Algebra & Number Theory

Volume 4
2010
No. 5

On the Spiegelungssatz for the 4-rank

Étienne Fouvry and Jürgen Klüners

mathematical sciences publishers

On the Spiegelungssatz for the 4-rank

Étienne Fouvry and Jürgen Klüners

Let d be a nonsquare positive integer. We give the value of the natural probability that the narrow ideal class groups of the quadratic fields $\mathbb{Q}(\sqrt{d}\,)$ and $\mathbb{Q}(\sqrt{-d}\,)$ have the same 4-ranks.

1. Introduction

Conventions and notations. Throughout this work, the letter D is reserved for the number 1 or a fundamental discriminant, that is, the discriminant of a linear or quadratic extension of $\mathbb{Q}$. Let $K = \mathbb{Q}(\sqrt{D})$. On the set of nonzero fractional ideals of the ring of integers of K we say that two fractional ideals $\mathfrak{I}$ and $\mathfrak{J}$ are *equivalent in the narrow sense*, if there is an element $a \in K$, such that $\mathfrak{I} = (a)\mathfrak{J}$ and a has positive norm. By the multiplication of the ideal classes, we obtain the *(narrow) class group of* K, that we denote by C_D. This is a finite abelian group.

We extend this definition of C_D in the following way: if d is a nonsquare integer, not necessarily a fundamental discriminant, we also denote by C_d the class group of the quadratic field $\mathbb{Q}(\sqrt{d})$. When d is a nonzero perfect square, we define $\mathrm{C}_d = \mathrm{C}_1$ to be the trivial group.

We reserve the letter p for prime numbers. For positive integers n we denote by $\omega(n)$ the number of distinct prime divisors of n.

If A is a finite multiplicative abelian group and p is a prime number, the *p-rank* is, by definition, $\mathrm{rk}_p(A) := \dim_{\mathbb{F}_p}(A/A^p)$. More generally, if k is an integer ≥ 1, we define the p^k*-rank* of A by $\mathrm{rk}_{p^k}(A) := \dim_{\mathbb{F}_p}(A^{p^{k-1}}/A^{p^k})$.

Scholz's Theorem. The original *Spiegelungssatz* concerned the 3-rank of C_D and was proved by Scholz [1932] in the form of the double inequality

$$\mathrm{rk}_3(\mathrm{C}_d) \leq \mathrm{rk}_3(\mathrm{C}_{-3d}) \leq \mathrm{rk}_3(\mathrm{C}_d) + 1 \tag{1}$$

for any nonsquare $d \geq 1$. With the convention above, it is straightforward to extend (1) to any $d \geq 1$, since the group C_{-3} is trivial.

Hence, when $d \geq 1$ is given, the integer $\mathrm{rk}_3(\mathrm{C}_{-3d})$ can only take two values: $\mathrm{rk}_3(\mathrm{C}_d)$ or $\mathrm{rk}_3(\mathrm{C}_d) + 1$. Each of these possibilities is well described in algebraic

MSC2000: primary 11R29; secondary 11R11.
Keywords: quadratic fields, class groups, reflection principle.

terms. But the natural question is to know the frequency of each of these events. Dutarte [1984], further pushing the probabilistic model leading to the heuristics of Cohen–Lenstra [1984], proposed a value of the second frequency, namely:

Conjecture 1.1 [Dutarte 1984, Formula 3, p. 8] . *For every integer $r \geq 0$ we have*

$$\lim_{X\to+\infty} \frac{\sharp\{D : 0 \leq D \leq X,\ \mathrm{rk}_3(\mathrm{C}_D) = r,\ \mathrm{rk}_3(\mathrm{C}_{-3D}) = r+1\}}{\sharp\{D : 0 \leq D \leq X,\ \mathrm{rk}_3(\mathrm{C}_D) = r\}} = 3^{-(r+1)}. \tag{2}$$

This conjectural equality can be seen as a conditional probability under the following convention: Let $\mathscr{A}$ be a subset of the set $\mathfrak{D}^+$ of positive fundamental discriminants D. We define the probability of the event $D \in \mathscr{A}$ as being equal to the following limit, if it exists:

$$\mathrm{Prob}^+(\mathscr{A}) := \lim_{X\to+\infty} \Bigl(\sum_{\substack{0<D\leq X\\ D\in\mathscr{A}}} 1 \Bigm/ \sum_{0<D\leq X} 1 \Bigr). \tag{3}$$

In an analogous way we define Prob^- to be the natural density within the set $\mathfrak{D}^-$ of negative fundamental discriminants.

We now formulate in this notation certain statements of the Cohen–Lenstra heuristics [1984, C5 and C9, pp. 56–57], extended by Gerth to $p = 2$:

Conjecture 1.2. *Let p be prime and $r \geq 0$, and for all $k \in \mathbb{N} \cup \{\infty\}$ and $t > 1$ define*

$$\eta_k(t) := \prod_{j=1}^{k} \bigl(1 - t^{-j}\bigr). \tag{4}$$

Then:

(i) $\mathrm{Prob}^-(\{D \in \mathfrak{D}^- : \mathrm{rk}_p(\mathrm{C}_D^2) = r\}) = a_p^-(r) := p^{-r^2}\eta_\infty(p)\eta_r(p)^{-2}$.

(ii) $\mathrm{Prob}^+(\{D \in \mathfrak{D}^+ : \mathrm{rk}_p(\mathrm{C}_D^2) = r\}) = a_p^+(r) := p^{-r^2-r}\eta_\infty(p)\eta_r(p)^{-1}\eta_{r+1}(p)^{-1}$.

With these definitions, Conjecture 1.1 is just a statement concerning the existence and value of a conditional probability. In other words, Dutarte believes that for any $r \geq 0$ we have the equality

$$\mathrm{Prob}^+\bigl(\mathrm{rk}_3(\mathrm{C}_{-3D}) = r+1 \text{ and } \mathrm{rk}_3(\mathrm{C}_D) = r \bigm| \mathrm{rk}_3(\mathrm{C}_D) = r\bigr) = 3^{-r-1}. \tag{5}$$

Conjectures 1.1 and 1.2 imply, for every $r \geq 0$, the equalities

$$\mathrm{Prob}^+\bigl(\mathrm{rk}_3(\mathrm{C}_{-3D}) - 1 = \mathrm{rk}_3(\mathrm{C}_D) = r\bigr) = 3^{-(r+1)} a_3^+(r), \tag{6}$$

$$\mathrm{Prob}^+\bigl(\mathrm{rk}_3(\mathrm{C}_{-3D}) = \mathrm{rk}_3(\mathrm{C}_D) = r\bigr) = \bigl(1 - 3^{-(r+1)}\bigr)\, a_3^+(r). \tag{7}$$

