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Claude Cibils, María Julia Redondo and Andrea Solotar

The main purpose of this paper is to provide explicit computations of the fundamental groups of several algebras. For this purpose, given a k-algebra A, we consider the category of all connected gradings of A by a group G and we study the relation between gradings and Galois coverings. This theoretical tool gives information about the fundamental group of A, which allows its computation using complete lists of gradings.

1. Introduction

We provide explicit computations of the intrinsic fundamental groups of some algebras. For this, we study in detail the relation between gradings and Galois coverings of each algebra considered as a k-linear category with one object. Particular attention is paid to matrix algebras, since the problem of classifying gradings of these algebras has been extensively treated in the literature [Aljadeff et al. 2010; Bahturin et al. 2001; Bahturin and Zaicev 2002; Bahturin and Shestakov 2001; Boboc 2003; Boboc and Dăscălescu 2001; 2006; 2007; Caenepeel et al. 2002; Chun and Lee 2007; Dăscălescu et al. 1999; Khazal et al. 2003].

We recall that the intrinsic fundamental group of an algebra was defined in [Cibils et al. 2007] using Galois coverings. We make use of an equivalence between the category of Galois coverings and its full subcategory with objects obtained from the smash product construction, which is deeply attached to connected gradings. We replace algebras by linear categories over a base ring: a category over a ring k is considered as an algebra with several objects [Mitchell 1972], and a k-algebra A can be viewed as a k-category with a single object and endomorphism ring equal to A. Note that in [Green 1983; Green and Marcos 1994], a relation between gradings

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and coverings is established for quivers with relations. In this paper we consider an intrinsic context, where the categories are not given by a presentation.

When computing the fundamental group of an algebra, one faces the problem of classifying and organizing its connected gradings. The methods we introduce allow the computation of the fundamental groups of matrix algebras, triangular matrix algebras, group algebras and diagonal algebras. We restrict to connected gradings and prove that the matrix algebras do not admit a universal grading. Indeed, there exist at least two nonisomorphic Galois coverings or, equivalently, two nonisomorphic connected gradings that are simply connected, in the sense that they have no nontrivial Galois coverings. In particular, this confirms that the fundamental group of an algebra takes into account the matrix structure; in other words, it is not a Morita invariant.

In Section 2, we show that the connectedness of gradings is the right notion that corresponds to the connectedness of the associated smash product. We recall the concept of Galois covering and observe that the smash product construction gives examples of Galois coverings. We describe in detail the morphisms between smash coverings.

In Section 3, we make an explicit comparison between Galois coverings and smash coverings of a k-category $\mathcal{B}$. More precisely, we provide an equivalence between the category $\mathrm{Gal}(\mathcal{B}, b_0)$ of Galois coverings of $\mathcal{B}$ and its full subcategory $\mathrm{Gal}^{\#}(\mathcal{B}, b_0)$, whose objects are the smash product coverings. We consider the fundamental group defined in [Cibils et al. 2007] using Galois coverings and show that we can restrict to smash coverings when computing the fundamental group $\pi_1(\mathcal{B}, b_0)$.

In the subsequent sections, we focus on the description of connected gradings of certain algebras in order to compute their fundamental groups. As a rule, we wonder about the existence of a universal grading, since when such a grading exists, the grading group is isomorphic to the fundamental group of the algebra.

In Section 4, we consider matrix algebras, proving that there is no universal covering by providing two nonisomorphic simply connected gradings. Despite the fact that they appear to be very different in nature, we show that they have a unique largest common nontrivial quotient. Using the classification of gradings of $M_2(k)$ given in [Khazal et al. 2003] and of $M_3(k)$ given in [Boboc and Dăscălescu 2007], we compute the fundamental groups of these algebras in the cases where the field is algebraically closed of characteristic different from 2 and 3, respectively. Using analogous methods and the classification in [Bahturin and Zaicev 2002], we compute the fundamental group of $M_p(k)$, where p is prime and k an algebraically closed field of characteristic zero, which is the direct product of the free group on $p-1$ generators with the cyclic group of order p. We compute the fundamental group of triangular matrix algebras, using results in [Valenti and Zaicev 2007], without any hypothesis on the characteristic of the field k. The fundamental group in this case is the free group on $n-1$ generators.

In Section 5, we first prove that the natural grading of a group algebra is simply connected. Next we consider in detail the group algebra of the cyclic group of order p, where p is a prime, in the case of a field of characteristic p. This algebra is isomorphic to the truncated polynomial algebra $k[x]/(x^p)$, and we show that it does not admit a universal grading. Nevertheless, we provide a complete description of its connected gradings, and we conclude that the fundamental group of the truncated polynomial algebra in characteristic p is the product of the infinite cyclic group and the cyclic group of order p.

Finally, in Section 6 we consider the group algebra kG, for G an abelian group of order n and k a field with enough n-th roots of unity or, equivalently, the algebra k^E of all maps from E to k, where E is a set with n elements. In the case where n is not square-free, we show that k^n has no universal covering. A special case occurs when $n=2$ and k is a field of characteristic different from 2: there exists a universal covering. More precisely, we prove that there is only one nontrivial group providing a connected grading of the set algebra k^2, namely the cyclic group of order 2, which in turn is the fundamental group of this algebra.

We end the paper by computing the fundamental group of the set algebras k^3 and k^4, using a description of all the gradings of k^E given in [Dăscălescu 2008]. In the case where k is a field containing all roots of unity of order 2 and 3, we prove that $\pi_1(k^3) = C_2 \times C_3$, while if k contains all roots of unity of order 3 and 4, we obtain

$$\pi_1(k^4) = (C_2 * C_2) \times C_6 \times C_4 \times C_2.$$

A detailed study of Dăscălescu's classification and the relations among the grading groups, together with the techniques presented in this paragraph, should lead to the computation of the fundamental group for arbitrary diagonal algebras.

2. Gradings and coverings

Let k be a commutative ring and let $\mathcal{B}$ be a small category such that each morphism set ${}_y\mathcal{B}_x$ from an object x to an object y is endowed with a k-module structure such that composition of morphisms is k-bilinear. Such a category is called a k-category; note that each endomorphism k-module ${}_x\mathcal{B}_x$ is a k-algebra and ${}_y\mathcal{B}_x$ is a ${}_y\mathcal{B}_y$-${}_x\mathcal{B}_x$-bimodule. Each k-algebra A provides in this way a single object k-category.

In [Cibils and Marcos 2006; Green 1983] it was shown that connected group gradings and Galois coverings are in one-to-one correspondence. We recall the definition of these categories and, even if they are not equivalent, we make precise the relation between them.

Definition 2.1. A *grading* X of a k-category $\mathcal{B}$ by a group Γ is given by a direct sum decomposition of each k-module of morphisms

$$ {}_y\mathcal{B}_x = \bigoplus_{s\in\Gamma} X^s({}_y\mathcal{B}_x), $$

such that $X^t({}_z\mathcal{B}_y)X^s({}_y\mathcal{B}_x) \subset X^{ts}({}_z\mathcal{B}_x)$. The *homogeneous component* of degree s from x to y is the k-module $X^s({}_y\mathcal{B}_x)$.

Next we consider *connected* gradings, in order to establish the correspondence with Galois coverings. We use the following notation: given a morphism f, its source object is denoted by $s(f)$ and its target object by $t(f)$.

We will also make use of walks. For this purpose we consider the set of formal pairs (f, ε) as *morphisms with sign*, where f is a morphism in $\mathcal{B}$ and $\varepsilon \in \{-1, 1\}$. We extend source and target maps to this set:

$$ s(f,1) = s(f), s(f,-1) = t(f), t(f,1) = t(f), t(f,-1) = s(f). $$

Definition 2.2. Let $\mathcal{B}$ be a k-category. A nonzero *walk* in $\mathcal{B}$ is a sequence of nonzero morphisms with signs $(f_n, \varepsilon_n)\dots(f_1, \varepsilon_1)$ such that

$$ s(f_{i+1}, \varepsilon_{i+1}) = t(f_i, \varepsilon_i). $$

We say that this walk goes from $s(f_1, \varepsilon_1)$ to $t(f_n, \varepsilon_n)$.

