Algebra & Number Theory

Volume 4

2010

No. 7

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mathematical sciences publishers

ALGEBRA AND NUMBER THEORY 4:7(2010)

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In previous work we showed that two apparently unrelated formulas for the Hall–Littlewood polynomials of type A are, in fact, closely related. The first is the tableau formula obtained by specializing $q = 0$ in the Haglund–Haiman–Loehr formula for Macdonald polynomials. The second is the type A instance of Schwer's formula (rephrased and rederived by Ram) for Hall–Littlewood polynomials of arbitrary finite type; Schwer's formula is in terms of so-called alcove walks, which originate in the work of Gaussent and Littelmann and of the author with Postnikov on discrete counterparts to the Littelmann path model. We showed that the tableau formula follows by "compressing" Ram's version of Schwer's formula. In this paper, we derive new tableau formulas for the Hall–Littlewood polynomials of type B and C by compressing the corresponding instances of Schwer's formula.

1. Introduction

Hall–Littlewood polynomials are at the center of many recent developments in representation theory and algebraic combinatorics. They were originally defined in type A, as a basis for the algebra of symmetric functions depending on a parameter t. This basis interpolates between two fundamental bases: the one of Schur functions, at $t = 0$, and the one of monomial functions, at $t = 1$. Besides the original motivation for defining Hall–Littlewood polynomials, which comes from the Hall algebra [Littlewood 1961], there are many other applications; see for example [Lenart 2011] and the references therein.

Macdonald [1971] showed that there is a formula for the spherical functions corresponding to a Chevalley group over a p-adic field that generalizes the formula for

Cristian Lenart was partially supported by the National Science Foundation grant DMS-0701044.
MSC2000: primary 05E05; secondary 33D52.
Keywords: Hall–Littlewood polynomials, Macdonald polynomials, alcove walks, Schwer's formula, the Haglund–Haiman–Loehr formula.

the Hall–Littlewood polynomials. Thus, the Macdonald spherical functions generalize the Hall–Littlewood polynomials to all root systems, and the two names are used interchangeably in the literature. There are two families of Hall–Littlewood functions of arbitrary type, called P and Q, which form dual bases for the Weyl group invariants. The P-polynomials specialize to the Weyl characters at $t=0$. The transition matrix between Weyl characters and P-polynomials is given by Lusztig's t-analog of weight multiplicities (Kostka–Foulkes polynomials of arbitrary type), which are certain affine Kazhdan–Lusztig polynomials [Kato 1982; Lusztig 1983]. On the combinatorial side, we have the Lascoux–Schützenberger formula [1979] for the Kostka–Foulkes polynomials in type A, but no generalization of this formula to other types is known. Other applications of the type A Hall–Littlewood polynomials that extend to arbitrary type are those related to fermionic multiplicity formulas [Ardonne and Kedem 2007] and affine crystals [Lecouvey and Shimozono 2007]. We refer to [Nelsen and Ram 2003; Stembridge 2005] for surveys on Hall–Littlewood polynomials of arbitrary type.

Macdonald [1992; 2000] defined a remarkable family of orthogonal polynomials depending on parameters q, t, which bear his name. These polynomials generalize the spherical functions for a p-adic group, the Jack polynomials, and the zonal polynomials. At $q = 0$, the Macdonald polynomials specialize to the Hall–Littlewood polynomials, and thus they further specialize to the Weyl characters (upon setting $t = 0$ as well). There has been considerable interest recently in the combinatorics of Macdonald polynomials. This stems in part from a combinatorial formula for the ones corresponding to type A, which is due to Haglund, Haiman, and Loehr [Haglund et al. 2005]. This formula is in terms of fillings of Young diagrams, and uses two statistics, called inv and maj, on such fillings. The Haglund–Haiman–Loehr formula has already found important applications, such as new proofs of the positivity theorem for Macdonald polynomials, which states that the two-parameter Kostka–Foulkes polynomials have nonnegative integer coefficients. One of these proofs is based on Hecke algebras [Grojnowski and Haiman 2007], while the other is purely combinatorial and leads to a positive formula for the two-parameter Kostka–Foulkes polynomials [Assaf 2010]. Moreover, in the one-parameter case (that is, when $q = 0$), the Haglund–Haiman–Loehr formula was used to give a concise derivation of the Lascoux–Schützenberger formula for the Kostka–Foulkes polynomials of type A [Haglund et al. 2005, Section 7].

An apparently unrelated development, at the level of arbitrary finite root systems, led to Schwer's formula [2006], rephrased and rederived by Ram [2006], for the Hall–Littlewood polynomials of arbitrary type. The latter formulas are in terms of so-called alcove walks, which originate in the work of Gaussent and Littelmann [2005] and of the author with Postnikov [Lenart and Postnikov 2007; 2008] on discrete counterparts to the Littelmann path model [Littelmann 1994; 1995]. Schwer's formula was recently generalized by Ram and Yip [2011] to a similar formula for the Macdonald polynomials. The generalization consists in the fact that the latter formula is in terms of alcove walks with both "positive" and "negative" foldings, whereas in the former only "positive" foldings appear.

In [Lenart 2011], we related Schwer's formula to the Haglund–Haiman–Loehr formula. More precisely, we showed that we can group the terms in the type A instance of Schwer's formula (in fact, we used Ram's version of it) for $P_\lambda(x;t)$ into equivalence classes, such that the sum in each equivalence class is a term in the Haglund–Haiman–Loehr formula for $q=0$. An equivalence class consists of all the terms corresponding to alcove walks that produce the same filling of a Young diagram λ (indexing the Hall–Littlewood polynomial) via a simple construction. In fact, we first considered the case when the partition λ has no two parts identical (that is, it is a regular weight); the general case, which displays additional complexity, was considered in the Appendix to the same paper, written with Lubovsky. The work referring to a regular weight λ was then extended in [Lenart 2009], by showing that the type A instance of the Ram–Yip formula for Macdonald polynomials compresses, in a similar way, to a formula analogous to the Haglund–Haiman–Loehr one, but with fewer terms.