Let $\mathfrak{D}^+(X)$ be the cardinality of the set $\mathfrak{D}^+ \cap [1, X]$, and let R be a fixed parameter. Then summing (6) and (7) for all $0 \leq r \leq R$, we get the two lower bounds

$$\sharp\{D\in\mathfrak{D}^+ : D\le X,\ \mathrm{rk}_3(\mathrm{C}_{-3D})=\mathrm{rk}_3(\mathrm{C}_D)+1\} \ \ge \Big(\sum_{r=0}^{R} 3^{-(r+1)}a_3^+(r)-o_R(1)\Big)\,\mathfrak{D}^+(X),$$

$$\sharp\{D\in\mathfrak{D}^+ : D\le X,\ \mathrm{rk}_3(\mathrm{C}_{-3D})=\mathrm{rk}_3(\mathrm{C}_D)\} \ \ge \Big(\sum_{r=0}^{R} (1-3^{-(r+1)})a_3^+(r)-o_R(1)\Big)\,\mathfrak{D}^+(X),$$

where $X\to\infty$. However, since the two sets appearing on the left side of these two inequalities form a partition of $\mathfrak{D}^+\cap[1, X]$, we obtain the double inequality

$$\sum_{r=0}^{R} 3^{-(r+1)}a_3^+(r)-o_R(1)\le \frac{\sharp\{D\in\mathfrak{D}^+ : D\le X,\ \mathrm{rk}_3(\mathrm{C}_{-3D})=\mathrm{rk}_3(\mathrm{C}_D)+1\}}{\mathfrak{D}^+(X)} \le 1-\sum_{r=0}^{R}(1-3^{-(r+1)})\,a_3^+(r)+o_R(1). \tag{8}$$

The relation

$$\sum_{r=0}^{\infty} a_p^+(r)=1 \tag{9}$$

implies the equality

$$\sum_{r=0}^{\infty} 3^{-(r+1)}a_3^+(r)=1-\sum_{r=0}^{\infty}(1-3^{-(r+1)})a_3^+(r). \tag{10}$$

Hence, letting $R\to\infty$ in (8), we obtain the equality

$$\begin{aligned}\mathrm{Prob}^+\big(\mathrm{rk}_3(\mathrm{C}_{-3D})=\mathrm{rk}_3(\mathrm{C}_D)+1\big) &= \lim_{R\to\infty}\sum_{r=0}^{R} 3^{-(r+1)}a_3^+(r)\\ &=\eta_\infty(3)\sum_{r=0}^{\infty} 3^{-(r+1)^2}\,\eta_r^{-2}(3)\,(1-3^{-(r+1)})^{-1} \qquad (11)\\ &=0.29765117\ldots.\end{aligned}$$

But this equality is conjectural for the moment. It has been tested on a computer by Dutarte [1984, §4.2]. We ran similar experiments and the constants are close, but not too close. We remark that similar problems occur in experiments when we check proved results for the 4-rank in this way, or when one wishes to test one of the Cohen–Lenstra heuristics. For example, similar problems for experiments occur in [Heath-Brown 1994, p. 336] and in [Stevenhagen 1993]. Usually the problem is that the second expected main term in the asymptotic expansion is close to the main term (see [Roberts 2001] for the case $p=3$).

As far as we know, the only result concerning the conjectural value (11) is due to Belabas [1999, Theorem 2.1; 2004], who proved the equality

$$\frac{\displaystyle\sum_{\substack{0<D\le X\\ \mathrm{rk}_3(\mathrm{C}_{-3D})=\mathrm{rk}_3(\mathrm{C}_D)+1}} 3^{\mathrm{rk}_3(\mathrm{C}_D)}}{\displaystyle\sum_{0<D\le X} 3^{\mathrm{rk}_3(\mathrm{C}_D)}}=\tfrac{1}{4}+O\big(\exp(-\tfrac{1}{5}(\log X\log\log X)^{1/2})\big) \tag{12}$$

as X tends to $+\infty$. This equality can be seen as a weighted version of (11). These weights are chosen in order to easily apply the seminal work of Davenport and Heilbronn [1971] concerning the average behavior of the 3-part of C_D.

1.1. *The Damey–Payan Theorem and Gerth's contribution.* Damey and Payan [1970, Theorems II.9 and II.10] have proved a similar phenomenon for the 4-rank:

Theorem 1.3 (Spiegelungssatz for the 4-rank). *For every $d \geq 1$ we have*

$$\mathrm{rk}_4(\mathrm{C}_d) \leq \mathrm{rk}_4(\mathrm{C}_{-d}) \leq \mathrm{rk}_4(\mathrm{C}_d) + 1. \tag{13}$$

Note the equality $\mathbb{Q}(\sqrt{-d}\,) = \mathbb{Q}(\sqrt{-4d}\,)$. We shall say that the fields $\mathbb{Q}(\sqrt{d}\,)$ and $\mathbb{Q}(\sqrt{-d}\,)$ are *reflected*. Note that $\mathbb{Q}$ is reflected to $\mathbb{Q}(\sqrt{-1}\,)$ by definition.

As for the 3-rank, the natural question is to evaluate the frequency of each of the events "$\mathrm{rk}_4(\mathrm{C}_{-d}) = \mathrm{rk}_4(\mathrm{C}_d)$" and "$\mathrm{rk}_4(\mathrm{C}_{-d}) = \mathrm{rk}_4(\mathrm{C}_d) + 1$". The only paper concerning this question is [Gerth 2001]. To present its results we introduce several notations. For $x \geq 1$ and integers $r, t \geq 0$ we introduce the two sets

$$A_{t;x} := \left\{m \in [1, x] : m \text{ squarefree and exactly } t \text{ primes ramify in } \mathbb{Q}(\sqrt{-m}\,)/\mathbb{Q}\right\}$$

and

$$A^{=}_{t,r;x} := \left\{m : m \in A_{t,x},\ \mathrm{rk}_4(\mathrm{C}_{-m}) = \mathrm{rk}_4(\mathrm{C}_m) = r\right\}.$$

Theorem 1.4 [Gerth 2001, p. 2551]. *For every integer $r \geq 0$ we have*

$$\lim_{t\to\infty} \lim_{x\to\infty} \frac{\sharp\ A^{=}_{t,r;x}}{\sharp\ A_{t;x}} = 2^{-r}\, 2^{-r^2} \eta_\infty(2)\eta_r(2)^{-2} = 2^{-r} a_2^{-}(r). \tag{14}$$

In this statement, Gerth has chosen to list all imaginary quadratic fields in the form $\mathbb{Q}(\sqrt{-m}\,)$ with m squarefree. Gerth could have adopted the other point of view of writing these imaginary fields in the form $\mathbb{Q}(\sqrt{D}\,)$ with D as a negative fundamental discriminant. This is the point of view that we prefer to adopt in this paper. Also remember that $D = -m$ or $D = -4m$ according to the cases $m \equiv 3 \bmod 4$ or $m \equiv 1$ or $2 \bmod 4$, and that exactly $\omega(|D|)$ primes ramify in $\mathbb{Q}(\sqrt{D}\,)$.

