\psi_i : \mathcal{A}_i \to A_{\mathbf{C}}(r_i, 1)$ for some $r_i \in \mathcal{R}$ with $r_i < 1$. Let T be the natural parameter on $A_{\mathbf{C}}(r_1, 1)$. Then viewing $\tilde{f} := \psi_2 \circ f \circ \psi_1^{-1}$ as an invertible function on $A_{\mathbf{C}}(r_1, 1)$, we may write $\tilde{f}$ as either $cT^d(1+g(T))$ or $cT^{-d}(1+g(T))$, where $g(T) \in A^+(A_{\mathbf{C}}(r_1, 1))$. In the first case, for any $t \in A_{\mathbf{C}}(r_1, 1)$, we clearly have $|\tilde{f}(t)| = |c| \cdot |t|^d$. So by surjectivity of f, this implies that $|c| = 1$ and $r_1^d = r_2$. Thus, $\log_p(1/r_1) = (1/d)\log_p(1/r_2)$. The second case is very similar. □

Definition 2.3. For any $r \in \mathbb{R}_+^* \setminus \mathcal{R}$, we let

$$K_r = \left\{ \textstyle\sum_{n \in \mathbb{Z}} a_n T^n : a_n \in K,\ \lim_{|n| \to \infty} |a_n| r^n = 0 \right\}.$$

Then K_r is a field, and $f \mapsto \max\{|a_n|r^n\}$ is a valuation[2] if $f(T) = \sum_{n\in\mathbb{Z}} a_n T^n$. If $r_1, \dots, r_n \in \mathbb{R}_+^*$ have linearly independent images in the $\mathbb{Q}$-vector space $\mathbb{R}_+^*/\mathcal{R}$, we let $K_{r_1,\dots,r_n} := (K_{r_1,\dots,r_{n-1}})_{r_n}$ and $K_\varnothing = K$.

Then $K_{r_1,\dots,r_n} \cong K_{r_1} \hat{\otimes}_K \dots \hat{\otimes}_K K_{r_n}$, and its value group is generated by $\mathcal{R}$ and $\{r_1, \dots, r_n\}$ [Berkovich 1990, pp. 21–22]. If m is a positive integer, the map $f(T) \mapsto f(T^m)$ gives an injection from K_{r^m} into K_r for any r.

Lemma 2.4. *The group* $\operatorname{Aut}_{\text{cont}}(K_{r_1,\dots,r_n}/K)$ *contains a subgroup* H, *for which* $h \mapsto h^d$ *is a bijection whenever* $p \nmid d$, *and whose fixed field is* K.

Proof. It suffices to do the case $n = 1$. Let $r = r_1$. Suppose $\alpha \in K$ such that $|\alpha| < r$. If $f \in K_r$ and $f(T) = \sum_{n\in\mathbb{Z}} a_n T^n$, we set

$$f^{\sigma_\alpha}(T) = \sum_{n>0} a_{-n}\left(\frac{T^{-1}}{1-\alpha T^{-1}}\right)^n + \sum_{n\geq 0} a_n (T-\alpha)^n.$$

Then $\sigma_\alpha \in \operatorname{Aut}_{\text{cont}}(K_r/K)$, and if $a_n = 0$ for large $|n|$, then $f^{\sigma_\alpha}(T)$ is the image of the rational function $f(T-\alpha)$ in K_r. It follows, by continuity, that the map $\alpha \mapsto \sigma_\alpha$ is an injective homomorphism from the subgroup $B_r := \{\alpha \in K : |\alpha| < r\}$ of K^+ into $\operatorname{Aut}_{\text{cont}}(K_r/K)$. Since $p \nmid d$, $\alpha \mapsto d\alpha$ is a bijection on B_r.

Now, if $f^{\sigma_\alpha}(T) = \sum_{n\in\mathbb{Z}} b_n T^n$, then

$$b_n = \sum_{m\geq n}\left((-1)^{m-n}\binom{m}{n} + \binom{-n-1}{-m-1}\right) a_m \alpha^{m-n},$$

where we set $\binom{a}{b} = 0$ if $a < 0$ or $b < 0$. Suppose $f^{\sigma_\alpha} = f$ for all $\alpha \in B_r$. Then in the formula above, we must have $b_n = a_n$ for all $\alpha \in B_r$. This can only happen if $a_n = 0$ for all $n \neq 0$. Therefore $f \in K$, and we may take H to be the image of B_r in $\operatorname{Aut}_{\text{cont}}(K_r/K)$. $\square$

Lemma 2.5. *Let* X *be a reduced affinoid over* K, *and let* $f : X \to A_K[a,b]$ *be finite*, *flat*, *and of degree* d, *where* $p \nmid d$ *and* $a, b \in \mathcal{R}$ *with* $a < 1 < b$. *Let* T *be the natural parameter on* $A_K[a,b]$. *Suppose there exists a function* G *on* $X[1] := f^{-1}C_K[1]$ *such that* $\|G^d - f^*T\|_{X[1]} < 1$. *Then there exist* $a_1, b_1 \in \mathcal{R}[a,b]$ *with* $a_1 < 1 < b_1$, *and a function* S *on* $f^{-1}A_K[a_1,b_1]$, *such that* $S^d = f^*T$.

Proof. Setting $s = \overline{G}$ and $t = \overline{T}$, we have $\mathbb{O}(\overline{X[1]}) = \mathbb{F}_K[s, s^{-1}]$ and $\mathbb{O}(\overline{C[1]}) = \mathbb{F}_K[t, t^{-1}]$, and $\bar{f} : \overline{X[1]} \to \overline{C[1]}$ is given by $t = s^d$. Let $V = A_K[a,1]$ and $U = f^{-1}(V)$. Then $\overline{C_K[1]}$ is an affine open in $\overline{V}$. Therefore, identifying $\mathbb{O}(\overline{U})$ with its image in $\mathbb{O}(\overline{X[1]})$, we have

$$\mathbb{O}(\overline{X[1]}) = \mathbb{O}(\overline{U}) \otimes_{\mathbb{O}(\overline{V})} \mathbb{O}(\overline{C_K[1]}).$$

[2]Some authors call this an absolute value.

Thus s is in the image of $\mathcal{O}(\overline{U})$. So we may lift s to a function $S_0 \in A^o(U)$ such that

$$\|S_0^d - f^*T\|_{X[1]} < 1.$$

Now, choose $a_1 \in \mathfrak{R}[a, 1)$ such that $|S_0^d - f^*T| < |f^*T|$ on $U_1 := f^{-1}A_K[a_1, 1]$. Let $p(x) = x^d - (f^*T/S_0^d)$, considered as a polynomial over $A^o(U_1)$. Then $x_0 := 1$ satisfies $|p(x_0)| < 1$ and $|p'(x_0)| = 1$ over all of U_1. Therefore, by the usual Hensel's lemma argument, there exists a unique $x \in A^o(U_1)$ with $p(x) = 0$ and $\|x - 1\|_{U_1} < 1$. Letting $S_1 = S_0x$, we have an $S_1 \in A(U_1)$ whose restriction to $X[1]$ is a lift of s, and for which $S_1^d = f^*T$.

By precisely the same argument, there is a function $S_2 \in A(U_2)$ that reduces to s on $X[1]$ and satisfies $S_2^d = f^*T$, where $U_2 = f^{-1}A_K[1, b_1]$ for some $b_1 \in \mathfrak{R}(1, b]$. Also, since X is reduced, $(S_1/S_2)^d = 1$ on $X[1]$ (with $p \nmid d$), and $\|S_i - G\|_{X[1]} < 1$, we must have $S_1 = S_2$ on $X[1]$. Therefore, S_1 and S_2 patch to a function S on $f^{-1}A_K[a_1, b_1]$ with $S^d = f^*T$. $\square$

Theorem 2.6. *Suppose $a < b \in \mathfrak{R}$. Any finite connected étale cover over K of the annulus $A_K[a, b]$ (respectively $A_K(a, b)$) of degree d, where $d < p$ if $p \neq 0$, is an annulus isomorphic over K to $A_K[a^{1/d}c, b^{1/d}c]$ (respectively $A_K(a^{1/d}c, b^{1/d}c)$) for some $c \in |K^*|^{1/d}$.*

Proof. We will first prove the statement for closed annuli.

Let W be a connected rigid space over K, and let $f : W \to A_K[a, b]$ be finite and étale of degree $d < p$ (if $p \neq 0$). Initially, we also assume that $a, b \in |K^*|$. For each $r \in |K^*| \cap [a, b]$, let W_r be the inverse image in W of the circle $C_K[r]$. Then the connected components of W_r, which we denote by $\{V_{r1}, \ldots, V_{rm_r}\}$, are affinoids over K, with each V_{ri} finite and étale of degree d_{ri} over $C_K[r]$, such that $\sum d_{ri} = d$. As $d < p$ or $p = 0$, each $\overline{V}_{ri}$ must be finite and étale of degree d_{ri} over $\overline{C_K[r]} \cong \mathbf{G}_m$. Thus, there must exist an isomorphism $\sigma_{ri} : V_{ri} \to C_K[r^{1/d_{ri}}]$ such that $f \circ \sigma_{ri}^{-1}$ reduces to $x \mapsto x^{d_{ri}}$ on $\mathbf{G}_m$ (with respect to the standard parameters). Moreover, this implies by Lemma 2.5 that for each $r \in |K^*| \cap (a, b)$ there exist $\alpha_r, \beta_r \in \mathfrak{R}[a, b]$ with $\alpha_r < r < \beta_r$, and an embedding

$$F_r : \coprod_{i=1}^{m_r} A_K(\alpha_r^{1/d_{ri}}, \beta_r^{1/d_{ri}}) \hookrightarrow W$$

such that $\mathrm{Im}(F_r) = f^{-1}A_K(\alpha_r, \beta_r)$. In fact, F_r^{-1} can be defined on the i-th component of $f^{-1}A_K(\alpha_r, \beta_r)$ by a parameter S_{ri} such that $S_{ri}^{d_{ri}} = f^*T$ (where T is the natural parameter on $A_K(\alpha_r, \beta_r)$). Similarly, we have embeddings F_a and F_b, each of a disjoint union of semiopen annuli into W, with images $f^{-1}A_K[a, \beta_a)$ and $f^{-1}A_K(\alpha_b, b]$.

Suppose further that $[a, b] = [a, \beta_a) \cup (\alpha_b, b] \cup \bigcup_{r \in |K^*| \cap (a,b)} (\alpha_r, \beta_r)$. Then by compactness of $[a, b]$, we may choose a finite set $\{r_1, \ldots, r_n\} \subset |K^*| \cap (a, b)$ such that $[a,b]$ is covered by $[a,\beta_a)$, $(\alpha_b,b]$ and the intervals $(\alpha_{r_i},\beta_{r_i})$ for $1\le i\le n$. Whenever two of these intervals overlap, it is clear from the properties of F_r that the inverse images in W of the corresponding subannuli of $A_K[a,b]$ must have the same number of connected components. Therefore, as W is connected, it follows that $m_r=1$ for all $r\in|K^*|\cap[a,b]$. Thus, F_r is an isomorphism of $A_K(\alpha_r^{1/d},\beta_r^{1/d})$ onto $f^{-1}A_K(\alpha_r,\beta_r)$, given by a parameter S_r with $S_r^d=f^*T$ (for $r\in|K^*|\cap(a,b)$, and similarly for $r=a$ or b). We claim that F_a, F_b, and the F_{r_i} can be used to construct an isomorphism of $A_K[a^{1/d},b^{1/d}]$ onto W. Indeed, whenever $(\alpha_{r_i},\beta_{r_i})\cap(\alpha_{r_j},\beta_{r_j})=(\alpha_{r_j},\beta_{r_i})$, we have a parameter S_{r_i} on $f^{-1}A_K(\alpha_{r_i},\beta_{r_i})$ such that $S_{r_i}^d=f^*T$, and likewise for r_j. After adjusting by a d-th root of unity in K if necessary, S_{r_i} and S_{r_j} agree on $f^{-1}A_K(\alpha_{r_j},\beta_{r_i})$. Therefore the two parameters patch to a parameter S_{ij} that identifies $f^{-1}A_K(\alpha_{r_i},\beta_{r_j})$ with $A_K(\alpha_{r_i}^{1/d},\beta_{r_j}^{1/d})$. After finitely many such patching steps, we have constructed a parameter S on W over K such that S^d equals f^*T and thus defines an isomorphism from W onto $A_K[a^{1/d},b^{1/d}]$.

More generally, without making the above two suppositions, for each $r\in[a,b]$ take M_r to be a finite Galois extension of K such that $r\in|M_r^*|$ if $r\in\mathfrak{R}$ and K_r (defined as above) otherwise. Then we may choose $\alpha_r,\beta_r\in\mathfrak{R}_{M_r}[a,b]$ and an embedding F_r that is defined over M_r, precisely as was done over K. Now, we know that $[a,b]$ is covered by $[a,\beta_a)$, $(\alpha_b,b]$, and $\{(\alpha_r,\beta_r):r\in(a,b)\}$. So by compactness, there exists a finite set $t_1,\ldots,t_m\in(a,b)$ such that $[a,b]$ is covered by $[a,\beta_a)$, $(\alpha_b,b]$, and $\{(\alpha_{t_i},\beta_{t_i}):1\le i\le m\}$. Choose a finite Galois extension L of K so that the images of the t_i in $\mathbb{R}_+^*/|L^*|$ generate a torsion-free abelian group. Then choose $r_1,\ldots,r_n\in\mathbb{R}_+^*$ so that their images form a basis for this group. Then the argument above can be applied to produce a parameter S on W, which is defined over $L_{r_1,\ldots,r_n}$ such that $S^d=f^*T$.

Now, if $\sigma\in\mathrm{Aut}_{\mathrm{cont}}(L_{r_1,\ldots,r_n}/L)$, the map $\sigma\mapsto\zeta(\sigma):=S^\sigma/S$ is a 1-cocycle with values in $\mu_d(A(W_{L_{r_1,\ldots,r_n}}))$. Since W is connected, this equals $\mu_d(L_{r_1,\ldots,r_n})$, which is $\mu_d(L)$. It follows from Lemma 2.4 that $\zeta(\sigma)=1$ for all σ in a subgroup whose fixed field is L. Thus S is defined over L. Then, for $\sigma\in\mathrm{Gal}(L/K)$, $S^\sigma=h(\sigma)S$, where h is a 1-cocycle. So by Hilbert's Theorem 90 there exists $\gamma\in L^*$ such that $h(\sigma)=\gamma^\sigma/\gamma$. Then $H:=S/\gamma$ is defined over K and $H^d=\alpha T$ for some $\alpha\in K^*$. Therefore H defines an isomorphism of W onto $A_K[a^{1/d}c,b^{1/d}c]$, where $c=|\alpha|^{1/d}$.

To deal with open annuli $A_K(a,b)$, choose sequences $\{a_n\}$ and $\{b_n\}$ in $\mathfrak{R}_K(a,b)$ such that $a_n<b_n$, $a_n\to a$ and $b_n\to b$. For large n, $W_{[a_n,b_n]}:=f^{-1}A_K[a_n,b_n]$ is connected, and it is finite étale over $A_K[a_n,b_n]$ of degree d. Therefore, it is isomorphic to $A_K[a_n^{1/d}c_n,b_n^{1/d}c_n]$ by what we have proven. The theorem follows when we let n go to infinity. $\square$

Remark 2.7. (i) When K is algebraically closed, there exists $a = c_0 < \cdots < c_{n+1} = b$ in $\mathcal{R}$ such that $f^{-1}A(c_i, c_{i+1})$ is a disjoint union of open annuli [Lütkebohmert 1993, Lemma 2.3]. One could then use Hilbert's Theorem 90 and Lemma 2.5, as in the proof above, to give another proof of the theorem.

(ii) One can obtain the same conclusion about W, for any finite étale surjection f whose Galois closure has degree prime to p when $p \neq 0$.

If X is a reduced affinoid over K and $P \in \overline{X}(\mathbb{F}_K)$, we let $R_X(P)$ denote the *residue class* of P. When the context makes it clear, we will drop the subscript X. This is the open rigid subspace of X whose $\mathbf{C}$-valued points reduce to P, or equivalently, the subspace $\mathrm{Red}^{-1} P$, where P is naturally identified with a subscheme of $\overline{X}$. Alternatively, suppose $f_1, \ldots, f_m \in A^o(X)$ are such that $\bar{f}_1, \ldots, \bar{f}_m$ generate the maximal ideal of P. Then $R(P)$ is admissibly covered by the increasing sequence of affinoids whose $\mathbf{C}$ valued points are

$$\{x \in X(\mathbf{C}) : |f_i(x)| \leq r_n,\ 1 \leq i \leq m\},$$

where $r_n \in \mathcal{R}$, $r_n < r_{n+1}$ and $\lim_{n\to\infty} r_n = 1$. If x is a point of X such that $\bar{x} = P$ (which always exists by [Tate 1971, Theorem 6.4]), this is naturally isomorphic to the formal fiber $X_+(x)$ of Bosch (by [1977a, Satz 6.1]).

Proposition 2.8. *Let K be a stable field. Suppose X is a reduced pure d-dimensional affinoid over K, $\|A(X)\| = |K|$ (equivalently, $A^o(X) \otimes_{R_K} \mathbb{F}_K$ is reduced), and $P \in \overline{X}(\mathbb{F}_K)$. Then $\overline{A(R(P))} \cong \hat{\mathcal{O}}_{\overline{X},P}$.*

Proof. Let $I(P)$ be the closure of $m_K A^o(R(P))$ in $A^o(R(P))$. Bosch [1977a, p. 44] proved that $A^o(R(P))/I(P) \cong \hat{\mathcal{O}}_{\overline{X},P}$ when there exists a surjective map $\phi : T_n \to A(X)$ such that $\mathring{\phi}$ is surjective.[3] That such a map exists when K is stable and $\|A(X)\|_X = |K|$ follows from [Bosch et al. 1984, Corollary 6.4.3/6]. It is clear that $I(P) \subseteq A^+(R(P)) \subseteq \mathrm{rad}(I(P))$. Since $\overline{X}$ is reduced, so is $\hat{\mathcal{O}}_{\overline{X},P}$, and hence $I(P) = A^+(R(P))$. The proposition follows. □

Definition 2.9. Let P be a point on a curve C over a field k. We say that P is an ordinary double point over k if $\hat{\mathcal{O}}_{C,P} \cong k[\![u, v]\!]/(uv)$.

Hypothesis B. R_K contains a bald subring [Bosch et al. 1984, Definition 1.7.2/1] with the same residue field.

K satisfies Hypothesis B if it is discretely valued, if its residue field is perfect, or if its residue field lifts to a subfield. In particular, this is the case if K satisfies Hypothesis T. We do not know if all complete, nonarchimedean-valued fields K satisfy Hypothesis B.

[3] As the example at the end of [Bosch et al. 1984, §6.4] implies, ϕ need not be distinguished; see [Bosch et al. 1984, Definition 6.4.3/2].

Proposition 2.10. *Let X be a reduced, irreducible affinoid over a stable field K satisfying Hypothesis B. Suppose that $\overline{X}$ is a reduced curve and $P \in \overline{X}(\mathbb{F}_K)$. Then P is an ordinary double point over $\mathbb{F}_K$ if and only if the residue class $R(P)$ is isomorphic to $A_K(r, 1)$ for some $r \in |K^*|$.*

This was proven in [BL 1985, Proposition 2.3] when K is algebraically closed, and we adapt their proof to our case here.

Lemma 2.11. *Let I be a bald subring of R_K, and $\{r_1, r_2, \dots\}$ a zero sequence in R_K. Then there exists a bald subring of R_K containing I and r_n for all $n \geq 1$.*

Proof. The proof is almost identical to that of [Bosch et al. 1984, Corollary 1.7.2/5]; just replace the I in the proof of 1.7.2/4 with this I. □

Lemma 2.12. *Let X be a reduced, one-dimensional affinoid, with reduced reduction, over a stable field K satisfying Hypothesis B. Suppose that $f, g \in C := A(\overline{X})$ generate a maximal ideal $\mathcal{M} = (f, g)$, such that $C/\mathcal{M} \cong \mathbb{F}_K$ and $fg \in \mathcal{M}^3$. Let*

$$U = \begin{cases} X & \text{if } fg = 0, \\ \{x \in X : f(\bar{x}) = g(\bar{x}) = 0\} & \text{otherwise.} \end{cases}$$

Then there exist $F, G \in A^o(U)$ and $c \in R_K$ such that

$$\overline{F} - f \in \mathcal{M}^2\overline{A(U)}, \quad \overline{G} - g \in \mathcal{M}^2\overline{A(U)}, \quad \text{and} \quad FG = c,$$

where we use Proposition 2.8 to identify $\overline{A(U)}$ with $\hat{\mathcal{O}}_{\overline{X},\overline{U}}$.

Proof. Suppose that $f_1, g_1 \in A^o(X)$ are such that $f = \bar{f}_1$ and $g = \bar{g}_1$, and that $\alpha : X \to B_K[1]$ is a finite morphism. That $\overline{X}$ is reduced implies $\|A(X)\|_X = |K|$. So by [Bosch et al. 1984, Corollary 6.4.1/4], $\alpha^* : A^o(B_K[1]) \to A^o(X)$ is finite. Now suppose $\alpha^*(A^o(B_K[1])) = R_K\langle T\rangle$. As C is torsion-free (because $\overline{X}$ is flat over $\mathbf{A}^1$) and finitely generated over $\mathbb{F}_K[T]$, it is free. Choose $h_1, \dots, h_n \in A^o(X)$ so that $\bar{h}_1, \dots, \bar{h}_n$ is a basis for C over $\mathbb{F}_K[T]$. Then $h_1, \dots, h_n$ is a basis for $A^o(X)$ over $R_K\langle T\rangle$. Thus $B := \{h_i T^j : 1 \leq i \leq n,\ j \in \mathbb{N}_0\}$ is an orthonormal Schauder basis [ibid., Definition 2.7.2/1] for $A(X)$. As $C = \mathcal{M} \oplus \mathbb{F}_K$, the ring C has a basis over $\mathbb{F}_K$ of the form $\{1,\ \bar{\alpha}_i f,\ \bar{\beta}_j g : i, j \in \mathbb{N}\}$ for some α_i, β_j in a subring of $A^o(X)$ finitely generated over a bald subring of R_K with residue field $\mathbb{F}_K$. It follows from Lemma 2.11 that the change of basis matrix from B to $\{1,\ \alpha_i f_1,\ \beta_j g_1 : i, j \in \mathbb{N}\}$ has entries in a bald subring of R_K [ibid., Definition 1.7.2/1]. Hence by the lifting theorem of [ibid., Theorem 2.7.3/2], this is also an orthonormal Schauder basis. Hence $A^o(X) = R_K + M$, where $M = (f_1, f_2)$.

We have

$$f_1 g_1 = \pi c_1 + f_1(\pi a_1 + b_1) + g_1(\pi a_2 + b_2),$$

with $c_1 \in R_K$, $a_i \in A^o(X)$, $b_i \in M^2$ (and $b_i = 0$ if $fg = 0$) for some $\pi \in R_K$, $|\pi| < 1$. Let $I = \pi R_K + f_1 A^o(X) + g_1 A^o(X) = \pi A^o(X) + M$, and let $J = \pi A^o(X)$

if $fg = 0$ and I otherwise. Let $f_2 = f_1 - (\pi a_2 + b_2)$, and $g_2 = g_1 - (\pi a_1 + b_1)$. Then

$$\begin{aligned} f_2 g_2 &= \pi c_1 + (\pi a_1 + b_1)(\pi a_2 + b_2) \\ &\equiv \pi c_1 + \pi^2 c_2' \mod (\pi A^o(X) + M)^3 \\ &\equiv \pi c_1 + \pi^2 c_2' \mod \pi^2 M \quad \text{if } fg = 0, \end{aligned}$$

for some $c_2' \in R_K$. Now $I^n = \pi^n R_K + f_1 I^{n-1} + g_1 I^{n-1}$, so this implies

$$f_2 g_2 = \pi c_1 + \pi^2 c_2 + f_1 r_{2,1} + g_1 r_{2,2},$$

where $c_2 \in R_K$, $r_{2,i} \in J^2$. Let $k_n = 2^{n-2} + 1$ for $n \geq 2$ and $k_1 = 1$. Suppose

$$f_n g_n = \pi c_1 + \pi^2 c_2 + \pi^4 c_3 + \cdots + \pi^{2k_{n-1}} c_n + f_1 r_{n,1} + g_1 r_{n,2}$$

for some $r_{n,i} \in J^{k_n}$. Set $f_{n+1} = f_n - r_{n,2}$ and $g_{n+1} = g_n - r_{n,1}$. Then

$$\begin{aligned} f_{n+1} g_{n+1} &= \pi c_1 + \pi^2 c_2 + \pi^4 c_3 + \cdots + \pi^{2k_{n-1}} c_n + r_{n,1} r_{n,2} \\ &= \pi c_1 + \pi^2 c_2 + \pi^4 c_3 + \cdots + \pi^{2k_{n-1}} c_n + \pi^{2k_n} c_{n+1} + f_1 r_{n+1,1} + g_1 r_{n+1,2}, \end{aligned}$$

where $r_{n+1,i} \in J^{k_{n+1}}$.

Finally, let $r_{1,1} = \pi a_1 + b_1$ and $r_{1,2} = \pi a_2 + b_2$. Set $F = f_1 - \sum_{n \geq 1} r_{n,2}$ and $G = g_1 - \sum_{n \geq 1} r_{n,1}$. Then these are elements of $A^o(U)$ that satisfy

$$FG = c := \pi c_1 + \pi^2 c_2 + \textstyle\sum_{n \geq 3} \pi^{2^{n-2}+2} c_n. \qquad \square$$

Proof of Proposition 2.10. Suppose $P \in \overline{X}(\mathbb{F}_K)$ is an ordinary double point. We can apply Lemma 2.12 to conclude that there exist $F, G \in A^o(R(P))$ and $c \in R_K$ such that $(\overline{F}, \overline{G}) = \mathcal{M}_P$ and $FG = c$. Thus we have a morphism $R(P) \to A_K(|c|, 1)$. That this is an isomorphism follows, as in the proof of [BL 1985, Proposition 2.3].

Conversely, suppose that $R(P)$ is isomorphic to the annulus $A_K(r, 1)$ for some $r \in |K^*|$ with $r < 1$. Then $A^o(R(P)) \cong R_K[\![T, cT^{-1}]\!]$, where $c \in K$ with $|c| = r$. So applying Proposition 2.8 we have

$$\hat{\mathcal{O}}_{\overline{X},P} \cong \overline{A(R(P))} \cong \mathbb{F}_K[\![x, y]\!]/(xy),$$

and hence P is an ordinary double point of $\overline{X}$. $\square$

For a rigid space W over K, set

$$D^i(W/K) = \mathrm{Ker}(\mathrm{d} : \Omega^i_{W/K}(W) \to \Omega^{i+1}_{W/K}(W))/\mathrm{d}\Omega^{i-1}_{W/K}(W),$$

where if $A(W)$ is the ring of rigid functions on W, then $\Omega^i_{W/K}(W)$ is the $A(W)$ module of rigid i-forms on W.

Lemma 2.13. *Suppose $W = A_K(r, s)$ or $B_K(r) \setminus \{0\}$, where $r, s \in |L^*|$ for some finite extension L/K. Then*

$$D^0(W/K) \cong D^1(W/K) \cong K.$$

Proof. If $r, s \in |K^*|$, the lemma is clear. For in this case, we may choose $a, b \in K$ with $|a| = r$ and $|b| = s$, and let $x = T/b$ and $y = a/T$, where T is the natural parameter on $\mathbf{A}^1_K$. Then $A_K(W)$ is equal to the set of functions represented by

$$\sum_{n=0}^{\infty} a_n x^n + \sum_{n=1}^{\infty} b_n y^n,$$

where $a_n, b_n \in K$, $a_n t^n \to 0$ and $b_n t^n \to 0$ as $n \to \infty$, for $|t| < 1$. There is a natural continuous linear map ρ_K from $\Omega^1_{W/K}(W)$ onto K such that $\rho_K(\mathrm{d}v/v) = 1$ for any parameter v on W_K such that $|v(u)| > |v(w)|$ if $|u| > |w|$ and $u, w \in A_K(r, s)(\mathbf{C})$. Moreover, for any $\omega \in \Omega^1_{W/K}(W)$, $\omega \in \mathrm{d}A_K(W)$ if and only if $\rho_K(\omega) = 0$.

More generally, suppose L is a finite Galois extension of K with Galois group G, and that $r, s \in |L^*|$. Then G acts on $\Omega^1(A_L(r, s))$ such that $\Omega^1(A_L(r, s))^G = \Omega^1(A_K(r, s))$ and $\rho_L(\omega^\sigma) = \rho_L(\omega)^\sigma$. Also, if $r, s \in |K^*|$, then $\rho_L|_{\Omega^1(A_K)} = \rho_K$. Suppose $\omega \in \Omega^1(A_K(r, s))$ and $\rho_L(\omega) = 0$. Then Hilbert's additive Theorem 90 gives $\omega \in \mathrm{d}A(A_K(r, s))$. Thus we have an injective K-linear map $D^1(W/K) \to L^G = K$. If $\omega \in \Omega^1(A_L(r, s))$, $\rho_L(\omega) = 1$ and $\nu = \sum_{\sigma \in G} \omega^\sigma$, then $\nu \in \Omega^1(A_K(r, s))$ and $\rho_L \nu = [L : K]$. So this map is an isomorphism. ☐

From the proof we see that for any open annulus W over K, there are two residue maps from Ω^1_W onto K. In particular, they are $\mathrm{res}_{r,s} \circ f^*$ and $-\mathrm{res}_{r,s} \circ f^*$, where $f : A_L(r, s) \to W_L$ is an isomorphism and $\mathrm{res}_{r,s} = \rho_L|_{\Omega^1(A_K)}$ for any extension L of K such that $r, s \in |L^*|$. By an oriented annulus over K, we mean a pair (W, ρ), where W is an open annulus and ρ is a choice of one of the residue maps.