A nonzero walk $\alpha = (f_n, \varepsilon_n)\dots(f_1, \varepsilon_1)$ is called *homogeneous* if each f_i is a homogeneous morphism in the graded category $\mathcal{B}$. We shall denote by $\deg f$ the degree of a homogeneous morphism f. We define the *degree* of a homogeneous nonzero walk α:

$$ \deg\alpha = (\deg f_n)^{\varepsilon_n} \dots (\deg f_1)^{\varepsilon_1}. $$

As expected, a k-category $\mathcal{B}$ is called *connected* if any two objects of $\mathcal{B}$ can be joined by a nonzero walk. Moreover, a Γ-grading of $\mathcal{B}$ is *connected* if given any two objects in $\mathcal{B}$ and any element $g \in \Gamma$, they can be joined by a nonzero homogeneous walk of degree g. Of course, if a grading of a k-category is connected, then the underlying category is connected. Conversely, the following easy result holds.

Lemma 2.3. *Let $\mathcal{B}$ be a connected k-category equipped with a Γ-grading and let x_0 be an object of $\mathcal{B}$. Assume there exist homogeneous walks of any degree from x_0 to itself. Then the grading is connected.*

The definition of a connected grading restricts to algebras as follows. First recall that the *support* of a grading X of a k-algebra A by a group Γ is

$$ \operatorname{Supp} X = \{s \in \Gamma \mid X^s A \neq 0\}. $$

If the category has only one object, the following result describes the notion of a connected grading of an algebra. Note that [Dăscălescu 2008] gives the name *faithful* to this kind of grading.

Proposition 2.4. *Let A be a k-algebra and X be a Γ-grading of A. The grading is connected if and only if* $\operatorname{Supp} X$ *is a set of generators of* Γ.

Proof. Consider the k-category $\mathcal{B}_A$ with a single object $*$ such that ${}_*(\mathcal{B}_A)_* = A$. Assume that the grading is connected. Then for any element g of Γ, there is a homogeneous nonzero walk $\alpha = (f_n, \varepsilon_n) \dots (f_1, \varepsilon_1)$ such that $\deg \alpha = g$, which precisely means that $\operatorname{Supp} X$ generates Γ. Conversely, let $g \in \Gamma$. Since $\operatorname{Supp} X$ generates Γ, we have that $g = g_n^{\varepsilon_n} \dots g_1^{\varepsilon_1}$ where $g_i \in \operatorname{Supp} X$ and $\varepsilon_i = \pm 1$. Let $a_n, \dots, a_1$ be nonzero homogeneous elements of A such that $\deg a_i = g_i$. Then $(a_n, \varepsilon_n) \dots (a_1, \varepsilon_1)$ is a nonzero closed homogeneous walk from $*$ to itself, of degree g. □

Remark 2.5. Clearly each Γ-grading of an algebra provides a unique connected grading by restricting Γ to the subgroup generated by the support.

We recall now the smash product category associated to a grading, as defined in [Cibils and Marcos 2006]. This construction is compatible with the one in the algebra case, in the sense that for a finite group Γ and a Γ-graded algebra A, we recover the smash product $A\#\Gamma$ given in that reference.

Definition 2.6. Let X be a Γ-grading of the k-category $\mathcal{B}$. The objects of the *smash product category* $\mathcal{B}\#\Gamma$ are $\mathcal{B}_0 \times \Gamma$, while the module of morphisms from (b, g) to (c, h) is $X^{h^{-1}g}{}_c\mathcal{B}_b$. In other words, morphisms are provided by homogeneous components, and composition in $\mathcal{B}\#\Gamma$ is given by the original composition in $\mathcal{B}$. The composition of morphisms is well-defined, as an immediate consequence of the definition of a graded category.

Remark 2.7. Consider this definition for a single object k-category $\mathcal{B}_A$ associated to a k-algebra A, and write $A\#\Gamma = \mathcal{B}_A\#\Gamma$. Then the set of objects of $A\#\Gamma$ is Γ, while the morphisms from g to h are the homogeneous elements of degree $h^{-1}g$. If Γ is finite, the matrix algebra obtained as the direct sum of all the morphisms of this category is precisely the smash product algebra of [Cibils and Marcos 2006].

Proposition 2.8. $\mathcal{B}\#\Gamma$ *is a connected category if and only if the* Γ*-grading of* $\mathcal{B}$ *is connected.*

Proof. Note first that there is a canonical functor $F : \mathcal{B}\#\Gamma \to \mathcal{B}$ given on objects by $F(b, g) = b$, while, on morphisms, F is the inclusion map of homogeneous components. Assume that $\mathcal{B}\#\Gamma$ is connected and let b and c be objects of $\mathcal{B}$ and g in Γ. Consider the objects $(b, 1_\Gamma)$ and (c, g) in $\mathcal{B}\#\Gamma$. Let $\alpha = (f_n, \varepsilon_n) \dots (f_1, \varepsilon_1)$ be a nonzero walk from $(b, 1_\Gamma)$ to (c, g). Each f_i is a homogeneous morphism in $\mathcal{B}$, by definition of $\mathcal{B}\#\Gamma$.

Note also that the target in $\mathcal{B}\#\Gamma$ of (f_1, ε_1) is $(t(f_1, \varepsilon_1), (\deg f_1)^{-\varepsilon_1})$. Moreover, the target in $\mathcal{B}\#\Gamma$ of (f_2, ε_2) is $(t(f_2, \varepsilon_2), (\deg f_1)^{-\varepsilon_1}(\deg f_2)^{-\varepsilon_2})$. Thus we get

$$g = (\deg f_1)^{-\varepsilon_1}(\deg f_2)^{-\varepsilon_2} \dots (\deg f_n)^{-\varepsilon_n},$$

so α is a homogeneous nonzero walk from b to c of degree g.

Conversely, assume that the Γ-grading of $\mathcal{B}$ is connected. Let (b, g) and (c, h) be objects of $\mathcal{B}\#\Gamma$, and consider $\alpha = (f_n, \varepsilon_n) \dots (f_1, \varepsilon_1)$ a homogeneous nonzero walk in $\mathcal{B}$ from b to c of degree $h^{-1}g$. Then α provides a nonzero walk from (b, g) to (c, h). □

Coverings of k-categories were introduced in [Bongartz and Gabriel 1982] in order to study representation theory. We recall the definition given in [Cibils et al. 2007]. First we define the star $\mathrm{St}_{b_0}\mathcal{B}$ at an object b_0 of a k-category $\mathcal{B}$ as the direct sum of all k-modules of morphisms with source or target b_0. A k-functor $F : \mathcal{C} \to \mathcal{B}$ induces a k-linear map $F : \mathrm{St}_x\mathcal{C} \to \mathrm{St}_{Fx}\mathcal{B}$ for any object x of $\mathcal{C}$.

Definition 2.9. Let $\mathcal{C}$ and $\mathcal{B}$ be k-categories. A k-functor $F : \mathcal{C} \to \mathcal{B}$ is a *covering* if it is surjective on objects and if F induces k-isomorphisms between the corresponding stars. More precisely, for each $b_0 \in \mathcal{B}_0$ and each x in the nonempty fiber $F^{-1}(b_0)$, the map

$$F^x_{b_0} : \mathrm{St}_x\mathcal{C} \to \mathrm{St}_{b_0}\mathcal{B}$$

provided by F is a k-isomorphism.

A morphism from a covering $F : \mathcal{C} \to \mathcal{B}$ to a covering $G : \mathcal{D} \to \mathcal{B}$ is a pair of k-linear functors (H, J), where $H : \mathcal{C} \to \mathcal{D}$, $J : \mathcal{B} \to \mathcal{B}$ are such that J is an isomorphism, J is the identity on objects, and $GH = JF$.

We consider, within the group of automorphisms of a covering $F : \mathcal{C} \to \mathcal{B}$, the subgroup $\mathrm{Aut}_1 F$ of invertible endofunctors G of $\mathcal{C}$ such that $FG = F$.

Let $b \in \mathcal{B}$ and let $F^{-1}(b)$ be the corresponding fiber. This fiber is nonempty by definition, and $\mathrm{Aut}_1 F$ acts freely on it [Le Meur 2007; Cibils et al. 2007].