More precisely, here is the variant of Theorem 1.4 that we have in mind and that could have been equally proved by Gerth in [2001]:

Theorem 1.5. *For every integer $r \geq 0$ we have*

$$\lim_{t\to\infty} \lim_{X\to\infty} \frac{\sharp\left\{D\,;\ 0 < -D \leq X,\ \omega(|D|) = t,\ \mathrm{rk}_4(\mathrm{C}_D) = \mathrm{rk}_4(\mathrm{C}_{-D}) = r\right\}}{\sharp\left\{D : 0 < -D \leq X,\ \omega(|D|) = t\right\}} = 2^{-r} a_2^{-}(r).$$

Theorems 1.4 and 1.5 deserve several remarks. By mixing Theorem 1.5 with the central result of [Gerth 1984, Formula 1.5], we get:

Corollary 1.6 [Gerth 2001, p. 2551]. *For every integer* $r \geq 0$ *we have*

$$\lim_{t\to\infty}\ \lim_{X\to\infty} \frac{\sharp\,\{D\,;\ 0 < -D \leq X,\ \omega(|D|) = t,\ \mathrm{rk}_4(\mathrm{C}_D) = \mathrm{rk}_4(\mathrm{C}_{-D}) = r\}}{\sharp\,\{D : 0 < -D \leq X,\ \omega(|D|) = t,\ \mathrm{rk}_4(\mathrm{C}_D) = r\}} = 2^{-r}.$$

This corollary, roughly speaking, asserts that for an imaginary quadratic field with 4-rank equal to r, the probability (in the special sense introduced by Gerth) that its reflected field has the same 4-rank is equal to 2^{-r}.

Secondly, if we sum the equality contained in Theorem 1.5 for all $r \geq 0$ and appeal to the same trick already used in the proof of the equality (11), we obtain:

Corollary 1.7 [Gerth 2001, Theorem 1].

$$\lim_{t\to\infty}\ \lim_{X\to\infty} \frac{\sharp\,\{D : 0 < -D \leq X,\ \omega(-D) = t,\ \mathrm{rk}_4(\mathrm{C}_D) = \mathrm{rk}_4(\mathrm{C}_{-D})\}}{\sharp\,\{D : 0 < -D \leq X,\ \omega(|D|) = t\}} = \sum_{r=0}^{\infty} 2^{-r}\, a_2^-(r) = 0.610321\ldots .$$

The third remark is that Gerth could have equally stated Theorem 1.4 by first considering the value r of $\mathrm{rk}_4(\mathrm{C}_m)$ instead of $\mathrm{rk}_4(\mathrm{C}_{-m})$. Then the value of the second part of the equalities contained in Theorems 1.4, 1.5 and Corollary 1.6 would have been modified. Of course, the numerical constant appearing in Corollary 1.7 would have been unchanged.

The purpose of this paper is to prove the statements of Theorem 1.5 and Corollaries 1.6 and 1.7, but in the context of the more natural probability space, as defined in (3). This is far from being a simple transposition of the original proofs of Gerth, since he writes [2001, p. 2547]: "However, computing these limits appears to be very difficult." We will make an explicit comparison at the bottom of the next page. The limits given in the results above by Gerth are those that will appear in Theorem 1.8 below.

Statement of the results. The next theorem states the main result for the following natural densities, where Prob is defined in (3):

Theorem 1.8. *For every integer* $r \geq 0$,

$$\mathrm{Prob}^-(\{D \in \mathfrak{D}^- : \mathrm{rk}_4(\mathrm{C}_D) = \mathrm{rk}_4(\mathrm{C}_{-D}) = r\}) = a_2^-(r)\, 2^{-r}, \tag{15}$$

$$\mathrm{Prob}^-(\{D \in \mathfrak{D}^- : \mathrm{rk}_4(\mathrm{C}_D) = \mathrm{rk}_4(\mathrm{C}_{-D}) + 1 = r\}) = a_2^-(r)\, (1 - 2^{-r}), \tag{16}$$

$$\mathrm{Prob}^+(\{D \in \mathfrak{D}^+ : \mathrm{rk}_4(\mathrm{C}_D) = \mathrm{rk}_4(\mathrm{C}_{-D}) = r\}) = a_2^+(r)\, (1 - 2^{-(r+1)}), \tag{17}$$

$$\mathrm{Prob}^+(\{D \in \mathfrak{D}^+ : \mathrm{rk}_4(\mathrm{C}_D) = \mathrm{rk}_4(\mathrm{C}_{-D}) - 1 = r\}) = a_2^+(r)\, 2^{-(r+1)}. \tag{18}$$

The given densities are the same, if we further restrict to the negative (positive) fundamental D *congruent to* 1 mod 4, 0 mod 8, *or* 4 mod 8.

These results can be easily stated in conditional probabilities:

Corollary 1.9. *For $D < 0$ we get the conditional probabilities*

$$\mathrm{Prob}^-(\{\mathrm{rk}_4(\mathrm{C}_{-D}) = s \mid \mathrm{rk}_4(\mathrm{C}_D) = r\}) = \begin{cases} 2^{-r} & \text{if } r = s, \\ (1-2^{-r}) & \text{if } s+1 = r, \\ 0 & \text{otherwise.} \end{cases}$$

For $D > 0$ we get the conditional probabilities

$$\mathrm{Prob}^+(\{\mathrm{rk}_4(\mathrm{C}_{-D}) = s \mid \mathrm{rk}_4(\mathrm{C}_D) = r\}) = \begin{cases} (1-2^{-(r+1)}) & \text{if } r = s, \\ 2^{-(r+1)} & \text{if } s-1 = r, \\ 0 & \text{otherwise.} \end{cases}$$

This follows as an obvious byproduct of Theorem 1.8 as soon as one applies the central result of [Fouvry and Klüners 2007], which is recalled as Theorem 2.3 below. We remark that the values of these conditional probabilities, in the case of positive D, coincide with the values suggested by Dutarte (Conjecture 1.1), with the natural replacement of 2 by 3.