An *end* of a rigid space W over K is an element of the inverse limit of the set of connected components of $W \setminus Z$, where Z ranges over finite unions of subaffinoids of W defined over K (ordered by containment). We let $\mathscr{E}(W)$ denote the set of ends of W, and we let $e(W) = |\mathscr{E}(W)|$ (which may be infinite). For example, $e(W) = 2$ whenever W is an open annulus. If W is admissibly covered by a countable number of affinoids, and f is a real-valued function of $W(\mathbf{C})$, it makes sense to compute the limit of f at an end $e \in \mathscr{E}(W)$. In particular, we define $\lim_{x \to e} f(x) = \lim_{n \to \infty} f(x_n)$, where $\{x_n\}$ is any sequence in $W(\mathbf{C})$ such that for any Z as above, x_n is contained in a connected component of $W \setminus Z$ that maps to e for sufficiently large n (provided this limit exists and is independent of the sequence).

The following result is used in the proof of [CM 2006, Theorem 5.2].

Proposition 2.14. *Suppose K is discretely valued and U is a rigid space over K with two ends such that for some finite extension L of K, U_L is isomorphic to the open annulus $A_L(|u|, 1)$, where $u \in K^*$ and $|u| < 1$. Then $U \cong A_K(|u|, 1)$.*

Proof. We may suppose that $e(L/K) > 1$ and L is a Galois extension of K with Galois group G. Let $M = A(U_L)$. Then G naturally acts on M, and $M^G = A(U)$. Let $R = R_L$, and let $\mathbb{F}_L$ and $\mathbb{F}_K$ denote the residue fields of L and K. Let π be a uniformizing parameter on L.

Let a and b denote the ends of U, and suppose $F \in M$ is an isomorphism from U_L onto $A_L(|u|, 1)$ such that $\lim_{x\to a}|F(x)| = 1$. Then we may use F to identify M with $L\{\{T, u/T\}\}$, and the group M_a of orientation-preserving automorphisms of $A_L(|u|, 1)$ with (under composition)

$$\left\{T\Bigl(\sum_{i=0}^{\infty} a_i T^i + \sum_{i=1}^{\infty} b_i (u/T)^i\Bigr) : a_i, b_i \in R \text{ and } a_0 \in R^*\right\}.$$

The group G preserves M_a. For $\sigma \in G$, set $\sigma(T) = G_\sigma(T)$. For

$$h(T) = \sum_{i=0}^{\infty} a_i T^i + \sum_{i=1}^{\infty} b_i (u/T)^i \in L\{\{T, u/T\}\},$$

set

$$h^\sigma(T) = \sum_{i=0}^{\infty} a_i^\sigma T^i + \sum_{i=1}^{\infty} b_i^\sigma (u/T)^i.$$

Then

$$G_\tau^\sigma \circ G_\sigma = G_{\sigma\tau}. \tag{1}$$

We will show that there exists $F \in M_a$ such that

$$F^\sigma \circ F^{-1} = G_\sigma. \tag{2}$$

This will imply that $F^{-1}(T) \in A(U)$, and as $F^{-1}(T)$ is a parameter on U, it will then follow that U is an annulus over K.

So first let I be the ideal in $C := R[[T, u/T]]$ generated by π, T and u/T, and suppose that

$$G_\sigma(T) \equiv a(\sigma)T \mod TI, \quad \text{where } a(\sigma) \in R^*.$$

Then, from (1), we have $a(\sigma)^\tau a(\tau) \equiv a(\sigma\tau) \bmod \pi$. Using Hilbert's Theorem 90 applied to $\mathbb{F}_L/\mathbb{F}_K$, one can show there exists a $c \in R^*$ such that $c^\sigma/c \equiv a(\sigma) \bmod \pi$. Let $h(T) = cT$. Then we have

$$(h^{-\sigma} \circ G_\sigma \circ h)(T) \equiv T \mod TI.$$

Now, suppose $G_\sigma(T) = T(1 + h_\sigma(T))$, where $h_\sigma(T) \in I^k$.

Lemma 2.15. *Suppose* $h(T) := \sum_{i=1}^{\infty} B_{-i}(u/T)^i + \sum_{i=0}^{\infty} B_i T^i$ *is in* C. *Then* $h(T) \in I^k$ *if and only if* $B_i \equiv 0 \bmod \pi^{k-|i|} R$.

Proof. Let S_k be the R-module of series whose coefficients satisfy the bounds above. The lemma is clearly true for $k = 0$. Suppose it is true for k. Let $\mathcal{T}$ be the continuous involution of the R-algebra C that takes T to u/T. Then I and S_k are preserved by $\mathcal{T}$. As $\pi^{k-i}T^i \in I^k$ for $0 \le i \le k$, and $T^{k+1}R[[T]] \subseteq I^{k+1}$, it follows that $S_{k+1} \subseteq I^{k+1}$. We have

$$\pi(\pi^{k-i}T^i) = \pi^{k+1-i}T^i \quad \text{and} \quad T(\pi^{k-i}T^i) = \pi^{k+1-(i+1)}T^{i+1},$$

and because $v(u) \ge 2v(\pi)$,

$$T(\pi^{k-i}(u/T)^i) = u\pi^{k-i}(u/T)^{i-1} \in \begin{cases} \pi^{k+1-(i-1)}(u/T)^{i-1}R & \text{if } i > 0, \\ \pi^{k+1-1}TR & \text{if } i = 0. \end{cases}$$

Thus $I^{k+1} \subseteq S_{k+1}$. □

Now suppose

$$h_\sigma(T) = \sum_{i=1}^{\infty} B_{-i}(\sigma)(u/T)^i + \sum_{i=0}^{\infty} B_i(\sigma)T^i.$$

Then, since

$$T(1 + h_\tau(T))\big(1 + h_\sigma^\tau(T(1 + h_\tau(T)))\big) \equiv T(1 + h_\sigma^\tau(T) + h_\tau(T)) \mod TI^{2k},$$

it follows that

$$G_\sigma^\tau \circ G_\tau(T) \equiv \\ T\left(1 + \sum_{i=1}^{2k}(B_{-i}(\tau) + B_{-i}(\sigma)^\tau)(u/T)^i + \sum_{i=0}^{2k}(B_i(\tau) + B_i(\sigma)^\tau)T^i\right) \mod TI^{2k}.$$

Therefore, by Lemma 2.15 we have

$$B_i(\sigma\tau) \equiv B_i(\tau) + B_i(\sigma)^\tau \mod \pi^{2k-|i|}.$$

Finally, using Hilbert's Theorem 90 again, we can find $C_i \in \pi^{k-|i|}R \cap R$ such that

$$C_i^\tau - C_i \equiv B_i(\tau) \mod \pi^{2k-|i|} \quad \text{for } -2k \le i \le 2k.$$

So let

$$H(T) = T\left(1 + \sum_{i=1}^{2k} C_{-i}(u/T)^i + \sum_{i=0}^{k^2-1} C_i T^i\right).$$

Then $H \in TI^k$ and $H^\sigma \circ H^{-1} \equiv G_\sigma \bmod TI^{2k}$. Thus we can find a convergent sequence $F_k \in M_a$ such that $F_k^\sigma \circ F_k^{-1} \to G_\sigma$ in M_a. The limit, $F \in M_a$, must satisfy (2). □

Remark 2.16. Suppose K is discretely valued and U is a rigid space with one end over K, such that U_L is isomorphic to the open disk $B_L(1)$ for some finite extension L of K. Then it follows from a similar argument that $U \cong B_K(1)$.

2B. *Wide open spaces.* In [Coleman 1989, §III] we defined wide open spaces over $\mathbb{C}_p$. Now we need to use them over more general fields. Suppose W is a one-dimensional smooth rigid space over K. Then W is a wide open space, or wide open, over K if it contains affinoid subdomains X and Y such that

(i) $W \setminus X$ is a disjoint union of finitely many open annuli,

(ii) X is relatively compact in Y, and

(iii) $Y \cap V$ is a semiopen annulus for all connected components V of $W \setminus X$.

We call X an underlying affinoid of W. From the definition, it is immediate that there is a natural bijection between $\mathscr{E}(W)$, the set of ends of W, and $\mathrm{CC}(W \setminus X)$, the set of connected components of $W \setminus X$. And X is connected to each element of $\mathrm{CC}(W \setminus X)$. So $e(W)$ is finite in this case. We call the connected component of $W \setminus X$ that corresponds to an element e of $\mathscr{E}(W)$ an *annulus at* e.

Remark 2.17. It is not immediate that the intrinsic definition of a wide open space given above is equivalent to the one given in [Coleman 1989, §III] when $K = \mathbb{C}_p$. However, this will follow in one direction from Theorem 2.18 and in the other from Theorem 2.40.

Theorem 2.18. *Let W be a wide open over K with underlying affinoid X. Then W may be completed to a proper algebraic curve C over K by gluing open disks onto the connected components of $W \setminus X$.*

Proof. More specifically, let $\mathscr{S}$ be the set of connected components of $W \setminus X$. For each open annulus $V \in \mathscr{S}$, let $\alpha_V : V \to B_V$ be an embedding of V into an open disk over K such that $B_V \setminus \alpha_V(V')$ is connected for any concentric annulus $V' \subseteq V$ that is connected to X. We will show that

$$C := \Big(W \cup \coprod_{V \in \mathscr{S}} B_V\Big) \Big/ \{\alpha_V(V) = V\}_{V \in \mathscr{S}}$$

is isomorphic to a complete algebraic curve.

It is clear that C is smooth of dimension one. Therefore, to establish the claim, by the Riemann existence theorem (Theorem A.2), we need only show that C is proper [Bosch et al. 1984, Definition 9.6.2/2]. The number of connected components of W is finite and equals the number of connected components of X, and so we may assume without loss of generality that W is connected. In this case X is contained in a residue class $R(P)$ of Y (where P is the image of $\overline{X}$ in $\overline{Y}$). Choose an $f \in A^o(Y)$ such that P is an isolated zero of $\bar{f}$. This can be done by first passing to a finite extension L of K so that $\overline{Y}_L$ is reduced and so that there is such a function

$g \in A^o(Y_L)$. Then let f be the norm of g. Now by [BL 1985, Lemma 2.4], if $\alpha \in \mathcal{R}$, and α is less than and sufficiently close to 1, then $\{x \in R(P) : |f(x)| \geq \alpha\}$ is the set of $\mathbf{C}$-valued points in a subdomain U_α of W which, after a finite extension (the field in that lemma is algebraically closed), becomes isomorphic to a finite union of semiopen annuli. In fact, for α sufficiently close to 1, U_α must decompose as $\coprod_{V \in \mathcal{S}} A_{\alpha,V}$, where each $A_{\alpha,V}$ is a concentric semiopen annulus in V. Thus $B_{1,V} := B_V \setminus \alpha_V(V \cap Y)$ and $B_{\alpha,V} := B_{1,V} \cup \alpha_V(A_{\alpha,V})$ are closed disks. Also, we may define X_α to be the rigid subdomain of W whose $\mathbf{C}$-valued points are $\{x \in R(P) : |f(x)| \leq \alpha\}$.

Then, $\mathcal{U} := \{X_\beta, B_{\beta,V} : V \in \mathcal{S}\}$ and $\mathcal{V} := \{Y, B_{\alpha,V} : V \in \mathcal{S}\}$, for any $\beta \in \mathcal{R}$ with $\alpha < \beta < 1$, are two finite admissible affinoid coverings of C such that each element of $\mathcal{U}$ is relatively compact in an element of $\mathcal{V}$. So C is proper if it is separated. To verify that C is separated, we must show that the diagonal map $\Delta : C \to C \times_K C$ is a closed immersion. This can be checked locally using the admissible affinoid cover of $C \times_K C$ given by $\{Z \times_K Z' : Z, Z' \in \mathcal{U}\}$. Indeed, for every $Z, Z' \in \mathcal{U}$, $\Delta^{-1}(Z \times_K Z') = Z \cap Z'$ is an affinoid and $\Delta^* : A(Z \times_K Z') \to A(Z \cap Z')$ is surjective. This is obvious when $Z = Z'$. Otherwise, when $Z \cap Z' \neq \varnothing$ we must have $\{Z, Z'\} = \{X_\beta, B_{\beta,V}\}$ for some $V \in \mathcal{S}$. So in this case, $Z \cap Z'$ is a circle over K, and in particular the concentric circle in $V \cap Y$ defined by $|f(x)| = \beta$. To obtain surjectivity, first make a finite base extension L of K so that $\overline{(X_\beta)_L}$ and $\overline{(B_{\beta,V})_L}$ are reduced. Then $\mathbb{O}(\overline{(X_\beta)_L})$ is isomorphic to a subring of

$$\mathbb{F}_L[t_1, \ldots, t_N]/(t_i t_j)_{i \neq j}$$

that contains a power of the ideal $(t_1, \ldots, t_N)$. Also, if t_i is the particular parameter corresponding to V, then $\mathbb{O}(\overline{(B_{\beta,V})_L})$ can be identified via the gluing map with $\mathbb{F}_L[t_i^{-1}]$. So Δ^* is surjective, as $\mathbb{O}(\overline{Z \cap Z'}) = \mathbb{F}_L[t_i, t_i^{-1}]$. Thus, C is separated over K [Bosch et al. 1984, Definition 9.6.1/1], and hence proper [Bosch et al. 1984, Definition 9.6.2/2]. Therefore, C is an algebraic curve by the Riemann existence theorem. □

When a wide open W is completed to a curve C as above, the underlying affinoid X is the complement in C of a finite union of open disks. As we will now show, this results in a close connection between the reductions of C and the canonical reduction of X. Of particular interest will be the case when (W, X) is basic (defined below), in which case, provided K is stable and assuming Hypothesis T, we show that X is the minimal underlying affinoid and C has a model that reduces to $\overline{X}^c$.

Lemma 2.19. *Assume Hypothesis T. Let C be a smooth complete curve over K, and let Z be a nonempty subset of $C(\overline{K})$ that is Galois stable over K and open in the canonical topology* [Bosch et al. 1984, §7.2.1]. *If Q is a point in $C(K)$, there* *exists a function f on C, defined over K, with a pole only at Q and zeroes only in Z.*

Proof. We can assume $g = g(C) > 0$ and $Q \notin Z$. Identify C with its image in its Jacobian J by $x \mapsto (x) - (Q)$. Then $U := [g]_J Z = Z[+]_J \cdots [+]_J Z$ is open in $J(\overline{K})$. Let $P \neq 0 \in U$. We claim that there is a sequence $m_1, m_2, \ldots$ of positive integers such that $[m_n]_J P \to 0$.

By [BL 1984, Theorems 5.1, 6.6, 7.4 and 7.5] (see [Cherry 1994, Theorems 2.1 and 2.2] also), there is a finite extension L of K,[4] a commutative rigid analytic group $\hat{J}$, and formal analytic groups J^{fm} and B over L [Bosch 1977b, Definition 1.4] (see also [Bosch 1976, introduction] and [Cherry 1994, p. 397]), such that B is proper and we have an injective composition $(J^{\mathrm{fm}})^{\mathrm{rg}} \to \hat{J} \to J_L^{\mathrm{rg}}$ such that the image of $(J^{\mathrm{fm}})^{\mathrm{rg}}$ in J_L^{rg} is a maximal connected subgroup with a formal analytic structure.[5] Moreover, there is a diagram with exact rows and columns

$$\begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & & \downarrow & & & & \\ & & & & \mathbb{Z}^t & & & & \\ & & & & \downarrow & & & & \\ 0 & \to & (\mathbf{G}_m^{\mathrm{rg}})^t & \longrightarrow & \hat{J} & \longrightarrow & B^{\mathrm{rg}} & \to & 0 \\ & & & & \downarrow & & & & \\ & & & & J_L^{\mathrm{rg}} & & & & \\ & & & & \downarrow & & & & \\ & & & & 0 & & & & \end{array}$$

where $t \in \mathbb{N}$ (the toric rank) and the image of $\mathbb{Z}^t$ is a discrete closed subgroup. This induces an exact sequence $0 \to (\mathbf{G}_m^{\mathrm{fm}})^t \to J^{\mathrm{fm}} \to B \to 0$, of formal analytic groups[6] and implies that $\hat{J}(L)/J^{\mathrm{fm}}(L)$ is isomorphic to $(\mathbf{G}_m^{\mathrm{rg}}(L)/\mathbf{G}_m^{\mathrm{fm}}(L))^t$ and the reduction of J^{fm} over the residue field of L is semiabelian.[7]

So $J(L)/J^{\mathrm{fm}}(L)$ is isomorphic to $(\mathbf{G}_m^{\mathrm{rg}}(L)/\mathbf{G}_m^{\mathrm{fm}}(L))^t/\,\Gamma$, where Γ is the injective image of $\mathbb{Z}^t \to \hat{J}(L)/J^{\mathrm{fm}}(L) \to (\mathbf{G}_m^{\mathrm{rg}}(L)/\mathbf{G}_m^{\mathrm{fm}}(L))^t$. Assuming Hypothesis T for L, $\mathbf{G}_m^{\mathrm{rg}}(L)/\mathbf{G}_m^{\mathrm{fm}}(L) = L^*/R_L^*$ is isomorphic to a subgroup of $\mathbb{Q}$, and hence it follows that $J(L)/J^{\mathrm{fm}}(L)$ is torsion. As all elements on a semiabelian variety over a finite field are torsion, some multiple $[k]_J P$ of P lies in the image of the kernel of reduction of J^{fm}, and then $[p^n k]_J P \to 0$.

Now, since U is open and $[m_n]_J P \to 0$, there is a positive integer m such that $-[m-1]_J P = P\,[-]_J [m]_J P \in U$. Thus $0 \in [mg]_J Z$. More specifically, there is

[4] While the field is assumed to be algebraically closed in [BL 1984], it is explained on [BL 1984, p. 257] how to show that $\hat{J}$ may be defined over a finite extension.

[5] If Y is a scheme or formal analytic space, Y^{rg} will denote the associated rigid space.

[6] $\mathbf{G}_m^{\mathrm{fm}}$ denotes the formal completion of $\mathbf{G}_m$ along its reduction.

[7] Formal analytic spaces have canonical reductions.

a principal divisor D of the form

$$(m-1)\sum_{i=1}^{g}(P_i)+\sum_{i=1}^{g}(R_i)-mg(Q),$$

where P_i and $R_i \in Z$. If g is a function over L with this divisor, we can take $f=\prod_\sigma g^\sigma$, where σ ranges over embeddings of L/K into $\mathbf{C}/K$. $\square$

Lemma 2.20. *Suppose C is a complete curve over K and U is an open disk in C. Then $Y := C \setminus U$ is a nonempty open in the canonical topology.*

Proof. Let P be any point in Y, which is nonempty since U is not proper and so cannot equal C. By Riemann–Roch, we can choose a meromorphic function g on C with a pole only at P. Then because $g|_U$ is holomorphic and finite to one, $g(U)$ is an open disk in $\mathbf{A}^1$. Let E be an open disk around infinity in $\mathbf{P}^1 \setminus g(U)$. Then $g^{-1}(E)$ is an open neighborhood of P in Y, in the canonical topology. $\square$

Proposition 2.21. *Assume Hypothesis T. Suppose C is a smooth complete curve over K. Let L be a finite Galois extension of K, and let T be a finite, nonempty, Galois stable subset of $C(L)$. Suppose $\mathfrak{D}=\{D_t : t\in T\}$ is a Galois stable collection of disjoint open disks over L in C, such that $D_t \cap T=\{t\}$ for all $t\in T$. Then if $U=\bigcup \mathfrak{D}$, then $X := C\setminus U$ is a one-dimensional affinoid over K, and the image of the ring of algebraic functions, $\mathbb{O}_C(C\setminus T)$, is dense in $A(X)$.*

Proof. X is nonempty, since U is not proper, and X is open in the canonical topology by Lemma 2.20. Therefore, Lemma 2.19 implies that for each Galois orbit $S\subseteq T$ there exists a function $f_S \in \mathbb{O}_C(C\setminus S)$, defined over K, that has a pole at each $s\in S$ and zeroes only on X. Set $\mathfrak{D}_0=\mathfrak{D}$, $X_0=X$, and $U_0=U$. Then for each $n\geq 1$, choose a Galois stable collection $\mathfrak{D}_n$ of $|T|$ open disks over L in C, such that $\mathfrak{D}_{n+1}\subseteq \mathfrak{D}_n$ for all $n\geq 0$ and $\bigcap_n \mathfrak{D}_n = T$. Set $U_n=\bigcup \mathfrak{D}_n$ and $X_n = C\setminus U_n$. Let D_{tn} be the disk in $\mathfrak{D}_n$ that contains any particular $t\in T$, and for any Galois orbit $S\subseteq T$, set $M_{Sn}=\inf\{|f_S(x)| : x\in \bigcup_{s\in S} D_{sn}\}$. (Note that this infimum is positive and does not belong to the set, since $|g|_{\sup}$ exists and is not equal to $|g(x)|$ for any $x\in D$ when g is a rigid function on an open disk D that vanishes at only finitely many points.) We claim

$$X_n = Z_n := \{x\in C : |f_S(x)| \leq M_{Sn} \text{ for all Galois orbits } S\subseteq T\}.$$

It is clear that $Z_n \subseteq X_n$ since Z_n cannot intersect D_{tn} for any $t\in T$. For the other direction, note that f_S is defined over K and has poles only on S, and so $f_S : C\to \mathbf{P}^1$ has degree $|S|d_S$ where $d_S := -\mathrm{ord}_s f_S$ for any $s\in S$. Moreover, since f_S has no zeroes on $\mathfrak{D}$, $f_S|_{D_{sn}}$ is a d_S to 1 map onto the disk $\mathbf{P}^1\setminus B_K[M_{Sn}]$. It follows that $M_{Sn}\in \mathfrak{R}$ and $\|f_S\|_{X_n}=M_{Sn}$. Thus, $X_n\subseteq Z_n$. So $X_n=Z_n$, and in particular X_n is an affinoid.

For each n and S, we may choose $e_{Sn} \in \mathbb{N}$ and $a_{Sn} \in K^*$ such that $|a_{Sn}| = M_{Sn}^{e_{Sn}}$. Then, using the notation of [Bosch et al. 1984, §7.2.3], we have

$$Z_n = Z_{n+1}(f_S^{e_{Sn}}/a_{Sn} : S \text{ is a Galois orbit in } T).$$

It follows from [Bosch et al. 1984, Proposition 7.2.3/1] that the image of $A(X_{n+1})$ is dense in $A(X_n)$. Suppose $g \in A(X)$ and $\epsilon > 0$. Then there exist functions $h_n \in A(X_n)$ such that $\|h_1 - g\|_X < \epsilon$ and $\|h_{n+1} - h_n\|_{X_n} < \epsilon/n$ for $n \geq 1$. It follows that the sequence h_n converges to an element $h_\epsilon \in A(C \setminus T)$ such that $\|h_\epsilon - g\|_X < \epsilon$. The proposition follows from the fact that $\mathcal{O}_C(C \setminus T)$ is dense in $A(C \setminus T)$. □

Corollary 2.22. *Assume Hypothesis T. Let W be a wide open over K. Then the image of $A(W)$ is dense in $A(X)$ for each underlying affinoid X.*

Proof. Glue open disks B_V to W to make a complete curve C as in the proof of Theorem 2.18. For each $V \in \mathcal{S}$, choose a point $t_V \in B_V \setminus W$ defined over K. Then let $T = \{t_V : V \in \mathcal{S}\}$ and follow the procedure above, noting that the map from $\mathcal{O}_C(C \setminus T)$ to $A(X)$ factors through $A(W)$. □

Corollary 2.23. *With the same hypotheses and notation as Proposition 2.21, set $A_0 = \{f \in \mathcal{O}_C(C \setminus T) : \|f\|_X \leq 1\}$. If $A_0 \otimes \mathbb{F}_K$ is reduced, then* Spec $A_0 \otimes \mathbb{F}_K \cong \overline{X}$.

A *basic wide open pair* over K is a pair (W, X) where W is a connected wide open over K and X is an underlying affinoid. In addition, we require that $\|A(X)\|_X = |K|$, that X have irreducible reduction with at worst ordinary double points as singularities, and that the components of $W \setminus X$ be isomorphic to annuli of the form $A_K(1, s)$. If (W, X) is a basic wide open pair for some X, we say that W is a *basic wide open*. By Proposition 2.21 and Corollary 2.23, basic wide open pairs can be constructed by taking $(W, X) = (C \setminus \bigcup_{i=1}^n D_i, C \setminus \bigcup_{i=1}^n U_i)$. Here C is a connected smooth complete curve over K that has a model $\mathscr{C}$ over R_K whose reduction is irreducible and has at worst ordinary double points as singularities, $\{U_1, \ldots, U_n\}$ is a finite collection of distinct residue classes of smooth points, and each D_i is an affinoid disk in U_i. The converse, that all basic wide open pairs can be constructed in this manner, follows, when K is stable and assuming Hypothesis T, from Theorem 2.27 (and thus the two notions are equivalent in this case).

Lemma 2.24. *Assume Hypothesis T. Suppose $f : X \to Y$ is a map between smooth one-dimensional affinoids over K, and $\overline{X}$ is irreducible.*

(i) *If $\bar{f} : \overline{X} \to \overline{Y}$ is a surjection, then f is a surjection.*

(ii) *If $\bar{f}(\overline{X}) \subseteq \overline{Y}$ is an open affine and $X(\mathbf{C}) \to Y(\mathbf{C})$ is an injection, then $\bar{f}$ is an injection.*

Proof. For both parts, we may extend scalars to $\mathbf{C}$. To prove (i), suppose $\bar{f}$ is a surjection but that there exists a $y \in Y$ that is not in the image of f. Let $\lambda \in A^o(Y)$ be a function that vanishes only at y and such that $\|\lambda\|_Y = 1$.[8] On the one hand, if we let $L = f^*(\lambda)$, the fact that $\bar{f}$ is a surjection implies that $\|L\|_X = 1$. On the other hand, since L does not vanish on X, L^{-1} exists and we may choose $c \in \mathbf{C}$ such that $|c| = \|L^{-1}\|_X$. Now the fact that $\bar{f}$ is a surjection implies that $|c| > 1$. Thus, if we let $M = c^{-1}L^{-1}$, we have $\overline{L}, \overline{M} \neq 0$ but $\overline{L}\,\overline{M} = 0$. So $\overline{X}$ must be reducible.

For (ii), let $Y' \subseteq Y$ be the Zariski subaffinoid for which $\bar{f}(\overline{X}) = \overline{Y'}$. Suppose there are distinct points $x_1, x_2 \in \overline{X}$ for which $\bar{f}(x_1) = \bar{f}(x_2)$. Let $X' = X \setminus R(x_2)$. Then f restricts to a map $f' : X' \to Y'$ that reduces to a surjection. Thus, by (i), f' is a surjection. But this is a contradiction since $f(R(x_2)) \subseteq Y'$ and f is an injection. Therefore, $\bar{f}$ must also be an injection. □

Lemma 2.25. *Suppose $h : B \to Y$ is an analytic map from an open annulus or open disk into a reduced affinoid. Then the image of B is contained in a residue class of Y.*

Proof. This is clear when Y is an affinoid disk. The general case follows. □

Remark 2.26. The same statement is true with B a connected wide open in place of an open annulus.

Theorem 2.27. *Suppose K is stable and satisfies Hypothesis T. Let (W, X) be a basic wide open pair over K. Attach disks B_V to W to obtain a complete curve C, as in the proof of Theorem 2.18. Then C has a model over R_K whose reduction is $\overline{X}^c$. Also, if x is a point at ∞ in $\overline{X}^c(\mathbb{F})$, then $x \in \overline{X}^c(\mathbb{F}_K)$ and $\{P \in C(\mathbf{C}) : \overline{P} = x\}$ is equal to $B_V(\mathbf{C})$ for some $V \in \mathcal{S} = \mathrm{CC}(W \setminus X)$.*

Proof. Choose a finite, Galois stable set of points $Y \subset X(L)$, for some finite extension L of K, that reduce to distinct smooth points in $\overline{X}_L(\mathbb{F}_L)$. The set $\{R(\bar{y}) : y \in Y\}$ of residue classes of X_L is a Galois stable set of open disks in C over L. Therefore, by Proposition 2.21, $Z := C \setminus \bigcup_{y \in Y} R(\bar{y})$ is an affinoid over K. Moreover, $X_1 := X \cap Z$ is a formal subdomain of X [BL 1985, p. 351], whose reduction is $\overline{X} \setminus \{\bar{y} : y \in Y\}$. We will show that X_1 is also a formal subdomain of Z, and hence $\mathcal{C} := \{X, Z\}$ is a formal covering of C.

To do this, let $Z_{\mathcal{T}} := Z \setminus \bigcup_{V \in \mathcal{T}} B_V$ for any $\mathcal{T} \subseteq \mathcal{S}$. This is an affinoid over K by Proposition 2.21. We claim that $Z_{\mathcal{T}}$ has irreducible reduction, and that B_V is a residue class of $Z_{\mathcal{T}}$ for each $V \in \mathcal{S} \setminus \mathcal{T}$. This is clearly true for $\mathcal{T} = \mathcal{S}$, because $Z_{\mathcal{S}} = X \setminus \bigcup_{y \in Y} R(\bar{y})$ is a Zariski subaffinoid of X and $\mathcal{S} \setminus \mathcal{T}$ is empty. Suppose it holds for some $\mathcal{T}$, and suppose also that $V \in \mathcal{T}$. Let $\mathcal{T}' = \mathcal{T} \setminus \{V\}$, so

[8] This can be done by embedding Y in a smooth, complete curve [Van der Put 1980, Theorem 1.1] and applying Lemma 2.19.

that $Z_{\mathcal{T}'} = Z_{\mathcal{T}} \coprod B_V$. By Lemma 2.25, applied to the inclusion of B_V into $Z_{\mathcal{T}'}$, B_V is contained in the residue class $R(\bar{t}_V)$. If $B_V \neq R(\bar{t}_V)$, the map $\overline{Z}_{\mathcal{T}} \to \overline{Z}_{\mathcal{T}'}$ is a surjection. But this is impossible by Lemma 2.24 since $Z_{\mathcal{T}}$ has irreducible reduction and $Z_{\mathcal{T}} \neq Z_{\mathcal{T}'}$. Therefore, B_V is a residue class of $Z_{\mathcal{T}'}$, $Z_{\mathcal{T}}$ is a Zariski subaffinoid of $Z_{\mathcal{T}'}$, and in particular $Z_{\mathcal{T}'}$ has irreducible reduction. The claim now follows for all $\mathcal{T}$ by induction. Taking $\mathcal{T} = \varnothing$, we see that Z has irreducible reduction, and that each disk B_V is a residue class of Z. Thus, X_1 is a formal subdomain of Z, and $\mathcal{C}$ is a formal covering of C. Moreover, by Proposition 2.8, the reduction of Z is the disjoint union of $\overline{X_1}$ and finitely many smooth points. Thus C has semistable reduction with respect to $\mathcal{C}$ [BL 1985, Definition 1.5]. So using the argument of [BL 1985, p. 377], it follows that C has a model with reduction $\overline{X}^c$. Moreover, the residue classes of the points at infinity on $\overline{X}^c$ are precisely the disks B_V over K. □

It may be proven that over $\mathbf{C}$, all wide opens that are not disks or annuli have minimal underlying affinoids. In fact, one can show that if W is a wide open over K that is not a disk or an annulus, W has an affinoid subdomain that becomes the minimal underlying affinoid of W_L, where L is a finite extension of K; see Remark 2.41. However, this fact is not used in this paper.