Definition 2.10. A covering $F : \mathcal{C} \to \mathcal{B}$ of k-categories is a *Galois covering* if $\mathcal{C}$ is connected and if $\mathrm{Aut}_1 F$ acts transitively on some fiber.

Remark 2.11. One can prove that for a Galois covering F, the group $\mathrm{Aut}_1 F$ acts transitively on any fiber [Le Meur 2007; Cibils et al. 2007].

As an example of Galois coverings, we have those coming from the smash product construction: if X is a Γ-grading of the k-category $\mathcal{B}$, the functor $\mathcal{B}\#\Gamma \to \mathcal{B}$, given by $(b, g) \mapsto b$ and the inclusion on morphisms, is a Galois covering with Γ as group of automorphisms.

It is useful to observe that the evident action of Γ on the smash product category $\mathcal{B}\#\Gamma$ is given as follows. The action on objects is given by the left action of Γ on itself. It is a free action. Observe that for any $u \in \Gamma$, a morphism from (b, g) to (c, h) is also a morphism from (b, ug) to (c, uh) since $h^{-1}g = (uh)^{-1}ug$.

We now consider Galois coverings together with a fixed object as follows. Given a k-category $\mathcal{B}$ and a fixed object b_0 of $\mathcal{B}$, the objects of the category $\mathrm{Gal}(\mathcal{B}, b_0)$ are Galois coverings $F : \mathcal{C} \to \mathcal{B}$. Morphisms are Galois covering morphisms $(H, J) : F_1 \to F_2$, where $H : \mathcal{C}_1 \to \mathcal{C}_2$ and $J : \mathcal{B} \to \mathcal{B}$ is an isomorphism that is the identity on objects.

We proved in [Cibils et al. 2007] that a morphism (H, J) induces a unique group epimorphism $\lambda_H : \mathrm{Aut}_1 F_1 \to \mathrm{Aut}_1 F_2$ verifying $Hf = \lambda_H(f)H$, for all $f \in \mathrm{Aut}_1 F_1$.

The following proposition describes morphisms of smash coverings in terms of the corresponding λ.

Proposition 2.12. *Let $b_0 \in \mathcal{B}$, and let $F_1 : \mathcal{B}\#G_1 \to \mathcal{B}$ and $F_2 : \mathcal{B}\#G_2 \to \mathcal{B}$ be Galois coverings associated to connected gradings X_1 and X_2 of $\mathcal{B}$ with groups G_1 and G_2. Given a morphism of coverings $(H, J) : F_1 \to F_2$ in* $\mathrm{Gal}(\mathcal{B}, b_0)$, *there exists a map $h : G_1 \to G_2$ such that $H(b_0, g) = (b_0, h(g))$ for all $g \in G_1$. Moreover, h is a G_1-morphism and $h(g) = \lambda_H(g)h(1)$, where $\lambda_H : G_1 \to G_2$ is the group morphism associated to H.*

Proof. It is clear that $H(b_0, g) = (b_0, g')$ for some $g' \in G_2$, since $b_0 = JF(b_0, g) = FH(b_0, g)$. We write $h(g) = g'$.

We have thus obtained that given $b_0 \in \mathcal{B}$, the morphism H induces a map

$$h : F_1^{-1}(b_0) \to F_2^{-1}(b_0).$$

Moreover, $F_i^{-1}(b_0)$ is a G_i-set ($i = 1, 2$), by identifying G_i with $\mathrm{Aut}_1 F_i$, and λ_H makes $F_2^{-1}(b_0)$ a G_1-set; more precisely, if $f \in G_1$ and $y \in F_2^{-1}(b_0)$, then $f \cdot y = \lambda_H(f) \cdot y$. We assert that h is a morphism of G_1-sets. For this purpose, take $x \in F_1^{-1}(b_0)$ and $f \in G_1$; then

$$\begin{aligned}(b_0, h(f \cdot x)) &= H(b_0, f \cdot x) = Hf(b_0, x)\\ &= \lambda_H(f)H(b_0, x) = \lambda_H(f)(b_0, h(x)) = (b_0, \lambda_H(f)h(x)).\end{aligned}$$

Finally, $h(g) = h(g \cdot 1) = \lambda_H(g)h(1)$. □

3. The fundamental group

In [Cibils et al. 2007], we defined the fundamental group of a connected k-category using Galois coverings. Our purpose is to relate this fundamental group to connected gradings. Let us recall the definition given in [Cibils et al. 2007]. Considering the fiber functor

$$\Phi : \mathrm{Gal}(\mathcal{B}, b_0) \to \mathsf{Sets}$$

given by $\Phi(F) = F^{-1}(b_0)$, we have defined $\pi_1(\mathcal{B}, b_0) = \mathrm{Aut}\,\Phi$.

To study the fundamental group we introduce the full subcategory $\mathrm{Gal}^{\#}(\mathcal{B}, b_0)$ of $\mathrm{Gal}(\mathcal{B}, b_0)$ whose objects are the smash product Galois coverings $F : \mathcal{B}\#\Gamma \to \mathcal{B}$.

Theorem 3.1. *The categories* $\mathrm{Gal}^{\#}(\mathcal{B}, b_0)$ *and* $\mathrm{Gal}(\mathcal{B}, b_0)$ *are equivalent.*

Proof. It is immediate from [Cibils and Marcos 2006], since any Galois covering $F : \mathcal{C} \to \mathcal{B}$ is isomorphic to the Galois covering $\mathcal{B}\#\mathrm{Aut}_1 F \to \mathcal{B}$. Note that the grading of $\mathcal{B}$ by $\mathrm{Aut}_1 F$ is not canonical; it depends on a choice of an object in each fiber. □

The next proposition shows that we can restrict to the subcategory $\mathrm{Gal}^{\#}(\mathcal{B}, b_0)$ of $\mathrm{Gal}(\mathcal{B}, b_0)$ when considering the fundamental group $\pi_1(\mathcal{B}, b_0)$.

Proposition 3.2. *Let* $F : \mathcal{C} \to \mathcal{D}$ *be an equivalence of categories*, $\Phi_{\mathcal{C}} : \mathcal{C} \to \mathsf{Sets}$, $\Phi_{\mathcal{D}} : \mathcal{D} \to \mathsf{Sets}$ *such that* $\Phi_{\mathcal{D}} F = \Phi_{\mathcal{C}}$. *Then there exists an isomorphism* $F^* : \mathrm{Aut}\,\Phi_{\mathcal{D}} \to \mathrm{Aut}\,\Phi_{\mathcal{C}}$.

Proof. Recall that an element $\tau \in \mathrm{Aut}\,\Phi_{\mathcal{D}}$ is an invertible natural transformation, that is, a family of invertible set maps $\tau_d : \Phi_{\mathcal{D}}(d) \to \Phi_{\mathcal{D}}(d)$ for every object d in $\mathcal{D}$, which are compatible with morphisms in $\mathcal{D}$. Since F is a functor, it is clear that $F^*(\tau)$ defined by $F^*(\tau)_c = \tau_{F(c)}$ is an element in $\mathrm{Aut}\,\Phi_{\mathcal{C}}$.

Let $\tau \in \mathrm{Aut}\,\Phi_{\mathcal{D}}$ be such that $F^*(\tau) = \mathrm{id}$. Since F is dense, for any object d in $\mathcal{D}$ there exists c in $\mathcal{C}$ with an isomorphism $\alpha : d \to F(c)$; the naturality of τ induces the commutative diagram

$$\begin{tikzcd}
\Phi_{\mathcal{D}}(d) \arrow[r, "\tau_d"] \arrow[d, "\Phi_{\mathcal{D}}(\alpha)"] & \Phi_{\mathcal{D}}(d) \arrow[d, "\Phi_{\mathcal{D}}(\alpha)"] \\
\Phi_{\mathcal{D}}(F(c)) \arrow[r, "\tau_{F(c)}"] & \Phi_{\mathcal{D}}(F(c)).
\end{tikzcd}$$

Since $\tau_{F(c)} = \mathrm{id}$ for all $c \in \mathcal{C}$, this implies that $\tau_d = \mathrm{id}$, and hence $\tau = \mathrm{id}$.