In Section 3 we also prove:

Corollary 1.10. *We have the equalities*

$$\mathrm{Prob}^-(\{D \in \mathfrak{D}^- : \mathrm{rk}_4(\mathrm{C}_D) = \mathrm{rk}_4(\mathrm{C}_{-D})\}) = \sum_{r=0}^{\infty} 2^{-r}\, a_2^-(r), \tag{19}$$

$$\mathrm{Prob}^-(\{D \in \mathfrak{D}^- : \mathrm{rk}_4(\mathrm{C}_D) = \mathrm{rk}_4(\mathrm{C}_{-D}) + 1\}) = \sum_{r=0}^{\infty} (1-2^{-r})\, a_2^-(r), \tag{20}$$

$$\mathrm{Prob}^+(\{D \in \mathfrak{D}^+ : \mathrm{rk}_4(\mathrm{C}_D) = \mathrm{rk}_4(\mathrm{C}_{-D})\}) = \sum_{r=0}^{\infty} (1-2^{-(r+1)})\, a_2^+(r), \tag{21}$$

$$\mathrm{Prob}^+(\{D \in \mathfrak{D}^+ : \mathrm{rk}_4(\mathrm{C}_D) = \mathrm{rk}_4(\mathrm{C}_{-D}) - 1\}) = \sum_{r=0}^{\infty} 2^{-(r+1)}\, a_2^+(r). \tag{22}$$

The given densities are the same, if we further restrict to the negative (positive) fundamental D congruent to 1 mod 4, 0 mod 8, *or* 4 mod 8.

It is important to remark that the values appearing on the right sides of Equations (15) and (19) coincide with the values appearing in Theorem 1.5 and Corollary 1.7, but the probabilistic models are not the same at all. However, these coincidences confirm an intuition of Gerth [2001, p. 2547]: "Although the limits we compute are not guaranteed to equal the limits above, our results do provide some insight into this question."

Comparison of our result with Gerth's approach. It is useful to compare Gerth's approach [2001] to ours, and the same comments apply to [Fouvry and Klüners 2007] when compared with [Gerth 1984]. To summarize the situation, let $\mathcal{Y}$ be a set of positive integers. We consider two ways of measuring the density of $\mathcal{Y}$:

- the natural one, defined by

$$\mathrm{dens}_{\mathrm{nat}}(\mathcal{Y}) := \lim_{X\to\infty} \frac{\#\{m \le X : m \in \mathcal{Y}\}}{\#\{m \le X\}},$$

- the density introduced by Gerth and defined by

$$\mathrm{dens}_{\mathrm{Gerth}}(\mathcal{Y}) := \lim_{t\to\infty} \lim_{X\to\infty} \frac{\#\{m \le X : m \in \mathcal{Y},\ \omega(m) = t\}}{\#\{m \le X : \omega(m) = t\}}.$$

These densities may not exist. There is no reason, generally speaking, that there exists a link between the existence of these two densities or between their values, as can be seen in the next two examples. First, let

$$\mathcal{Y} := \{m \ge 1 : \omega(m) \ge \tfrac{1}{2} \log\log(m+1)\}.$$

In this case, both densities exist and we have the equality

$$\mathrm{dens}_{\mathrm{nat}}(\mathcal{Y}) = 1.$$

This is a consequence of the well known fact that the function $m \mapsto \log\log(m+1)$ is a normal order of the additive function $m \mapsto \omega(m)$. For this notion, see for instance [Tenenbaum 2008, Chapter III.3]. We also have the trivial equality

$$\mathrm{dens}_{\mathrm{Gerth}}(\mathcal{Y}) = 0.$$

The second example consists in now defining $\mathcal{Y}$ as

$$\mathcal{Y} := \{m \ge 1 : \omega(m) \equiv 0 \bmod 2\}.$$

By the prime number theorem, we know that $\mathrm{dens}_{\mathrm{nat}}(\mathcal{Y}) = \frac{1}{2}$ and we trivially see that $\mathrm{dens}_{\mathrm{Gerth}}(\mathcal{Y})$ does not exist.

However, a link could be established between these two densities if the situation is such that one can ensure some uniformity in the double limit $\lim_{t\to\infty} \lim_{X\to\infty}$.

Gerth [1984; 2001] builds his proofs on the theory of Rédei matrices with dimension t, and it seems quite difficult to introduce the required uniformity in such an approach. In [Fouvry and Klüners 2007] we draw a new way of attacking these questions by replacing the theory of Rédei matrices by the study of oscillations of Jacobi symbols, without any restriction on the number of prime factors in the numerator and denominator. Note also that our proofs can be adapted to recover Gerth's results.

2. General results on the 4-rank

We have already seen that our problem is deeply connected to the Cohen–Lenstra heuristics, which were extended by Gerth to the case $p = 2$. In this section we collect the statements needed for our proofs.

As usual for quadratic fields, we need to distinguish between positive and negative D, corresponding to totally real and totally imaginary quadratic fields. For each case we also need to consider the behavior at 2, that is, if $D \equiv a \bmod q$ for the cases

$$(a, q) \in \{(1,4),\ (0,8),\ (4,8)\}. \tag{23}$$

Therefore we introduce six counting functions:

$$\mathfrak{D}^{\pm}(X, a, q) := \sum_{\substack{0 < \pm D \le X \\ D \equiv a \bmod q}} 1. \tag{24}$$

These are the cardinalities of positive (negative) fundamental discriminants (including 1) up to X, which are congruent to $a \bmod q$. These cardinalities are well known since we have

$$\mathfrak{D}^{-}(X, 1, 4),\ 4 \cdot \mathfrak{D}^{-}(X, 0, 8),\ 4 \cdot \mathfrak{D}^{-}(X, 4, 8) = \frac{2}{\pi^2} X + O(\sqrt{X}), \tag{25}$$

$$\mathfrak{D}^{+}(X, 1, 4),\ 4 \cdot \mathfrak{D}^{+}(X, 0, 8),\ 4 \cdot \mathfrak{D}^{+}(X, 4, 8) = \frac{2}{\pi^2} X + O(\sqrt{X}), \tag{26}$$

uniformly for $X \ge 2$. The equalities (25) and (26) are just variations of the classical formula

$$\sum_{n \le X} \mu^2(n) = \frac{6}{\pi^2} X + O(\sqrt{X}),$$

which counts the number of squarefree numbers up to X, where $\mu(n)$ is the Möbius function.

In [Fouvry and Klüners 2007, Theorem 3] we proved that Conjecture 1.2 is true for $p = 2$ and all $r \ge 0$. For our main result we need the stronger result that the densities above are the same when we restrict the fundamental discriminants to the cases $D \equiv a \bmod q$ for $(a, q) \in \{(1,4),\ (0,8),\ (4,8)\}$. We could easily have stated this extension in that paper, but unfortunately we did not. We explain briefly how to get the stronger result. In [Fouvry and Klüners 2007] we introduced the following sums, which are moments of order k of the arithmetic function $2^{\mathrm{rk}_4(\mathrm{C}_D)}$:

$$S^{\pm}(X, k, a, q) := \sum_{\substack{0 < \pm D \le X \\ D \equiv a \bmod q}} 2^{k\, \mathrm{rk}_4(\mathrm{C}_D)}, \tag{27}$$

where $X \ge 2$ is a real number, $k \ge 0$ is an integer, and (a, q) is one of $(1, 4)$, $(0, 8)$, and $(4, 8)$. Then we proved in Theorems 6–11 of the same reference the following results, where $\mathcal{N}(k, 2)$ denotes the number of $\mathbb{F}_2$-vector subspaces of $\mathbb{F}_2^k$.