Lemma 2.28. *Suppose K is stable and assume Hypothesis T. If (W, Z) is a basic wide open pair over K, and W is not a disk or annulus, then Z is a minimal underlying affinoid of W.*

Proof. Suppose there are e ends. Glue in disks, as above, to get a smooth complete curve C, so that $C \backslash Z$ is the union of e open disks $U_1 \dots U_e$. Then by Theorem 2.27, C will have a model $\mathcal{C}$ with reduction isomorphic to the completion of $\overline{Z}$.

We can and will extend scalars to $\mathbf{C}$. Suppose V is any underlying affinoid of W and A is a component of $W \setminus V$. Then $A \cap U_i \neq \varnothing$ for some i. Set $U = U_i$. We claim that A is contained in U.

Identify A with $A_{\mathbf{C}}(r, s)$ so that $A_{\mathbf{C}}(t, s)$ is connected to V for any $t \in \mathcal{R}(r, s)$. It follows from [BL 1985, Proposition 5.4(c)] that every circle in A that intersects a residue class of C is contained in that class. Hence $A \cap U$ contains $C_{\mathbf{C}}[t]$ for any $t \in \mathcal{R}(r, s)$ with $C_{\mathbf{C}}[t] \cap U \neq \varnothing$. In fact, $A \cap U \supset A_{\mathbf{C}}(r, t]$ for any such t. Let $q = LUB\{t \in \mathcal{R}(r, s) : A_{\mathbf{C}}(r, t) \subseteq U\}$. Suppose that $q < s$, and let

$$v = LUB\{t \in \mathcal{R}[q, s] : C_{\mathbf{C}}[t] \cap Z \neq \varnothing\} = GLB\{t \in \mathcal{R}[q, s] : C_{\mathbf{C}}[t] \cap Z = \varnothing\}.$$

The number v exists since U is disconnected from U_j for $j \neq i$. For $u \in \mathcal{R}[q, v)$, $C_{\mathbf{C}}[u] \subseteq Z$ (again by [BL 1985, Proposition 5.4(c)]). Let $w = q$ if $q \in \mathcal{R}$, and $w \in \mathcal{R}[q, u)$ otherwise, and set $Y = A_{\mathbf{C}}[w, u]$. We have a rigid morphism $Y \to Z$. Since $\overline{Y}$ is either a line or two lines crossing at a point, and $\overline{Z}$ is irreducible, not isomorphic to $\mathbf{A}^1$ or $\mathbf{G}_m$, with only ordinary double points as singularities, it follows that the map $\overline{Y} \to \overline{Z}$ is constant. This means $A \setminus U$ is contained in a residue class R of Z. Thus $\{U, R\}$ is a disjoint admissible cover of A. This is impossible as A is connected.

From the contradiction, we know that $q = s$, and thus $A \subseteq U$. Now, since each component of $W \setminus V$ is contained in $W \setminus Z$, we have shown that $Z \subseteq V$. □

The final two results of this section provide useful criteria for determining when a rigid space is a wide open.

Theorem 2.29. *Suppose X is a smooth, one-dimensional affinoid over a stable field K satisfying Hypothesis B, and x is a point of degree one on $\overline{X}$. Then, if $U = R_X(x)$, there is a finite extension L of K such that U_L is a connected wide open over L. Moreover, the number of ends of U_L equals the number of branches of $\overline{X}_L$ through x.*

This is a consequence of the following lemma. Recall that $\mathbb{F}_K \subseteq \overline{\mathbb{F}}$.

Lemma 2.30. *Suppose X is a pure, one-dimensional reduced affinoid over a stable field K satisfying Hypothesis B, with reduced reduction, and $x \in \overline{X}(\mathbb{F}_K)$ is a degree one point. Choose any $f \in A^o(X)$ such that $\bar{f}$ has an isolated zero at x, that is, such that x is the only zero in a Zariski neighborhood. For $r \in \mathcal{R}(0, 1)$, let $V(r)$ be the subspace of X such that*

$$V(r)(\mathbf{C}) = \{y \in R(x)(\mathbf{C}) : r < |f(y)| < 1\}.$$

Then for r sufficiently close to 1, there is a finite extension L of K such that $V_L(r)$ is a disjoint union of $m := |(n^{-1}x)(\overline{\mathbb{F}})|$ open annuli, where $n : Y \to X_{\overline{\mathbb{F}}}$ is the normalization of $X_{\overline{\mathbb{F}}} := \overline{X} \otimes_{\mathbb{F}_K} \overline{\mathbb{F}}$.

Proof. Without loss of generality, we may assume that x is the only zero of $\bar{f}$ (otherwise replace X with a suitable Zariski subaffinoid). Let $Z := Z_r$ be the subaffinoid of X whose $\mathbf{C}$-valued points are $\{y \in X(\mathbf{C}) : |f(y)| \geq r\}$. Let X_x be the curve obtained from Y by identifying $n^{-1}(x')$ to a point for each $x' \in X_{\overline{\mathbb{F}}}(\overline{\mathbb{F}}) \setminus \{x\}$ (thus, X_x is the minimal finite surjective cover of $X_{\overline{\mathbb{F}}}$ that is smooth at all points above x). It is proven in the remark after [BL 1985, Lemma 2.4] that for $r \in \mathcal{R}(0, 1)$ sufficiently close to 1, the reduction of $Z_{\mathbf{C}}$ is isomorphic to the union of X_x and m lines, each crossing a single point above x normally.[9]

There is a finite extension M of K such that $\overline{Z}_M$ is reduced, so $\overline{Z}_{\mathbf{C}} \cong (\overline{Z}_M)_{\overline{\mathbb{F}}}$. Thus, there is a finite extension L of K such that $\overline{Z}_L$ is isomorphic to the union of a finite surjective cover of $\overline{X}_{\mathbb{F}_L}$ that is smooth at all points above x, and m lines each crossing a single point above x normally. Now apply Proposition 2.10. □

[9] This is a minor correction of the statement in [BL 1985].

Proposition 2.31. *Assume Hypothesis T. Suppose $f : U \to W$ is a finite surjective morphism over K of a smooth, one-dimensional rigid space onto a wide open, with finitely many branch points all defined over K. If f has degree strictly less than p, then U is a wide open over K.*

Proof. First we claim that an underlying affinoid $X \subseteq W$ can be chosen so that it contains the set $\mathcal{B}$ of branch points of f. Indeed, let X_1 be *any* underlying affinoid of W. Glue disks B_V onto W for each annulus $V \in W \setminus X_1$, as in the proof of Theorem 2.18, to obtain a complete curve C. Then for each V, choose an open disk D_V over K such that $B_V \setminus \alpha_V(V) \subset D_V \subseteq B_V$ and $D_V \cap \mathcal{B} = \varnothing$. The rigid subspace $X := C \setminus \bigcup D_V$ is, by Proposition 2.21, an affinoid that is disjoint from $\mathcal{B}$ and easily shown to be underlying in W.

Now suppose X is relatively compact in some affinoid $Y \subseteq W$. Then $f^{-1}(X)$ and $f^{-1}(Y)$ are affinoids in U. Moreover, as f is finite, and the image of $\overline{X}$ in $\overline{Y}$ is finite, it follows that the image of $\overline{f^{-1}(X)}$ in $\overline{f^{-1}(Y)}$ is finite. So $f^{-1}(X)$ is relatively compact in $f^{-1}(Y)$. All that remains is to check that $U \setminus f^{-1}(X)$ is the disjoint union of open annuli, and for this Theorem 2.6 suffices. □

2C. *Semistable coverings.* For a wide open W over K, let

$$H^i_{DR}(W/K) = D^i(W/K).$$

Using Lemma 2.13, the arguments in the proof of [Coleman 1989, Theorem 4.2] generalize and allow us to conclude that $H^i_{DR}(W/K)$ is finite-dimensional over K. We define the genus of W, which we denote by $g(W)$, to be

$$\tfrac{1}{2}(\dim_K H^1_{DR}(W/K) - e(W) + 1).$$

Then $2g(W)$ can be interpreted as the dimension of the first compactly supported de Rham cohomology group of W. For example, in Corollary 2.33, we show that

$$2g(W) = \dim_K(\ker(H^1_{DR}(W/K) \to H^1_{DR}((W \setminus X)/K))),$$

where X is any underlying affinoid of W. We also show in Proposition 2.32 that if a wide open W is completed to a projective curve C by attaching disks at the ends, as in Theorem 2.18, then $g(W) = g(C)$. As an immediate corollary of this and of Theorem 2.27, if (W, X) is a basic wide open pair over a complete, stable field K satisfying Hypothesis T, and X has good reduction $\overline{X}$, then $(\overline{X})^c$ will also have genus $g(W)$.

Proposition 2.32. *Let W be a connected wide open over K. Suppose C is a smooth, complete curve (over K) obtained by attaching disks at the ends of W, as in Theorem 2.18. Then $g(W) = g(C)$.*

Proof. The main idea is to view W and the attached disks as an admissible covering of C, and then to apply the (generalized) Mayer–Vietoris sequence of de Rham cohomology (over K). So first suppose $D_1, \ldots, D_n$ are the disks, and set $D_0 = W$. Then Mayer–Vietoris gives us the exact sequence

$$0 \to H^0_{DR}(C) \to \bigoplus_i H^0_{DR}(D_i) \to \bigoplus_{i \neq j} H^0_{DR}(D_i \cap D_j) \to$$
$$H^1_{DR}(C) \to \bigoplus_i H^1_{DR}(D_i) \to \bigoplus_{i \neq j} H^1_{DR}(D_i \cap D_j) \to H^2_{DR}(C) \to 0.$$

Using Lemma 2.13, the definition above of $g(W)$, and the fact that $H^1_{DR}(D_i) = 0$ for $i > 0$, we count dimensions to obtain

$$1 - (e(W) + 1) + e(W) - 2g(C) + (2g(W) + e(W) - 1) - e(W) + 1 = 0.$$

From this we conclude that $g(C) = g(W)$. $\square$

Corollary 2.33. *Suppose W is a wide open over K, and X is an underlying affinoid of W. Then*

$$2g(W) = \dim_K(\ker(H^1_{DR}(W/K) \to H^1_{DR}((W \setminus X)/K))).$$

Proof. Suppose C is a smooth complete curve obtained by gluing disks to the ends of W. Then arguing from Mayer–Vietoris exactly as in the above proof, we have the exact sequence

$$0 \to H^1_{DR}(C) \to H^1_{DR}(W) \to H^1_{DR}(W \setminus X) \to K \to 0.$$

Now apply Proposition 2.32. $\square$

Let C be a wide open or a smooth proper curve over K. Let $\mathscr{C}$ be a finite set of basic wide open pairs (U, U^u) over K such that $\mathscr{C}^w := \{U, (U, U^u) \in \mathscr{C}\}$ is an admissible covering of C. Then we call $\mathscr{C}$ a *semistable covering* over K if the following conditions hold:

(i) If $U, V \in \mathscr{C}^w$ and $U \neq V$, the intersection of U and V is a disjoint union of connected components of $U \setminus U^u$ (by definition, annuli of the form $A_K(1, s)$).

(ii) If U, V and W are three distinct elements of $\mathscr{C}^w$, their intersection is empty.

We say that a semistable covering $\mathscr{C}$ is *stable* if none of the elements of $\mathscr{C}^w$ are disks or annuli. Having a semistable covering is not immediately equivalent to having a *semistable reduction* in the sense of [BL 1985, Definition 1.5]. When the context is clear, we will abuse notation by dropping the superscript w and writing $U \in \mathscr{C}$ to mean $U \in \mathscr{C}^w$.

Proposition 2.34. *Suppose $\mathscr{C}$ is a semistable covering of a smooth proper curve C over K. Let $\Gamma_{\mathscr{C}}$ be the unoriented graph without loops whose vertices correspond to the elements of $\mathscr{C}$ and whose edges with endpoints corresponding to distinct $U, V \in \mathscr{C}$ correspond to the connected components of $U \cap V$. Then*

$$g(C) = \sum_{U \in \mathscr{C}} g(U) + \mathrm{Betti}(\Gamma_{\mathscr{C}}).$$

Proof. Again, we begin with the Mayer–Vietoris sequence (of de Rham cohomology over K) associated to this covering.

$$\cdots \to \bigoplus_{U,V \in \mathscr{C}} H^{i-1}_{DR}(U \cap V) \to H^i_{DR}(C) \to \bigoplus_{U \in \mathscr{C}} H^i_{DR}(U) \to \cdots$$

It is immediate that

$$H^0_{DR}(C) \cong H^2_{DR}(C) \cong K, \quad \bigoplus_{U \in \mathscr{C}} H^2_{DR}(U) = 0, \quad \text{and} \quad \bigoplus_{U \in \mathscr{C}} H^0_{DR}(U) \cong K^{\#\mathscr{C}}.$$

Also, by applying Lemma 2.13 and condition (i) from above, we see that

$$\bigoplus_{U,V \in \mathscr{C}} H^0_{DR}(U \cap V) \cong \bigoplus_{U,V \in \mathscr{C}} H^1_{DR}(U \cap V) \cong K^{\#\mathscr{E}},$$

where $\mathscr{E}$ is the edge set of $\Gamma_{\mathscr{C}}$. Now to prove the proposition, we simply count dimensions over K and compute the dimension of $H^1_{DR}(C)$ using the exact sequence. We have

$$\begin{aligned} 2g(C) &= \sum_{U \in \mathscr{C}} (2g(U) + e(U) - 1) - \#\mathscr{C} + 2 = 2\Big(\sum_{U \in \mathscr{C}} g(U) + \#\mathscr{E} - \#\mathscr{C} + 1\Big) \\ &= 2\Big(\sum_{U \in \mathscr{C}} g(U) + \mathrm{Betti}(\Gamma_{\mathscr{C}})\Big). \end{aligned}$$

□

Definition 2.35. A *semistable model* $\mathscr{B}$ of a curve C over K is a flat, proper scheme over R_K whose generic fiber is C, such that all of the singular points of the special fiber of $\mathscr{B}$ have degree 1 and are ordinary double points. We say that $\mathscr{B}$ is *stable* if it is the final object in the category of semistable models over K.[10]

See [BL 1985] and [Van der Put 1984] for a rigid analytic treatment of the theory of stable models of curves over complete nonarchimedean fields, and in particular, for a rigid analytic proof of the generalization to arbitrary complete

[10] This weakens the definition of the semistable model in [Mumford 1977] since it allows smooth rational components that meet the other components in only one point. Requiring the singular points to have degree 1 means that $X_0(p)$ usually does not have a stable model over $\mathbb{Q}_p$, but does over $W(\mathbb{F}_{p^2}) \otimes \mathbb{Q}_p$.

nonarchimedean fields [Van der Put 1984, Corollary 3.3] (see also [BL 1985][11]) of the existence theorem of Deligne and Mumford. Moreover, we will use the results of [BL 1984; 1985] to prove the following theorem, which relates stable coverings to stable models. This result generalizes [Coleman 2003, Proposition 2.1], and the proof we give is more complete than the one given there.

Theorem 2.36. *Let C be a smooth complete curve over a stable field K satisfying Hypothesis B.*

(i) *If C has a semistable model over R_K whose reduction has at least two components, then C has an associated semistable covering over K.*

(ii) *If K satisfies Hypothesis T, and C has a semistable covering over K, then C has an associated semistable model over R_K whose reduction has at least two components.*[12]

Stable coverings are precisely those that correspond to stable models whose reductions have at least two components.

Proof. Suppose $\mathscr{C}$ is a semistable model for C over R_K, and let $\mathscr{I}_{\mathscr{C}}$ be the set of irreducible components in the reduction of $\mathscr{C}$. For each $\Gamma \in \mathscr{I}_{\mathscr{C}}$, let

$$\Gamma^o = \Gamma \setminus \bigcup_{\Gamma' \in \mathscr{I}_{\mathscr{C}}, \Gamma' \neq \Gamma} \Gamma'.$$

Assume, without loss of generality, that $\overline{\mathscr{C}}$ is connected.

For each affine open $U \subseteq \overline{\mathscr{C}}$, there is a natural affinoid subdomain of C^{rg}, which we denote by $\mathrm{Red}^{-1}\,U$, whose points are all the points of C^{rg} that reduce to points of U. To see this, let $\operatorname{Spec} S$ be any affine open subscheme of $\mathscr{C}$ that reduces to U and $\hat{S} = \varprojlim_n S/\pi^n S$ for some $\pi \in R_K$ with $0 < |\pi| < 1$. Then $\hat{S}$ is an admissible R_K-algebra in the sense of [BL 1993, p. 293], as can be seen from [BL 1993, Lemma 1.2]. Then $\hat{S} \otimes_{R_K} K$ is an affinoid algebra over K [BL 1993, §4] that up to canonical isomorphism does not depend on the choices. We refer to the affinoid $\mathrm{Sp}(\hat{S} \otimes_{R_K} K)$ as $\mathrm{Red}^{-1}\,U$. Because U is reduced, $\overline{\mathrm{Red}^{-1}\,U} \cong U$. More generally, suppose V is the union of finitely many subschemes W of $\overline{\mathscr{C}}$, with each contained in some affine open U_W. Then we let $\mathrm{Red}^{-1}\,V$ be the open rigid subspace that is the union in C^{rg} of the subspaces $\mathrm{Red}^{-1}\,W \subseteq \mathrm{Red}^{-1}\,U_W$, as was defined in the beginning of Section 2. This subspace is independent of the choices of U_W.

If $\Gamma \in \mathscr{I}_{\mathscr{C}}$, let $W_\Gamma = \mathrm{Red}^{-1}\,\Gamma$ and $X_\Gamma = \mathrm{Red}^{-1}\,\Gamma^o$. We claim that $\{(W_\Gamma, X_\Gamma) : \Gamma \in \mathscr{I}_{\mathscr{C}}\}$ is a semistable covering. First, W_Γ is a smooth, one-dimensional rigid

[11] Bosch and Lütkebohmert [BL 1985, p. 377], while proving the theorem of Deligne and Mumford, remark that their argument does not require the field to be discretely valued.

[12] In fact, when K satisfies Hypothesis T we have a natural one-to-one correspondence between semistable coverings and semistable models whose reductions have at least two components. It would be interesting to know if this is true more generally.

space over K, and X_Γ is an affinoid subdomain, such that $W_\Gamma \setminus X_\Gamma$ is a disjoint union of a finite number of annuli of the form $A_K(1, s)$ by Proposition 2.10. Also, X_Γ has absolutely irreducible reduction with at worst ordinary double points as singularities. Moreover, if $\Gamma, \Gamma', \Gamma'' \in \mathcal{I}_{\mathscr{C}}$, then $W_\Gamma \cap W_{\Gamma'}$ is a union of connected components of $W_\Gamma \setminus X_\Gamma$ if $\Gamma \neq \Gamma'$, and $W_\Gamma \cap W_{\Gamma'} \cap W_{\Gamma''} = \varnothing$ if Γ, Γ' and Γ'' are all distinct.

What remains to be shown for (i) is that (W_Γ, X_Γ) is a basic wide open pair for each $\Gamma \in \mathcal{I}_{\mathscr{C}}$, and for that, all we have to show is that there exists an affinoid subdomain Y of W_Γ such that X_Γ is relatively compact in Y and $Y \cap V$ is a semiopen annulus for each connected component of $W_\Gamma \setminus X_\Gamma$. (That W_Γ is connected will follow from the absolute irreducibility of $\overline{X}_\Gamma$.) For this, let $\mathcal{S}_\Gamma$ be the set of singular points in $\overline{\mathscr{C}}$ where Γ intersects some other component. Blow up $\mathscr{C}$ at every point in $\mathcal{S}_\Gamma$ to obtain a new model $\mathscr{C}_\Gamma$ that is defined over K and becomes semistable over an, at worst, quadratic extension L. Let $\hat{\Gamma}$ be the proper transform of Γ in $\mathscr{C}_\Gamma$, and let $\mathcal{I}_{\mathscr{C}_\Gamma}$ be the set of irreducible components in the reduction of $\mathscr{C}_\Gamma$. Set

$$\widetilde{Y}_\Gamma = \overline{\mathscr{C}}_\Gamma \setminus \bigcup_{\substack{\Gamma' \in \mathcal{I}_{\mathscr{C}_\Gamma}, \\ \Gamma' \cap \hat{\Gamma} = \varnothing}} \Gamma',$$

and let $Y_\Gamma = \mathrm{Red}^{-1}(\widetilde{Y}_\Gamma)$. It is clear that $(X_\Gamma)_L \subseteq Y_\Gamma \subseteq (W_\Gamma)_L$ and Y_Γ is naturally defined over K. Although $\widetilde{Y}_\Gamma$ is not an affine open in $\overline{\mathscr{C}}_\Gamma$, Y_Γ is the reduction inverse of an affine open in the model obtained from $\mathscr{C}_\Gamma$ by blowing down $\hat{\Gamma}$. This affine open will consist of $|\mathcal{S}_\Gamma|$ lines intersecting in a single singular point that contains the reduction of X_Γ. Thus, not only is Y_Γ also an affinoid subdomain of W_Γ, but X_Γ is relatively compact in Y_Γ. Finally, by applying Proposition 2.10 again, we see that the intersection of Y_Γ with each component of $W_\Gamma \setminus X_\Gamma$ is a semiopen annulus. Therefore, we are done with (i).

To prove (ii), suppose $\mathscr{C}$ is a semistable covering of C. Then by Theorem 2.27, there is a natural one-to-one correspondence between $(\overline{U^u})^c \setminus \overline{U^u}$, $\mathrm{CC}(U \setminus U^u)$, and $\mathscr{E}(U)$, for each $U \in \mathscr{C}$. If $e \in \mathscr{E}(U)$, let $x(e)$ denote the corresponding point on $(\overline{U^u})^c$ and let $A(e)$ denote the corresponding connected component of $U \setminus U^u$ (an annulus). If $e \in \mathscr{E}(U)$ and $f \in \mathscr{E}(V)$ for $U, V \in \mathscr{C}$, we say that $e \sim f$ whenever $A(e) = A(f)$ (equivalently, $A(e) \cap A(f) \neq \varnothing$). Let $\mathscr{E}$ denote the quotient of $\coprod_{U \in \mathscr{C}} \mathscr{E}(U)$ by this equivalence relation. We define $\overline{\mathscr{C}}$ to be the curve over $\mathbb{F}_K$ obtained from $\coprod_{U \in \mathscr{C}} (\overline{U^u})^c$ by identifying the points $x(e)$ and $x(f)$ whenever $e \sim f$. The reduction maps from $U(\mathbf{C}) \to (\overline{U})^c(\overline{\mathbb{F}})$ for each $U \in \mathscr{C}$, which are guaranteed by Theorem 2.27, patch together to form a natural Galois equivariant reduction map from $C(\mathbf{C}) \to \overline{\mathscr{C}}(\overline{\mathbb{F}})$. We will show that in fact there is a model over R_K whose reduction is $\overline{\mathscr{C}}$.

Let T be a finite Galois stable set of points of $C(\overline{K})$ that, by the above reduction map, injects into the smooth locus of $\overline{\mathscr{C}}$, and such that $T \cap U \neq \varnothing$ for each $U \in \mathscr{C}$. Since each $t \in T$ lies on a unique U^u, the residue class, $R(\bar{t}) := R_{U^u}(\bar{t})$, is well defined and can be viewed as an open disk in C over $\overline{K}$. Moreover, as $\mathscr{C}$ is defined over K, $R(T) := \bigcup_{t\in T} R(\bar{t})$ is Galois stable over K. So by Proposition 2.21, $Z_T := C \setminus R(T)$ is an affinoid over K. We want to show that $\overline{Z}_T = \overline{\mathscr{C}} \setminus \overline{T}$, where $\overline{T} = \{\bar{t} : t \in T\}$.

For each $U \in \mathscr{C}$, we let $U_T = U^u \setminus R(T)$, a Zariski subaffinoid of U^u. Then the affinoid Z_T is the disjoint union of $\coprod_{U\in\mathscr{C}} U_T$ and $\coprod_{e\in\mathscr{E}} A(e)$. Now fix U and consider the natural inclusion map $U_T \hookrightarrow Z_T$. Since the reduction of U_T is irreducible, it follows that $\overline{U}_T$ maps to a point or onto an affine open of some irreducible component Γ_U of $\overline{Z}_T$. As U_T and Z_T are both connected to $R(T)$, the first case is not possible. Therefore, by Lemma 2.24(ii), $\overline{U}_T$ must inject into some such Γ_U. Let U'_T be the subaffinoid of Z_T that lies above the image of $\overline{U}_T$. By Lemma 2.24(i), the inclusion of U_T into U'_T is surjective, and therefore an equality. As the U_T don't intersect, and $U_T = U'_T$ for each $U \in \mathscr{C}$, the Γ_U must be distinct components.

Now suppose $e \in \mathscr{E}$. By applying Lemma 2.25 to the inclusion of $A(e)$ into Z_T, we see that $A(e)$ must be contained in a residue class, $R(y_e) := R_{Z_T}(y_e)$, for some point $y_e \in \overline{Z}_T(\mathbb{F}_K)$. Thus there can be no irreducible components of $\overline{Z}_T$ other than $\{\Gamma_U : U \in \mathscr{C}\}$. Moreover, it is clear that $\bigcup_{e\in\mathscr{E}} A(e) = \bigcup_{e\in\mathscr{E}} R(y_e)$. So using the fact that residue classes of an affinoid are connected,[13] it follows that the y_e are distinct and hence $A(e) = R(y_e)$ for each $e \in \mathscr{E}$. From connectivity, we also have that $y_e \in \Gamma_U \cap \Gamma_V$ whenever $U \neq V$ and $A(e) \subseteq U \cap V$, and by Proposition 2.10 this must be a normal crossing. Therefore, as claimed, we have shown that $\overline{Z}_T = \overline{\mathscr{C}} \setminus \overline{T}$, and we use equality here to emphasize that the canonical reduction map on the $Z_T(\mathbf{C})$ is compatible with the previously defined reduction map on $C(\mathbf{C})$.

To finish the proof, let T_1 and T_2 be two finite Galois stable sets of points of $C(\overline{K})$ satisfying the above conditions on T, and such that $\overline{T}_1 \cap \overline{T}_2 = \varnothing$. Then $Z := Z_{T_1} \cap Z_{T_2}$ is equal to $Z_{T_1 \cup T_2}$. Therefore, Z is a formal subdomain of both Z_{T_1} and Z_{T_2} by the compatibility of reduction maps. So C has semistable reduction $\overline{Z}_{T_1} \cup \overline{Z}_{T_2} = \overline{\mathscr{C}}$ with respect to the formal covering $\{Z_{T_1}, Z_{T_2}\}$; see [BL 1985, Definition 1.5]. Then, by the same argument as used in the proof of Theorem 2.27, C has a semistable model over R_K whose reduction is isomorphic to $\overline{\mathscr{C}}$. □

Remark 2.37. As a consequence of Theorem 2.36 we have the result that whenever K is stable and satisfies Hypothesis B, every semistable curve over R_K can be constructed by gluing together wide opens taken out of curves with *good reduction* over R_K. Crossings of distinct irreducible components are created by gluing two

[13] If R is a residue class, $A^o(R)$ is a local ring.

annuli at the ends of two distinct wide opens, while self-intersections within a component are created by gluing two annuli at distinct ends of a single wide open.

Lemma 2.38. *Suppose D is a closed disk and U is either an open disk or open annulus in a smooth complete curve C, all defined over K, such that $D \cap U \neq \varnothing$. Then either $D \subseteq U$, $U \subseteq D$, $D \cup U$ is an open disk, or $D \cup U = C \cong \mathbf{P}^1$ and U is an open disk.*

Proof. We can assume $K = \mathbf{C}$ and $g(C) > 0$. When U is an open disk the lemma follows from [BL 1985, Proposition 5.4(a)]. So suppose U is an open annulus, with $U \not\subseteq D$ and $D \not\subseteq U$. We first show that every concentric circle R of U that intersects D must be contained in D. Indeed, applying [BL 1985, Proposition 5.4(c)] to the height 1 annulus R and the disk D, and using $D \not\subseteq R$, we can conclude that R is contained in some closed disk E. Then by [BL 1985, Proposition 5.4(a)], we have $D \subseteq E$ or $E \subseteq D$. Either way, it follows that $R \subseteq D$.

Now choose a parametrization $\psi : A_K(r, s) \xrightarrow{\simeq} U$. By the preceding argument, we can then choose $t \in \mathcal{R}(r, s)$ such that $Y_t := \psi(C_K[t]) \subseteq D$. Then $C \setminus Y_t$ and $U \setminus Y_t$ have two connected components each. Since U is connected, $U \setminus Y_t \not\subseteq C \setminus D$. Thus there exists $u \in \mathcal{R}(r, s)$ such that $u \neq t$ and $Y_u \subseteq D$. We can assume that $u < t$ and Y_u is contained in the connected component Z of $C \setminus Y_t$ that lies inside D. Because $A_K[u, t)$ is connected, it follows that $\psi(A_K[u, t)) \subseteq Z$.