In this paper we extend the results in [Lenart 2011] to types B and C. More precisely, we derive new formulas for the Hall–Littlewood polynomials of type B and C indexed by regular weights in terms of fillings of Young diagrams; we do this by compressing the corresponding instances of Schwer's formula (in fact, we again use Ram's version of it). Note that no tableau formula for the Hall–Littlewood or Macdonald polynomials exists beyond type A so far. Our approach provides a natural way to obtain such formulas, and suggests that this method could be further extended to type D (this case is slightly more complex than types B and C, as seen below), as well as to Macdonald polynomials; these problems are currently explored, as is the compression in the case of a Hall–Littlewood polynomial indexed by a nonregular weight (by extending the type A result in the Appendix of [Lenart 2011]). Our formula is more complex than the corresponding one in type A (that is, the Haglund–Haiman–Loehr formula at $q=0$). However, the statistic we use is, in the case of some special fillings, completely similar to the Haglund–Haiman–Loehr inversion statistic (which is the more intricate of their two statistics). The naturality of our formula is also supported by the fact that the Kashiwara–Nakashima tableaux [1994] of type B and C are, essentially, the surviving fillings in this formula when we set $t=0$. We also note that the passage from (Ram's version of) Schwer's formula to ours results in a considerably larger reduction in the number of terms in type B and C compared to type A. In terms of applications, it would be very interesting to see whether our formula could be used to derive, in the spirit of [Haglund et al. 2005, Section 7], a positive combinatorial formula for Lusztig's t-analog of weight multiplicities in type B and C, which has been long sought.

2. The tableau formula in type C

Let us start by recalling the Weyl group of type B/C, viewed as the group of signed permutations B_n. Such permutations are bijections w from

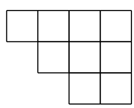
$$[\bar{n}] := \{1 < 2 < \cdots < n < \bar{n} < \overline{n-1} < \cdots < \bar{1}\}$$

to $[\bar{n}]$ satisfying $w(\bar{\imath}) = \overline{w(i)}$. Here $\bar{\imath}$ is viewed as $-i$, so $\bar{\bar{\imath}} = i$. We use the window notation $w = w(1)\ldots w(n)$. Given $1 \le i < j \le n$, we denote by (i, j) the reflection that transposes the entries in positions i and j (upon right multiplication). Similarly, we denote by $(i, \bar{\jmath})$, again for $i < j$, the transposition of entries in positions i and j followed by the sign change of those entries. Finally, we denote by $(i, \bar{\imath})$ the sign change in position i. Given w in B_n, we define

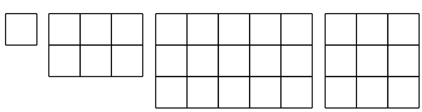
$$\begin{aligned} \ell_+(w) &:= \left|\{(k,l) : 1 \le k < l \le n,\ w(k) > w(l)\}\right|, \\ \ell_-(w) &:= \left|\{(k,l) : 1 \le k \le l \le n,\ w(k) > \overline{w(l)}\}\right|. \end{aligned} \tag{2-1}$$

Then the length of w is given by $\ell(w) := \ell_+(w) + \ell_-(w)$.

Let λ be a partition corresponding to a regular weight in type C_n for $n \ge 2$, that is, $\lambda = (\lambda_1 > \lambda_2 > \cdots > \lambda_n > 0)$ with $\lambda_i \in \mathbb{Z}$. We identify λ with its Young (or Ferrers) diagram, as usual, but we draw this diagram in "Japanese style" (as opposed to the more common English or French styles), that is, we embed it in the third quadrant, where $n = 3$:

$\lambda = (4, 3, 2) =$  .

Consider the shape $\hat{\lambda}$ obtained from λ by replacing each column of height k with k or $2k-1$ (adjacent) copies of it, depending on the given column being the rightmost one or not. In this example, we have

$\hat{\lambda} = (12, 11, 8) =$  .

Here $\hat{\lambda}$ is shown divided into rectangular blocks, each of which corresponds to a column of λ; the heights of the blocks (from right to left) are given by the conjugate partition $\lambda' = (3, 3, 2, 1)$.

We are representing a filling σ of $\hat{\lambda}$ as a concatenation of columns C_{ij} and C'_{ik}, where $i = 1, \ldots, \lambda_1$, while for a given i we have $j = 1, \ldots, \lambda'_i$ if $i > 1$, $j = 1$ if

$i=1$, and $k=2,\dots,\lambda_i'$; the columns C_{ij} and C_{ik}' have height λ_i'. More precisely, we let

$$\sigma=\mathscr{C}^{\lambda_1}\dots\mathscr{C}^1, \tag{2-2}$$

where

$$\mathscr{C}^i:=\begin{cases} C_{i2}'\dots C_{i,\lambda_i'}'C_{i1}\dots C_{i\lambda_i'} & \text{if } i>1,\\ C_{i2}'\dots C_{i,\lambda_i'}'C_{i1} & \text{if } i=1.\end{cases}$$

Note that the leftmost column is $C_{\lambda_1,1}$ and the rightmost column is C_{11}.

Example 2.1. The following is a filling for the partition considered above, where we use the same division into blocks as above:

$$\begin{array}{c|c|ccc|ccccc|ccc}
 & \mathscr{C}^4 & & \mathscr{C}^3 & & & & \mathscr{C}^2 & & & & \mathscr{C}^1 & \\
 & C_{41} & C_{32}' & C_{31} & C_{32} & C_{22}' & C_{23}' & C_{21} & C_{22} & C_{23} & C_{12}' & C_{13}' & C_{11}\\
\hline
\sigma= & \bar{1} & \bar{1} & \bar{1} & 3 & 2 & 2 & 1 & 1 & 1 & 1 & 1 & 1\\
 & & \bar{2} & \bar{2} & \bar{2} & 3 & 3 & 2 & 2 & 2 & 2 & 2 & 2\\
 & & & & & \bar{1} & \bar{1} & \bar{3} & \bar{3} & \bar{3} & 3 & 3 & 3
\end{array}.$$

Essentially, the description (2-2) of a filling of $\hat{\lambda}$ says that the column to the right of C_{ij} is $C_{i,j+1}$, whereas the column to the right of C_{ik}' is $C_{i,k+1}'$. Here we are assuming that the mentioned columns exist, up to the conventions

$$C_{i,\lambda_i'+1}=\begin{cases} C_{i-1,2}' & \text{if } i>1 \text{ and } \lambda_{i-1}'>1,\\ C_{i-1,1} & \text{if } i>1 \text{ and } \lambda_{i-1}'=1,\end{cases} \qquad C_{i,\lambda_i'+1}'=C_{i1}. \tag{2-3}$$

The entry in position i, counted from the top, in some column C is denoted by $C(i)$. We also write $C[i,j]$ for the portion of column C consisting of the entries in positions $i,i+1,\dots,j$; this is empty if $i>j$.