Theorem 2.1. *Let* $(a, q) \in \{(1,4),\ (0,8),\ (4,8)\}$. *For every positive integer k and every positive ε we have*, *uniformly for* $X \ge 2$,

$$S^{-}(X, k, a, q) = \mathcal{N}(k, 2)\, \mathfrak{D}^{-}(X, a, q) + O_{\varepsilon,k}(X (\log X)^{-2^{-k}+\varepsilon})$$

and

$$S^+(X,k,a,q) = \frac{1}{2^k}(\mathcal{N}(k+1,2)-\mathcal{N}(k,2))\,\mathfrak{D}^+(X,a,q) + O_{\varepsilon,k}(X(\log X)^{-2^{-k}+\varepsilon}).$$

Using the same proof as in [Fouvry and Klüners 2007, Proposition 1] and applying it to Theorem 2.1, we get the following result for our six families.

Theorem 2.2. *Let* $(a,q) \in \{(1,4),\ (0,8),\ (4,8)\}$. *For every positive integer* k,

$$\lim_{X\to\infty} \frac{\sum\limits_{\substack{0<-D\le X\\ D\equiv a \bmod q}} \prod\limits_{0\le i<k} (2^{\mathrm{rk}_4(\mathrm{C}_D)}-2^i)}{\mathfrak{D}^-(X,a,q)} = 1, \quad \lim_{X\to\infty} \frac{\sum\limits_{\substack{0<D\le X\\ D\equiv a \bmod q}} \prod\limits_{0\le i<k} (2^{\mathrm{rk}_4(\mathrm{C}_D)}-2^i)}{\mathfrak{D}^+(X,a,q)} = 2^{-k}.$$

We remark that this theorem is a positive answer to [Cohen and Lenstra 1984, C6 and C10, p. 56f] for the case $p = 2$. We can use the same approach for our six subfamilies as in the proofs of [Fouvry and Klüners 2006, Theorems 1 and 2]. Altogether, we get the following result, which extends [Fouvry and Klüners 2007, Theorem 3] to the six families.

Theorem 2.3. *Let* (a,q) *satisfy* (23). *For every* $r \ge 0$ *we have*

$$\lim_{X\to\infty} \frac{\sharp\left\{D : 0 < -D \le X,\ D \equiv a \bmod q,\ \mathrm{rk}_4(\mathrm{C}_D) = r\right\}}{\mathfrak{D}^-(X,a,q)} = a_2^-(r),$$

$$\lim_{X\to\infty} \frac{\sharp\left\{D : 0 < D \le X,\ D \equiv a \bmod q,\ \mathrm{rk}_4(\mathrm{C}_D) = r\right\}}{\mathfrak{D}^+(X,a,q)} = a_2^+(r).$$

3. Proofs of our main results

We start with some formulas between the densities occurring in the Cohen–Lenstra heuristics. In this paper we are using them only for $p = 2$ and $p = 3$, but it is easy to give them for every prime p.

Lemma 3.1. *Let* p *be prime and* $a_p^{\pm}(r)$ *be defined as in Conjecture 1.2. Then*

(i) $a_p^+(r) = \dfrac{p}{p^{r+1}-1}\, a_p^-(r)$ *for all* $r \ge 0$,

(ii) $a_p^-(r+1) = \dfrac{p}{(p^{r+1}-1)^2}\, a_p^-(r)$ *for all* $r \ge 1$.

Proof.

(i) $a_p^+(r) = p^{-r}\left(1-\dfrac{1}{p^{r+1}}\right)^{-1} a_p^-(r) = \dfrac{p}{p^{r+1}-1}\, a_p^-(r)$,

(ii) $a_p^-(r) = p^{-2r-1}\left(1-\dfrac{1}{p^{r+1}}\right)^{-2} a_p^-(r) = \dfrac{p}{(p^{r+1}-1)^2}\, a_p^-(r)$. □

Now we define the quantities which, for $p = 2$ and $p = 3$, appear quite naturally in the reflection principle.

Definition 3.2. For $r, s \geq 0$ we recursively define

(i) $c_p(0,0) := a_p^-(0), c_p(0,1) := a_p^+(0) - c(0,0) = a_p^+(0) - a_p^-(0)$;

(ii) $c_p(r,r) := a_p^-(r) - c_p(r-1,r)$ and $c_p(r,r+1) := a^+(r) - c_p(r,r) = a^+(r) - a_p^-(r) + c(r-1,r)$ for all $r \geq 1$;

(iii) $c(r,s) = 0$ in all other cases, that is, when $s - r \notin \{0,1\}$.

We have the two easy identities

$$a_p^-(r) = c_p(r-1,r) + c_p(r,r) \quad \text{for } r \geq 1,$$
$$a_p^+(r) = c_p(r,r) + c_p(r,r+1) \quad \text{for } r \geq 0.$$

Lemma 3.3. *Let p be a prime.*

(i) *For all $r \geq 0$ we have $c_p(r,r)/a_p^-(r) = p^{-r}$.*

(ii) *For all $r \geq 1$ we have $c_p(r-1,r)/a_p^-(r) = 1 - p^{-r}$.*

(iii) *For all $r \geq 0$ we have $c_p(r,r)/a_p^+(r) = 1 - p^{-(r+1)}$.*

(iv) *For all $r \geq 0$ we have $c_p(r,r+1)/a_p^+(r) = p^{-(r+1)}$.*

Proof. We prove (i) by induction, the case $r = 0$ being trivial. Now

$$\frac{c_p(r+1,r+1)}{a_p^-(r+1)} = \frac{a_p^-(r+1) - c_p(r,r+1)}{a_p^-(r+1)} = 1 - \frac{a_p^+(r) - c_p(r,r)}{a_p^-(r+1)}.$$

Using Lemma 3.1 twice we reduce this expression to

$$1 - \frac{p/(p^{r+1}-1)\, a_p^-(r) - c_p(r,r)}{a_p^-(r)\; p/(p^{r+1}-1)^2} = 1 - (p^{r+1}-1) + \frac{1}{p^r}\frac{(p^{r+1}-1)^2}{p},$$

the equality being checked by induction. But this equals $p^{-(r+1)}$, which proves (i).

Part (ii) follows easily from (i) and $c_p(r-1,r) + c_p(r,r) = a_p^-(r)$.