Now choose a $P \in Z \setminus U$, and let $\phi : B_K[1] \xrightarrow{\simeq} D$ be any parametrization such that $\phi(0) = P$. We may assume that $\phi(C_K[v]) = Y_v$ whenever $Y_v \subseteq D$. Thus, $\phi(A_K(r, 1]) = U \cap D$ and $s > 1$. Finally, we let $V = D \cup U$. Then V is a wide open with one end and $\phi^{-1} B_K[t]$ is an underlying affinoid for $t \in \mathcal{R}(r, 1]$. Hence $g(V) = 0$, and so by the Riemann existence theorem, V is isomorphic to $\mathbf{P}^1$ minus a closed disk (in particular, an open disk). $\square$

If $\mathscr{C}$ is a semistable covering of C, we define a *residue class* of $\mathscr{C}$ to be either a residue class of U^u or a component of $U \setminus U^u$ for some $U \in \mathscr{C}$.

Corollary 2.39. *Suppose $\mathscr{C}$ is a stable covering of C. Then every closed disk D in C is contained in a residue class of $\mathscr{C}$.*

Proof. Extend scalars to $\mathbf{C}$. The curve C is not isomorphic to $\mathbf{P}^1$ as $\mathbf{P}^1$ does not have a stable covering. Let R be a residue class of $\mathscr{C}$ such that $D \cap R \neq \varnothing$, and suppose $D \not\subseteq R$. First suppose R is an open disk. If necessary, refine $\mathscr{C}$ to a semistable covering $\mathscr{C}'$ for which all underlying affinoids have smooth reduction, none are closed disks, and R is a residue class of U^u for some $U \in \mathscr{C}'$. By Lemma 2.38, we have $R \subseteq D$.

This latter containment implies that $U^u \cap D$ is a nonempty affinoid with good reduction. Every such affinoid is a Zariski subaffinoid of a closed disk E_1 in D, because D is a closed disk. Since U^u has good reduction, E_1 is a disk, and $C \ncong \mathbf{P}^1$, it follows that U^u is a Zariski subaffinoid of E_1.

Set $U_1 = U$. Then U_1^u is not equal to E_1 since none of the underlying affinoids in $\mathscr{C}'$ are disks. Therefore, there exists a residue disk R_1 of E_1 such that $A_1 := R_1 \cap U_1$ is a component of $U_1 \setminus U_1^u$ (an open annulus). Now $E_2 := R_1 \setminus A_1$ is a closed disk. Let U_2 be the other element of $\mathscr{C}'$ containing A_1. By the same argument as above, and the fact that both U_2^u and E_2 are connected to A_1, it follows that U_2^u must be a Zariski subaffinoid of E_2. Again (for $i = 2$ now), we must have $U_i^u \neq E_i$. Proceeding in this manner, we eventually exhaust the underlying affinoids of $\mathscr{C}'$ or find a $V \in \mathscr{C}'$ such that V^u is a closed disk. Thus, we have a contradiction.

Now suppose R is an annulus. If an annulus at one end of R is contained in D, and U^u is connected to R at that end for some $U \in \mathscr{C}$, then D intersects every residue class of U^u. In particular, it intersects an open disk. Now apply the above argument. □

Theorem 2.40. *Suppose C is a smooth complete curve over a stable field K satisfying Hypothesis B, and $\mathscr{D}$ is a finite (possibly empty) collection of disjoint closed disks in C all defined over K. Then there exists a semistable covering $\mathscr{C}$ of C over some finite extension of K such that*

(1) *$D = U_D^u$ for some $U_D \in \mathscr{C}$ for each $D \in \mathscr{D}$, and*

(2) *$\mathscr{C} \setminus \{(U_D, D) : D \in \mathscr{D}\}$ is a semistable covering of $W := C \setminus \bigcup_{D \in \mathscr{D}} D$.*

Proof. If $\mathscr{D}$ is empty, or if $|\mathscr{D}| = 1$ and $g(C) = 0$, the theorem follows directly from Theorem 2.36 and [Deligne and Mumford 1969; Van der Put 1984].

Otherwise, suppose we have a semistable covering $\mathscr{C}$ of C that is *compatible* with $\mathscr{D}$, in the sense that each $D \in \mathscr{D}$ is either contained in a residue class of $\mathscr{C}$ or equal to U^u for some $U \in \mathscr{C}$. Then we can refine $\mathscr{C}$ to obtain a covering that satisfies the conclusions of the theorem. Indeed, suppose $D \in \mathscr{D}$ and $D \neq U^u$ for any $U \in \mathscr{C}$. Then D is contained in a residue class R of $\mathscr{C}$, and there are three possibilities to consider. First, D could be contained in a residue disk R of U^u for some $U \in \mathscr{C}$. In this case we refine our covering to

$$\mathscr{C}_D := \mathscr{C} \setminus \{(U, U^u)\} \cup \{(U \setminus D, U^u \setminus R), (R, D)\}.$$

The second possibility is that D is contained in a residue annulus R of some U^u. Applying Lemma 2.38 from above, there must then be a concentric circle T in R, and a residue disk S in T, such that $D \subseteq S$. In this case, we let

$$\mathscr{C}_D := \mathscr{C} \setminus \{(U, U^u)\} \cup \{(U \setminus T, U^u \setminus R), (R \setminus D, T \setminus S), (S, D)\}.$$

Finally, the residue class R that contains D may be a connected component of $U \cap V$ for two distinct $U, V \in \mathscr{C}$. Again there must be a concentric circle T in R and a residue disk S in T such that $D \subseteq S$. Let R_U and R_V be the connected components of $R \setminus T$ that are connected to U and V, respectively. Let $\hat{U} = (U \setminus R) \cup R_U$ and $\hat{V} = (V \setminus R) \cup R_V$. Then we may take as our refined cover

$$\mathscr{C}_D := \mathscr{C} \setminus \{(U, U^u), (V, V^u)\} \cup \{(\hat{U}, U^u), (\hat{V}, V^u), (R \setminus D, T \setminus S), (S, D)\}.$$

After applying this procedure finitely many times, we are done.

The only issue remaining is that of finding a compatible covering as a starting point. If $\mathscr{C}$ is a stable covering, then it is compatible with $\mathscr{D}$ by Corollary 2.39. So when $g \geq 2$ we are done by Theorem 2.36. If $g = 0$, and $D_1, D_2 \in \mathscr{D}$ with $D_1 \neq D_2$, then $\mathscr{C} := \{(C \setminus D_1, D_2), (C \setminus D_2, D_1)\}$ is compatible with $\mathscr{D}$. If $g(C) = 1$, $D \in \mathscr{D}$, and U is the largest open disk in C containing D, then $\mathscr{C} := \{(C \setminus D, C \setminus D), (U, D)\}$ is compatible with $\mathscr{D}$. So in each case we are able to construct the desired covering of C. □

Remark 2.41. If $g(C) \geq 2$, or $g(C) = 0$ and $|\mathscr{D}| \geq 3$, or $g(C) = 1$ and $|\mathscr{D}| \geq 1$, there exists a final object $\mathscr{C}_{\mathscr{D}}$ in the category of such coverings. In these cases, $C \setminus \bigcup_{D \in \mathscr{D}} U_D$ is the minimal underlying affinoid of $W := C \setminus \bigcup_{D \in \mathscr{D}} D$.

Corollary 2.42. *Let f be a meromorphic function with finitely many zeroes and poles on a wide open W over a stable field K satisfying Hypothesis B. Then there is a semistable covering $\mathscr{C}$ of W over a finite extension of K such that for each $U \in \mathscr{C}$, U^u has good reduction and all the zeroes and poles of f are contained in $\bigcup_{U \in \mathscr{C}} U^u$.*

Proof. Glue in disks to get a complete curve C. Let $\mathscr{D}$ be the union of $\mathrm{CC}(C \setminus W)$ with a finite collection of disjoint closed disks in W that contain the support of f. Apply the theorem to get a semistable covering $\mathscr{C}_1$ of C over some finite extension of K, and then throw out those $U \in \mathscr{C}_1$ for which $U^u \in \mathrm{CC}(C \setminus W)$. This yields a semistable covering $\mathscr{C}_2$ of W such that all the zeroes and poles of f are contained in $\bigcup_{U \in \mathscr{C}_2} U^u$. Let $\mathscr{S}$ be the collection of singular residue classes in U^u for all $U \in \mathscr{C}_2$. For each $R \in \mathscr{S}$, choose a concentric circle $A_R \subset R$ (such an R is an open annulus). Then

$$\mathscr{C} := \left\{ \left(U \setminus \bigcup_{R \in \mathscr{S}} A_R, U^u \setminus \bigcup_{R \in \mathscr{S}} R \right) : U \in \mathscr{C}_2 \right\} \cup \{(R, A_R) : R \in \mathscr{S}\}$$

satisfies the requirements of the corollary. □

Our final result of this section is a lemma that will play a key role in the proof of our main theorem, Theorem 9.2.

Lemma 2.43. *Suppose W is a connected wide open over a stable field K satisfying Hypothesis B, with minimal underlying affinoid W^u, and let $X \subset W$ be an affinoid subdomain with smooth irreducible (connected) reduction such that $g(W) = g(\overline{X}^c) > 0$. If X is connected to all but at most one component of $W \setminus W^u$, then W is a basic wide open and X is a Zariski subaffinoid of W^u.*

Proof. First glue disks to W to obtain a smooth connected complete curve C over K. Then by [Coleman 2005, Theorem A1][14], there exists a semistable model $\mathcal{T}$ of C over a finite extension E of K, and a subset S of the set of components of $\overline{\mathcal{T}}$ such that $X_E = X(\mathcal{T}, S)$. Moreover, there exists an $s \in S$ such that $X(\mathcal{T}, s)$ is a Zariski subaffinoid of X_E.

Let $\mathscr{C} := \mathscr{C}_{\mathcal{T}}$ be the semistable covering of C_E associated to $\mathcal{T}$ by Theorem 2.36. This implies by Proposition 2.34 that $\mathrm{Betti}(\Gamma_{\mathscr{C}}) = 0$ and $g(z) = 0$ for all $z \in S$ different from s. It follows that C_E has good reduction isomorphic to $\overline{X}^c_E$, that X_E is a Zariski subaffinoid of C_E, and that each affinoid disk in $C \setminus W$ is contained in a residue class of C_E. Furthermore, the statement that X is connected to all but at most one end of W implies that the elements of $C \setminus W$ lie in distinct residue classes of C_E, and that the complement of these residue classes is the minimal underlying affinoid of W_E that equals W^u_E.

We now know that $A^o(W^u_E) \cong A(W^u_E) \cap A^o(X_E)$, under restriction ρ, and that $A(W^u_E) \cong A(W^u) \otimes_K E$. Also, $A^o(W^u) = A^o(W^u_E) \cap A(W^u)$ and $A^o(X_E) = A^o(X) \otimes_{R_K} R_E$ because X has good reduction.

It follows that there is some nonzero element $m \in R_K$ with $|m| < 1$, such that $mA^o(W^u_E) \subset A^o(W^u) \otimes_{R_K} R_E$. So if $c \in A^o(W^u_E)$, then $\rho(c) = \sum_i r_i b_i$, where $r_1, \dots, r_n$ is a basis for R_E over R_K and $b_i \in A^o(X)$. It follows that $mb_i = \rho(a_i)$ for some $a_i \in \rho(A^o(W^u))$. Thus $a_i/m \in A^o(W^u_E)$, and so $a_i/m \in A^o(W^u)$. Therefore $A^o(W^u_E) = A^o(W^u) \otimes_{R_K} R_E$. This implies that W^u has good reduction, which completes the proof. □

2D. ***Riemann–Hurwitz for wide opens.*** For this entire section we assume that K is a stable field satisfying Hypothesis B. Let $\mathscr{A}$ be an *oriented* annulus over K. Suppose f is a function on $\mathscr{A}$, and ω a differential, each with no zeroes or poles in $\mathscr{A}(\mathbf{C})$. Then we define $\mathrm{ord}_{\mathscr{A}} f = \mathrm{res}_{\mathscr{A}} (\mathrm{d}f/f)$, which is an integer (see the proof of [Coleman 1989, Lemma 2.1]), and $\mathrm{ord}_{\mathscr{A}}\, \omega = \mathrm{ord}_{\mathscr{A}} (\omega/\mathrm{d}z)$ for any $z \in A(\mathscr{A})^*$ with $\mathrm{ord}_{\mathscr{A}} z = 1$ (which is independent of the choice of z). Using this definition, we can also define ord_e at any end e of a wide open W/K. Indeed, suppose ν is either a meromorphic function or differential on W, with finitely many zeroes and poles in $W(\mathbf{C})$. Over some finite extension L of K, W_L will have an underlying affinoid X_L containing the support of ν. Let $\mathscr{A}$ be the component of $W_L \setminus X_L$ corresponding to a fixed $e \in \mathscr{E}(W)$, and let $\psi : A_K(r, s) \to \mathscr{A}$ be an isomorphism such that $\psi(A_K(t, s))$ is connected to X whenever $r < t < s$. Then we define the *inherited orientation* on $\mathscr{A}$ by $\mathrm{res}_{\mathscr{A}} = \mathrm{res}_{r,s} \circ \psi^*$, and we set $\mathrm{ord}_e\, \nu = \mathrm{ord}_{\mathscr{A}}\, \nu$.

[14] The proof of this result was based on [Coleman 2003, Proposition 2.1], which is now a special case of Theorem 2.36.

Let $\mathrm{Div}(W) := \mathbb{Z}^{W(\mathbf{C})\cup\mathscr{E}(W)}$, and for any $D \in \mathrm{Div}(W)$ let

$$\deg D = \sum_{P\in W(\mathbf{C})} D(P) + \sum_{e\in\mathscr{E}(W)} D(e).$$

Then for ν as above, set $(\nu) = (\nu)_{\mathrm{fin}} + (\nu)_{\mathrm{inf}}$, where

$$(\nu)_{\mathrm{fin}} = \sum_{P\in W(\mathbf{C})} \mathrm{ord}_P\,\nu \quad \text{and} \quad (\nu)_{\mathrm{inf}} = \sum_{e\in\mathscr{E}(W)} \mathrm{ord}_e\,\nu.$$

Lemma 2.44. *Suppose f is a meromorphic function and ω is a meromorphic differential on $B(1) := B_{\mathbf{C}}(1)$, each with finitely many zeroes and poles, and each supported on $B[r] := B_{\mathbf{C}}[r]$ for some $r < 1$. Let $\mathscr{A} = A_{\mathbf{C}}(r, 1)$, oriented so that* $\mathrm{res}_{\mathscr{A}} = \mathrm{res}_{r,1}$ *(so* not *the inherited orientation from $B(1)$ as a wide open). Then*

$$\mathrm{ord}_{\mathscr{A}}\, f = \sum_{P\in B(1)} \mathrm{ord}_P\, f, \quad \mathrm{ord}_{\mathscr{A}}\,\omega = \sum_{P\in B(1)} \mathrm{ord}_P\,\omega, \quad \mathrm{res}_{\mathscr{A}}\,\omega = \sum_{P\in B(1)} \mathrm{res}_P\,\omega.$$

Proof. Let z be the natural parameter on $B(1)$. For the first equation, suppose f is supported on $\{P_1, \ldots, P_n\}$ with $\mathrm{ord}_{P_i} f = e_i$ and $z(P_i) = \alpha_i$. By the Weierstrass preparation theorem, we may write $f(z) = \prod_{i=1}^n (z-\alpha_i)^{e_i}\cdot u(z)$, where u is a unit. Then

$$\mathrm{ord}_{\mathscr{A}}\, f = \sum_{i=1}^{n} \mathrm{res}_{\mathscr{A}}\Big(\frac{e_i\,\mathrm{d}z}{z-\alpha_i}\Big) = \sum_{i=1}^{n} e_i = \sum_{P\in B(1)} \mathrm{ord}_P\, f.$$

The other two equations follow from essentially the same argument. ☐

Theorem 2.45. *Let f be a rigid function and ω a differential on W, each with finitely many poles and zeroes in $W(\mathbf{C})$. Then*

(i) $\deg(f) = 0$,

(ii) $\deg(\omega) = 2g(W) - 2$, *and*

(iii) $\sum_{P\in W(\mathbf{C})} \mathrm{res}_P\,\omega + \sum_{e\in\mathscr{E}(W)} \mathrm{res}_e\,\omega = 0$.

Proof. Attach disks at the ends of W to obtain a smooth projective curve C. For any rational function g on C, it follows from Lemma 2.44 that $\deg(g|_W) = 0$.

For more general f, suppose first that (W, X) is a basic wide open pair, X has good reduction, and $(f)_{\mathrm{fin}}$ is supported on X. In this case, there exists a $g \in \mathbb{O}_C$ and a Zariski subaffinoid Y of X such that f/g is regular on Y and $|(f/g) - 1|_Y < 1$ (in particular, we could choose Y so that f and g have no poles or zeroes on Y). It follows that there is a wide open V, with $Y \subset V \subseteq W$, such that (V, Y) is a basic wide open pair and $|(f/g) - 1|_V < 1$. Hence, $(f|_V) = (g|_V)$. Now, we have a natural map $\beta : \mathrm{Div}(W) \to \mathrm{Div}(V)$. Indeed, the elements of $\mathscr{E}(V)$ are in one-to-one correspondence with the connected components of $V\setminus Y$, which in turn are in one-to-one correspondence (by intersection) with the connected components of $W\setminus Y = (W\setminus X)\cup(X\setminus Y)$. Thus, as $e \in \mathscr{E}(W)$ corresponds to a unique component of

$W\backslash X$, it then corresponds to a unique end of V that we take to be $\beta(e)$. Similarly, if $P \in W(\mathbf{C})$, we let $\beta(P) = P$ if $P \in V(\mathbf{C})$, and the element of $\mathscr{E}(V)$ corresponding to the component of $X \setminus Y$ that contains P otherwise. Extend this map by linearity. Since $\deg \beta(D) = \deg D$ and $(f|_V) = \beta(f)$, we have

$$\deg(f) = \deg(f|_V) = \deg(g|_V) = 0.$$

To complete the proof that $\deg(f) = 0$, let $\mathscr{C}$ be a semistable covering of W such that U^u has good reduction and $(f|_U)_{\text{fin}}$ is supported on U^u for each $U \in \mathscr{C}$ (which exists by Corollary 2.42). Then

$$(f) = \sum_{U \in \mathscr{C}} (f|_U),$$

where we regard both sides as elements of

$$\{D \in \mathbb{Z}^{W(\mathbf{C}) \cup \bigcup_{U \in \mathscr{C}} \mathscr{E}(U)} : D(a) = -D(b) \text{ if } a \in \mathscr{E}(U), b \in \mathscr{E}(V), U \neq V, U_a = V_b\}.$$

Therefore,

$$\deg(f) = \sum_{U \in \mathscr{C}} \deg(f|_U) = 0.$$

Statements (ii) and (iii) are clearly true whenever $\omega = \eta|_W$ for $\eta \in \Omega^1_C$, by Lemma 2.44. Moreover, the general case of (ii) then follows from (i) and the fact that $(f\omega) = (f) + (\omega)$. Finally, the general case of (iii) will follow once we know it for basic wide opens, by an argument similar to that above.

So suppose (W, X) is a basic wide open pair, with X and C as above. For a reduced affinoid X over K and $\omega \in \Omega^1_{X/K}$, we set

$$|\omega|_X = \inf\{|a| : a \in K, \omega \in aA^o(X)\mathrm{d}A^o(X)\}.$$

Using Riemann–Roch, we can find $\eta \in \Omega^1_C$ such that $\omega - \eta$ has no poles on X and $|\omega - \eta|_X < \epsilon$. Note that $\omega - \eta$ extends to a regular differential on a wide open neighborhood V of X in W. Then statement (iii) for W follows from the general fact that if (V, X) is a basic wide open pair, $\omega \in \Omega^1_{V/K}$, $|\omega|_X < \epsilon$ and $e \in \mathscr{E}(V)$, then $|\mathrm{res}_e\, \omega| < \epsilon$. Indeed, let $T : U \to A(1, \infty)$ be a parameter on the component U of $V \setminus X$ corresponding to e, such that $|T(x)| \to 1$ as $x \to X$. Suppose on U

$$\omega = \sum_{n=-\infty}^{\infty} a_n T^n \,\mathrm{d}T.$$

Then $|\omega|_X = \max\{|a_n| : -\infty < n < \infty\}$. So $|\mathrm{res}_e\, \omega| = |a_{-1}| < \epsilon$. $\square$

Suppose $f : W \to V$ is a finite map. As f is finite, f naturally maps $\mathscr{E}(W)$ to $\mathscr{E}(V)$. For $a \in W(\mathbf{C}) \cup \mathscr{E}(W)$, let $\delta_f(a) = \mathrm{ord}_a\, f^*\mathrm{d}T$, where T is a parameter at $b := f(a)$ such that $\mathrm{ord}_b\, T = 1$. When a and b are ends, there exist annuli $\mathscr{A}$

and $\mathcal{B}$ at a and b such that f restricts to a finite étale map from $\mathcal{A}$ onto $\mathcal{B}$. Let $e_f(a)$ be the degree of this map. Otherwise, at a point in $W(\mathbf{C})$, let $e_f(a)$ denote the usual ramification index.

Lemma 2.46. *With notation as above, if ω is a differential with finitely many zeroes and poles on W, then*

$$\operatorname{ord}_a f^*\omega = e_f(a)\operatorname{ord}_b \omega + \delta_f(a).$$

Proof. First suppose $a \in \mathcal{E}(W)$, and let $\mathcal{A}$ and $\mathcal{B}$ be annuli at a and b such that ω is regular and nonvanishing on $\mathcal{B}$. Choose parameters S and T on $\mathcal{A}$ and $\mathcal{B}$ respectively, such that $\operatorname{ord}_a S = \operatorname{ord}_b T = 1$. Then $f^*T|_{\mathcal{A}} = S^e g(S)$ and $\omega|_{\mathcal{B}} = T^d h(T)\,\mathrm{d}T$, where g is a unit on $\mathcal{A}$ with $\operatorname{ord}_a g = 0$, h is unit on $\mathcal{B}$ with $\operatorname{ord}_b h = 0$, $e = e_f(a)$, and $d = \operatorname{ord}_b \omega$. So

$$(f^*\omega)|_{\mathcal{A}} = (S^e g(S))^d h(S^e g(S))\, f^*\,\mathrm{d}T,$$

from which the lemma follows.

The proof when $a \in W(\mathbf{C})$ is very similar. Let $\mathcal{A}$ and $\mathcal{B}$ be the stalks at a and b. Then after choosing uniformizers S and T, respectively, the map $f : \mathcal{A} \to \mathcal{B}$ is given by a homomorphism between formal power series rings over $\mathbf{C}$. Thus, we have $\omega = T^d h(T)\,\mathrm{d}T$ and $f^*T = S^e g(S)$, where $e = e_f(a)$, $d = \operatorname{ord}_b \omega$, and g and h are formal power series with nonzero constant terms. The lemma follows from the computation above and the fact that $h(S^e g(S))$ will again have nonzero constant term. □

Corollary 2.47. *Suppose that $|e_f(a)| = 1$ in K, or that $e_f(a) \neq 0$ in K and $a \in W(\mathbf{C})$. Then $\delta_f(a) = e_f(a) - 1$.*

Proof. Keeping the same notation as above, we compute $\delta_f(a)$ directly from the definition

$$\delta_f(a) = \operatorname{ord}_a \mathrm{d}T = \operatorname{ord}_a \mathrm{d}(S^e g(S)) = \operatorname{ord}_a(eg(S) + Sg'(S)) + e - 1.$$

When $a \in W(\mathbf{C})$ and $e_f(a) \neq 0$ in K, this equals $e-1$, since the power series $eg(S) + Sg'(S)$ must have nonzero constant term. On the other had, if $a \in \mathcal{E}(W)$ and $|e_f(a)| = 1$, it is straightforward to show that $eg(S) + Sg'(S)$ has constant absolute value on $\mathcal{A}$. So either way we are done. □

Theorem 2.48. *Let $f : W \to V$ be a finite map of wide opens of degree d. Then*

$$2g(W) - 2 = d(2g(V) - 2) + \sum_{a \in W(\mathbf{C}) \cup \mathcal{E}(W)} \delta_f(a).$$

Furthermore, under the hypotheses of Corollary 2.47, this is

$$d(2g(V) - 2) + \sum_{a \in W(\mathbf{C}) \cup \mathcal{E}(W)} (e_f(a) - 1).$$

Proof. Let ω be a nonzero meromorphic differential on V. Then the degree of $f^*\omega$ must be $2g(W)-2$ by Theorem 2.45. On the other hand, we obtain the right side of the equation if we compute $\deg(f^*\omega)$ using Lemma 2.46, Corollary 2.47, and

$$\sum_{\substack{x\in W(\mathbf{C})\\ f(x)=y}} e_f(x) = \sum_{\substack{a\in\mathscr{E}(W)\\ f(a)=b}} e_f(a) = d. \qquad \square$$

Proposition 2.49. *Suppose (W, W^u) and (V, V^u) are basic wide open pairs, and $f : W \to V$ is a finite map such that $f(W^u) = V^u$. Let X and Y be the completions of $\overline{W^u}$ and $\overline{V^u}$, and let $\bar{f} : X \to Y$ be the induced map. If $\bar{f}$ is separable, then*

$$\delta_f(a) = \operatorname{length}(\Omega_{X/Y})_a.$$

Proof. First, we can lift $\bar{f}$ to a map $g : C_X \to C_Y$ between complete liftings of X and Y. There must exist wide open neighborhoods $W' \subseteq W$ and $V' \subseteq V$ of W^u and V^u, and embeddings $\phi_X : W' \to C_X$ and $\phi_Y : V' \to C_Y$, such that $f(W') = V'$, $\overline{\phi_X|_{W^u}}$ and $\overline{\phi_Y|_{V^u}}$ are the natural inclusions, and

$$\overline{g \circ \phi_X|_{W^u}} = \overline{\phi_Y \circ f|_{W^u}}.$$

Now, Hartshorne's version [1977, Corollary 2.4] of Hurwitz's theorem implies the proposition for

$$h := \phi_Y^{-1} \circ g \circ \phi_X : W' \to V'.$$

The proposition follows because $\delta_f(a) = \delta_f(a') = \delta_g(a'')$, where a' is the component of $W' \setminus W^u$ corresponding to a and $a'' = \phi_X(a')$. $\square$

3. $X_0(p^n)$ and its subspaces

Now that the rigid analytic foundation has been laid, we turn our focus specifically to the curve $X_0(p^n)$, which we will always think of in moduli-theoretic terms. More precisely, we think of $X_0(p^n)$ as the rigid analytic curve over $\mathbb{Q}_p$ whose points over $\mathbb{C}_p$ are in a one-to-one correspondence with (isomorphism classes of) pairs (E, C), where $E/\mathbb{C}_p$ is a generalized elliptic curve and C is a cyclic subgroup of order p^n. We implicitly make use of this correspondence when we speak loosely of "the point (E, C)". There are various natural maps from $X_0(p^n)$ to $X_0(p^m)$ that can be defined by way of this moduli-theoretic interpretation of points, and we begin this section by fixing notation for these fundamental maps.

Definition 3.1. First let

$$\pi_f, \pi_v : \coprod_{n\geq 1} X_0(p^n) \to \coprod_{n\geq 0} X_0(p^n)$$

be the maps given by $\pi_f(E, C) = (E, pC)$ and $\pi_v(E, C) = (E/C[p], C/C[p])$, where $C[p^i]$ is the kernel of multiplication by p^i in C. Then by letting $\pi_{ab} =$

$\pi_f^b \circ \pi_v^a$, we get maps

$$\pi_{ab} : \coprod_{n \geq a+b} X_0(p^n) \to \coprod_{n \geq 0} X_0(p^n).$$

Remark 3.2. This definition is identical to the definition in [Coleman 2005, §1]. We also note that over $\mathbb{C}$, π_{ab} corresponds to the map on the upper half plane that takes z to $p^a z$.

Another map crucial to this paper is the Atkin–Lehner involution,

$$w : \coprod_{n \geq 0} X_0(p^n) \to \coprod_{n \geq 0} X_0(p^n),$$

which is defined by the formula

$$w_n(E, C) = (E/C, E[p^n]/C), \quad \text{where } w_n := w|_{X_0(p^n)}.$$

The Atkin–Lehner involution is compatible with the level-lowering maps in the sense that $\pi_f \circ w = w \circ \pi_v$ (or equivalently, $w \circ \pi_f = \pi_v \circ w$, since w is an involution).

3A. ***Canonical subgroups and supersingular annuli.*** We now introduce some natural rigid subspaces of $X_0(p^n)$ over finite extensions of $\mathbb{Q}_p$ using the theory of the canonical subgroup, which we now review and extend [Buzzard 2003, §3].[15] If E is an elliptic curve over $\mathbb{C}_p$, we let $h(E)$ denote the minimum of 1 and the valuation of a lifting of the Hasse invariant of the reduction of a nonsingular model of E mod p, if it exists, and 0 otherwise[16] (this is denoted by $v(E)$ in [Buzzard 2003]). Katz [1973, §3] constructed a rigid analytic section s_1 of $\pi_f : X_0(p) \to X(1)$ over the wide open W_1 whose $\mathbb{C}_p$-valued points are represented by generalized elliptic curves E such that $h(E) < p/(p+1)$, when $p \geq 5$. Both W_1 and s_1 are defined over $\mathbb{Q}_p$. Changing notation slightly from [Buzzard 2003], we let $K_1(E) \subseteq E$ denote the subgroup of order p for which $s_1(E) = (E, K_1(E))$, and we call $K_1(E)$ the canonical subgroup of order p.

Using [Buzzard 2003, Theorem 3.3], we can also define canonical subgroups of higher order. For $n \geq 1$, we generalize W_1 by taking W_n to be the wide open in $X(1)$ where $h(E) < p^{2-n}/(p+1)$ (the complement of finitely many affinoid disks, one in each supersingular residue class). For $E \in W_n$ we then define $K_n(E)$ inductively, as in [ibid., Definition 3.4], as the preimage of $K_{n-1}(E/K_1(E))$ under the natural projection from $E \to E/K_1(E)$. This is a cyclic subgroup when $E \in W_n$

[15] Although Buzzard works over a complete discrete valuation ring, all of his results can be extended to complete local rings.