We consider the set $\mathscr{F}(\lambda)$ of fillings of $\hat{\lambda}$ with entries in $[\bar{n}]$ that satisfy the following conditions:

(1) The rows are weakly decreasing from left to right.

(2) No column contains two entries a,b with $a=\pm b$.

(3) Each column after the first is related to its left neighbor as indicated in the next paragraph. (Essentially, consecutive columns differ by a *signed cycle*, that is, a composition $(r_1,\bar{\jmath})\dots(r_p,\bar{\jmath})$, where $1\le r_1<\dots<r_p<j$; furthermore, j varies from 1 to the length of the column in question, as we consider the columns from left to right.)

Here we let the reflections in B_n act on columns C like they do on signed permutations; for instance, $C(a,\bar{b})$ is the column obtained from C by transposing the entries in positions a,b and by changing their signs. Let us first explain the passage from some column C_{ij} to $C_{i,j+1}$. There exist positions $1\le r_1<\dots<r_p<j$ (possibly $p=0$) such that $C_{i,j+1}$ differs from $D=C_{ij}(r_1,\bar{\jmath})\dots(r_p,\bar{\jmath})$ only in

position j, while $C_{i,j+1}(j) \le D(j)$. To include the case $j = \lambda'_i$ in this description, just replace $C_{i,j+1}$ everywhere by $C_{i,j+1}[1, \lambda'_i]$ and use the conventions (2-3). Let us now explain the passage from some column C'_{ik} to $C'_{i,k+1}$. There exist positions $1 \le r_1 < \cdots < r_p < k$ (possibly $p = 0$) such that $C'_{i,k+1} = C'_{ik}(r_1, \overline{k}) \ldots (r_p, \overline{k})$. This description includes the case $k = \lambda'_i$, based on the conventions (2-3).

Note that the filling σ in Example 2.1 satisfies the above conditions. Indeed, conditions (1) and (2) are clearly verified. Then compare, for instance,

$$C_{33}[1,2] = C'_{22}[1,2] = \begin{array}{|c|}\hline 2 \\ \hline 3 \\ \hline\end{array} \quad \text{and} \quad D = C_{32}(1,\overline{2}) = \begin{array}{|c|}\hline 3 \\ \hline 2 \\ \hline\end{array}(1,\overline{2}) = \begin{array}{|c|}\hline 2 \\ \hline 3 \\ \hline\end{array};$$

they only differ in position 2, while $C'_{22}(2) = 3 < D(2) = \overline{3}$. Similarly, we have

$$C'_{24} = C_{21} = C'_{23}(1,\overline{3})(2,\overline{3}).$$

Also note that, while the rows are weakly decreasing (from left to right), the columns need not be always increasing or always decreasing (compare $C'_{32} = C_{31}$ with the other columns).

Let us now define the content of a filling. For this purpose, we first associate with a filling σ a *compressed* version of it, namely the filling $\overline{\sigma}$ of the partition 2λ. This is defined as follows:

$$\overline{\sigma} = \overline{\mathscr{C}}^{\lambda_1} \ldots \overline{\mathscr{C}}^{1}, \quad \text{where} \quad \overline{\mathscr{C}}^{i} := C'_{i2} C_{i1}, \tag{2-4}$$

where the conventions (2-3) are used again. Now define $\operatorname{ct}\sigma = (c_1, \ldots, c_n)$, where c_i is half the difference between the number of occurrences of the entries i and $\bar{\imath}$ in $\overline{\sigma}$. Sometimes, this vector is written in terms of the coordinate vectors ε_i:

$$\operatorname{ct}\sigma = c_1\varepsilon_1 + \cdots + c_n\varepsilon_n = \tfrac{1}{2}\sum_{b \in \overline{\sigma}} \varepsilon_{\overline{\sigma}(b)}; \tag{2-5}$$

here the last sum is over all boxes b of $\overline{\sigma}$, and we set $\varepsilon_{\bar{\imath}} := -\varepsilon_i$. In our running example, we have

$$\overline{\sigma} = \begin{array}{|c|c|c|c|c|c|c|c|}\hline \overline{1} & \overline{1} & \overline{1} & \overline{1} & 2 & 1 & 1 & 1 \\ \hline \multicolumn{2}{c|}{} & \overline{2} & \overline{2} & 3 & 2 & 2 & 2 \\ \cline{3-8} \multicolumn{4}{c|}{} & \overline{1} & \overline{3} & 3 & 3 \\ \cline{5-8}\end{array},$$

so $\operatorname{ct}\sigma = (-1, 1, 1)$.

We now define two statistics on fillings in $\mathscr{F}(\lambda)$ that will be used in our compressed formula for Hall–Littlewood polynomials. Intervals refer to the totally ordered set $[\overline{n}]$. Let

$$\sigma_{ab} := \begin{cases} 1 & \text{if } a, b \ge \overline{n}, \\ 0 & \text{otherwise.} \end{cases} \tag{2-6}$$

Given a word w, we use the notation $N_{ab}(w)$ for the number of entries in w contained in the interval (a, b).

Given two columns D, C of the same height d such that $D \geq C$ componentwise, we will define two statistics $N(D, C)$ and $\operatorname{des}(D, C)$ in some special cases, as specified below.

Case 0. If $D = C$, then $N(D, C) := 0$ and $\operatorname{des}(D, C) := 0$.

Case 1. Assume that $C = D(r, \bar{\jmath})$ with $r < j$. Let $a := D(r)$ and $b := D(j)$. In this case, we set

$$N(D, C) := N_{\bar{b}a}(D[r+1, j-1]) + \left|(\bar{b}, a) \setminus \{\pm D(i) \,:\, i = 1, \dots, j\}\right| + \sigma_{ab},$$

and $\operatorname{des}(D, C) := 1$.