By part (i) and by Lemma 3.1 we have

$$\frac{c_p(r,r)}{a_p^+(r)} = \frac{c_p(r,r)}{a_p^-(r)\; p/(p^{r+1}-1)} = p^{-r}\frac{p^{r+1}-1}{p} = 1 - p^{-(r+1)},$$

which proves part (iii).

The last part follows from (iii) and $a_p^+(r) = c(r,r) + c(r,r+1)$. □

The main step. Now we are able to prove the main result, which gives the natural density of the set of negative D, such that the 4-rank of C_D and C_{-D} have prescribed values. To state this result, for integers a, q, nonnegative integers r, s, and $X \geq 1$, we introduce

$$B^{\pm}(X,a,q,r,s) := \sharp\big\{D : 0 < \pm D \leq X,\ D \equiv a \bmod q,\ \mathrm{rk}_4(\mathrm{C}_{-D}) = r,\ \mathrm{rk}_4(\mathrm{C}_D) = s\big\}.$$

Theorem 3.4. *Let* $(a, q) \in \{(1,4), (0,8), (4,8)\}$. *For every* r *and* $s \geq 0$ *we have*

$$\lim_{X\to\infty} \frac{B^-(X,a,q,r,s)}{\mathfrak{D}^-(X,a,q)} = c_2(r,s).$$

Proof. We shall fix the case $(a,q) = (1,4)$ and give indications for the other two cases. A direct application of Theorems 1.3 and 2.3 leads to the following asymptotic behaviors for $X \to \infty$:

$$B^-(X,1,4,0,0) \sim a_2^-(0)\mathfrak{D}^-(X,1,4), \tag{28}$$

$$B^-(X,1,4,s,s) + B^-(X,1,4,s-1,s) \sim a_2^-(s)\mathfrak{D}^-(X,1,4) \quad \text{if } s \geq 1. \tag{29}$$

Note that when $D<0$ is congruent to 1 mod 4, the reflected field $\mathbb{Q}(\sqrt{-D})$ has discriminant $-4D$; hence the reflection creates a one-to-one correspondence between negative discriminants congruent to 1 mod 4 and not less than $-X$, on the one hand, and positive discriminants congruent to 4 mod 8 and not exceeding $\leq 4X$, on the other. We use this bijection in the form of the equalities

$$\mathfrak{D}^-(X,1,4) = \mathfrak{D}^+(4X,4,8), \quad B^-(X,1,4,r,s) = B^+(4X,4,8,s,r), \tag{30}$$

which are true for any integers r and s. Using Theorems 1.3 and 2.3 once more we have

$$B^+(4X,4,8,s,s) + B^+(4X,4,8,s+1,s) \sim a_2^+(s)\mathfrak{D}^+(4X,4,8) \tag{31}$$

as $X \to \infty$ for any $s \geq 0$. We reinterpret this relation by appealing to (30), obtaining

$$B^-(X,1,4,s,s) + B^-(X,1,4,s,s,+1) \sim a_2^+(s)\mathfrak{D}^-(X,1,4). \tag{32}$$

Let $b^-(X,r,s) := B^-(X,1,4,r,s)/\mathfrak{D}^-(X,1,4)$. The relations (28), (29) and (32) are written as

$$\begin{aligned} b^-(X,0,0) &\sim a_2^-(0), \\ b^-(X,s,s) + b^-(X,s-1,s) &\sim a_2^-(s) \quad \text{for } s \geq 1, \\ b^-(X,s,s) + b^-(X,s,s+1) &\sim a_2^+(s) \quad \text{for } s \geq 0, \end{aligned} \tag{33}$$

as $X \to \infty$. (Recall that $b^-(X,r,s) = 0$ when $s - r \notin \{0,1\}$.) An easy induction applied to the asymptotics (33) proves that each $b^-(X,r,s)$ has a limit as $X \to \infty$, which is denoted by $b^-(r,s)$. We then get from (33) the following equalities among these limits:

$$\begin{aligned} b^-(0,0) &\sim a_2^-(0), \\ b^-(s,s) + b^-(s-1,s) &\sim a_2^-(s) \quad \text{for } s \geq 1, \\ b^-(s,s) + b^-(s,s+1) &\sim a_2^+(s) \quad \text{for } s \geq 0. \end{aligned} \tag{34}$$

We exactly recognize the identities satisfied by the coefficients $c_2(r,s)$ for all r and s. By an easy induction, we deduce that $b^-(r,s) = c_2(r,s)$. This completes the proof of Theorem 1.8 when $(a, q) = (1, 4)$. It remains to give some hints on the other cases.

- When $(a, q) = (4, 8)$, the reflection creates a bijection between the set of negative discriminants $\geq -X$ and congruent to 4 mod 8 with the set of positive discriminants (including 1) $\leq X/4$ and congruent to 1 mod 4.
- When $(a, q) = (0, 8)$, the reflection creates a bijection between the set of negative discriminants $\geq -X$ and congruent to 0 mod 8 with the set of positive discriminants $\leq X$ and congruent to 0 mod 8.

With these remarks, the counting process is the same. □

Proof of Theorem 1.8. By Theorem 3.4 we see that (15) and (16) are obvious when we use the first two formulas of Lemma 3.3 for $p = 2$ in the three cases of $D \equiv 1 \bmod 4$, $D \equiv 4 \bmod 8$ and $D \equiv 0 \bmod 8$. For the equalities (17) and (18), we shall restrict ourselves to the case $D \equiv 1 \bmod 4$ since the other cases are similar. So we are concerned with the limit of the ratio

$$\frac{\sharp\{D : 0 < D \leq X,\ \mathrm{rk}_4(\mathrm{C}_D) = \mathrm{rk}_4(\mathrm{C}_{-D}) = r,\ D \equiv 1 \bmod 4\}}{\mathcal{D}^+(X, 1, 4)}.$$

By the reflection map, this ratio is equal to

$$\frac{\sharp\{D : 0 < -D \leq 4X,\ \mathrm{rk}_4(\mathrm{C}_D) = \mathrm{rk}_4(\mathrm{C}_{-D}) = r,\ D \equiv 4 \bmod 8\}}{\mathcal{D}^+(4X, 4, 8)}.$$

By Theorem 3.4 as $X \to \infty$, this ratio tends to

$$c_2(r, r) = a_2^+(r)(1 - 2^{-(r+1)}),$$

by Lemma 3.3(iii). □

Proof of Corollary 1.10. Heuristically, we want to sum up the results of Theorem 1.8. But this is an infinite summation of all the probabilities corresponding to $0 \leq r < \infty$. Following the technique used in the proof of (11), we can perform this infinite series. Hence we can pass from each of the four equalities of Theorem 1.8 to each of the four equalities of Corollary 1.10. □

The case $p = 3$. Analyzing the proof of Theorem 1.8, we see that everything works for $p = 3$ as soon as we have a suitable proven version of Conjecture 1.2. For $p = 2$ we used Theorem 2.3, which gives the corresponding densities for the cases $D \equiv a \bmod q$ and (a, q) as in (23). It is important for our argument in the proof of Theorem 3.4 that the reflection from $\mathbb{Q}(\sqrt{d})$ to $\mathbb{Q}(\sqrt{-d})$ is order-preserving and is a permutation of the set consisting of the three congruence classes defined in (23).