[16] As pointed out in [BL 1985, Remark 6.4], the good reduction of E is well defined if it exists.

by [ibid., Theorem 3.3], and we call it the canonical subgroup of order p^n.[17] Thus, when E has supersingular reduction, either $h(E) \geq p/(p+1)$ (and E is too supersingular in the language of [ibid.]), or there is a largest $n \geq 1$ for which $K_n(E)$ can be defined. In the first case, we define the *canonical subgroup* of E, written $K(E)$, to be the trivial subgroup, and in the second we let $K(E) = K_n(E)$ for this largest n. Whenever $E/\mathbb{C}_p$ has ordinary reduction (by this we mean ordinary good or multiplicative[18]) we let $K(E)$ be the p-power torsion of E that is contained in the kernel of reduction, which does not depend on the good or multiplicative model.

It is important to note that s_1 also generalizes, in the sense that the map defined by $s_n(E) = (E, K_n(E))$ is also a rigid analytic section of $\pi_{0n} : X_0(p^n) \to X(1)$ over the wide open W_n. To see this, first regard $X_0(p^n)$ for $n > 1$ as the normalization of the fiber product of $X_0(p^{n-1})$ with itself over $X_0(p^{n-2})$ via the maps π_f and π_v. More specifically, let

$$\psi_n : X_0(p^n) \to X_0(p^{n-1}) \times_{\pi_f, \pi_v} X_0(p^{n-1})$$

be the isomorphism described by $\psi_n = (\pi_v, \pi_f)$ (after normalization of the right side). Now assume that s_{n-1} is rigid analytic. With [Buzzard 2003, Theorem 3.3], it is straightforward to verify that over W_n, we have

$$\pi_f \circ s_{n-1} \circ \pi_{1n-2} \circ s_{n-1} = \pi_v \circ s_{n-1}.$$

Thus we may define a rigid analytic map from W_n to $X_0(p^n)$ by

$$s_n := \psi_n^{-1} \circ (s_{n-1} \circ \pi_{1n-2} \circ s_{n-1}, s_{n-1}).$$

Again by the same theorem, we see that this map does indeed take E to $(E, K_n(E))$. So by induction we are done. Note that both W_n and s_n are defined over $\mathbb{Q}_p$.

Another way to focus on rigid subspaces of $X_0(p^n)$ is to fix the isomorphism class of the reduction of E. In particular, we make the following definition.

Definition 3.3. For a fixed elliptic curve A over a finite field $\mathbb{F}$, we let $W_A(p^n)$ represent the rigid subspace of $X_0(p^n)$ (over $\mathbb{Q}_p \otimes W(\mathbb{F})$) whose points over $\mathbb{C}_p$ are represented by pairs (E, C) with $\overline{E} \cong A$.

Of course, $W_A(1)$ (for any A) is just a residue disk of the j-line. When A is a supersingular curve, it is well known that $W_A(p)$ is isomorphic, over $\mathbb{Q}_{p^2} := W(\mathbb{F}_{p^2}) \otimes \mathbb{Q}_p$, to an open annulus of width $i(A) := |\mathrm{Aut}(A)|/2$. This means that one can choose a parameter x_A on $W_A(p)$ over $\mathbb{Q}_{p^2}$ that identifies it with the (open)

[17]Thinking of the kernel of reduction of E as a disk, the set of points of order p^n are not always equidistant from the identity. When $E \in W_n$, $K_n(E)$ is the union over $i \leq n$ of those points of order p^i that are closest to the identity.

[18]Equivalently, $j(E)$ is not congruent to a supersingular j-invariant modulo $m_{\mathbb{R}_p}$.

annulus $A_{\mathbb{Q}_{p^2}}(p^{-i(A)},1)$. In fact, we can and will always do this in such a way that $v(x_A(E,C))=i(A)h(E)$ when $C=K_1(E)$, and $i(A)(1-h(E/C))$ otherwise (this is justified in [Buzzard 2003, Theorem 3.3 and §4]).

Now, inside the annulus, $W_A(p)$, there are three concentric circles that will be essential for our analysis of $X_0(p^2)$ and $X_0(p^3)$. First there is the *too-supersingular circle*, which we denote by $\mathbf{TS}_A$, whose points correspond to pairs (E,C), where the canonical subgroup of E is trivial. Equivalently, these are the points with $h(E)\geq p/(p+1)$. Next there is the *self-dual circle*, denoted by $\mathbf{SD}_A$, whose points correspond to pairs (E,C), where the subscheme C of order p is potentially self-dual, that is, isomorphic to its Cartier dual after finite base extension. Equivalently, $\mathbf{SD}_A$ consists of those points that satisfy $h(E)=1/2$ and $C=K_1(E)$. When $A/\mathbb{F}_p$, $\mathbf{SD}_A$ can also be described as the unique circle in $W_A(p)$ that is fixed by the involution w_1, and hence we call it the *Atkin–Lehner circle*. Finally, we must also consider what might be called the *anti-Atkin–Lehner circle*. It is the subspace $\mathbf{C}_A\subseteq W_A(p)$ whose points correspond to pairs (E,C') for which there exists a C such that $(E,C)\in\mathbf{SD}_A$ but $C'\neq C$. We let $\tau_f:\mathbf{C}_A\to\mathbf{SD}_A$ be the map that corresponds to replacing the cyclic subgroup C' with $K_1(E)$. Then τ_f is rigid analytic since it is the restriction of $s_1\circ\pi_f$.

Remark 3.4. The fact that the regions above are circles follows from Buzzard's discussion of rigid subspaces of $X_1(p)$ [2003, §4]. Using a parameter x_A chosen as above, the circles $\mathbf{TS}_A$, $\mathbf{SD}_A$ and $\mathbf{C}_A$ are those where $v(x_A)/i(A)$ equals $p/(p+1)$, $1/2$, and $1-1/(2p)$, respectively.

From above, whenever $A/\mathbb{F}_p$ is supersingular, $W_A(p)$ is an annulus preserved by the Atkin–Lehner involution w_1, and which is mapped onto the residue disk $W_A(1)$ via π_f. In our analysis of the stable models of $X_0(p^2)$ and $X_0(p^3)$, we will need to work with fairly explicit approximations for the restrictions of π_f and w_1 to these subspaces:

Theorem 3.5. *Let $\mathbb{Z}_{p^2}:=W(\mathbb{F}_{p^2})$ and let $A/\mathbb{F}_p$ be a supersingular curve with $j(A)\neq 0, 1728$. Then there are parameters s and t over $\mathbb{Z}_{p^2}$ that identify $W_A(1)$ with the disk $B_{\mathbb{Q}_{p^2}}(1)$ and $W_A(p)$ with the annulus $A_{\mathbb{Q}_{p^2}}(p^{-1},1)$, and there are series $F(T), G(T)\in T\mathbb{Z}_{p^2}[[T]]$ such that*

(i) $w_1^*(t)=\kappa/t$ *for some* $\kappa\in\mathbb{Z}_{p^2}$ *with* $v(\kappa)=1$, *and*

(ii) $\pi_f^*s=F(t)+G(\kappa/t)$, *where* $F'(0)\equiv 1 \pmod{p}$ *and* $G(T)\equiv(F(T))^p \pmod{p}$.

Proof. One only has to translate results in [de Shalit 1994, §2, §3]. Our t and κ are de Shalit's y and π. Then our π_f^*s is de Shalit's $\psi(y)-\beta_0$. The theorem follows from [de Shalit 1994, (4) of §2, Lemma 1 and Corollaries 2–4 of §3]. ☐

Note that the parameter t from Theorem 3.5 is a suitable choice for x_A. This follows from condition (ii), which guarantees that π_f has degree $p+1$ on the circle where $v(t) = p/(p+1)$ and has degree 1 or p on all other concentric circles.

3B. ***Neighborhoods of the ordinary locus.*** The finitely many subspaces $W_A(p^n)$ (defined above), where A runs over supersingular curves over $\mathbb{F}_{p^2}$, cover the supersingular locus of $X_0(p^n)$ over $\mathbb{Q}_{p^2}$, that is, the subspace whose points over $\mathbb{C}_p$ correspond to pairs (E, C), where E has supersingular reduction. Furthermore, these subspaces become connected wide opens over $\mathbb{C}_p$ by Theorem 2.29. We will now describe a finite collection $W^{\pm}_{ab} \subseteq X_0(p^n)$ of subspaces that cover the ordinary locus. These will, in fact, be shown to be basic wide opens when $n \leq 3$, and we do expect this to hold more generally. Essentially, we extend the irreducible affinoids, $\mathbf{X}^{\pm}_{ab}$ (introduced in [Coleman 2005][19]), to wide open neighborhoods, by considering points (E, C) that are *nearly* ordinary in the sense that either $K(E)$ or $K(E/C)$ is large.

More precisely, for $a \geq b \geq 0$ with $a + b = n$, we start by letting

$$W_{ab} = \{(E, C) : |K(E)| \geq p^n, |K(E) \cap C| = p^a\}.$$

For $b > a \geq 0$ with $a+b=n$, we then define $W_{ab} = w_n(W_{ba})$. Now we show that the pairing on $K_a(E)$, which was defined in [Coleman 2005] for points $(E, C) \in W_{ab}$ where E has ordinary reduction, carries over to all points in W_{ab}. Let (E, C) be a point of W_{ab} with $a \geq b$, and let $A, B \in K_a(E)$. Then by the definition of W_{ab}, we can choose $P \in C$ and $Q \in K_n(E)$ such that $p^b P = A$ and $p^b Q = B$. Now set $\mathcal{P}_{E,C}(A, B) = e_n(P, Q)$, where $e_n(\cdot\,, \cdot)$ denotes the Weil pairing on $E[p^n]$. This gives a well-defined pairing of $K_a(E)$ with itself onto μ_{p^b}. Furthermore, if $p > 2$, there are exactly two isomorphism classes of pairings on $\mathbb{Z}/p^a\mathbb{Z}$ onto μ_{p^b} whenever $b > 0$. Let $e^+(\cdot\,, \cdot)$ and $e^-(\cdot\,, \cdot)$ be representatives for these classes. Then, essentially repeating the argument from [Coleman 2005] for the ordinary affinoids, $X^{\pm}_{ab}$, there is a rigid subspace $W^{\pm}_{ab}$ of $X_0(p^n)$ defined over $\mathbb{Q}_p(\sqrt{(-1)^{(p-1)/2}})$ whose $\mathbb{C}_p$-valued points are

$$\{(E, C) \in W_{ab} \mid (K_a(E), \mathcal{P}_{E,C}) \cong (\mathbb{Z}/p^a\mathbb{Z}, e^{\pm})\}.$$

Set $W^+_{n0} = W^-_{n0} = W_{n0}$, and for $b > a \geq 0$, set $W^{\beta}_{ab} = w_n(W_{ba}^{(\frac{-1}{p})\beta})$.

Thus, $\mathbf{X}^{\pm}_{ab}$ is just the affinoid whose points are those $(E, C) \in W^{\pm}_{ab}$ for which E has ordinary or multiplicative reduction. It is *not* immediate that $W^{\pm}_{ab}$ is a basic wide open with $\mathbf{X}^{\pm}_{ab}$ as a minimal underlying affinoid. We will show that this is the case, however, when $n \leq 3$, and we do expect it to hold for arbitrary n as well.

[19]When $a < b$, the X^{β}_{ab} here is the same as $X_{ab}^{(\frac{-1}{p})\beta}$ from [Coleman 2005].

The affinoid $\mathbf{X}_{ab}^{\pm}$ is well understood from results of [Coleman 2005]. In particular, we have the following result, proven but not made explicit therein.

Proposition 3.6. *The affinoid* $\mathbf{X}_{ab}^{\pm}$ *with* $a \geq b > 0$ *is defined and has good reduction over* $\mathbb{Q}_p(\mu_{p^b})$.

Proof. It was proven in [Coleman 2005, §0] that $\mathbf{X}_{ab}^{\pm}$ is an affinoid defined over the quadratic subfield of $\mathbb{Q}_p(\mu_p)$. For $\zeta \in \mu_{p^b}$, we can define an embedding a_ζ of $\mathbf{X}_{ab}^{\pm}$ onto an affinoid in $X_1(p^b)^{\mathrm{bal}}$ by taking $a_\zeta(E, C)$ to be the point that is represented by the balanced $\Gamma_1(p^b)$-structure [Katz and Mazur 1985, (3.3)]

$$P, E \underset{\check{\alpha}}{\overset{\alpha}{\rightleftarrows}} E/C, P'.$$

Here we have $P \in K_b(E)$, $\mathcal{P}_{E,C}(P, P) = \zeta$, and $P' = \alpha(Q)$ for some $Q \in E[p^b]$ such that $(P, Q) = \zeta$. This image affinoid reduces to $\mathrm{Ig}(p^b)$ by (the extension to level 1 of) [Katz and Mazur 1985, p. 450].[20] $\square$

Corollary 3.7. *The affinoid* $\mathbf{X}_{ab}^{\pm}$ *with* $a + b = n$ *is defined and has good reduction over* $\mathbb{Q}_p(\mu_{p^{\lfloor n/2 \rfloor}})$.

Proof. When $a \geq b$, this follows immediately from Proposition 3.6. Otherwise, apply w_n first. $\square$

4. Formal groups

In the previous section we defined a finite collection $W_A(p^n)$ of connected wide opens that cover the supersingular locus of $X_0(p^n)$. Unfortunately, $W_A(p^n)$ is only basic when $n \leq 2$. Therefore, to arrive at a stable covering of $X_0(p^n)$, it is necessary to use smaller subspaces of $W_A(p^n)$. One approach is to use canonical subgroup considerations, as in Section 3A. Another is to use the interpretation from [Lubin et al. 1964] of an elliptic curve, over a complete local ring R with residue characteristic p, as a lifting of some formal group in characteristic p. In particular, this will enable us to use explicit formulas of Hopkins and Gross, which we recall in Section 4B.

Theorem 4.1 (Woods Hole theory). *Suppose R is the ring of integers in a complete subfield of $\mathbb{C}_p$, with residue field $\mathbb{F}$. The category of elliptic curves over R is equivalent to the category of triples (F, A, α), where F is a formal group over R, A is an elliptic curve over $\mathbb{F}$, and $\alpha : \overline{F} \to \hat{A}$ is an isomorphism. A morphism between two triples, (F, A, α) and (F', A', β), is a pair (σ, τ), where $\sigma : F \to F'$*

[20] $\mathrm{Ig}(p^b)$ is the Igusa curve in characteristic p that classifies pairs (E, ψ), where E is an elliptic curve and $\psi : \mu_{p^b} \hookrightarrow E$ (studied in [Igusa 1968]).

and $\tau : A \to A'$ are homomorphisms such that the following diagram commutes.

$$\begin{tikzcd} \bar{F} \arrow[r, "\bar{\sigma}"] \arrow[d, "\alpha"'] & \bar{F}' \arrow[d, "\beta"] \\ \hat{A} \arrow[r, "\hat{\tau}"'] & \hat{A}' \end{tikzcd}$$

Proof. If E is an elliptic curve over R, let $\mathcal{F}_R(E) = (\hat{E}, \bar{E}, \iota)$, where $\iota : \bar{\hat{E}} \to \hat{\bar{E}}$ is the natural isomorphism. This is a functor, compatible with changing R, from the first category to the second. We claim this is an equivalence of categories. The analogous statement is proven when R is a local Artinian ring with residue field of characteristic p in [Lubin et al. 1964, §6]. Then on [Lubin et al. 1964, p. 7], it is explained that by "passing to the limit… one sees that it continues to hold over a complete local Noetherian ring". Thus the theorem is true when R is the ring of integers in a complete discretely valued subfield of $\mathbb{C}_p$.

To obtain it more generally, we apply [Lubin et al. 1964, Theorem 1]. This theorem implies that given an elliptic curve A over an algebraic extension of $\mathbb{F}_p$, the collection of liftings of $\hat{A}$ to $\mathbb{R}_p$ is naturally the set of points in a wide open disk $\mathcal{D}$. On the other hand, the set of liftings of A to $\mathbb{R}_p$ is the set of points in a residue disk $\mathcal{R}$ of $X(1)$ and $\mathcal{F}$ yields a degree one rigid analytic map from $\mathcal{R}$ to $\mathcal{D}$ with dense image. Hence it is an isomorphism. $\square$

In light of this theorem, we may think of points $(E, C) \in W_A(p^n)$ as triples (F, C, α), where F is a formal group, $C \subseteq F$ is a cyclic subgroup of order p^n, and $\alpha : \bar{F} \to \hat{A}$ is an isomorphism. We then refer to such a triple as a Woods Hole representation of (E, C). There are two specific ways in which we apply this theory. First of all, from the fact that all supersingular elliptic curves are p-prime isogenous, we are able to show that all supersingular regions $W_A(p^n)$ for a fixed p and n are *nearly* isomorphic. Along with the result in Appendix B, this enables us to do all of our calculations under the simplifying assumption that $A/\mathbb{F}_p$ and $j(A) \neq 0, 1728$. Secondly, we use extensively the natural action of the p-adic group $\mathrm{Aut}(\hat{A})$ on $W_A(p)$, which was studied in detail in [Hopkins and Gross 1994].

4A. *All supersingular regions are (nearly) isomorphic.*

Proposition 4.2. *Let A and $A'/\mathbb{F}_{p^2}$ be two supersingular elliptic curves, with $j(A)$ not equal to 0 or 1728. Let $\mathbb{F}/\mathbb{F}_{p^2}$ be a finite extension over which A and A' are p-prime isogenous (which always exists). Then the wide open $W_{A'}(p^n)$ is isomorphic over $W(\mathbb{F}) \otimes \mathbb{Q}_p$ to the quotient of $W_A(p^n)$ by a faithful action of* $\mathrm{Aut}(A')/\{\pm 1\}$.

Proof. Suppose $\iota : A \to A'$ is an isogeny of degree prime to p over $\mathbb{F}$. Since $(\deg \iota, p) = 1$, the induced map $\hat{\iota} : \hat{A} \to \hat{A}'$ is an isomorphism of formal groups.

So in Woods Hole terms we may define a map $\psi_\imath : W_A(p^n) \to W_{A'}(p^n)$ by taking

$$\psi_\imath(F, C, \alpha) = (F, C, \hat{\imath} \circ \alpha).$$

To show that the map is, in fact, well defined, suppose that the triples (F_1, C_1, α_1) and (F_2, C_2, α_2) represent the same point of $W_A(p^n)$. This means that there are isomorphisms $\gamma : F_1 \to F_2$ (mapping C_1 to C_2) and $\tau : A \to A$ such that $\alpha_2 \circ \overline{\gamma} = \hat{\tau} \circ \alpha_1$. Because $j(A) \neq 0, 1728$, we know that $\tau = \pm 1$. Therefore $\hat{\tau}$ commutes with all isogenies. In particular, composing with $\hat{\imath}$ on both sides, we have

$$\hat{\imath} \circ \alpha_2 \circ \overline{\gamma} = \hat{\tau} \circ \hat{\imath} \circ \alpha_1.$$

Therefore $(F_1, C_1, \hat{\imath} \circ \alpha_1)$ and $(F_2, C_2, \hat{\imath} \circ \alpha_2)$ are Woods Hole representations of the same point in $W_{A'}(p^n)$, and $\psi_\imath$ is well defined.

To show that $\psi_\imath$ is onto, choose any point of $W_{A'}(p^n)$ and let (F, C, β) be one of its Woods Hole representations (so $\beta : \overline{F} \to \hat{A}'$ is an isomorphism). Since $\hat{\imath}$ is an isomorphism, we can choose a point of $W_A(p^n)$ by taking $(F, C, \hat{\imath}^{-1} \circ \beta)$, and this point maps onto our chosen point of $W_{A'}(p^n)$ by definition. (Note, however, that this does *not* define a map from $W_{A'}(p^n)$ to $W_A(p^n)$, since our original choice of triple was noncanonical.)

Finally, suppose that two points of $W_A(p^n)$, represented by (F_1, C_1, α_1) and (F_2, C_2, α_2), have the same image in $W_{A'}(p^n)$. Then there must be isomorphisms $\gamma : F_1 \to F_2$ (taking C_1 to C_2) and $\tau : A' \to A'$ such that

$$\hat{\imath} \circ \alpha_2 \circ \overline{\gamma} = \hat{\tau} \circ \hat{\imath} \circ \alpha_1 \quad \text{and} \quad \alpha_2 \circ \overline{\gamma} = \widehat{\mathrm{id}} \circ (\hat{\imath}^{-1} \circ \hat{\tau} \circ \hat{\imath}) \circ \alpha_1.$$

In particular, $\tau \mapsto ((F, C, \alpha) \mapsto (F, C, \hat{\imath}^{-1} \tau \hat{\imath} \circ \alpha))$ gives a faithful action of $\mathrm{Aut}(A')/\{\pm 1\}$ on the fibers of ψ. □

Remark 4.3. Suppose now that $\mathbb{F} \supseteq \mathbb{F}_{p^2}$ is a field over which all supersingular curves are p-prime isogenous. It follows, then, that all of the regions $W_A(p^n)$ are nearly isomorphic over $W(\mathbb{F}) \otimes \mathbb{Q}_p$. We showed in [CM 2006, Theorem 5.5] that this $\mathbb{F}$ can always be taken to be $\mathbb{F}_{p^{24}}$.

4B. ***Woods Hole action and Gross–Hopkins theory.*** The other way we use Woods Hole theory is to define a continuous action of a p-adic group on $W_A(p^n)$. In particular, when A is a supersingular elliptic curve, it is well known [Tate 1966, Main Theorem] that

$$B := \mathrm{End}(\hat{A}) \cong \mathbb{Z}_p[i, j, k],$$

where i^2 is a quadratic nonresidue, $j^2 = -p$, and $ij = -ji = k$. Furthermore, we may take j to be the Frobenius endomorphism whenever A is defined over $\mathbb{F}_p$. Then $B^* = \mathrm{Aut}(\hat{A})$ acts on $W_A(p^n)$ by

$$\rho(F, C, \alpha) = (F, C, \rho \circ \alpha) \quad \text{for } \rho \in B^*.$$

Remark 4.4. The subgroup $\mathbb{Z}_p^* \subseteq B^*$ acts trivially on $W_A(p^n)$. Indeed, for $\rho \in \mathbb{Z}_p^*$, just take $\sigma = \rho^{-1}$ and $\tau = \mathrm{id}$ in Theorem 4.1. Not only does this define an isomorphism between (F, α) and $(F, \rho \circ \alpha)$, but in fact the isomorphism leaves invariant the subgroups of F of order p^n.

Hopkins and Gross studied the analogous action for deformation spaces of finite height formal groups, and explicitly computed the action in the height 2 case in [1994, §25]. To better understand their results (and translate them into our setting), we now offer a brief review of their theory under suitable simplifying assumptions. First, let K be a finite unramified extension of $\mathbb{Q}_p$ with residue field $\mathbb{F} \supseteq \mathbb{F}_{p^2}$, and let $F_0/\mathbb{F}$ be a fixed height 2 formal group. They show that there is a rigid space over K, denoted by X_K, whose L-valued points for any finite extension L of K correspond to liftings of F_0 to a formal group over $\mathbb{O}_L$. Here two liftings are equivalent (say, (G_1, γ_1) and (G_2, γ_2) with $\gamma_i : \overline{G}_i \xrightarrow{\sim} F_0$) if there is an isomorphism between them that induces the identity on F_0. Then $\mathrm{Aut}(F_0)$ acts (rigid analytically) on X_K in the same manner as above, and Hopkins and Gross make this action completely explicit with their crystalline period mapping

$$\Phi : X_K \to \mathbf{P}^1_K,$$

which can be understood as follows. Again, it is well known that $B := \mathrm{End}(F_0)$ is isomorphic to the maximal order of some quaternion algebra over $\mathbb{Q}_p$, and hence $B \otimes K$ is (noncanonically) isomorphic to $M_{2\times 2}(K)$. Since the image of B^* in $M_{2\times 2}(K)$ must take lines to lines, we thus obtain an action of B^* on $\mathbf{P}^1_K$. Hopkins and Gross define the (rigid analytic) map Φ and decompose $B \otimes K$ in such a way that $\Phi(\rho x) = \Phi(x)^\rho$ for all $\rho \in B^*$, that is, Φ is B^*-equivariant. So the beauty of their theory is that the action of B^* on X_K can be concretely expressed in terms of linear algebra.

Indeed, suppose $A/\mathbb{F}_p$ with $j(A) \neq 0, 1728$. Then X_K and $W_A(1)$ are naturally isomorphic over the unramified quadratic extension K of $\mathbb{Q}_p$, and we may decompose B as $R \oplus Rj$, where

$$R = \mathbb{Z}_p[i] \cong \mathbb{O}_K$$

and j is the Frobenius endomorphism of A (as above). Then by [Hopkins and Gross 1994, §25], $\rho = \alpha + j\beta \in B^*$ (with $\alpha, \beta \in \mathbb{Z}_p[i]$) acts on $\mathbf{P}^1(K) = \Phi(X_K)$ via multiplication on the *right* by the matrix

$$\begin{bmatrix} \alpha & -p\bar{\beta} \\ \beta & \bar{\alpha} \end{bmatrix}. \tag{3}$$

Of course, this formula only completely defines the action of B^* on B^*-stable subspaces of $W_A(1)$ on which Φ is an injection (for example, the canonical liftings of [Gross 1986, 2.1]). Hopkins and Gross [1994, 25.12] specify an affinoid disk $Y \subseteq W_A(1)$ for which this is the case, and which maps via Φ onto the disk $v(t) \geq 1/p$, where t is the parameter on $\mathbf{P}^1$ corresponding to the row vector $[1, t]$. This parameter is distinct from the parameter on $W_A(p)$ from Theorem 3.5. However, from the explicit action of B^* on $\Phi(Y)$ and the fact that this t vanishes at some canonical lifting (which is necessarily too supersingular), there is significant compatibility between the two. In particular, it is clear that the canonical section of $\pi_f : W_A(p) \to W_A(1)$ exists over the annulus in $Y \subseteq W_A(1)$ described by $1/p < v(t) < p/(p+1)$ and preserves valuations with respect to the two parameters. As B^* acts equivariantly with respect to π_f, the upshot of all this is that B^* acts on the subannulus of $W_A(p)$ that is identified via $\Phi \circ \pi_f$ with the annulus $1/p < v(t) < p/(p+1)$ according to

$$\rho(t) = \frac{-p\bar{\beta} + \bar{\alpha}t}{\alpha + \beta t}, \quad \text{where } \rho = \alpha + j\beta. \tag{4}$$

In particular, we are most interested in the action of B^* on the Atkin–Lehner circle (equivalently, where $v(t) = 1/2$). The following proposition and remark summarize the specific results (still assuming that $A/\mathbb{F}_p$ and $j(A) \neq 0, 1728$) which we will need for our explicit analysis of $X_0(p^3)$.

Definition 4.5. For $\rho = \alpha + j\beta$ (as above), let $\rho' = \bar{\alpha} + j\bar{\beta}$, and let B' be the set of all $\rho \in B^*$ such that $\rho\rho' \in \mathbb{Z}_p^*$. Alternatively, B' is just the set of all $\rho \in B^*$ with $\rho = a + bi + dk$.

Proposition 4.6. *Let $j(A) \neq 0, 1728$. For any $\rho \in B^*$, let $w_\rho := \rho \circ w_1$. Then w_ρ is an automorphism of $\mathbf{SD}_A$ with two fixed points and is an involution exactly when $\rho \in B'$.*

Proof. If $(E_2, K(E_2)) = w_1(E_1, K(E_1))$ are two points of $\mathbf{SD}_A$, this means that there is a degree p isogeny $f : E_1 \to E_2$ with kernel $K(E_1)$. Since A is supersingular with $\mathrm{Aut}(A) = \pm 1$, f can only induce $\pm j$ in $\mathrm{End}(A)$ (and hence in $B = \mathrm{End}(\hat{A})$). Now, $j \notin B^*$, but the full group, $(B \otimes K)^\times$, acts equivariantly on $\Phi(X_K)$ by [Hopkins and Gross 1994, 23.11]. So this means that on $\mathbf{SD}_A$ we may identify w_1 with $\pm j$ (and the sign is irrelevant).

To determine when w_ρ is an involution, we first verify that $\rho \circ w_1 = w_1 \circ \rho'$ (equivalently, $\rho j = j\rho'$). This shows that w_ρ^2 acts like $\rho\rho'$, and only $\mathbb{Z}_p^* \subseteq B^*$ acts trivially from (4). So w_ρ is an involution exactly when $\rho \in B'$. In particular, w_ρ is given by

$$w_\rho(t) = \frac{-p\bar{\alpha} - p\bar{\beta}t}{-p\beta + \alpha t}, \tag{5}$$

and the explicit formula shows that in any case w_ρ has two fixed points. □

Remark 4.7. To better understand how $\rho \in B^*$ and w_ρ act on $\overline{\mathbf{SD}}_A$, we could choose the parameter $u = t/\sqrt{-p}$ that identifies $\mathbf{SD}_A$ with $C[1]$. Then, by reducing equations (4) and (5) from above, on $\overline{\mathbf{SD}}_A \cong \mathbf{G}_m$ we have

$$\rho\bar{u} = \bar{\alpha}\alpha^{-1}\bar{u} = \zeta\bar{u} \quad \text{and} \quad w_\rho\bar{u} = \frac{\bar{\alpha}\alpha^{-1}}{\bar{u}} = \frac{\zeta}{\bar{u}} \quad \text{for some } \zeta \in \mu_{p+1} \subseteq \mathbb{F}_{p^2}^*.$$

So on $\overline{\mathbf{SD}}_A$, the w_ρ reduce to $p+1$ distinct involutions with $2(p+1)$ distinct fixed points (in a $\mu_{2(p+1)}$ orbit). Furthermore, each of these involutions of $\overline{\mathbf{SD}}_A$ lifts to an involution of $\mathbf{SD}_A$.

Another way to think of the fixed points of the automorphisms $\{w_\rho\}$ is that they correspond to elliptic curves whose formal groups have complex multiplication by the ring of integers in a ramified quadratic extension of $\mathbb{Q}_p$ (see Proposition 4.9 below). This point of view becomes crucial when we determine the field of definition of our stable model, because it ties our construction to the arithmetic theory of CM elliptic curves. To this end, we make the following definition.