Case 2. Assume that $C = D(r_1, \bar{\jmath}) \dots (r_p, \bar{\jmath})$, where $1 \leq r_1 < \cdots < r_p < j$. Let $D_i := D(r_1, \bar{\jmath}) \dots (r_i, \bar{\jmath})$ for $i = 0, \dots, p$, so that $D_0 = D$ and $D_p = C$. We define

$$N(D, C) := \sum_{i=1}^{p} N(D_{i-1}, D_i), \quad \operatorname{des}(D, C) := p.$$

For instance, in the example above, we have

$$N(C'_{23}C_{21}) = N\left(\begin{array}{|c|c|}\hline 2 & 1 \\ \hline 3 & 2 \\ \hline \bar{1} & \bar{3} \\ \hline\end{array}\right) = N\left(\begin{array}{|c|c|}\hline 2 & 1 \\ \hline 3 & 3 \\ \hline \bar{1} & \bar{2} \\ \hline\end{array}\right) + N\left(\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 & 2 \\ \hline \bar{2} & \bar{3} \\ \hline\end{array}\right) = N_{12}\left(\boxed{3}\right) + N_{23}(\varnothing) = 0,$$

and $\operatorname{des}(C'_{23}C_{21}) = 2$.

Case 3. Assume that C differs from $D' := D(r_1, \bar{\jmath}) \dots (r_p, \bar{\jmath})$ with $1 \leq r_1 < \cdots < r_p < j$ (possibly $p = 0$) only in position j, while $C(j) < D'(j)$. We define

$$N(D, C) := N(D, D') + N_{C(j), D'(j)}(D[j+1, d]), \quad \operatorname{des}(D, C) := p + 1.$$

For instance, in our running example, we have

$$N(C_{31}C_{32}) = N\left(\begin{array}{|c|c|}\hline \bar{1} & 3 \\ \hline \bar{2} & \bar{2} \\ \hline\end{array}\right) = N_{3\bar{1}}\left(\boxed{\bar{2}}\right) = 1,$$

and $\operatorname{des}(C_{31}C_{32}) = 1$.

If the height of C is larger than the height d of D (necessarily by 1), and $N(D, C[1, d])$ can be computed as above, we let $N(D, C) := N(D, C[1, d])$ and $\operatorname{des}(D, C) := \operatorname{des}(D, C[1, d])$. For instance, we have

$$N(C_{32}C'_{22}) = N\left(\begin{array}{|c|c|}\hline 3 & 2 \\ \hline \bar{2} & 3 \\ \hline\end{array}\right) = N\left(\begin{array}{|c|c|}\hline 3 & 2 \\ \hline \bar{2} & \bar{3} \\ \hline\end{array}\right) + N_{3\bar{3}}(\varnothing) = N_{23}(\varnothing) = 0,$$

and $\operatorname{des}(C_{32}C'_{22}) = 2$.

Given a filling σ in $\mathcal{F}(\lambda)$ with columns $C_m, \ldots, C_1$, we set

$$N(\sigma) := \sum_{i=1}^{m-1} N(C_{i+1}, C_i) + \ell_+(C_1);$$

here $\ell_+(C_1)$ is defined as in (2-1). We also set

$$\operatorname{des}\sigma := \sum_{i=1}^{m-1} \operatorname{des}(C_{i+1}, C_i).$$

Note that $\operatorname{des}\sigma$ essentially counts the descents in the rows of σ. In our running example, we have $N(\sigma) = 1$ and $\operatorname{des}\sigma = 6$.

We can now state our new formula for the Hall–Littlewood polynomials of type C, which follows as a corollary of our main result, Theorem 4.6. A completely similar formula in type B is discussed in Section 5. We refer to Proposition 2.4 and Remarks 4.7 for more insight into our formula. In particular, the Kashiwara–Nakashima tableaux of type C are, essentially, the surviving fillings in this formula when we set $t = 0$. Furthermore, in some special cases, the statistic $N(\sigma)$ is completely similar to the Haglund–Haiman–Loehr inversion statistic (the more intricate of their two statistics); more precisely, this happens when the related chains in Bruhat order contain no reflections of type B, that is $(i, \bar{j})$, where i and j are less than the height of the corresponding column of the filling (see Proposition 2.4).

Theorem 2.2. *Given a regular weight λ, we have*

$$P_\lambda(X; t) = \sum_{\sigma \in \mathcal{F}(\lambda)} t^{N(\sigma)} (1-t)^{\operatorname{des}\sigma} x^{\operatorname{ct}\sigma}, \tag{2-7}$$

where $x^{(c_1,\ldots,c_n)} := x_1^{c_1} \ldots x_n^{c_n}$.

Example 2.3. Consider the simplest case, namely $n = 2$ and $\lambda = (2, 1)$. This leads to considering fillings of the shape $(3, 2)$ with elements in $[\bar{2}]$, namely

$$\begin{array}{|c|c|c|} \hline e & c & a \\ \hline \end{array}$$
$$\begin{array}{c|c|c|} \cline{2-3} & d & b \\ \cline{2-3} \end{array}\,.$$

The fillings need to satisfy the following conditions:

- $a \le c \le e,\ b \le d$.
- $a \ne \pm b$.
- either $c = a$ and $d = b$, or $c = \bar{b}$ and $d = \bar{a}$.