The latter is true for $p = 3$ when we restrict to the cases $D \equiv 0 \bmod 3$ and $D \equiv 1, 2 \bmod 3$. Indeed, if $D > 0$ is a fundamental discriminant, then $-3D$ is a fundamental discriminant, when $D \not\equiv 0 \bmod 3$. In case that $D \equiv 0 \bmod 3$, the reflected field has discriminant $-D/3$. Since $-3 \equiv 1 \bmod 4$, we have no problems with ramification at 2 in this case.

Here also we recognize a permutation of the set consisting of the two subsets $\{D \equiv 0 \bmod 3\}$ and $\{D \not\equiv 0 \bmod 3\}$. Denote by

$$\mathcal{A}_0^{\pm} := \{D \in \mathfrak{D}^{\pm} : D \equiv 0 \bmod 3\}, \quad \mathcal{A}_{\neq 0}^{\pm} := \{D \in \mathfrak{D}^{\pm} : D \not\equiv 0 \bmod 3\}$$

four different sets. The proof of the following theorem is now obvious using the remarks above.

Theorem 3.5. *Assume that the following four equations are true, where* $* \in \{+, -\}$ *and b can be* 0 *or* $\neq 0$:

$$\lim_{X\to\infty} \frac{\sharp\left\{D \in \mathcal{A}_b^* : 0 < |D| \le X,\ \mathrm{rk}_3(\mathrm{C}_D) = r\right\}}{\sharp\{D \in \mathcal{A}_b^* : 0 < |D| \le X\}} = a_3^*(r).$$

Then the corresponding result of Theorem 1.8 is true. Especially, Conjecture 1.1 is true.

The four statements assumed in the hypotheses of Theorem 3.5 are only extensions of some Cohen–Lenstra heuristics (see Conjecture 1.2 above, with $p = 3$) to congruence classes modulo 3.

A weighted version. We already said in (12) that in [Belabas 1999; 2004] a weaker result for $p = 3$ is proved. Here the density is considered with some weight, which makes it possible to deduce this result by knowing only the following averages for $p = 3$:

$$\lim_{X\to\infty} \frac{\sum\limits_{0<D\le X} p^{\mathrm{rk}_p(\mathrm{C}_D^2)}}{\sum\limits_{0<D\le X} 1} = 1 + 1/p, \qquad \lim_{X\to\infty} \frac{\sum\limits_{0<-D\le X} p^{\mathrm{rk}_p(\mathrm{C}_D^2)}}{\sum\limits_{0<-D\le X} 1} = 2.$$

Knowing these averages (and some proven error term) for all discriminants divisible by 3 and not divisible by 3, respectively, for $p = 3$, Equation (12) can be deduced.

We mention this type of result for two reasons. First, it can be proven for $p = 3$ and second we get rational constants for this weighted density. On the other hand, this weighted density is not the one we want. As in Theorem 1.8 we have four different points of view to express this result. It is clear that in [Belabas 1999; 2004] all of these four viewpoints could have been proved. For $p = 2$, that is, the reflection principle for 4-ranks, we can easily prove similar statements. Let us define (if they exist) the following weighted densities for $p \in \{2, 3\}$, $a \in \{0, 1\}$:

$$d_{3,a,\pm} := \lim_{X\to\infty} \frac{\sum_{\substack{0<\pm D\le X\\ \mathrm{rk}_3(\mathrm{C}^2_{-3D})=\mathrm{rk}_3(\mathrm{C}^2_D)\mp a}} 3^{\mathrm{rk}_3(\mathrm{C}^2_D)}}{\sum_{0<\pm D\le X} 3^{\mathrm{rk}_3(\mathrm{C}^2_D)}}, \tag{35}$$

$$d_{2,a,\pm} := \lim_{X\to\infty} \frac{\sum_{\substack{0<\pm D\le X\\ \mathrm{rk}_2(\mathrm{C}^2_{-D})=\mathrm{rk}_2(\mathrm{C}^2_D)\mp a}} 2^{\mathrm{rk}_2(\mathrm{C}^2_D)}}{\sum_{0<\pm D\le X} 2^{\mathrm{rk}_2(\mathrm{C}^2_D)}}. \tag{36}$$

Theorem 3.6. *Let $p = 2$ or 3. Then the weighted densities exist and are given by*

$$d_{p,1,+} = \frac{1}{p+1}, \quad d_{p,0,+} = \frac{p}{p+1}, \quad d_{p,1,-} = \frac{1}{2}, \quad d_{p,0,-} = \frac{1}{2}.$$

Proof. Let us start with $d_{2,0,-}$. We multiply (15) in Theorem 1.8 by 2^r and, using the same arguments as in the proof of Corollary 1.10, we perform the summation:

$$\sum_{r=0}^{\infty} a_2^-(r)2^{-r}2^r = \sum_{r=0}^{\infty} a_2^-(r) = 1.$$

We know that the denominator of (36) has average 2 by Theorem 2.1, and therefore we get $\frac{1}{2}$ as the weighted density. The result $d_{2,1,-} = 1 - \frac{1}{2} = \frac{1}{2}$ is now obvious.

Now we look at $d_{2,1,+}$ and we are led to the sum

$$\sum_{r=0}^{\infty} a_2^+(r)2^{-(r+1)}2^r = \frac{1}{2}\sum_{r=0}^{\infty} a_2^+(r) = \frac{1}{2}.$$

The denominator has average $\frac{3}{2}$ by Theorem 2.1 and we get $\frac{1}{2}/\frac{3}{2} = \frac{1}{3}$ as the weighted density.