Definition 4.8. For K a complete subfield of $\mathbb{C}_p$, an elliptic curve E/K has fake CM if $\mathrm{End}_K \hat{E} \neq \mathbb{Z}_p$ and potential fake CM if $\mathrm{End}_{\mathbb{C}_p} \hat{E} \neq \mathbb{Z}_p$.

Proposition 4.9. *Let (E, C) be any point of $\mathbf{SD}_A$. Then the following statements are equivalent.*

(i) *(E, C) is fixed by w_ρ for some $\rho \in B'$.*

(ii) *(E, C) is fixed by w_ρ for some $\rho \in B^*$.*

(iii) *E has potential fake CM by $\mathbb{Z}_p[\pi]$, where $\pi \in \mathrm{End}(\hat{E})$ and $C = \ker\pi$.*

Proof. We will show that (ii) is equivalent to both (i) and (iii) with a Woods Hole argument. So before we begin we must reinterpret condition (ii) in the language of Theorem 4.1. Let (F, α, C) be a Woods Hole representation of (E, C), and let $\iota_C : F \to F/C$ be the natural map. Then (E, C) is a fixed point of w_ρ if and only if there is an isomorphism $\sigma : F/C \to F$ that makes the following diagram commute.

$$\begin{tikzcd} \bar{F} \arrow[r, "\bar{\iota}_C"] \arrow[d, "\alpha"'] & \overline{F/C} \arrow[r, "\mathrm{id}"] \arrow[d, "\beta"'] & \overline{F/C} \arrow[r, "\bar{\sigma}"] \arrow[d, "\rho\circ\beta"'] & \bar{F} \arrow[d, "\alpha"'] \\ \hat{A} \arrow[r, "j"'] & \hat{A} \arrow[r, "\rho"'] & \hat{A} \arrow[r, "\mathrm{id}"'] & \hat{A} \end{tikzcd}$$

Note that the first commuting square represents the isogeny (of elliptic curves) $E \to E/C$. The pair $(F/C, \rho\circ\beta)$ then corresponds to the elliptic curve $\rho(E/C)$.

Now, to show (iii) implies (ii), suppose first that we are given $\pi \in \mathrm{End}(F)$ with $\ker(\pi) = C$. Then π must factor as $\sigma \circ \iota_C$ for some isomorphism $\sigma : F/C \to F$, and we may take $\rho = \alpha \circ \bar{\sigma} \circ \beta^{-1} \in B^*$ in the diagram above. Conversely, if (E, C) is a fixed point of w_ρ for some $\rho = a + bi + cj + dk \in B^*$, and hence we have a commutative diagram as above, $\mathrm{End}(F)$ must contain both $\pi := \sigma \circ \iota_C$ and $\pi_0 := \pi + pc$. Using the diagram to compute inside $\mathrm{End}(\hat{A})$, we then have

$$\alpha \circ \bar{\pi}_0 \circ \alpha^{-1} = \rho j - cj^2 = (\rho - cj) j.$$

Note that $\rho - cj \in B'$. So $\pi_0^2 \in p\mathbb{Z}_p^*$, which means that $\mathbb{Z}_p[\pi_0] = \mathbb{Z}_p[\pi]$ is already the maximal order in a ramified quadratic extension of $\mathbb{Q}_p$ (and hence all of $\mathrm{End}(F)$). Thus we have shown that (ii) implies (iii). We also get for free, however, that (ii) implies (i), since (E, C) is now also fixed by w_{ρ_0}, where $\rho_0 := \rho - cj \in B'$. $\square$

Corollary 4.10. *If $(E, C) \in \mathbf{SD}_A$ is fixed by w_{ρ_0} for some $\rho_0 \in B'$, then w_ρ fixes (E, C) precisely when $\rho = a\rho_0 + bj$ for $a \in \mathbb{Z}_p^*$ and $b \in \mathbb{Z}_p$.*

Remark 4.11. With notation as above, suppose that $(E, C) \in \mathbf{SD}_A$ is fixed by w_ρ and H is one of the noncanonical subgroups of E of order p (so $(E, H) \in \mathbf{C}_A$). Since $\pi_0^2 \in p\mathbb{Z}_p^*$ and $\ker(\pi_0) = C$, we must have $\pi_0(H) = C$. This implies that $a + b\pi_0 \in (\mathbb{Z}_p[\pi_0])^* \cong \mathrm{Aut}(F)$ fixes the noncanonical subgroups of order p when $p \mid b$, and transitively permutes them otherwise.

Remark 4.12. We showed in [CM 2006, Remark 3.11] that the points that satisfy the conditions of Proposition 4.9 are precisely the canonical liftings of $\hat{A}$ in the sense of [Gross 1986, 2.1], where K is one of the ramified quadratic extensions of $\mathbb{Q}_p$ and $\hat{A}$ is given the structure of a formal $\mathcal{O}_K$-module.

5. Stable reduction of $X_0(p^2)$

At this point we have done enough groundwork to prove a rigid analytic reformulation of Edixhoven's result [1990, Theorem 2.1.2] on the stable reduction of $X_0(p^2)$. Most of the work is in computing the reduction of $\mathbf{Y}_A$, the underlying affinoid of $W_A(p^2)$. This is done by first embedding $\mathbf{Y}_A$ into the product of two circles (specifically $\mathbf{TS}_A \times \mathbf{TS}_A$) and then applying the explicit formula of Theorem 3.5. After that, we use results from Section 2 to show that the wide opens in

$$\{W_{20}, W_{11}^{+}, W_{11}^{-}, W_{02}\} \cup \{W_A(p^2) : A \text{ is supersingular}\}$$

intersect properly and comprise a stable covering of $X_0(p^2)$.

Lemma 5.1. *Let $\mathbf{Y}_A = \pi_v^{-1}(\mathbf{TS}_A)$. If $A/\mathbb{F}_p$, $\mathbf{Y}_A$ is naturally isomorphic to*

$$S := \{(x, y) \in \mathbf{TS}_A \times \mathbf{TS}_A \mid x \neq y,\ \pi_f(x) = \pi_f(y)\}.$$

Proof. If $(x, y) \in S$, then $x = (E, C_1)$ and $y = (E, C_2)$ for E some too supersingular curve, and C_1 and C_2 are two distinct subgroups of order p. So we can define a map $\psi : S \to \mathbf{Y}_A$ by taking (x, y) to $(E/C_1, p^{-1}C_2/C_1)$. It is immediate that this takes values in $\mathbf{Y}_A$ since $\pi_v \circ \psi(x, y)$ is then

$$(E/E[p], p^{-1}C_2/E[p]) \cong (E, C_2).$$

Also, we can define a map going the other way, say ϕ, by taking $(E, C) \in \mathbf{Y}_A$ to the pair $(x, y) \in S$ with $x = (E/pC, E[p]/pC)$ and $y = (E/pC, C/pC)$. This takes values in S precisely because $\pi_\nu(E, C) = (E/pC, C/pC) \in \mathbf{TS}_A$, and it is straightforward to check that $\psi \circ \phi$ and $\phi \circ \psi$ are the respective identities. $\square$

Proposition 5.2. *Let A be as in Theorem 3.5. Then if K is any extension of $W(\mathbb{F}_{p^2}) \otimes \mathbb{Q}_p$ such that $(p+1) \mid e(K)$, $\overline{\mathbf{Y}}_A := \overline{(\mathbf{Y}_A)}_K$ is a smooth, affine curve of genus $(p-1)/2$ with 4 points at infinity (equation given below).*

Proof. Let x and y be parameters on $\mathbf{TS}_A$ that are specializations of the parameter t on $W_A(p)$ from Theorem 3.5. Then by Lemma 5.1, $\mathbf{Y}_A$ can be described by the equation

$$F(x) + G(\kappa/x) = F(y) + G(\kappa/y),$$

where $v(x) = v(y) = p/(p+1)$. Now choose any $\alpha \in K$ with $v(\alpha) = 1/(p+1)$ and substitute $u = \alpha^p/x$ and $\upsilon = \alpha^p/y$ into the above equation for $\mathbf{Y}_A$ (so that $v(u) = v(\upsilon) = 0$). Dividing through by α^p, we obtain an equation for $\mathbf{Y}_A$ in u and υ that has integral coefficients and satisfies the congruence

$$u^{-1} - \upsilon^{-1} \equiv (\upsilon^p - u^p)(\kappa/\alpha^{p+1})^p \pmod{\alpha}.$$

Now let $b = (\kappa/\alpha^{p+1})^p$ (a unit), and we obtain

$$1 \equiv bu\upsilon(\upsilon - u)^{p-1} \pmod{\alpha}$$

as an equation for $\overline{\mathbf{Y}}_A$.

Strictly speaking, the above curve has three infinite points, with projective coordinates (0:1:0), (1:0:0), and (1:1:0). However, while the first two are nonsingular, the third splits into two points in the normalization. The genus can easily be computed by applying Riemann–Hurwitz to the equation

$$s^{p+1} = \frac{b}{4}(r^2 - 1),$$

where $s = 1/(\upsilon - u)$ and $r = (\upsilon + u)/(\upsilon - u)$. $\square$

Theorem 5.3. *Let $p \geq 13$ be a prime, and K an extension of $W(\mathbb{F}_{p^{24}}) \otimes \mathbb{Q}_p(\mu_p)$ with $(p+1) \mid e(K)$. The following is a semistable covering of $X_0(p^2)$ over K:*

$$\mathscr{C}_0(p^2) := \{W_{20}, W_{11}^+, W_{11}^-, W_{02}\} \cup \{W_A(p^2) : A \text{ is supersingular}\}.$$

The affinoids $\mathbf{X}_{ab}^{\pm}$ and $\mathbf{Y}_A$ are minimal underlying affinoids in $W_{ab}^{\pm}$ and $W_A(p^2)$.

Proof. The wide opens $W_{ab}^{\pm}$, which cover the ordinary locus, are disjoint from each other, and we have

$$\mathbf{Y}_A = W_A(p^2) \setminus \bigcup W_{ab}^{\pm}.$$

By Proposition 3.6, all four ordinary affinoids have good reduction over K. Therefore, it suffices to show that each $\mathbf{Y}_A$ has good reduction, and that $W_A(p^2)\cap W_{ab}^{\pm}$ is always an annulus.

First we demonstrate that the wide open intersections are annuli over K (where we still assume $(p+1)\mid e$). In the case of W_{20} this is immediate, as $W_{20}\cap W_A(p^2)$ maps isomorphically onto the annulus

$$x_A^{-1}(A(p^{-i(A)/(p+1)},1))\subseteq W_A(p)$$

over K, via π_f. Similarly, $W_{11}^{\pm}\cap W_A(p^2)$ maps onto the same annulus via π_f, but with degree $(p-1)/2$. Then Theorem 2.6 implies that this too is an annulus over K. Finally, $W_{02}\cap W_A(p^2)$ must be an annulus since it is isomorphic to the region $W_{20}\cap W_{A^\sigma}(p^2)$, by the Atkin–Lehner involution w_2.

Next we consider the reductions of the affinoids $\mathbf{Y}_A$. Since $p\geq 13$, Theorem B.1 guarantees us a supersingular elliptic curve A_0 for which Proposition 5.2 directly applies. Then for any other supersingular curve A we use Proposition 4.2. In particular, we choose a surjection ψ_ι that maps $W_{A_0}(p^2)$ onto $W_A(p^2)$ with degree $i(A)$. If $i(A)=1$, the two regions are isomorphic and we are done. In any case, ψ_ι necessarily takes $\mathbf{Y}_{A_0}$ to $\mathbf{Y}_A$ and is étale. Therefore $\overline{\mathbf{Y}}_A$ is isomorphic to the quotient of $\overline{\mathbf{Y}}_{A_0}$ by an automorphism of degree $i(A)$ (which fixes the four infinite points). Hence $\mathbf{Y}_A$ has good reduction and we are done. □

Corollary 5.4. *For any supersingular curve A, the reduction of $\mathbf{Y}_A$ must have* (*with the correct choice of parameters*) *the equation*

$$y^{(p+1)/i(A)}=x^2-1,$$

and genus $(p+1)/(2i(A))-1$.

Proof. After a change of coordinates, the reduction of $\mathbf{Y}_{A_0}$ has the equation $y^{p+1}=x^2-1$, with two of the four infinite points moved to $(\pm1,0)$ and two still at infinity. Now, any automorphism of order $i(A)$ that acts on this curve and fixes these four points must fix x and take y to ζy, where $\zeta^{i(a)}=1$. □

Remark 5.5. Let K be as in Theorem 5.3, with $e_p(K)=(p^2-1)/2$. By computing the widths of the annuli in the stable covering (see Section 9A for more details), one finds intersection multiplicities of $i(A)$ where $\mathbf{X}_{11}^{\pm}$ meets $\mathbf{Y}_A$ and of $i(A)\cdot(p-1)/2$ where $\mathbf{X}_{20}$ and $\mathbf{X}_{02}$ meet $\mathbf{Y}_A$.

The following implies [Coleman 2005, Theorem 3.1].

Corollary 5.6. *The point (E,C) is not in $S:=W_{20}\cup W_{11}^{+}\cup W_{11}^{-}\cup W_{02}$ if and only if $pC=K(E)$ and $E[p]/pC=K(E/C)$, or equivalently $K(E/pC)=0$.*

Proof. (E, C) is not in S if and only if it is in some $\mathbf{Y}_A$, which by definition means that E/pC has trivial canonical subgroup. This is equivalent to $pC = K(E)$ and $E[p]/pC = K(E/C)$ by [Buzzard 2003, Theorem 3.3(vi)]. □

Corollary 5.7. *The Hecke correspondence T_ℓ takes a divisor supported on S to a divisor supported on S if and only if $\ell \neq p$.*

Proof. This follows from the fact that

$$T_\ell(E, C) = \sum_{\substack{\deg \alpha = \ell \\ |\alpha C| = |C|}} (\alpha E, \alpha C). \qquad \square$$

Remark 5.8. Using the fact that $X(p) \cong X_0(p^2) \times_{X_0(p)} X_1(p)$, Jared Weinstein and the second author have used the results of this section to determine a stable model of $X(p)$.

6. Outline of $X_0(p^3)$ analysis

At this point we would like to construct a stable covering for $X_0(p^3)$ in much the same way as was just done for $X_0(p^2)$. By analogy, the natural starting point would be the covering consisting of

$$\{W_{30}, W_{21}^+, W_{21}^-, W_{12}^+, W_{12}^-, W_{03}\} \cup \{W_A(p^3) : A \text{ is supersingular}\}.$$

This is not stable, however, because $W_A(p^3)$ is not a basic wide open. This can actually be seen immediately from the fact that each $W_A(p^3)$ at least contains the affinoids $\mathbf{E}_{1A} := \pi_f^{-1}(\mathbf{Y}_A)$ and $\mathbf{E}_{2A} := \pi_v^{-1}(\mathbf{Y}_{A^\sigma})$ (which are nontrivial from Section 5). So our covering for $X_0(p^3)$ must at least be refined to take these regions into account. In fact, things are much more complicated.

For simplicity, suppose that $A/\mathbb{F}_p$ with $j(A) \neq 0, 1728$ (other $W_A(p^3)$ can be handled by Proposition 4.2). Since π_{11} maps $W_A(p^3)$ onto the width 1 annulus, $W_A(p)$, this gives us a convenient way to keep track of where various subspaces are in relation to each other. For example, it follows from Section 5 that the above affinoids, $\mathbf{E}_{1A}$ and $\mathbf{E}_{2A}$, lie over the circles described by $v(x_A) = p/(p+1)$ and $v(x_A) = 1/(p+1)$ respectively (with parameter x_A as in Section 3). The former is the too-supersingular circle, and the latter is what was called the nearly too-supersingular circle in [Coleman 2005, §3]. Lying in between these two circles is the Atkin–Lehner circle, $\mathbf{SD}_A$, where $v(x_A) = 1/2$. So lying "in between" $\mathbf{E}_{1,A}$ and $\mathbf{E}_{2,A}$ in some sense is the affinoid $\mathbf{Z}_A := \pi_{11}^{-1}(\mathbf{SD}_A)$. It turns out that this affinoid is where all of the new complication arises at the p^3 level. We now give a brief summary of the analysis of $\mathbf{Z}_A$ that will follow in Sections 7 and 8.

Much of our analysis of $\mathbf{Z}_A$ is explicit (see Section 8), and is based on an embedding into the product of two circles as in Lemma 5.1. More specifically, let

$\tau_f : \mathbf{C}_A \to \mathbf{SD}_A$ be as in Section 3. Then $\mathbf{Z}_A$ can be identified with

$$S := \{(x, y) \in \mathbf{C}_A \times \mathbf{C}_A \mid \tau_f(x) = w_1 \circ \tau_f(y)\}.$$

Since $\pi_f \circ \tau_f = \pi_f$, this identification along with de Shalit's result (Theorem 3.5) gives us a way to explicitly compute the reduction of $\mathbf{Z}_A$ as

$$X^{p+1} + X^{-(p+1)} = Z^p.$$

So $\overline{\mathbf{Z}}_A$ has $2(p+1)$ cuspidal singular points, and its normalization is a copy of the affine line whose completion is what we will call a "bridging component". Basically, we want to show that the $2(p+1)$ singular residue classes of $\mathbf{Z}_A$ are basic wide open subspaces, with underlying affinoids that reduce to $y^2 = x^p - x$.

To motivate and explain this, consider the identity $\pi_{11} \circ w_3 = w_1 \circ \pi_{11}$, relating the Atkin–Lehner involutions on $X_0(p^3)$ and $X_0(p)$. It follows immediately that w_3 preserves $\mathbf{Z}_A$, as well as $\widetilde{\mathfrak{D}} := \pi_{11}^{-1}(\mathfrak{D})$, where $\mathfrak{D}$ is either of the residue disks of $\mathbf{SD}_A$ preserved by w_1. Furthermore, a moduli-theoretic argument shows that w_3 has $2p$ fixed points that lie $p:1$ over the w_1 fixed points in $\mathbf{SD}_A$. So $\widetilde{\mathfrak{D}} \subseteq \mathbf{Z}_A$ is a wide open with one end upon which the involution w_3 acts with p fixed points. We show that $\widetilde{\mathfrak{D}}$ is in fact isomorphic to the complement of an affinoid disk near infinity in a hyperelliptic curve that reduces to $y^2 = x^p - x$ (w_3 is the hyperelliptic involution). Such an argument, however, would only account for two of the singular residue classes of $\mathbf{Z}_A$. To handle all of them, we use the action of $B^* = \mathrm{Aut}(\hat{A})$ to generalize the pair (w_1, w_3) to a pair $(w_\rho, \widetilde{w}_\rho)$, as was done in Proposition 4.6. Thus we are able to handle all $2(p+1)$ residue classes because of Remark 4.7.

Once we have actually constructed all of the nontrivial components in the stable reduction of $X_0(p^3)$, the argument is reduced to showing that nothing else interesting can happen. We do this in Section 9, with a total genus calculation playing a key role. Again we first use the fact that all supersingular regions are (nearly) isomorphic along with the result of Appendix B, so that calculations only need to be done for a supersingular curve with $A/\mathbb{F}_p$ and $j(A) \neq 0, 1728$. The remaining cases of $p \leq 11$ were handled explicitly in [CM 2006, §6], which we hope makes our construction more understandable, and which completes Theorem 9.2.

7. The bridging component

Fix a supersingular elliptic curve $A/\mathbb{F}_p$ with $j(A) \neq 0, 1728$. In this section we begin our analysis of the affinoid $\mathbf{Z}_A := \pi_{11}^{-1}(\mathbf{SD}_A) \subseteq W_A(p^3)$. In particular, we show by a moduli-theoretic argument that $\mathbf{Z}_A$ can be embedded into $\mathbf{C}_A \times \mathbf{C}_A$. Using the embedding, we then construct a family of involutions on $\mathbf{Z}_A$. These involutions are compatible (with respect to π_{11}) with the involutions of $\mathbf{SD}_A$ that were introduced in Proposition 4.6.

Proposition 7.1. *Let $\mathbf{C}_A$ and $\tau_f : \mathbf{C}_A \to \mathbf{SD}_A$ be as in Section 3. There is a natural isomorphism ψ from*

$$S := \{(x, y) \in \mathbf{C}_A \times \mathbf{C}_A \mid \tau_f(x) = w_1 \circ \tau_f(y)\}$$

to $\mathbf{Z}_A$, such that $w_3(\psi(x, y)) = \psi(y, x)$ and $\pi_{11}(\psi(x, y)) = \tau_f(x)$.

Proof. Suppose $(x, y) \in S$. Then there exists an $(E, C) \in \mathbf{SD}_A$ such that $x = (E, H)$ for some $H \neq C$. The p noncanonical subgroups of E/C are precisely the subgroups D/C, where $D \subseteq E$ is a cyclic subgroup of order p^2 with $pD = C$ [Buzzard 2003, 3.3]. Therefore, since $\tau_f(x) = w_1(\tau_f(y)) = (E, C)$, there is a unique D such that $y = (E/C, D/C)$. Hence we can define a map $\psi : S \to W_A(p^3)$ by

$$\psi(x, y) = (E/H, (p^{-1}D)/H).$$

Note that $(p^{-1}D)/H$, and hence ψ, is well defined since $pD = C$ and H span $E[p]$. The key fact to check is that $\psi(x, y)$ lies in $\mathbf{Z}_A$, that is, $\pi_{11}(\psi(x, y)) \in \mathbf{SD}_A$.

$$\begin{aligned}\pi_{11}(E/H, (p^{-1}D)/H) &= (E/\langle H, pD\rangle, D/\langle H, pD\rangle)\\ &= (E/E[p], D/E[p])\\ &\equiv (E, pD) = (E, C) \in \mathbf{SD}_A.\end{aligned}$$

This calculation shows that $\psi(x, y) \in \mathbf{Z}_A$, that $\pi_{11}(\psi(x, y)) = \tau_f(x)$, and more. Once a point $(E, C) \in \mathbf{SD}_A$ is fixed, there are p independent choices for both H and D. Therefore we have produced p^2 points of $\mathbf{Z}_A$ that are in the image of ψ and in the π_{11}-fiber over that particular $(E, C) \in \mathbf{SD}_A$. Since the total degree of $\pi_{11} : X_0(p^3) \to X_0(p)$ is only p^2, we can conclude that ψ maps *onto* $\mathbf{Z}_A$, and hence is an isomorphism. We now describe its inverse. For an arbitrary $(E, K) \in \mathbf{Z}_A$, let $x(E, K) = (E/p^2K, E[p]/p^2K)$, $y(E, K) = (E/pK, K/pK)$, and $\phi(E, K) = (x(E, K), y(E, K))$. To show that $\phi = \psi^{-1}$, it suffices to check that $\phi \circ \psi$ is the identity on S. We have

$$\begin{aligned}x(E/H, (p^{-1}D)/H) &= (E/\langle H, C\rangle, p^{-1}H/\langle H, C\rangle)\\ &= (E/E[p], p^{-1}H/E[p]) \equiv (E, H)\end{aligned}$$

and

$$\begin{aligned}y(E/H, (p^{-1}D)/H) &= (E/\langle H, D\rangle, p^{-1}D/\langle H, D\rangle)\\ &= (E/\langle E[p], D\rangle, p^{-1}D/\langle E[p], D\rangle)\\ &\equiv (E/pD, D/pD) = (E/C, D/C).\end{aligned}$$

Now that we have determined ψ^{-1}, we can verify the claim regarding w_3 by applying $\psi^{-1} \circ w_3 \circ \psi$ to the pair (x, y), where $x = (E, H)$ and $y = (E/C, D/C)$.

We have

$$
\begin{aligned}
w_3 \circ \psi(x, y) &= w_3(E/H, (p^{-1}D)/H) \\
&= (E/\langle H, p^{-1}D\rangle, p^{-3}H/\langle H, p^{-1}D\rangle) \\
&= (E/\langle E[p], p^{-1}D\rangle, p^{-3}H/\langle E[p], p^{-1}D\rangle) \\
&\equiv (E/D, p^{-2}H/D), \\
x(E/D, p^{-2}H/D) &= (E/\langle D, H\rangle, p^{-1}D/\langle D, H\rangle) \\
&= (E/\langle E[p], D\rangle, p^{-1}D/\langle E[p], D\rangle) \\
&\equiv (E/pD, D/pD) = (E/C, D/C) = y, \\
y(E/D, p^{-2}H/D) &= (E/\langle D, p^{-1}H\rangle, p^{-2}H/\langle D, p^{-1}H\rangle) \\
&= (E/E[p^2], p^{-2}H/E[p^2]) \equiv (E, H) = x. \qquad \square
\end{aligned}
$$

Proposition 7.2. *For each $\rho \in B^*$, we can define an automorphism $\tilde{w}_\rho$ of $\mathbf{Z}_A$ (identified with S) by*

$$\tilde{w}_\rho(x, y) = (\rho y, \rho' x).$$

Furthermore, $\widetilde{w}_\rho$ is compatible with w_ρ, in the sense that $\pi_{11} \circ \widetilde{w}_\rho = w_\rho \circ \pi_{11}$, and is an involution of $\mathbf{Z}_A$ whenever $\rho \in B'$.

Proof. The action of B^* on $W_A(p)$ preserves circles. So at least this defines a map from $\mathbf{C}_A \times \mathbf{C}_A$ to itself. To verify that it preserves the subspace S we need to check that

$$\tau_f(x) = w_1 \circ \tau_f(y) \Rightarrow \tau_f(\rho y) = w_1 \circ \tau_f(\rho' x).$$

But τ_f commutes with B^*. So this follows from the identity $\rho w_1 = w_1 \rho'$, which was shown in the proof of Proposition 4.6.

By Remark 4.4, the inverse of $\widetilde{w}_\rho$ is given by $\widetilde{w}_\xi$ for any $\xi \in B^*$ with $\xi\rho' \in \mathbb{Z}_p^*$. In particular, $\widetilde{w}_\rho$ is an involution exactly when $\rho \in B'$. Finally, the compatibility relation follows easily from the fact that $\pi_{11}(x, y) = \tau_f(x)$. $\square$

Corollary 7.3. *Every fixed point of $\widetilde{w}_\rho$ lies (via π_{11}) over a fixed point of w_ρ. If $\mathfrak{D}_\rho \subseteq \mathbf{SD}_A$ is one of the two residue disks that are preserved by w_ρ, then $\widetilde{\mathfrak{D}}_\rho := \pi_{11}^{-1}(\mathfrak{D}_\rho)$ is invariant under $\widetilde{w}_\rho$.*

Proof. These are immediate consequences of $\pi_{11} \circ \widetilde{w}_\rho = w_\rho \circ \pi_{11}$. $\square$

Remark 7.4. Let 1_B be the multiplicative identity in B. Then w_{1_B} is $w_1|_{\mathbf{SD}_A}$ and $\widetilde{w}_{1_B}$ is $w_3|_{\mathbf{Z}_A}$.

Recall from Proposition 4.9 that the fixed points of w_ρ correspond to pairs (E, C), where E has fake CM by $\mathbb{Z}_p[\pi]$ and $\ker(\pi) = C$. The points of $\mathbf{Z}_A$ that lie over such a fixed point then correspond to pairs $(E/H, p^{-1}D/H)$, where H and D are as in the proof of Proposition 7.1. In particular, $H \subseteq E$ is a noncanonical subgroup of order p, and $D \subseteq E$ is cyclic of order p^2 such that $pD = C$. Combining these facts with Corollary 7.3 gives us a convenient way to describe (and count) the fixed points of $\widetilde{w}_\rho$.

Proposition 7.5. *Let (E, C) be a fixed point of w_ρ for some $\rho \in B^*$, such that $\mathrm{End}(\hat{E}) = \mathbb{Z}_p[\pi]$ with $\ker(\pi) = C$. If $\rho \in B'(1 + pjB)$, there are p fixed points of $\widetilde{w}_\rho$ lying over (E, C), specifically those pairs $(E/H, p^{-1}D/H)$ with $\pi(D) = H$. Otherwise, $\widetilde{w}_\rho$ has no fixed points.*

Proof. Fix a Woods Hole triple (F, α, C) corresponding to (E, C). Then E/C is equivalent to some pair $(F/C, \beta)$, such that the diagram from the proof of Proposition 4.9 commutes. Note that an explicit isomorphism from $\rho(E/C)$ to E is then given by the pair (σ, id). To determine the $\widetilde{w}_\rho$ fixed points, it will be useful to similarly describe the isomorphism from $\rho'(E)$ to E/C, which exists by $\rho \circ w_1 = w_1 \circ \rho'$. This can be done by replacing $\rho \circ j$ with $j \circ \rho'$ in the diagram, and repeating the first isogeny, to obtain the following.

$$\begin{tikzcd}
\overline{F} \arrow[r, "\bar{\iota}_C"] \arrow[d, "\rho' \circ \alpha"'] & \overline{F/C} \arrow[r, "\bar{\sigma}"] \arrow[d, "\rho \circ \beta"'] & \overline{F} \arrow[r, "\bar{\iota}_C"] \arrow[d, "\alpha"'] & \overline{F/C} \arrow[d, "\beta"'] \\
\hat{A} \arrow[r, "j"'] & \hat{A} \arrow[r, "\mathrm{id}"'] & \hat{A} \arrow[r, "j"'] & \hat{A}
\end{tikzcd}$$

Since $j^2 = -p$, this diagram shows that an isomorphism from $\rho'(E)$ to E/C is given by the pair (γ, id), where $\gamma = -p^{-1}\iota_C \circ \sigma \circ \iota_C$.

Now, choose a point lying over (E, C) by taking $x = (E, H) = (F, \alpha, H)$ and $y = (E/C, D/C) = (F/C, \beta, D/C)$. We must determine when

$$\widetilde{w}_\rho(x, y) = (\rho y, \rho' x) = (x, y).$$

Since an isomorphism from $\rho(E/C)$ to E is given by (σ, id), the condition $\rho y = x$ is equivalent to $\sigma(D/C) = H$. Similarly, the condition $\rho' x = y$ is equivalent to $\gamma(H) = D/C$. Putting these in terms of π, the first condition is $\pi(D) = H$ and the second is $\pi(D) = -(\pi^2/p)(H)$. By Remark 4.11, these two conditions are equivalent when $\rho \in B'(1 + pjB)$, and incompatible otherwise. □

Remark 7.6. If (E, C) is *any* point lying over a fixed point of w_ρ via π_{11}, it is a fake Heegner point in the sense that E has fake CM and $\mathrm{End}(\widehat{E/C})$ is isomorphic to $\mathrm{End}(\hat{E})$. In fact, one can show in this case that $\mathrm{End}(\hat{E}) \cong \mathbb{Z}_p[\lambda]$ for some λ such that $\ker(\lambda) = C$.