For $i \in \{1, 2\}$, let n_i be half the difference between the number of i's and $\bar{i}$'s in the multiset $\{a, b, c, d, e, e\}$. Given a proposition A, we let $\chi(A)$ be 1 or 0, depending on the logical value of A being true or false. Then

$$P_{(2,1)}(x_1, x_2; t) = \sum_{(a,b,c,d,e)} t^{\chi(a>b)+\chi(a,b\leq 2,\, a\neq c)} (1-t)^{\chi(a\neq c)+\chi(c\neq e)} x_1^{n_1} x_2^{n_2}.$$

It turns out that there are 27 terms in this sum, versus 70 terms in (Ram's version of) Schwer's formula. For instance, the terms contributing to the coefficient of x_2 correspond to the fillings

$$\begin{array}{|c|c|c|} \hline \bar{1} & 1 & 1 \\ \hline \multicolumn{1}{c|}{} & 2 & 2 \\ \cline{2-3} \end{array}\,, \qquad \begin{array}{|c|c|c|} \hline \bar{1} & 2 & 2 \\ \hline \multicolumn{1}{c|}{} & 1 & 1 \\ \cline{2-3} \end{array}\,, \qquad \begin{array}{|c|c|c|} \hline 2 & 2 & 1 \\ \hline \multicolumn{1}{c|}{} & \bar{1} & \bar{2} \\ \cline{2-3} \end{array}\,;$$

the associated polynomials in t are

$$1-t, \quad t(1-t), \quad 1-t,$$

respectively. Note that these polynomials are obtained by compressing 3, 2, and 2 terms in Schwer's formula, respectively. By symmetry, the coefficients of x_1, x_2, x_1^{-1}, and x_2^{-1} in $P_{(2,1)}(x_1, x_2; t)$ are all $(t+2)(1-t)$. Other fillings have an even larger number of terms in Schwer's formula corresponding to them, such as

$$\begin{array}{|c|c|c|} \hline \bar{1} & \bar{2} & \bar{2} \\ \hline \multicolumn{1}{c|}{} & \bar{1} & \bar{1} \\ \cline{2-3} \end{array}\,,$$

which has 7; in other words, the associated polynomial in t, namely $1-t$, which contributes to the coefficient of $x_1^{-2}x_2^{-1}$, is the sum of 7 polynomials of the form $t^r(1-t)^s$ in Schwer's formula. In conclusion, we have

$$\begin{aligned} P_{(2,1)}(x_1, x_2; t) = {} & x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_2^{-1} + x_1 x_2^{-2} + x_1^{-1} x_2^2 + x_1^{-2} x_2 \\ & + x_1^{-1} x_2^{-2} + x_1^{-2} x_2^{-1} + (t+2)(1-t)(x_1 + x_2 + x_1^{-1} + x_2^{-1}). \end{aligned}$$

In order to relate our statistic $N(\sigma)$ to the Haglund–Haiman–Loehr *inversion statistic* and to compare our formula to its type A counterpart (see [Haglund et al. 2005, Proposition 8.1] or [Lenart 2011, Theorem 2.10]), let us recall some definitions from [Haglund et al. 2005; Lenart 2011]. We start by considering fillings τ of the shape λ with entries in $[\bar{n}]$, which are again displayed in Japanese style, as a sequence of columns $\tau = C_{\lambda_1} \ldots C_1$; here C_i is a sequence $(C_i(1), \ldots, C_i(\lambda_i'))$, so the entry in cell $u = (i, j)$ is $\tau(u) = C_j(i)$. Two cells $u, v \in \lambda$ are said to *attack* each other if they are in one of the following two relative positions:

$$\begin{array}{|c|c|} \hline \bullet & \\ \hline & \\ \hline \bullet & \\ \hline \multicolumn{1}{c|}{} & \\ \cline{2-2} \end{array}\,, \qquad \begin{array}{|c|c|} \hline & \bullet \\ \hline & \\ \hline \bullet & \\ \hline \multicolumn{1}{c|}{} & \\ \cline{2-2} \end{array}\,.$$

An *inversion* of τ is a pair of attacking cells (u, v) that have one of the following two relative positions, where $a := \tau(u) < b := \tau(v)$:

$$\begin{array}{|c|c|}\hline a & \\ \hline & \\ \hline b & \\ \hline \multicolumn{1}{c|}{} & \\ \cline{2-2}\end{array}\,, \qquad \begin{array}{|c|c|}\hline & b \\ \hline & \\ \hline a & \\ \hline \multicolumn{1}{c|}{} & \\ \cline{2-2}\end{array}\,.$$

The Haglund–Haiman–Loehr statistic $\operatorname{inv} \tau$ is defined as the number of inversions of τ. The *descent statistic*, denoted $\operatorname{des} \tau$ (which is similar to des for fillings of $\hat{\lambda}$ defined above, as seen below), is the number of cells $u = (i, j)$ with $j \neq 1$ and $\tau(u) > \tau(v)$, where $v = (i, j-1)$. As usual, let

$$n(\lambda) := \sum_i (i-1)\lambda_i,$$

and assume that τ has the following two properties: (i) $\tau(u) \neq \tau(v)$ whenever u and v attack each other; and (ii) τ is weakly decreasing in rows. Then it was shown in [Lenart 2011, Proposition 2.12] that the so-called *complementary inversion statistic* $\operatorname{cinv} \tau := n(\lambda) - \operatorname{inv} \tau$ counts the triples of cells filled with $a < b < c$ that have the following relative position (here the cell supposed to contain c might be outside the shape λ, in which case we only require $a < b$):

$$\begin{array}{|c|c|}\hline & b \\ \hline & \\ \hline c & a \\ \hline \multicolumn{1}{c|}{} & \\ \cline{2-2}\end{array}\,.$$

Proposition 2.4. *Let σ in $\mathcal{F}(\lambda)$ be a filling satisfying the properties that $C'_{i,j+1} = C'_{i,j}$ for all i and $j = 2, \ldots, \lambda'_i$; and that $C_{i,j+1}$ differs from C_{ij} at most in position j, for all i and $j = 1, \ldots, \lambda'_i$. Let $\tilde{\sigma}$ be the filling of λ given by*

$$\tilde{\sigma} := C_{\lambda_1,1} C_{\lambda_1-1,1} \ldots C_{11}.$$

Then $N(\sigma) = \operatorname{cinv} \tilde{\sigma}$ and $\operatorname{des} \sigma = \operatorname{des} \tilde{\sigma}$.

Before presenting the proof, let us exhibit an example.