In order to prove that F^* is surjective, let $\sigma \in \mathrm{Aut}\,\Phi_{\mathcal{C}}$ and consider $\hat{\sigma}$ defined in the following way. For any object d in $\mathcal{D}$, we choose c and an isomorphism $\alpha : d \to F(c)$; in the case where $d = F(c)$, we choose $\alpha = \mathrm{id}$. Now we define $\hat{\sigma}_d$ such that the following diagram is commutative:

$$\begin{tikzcd}
\Phi_{\mathcal{D}}(d) \arrow[r, "\hat{\sigma}_d"] \arrow[d, "\Phi_{\mathcal{D}}(\alpha)"] & \Phi_{\mathcal{D}}(d) \arrow[d, "\Phi_{\mathcal{D}}(\alpha)"] \\
\Phi_{\mathcal{C}}(c) = \Phi_{\mathcal{D}}(F(c)) \arrow[r, "\sigma_c"] & \Phi_{\mathcal{D}}(F(c)) = \Phi_{\mathcal{C}}(c).
\end{tikzcd}$$

Since F is full, we have that $\hat{\sigma}$ is a natural transformation and $F^*(\hat{\sigma}) = \sigma$. □

Corollary 3.3. *Let* $\Phi^{\#} : \mathrm{Gal}^{\#}(\mathcal{B}, b_0) \to \mathsf{Sets}$ *be the functor given by*

$$\Phi(F : \mathcal{B}\#G \to \mathcal{B}) = F^{-1}(b_0) = G.$$

Then $\pi_1(\mathcal{B}, b_0) \cong \operatorname{Aut} \Phi^{\#}$.

Corollary 3.4. *If* $\mathcal{B}$ *only admits the trivial connected grading, then* $\pi_1(\mathcal{B}, b_0) = 1$.

An advantage of considering $\mathrm{Gal}^{\#}(\mathcal{B}, b_0)$ instead of $\mathrm{Gal}(\mathcal{B}, b_0)$ is explained by the following proposition, which describes the automorphisms of the fiber functor.

Proposition 3.5. *Let* $\sigma \in \operatorname{Aut} \Phi^{\#}$, *and let* G *be a group grading the category* $\mathcal{B}$ *in a connected way. The map* $\sigma_G : G \to G$ *is given by* $\sigma_G(x) = xg$, *where* $g \in G$ *is uniquely determined.*

Proof. Consider a covering $F : \mathcal{B}\#G \to \mathcal{B}$. Each $g \in G$ induces an automorphism of the covering F, which is the identity on $\mathcal{B}$ and the left action of G on itself. We shall denote it by l_g. Given $\sigma \in \operatorname{Aut} \Phi^{\#}$, we get a map $\sigma_G : G \to G$. It must make the diagram

$$\begin{tikzcd} G \arrow[r, "\sigma_G"] \arrow[d, "\widetilde{l_g}"'] & G \arrow[d, "\widetilde{l_g}"] \\ G \arrow[r, "\sigma_G"] & G \end{tikzcd}$$

commutative, where $\widetilde{l_g}$ is induced by l_g. So, for all $x \in G$, we get $\sigma_G(g_0 x) = g_0 \sigma_G(x)$. Taking $x = 1$ we obtain $\sigma_G(g_0) = g_0 \sigma_G(1)$. Note that g_0 is an arbitrary element of G. □

4. The fundamental group of matrix and triangular algebras

Let k be a field containing a primitive n-th root of unity q, and let $M_n(k)$ be the k-algebra of $n \times n$ matrices. The problem of classifying all the gradings of $M_n(k)$ is not solved. Lists of gradings have been described by several authors [Aljadeff et al. 2010; Bahturin et al. 2001; Bahturin and Zaicev 2002; Boboc 2003; Boboc and Dăscălescu 2001; 2006; Caenepeel et al. 2002; Dăscălescu et al. 1999], and the complete lists for $n = 2$ and $n = 3$ are obtained in [Khazal et al. 2003; Boboc and Dăscălescu 2007].

We consider connected gradings of the algebra $M_n(k)$. In the case of a nonconnected grading, we shall restrict to the subgroup generated by the support, in order to study the unique associated connected grading.

We briefly recall the definition of the universal covering of a k-category.

Definition 4.1. A *universal covering* $U : \mathcal{U} \to \mathcal{B}$ is an object in $\mathrm{Gal}(\mathcal{B})$ such that for any Galois covering $F : \mathcal{C} \to \mathcal{B}$, and for any $u_0 \in \mathcal{U}_0$, $c_0 \in \mathcal{C}_0$ with $U(u_0) = F(c_0)$, there exists a unique morphism $(H, 1)$ from U to F verifying $H(u_0) = c_0$.

Theorem 4.2 [Cibils et al. 2007, Theorem 4.6]. *Suppose that a connected* k-*category* $\mathcal{B}$ *admits a universal covering* U. *Then*

$$\pi_1(\mathcal{B}, b_0) \simeq \mathrm{Aut}_1 U.$$

Definition 4.3. A connected k-category is *simply connected* if its only connected grading is the trivial grading. A connected grading is *simply connected* if the corresponding Galois covering is simply connected.

We will prove that there is no universal cover for $M_n(k)$. Indeed there exist at least two nonisomorphic connected gradings of $M_n(k)$ that provide simply connected Galois coverings. Recall that a covering is simply connected if it admits no proper Galois covering.

Proposition 4.4 [Bahturin et al. 2001; Chun and Lee 2007]. *There exists a connected $C_n \times C_n$-grading of $M_n(k)$.*

Proof. The algebra $M_n(k)$ has a well-known presentation

$$M_n(k) = k\{x, y\}/\langle x^n = 1,\ y^n = 1,\ yx = qxy\rangle,$$

where

$$x = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}, \qquad y = \begin{pmatrix} q & 0 & 0 & \cdots & 0 \\ 0 & q^2 & 0 & \cdots & 0 \\ 0 & 0 & q^3 & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & q^n \end{pmatrix},$$

with q a primitive n-th root of unity. We provide a connected grading of $k\{x, y\}$ by assigning degree $(t, 1)$ to x and degree $(1, t)$ to y, where t is a generator of C_n. The group is abelian and the order of the generators is n, and hence the ideal of relations is homogeneous. Since the support coincides with $C_n \times C_n$, the grading is connected. □

Proposition 4.5. *Let $\mathscr{C}$ be a k-category with a finite set of objects and one-dimensional vector spaces of morphisms between any pair of objects b and c, denoted by ${}_c\mathscr{C}_b = k\ {}_cf_b$, verifying*

$$({}_df_c)({}_cf_b) = q_{d,c,b}\ ({}_df_b)$$

for any triple of objects of $\mathscr{C}$, where $q_{d,c,b} \in k^*$ are the structure constants. Then $\mathscr{C}$ is simply connected.

Proof. Let G be a group providing a grading of the category $\mathscr{C}$. As noted above, we consider connected gradings. Since all the k-vector spaces of morphisms are one-dimensional, they are homogeneous. Let ${}_cs_b$ be the degree of ${}_c\mathscr{C}_b$. Note that for each object b we have ${}_b\mathscr{C}_b = k$; hence ${}_bs_b = 1$ and ${}_bs_c = {}_cs_b^{-1}$. We assert that any nonzero homogeneous closed walk has degree 1. Indeed, since composition of nonzero morphisms is nonzero in $\mathscr{C}$, and since ${}_bs_c = {}_cs_b^{-1}$, a nonzero homogeneous closed walk at b can be replaced by a nonzero endomorphism of b with the same degree. Since endomorphisms of b have degree 1, the assertion is proved. Recall that a grading is connected if for any pair of objects, any group element appears as the degree of a nonzero homogeneous walk between them. Since the grading is connected, the group is trivial. □

Corollary 4.6. *Let $\mathscr{C}$ be a category as above. Then $\pi_1(\mathscr{C}) = 1$.*

Let ${}_jE_i$ be the matrix whose entries are zero, except the (j,i) entry, which equals 1. We recall that a *good grading* of a matrix algebra is a grading where the elementary matrices ${}_jE_i$, also called *matrix units*, are homogeneous; see for instance [Dăscălescu et al. 1999]. Note that the k-category $\mathcal{M}_n(k)$ associated to a matrix algebra with respect to the idempotent elementary matrices ${}_iE_i$ is precisely a category as in the proposition above, where all the structure constants equal 1.