8. Explicit analysis

In this section, we use Proposition 7.1 and Theorem 3.5 to explicitly compute the reduction of $\mathbf{Z}_A$ (for $A/\mathbb{F}_p$ and $j(A) \neq 0, 1728$), in much the same way that the reduction of $\mathbf{Y}_A$ was computed in the proof of Proposition 5.2. We obtain

$$X^{p+1} + X^{-(p+1)} = Z^p.$$

Moreover, the residue classes of $\mathbf{Z}_A$ that have singular reduction on this model are shown to coincide with those regions $\widetilde{\mathfrak{D}}_\rho$ that were described in Corollary 7.3. From the previous section we know that $\widetilde{\mathfrak{D}}_\rho$ is acted on by the involution $\widetilde{w}_\rho$, with p fixed points. In addition, from the explicit equation for $\overline{\mathbf{Z}}_A$, we are able to deduce that $\widetilde{\mathfrak{D}}_\rho$ is a connected wide open with one end, and that $\widetilde{\mathfrak{D}}_\rho/\widetilde{w}_\rho$ is a disk. Putting all of this information together (and a little more), we are able to show in Section 8B that $\widetilde{\mathfrak{D}}_\rho$ is a basic wide open whose underlying affinoid reduces to $y^2 = x^p - x$.

8A. *Reduction of $\mathbf{Z}_A$.* Recall that Proposition 7.1 identifies $\mathbf{Z}_A$ with the subspace of $\mathbf{C}_A \times \mathbf{C}_A$ defined by $\tau_f(x) = w_1 \circ \tau_f(y)$. From this embedding we can obtain an explicit equation for $\mathbf{Z}_A$, provided we can derive approximation formulas for w_1 on $\mathbf{SD}_A$ and $\tau_f : \mathbf{C}_A \to \mathbf{SD}_A$. Such formulas follow readily from Theorem 3.5. However, while the formula in this theorem is given over $\mathbb{Q}_p \otimes W(\mathbb{F}_{p^2})$, we will ultimately need to work over a finite base extension. This extension can be generated by fixing a square root $\sqrt{\kappa}$ of κ in $\mathbb{C}_p$ (where κ is as in Theorem 3.5) and a $\beta \in \mathbb{C}_p$ satisfying

$$\beta^{p^2} \equiv \kappa \pmod{p^{3/2-1/2p^2}}. \tag{6}$$

Remark 8.1. For example, if $g(x) = x^{p^2} - \sqrt{\kappa}x$, and γ is a root of $g(g(x))/g(x)$, one may take $\beta = \gamma^{2(p^2-1)}$. Then, by Lubin–Tate theory, applied to the Lubin–Tate formal group over $F := \mathbb{Q}_p(\sqrt{\kappa}) \otimes_{\mathbb{Z}_p} W(\mathbb{F}_{p^2})$, with endomorphism $g(x)$, $F(\beta)$ is Galois over F with Galois group $C_p \times C_p$.

Proposition 8.2. *Over $R := \mathbb{Z}_p[\sqrt{\kappa}, \beta] \otimes W(\mathbb{F}_{p^2})$, the reduction of $\mathbf{Z}_A$ has the equation*

$$X^{p+1} + X^{-(p+1)} = Z^p.$$

Hence, over R, its reduction is a reduced, connected, affine curve of genus zero with only one branch through each singular point.

Proof. First we derive an approximation for $\tau_f : \mathbf{C}_A \to \mathbf{SD}_A$ in terms of the parameter t from Theorem 3.5. For any $P_1 \in \mathbf{SD}_A$ and $P_2 \in \mathbf{C}_A$, we note that $P_1 = \tau_f(P_2)$ if and only if $\pi_f(P_1) = \pi_f(P_2)$. Thus, an approximation for τ_f should follow from approximations for π_f on $\mathbf{SD}_A$ and $\mathbf{C}_A$. Now, we know from [Buzzard 2003, 3.3] that $\mathbf{SD}_A$ and $\mathbf{C}_A$ are the circles described by $v(t) = 1/2$ and $v(t) = 1 - 1/2p$. In particular, we must have $v(t(P_1)) = 1/2$ and $v(t(P_2)) =$

$1 - 1/2p$. Therefore, from Theorem 3.5 we can approximate π_f on $\mathbf{SD}_A$ and $\mathbf{C}_A$:

$$\begin{aligned} s(\pi_f(P_1)) &\equiv t(P_1) && \pmod{p}, \\ s(\pi_f(P_2)) &\equiv t(P_2) + (\kappa/t(P_2))^p && \pmod{p}. \end{aligned}$$

Hence an approximation for $\tau_f : \mathbf{C}_A \to \mathbf{SD}_A$ is given by

$$t(\tau_f(P)) \equiv t(P) + (\kappa/t(P))^p \pmod{p}.$$

To describe the reduction of $\mathbf{Z}_A$ via Proposition 7.1, we now choose parameters that identify $\mathbf{C}_A$ and $\mathbf{SD}_A$ with the unit circle, $C[1]$. For such a parameter on $\mathbf{SD}_A$ we let $U := t/\sqrt{\kappa}$, and on $\mathbf{C}_A$ we let $X := t/\alpha$, where

$$\alpha = (\beta^{(p^2+1)/2}/\sqrt{\kappa})^{p(2p-1)}$$

(note that $v(\alpha) = 1 - 1/2p$). In terms of these new parameters, the Atkin–Lehner involution is just given by $w_1^* U = 1/U$. Also, using the defining congruence for β, the approximation formula above for τ_f becomes

$$\tau_f^* U \equiv \alpha X/\sqrt{\kappa} + X^{-p} \pmod{\sqrt{p}}.$$

Now let Y and V be analogous parameters on copies of $\mathbf{C}_A$ and $\mathbf{SD}_A$, so that the equation $\tau_f(P) = w_1(\tau_f(Q))$ on $\mathbf{C}_A \times \mathbf{C}_A$ (which defined the subspace $S \cong \mathbf{Z}_A$) becomes $\tau_f^* U = 1/\tau_f^* V$. Then on S the parameters X and Y satisfy the congruence relations

$$\begin{aligned} (\alpha X/\sqrt{\kappa} + X^{-p})(\alpha Y/\sqrt{\kappa} + Y^{-p}) &\equiv 1 && \pmod{\sqrt{p}}, \\ \alpha X^{p+1}/\sqrt{\kappa} + \alpha Y^{p+1}/\sqrt{\kappa} + 1 &\equiv X^p Y^p && \pmod{\sqrt{p}}. \end{aligned} \tag{7}$$

Finally, we define a new parameter Z on $\mathbf{C}_A \times \mathbf{C}_A$ by $XY = \beta^{(p-1)/2} Z + 1$. Then $\mathbf{Z}_A$ is determined over $R \otimes \mathbb{Q}_p$ by $|X| \le 1$ and $|Z| \le 1$. The congruence

$$X^{p+1} + X^{-(p+1)} \equiv Z^p \pmod{m_R},$$

where m_R is the maximal ideal of R, follows from (7). □

Proposition 8.3. *The involutions $\widetilde{w}_p$ on $\mathbf{Z}_A$ reduce to the involutions on $\overline{\mathbf{Z}}_A$ given by*

$$t_\zeta : (X, Z) \mapsto (\zeta/X, Z),$$

where ζ varies over all $(p+1)$-st roots of unity. The $\widetilde{\mathfrak{D}}_p^i$ coincide with the singular residue classes of $\mathbf{Z}_A$, which are described by $X^{2p+2} \equiv 1$.

Proof. We use the compatibility relation in the proof of Proposition 7.2, namely $\pi_{11} \circ \widetilde{w}_p = w_p \circ \pi_{11}$. Recall from Proposition 7.1 that $\pi_{11}(x, y) = \tau_f(x)$ (with notation consistent with that of the previous proposition). So from the proof of the previous proposition, an explicit formula for π_{11} as a map from $\overline{\mathbf{Z}}_A$ to $\overline{\mathbf{SD}}_A$ is

$$U = \pi_{11}(X, Z) = X^{-p}.$$

Now, we know from Remark 4.7 that on $\overline{\mathbf{SD}}_A$ the involutions w_ρ reduce to those of the form $U \to \zeta/U$ (where ζ is any $(p+1)$-st root of unity). So fix a ρ and corresponding ζ. Choose any point (X_0, Z_0) on $\overline{\mathbf{Z}}_A$, and let $(X_1, Z_1) = \widetilde{w}_\rho(X_0, Z_0)$. We can compute both sides of the compatibility relation above:

$$w_\rho \circ \pi_{11}(X_0, Z_0) = w_\rho(X_0^{-p}) = \zeta X_0^p,$$
$$\pi_{11} \circ \widetilde{w}_\rho(X_0, Z_0) = \pi_{11}(X_1, Z_1) = X_1^{-p}.$$

Since $\zeta = \zeta^{-p}$, we must have $X_1 = \zeta/X_0$ and subsequently $Z_1 = Z_0$. In other words, we have shown that on $\overline{\mathbf{Z}}_A$ we have $\widetilde{w}_\rho(X, Z) = (\zeta/X, Z)$.

Keeping the same notation, the points of $\overline{\mathbf{SD}}_A$ that are fixed by w_ρ are the two described by $U^2 = \zeta$, and by definition $\widetilde{\mathfrak{D}}_\rho^1$ and $\widetilde{\mathfrak{D}}_\rho^2$ are π_{11}^{-1} of the corresponding residue classes. Since $\pi_{11} : \overline{\mathbf{Z}}_A \to \overline{\mathbf{SD}}_A$ is given by $U = X^{-p}$, this is equivalent to saying that $\widetilde{\mathfrak{D}}_\rho^i$ are the classes of $\mathbf{Z}_A$ described by $X^2 \equiv \zeta$. Letting ζ vary over all $(p+1)$-st roots of unity, we obtain all the residue classes described by $X^{2p+2} \equiv 1$, and these are easily verified to be the singular ones. □

Proposition 8.4. *For any $\rho \in B'$, the residue classes of the affinoid quotient $\mathbf{Z}_A/\widetilde{w}_\rho$, which are the images of the $\widetilde{\mathfrak{D}}_\rho^i$, are disks over $\mathbb{Z}_p[\sqrt{\kappa}, \beta] \otimes W(\mathbb{F}_{p^2})$.*

Proof. Let ζ be the $(p+1)$-st root of unity such that $\widetilde{w}_\rho$ reduces to t_ζ on $\overline{\mathbf{Z}}_A$. Let $f_\zeta(x)$ be the unique polynomial of degree $p+1$ such that

$$f_\zeta(X + \zeta/X) = X^{p+1} + X^{-(p+1)}.$$

Then $f_\zeta(x) = z^p$ is an equation for the reduction of $\mathbf{Z}_A/\widetilde{w}_\rho$. Also

$$f'_\zeta(X + \zeta/X) = \frac{X^{2p+2} - 1}{X^p(X^2 - \zeta)},$$

and the right side doesn't vanish at ϵ if $\epsilon^2 = \zeta$. Thus $f'_\zeta(2\epsilon) \neq 0 \bmod p$, and the two residue classes of $\mathbf{Z}_A/\widetilde{w}_\rho$ described by $X = \pm\epsilon$ are disks. □

From Theorem 2.29 and Proposition 2.31, we now conclude that (over a suitable field extension) $\widetilde{\mathfrak{D}}_\rho^i$ is a connected wide open with one end. Furthermore, using Theorem 2.48 and the fact that there are p branch points in the degree 2 quotient of $\widetilde{\mathfrak{D}}_\rho^i$ by $\widetilde{w}_\rho$, we compute the genus of $\widetilde{\mathfrak{D}}_\rho^i$ to be $(p-1)/2$. To summarize, we have the following corollary.

Corollary 8.5. *Let L be a complete stable subfield of $\mathbb{C}_p$ containing R, over which the fixed points of $\widetilde{w}_\rho$ are defined. Over L, the rigid spaces $\widetilde{\mathfrak{D}}_\rho^i$ for $i = 1$ or 2 are connected wide opens with one end of genus $(p-1)/2$.*

8B. ***The new components.*** We now show that over a suitable base extension, the $2(p+1)$ residue classes $\widetilde{\mathfrak{D}}^i_\rho \subseteq \mathbf{Z}_A$ are basic wide opens, and we compute the reductions of their underlying affinoids. The main idea is to construct an automorphism of order p on each $\widetilde{\mathfrak{D}}^i_\rho$ that transitively permutes the p fixed points of the involution $\tilde{w}_\rho$. This induces an automorphism on the quotient $\widetilde{\mathfrak{D}}^i_\rho/\tilde{w}_\rho$, a disk by Corollary 8.5, which then must be conjugate to a translation.

First we define automorphisms of order p on the disk $\tau_f^{-1}(\mathfrak{D}) \subseteq \mathbf{C}_A$, where $\tau_f : \mathbf{C}_A \to \mathbf{SD}_A$ is as in Section 3, and $\mathfrak{D}$ is either of the two residue disks of $\mathbf{SD}_A$ fixed by w_ρ. Recall that points of $\mathbf{SD}_A$ correspond to pairs (E, C) where $h(E) = 1/2$ and C is canonical. One of the key facts that we use in our construction is that over the residue disk $\mathfrak{D}$ one can analytically choose a generator up to sign for each of these canonical subgroups. This amounts to choosing a section σ of the forgetful map from $X_1(p)$ to $X_0(p)$ over $\mathfrak{D}$, given by

$$\sigma : (E, K_1(E)) \mapsto (E, P_\sigma(E)),$$

where $P_\sigma(E)$ is a pair consisting of a generator of $K_1(E)$ and its inverse. Such a section exists because this map is an étale map of annuli over $\mathbf{SD}_A$ (over any extension of $\mathbb{Q}_p$ whose ramification index is divisible by $2(p-1)$). In fact, the group $(\mathbb{Z}/p\mathbb{Z})^*/\{\pm 1\}$ acts simply transitively on the set of the sections over $\mathfrak{D}$. Once σ is chosen, automorphisms of $\tau_f^{-1}(\mathfrak{D})/\mathfrak{D}$ can be constructed by looking closely at the Weil pairing.

Lemma 8.6. *For any $\zeta \in \mu_p^*$ and σ (as above), we can define an analytic automorphism of $\tau_f^{-1}(\mathfrak{D})/\mathfrak{D}$ by $S_{\sigma,\zeta}(E, H) = (E, \langle R\rangle)$, where $R \in E[p]$ is chosen so that $e_p(P, R) = \zeta$ and $R - P \in H$ for some $P \in P_\sigma(E)$. Also,*

(i) $S^i_{\sigma,\zeta}(E, H) = (E, \langle R + (i-1)P\rangle)$ *for* $i \in \mathbb{Z}$;

(ii) $f_{\sigma,\zeta} : \mathbb{Z}/p\mathbb{Z} \to \mathrm{Aut}_{an}(\tau_f^{-1}(\mathfrak{D})/\mathfrak{D})$, *defined by* $f_{\sigma,\zeta}(i) = S^i_{\sigma,\zeta}$ *for* $i \neq 0$ *and the identity otherwise, is an injective homomorphism*;

(iii) $S_{a\sigma,\zeta^b} = S^{a^2/b}_{\sigma,\zeta}$ *for any* $a, b \in (\mathbb{Z}/p\mathbb{Z})^*$;

(iv) $S^\tau_{\sigma,\zeta} = S_{\sigma^\tau,\zeta^\tau}$ *for any* $\tau \in \mathrm{Aut}_{\mathrm{cont}}(\mathbb{C}_p)$ *that preserves* $\mathfrak{D}$.

Proof. Fix σ and $\zeta \in \mu_p^*$. For a given pair (E, H) and choice of $P \in P_\sigma(E)$, there is a unique $R \in E[p]$ that satisfies the two conditions. Note also that reversing the sign of P just reverses the sign of R. Since $\langle R\rangle = \langle -R\rangle$ is neither H nor the canonical subgroup, it follows that $S_{\sigma,\zeta}$ is at least a well-defined automorphism of $\tau_f^{-1}(\mathfrak{D})/\mathfrak{D}$ with no fixed points.

Fix $P \in P_\sigma(E)$. It is easy to verify (i) by induction, and then (ii) follows immediately. To prove (iii), we note that by definition $S_{a\sigma,\zeta^b}(E, H)$ is the pair $(E, \langle Q\rangle)$ where $e_p(aP, Q) = \zeta^b$ and $Q - aP \in H$. A simple Weil pairing calculation shows that Q is just $aP+(b/a)(R-P)$. So we verify (iii) by checking that

$$e_p(aP+(b/a)(R-P), R+(a^2/b-1)P)=1.$$

Finally, property (iv) follows from Galois properties of the Weil pairing and the fact that $\mathfrak{D}$ is connected. □

Proposition 8.7. *Let L be a finite extension of $\mathbb{Q}_p(\sqrt{\kappa},\beta)$ in $\mathbb{C}_p$, with $\sqrt{\kappa}$ and β as in Equation* (6), *over which the fixed points of $\widetilde{w}_\rho$ are defined. Then $\widetilde{\mathfrak{D}}^i_\rho$ is a basic wide open over a quadratic extension of L, whose underlying affinoid has good reduction, which can be described by $y^2=x^p-x$.*

Proof. As usual, let $\mathfrak{D}$ be either of the residue disks of $\mathbf{SD}_A$ fixed by the involution w_ρ, and let $\widetilde{\mathfrak{D}}$ be the wide open lying over $\mathfrak{D}$ via π_{11}. Then the embedding of $\mathbf{Z}_A=\pi_{11}^{-1}(\mathbf{SD}_A)$ into $\mathbf{C}_A\times\mathbf{C}_A$ embeds $\widetilde{\mathfrak{D}}$ into $\tau_f^{-1}(\mathfrak{D})\times\rho'\tau_f^{-1}(\mathfrak{D})$. Therefore, by the previous lemma, we can lift any automorphism $S:=S_{\sigma,\zeta}$ on $\tau_f^{-1}(\mathfrak{D})$ (for a fixed σ and ζ) to an automorphism $\widetilde{S}:=\widetilde{S}_{\sigma,\zeta}$ of $\widetilde{\mathfrak{D}}$ by taking

$$\widetilde{S}(x,y)=(S(x),\rho'S(\rho y)).$$

One easily checks that $\widetilde{S}$ also has order p, since $\widetilde{S}^i(x,y)=(S^i(x),\rho'S^i(\rho y))$. Furthermore, $\widetilde{S}$ commutes with $\widetilde{w}_\rho$:

$$\begin{aligned}\widetilde{S}\widetilde{w}_\rho(x,y)&=\widetilde{S}(\rho y,\rho' x)\\&=(S(\rho y),\rho' S\rho(\rho' x))=(S(\rho y),\rho' S(x)),\\ \widetilde{w}_\rho\widetilde{S}(x,y)&=\widetilde{w}_\rho(S(x),\rho' S(\rho y))\\&=(\rho\rho' S(\rho y),\rho' S(x))=(S(\rho y),\rho' S(x)).\end{aligned}$$

It follows that $\widetilde{S}$ passes to an automorphism of $\widetilde{\mathfrak{D}}/\widetilde{w}_\rho$ with order p and no fixed points, which acts transitively on the images of the p fixed points of $\widetilde{w}_\rho$. This is the key idea in the proof of the proposition.

To finish the argument, recall from Corollary 8.5 that $\widetilde{\mathfrak{D}}$ is a connected wide open with one end. The involution $\widetilde{w}_\rho$ acts on it with p fixed points, and the quotient space, say $U:=\widetilde{\mathfrak{D}}/\widetilde{w}_\rho$, is a disk, by Proposition 8.4. It follows that, over a quadratic extension of L, $\widetilde{\mathfrak{D}}$ can be described by

$$y_0^2=(x_0-\alpha_1)\cdots(x_0-\alpha_p),$$

where x_0 is a parameter for U and the α_i are the x_0 coordinates of the p fixed points. Without loss of generality, we choose x_0 so that U is identified with the disk, $v(x_0)>0$. Because $\widetilde{S}$ passes to an automorphism of a disk of order p and no fixed points, it must reduce to a translation, in the sense that there exists an $a\in R_p$ with $v(a)>0$ such that for all $x_0\in U$ we have

$$v(\widetilde{S}(x_0)-(x_0+a))>v(a).$$

Therefore, after possible reordering, the x_0 coordinates of the fixed points must satisfy

$$\beta_i := \frac{\alpha_i - \alpha_1}{a} \equiv i \pmod{m_p}.$$

So if we make the changes of variables $x = (x_0 - \alpha_1)/a$ and $y = y_0/a^{p/2}$, we identify $\widetilde{\mathfrak{D}}$ with the wide open

$$y^2 = (x - \beta_1)(x - \beta_2) \cdots (x - \beta_p), \quad \text{where } v(x) > -v(a),$$

whose minimal underlying affinoid, determined by $v(x) \geq 0$, reduces as claimed. □

Remark 8.8. The results of Section 8 were proven for $A/\mathbb{F}_p$ with $j(A) \neq 0, 1728$, but similar results now follow for any other supersingular A', by Proposition 4.2. Since $\mathbf{Z}_{A'}$ is an étale quotient of $\mathbf{Z}_A$ of degree $i(A)$, $\overline{\mathbf{Z}}_{A'}$ is a genus 0 curve with $2(p+1)/i(A')$ singular points, corresponding to basic wide opens that are isomorphic to those described in Proposition 8.7. Note, however, that one might need to replace the field L from Corollary 8.5 by a finite unramified extension in order to define the surjection from $W_A(p^3)$ onto $W_{A'}(p^3)$ and describe the underlying affinoids. In general, the reduction of the bridging component has the equation

$$X^{(p+1)/i(A)} + X^{-(p+1)/i(A)} = Z^p.$$

Lemma 8.9. *The Hecke correspondence T_ℓ takes a divisor supported on $\bigcup_A \mathbf{Z}_A$ to a divisor supported on $\bigcup_A \mathbf{Z}_A$ for all primes $\ell \neq p$.*

Proof. A point (E, C) lies on some $\mathbf{Z}_A$ if and only if C is cyclic of order p^3 and pC/p^2C is self-dual. If $f : E \to F$ is an isogeny such that $\ker(f) \cap C = 0$, the same is true for $(F, f(C))$. □

Remark 8.10. The analogous statement, for the union of all of the underlying affinoids in Proposition 8.7 (corresponding to new components), follows from the results of [CM 2006, §8].

9. Stable reduction of $X_0(p^3)$

In this section we give the stable covering of $X_0(p^3)$. In particular, we give a covering by *basic* wide opens, whose intersections are annuli as in Proposition 2.34. We already defined some of these wide opens, namely the $W^{\pm}_{ab}$, in Section 3. They cover the ordinary locus, and will be shown to be basic with the $\mathbf{X}^{\pm}_{ab}$ as underlying affinoids. From our analysis of $\mathbf{Z}_A$ in Section 8, we now know that $W_A(p^3)$ is not a basic wide open. So our next priority is to specify some new wide open subspaces that cover each $W_A(p^3)$ and that can ultimately be shown to be basic.

Now let A be any supersingular elliptic curve mod p (no restriction). Identify $W_{A^\sigma}(p)$, where σ is the Frobenius automorphism, with the annulus $A(p^{-i(A)}, 1)$, as explained in Section 3A. Then we can define three subspaces of $W_A(p^3)$:

$$\begin{aligned} V_1(A) &:= \pi_{11}^{-1} A(p^{-i(A)}, p^{-i(A)/2}),\\ V_2(A) &:= \pi_{11}^{-1} A(p^{-i(A)/2}, 1),\\ U(A) &:= \pi_{11}^{-1} A(p^{-pi(A)/(p+1)}, p^{-i(A)/(p+1)}). \end{aligned}$$

First we want to show that these subspaces are wide opens (over $\mathbb{C}_p$). Since $V_1(A)$ is a union of residue classes of the affinoid

$$\pi_{11}^{-1}(X_{01} \cup A(p^{-i(A)}, p^{-i(A)/2}]),$$

and since it is connected, it is in fact one residue class and therefore a wide open, by Theorem 2.29. The same argument applies to $V_2(A)$ and $U(A)$, the latter being a residue class of

$$\pi_{11}^{-1} A[p^{-pi(A)/(p+1)}, p^{-i(A)/(p+1)}].$$

Remark 9.1. The points of $A(p^{-pi(A)/(p+1)}, p^{-i(A)/(p+1)})$ are pairs (E, C), where C is the canonical subgroup of E and $E[p]/C$ is the canonical subgroup of E/C.

Two of these supersingular wide opens will in fact be shown to be basic. More specifically, $V_1(A)$ is a wide open neighborhood of the affinoid

$$\mathbf{E}_{1A} := \pi_{11}^{-1} C[p^{-pi(A)/(p+1)}],$$

which will be shown to be an underlying affinoid with good reduction. Points of $\mathbf{E}_{1A}$ are pairs (E, C), such that E/p^2C is too supersingular. Alternatively, $\mathbf{E}_{1A}$ can be described as $\pi_f^{-1}\mathbf{Y}_A$, which is a key point because it implies that $\mathbf{E}_{1A}$ is nontrivial. Similarly, $V_2(A)$ is a neighborhood of

$$\mathbf{E}_{2A} := \pi_{11}^{-1} C[p^{-i(A)/(p+1)}].$$

Points of $\mathbf{E}_{2A}$ are pairs (E, C) with E/pC too supersingular, and $\mathbf{E}_{2A}$ maps onto $\mathbf{Y}_{A^\sigma}$ via π_v. $U(A)$ is not basic, because its underlying affinoid $\mathbf{Z}_A$ has the $\widetilde{\mathfrak{D}}^i_\rho$ as (bad) residue classes. However, the $\widetilde{\mathfrak{D}}^i_\rho$ were shown to be basic in Proposition 8.7. So this problem can essentially be solved by removing the underlying affinoids of the $\widetilde{\mathfrak{D}}^i_\rho$ from $U(A)$ (obtaining a basic wide open) and then including the $\widetilde{\mathfrak{D}}^i_\rho$ in the overall covering. To be more precise, let $\mathcal{S}(A)$ denote the set of singular residue classes of $\mathbf{Z}_A$, and for each $S \in \mathcal{S}(A)$ let $\mathbf{X}_S$ be the underlying affinoid of S. Let $\hat{U}(A)$ denote the wide open given by

$$\hat{U}(A) := U(A) \setminus \bigcup_{S \in \mathcal{S}(A)} \mathbf{X}_S.$$

Theorem 9.2. *Let $p \geq 13$ be a prime. The covering $\mathscr{C}_0(p^3)$ of $X_0(p^3)$, which is made up of*

$$\{W_{ab}^{\pm} \mid a, b \geq 0, a+b=3\}$$

and the union over all supersingular curves A of

$$\{V_1(A), V_2(A), \hat{U}(A)\} \cup \mathscr{S}(A),$$

is stable (over $\mathbb{C}_p$).

Proof. We know that the elements of $\mathscr{C}_0(p^3)$ are wide opens, and that $(S, \mathbf{X}_S)$ is a basic wide open pair for each $S \in \mathscr{S}(A)$. It is also easy to verify that condition (ii) of Proposition 2.34 holds, by simply listing for each wide open the other members of the covering that intersect it nontrivially. In particular, the $W_{ab}^{\pm}$ are disjoint from each other, and each $W_{ab}^{\pm}$ intersects $W_A(p^3)$ only at $V_2(A)$ when $a > b$, and only at $V_1(A)$ otherwise. Similarly, while $V_1(A)$, $V_2(A)$, and the residue classes $S \in \mathscr{S}(A)$ are pairwise disjoint, each of these wide opens intersects $\hat{U}(A)$ nontrivially. This completely describes all adjacency relations of wide opens in the covering, and it follows immediately that every triple intersection is empty. The bulk of what we still have to show is that whenever two wide opens in the cover do intersect, the intersection is the disjoint union of annuli. Then we have to show that each wide open is basic, with an underlying affinoid that has good reduction.

We start by showing that

$$U_{ab}^{\pm}(A) := W_{ab}^{\pm} \cap W_A(p^3)$$

is a wide open annulus in all cases. For U_{30} and $U_{21}^{\pm}$ it suffices to consider the map π_{02} from $X_0(p^3)$ to $X_0(p)$. The restriction of π_{02} to U_{30} is an isomorphism onto the annulus

$$B := A(p^{-i(A)/(p(p+1))}, 1) \cong A(1, p^{i(A)/(p(p+1))})$$

(considered as a subspace of $W_A(p)$, which has been identified with $A(p^{-i(A)}, 1)$ as in Section 3A). So U_{30} is an annulus right away. U_{21}^{+} and U_{21}^{-} also map onto B via π_{02}, but each with degree $(p-1)/2$. To see that $U_{21}^{\pm}$ is at least connected, we look at how π_{02} reduces when restricted to a map between the affinoid regions $\mathbf{X}_{21}^{\pm}$ and $\mathbf{X}_{10}$. The latter is an isomorphic copy of the ordinary locus of $X(1)$, and by [Coleman 2005, p. 5] the reduction of $\mathbf{X}_{21}^{\pm}$ is isomorphic to the ordinary locus of $\mathrm{Ig}(p)$. Furthermore, by these identifications, π_{02} reduces to the forgetful map from $\mathrm{Ig}(p)$ to $X(1)$, which is totally ramified at the supersingular points. This implies that one of the ends of B totally ramifies in the restriction of π_{02} to $U_{21}^{\pm}$. Hence $U_{21}^{\pm}$ must be connected. Now it follows directly from Theorem 2.6 that $U_{21}^{\pm}$ is an annulus. Similar arguments can be made for $U_{12}^{\pm}$ and U_{03} using π_{20}. Alternatively one can use the fact that the Atkin–Lehner involution, w_3, switches W_{ab} with W_{ba}

and $W_A(p^3)$ with $W_{A^\sigma}(p^3)$. Note that from this argument we also deduce that each $(W^{\pm}_{ab}, \mathbf{X}^{\pm}_{ab})$ is a basic wide open pair.