The result for $p = 3$ is proven in [Belabas 1999; 2004] for $d_{3,1,+}$. Then $d_{3,0,+} = 1 - d_{3,1,+}$ and the other two densities can be proved analogously. □

4. Some remarks

In an earlier version of this paper we gave a much more complicated proof of Theorem 1.8. We defined for (a, q) in (23):

$$S^-_{\mathrm{mix}}(X, k, a, q) := \sum_{\substack{0<-D\le X\\ D\equiv a \bmod q}} 2^{k\,\mathrm{rk}_4(\mathrm{C}_D)} \cdot 2^{\mathrm{rk}_4(\mathrm{C}_{-D})}. \tag{37}$$

We then proved the following theorem using techniques similar to those of [Fouvry and Klüners 2007]:

Theorem 4.1. *Let* (a, q) *satisfy* (23). *For any integer* $k \geq 0$ *and for any* $\varepsilon > 0$ *we have the equality*

$$S^{-}_{\mathrm{mix}}(X, k, a, q) = \frac{\mathcal{N}(k+1, 2) + \mathcal{N}(k, 2)}{2} \cdot \mathfrak{D}^{-}(X, a, q) + O_{k,\varepsilon}\big(X(\log X)^{-2^{-k}+\varepsilon}\big),$$

uniformly for $X \geq 2$.

Then it was possible to deduce Theorem 1.8 from this theorem and the main result in [Fouvry and Klüners 2007].

Our new proof is simply a corollary of Theorem 2.3, which is a slight extension of [Fouvry and Klüners 2007, Theorem 3]. Unfortunately, we did not know about this possibility when we wrote that paper. We already mentioned the results of Gerth [1984; 2001], which prove these things by considering the number of prime factors. In the second of those papers Gerth also starts by reproving all the things in a similar way as he did in the first. It is possible to use the same procedure to derive the results in [Gerth 2001] from the earlier paper [1984], provided that it has been generalized to each of the congruence classes appearing in (23).

Dutarte [1984] checked the compatibility of different principles leading to the Cohen–Lenstra heuristics and to the probabilities occurring in the reflection principle. Theorem 3.5 shows that the corresponding probabilities in the reflection principle can be deduced when we know that Cohen–Lenstra heuristics are true for $p = 3$ in congruence classes modulo 3. This was not seen in [Dutarte 1984]. Nevertheless, he produces heuristics for the other direction.

References

[Belabas 1999] K. Belabas, "On the mean 3-rank of quadratic fields", *Compositio Math.* **118**:1 (1999), 1–9. MR 2000g:11102 Zbl 0929.11046

[Belabas 2004] K. Belabas, "Corrigendum: On the mean 3-rank of quadratic fields", *Compos. Math.* **140**:5 (2004), 1221. MR 2005b:11176

[Cohen and Lenstra 1984] H. Cohen and H. W. Lenstra, Jr., "Heuristics on class groups of number fields", pp. 33–62 in *Number theory* (Noordwijkerhout, 1983), edited by H. Jager, Lecture Notes in Math. **1068**, Springer, Berlin, 1984. MR 85j:11144 Zbl 0558.12002

[Damey and Payan 1970] P. Damey and J.-J. Payan, "Existence et construction des extensions Galoisiennes et non-abéliennes de degré 8 d'un corps de caractéristique différente de 2", *J. Reine Angew. Math.* **244** (1970), 37–54. MR 43 #6186 Zbl 0206.34401

[Davenport and Heilbronn 1971] H. Davenport and H. Heilbronn, "On the density of discriminants of cubic fields, II", *Proc. Roy. Soc. London Ser. A* **322**:1551 (1971), 405–420. MR 58 #10816 Zbl 0212.08101

[Dutarte 1984] P. Dutarte, "Compatibilité avec le Spiegelungssatz de probabilités conjecturales sur le p-rang du groupe des classes", in *Théorie des nombres* 1983/1984 (Exposé 4), Université de Franche-Comté, Besançon, 1984. MR 86m:11103

[Fouvry and Klüners 2006] É. Fouvry and J. Klüners, "Cohen–Lenstra heuristics of quadratic number fields", pp. 40–55 in *Algorithmic number theory* (Berlin, 2006), edited by F. Hess et al., Lecture Notes in Comput. Sci. **4076**, Springer, Berlin, 2006. MR 2008f:11125

[Fouvry and Klüners 2007] É. Fouvry and J. Klüners, "On the 4-rank of class groups of quadratic number fields", *Invent. Math.* **167**:3 (2007), 455–513. MR 2007k:11187

[Gerth 1984] F. Gerth, III, "The 4-class ranks of quadratic fields", *Invent. Math.* **77**:3 (1984), 489–515. MR 85j:11137 Zbl 0533.12004

[Gerth 2001] F. Gerth, III, "Comparison of 4-class ranks of certain quadratic fields", *Proc. Amer. Math. Soc.* **129**:9 (2001), 2547–2552. MR 2002c:11149 Zbl 1012.11097

[Heath-Brown 1994] D. R. Heath-Brown, "The size of Selmer groups for the congruent number problem, II", *Invent. Math.* **118**:1 (1994), 331–370. MR 95h:11064 Zbl 0815.11032

[Roberts 2001] D. P. Roberts, "Density of cubic field discriminants", *Math. Comp.* **70**:236 (2001), 1699–1705. MR 2002e:11142 Zbl 0985.11068

[Scholz 1932] A. Scholz, "Über die Beziehung der Klassenzahlen quadratischer Körper zueinander", *J. für Reine Angew. Math.* **166** (1932), 201–203. Zbl 0004.05104 JFM 58.0181.05

[Stevenhagen 1993] P. Stevenhagen, "The number of real quadratic fields having units of negative norm", *Experiment. Math.* **2**:2 (1993), 121–136. MR 94k:11120 Zbl 0792.11041

[Tenenbaum 2008] G. Tenenbaum, *Introduction à la théorie analytique et probabiliste des nombres*, 3rd ed., Belin, Paris, 2008.

Communicated by Bjorn Poonen
Received 2008-10-02 Revised 2009-11-30 Accepted 2010-01-19

Etienne.Fouvry@math.u-psud.fr *Université de Paris-Sud, Laboratoire de Mathématiques, 91405 Orsay Cedex, France*

CNRS, 91405 Orsay Cedex, France

klueners@math.uni-paderborn.de *Universität Paderborn, Institut für Mathematik, Warburger Straße 100, 33098 Paderborn, Germany*

Algebra & Number Theory

www.jant.org

See inside back cover or www.jant.org for submission instructions.

The subscription price for 2010 is US $140/year for the electronic version, and $200/year (+$30 shipping outside the US) for print and electronic. Subscriptions, requests for back issues from the last three years and changes of subscribers address should be sent to Mathematical Sciences Publishers, Department of Mathematics, University of California, Berkeley, CA 94720-3840, USA.

Algebra & Number Theory (ISSN 1937-0652) at Mathematical Sciences Publishers, Department of Mathematics, University of California, Berkeley, CA 94720-3840 is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

ANT peer review and production are managed by EditFlow™ from Mathematical Sciences Publishers.

PUBLISHED BY
mathematical sciences publishers
http://www.mathscipub.org
A NON-PROFIT CORPORATION
Typeset in LaTeX

Algebra & Number Theory

Volume 4 No. 5 2010