Clearly good gradings of $M_n(k)$ and gradings of the k-category $\mathcal{M}_n(k)$ coincide.

Corollary 4.7. *Let G be a group providing a good grading of a matrix algebra, and assume that the corresponding grading of the k-category $\mathcal{M}_n(k)$ is connected. Then G is trivial.*

Remark 4.8. A good grading by a nontrivial group G of a matrix algebra $M_n(k)$ can be connected when $M_n(k)$ is viewed as a category with a single object. This means that the support of the grading generates G. Corollary 4.7 makes precise that the corresponding grading of the k-category $\mathcal{M}_n(k)$ will not be connected.

Theorem 4.9. *The connected grading of the matrix algebra $M_n(k)$ by the group $C_n \times C_n$ of Proposition 4.4 is simply connected.*

Proof. We will prove that the Galois covering $\mathscr{C} = M_n(k)\#(C_n \times C_n)$ is simply connected. The category $\mathscr{C} = M_n(k)\#(C_n \times C_n)$ has set of objects $C_n \times C_n = \{a^i b^j \mid 0 \le i, j \le n-1\}$ and

$$ {}_{a^s b^l}\mathscr{C}_{a^i b^j} = X^{a^{i-s} b^{j-l}} M_n(k) = k(x^{i-s} y^{j-l}). $$

Hence the k-vector spaces of morphisms are one-dimensional with basis elements

$$ {}_{(s,l)}f_{(i,j)} = x^{i-s} y^{j-l} $$

and

$$ {}_{(u,v)}f_{(s,l)}\,{}_{(s,l)}f_{(i,j)} = x^{s-u} y^{l-v} x^{i-s} y^{j-l} = q^{l+i-v-s}\,{}_{(u,v)}f_{(i,j)}. $$

Finally, Proposition 4.5 asserts that such categories are simply connected. □

Each time a universal covering exists, the fundamental group is isomorphic to its Galois group. Clearly, if there exist at least two nonisomorphic simply connected coverings, there is no universal covering.

We will now show that this is the case for $M_n(k)$, that is, there exists at least another simply connected grading of $M_n(k)$. For this purpose we first provide another presentation of the matrix algebra as a quotient of a path algebra.

Proposition 4.10. *Let Q be the quiver with n vertices labelled $1,\dots,n$, and arrows x_i from i to $i+1$ as well as reverse arrows y_i from $i+1$ to i for $1 \le i < n$. We denote by $e_1,\dots,e_n$ the idempotents of the path algebra kQ corresponding to the vertices. Let I be the two-sided ideal of kQ generated by $y_i x_i - e_i$ and $x_i y_i - e_{i+1}$ for $1 \le i < n$. Then kQ/I is isomorphic to $M_n(k)$.*

Proof. Consider the morphism of algebras $\varphi : kQ \to M_n(k)$ given by $\varphi(e_i) = {}_iE_i$, $\varphi(x_i) = {}_{i+1}E_i$ and $\varphi(y_i) = {}_iE_{i+1}$, which is well-defined by the universal property of path algebras, which are in fact tensor algebras over the semisimple commutative algebra given by the length zero paths. This map is surjective, since the matrices ${}_jE_i$ are clearly images of paths of Q. Also $I \subset \operatorname{Ker}\varphi$ and $\dim_k kQ/I \le n^2$. □

Let F_{n-1} be the free group on $n-1$ generators $s_1,\dots,s_{n-1}$. First we introduce an F_{n-1} grading of kQ as follows: for $1 \le i \le n$, let $\deg e_i = 1$, while for $1 \le i \le n-1$ we set $\deg x_i = s_i$ and $\deg y_i = (s_i)^{-1}$. The path algebra is a free algebra on the set of arrows with respect to the semisimple subalgebra of vertices, so this provides a well-defined grading of kQ. More precisely, the degree of any path is the corresponding product of the degrees of the arrows. Since the ideal I is homogeneous with respect to this grading, we obtain a grading of kQ/I, and hence of $M_n(k)$. Note that this grading, considered as a grading of the algebra $M_n(k)$, that is, as a grading of the single object category with endomorphism algebra $M_n(k)$, is connected, since the generators of the free group are in the support.

Proposition 4.11. *The F_{n-1}-grading of $M_n(k)$ just described is simply connected.*

Proof. The set of objects of $M_n(k)\#F_{n-1}$ is F_{n-1}. For $j > i$, let ${}_js_i = s_{j-1}\dots s_{i+1}s_i$. There is a one-dimensional vector space of morphisms from a word w in F_{n-1} considered as an object of $M_n(k)\#F_{n-1}$ to each object ${}_js_i w$ with basis vector denoted by ${}_jE_i{}^w$. Similarly, for $j < i$ there is a one-dimensional vector space of morphisms from w to ${}_js_i^{-1}w$, with basis ${}_jE_i{}^w$. From w to w, the n-dimensional vector space of morphisms has basis $\{{}_1E_1{}^w,\dots,{}_nE_n{}^w\}$. Note that the endomorphism algebra of each object is the n-dimensional diagonal algebra $k({}_1E_1{}^w)\times\cdots\times k({}_nE_n{}^w)$. Consider now a grading of this category by a group G. Since the spaces of morphisms between different objects are one-dimensional, they are homogeneous. This fact implies that for each object w, the subvector space $k\,({}_iE_i{}^w)$ is homogeneous, since

$$ {}_iE_j{}^{({}_js_i w)}\,{}_jE_i{}^w = {}_iE_i{}^w. $$

Observe that an idempotent homogeneous element necessarily has degree 1, so each endomorphism algebra has trivial grading (all elements have degree 1). As a consequence,

$$ \deg\left({}_iE_j{}^{({}_js_i w)}\right) = \left(\deg {}_jE_i^w\right)^{-1}. $$

Moreover, for $j > i$,

$$\deg({}_jE_i{}^{w}) = \deg\big({}_jE_{j-1}{}^{({}_{j-1}s_i w)}\big) \ \cdots \ \deg({}_{i+2}E_{i+1}{}^{s_i w}) \ \deg({}_{i+1}E_i{}^{w}).$$

For $j < i$, the statement is analogous considering the inverses of the degrees above. This complete description of any possible grading of the smash category shows that any closed homogeneous nonzero walk has degree 1. Consequently the grading is connected only if the group is trivial. □

The complete list of good gradings of a matrix algebra is obtained in [Caenepeel et al. 2002]. In order to compute the fundamental groups of matrix algebras, we make this classification explicit using Proposition 4.10.

Theorem 4.12. *There is a one-to-one correspondence between good connected G-gradings of $M_n(k)$ and maps $\{1, \dots, n-1\} \to G$ such that the image generates G.*

Proof. Let m be a map from $\{1, \dots, n-1\}$ to G. We obtain a grading of the algebra kQ defined in Proposition 4.10 as before, namely $\deg({}_{i+1}E_i) = m(i)$ and $\deg({}_iE_{i+1}) = m(i)^{-1}$. The ideal of relations of Proposition 4.10 is homogeneous and we obtain a good grading of $M_n(k)$. If the image of m generates G, then the grading is connected. Conversely, consider a good connected grading of $M_n(k)$ by a group G. The image of the map $m : \{1, \dots, n-1\} \to G$ given by $m(i) = \deg({}_{i+1}E_i)$ generates G. □

Note that relaxing the connectedness requirement for good gradings is equivalent to removing the condition that the image of each map m generates G. In [Caenepeel et al. 2002], the algebra $M_n(k)$ is viewed as the endomorphism algebra of a vector space V, and good gradings are obtained from a grading of V, considering graded endomorphisms as homogeneous components.

Definition 4.13. The *quotient* of a G-grading X of a category $\mathcal{B}$ by a normal subgroup N of G is a G/N-grading X/N of $\mathcal{B}$, where the homogeneous component of degree α is

$$(X/N)^{\alpha}{}_c\mathcal{B}_b = \bigoplus_{g \in \alpha} X^g{}_c\mathcal{B}_b.$$

Observe that if X is connected then X/N is also connected.

The corresponding functor between the smash product coverings is precisely the canonical projection obtained through the quotient of $\mathcal{B}\#G \to \mathcal{B}$ by N.