Example 2.5. For the partition $\lambda = (4, 3, 2)$ considered above, a filling satisfying the conditions in Proposition 2.4 is

$$\begin{array}{c} \\ \\ \sigma = \end{array} \begin{array}{c} \mathscr{C}^4 \\ C_{41} \\ \begin{array}{|c|}\hline \bar{1} \\ \hline \end{array} \\ \\ \\ \end{array} \begin{array}{c} \mathscr{C}^3 \\ C'_{32}\; C_{31}\; C_{32} \\ \begin{array}{|c|c|c|}\hline \bar{2} & \bar{2} & 1 \\ \hline 3 & 3 & 3 \\ \hline \end{array} \\ \\ \end{array} \begin{array}{c} \mathscr{C}^2 \\ C'_{22}\; C'_{23}\; C_{21}\; C_{22}\; C_{23} \\ \begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 2 & 2 & 2 \\ \hline \bar{3} & \bar{3} & \bar{3} & \bar{3} & \bar{3} \\ \hline \end{array} \end{array} \begin{array}{c} \mathscr{C}^1 \\ C'_{12}\; C'_{13}\; C_{11} \\ \begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 2 & 2 & 2 \\ \hline 3 & 3 & 3 \\ \hline \end{array} \end{array}.$$

We have

$$\tilde\sigma = \begin{array}{|c|c|c|c|}\hline \bar1 & \bar2 & 1 & 1 \\ \hline \multicolumn{1}{c|}{} & 3 & 2 & 2 \\ \cline{2-4} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & \bar3 & 3 \\ \cline{3-4} \end{array}\,.$$

It is easy to check that $N(\sigma) = \operatorname{cinv}\tilde\sigma = 1$ and $\operatorname{des}\sigma = \operatorname{des}\tilde\sigma = 4$. Indeed, the only triple of cells in $\tilde\sigma$ contributing to statistic cinv, which in this case is missing one cell, is formed by the cells in the unique column of height 2.

Proof of Proposition 2.4. The equality $\operatorname{des}\sigma = \operatorname{des}\tilde\sigma$ is clear, so we concentrate on the other equality. Let $m := \lambda_1$ be the number of columns of λ, and let $C_m = C_{m1}, \dots, C_1 = C_{11}$ be the columns of $\tilde\sigma$, of lengths $c_m := \lambda'_m, \dots, c_1 := \lambda'_1$; let $C'_k := C_k[1, c_{k+1}]$, for $k = 1, \dots, m-1$. We refer to a pair (i, j) with $1 \le i < j \le c_k$ and $C_k(i) > C_k(j)$ as a (type A) inversion in C_k. It is easy to see that $\tilde\sigma$ satisfies the properties considered above: (i) $\tilde\sigma(u) \neq \tilde\sigma(v)$ whenever u and v attack each other; (ii) $\tilde\sigma$ is weakly decreasing in rows. We start by evaluating $N(\mathscr{C}^k C_{k-1,1})$, with $\mathscr{C}^k$ as in (2-2). By definition, $N(\mathscr{C}^k C_{k-1,1}) = \sum_{i=1}^{c_k-1} N_{C_{k-1}(i), C_k(i)}(C_k[i+1, c_k])$. This is the number of inversions (i, j) in C_k for which $C_{k-1}(i) < C_k(j)$. If (i, j) is an inversion in C_k not satisfying the previous condition, then $C_{k-1}(i) > C_k(j)$ (by property (i) of $\tilde\sigma$), and thus (i, j) is an inversion in C'_{k-1} (by property (ii) of $\tilde\sigma$). Moreover, the only inversions of C'_{k-1} that do not arise in this way are those counted by the statistic $\operatorname{cinv}(C_k C'_{k-1})$, so

$$N(\mathscr{C}^k C_{k-1,1}) = \ell_+(C_k) - (\ell_+(C'_{k-1}) - \operatorname{cinv}(C_k C'_{k-1})).$$

We conclude that

$$N(\sigma) - \ell_+(C_1) = \sum_{k=2}^{m} \ell_+(C_k) - \ell_+(C'_{k-1}) + \operatorname{cinv}(C_k C'_{k-1}).$$

Now recall that λ has no two parts identical. We clearly have $c_m = 1$, so $\ell_+(C_m) = 0$. Therefore,

$$\begin{aligned} N(\sigma) = \sum_{k=2}^{m} \ell_+(C_{k-1}) - \ell_+(C'_{k-1}) + \operatorname{cinv}(C_k C'_{k-1}) &= \sum_{k=2}^{m} \operatorname{cinv}(C_k C_{k-1}) \\ &= \operatorname{cinv}\tilde\sigma. \end{aligned}$$ □

3. Background on Ram's version of Schwer's formula

We recall some background information on finite root systems and affine Weyl groups.

3.1. ***Root systems.*** Let $\mathfrak{g}$ be a complex semisimple Lie algebra, and $\mathfrak{h}$ a Cartan subalgebra, whose rank is r. Let $\Phi \subset \mathfrak{h}^*$ be the corresponding irreducible *root system*, $\mathfrak{h}^*_{\mathbb{R}} \subset \mathfrak{h}^*$ the real span of the roots, and $\Phi^+ \subset \Phi$ the set of positive roots. Let $\alpha_1, \dots, \alpha_r \in \Phi^+$ be the corresponding *simple roots*. We denote by $\langle \cdot, \cdot \rangle$ the nondegenerate scalar product on $\mathfrak{h}_{\mathbb{R}}^*$ induced by the Killing form. Given a root α, we consider the corresponding *coroot* $\alpha^\vee := 2\alpha/\langle\alpha,\alpha\rangle$ and reflection s_α.

Let W be the corresponding *Weyl group*, whose Coxeter generators are denoted, as usual, by $s_i := s_{\alpha_i}$. The length function on W is denoted by $\ell(\cdot)$. The *Bruhat graph* on W is the directed graph with edges $u \to w$, where $w = us_\beta$ for some $\beta \in \Phi^+$, and $\ell(w) > \ell(u)$; we usually label such an edge by β and write $u \xrightarrow{\beta} w$. The *reverse Bruhat graph* is obtained by reversing the directed edges above. The *Bruhat order* on W is the transitive closure of the relation corresponding to the Bruhat graph.