Among the remaining intersections of wide opens in the covering, we also have $S \cap \hat{U}(A)$ for each $S \in \mathcal{S}(A)$. It is immediate, however, that this is an annulus, since S is a basic wide open with one end, and by definition $S \cap \hat{U}(A)$ is the complement in S of its underlying affinoid $\mathbf{X}_S$. So all that remains to be proven is that $V_i(A) \cap \hat{U}(A)$ is the disjoint union of annuli (in fact, one annulus), and that $(V_1(A), \mathbf{E}_{1A})$, $(V_2(A), \mathbf{E}_{2A})$, and $(\hat{U}(A), \mathbf{Z}_A)$ are basic wide open pairs. This essentially comes down to a genus computation and Proposition 2.34.

First shrink each $\hat{U}(A)$ to a basic wide open neighborhood $\hat{U}'(A)$ of $\mathbf{Z}_A$, and call the resulting covering $\mathcal{C}_1(p^3)$. Although we do not know that $\mathcal{C}_1$ is semistable (and in fact it isn't), Proposition 2.34 can still be applied as the wide opens in the covering intersect properly in the disjoint union of annuli. Moreover, we know that the intersection of $\hat{U}'(A)$ with $V_i(A)$ is just one annulus, because $\overline{\mathbf{Z}}_A$ has only two points at infinity (see Theorem 2.29). So the Betti number of the graph associated to $\mathcal{C}_1(p^3)$ is exactly $5(s_p - 1)$, where s_p is the number of supersingular j-invariants (mod p). To apply Proposition 2.34, we need to know the genera of the wide opens in $\mathcal{C}_1(p^3)$. The genus of W_{ab} is 0 when $ab = 0$ and $g(\mathrm{Ig}(p))$ otherwise, by [Coleman 2005, §1]. The genus of $\hat{U}'(A)$ is 0 and the genus of each $S \in \mathcal{S}(A)$ is $(p-1)/2$, by Proposition 8.2 and Corollary 8.5. The only genera that aren't immediately available are those of $V_1(A)$ and $V_2(A)$. We can, however, provide a lower bound for these genera. Recall that $\mathbf{E}_{1A}$ maps onto $\mathbf{Y}_A$ via π_f, and $\mathbf{E}_{2A}$ maps onto $\mathbf{Y}_{A^\sigma}$ via π_v. So by a Riemann–Hurwitz argument we know that $g(V_i(A)) \geq g(\overline{\mathbf{Y}}_A)$ (which we know from Corollary 5.4).

We now compute a lower bound for the genus of $X_0(p^3)$, using the above and Proposition 2.34. For brevity we only discuss the case $p = 12k + 5$. Then $s_p = k + 1$ and from [Igusa 1968, p. 103] we have $g(\mathrm{Ig}(p)) = 3k^2 - k$. There are k supersingular regions with $j(A) \neq 0, 1728$, each of which contributes two wide opens $V_1(A)$ and $V_2(A)$ of genus at least $g(\mathbf{Y}_A) = 6k + 2$, and $24k + 12$ residue classes $S \in \mathcal{S}(A)$ with genus $6k+2$. In addition, we have one supersingular region corresponding to $j(A) = 0$ that contributes two wide opens $V_1(A)$ and $V_2(A)$ of genus at least $g(\mathbf{Y}_A) = 2k$, and $8k + 4$ residue classes $S \in \mathcal{S}(A)$ of genus $6k + 2$. Summing up the Betti number and genera as in Proposition 2.34, we have

$$\begin{aligned} g(X_0(p^3)) &\leq 5k + 4(3k^2 - k) + 2(2k) + (8k + 4)(6k + 2) \\ &\qquad\qquad + k(2(6k + 2) + (24k + 12)(6k + 2)) \\ &\leq 144k^3 + 192k^2 + 73k + 8. \end{aligned}$$

This is now easily shown to be the *actual* genus of $X_0(p^3)$ using the well-known genus formula [Shimura 1971, Propositions 1.40 and 1.43]. Thus the inequalities above are actually equalities. Furthermore, since $g(V_i(A)) \geq g(\overline{\mathbf{E}}_{i\,A}) \geq g(\overline{\mathbf{Y}}_A)$, Lemma 2.43 implies that $V_1(A)$ and $V_2(A)$ are basic wide opens such that $\mathbf{E}_{1\,A}$ and $\mathbf{E}_{2\,A}$ are Zariski subaffinoids of the underlying affinoids. Then, since the reductions of these affinoids each have at least four points at infinity, and since $V_i(A)$ has only four ends, it follows that $\mathbf{E}_{1\,A}$ and $\mathbf{E}_{2\,A}$ are the underlying affinoids (with good reduction). Therefore $V_i(A) \cap \hat{U}(A)$ must be an annulus, and we have shown that $\mathscr{C}_0(p^3)$ is a stable covering. □

Remark 9.3. Since $\mathbf{E}_{1\,A} = \pi_f^{-1}(\mathbf{Y}_A)$, and since $\mathbf{E}_{1\,A}$ has good reduction with $g(\overline{\mathbf{E}}_{1\,A}) = g(\overline{\mathbf{Y}}_A)$, it follows that $\pi_f : \overline{\mathbf{E}}_{1\,A} \to \overline{\mathbf{Y}}_A$ is purely inseparable and factors as Frobenius followed by an isomorphism. Hence, $\overline{\mathbf{E}}_{1\,A} \cong \overline{\mathbf{Y}}_A^{\sigma}$, and similarly $\overline{\mathbf{E}}_{2\,A} \cong \overline{\mathbf{Y}}_{A^\sigma}^{\sigma}$.

9A. *Graphs and intersection data.* From Theorem 9.2, it is now straightforward to generate graphs for the stable reduction of $X_0(p^3)$ according to the four classes of p (mod 12), and we include these graphs below in Figures 2–5. To make the graphs more understandable, a brief description of how the various components are organized and labeled is in order. First of all, recall from Section 3B that there are

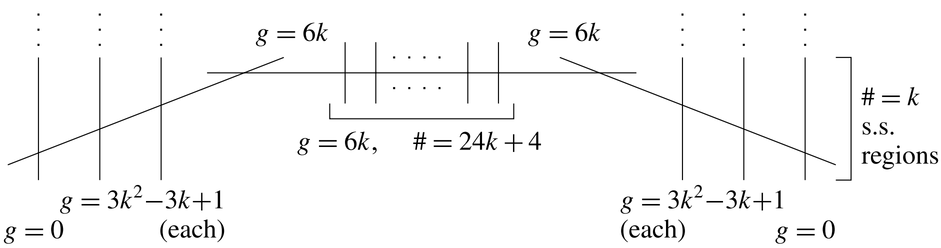

Figure 2. Graph of $X_0(p^3)$ when $p = 12k + 1$.

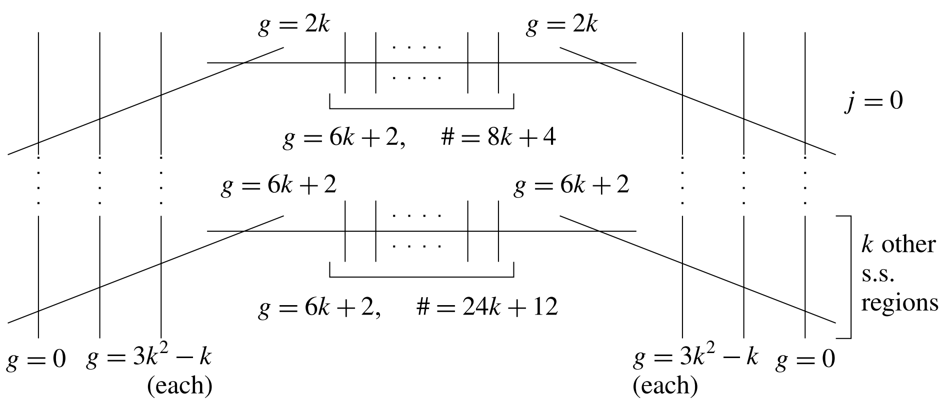

Figure 3. Graph of $X_0(p^3)$ when $p = 12k + 5$.

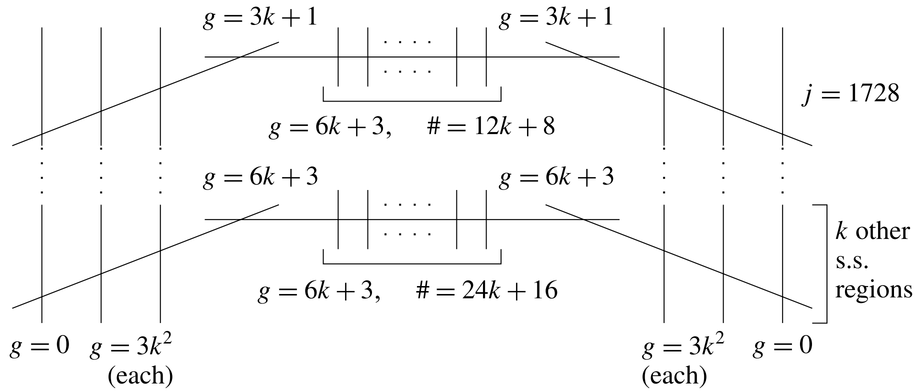

Figure 4. Graph of $X_0(p^3)$ when $p=12k+7$.

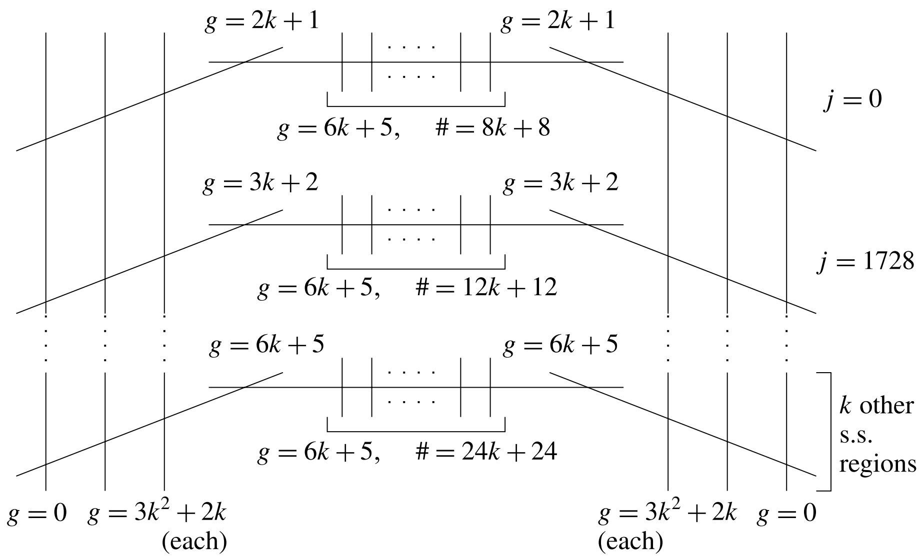

Figure 5. Graph of $X_0(p^3)$ when $p=12k+11$.

six ordinary components in every case, namely those corresponding to $\mathbf{X}_{30}$, $\mathbf{X}_{21}^{\pm}$, $\mathbf{X}_{12}^{\pm}$, and $\mathbf{X}_{03}$. These are always presented as vertical components and labeled explicitly with their genera. In addition to the six ordinary components, we have one connected, acyclic configuration of components for each supersingular elliptic curve A. This configuration is always presented as a horizontal chain of three components, corresponding to $\mathbf{E}_{2A}$, $\mathbf{Z}_A$, and $\mathbf{E}_{1A}$ (in that order), along with a number of unmarked vertical components intersecting the middle component. We explicitly label the genera of the reductions of $\mathbf{E}_{1A}$ and $\mathbf{E}_{2A}$, but not the central "bridging component", as it always has genus 0. Below the central horizontal component, we list the number of copies of $y^2 = x^p - x$ that intersect it, as well as the genus of each copy. Finally, we point out for clarification that the components corresponding to $\mathbf{X}_{30}$ and $\mathbf{X}_{21}^{\pm}$ meet each supersingular region in exactly one point in the reduction of $\mathbf{E}_{2A}$, while the same can be said for the other three ordinary components and $\mathbf{E}_{1A}$. In particular, one is reading the graph properly if the Betti number (equivalently the toric rank of the Jacobian) appears to be $5(ss-1)$, where ss is the number of supersingular j-invariants. This fact generalizes, as we show in the following theorem.

Theorem 9.4. *The toric rank of* $J_0(Np^n)$ *for* $(N, p) = 1$ *and* $n \geq 0$ *is given by* $(s(N) - 1)(2n - 1)$, *where* $s(N)$ *is the number of supersingular points on* $X_0(N)$ *mod* p.

Proof. For $N = 1$ and $n \leq 1$ this follows from [Deligne and Rapoport 1973, §VI.6]. After inverting isogenies, we have the exact sequence

$$0 \to J_0(p^{n-1}) \to J_0(p^n) \times J_0(p^n) \to J_0(p^{n+1}) \to J_0(p^{n+1})^{\text{new}} \to 0.$$

It follows from [Katz and Mazur 1985, Theorem 14.7.2] that $J_0(p^n)^{\text{new}}$ has potential good reduction for $n > 1$. Thus, by induction, the theorem is true for all $J_0(p^n)$. The result for more general N follows from essentially the same argument. □

To go with the stable reduction graphs, we include the intersection multiplicities in Table 1. These numbers have been obtained via a rigid analytic reformulation. In particular, suppose that X and Y are components of a curve with semistable reduction over some extension $K/\mathbb{Q}_p$, and that they intersect in an ordinary double point P. Then $R(P)$ is an annulus (by Proposition 2.10), say with width $w(P)$. In this case, the intersection multiplicity of X and Y at P can be found by

$$\mathrm{M}_K(P) = e_p(K) \cdot w(P).$$

Note that while intersection multiplicity depends on K, the width makes sense even over $\mathbb{C}_p$, which in some sense makes width a more natural invariant from the purely geometric perspective.

P	(X_{30}, E_{2A}), (X_{03}, E_{1A})	$(X_{21}^{\pm}, E_{2A})$, $(X_{12}^{\pm}, E_{1A})$	$(\mathbf{Z}_A, E_{2A})$, $(\mathbf{Z}_A, E_{1A})$	$(\mathbf{X}_S, \mathbf{Z}_A)$
$w(P)$	$\dfrac{i(A)}{p(p+1)}$	$\dfrac{2 \cdot i(A)}{p(p^2-1)}$	$\dfrac{(p-1) \cdot i(A)}{2p^2(p+1)}$	$\dfrac{1}{4p^2}$ *
$\mathrm{M}_K(P)$	$p(p-1) \cdot i(A)$	$2p \cdot i(A)$	$\dfrac{(p-1)^2 \cdot i(A)}{2}$	$\dfrac{p^2-1}{4}$ *

Table 1. Intersection multiplicity data for $X_0(p^3)$.

For our calculations on $X_0(p^3)$, we take $e_p(K) = p^2(p^2-1)$, since this is the ramification index over $\mathbb{Q}_p$ for the field of Krir (see [CM 2006, §5] for details). First we treat those singular points where $\mathbf{E}_{iA}$ meets either an ordinary component or the bridging component. The reduction inverse of any such singular point is an annulus in the supersingular locus that surjects via the forgetful map onto some subannulus of $W_A(p)$. Using Hasse invariant and canonical subgroup considerations, we can determine this subannulus and in particular its width. Then we apply Proposition 2.2. For example, the ordinary component corresponding to $\mathbf{X}_{30}$ intersects the one corresponding to (each) $\mathbf{E}_{2A}$ in a unique singular point. As we saw in the proof of Theorem 9.2, the corresponding annulus maps via π_{02} (the forgetful map) onto the subannulus of $W_A(p)$ described by

$$0 < v(x_A) < \frac{i(A)}{p(p+1)} \quad \text{with degree 1.}$$

The ordinary components corresponding to $\mathbf{X}_{21}^{\pm}$ also meet the reduction of $\mathbf{E}_{2A}$ in exactly one singular point (each). The corresponding two annuli surject onto this same subannulus, but with degree $(p-1)/2$. Using this line of reasoning, we arrive at most of the data in Table 1. Note that any two components intersect in at most one point, and so we may designate a singular point in the stable reduction unambiguously by listing a pair of intersecting components.

The only intersection multiplicities that do *not* follow readily from the above reasoning come from singular points where a copy of $y^2 = x^p - x$ (denoted $\mathbf{X}_S$ for $S \in \mathcal{S}(A)$ as in the theorem) intersects a bridging component. At such a singular point, the corresponding annulus maps via π_{11} onto an annulus that is the complement of an affinoid disk inside a residue disk of $\mathbf{SD}_A$. Unfortunately, it is not at all clear what the width of this image annulus is. We have some theoretical evidence and some computational evidence [McMurdy 2004, Remark on p. 27] that suggest that the width of the original annulus, that is, the annulus of intersection, is $1/4p^2$. Therefore, we have included this in Table 1 with an asterisk to indicate that it is our current best guess.

Appendix A: Riemann existence theorem

The p-adic Riemann existence theorem is well known, but not apparently in the literature.[21] Here we recall and adapt the proof of the existence of global meromorphic functions given in [Grauert and Remmert 1977, pp. 208–209], and then use results from [Kiehl 1967] and [Köpf 1974] to deduce the final result.

[21] This is possibly because it follows the same lines of reasoning as those used in “the” complex case; see [Springer 1957] for a history of the complex proofs and for a proof that has no obvious p-adic analogue.

Theorem A.1. *Suppose X is a proper*[22] *one-dimensional smooth rigid space over a complete local field K or a compact Riemann surface, $\mathcal{F} \neq 0$ is a locally free sheaf on X, and D is a divisor of positive degree. Then*

$$\lim_{n\to\infty} \dim_K \mathcal{F}(nD)(X) = \infty,$$

where $K = \mathbb{C}$ if X is a Riemann surface.

Proof. Let $E \leq E'$ be divisors on X, and let $\mathcal{T} = \mathcal{F}(E')/\mathcal{F}(E)$. Then

$$0 \to \mathcal{F}(E)(X) \to \mathcal{F}(E')(X) \to \mathcal{T}(X) \to H^1(X, \mathcal{F}(E)) \to H^1(X, \mathcal{F}(E')) \to 0$$

is exact. Moreover, if r is the rank of $\mathcal{F}$, we have

$$\dim_K \mathcal{T}(X) = r \deg(E' - E).$$

Now, for any coherent sheaf $\mathcal{S}$ on X, let

$$\chi(\mathcal{S}) = \dim_K H^0(X, \mathcal{S}) - \dim_K H^1(X, \mathcal{S}).$$

We deduce that

$$\chi(\mathcal{F}(D)) - \chi(\mathcal{F}) = r \deg D.$$

The theorem follows. □

Theorem A.2 (p-adic Riemann existence theorem). *Let X be a smooth proper rigid space of dimension one over a complete local field K. Then X is isomorphic to the analytification of a complete algebraic curve over K.*

Proof. By the previous theorem, there exists a nonconstant map $f : X \to \mathbf{P}^1_K$ that must be finite since X is proper of dimension one. By Kiehl's direct image theorem [1967, Theorem 3.3], it follows that $f_*\mathcal{O}_X$ is a coherent sheaf of analytic algebras on $\mathbf{P}^1_K$. Then, we know from [Köpf 1974, Sätze 4.11 and 5.1] that $f_*\mathcal{O}_X \cong g_*\mathcal{O}_Y$, where g is a finite morphism from some algebraic curve Y onto $\mathbf{P}^1_K$.

To complete the proof, let $\mathcal{C}$ be an admissible open covering of $\mathbf{P}^1_K$ by affinoids. Then $f^{-1}(\mathcal{C})$ and $g^{-1}(\mathcal{C})$ are admissible open coverings of X and Y by affinoids. Moreover, for each $U \in \mathcal{C}$, we have

$$A(f^{-1}U) = f_*\mathcal{O}_X(U) \cong g_*\mathcal{O}_Y(U) = A(g^{-1}U).$$

Thus $f^{-1}U \cong g^{-1}U$ for each $U \in \mathcal{C}$, and these isomorphisms are compatible, which implies that $X \cong Y$. □

[22] See [Bosch et al. 1984, 9.6.2] for definition.

Appendix B: Supersingular curves

by Everett W. Howe

Theorem B.1. *For $p \geq 13$ there is a supersingular elliptic curve E defined over $\mathbb{F}_p$ with $j(E) \neq 0, 1728$.*

Proof. Note that there is always at least one supersingular curve over $\mathbb{F}_p$, because the number of curves of trace 0 is given by the Kronecker class number $H(-4p)$, which is positive [Schoof 1987]. So if p is a prime for which neither $j = 0$ nor $j = 1728$ is supersingular, then there exists a supersingular curve over $\mathbb{F}_p$ with $j \neq 0, 1728$.

If p is a prime for which $j = 0$ is supersingular, then p is inert in the field $\mathbb{Q}(\sqrt{-3})$. But then the elliptic curve over $\mathbb{Q}$ with $j = 2^4 \cdot 3^3 \cdot 5^3 = 54000$ (which has CM by the order $\mathbb{Z}[\sqrt{-3}]$) reduces to a supersingular curve over $\mathbb{F}_p$. (If an elliptic curve over $\mathbb{F}_p$ is not supersingular then its endomorphism ring tensored with $\mathbb{Q}$ is an imaginary quadratic field in which p splits.) Note that 54000 is neither 0 nor 1728 modulo p for $p > 11$.

If p is a prime for which $j = 1728$ is supersingular, then p is inert in the field $\mathbb{Q}(i)$. Then the elliptic curve over $\mathbb{Q}$ with $j = 2^3 \cdot 3^3 \cdot 11^3 = 287496$ (which has CM by $\mathbb{Z}[2i]$) reduces to a supersingular curve over $\mathbb{F}_p$, and 287496 is neither 0 nor 1728 modulo p when $p > 11$. □

Appendix C: Concordance with [CM 2006]

Some of the references in [CM 2006] are no longer correct due to some shuffling of the material in this paper. This problem can be resolved by noting the following:

- The reference to §2 on page 265 should be to Section 2C.
- Theorem 2.6 is referred to as Lemma 3.3 on page 295, and as Lemma 2.3 on page 278.
- Proposition 2.14 is referred to as Proposition 3.14 on page 279.
- Proposition 2.34 is referred to as Proposition 2.5 on pages 267 and 278.
- Definition 2.35 and Theorem 2.36 are referred to as Definition 2.6 and Proposition 2.7 on pages 279, 292 and 293.
- Proposition 3.6 is referred to as Lemma 3.6 on page 278.
- Proposition 4.6 is referred to as Corollary 4.6 on page 270.
- Remark 4.7 and Proposition 4.9 are referred to as Remark 4.8 and Proposition 4.10 on page 272.
- Proposition 7.5 and Remark 7.6 are referred to as Proposition 7.4 and Remark 7.5 on pages 267, 275, 277 and 281.
- Theorem B.1 is cited as "results of E. Howe in §10" on page 262.

Index of important notation

K, complete nonarchimedean-valued field	Section 2
R_K, ring of integers of K	
$\mathbb{F}_K$, residue field of K	
$\mathbf{C}$, completion of an algebraic closure of K	
$\mathbf{R}$, ring of integers of $\mathbf{C}$	
$\overline{\mathbb{F}}$, residue field of $\mathbf{C}$ and algebraic closure of $\mathbb{F}_K$	
$W(\mathbb{F})$, Witt vectors of $\mathbb{F}$ for $\mathbb{F} \subseteq \overline{\mathbb{F}}$	
$\mathcal{R}_K$, value group of $\mathbf{C}^*$	
$\mathbb{C}_p$, completion of an algebraic closure of $\mathbb{Q}_p$	
$\mathbb{R}_p$, ring of integers in $\mathbb{C}_p$	
$\mathbf{\Omega}_p$, completion of an algebraic closure of $\overline{\mathbb{F}}_p((T))$	
$\mathbb{N} := \{n \in \mathbb{Z} : n \geq 1\}$ and $\mathbb{N}_0 := \{n \in \mathbb{Z} : n \geq 0\}$	
$B_K(r)$ and $B_K[r]$, wide open and affinoid disks around 0	
$A_K(r, s)$ and $A_K[r, s]$, wide open and affinoid annuli	
$C_K[s]$, circle $A_K[s, s]$	
$A(X) := \mathcal{O}_X(X)$	
$A^o(X)$ and $A^+(X)$, subrings of $A(X)$ where $\Vert f\Vert_X \leq 1$ and $\Vert f\Vert_X < 1$ (when X is a reduced affinoid)	
$\overline{A(X)} := A^o(X)/A^+(X)$	
$\overline{X}$, canonical reduction of X given by $\mathrm{Spec}(\overline{A(X)})$	
$\mathrm{Red} : X(\mathbf{C}) \to \overline{X}(\overline{\mathbb{F}})$, reduction map on $\mathbf{C}$-valued points	
$\mathrm{Red}^{-1}(\tilde{Y})$, Zariski subaffinoid of X corresponding to affine open $\tilde{Y} \subseteq \overline{X}$	
$\overline{X}^c$, completion of $\overline{X}$, nonsingular at infinity	
$R(P) := R_X(P)$, residue class in X of $P \in \overline{X}(\mathbb{F}_K)$	Section 2A
$\mathrm{res}_{r,s}$, canonical residue map on the annulus, $A_K(r, s)$	
$\mathcal{E}(W)$, $e(W)$, set of ends, and number of ends, for a rigid space W	
$\mathrm{CC}(W)$, set of connected components of a rigid space W	Section 2B
$H^i_{DR}(W/K)$, de Rham cohomology of a wide open	Section 2C
$g(W)$, genus of a wide open	
$\mathcal{C}$ and $\mathcal{C}^w$, semistable coverings of a wide open or curve	
U^u, underlying affinoid of a wide open U, in a basic wide open pair	
$\Gamma_{\mathcal{C}}$, graph associated with a semistable covering	
$\mathrm{ord}_{\mathcal{A}}\, \nu$, $\mathrm{ord}_e\, \nu$, ord of a function or differential at an annulus or end	Section 2D
$\mathrm{Div}(W)$, divisor group of a wide open	
π_f, π_ν and π_{ab}, level lowering maps from $X_0(p^n)$ to $X_0(p^m)$	Section 3

w_n, Atkin–Lehner involution on $X_0(p^n)$	
$K_n(E)$, canonical subgroup of E of order p^n	Section 3A
$K(E)$, (maximal) canonical subgroup of E	
$h(E)$, valuation of Hasse invariant of E (almost)	
s_n, rigid analytic section of π_{0n} over W_n	
$W_A(p^n)$, wide open subspace of $X_0(p^n)$ where $\bar{E} \cong A$	
x_A, parameter on $W_A(p)$	
$i(A) := \lvert\mathrm{Aut}(A)\rvert/2$	
$\mathbf{TS}_A$ and $\mathbf{SD}_A$, too-supersingular and self-dual circles inside $W_A(p)$	
$\mathbf{C}_A$ and τ_f, special circle of $W_A(p)$ and map to $\mathbf{SD}_A$	
$\mathbf{X}^{\pm}_{ab}$, ordinary affinoids	Section 3B
$W^{\pm}_{ab}$, wide open neighborhood of $\mathbf{X}^{\pm}_{ab}$	
$\mathrm{Ig}(p^n)$, level p^n Igusa curve	
(F, A, α), Woods Hole representation of an elliptic curve	Section 4
$\hat{A}$, formal group of A	
B, quaternionic order over $\mathbb{Z}_p$ isomorphic to $\mathrm{End}(\hat{A})$	Section 4B
Φ, Gross–Hopkins period map	
B', special subset of B^*	
w_ρ, generalized Atkin–Lehner involution of $\mathbf{SD}_A$ for $\rho \in B'$	
$\mathbf{Y}_A$, nontrivial affinoid in $W_A(p^2)$	Section 5
$\mathscr{C}_0(p^2)$, stable covering of $X_0(p^2)$	
$\mathbf{E}_{1,A}$ and $\mathbf{E}_{2,A}$, two pullbacks of $\mathbf{Y}_A$ to $X_0(p^3)$	Section 6
$\mathbf{Z}_A := \pi_{11}^{-1}(\mathbf{SD}_A)$, affinoid in $W_A(p^3)$ that corresponds to the "bridging component"	
$\tilde{w}_\rho$, generalized Atkin–Lehner involution of $\mathbf{Z}_A$ for $\rho \in B'$	Section 7
$\mathscr{D}^i_\rho$ and $\widetilde{\mathscr{D}}^i_\rho$, residue classes of $\mathbf{SD}_A$ and $\mathbf{Z}_A$ invariant under w_ρ and $\widetilde{w}_\rho$	
$S_{\sigma,\zeta}$, $\tilde{S}_{\sigma,\zeta}$, order p automorphisms of $\tau_f^{-1}(\mathscr{D}^i_\rho)$ and $\widetilde{\mathscr{D}}^i_\rho$	Section 8B
$V_i(A)$ and $U(A)$, wide open neighborhoods of $\mathbf{E}_{i,A}$ and $\mathbf{Z}_A$	Section 9
$\mathscr{S}(A)$, singular residue classes of $\mathbf{Z}_A$	
$\mathbf{X}_S$, underlying affinoid of $S \in \mathscr{S}(A)$	
$\hat{U}(A)$, basic wide open refinement of $U(A)$	
$\mathscr{C}_0(p^3)$, stable covering of $X_0(p^3)$	
$M_K(P)$, intersection multiplicity at an ordinary double point	Section 9A
$w(P)$, width of the annulus that lifts an ordinary double point	

Acknowledgements

We are grateful to Ken Ribet for explaining to us how potential good reduction of $J_0(p^3)^{\text{new}}$ follows from known results. This greatly simplified our search for the stable models. In Theorem 9.4 we show how the generalization of this result for $J_0(p^n)^{\text{new}}$ (which follows from work of Katz and Mazur [1985]) can be used to compute the toric rank of $J_0(p^n)$. We would also like to express our appreciation to Kevin Buzzard, Brian Conrad, Dino Lorenzini, and Jonathan Lubin for helpful communications. The referee also made a number of suggestions that led to significant improvements in the manuscript, particularly in Section 2.

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