Proposition 4.14. *Any good connected G-grading of $M_n(k)$ is a quotient of the F_{n-1}-grading considered before.*

Proof. Let $m_0 : \{1, \dots, n-1\} \to F_{n-1}$ be the map corresponding to this grading, given by $m_0(i) = s_i$, and let $m : \{1, \dots, n-1\} \to G$ be another map such that the image of m generates G. Then the group homomorphism given by $s_i \mapsto m(i)$ is a surjective group morphism. □

We recall that a simply connected grading is a grading that is maximal in the sense that it is not isomorphic to a proper quotient of a connected grading.

Proposition 4.15. *Let k be a field containing a primitive n-th root of unity. The grading by $C_n \times C_n$ of Proposition 4.4 and the grading by the free group of Proposition 4.11 have a unique maximal common quotient C_n-grading.*

Proof. We denote by X the grading by $C_n \times C_n$ and we observe that the vector space X^1 of homogeneous elements of trivial degree is one-dimensional. Let Y be the grading by F_{n-1}: observe that Y^1 is the n-dimensional subalgebra of diagonal matrices. Assume that Z is a common quotient of X and Y, and let N be the normal subgroup of $C_n \times C_n$ that provides Z as a quotient of X. Since Z is a quotient of Y, clearly Z^1 contains at least Y^1, the diagonal matrices. Observe that the elementary diagonal matrices are homogeneous for X, and consequently their degrees must be elements of N in order to become trivial. The set of degrees of the diagonal matrices for X is precisely $1 \times C_n$. Hence $1 \times C_n$ is the smallest subgroup of $C_n \times C_n$ that has a chance to meet a quotient of Y; let $N = 1 \times C_n$. In fact we assert that X/N is already a good grading; in other words, elementary matrices are homogeneous. Indeed, consider the n-dimensional subvector space E of $M_n(k)$ with basis $\{{}_2E_1, {}_3E_2, \ldots, {}_nE_{n-1}, {}_1E_n\}$. Recall that x is the circulant matrix, which is the sum of all the previous basis vectors of E, while y is the diagonal matrix made with powers of the primitive root of unity q. Then the set $\{x, xy, xy^2, \ldots, xy^{n-1}\}$ is clearly contained in E. Also the elements xy^i, for $0 \le i \le n-1$, are homogeneous for the grading X, of different degrees (t, t^i) where t is the generator of C_n. Hence they are linearly independent and they form a basis of E. Finally we observe that for X/N, all these elements have the same degree $\overline{(t, 1)}$, and hence E is contained in the set of homogeneous elements of degree $\overline{(t, 1)}$ of X/N.

Consequently, each elementary matrix is homogeneous for X/N. Considering Y, we obtain X/N as the quotient Y/M, where M is the smallest normal subgroup of F_{n-1} such that in F_{n-1}/M all the generators of F_{n-1} are equal, and this element is of order n. □

Theorem 4.16. *Let k be an algebraically closed field.*

(1) *If* $\operatorname{char}(k) \neq 2$, *then* $\pi_1 M_2(k) \simeq \mathbb{Z} \times C_2$.

(2) *If* $\operatorname{char}(k) \neq 3$, *then* $\pi_1 M_3(k) \simeq F_2 \times C_3$.

Proof. Under these assumptions, the classifications of [Khazal et al. 2003; Boboc and Dăscălescu 2001] show that all gradings are good gradings or quotients of the one given by Proposition 4.4. The latter and the grading by the free group have a common quotient described in Proposition 4.15. Recall that we have proved that all good gradings of $M_n(k)$ are quotients of the F_{n-1}-grading. We now prove the first assertion. We construct two inverse group morphisms between $\pi_1 M_2(k)$ and

$F_1 \times C_2$. Let $\sigma \in \operatorname{Aut}\Phi^{\#}$, where $\Phi^{\#} : \operatorname{Gal}^{\#}(M_2(k)) \to \text{Sets}$ is the fiber functor. Consider the good F_1-grading of $M_2(k)$. By Proposition 3.5, the map σ_{F_1} verifies $\sigma_{F_1}(x) = xg$ for some uniquely determined $g \in F_1$. Analogously, the $C_2 \times C_2$-grading provides $\sigma_{C_2\times C_2}$ and an element (t^a, t^b), where $C_2 = \langle t\rangle$. The compatibility condition obtained when considering the maximal common quotient C_2 says that t^a equals the class of g in C_2. We associate the pair (g, t^b) to σ.

Conversely, given $(g, t^b) \in F_1 \times C_2$, we will construct $\sigma \in \operatorname{Aut}\Phi^{\#}$ associated to it. One needs to have maps $\sigma_G : G \to G$ for each group G providing a connected grading of $M_2(k)$. Using the classification of the gradings given in [Khazal et al. 2003] and Proposition 4.15, it is sufficient to describe σ_{F_1} and $\sigma_{C_2\times C_2}$. Fix $\sigma_{F_1}(x) = xg$ and $\sigma_{C_2\times C_2}(x) = x(\overline{g}, t^b)$. Note that these maps satisfy the compatibility condition. Of course, for the other quotients G of F_1 or of $C_2 \times C_2$, the map σ_G is uniquely determined thanks to the quotient compatibility conditions.

The proof of the second statement is completely analogous. □

Next we prove a generalization of the preceding theorem for matrices of prime size. Consider an algebraically closed field k of characteristic zero. The main result — Theorem 5.1 — of [Bahturin and Zaicev 2002] states that any grading of $M_n(k)$ by a group G is a tensor product of gradings, in the sense that there exists a decomposition $n = n_1 n_2$, a fine grading of $M_{n_1}(k)$ by a subgroup G_1 of order n_1^2, and a good G-grading of $M_{n_2}(k)$ such that $M_n(k)$ is isomorphic as a G-graded algebra to the tensor product algebra $M_{n_1}(k) \otimes M_{n_2}(k)$ that is obtained as an induced grading. The construction of an induced grading resembles a tensor construction, but is well-defined only in the case where one of the graded algebras involved is a matrix algebra with a good grading [Bahturin and Zaicev 2002].

Proposition 4.17. *Let p be a prime and k be an algebraically closed field of characteristic zero. Let X be a maximal connected grading by a group G of $M_p(k)$. Then either the group G is isomorphic to $C_p \times C_p$ and the grading is fine as in Proposition 4.4, or the grading is a good grading given by $m : \{1, \dots, p-1\} \to G$ such that* $\operatorname{Im}(m)$ *generates G.*

Proof. Since p is a prime, [Bahturin and Zaicev 2002, Theorem 5.1] shows that the grading is either good, or fine with group of order p^2. We already know that good connected gradings are as described in Proposition 4.14. If the grading is fine, the order p^2 of the group is precisely the dimension of the matrix algebra, and hence $\operatorname{Supp} X = G$. Moreover, for fine gradings of matrix algebras, homogeneous nonzero elements are invertible by Corollary 2.7 of the same article. Then we assert that the group is not cyclic: indeed, if G has a generator t of order p^2, let x be a nonzero element of degree p^2, and thus invertible. Note that $X^1 M_p(k) = k$. Hence $x^{p^2} \in k$ and $x^{p^2} \neq 0$, and we can normalize x by dividing it by a scalar in order to obtain $x' \in X^t M_p(k)$ such that $x'^{p^2} = 1$. Then $M_p(k)$ would be isomorphic to the group algebra of the cyclic group of order p^2, which is false since (for instance) the former is commutative.

Consequently, a fine connected grading of $M_p(k)$ is given by $C_p \times C_p$. As before, $\operatorname{Supp} X = G$ for dimensional reasons. Let t be a generator of C_p and let x and y be nonzero elements of degree $(t, 1)$ and $(1, t)$ respectively. Again, x and y are invertible and we normalize them in order to have $x^p = y^p = 1$. They do not commute, since otherwise the algebra would be the commutative algebra $k(C_p \times C_p)$. In fact, xy and yx are both nonzero and have common degree (t, t). Hence they differ by a scalar: $yx = qxy$. Moreover $q^p = 1$, since $x = y^p x = q^p xy^p = q^p x$. Then q is a primitive root of unity and the grading corresponds to the grading of Proposition 4.4. $\square$

Theorem 4.18. *Let k be an algebraically closed field of characteristic zero, and let p be a prime. Then*

$$\pi_1 M_p(k) \simeq F_{p-1} \times C_p.$$

The proof is completely analogous to the proof of Theorem 4.16.