The *weight lattice* Λ is given by

$$\Lambda := \{\lambda \in \mathfrak{h}_{\mathbb{R}}^* : \langle\lambda, \alpha^\vee\rangle \in \mathbb{Z} \text{ for any } \alpha \in \Phi\}. \tag{3-1}$$

The weight lattice Λ is generated by the *fundamental weights* $\omega_1, \dots, \omega_r$, which form the dual basis to the basis of simple coroots, that is, $\langle\omega_i, \alpha_j^\vee\rangle = \delta_{ij}$. The set Λ^+ of *dominant weights* is given by

$$\Lambda^+ := \{\lambda \in \Lambda : \langle\lambda, \alpha^\vee\rangle \geq 0 \text{ for any } \alpha \in \Phi^+\}.$$

The subgroup of W stabilizing a weight λ is denoted by W_λ, and the set of minimum coset representatives in W/W_λ by W^λ. Let $\mathbb{Z}[\Lambda]$ be the group algebra of the weight lattice Λ, which has a $\mathbb{Z}$-basis of formal exponents $\{x^\lambda : \lambda \in \Lambda\}$ with multiplication $x^\lambda \cdot x^\mu := x^{\lambda+\mu}$.

Given $\alpha \in \Phi$ and $k \in \mathbb{Z}$, we denote by $s_{\alpha,k}$ the reflection in the affine hyperplane

$$H_{\alpha,k} := \{\lambda \in \mathfrak{h}_{\mathbb{R}}^* : \langle\lambda, \alpha^\vee\rangle = k\}. \tag{3-2}$$

These reflections generate the *affine Weyl group* W_{aff} for the *dual root system* $\Phi^\vee := \{\alpha^\vee : \alpha \in \Phi\}$. The hyperplanes $H_{\alpha,k}$ divide the real vector space $\mathfrak{h}_{\mathbb{R}}^*$ into open regions, called *alcoves.* The *fundamental alcove* $A_\circ$ is given by

$$A_\circ := \{\lambda \in \mathfrak{h}_{\mathbb{R}}^* : 0 < \langle\lambda, \alpha^\vee\rangle < 1 \text{ for all } \alpha \in \Phi^+\}.$$

3.2. ***Alcove walks.*** We say that two alcoves A and B are *adjacent* if they are distinct and have a common wall. Given two such alcoves, we write $A \xrightarrow{\beta} B$ if the common wall is of the form $H_{\beta,k}$ and the root $\beta \in \Phi$ points in the direction from A to B.

Definition 3.1. An *alcove path* is a sequence of alcoves such that any two consecutive ones are adjacent. We say that an alcove path $(A_0, A_1, \dots, A_m)$ is *reduced* if m is the minimal length of all alcove paths from A_0 to A_m.

We need the following generalization of alcove paths.

Definition 3.2. An *alcove walk* is a sequence

$$\Omega = (A_0, F_1, A_1, F_2, \ldots, F_m, A_m, F_\infty)$$

such that $A_0, \ldots, A_m$ are alcoves; F_i is a codimension-one common face of the alcoves A_{i-1} and A_i, for $i = 1, \ldots, m$; and F_∞ is a vertex of the last alcove A_m. The weight F_∞ is called the *weight* of the alcove walk, and is denoted by $\mu(\Omega)$.

The *folding operator* ϕ_i is the operator that acts on an alcove walk by leaving its initial segment from A_0 to A_{i-1} intact and by reflecting the remaining tail in the affine hyperplane containing the face F_i. In other words, we define

$$\phi_i(\Omega) := (A_0, F_1, A_1, \ldots, A_{i-1}, F'_i = F_i, A'_i, F'_{i+1}, A'_{i+1}, \ldots, A'_m, F'_\infty);$$

here $A'_j := \rho_i(A_j)$ for $j \in \{i, \ldots, m\}$, $F'_j := \rho_i(F_j)$ for $j \in \{i, \ldots, m\} \cup \{\infty\}$, and ρ_i is the affine reflection in the hyperplane containing F_i. Note that any two folding operators commute. An index j such that $A_{j-1} = A_j$ is called a *folding position* of Ω. Let $\mathrm{fp}(\Omega) := \{j_1 < \cdots < j_s\}$ be the set of folding positions of Ω. If this set is empty, Ω is called *unfolded*. We define the operator "unfold", producing an unfolded alcove walk, by

$$\mathrm{unfold}(\Omega) = \phi_{j_1} \ldots \phi_{j_s}(\Omega).$$

Definition 3.3. An alcove walk $\Omega = (A_0, F_1, A_1, F_2, \ldots, F_m, A_m, F_\infty)$ is called *positively folded* if, for any folding position j, the alcove $A_{j-1} = A_j$ lies on the positive side of the affine hyperplane containing the face F_j.

We now fix a dominant weight λ and a reduced alcove path

$$\Pi := (A_0, A_1, \ldots, A_m)$$

from $A_\circ = A_0$ to its translate $A_\circ + \lambda = A_m$. Assume that

$$A_0 \xrightarrow{\beta_1} A_1 \xrightarrow{\beta_2} \cdots \xrightarrow{\beta_m} A_m,$$

where $\Gamma := (\beta_1, \ldots, \beta_m)$ is a sequence of positive roots. This sequence, which determines the alcove path, is called a λ*-chain* (of roots). Two equivalent definitions of λ-chains (in terms of reduced words in affine Weyl groups, and an interlacing condition) can be found in [Lenart and Postnikov 2007, Definition 5.4] and [Lenart and Postnikov 2008, Definition 4.1 and Proposition 4.4]; note that the λ-chains considered in the these papers are obtained by reversing the ones in this paper. We also let $r_i := s_{\beta_i}$, and let $\hat{r}_i$ be the affine reflection in the common wall of A_{i-1} and A_i, for $i = 1, \ldots, m$; in other words, $\hat{r}_i := s_{\beta_i, l_i}$, where $l_i := |\{j \le i \,:\, \beta_j = \beta_i\}|$ is the cardinality of the corresponding set. Given

$$J = \{j_1 < \cdots < j_s\} \subseteq [m] := \{1, \ldots, m\},$$

we define the Weyl group element $\phi(J)$ and the weight $\mu(J)$ by

$$\phi(J) := r_{j_1} \dots r_{j_s}, \quad \mu(J) := \hat{r}_{j_1} \dots \hat{r}_{j_s}(\lambda). \tag{3-3}$$