We end this section with a computation of the fundamental group of triangular matrix algebras, based on [Valenti and Zaicev 2007].

A grading of an upper triangular matrix algebra $T_n(k)$ is *good* if the elementary matrices ${}_jE_i$ are homogeneous. Clearly any good grading is completely determined by assigning group elements to subdiagonal elementary matrices ${}_{i+1}E_i$, since the idempotents ${}_iE_i$ necessarily have trivial degree. In other words, a good grading is determined as before by a map $m : \{1, \dots, n-1\} \to G$. The grading is connected if and only if $\operatorname{Im} m$ generates G. As before, any good connected grading is a quotient of the grading given by the free group F_{n-1} on a set $\{s_1, \dots, s_{n-1}\}$ and a map m such that $\operatorname{Im} m = \{s_1, \dots, s_{n-1}\}$.

Theorem 7 of [Valenti and Zaicev 2007] states that any grading of a triangular algebra is good, without any hypothesis concerning the field. As an immediate consequence we obtain:

Theorem 4.19. *Let k be a field and let $T_n(k)$ be the algebra of triangular matrices of size n. Then*

$$\pi_1 T_n(k) \simeq F_{n-1}.$$

5. The fundamental group of truncated polynomial algebras

In this section we compute the fundamental group of the group algebra of the cyclic group of order p in characteristic p, that is, we compute the fundamental group of $k[x]/(x^p)$.

Proposition 5.1. *Let G be a finite group and let k be any field. The usual G-grading of the group algebra kG is simply connected.*

Proof. The Galois covering $kG\#G$ has G as set of objects and, given $s, t \in G$,

$$_t(kG\#G)_s = k(t^{-1}s).$$

The composition is given by the product of G. In other words, all k-vector spaces of morphisms are one-dimensional and all of the structure constants are 1. By Proposition 4.5, this category is simply connected. □

Remark 5.2. As a consequence of this proof, we recover the Cohen–Montgomery duality theorem for coactions [Cohen and Montgomery 1984]: the algebras $kG\#G$ and $M_{|G|}(k)$ are isomorphic. The algebra associated to a finite object category is obtained as the direct sum of all the vector spaces of morphisms. In particular, if all the vector spaces of morphisms are one-dimensional, we get the matrix algebra. Hence the algebra associated to the category $kG\#G$ is $M_{|G|}(k)$. On the other hand, it was proved in [Cibils and Marcos 2006] that the algebra corresponding to the categorical smash product by a finite group is precisely the usual smash product algebra.

Next we provide an example of a path k-algebra of a quiver with admissible relations, which does not admit a universal cover when the field is of characteristic p. The quiver is a loop, and the relation is given by the p-th power of the loop. There are at least two simply connected coverings by smash categories. One of them is not a covering of "quivers with relations" in the sense of [Gabriel 1981].

Proposition 5.3. *Let k be a field of characteristic p. The truncated polynomial algebra $k[x]/(x^p)$ does not admit a universal covering.*

Proof. First, note that $k[x]/(x^p)$ is isomorphic to the k-group algebra of the cyclic group C_p of order p, and hence the preceding proposition provides a simply connected covering with group C_p. Note that this covering is the category with p vertices, where all vector spaces of morphisms are one-dimensional and all the structure constants are 1.

On the other hand, consider the usual $\mathbb{Z}$-grading of $k[x]$. Since (x^p) is a homogeneous ideal — this holds in any characteristic — it induces a grading in $k[x]/(x^p)$. For this grading, $[k[x]/(x^p)]\#\mathbb{Z}$ is the category that has $\mathbb{Z}$ as set of objects, one-dimensional vector spaces of morphisms from i to j if $0 \leq j - i < p$, and 0 otherwise. In other words, the morphisms in the category are generated by morphisms from i to $i+1$ for each integer i, with relations such that any composition of p generators is zero. As a consequence of this description, each grading of $[k[x]/(x^p)]\#\mathbb{Z}$ is freely determined by assigning a degree to the one-dimensional vector space of morphisms from i to $i+1$. Hence, any homogeneous nonzero closed walk has trivial degree.

Recall that by the definition of a connected grading, any element of the group should be the degree of a homogeneous walk between objects. Then the unique group that grades this smash product category in a connected way is the trivial one. As a consequence, this covering category is simply connected. Finally note that the Galois coverings are not isomorphic, since their groups of automorphisms are not isomorphic. In this way we have constructed two nonisomorphic simply connected coverings. □

It is well-known and easy to prove that the trivial homogeneous component of any grading always contains the ground field k.

A grading is called *fine* if the dimension of each homogeneous component is at most one; see for instance [Bahturin et al. 2001].

Theorem 5.4. *Let k be a field of characteristic p and let $A = k[x]/(x^p)$. There are two types of connected gradings of A, with no common quotient except the trivial one. The first type corresponds to the group algebra case, and its grading group is C_p. In the second one, the grading group is either $\mathbb{Z}$ or any of its quotients.*

Proof. Let X be a connected basic grading of A. There are two cases, according to the existence of an invertible homogeneous element of nontrivial degree. First we suppose that there exists an invertible homogeneous element a of degree $s \neq 1$. We write $a = a_0 + a_+$, where $a_0 \in k^*$ and $a_+ \in (x)$, and we normalize a in order to have $a_0 = 1$. Since the characteristic of k is p, we obtain that $a^p = 1$ and p is the order of a. For $i < p$ we infer that $a^i \neq 0$, and thus $X^{s^i}A \neq 0$. Moreover, $X^{s^i}A \neq X^{s^j}A$ for $i \neq j$, $i, j < p$. Also $1 = a^p \in X^{s^p}A$ implies $s^p = 1$. Since the grading is connected, by computing dimensions we deduce that the group is cyclic of order p, and the grading is fine.

As a second case, assume that all homogeneous elements of nontrivial degree belong to the maximal ideal (x):

$$\bigoplus_{s \in G,\, s \neq 1} X^s A \subseteq (x).$$

Consider now the usual valuation ν on A: namely, for $f \neq 0$ we have that $\nu(f)$ is the smallest exponent of x appearing in f. Of course $\nu(f) = 0$ if and only if f is invertible. The valuation ν has the following properties:

- $\nu(f + g) \geq \inf\{\nu(f), \nu(g)\}$ for $f, g, f + g \neq 0$.
- $\nu(fg) = \nu(f) + \nu(g)$ for $f, g, fg \neq 0$.

Then for $f \neq 0$ we obtain $f = x^{\nu(f)}u$, where u is invertible.

Assume first that there exists a homogeneous $g_1 \in X^1 A$ of valuation 1, that is, $g_1 = x + u$ with $u \in (x^2)$. Since $g_1^{p-1} = x^{p-1}$ and g_1^{p-1} is homogeneous, we infer that x^{p-1} is homogeneous of degree 1. Now, $g_1^{p-2} = x^{p-2} + \lambda x^{p-1}$, so $x^{p-2} = g_1^{p-2} - \lambda x^{p-1}$ and thus x^{p-2} is homogeneous of degree 1. If we continue with this procedure, we finally get that x is homogeneous of degree 1 and the grading is trivial. Finally, assume $\nu(g_1) \geq 2$ for any homogeneous $g_1 \in X^1 A$. We claim that there exists a homogeneous f of valuation 1. If not, for any $g \in (x)$ we have $\nu(g) \geq 2$, by decomposing g as a sum of its homogeneous components and using the property of a valuation just discussed, which is clearly false since $\nu(x) = 1$. Now $\nu(f^i) = i$ for $i < p$. Since $f^i \in X^{s^i} A$, the latter is not zero. For dimensional reasons, we infer that the support of the grading is $\{1, s, \ldots, s^{p-1}\}$, which generates a cyclic group. □

Corollary 5.5. *Let k be a field of characteristic p. Then $\pi_1(k[x]/(x^p)) = \mathbb{Z} \times C_p$.*

6. The fundamental group of diagonal algebras

Let E be a finite set and k a field. The diagonal algebra k^E is the vector space of maps from E to k with pointwise multiplication. Next we consider connected gradings of diagonal algebras [Dăscălescu 2008; Bichon 2008]. The following result shows that any abelian group with the cardinality of a given set grades the diagonal algebra in a connected way, if the field contains enough roots of unity.

Proposition 6.1. *Let E be a finite set of order n, and let k be a field with enough n-th roots of unity. Let G be any abelian group of order n. Then there is a simply connected G-grading of k^E.*

Proof. We first sketch the proof of the following well-known result. Let G be any abelian group of order n, E a set of cardinal n, and k a field containing n different n-th roots of unity; then the algebras kG and k^E are isomorphic. First assume that G is cyclic. Let t be a generator of G and let μ_n be the set of n-th roots of unity in k. Note that under our assumptions p does not divide n in the case where k is a field of characteristic $p > 0$. Then the set

$$\left\{ e_\zeta = \frac{1}{n} \sum_{i=0}^{n-1} \zeta^i t^i \right\}_{\zeta \in \mu_n}$$

is a complete set of orthogonal idempotents of kG and has n elements. This set provides a new basis of kG, proving that kG is isomorphic to $\bigoplus_{\zeta \in \mu_n} k e_\zeta$, which in turn is identified with k^E through a bijection between E and μ_n by considering the Dirac masses in k^E.

For an arbitrary abelian group G of order n, note that G is a direct product of finite cyclic groups. Note also that a group algebra $k(G_1 \times G_2)$ is isomorphic to $kG_1 \otimes kG_2$, while the algebras $k^{E_1 \times E_2}$ and $k^{E_1} \otimes k^{E_2}$ are also isomorphic. The previous case provides the required isomorphism.

Next we prove the statement of the proposition. Consider the algebra k^E, an arbitrary abelian group G of order n, and an algebra isomorphism between kG and k^E as before. The usual G-grading of kG provides a grading of k^E by transporting the structure through the isomorphism. Consequently any abelian group of order n provides a simply connected grading of the algebra k^E. $\square$

Corollary 6.2. *Let n be a nonsquare free positive integer and let k be a field as above. The algebra k^n does not admit a universal covering.*

Proof. If n is not square-free, there exist at least two nonisomorphic groups of order n. Proposition 6.1 provides at least two nonisomorphic simply connected coverings, so k^n does not admit a universal cover. Moreover, each abelian group G of order n provides a simply connected grading through the isomorphism of k^n with kG. $\square$

The following result is based on the fact that $k\times k$ admits precisely one connected grading. We provide a proof of this, which is also a particular case of Dăscălescu's classification [2008] (see also [Bichon 2008]).

Proposition 6.3. *Let k be a field of characteristic different from 2. The fundamental group $\pi_1(k \times k)$ is cyclic of order 2.*

Proof. Let X be a connected G-grading of $k \times k$ for some group G. The trivial homogeneous component $X^1(k\times k)$ contains the unit of the algebra. If $X^1(k\times k) = k \times k$, then the group is trivial since the support of X is just the trivial element of G and the grading is connected. Otherwise there is exactly one more nonzero homogeneous component $X^s(k \times k)$ that is one-dimensional. Note that s has to generate G. We prove that s is of order 2. Let (x, y) be a nonzero element of degree s. Clearly $(x, y)^2 \neq 0$, and also $(x, y)^2 \in X^{s^2}(k \times k)$. Since there are only two homogeneous components, we infer that $X^{s^2}(k \times k) = X^1(k \times k)$ or $X^{s^2}(k \times k) = X^s(k \times k)$. In the first case $s^2 = 1$, while in the second case $s = 1$. Consequently, there are precisely two connected gradings and the fundamental group is cyclic of order two. $\square$

Lemma 6.4. *Let A and B be algebras with connected G_A and G_B-gradings X and Y. Then the algebra $C = A \times B$ has a natural $(G_A * G_B)$-connected grading Z. As a consequence, all quotients of $G_A * G_B$ grade C connectedly.*

Proof. Consider the following subspaces of C:

$$\begin{aligned}
&Z^1 C = X^1 A \times Y^1 B, \\
&Z^s C = X^s A \times 0, && \text{if } s \neq 1 \text{ and } s \in G_A, \\
&Z^t C = 0 \times Y^t B, && \text{if } t \neq 1 \text{ and } t \in G_B, \\
&Z^w C = 0, && \text{in the remaining cases.}
\end{aligned}$$

The support of the grading Z is the union of the supports of X and Y. These supports generate G_A and G_B respectively, and hence the support of Z generates $G_A * G_B$. $\square$

Example 6.5. Let E_5 be a set with five elements. There exists a connected C_6-grading of k^{E_5}.

Indeed, let E_2 and E_3 be sets with two and three elements respectively. Then $k^{E_5} \cong k^{E_2} \times k^{E_3}$ and we consider the previous fine and connected gradings given by C_2 and C_3 of k^{E_2} and k^{E_3} respectively. Lemma 6.4 shows that the product group $C_2 * C_3$ grades the product algebra k^{E_5} in a connected way, as well as any of its quotients, in particular C_6.

This example is the basis of the general procedure developed by Dăscălescu in order to describe all the connected gradings of a diagonal algebra. We rephrase one of his results.

Lemma 6.6 [Dăscălescu 2008, Lemma 1]. *Let k^n be a diagonal algebra. Any connected G-grading with one-dimensional trivial homogeneous component is given by the usual G-grading of kG, where G is any abelian group of order n.*

Note that Dăscălescu calls *ergodic* a grading with one-dimensional trivial homogeneous component. For $n = 2$, a nontrivial grading has to be ergodic, and hence we recover the fact that there is only one nontrivial grading of $k \times k$ as in Proposition 6.3.

Theorem 5 of [Dăscălescu 2008] provides a description of all the gradings of k^E, which is based on ergodic ones. We shall use it in order to compute $\pi_1(k^n)$ for small values of n. In order to state his result, we fist consider the following specific connected gradings of a diagonal algebra, modeled on Example 6.5.

Roughly speaking, the specific gradings are free product gradings of connected ergodic ones based on a product algebra decomposition of a diagonal algebra. Note that connected ergodic gradings of diagonal algebras are classified by Lemma 6.6.

More precisely, let $A = k^E$ be a diagonal algebra and let $M_1, \ldots, M_s$ be a partition of E. Let A_{M_i} be the algebra $A.e_{M_i}$, where

$$e_{M_i} = \sum_{x \in M_i} \delta_x$$

with δ_x the Dirac mass at x. It is easy to prove that any direct product decomposition of k^E is obtained in this way. Let H_i be an abelian group of order $\#(M_i)$, and finally let X_i be the corresponding H_i-ergodic grading of A_{M_i}. Then by Lemma 6.4, the group $H_1 * \cdots * H_s$ provides a connected grading of $A = A_{M_1} \times \cdots \times A_{M_s}$, which we call specific.

Theorem 6.7 [Dăscălescu 2008]. *Let E be a finite set and let k be a field containing all roots of unity of order less than or equal to $\#E$. Any connected grading of k^E is a quotient of a specific grading.*

Corollary 6.8. *Let k be a field containing all roots of unity of order* 2 *and* 3. *Then* $\pi_1(k^3) = C_2 \times C_3$.

Group	Dimension of the trivial component	Dimensions of other components
$\{1\}$	4	0
$C_2 * C_2$	2	1, 1
C_3	2	1, 1
C_2	3	1
C_4	1	1, 1, 1
$C_2 \times C_2$	1	1, 1, 1

Proof. The two nontrivial partitions of $\{1, 2, 3\}$ provide connected gradings by C_2 and C_3. Clearly they do not have nontrivial common quotients. □

Theorem 6.9. *Let k be a field containing all roots of unity of order* 2, 3 *and* 4. *Then* $\pi_1(k^4) = (C_2 * C_2) \times C_4 \times C_2 \times C_2 \times C_3 = (C_2 * C_2) \times C_6 \times C_4 \times C_2$.

Proof. The specific gradings of k^4 are given by the partitions of the set $\{1, 2, 3, 4\}$ as shown in the table above. An inspection of the possible common quotients, taking into account the structure of the groups and the dimension of the trivial homogeneous components, shows that the C_2-grading is a quotient of the $C_2 * C_2$-grading. Moreover, there is no other nontrivial common quotient. □

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