Given $w \in W$, we define the alcove path $w(\Pi) := (w(A_0), w(A_1), \dots, w(A_m))$. Consider the set of alcove paths

$$\mathcal{P}(\Gamma) := \{w(\Pi) \,:\, w \in W^\lambda\}.$$

We identify any $w(\Pi)$ with the obvious unfolded alcove walk of weight

$$\mu(w(\Pi)) := w(\lambda).$$

Let us now consider the set of alcove walks

$$\mathcal{F}_+(\Gamma) := \{\text{positively folded alcove walks } \Omega \,:\, \text{unfold}(\Omega) \in \mathcal{P}(\Gamma)\}.$$

We can encode an alcove walk Ω in $\mathcal{F}_+(\Gamma)$ by the pair (w, J) in $W^\lambda \times 2^{[m]}$, where

$$\text{fp}(\Omega) = J \quad \text{and} \quad \text{unfold}(\Omega) = w(\Pi).$$

Clearly, we can recover Ω from (w, J) with $J = \{j_1 < \cdots < j_s\}$ by

$$\Omega = \phi_{j_1} \dots \phi_{j_s}(w(\Pi)).$$

Let $\mathcal{A}(\Gamma)$ be the image of $\mathcal{F}_+(\Gamma)$ under the map $\Omega \mapsto (w, J)$. We call a pair (w, J) in $\mathcal{A}(\Gamma)$ an *admissible pair*, and the subset $J \subseteq [m]$ in this pair a *w-admissible subset*.

Proposition 3.4 [Lenart 2011]. *If $\Omega \mapsto (w, J)$, then $\mu(\Omega) = w(\mu(J))$. Moreover,*

$$\mathcal{A}(\Gamma) = \big\{(w, J) \in W^\lambda \times 2^{[m]} : J = \{j_1 < \cdots < j_s\},\\ w > wr_{j_1} > \cdots > wr_{j_1} \dots r_{j_s} = w\phi(J)\big\}; \tag{3-4}$$

where the decreasing chain is in the Bruhat order on the Weyl group, its steps not being covers necessarily.

The formula for the Hall–Littlewood P-polynomials in [Schwer 2006] was rederived in [Ram 2006] in a slightly different version, based on positively folded alcove walks. Based on Proposition 3.4, we now restate the latter formula in terms of admissible pairs.

Theorem 3.5 [Ram 2006; Schwer 2006]. *Given a dominant weight λ, we have*

$$P_\lambda(X; t) = \sum_{(w,J) \in \mathcal{A}(\Gamma)} t^{(1/2)(\ell(w)+\ell(w\phi(J))-|J|)} (1-t)^{|J|} x^{w(\mu(J))}. \tag{3-5}$$

4. Specializing Ram's version of Schwer's formula to type C

We now restrict ourselves to the root system of type C_n. We can identify the space $\mathfrak{h}^*_{\mathbb{R}}$ with $V := \mathbb{R}^n$, the coordinate vectors being $\varepsilon_1, \dots, \varepsilon_n$. The root system Φ can be represented as $\Phi = \{\pm\varepsilon_i \pm \varepsilon_j : 1 \le i < j \le n\} \cup \{\pm 2\varepsilon_i : 1 \le i \le n\}$. The simple roots are $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$, for $i = 1, \dots, n-1$ and $\alpha_n = 2\varepsilon_n$. The fundamental weights are $\omega_i = \varepsilon_1 + \cdots + \varepsilon_i$, for $i = 1, \dots, n$. The weight lattice is $\Lambda = \mathbb{Z}^n$. A dominant weight $\lambda = \lambda_1\varepsilon_1 + \cdots + \lambda_{n-1}\varepsilon_{n-1} + \lambda_n\varepsilon_n$ is identified with the partition $(\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_{n-1} \ge \lambda_n \ge 0)$ of length at most n. A dominant weight is regular if all these inequalities are strict: that is, the corresponding partition has all parts distinct and nonzero. We fix such a partition λ for the remainder of this paper.

The corresponding Weyl group W is the group of signed permutations B_n. For simplicity, we use the same notation for roots and the corresponding reflections (see Section 2). For instance, given $1 \le i < j \le n$, we denote by (i, j) the positive root $\varepsilon_i - \varepsilon_j$, by $(i, \bar{\jmath})$ the positive root $\varepsilon_i + \varepsilon_j$, and by $(i, \bar{\imath})$ the positive root $2\varepsilon_i$.

Let

$$\Gamma(k) := \Gamma'_2 \dots \Gamma'_k \Gamma_1(k) \dots \Gamma_k(k),$$

where

$$\begin{aligned}
\Gamma'_j &:= \big((1, \bar{\jmath}), \quad (2, \bar{\jmath}), \quad \dots, (j-1, \bar{\jmath})\big);\\
\Gamma_j(k) &:= \big((1, \bar{\jmath}), \quad (2, \bar{\jmath}), \quad \dots, (j-1, \bar{\jmath}),\\
&\qquad (j, \overline{k+1}), (j, \overline{k+2}), \dots, (j, \bar{n}), \quad (j, \bar{\jmath}),\\
&\qquad (j, n), \quad (j, n-1), \quad \dots, (j, k+1)\big).
\end{aligned}$$

Lemma 4.1. $\Gamma(k)$ *is an* ω_k*-chain.*

Proof. We use the criterion for λ-chains given in [Lenart and Postnikov 2008, Definition 4.1 and Proposition 4.4] (see also Proposition 10.2 of the same reference). This criterion says that a chain of roots Γ is a λ-chain if and only if it satisfies the following conditions:

(R1) The number of occurrences of any positive root α in Γ is $\langle \lambda, \alpha^\vee \rangle$.

(R2) For each triple of positive roots (α, β, γ) with $\gamma^\vee = \alpha^\vee + \beta^\vee$, the subsequence of Γ consisting of α, β, γ is a concatenation of pairs (γ, α) and (γ, β) (in any order).

Letting $\lambda = \omega_k = \varepsilon_1 + \cdots + \varepsilon_k$, the first condition is easily checked; for instance, a root $(a, \bar{b})$ appears twice in $\Gamma(k)$ if $a < b \le k$, once if $a \le k < b$, and zero times otherwise. For the second condition, we use a case by case analysis, as follows, where $a < b < c$:

(1) $\alpha = (a,b),\ \beta = (b,c),\ \gamma = (a,c)$; (2) $\alpha = (a,b),\ \beta = (b,\bar{c}),\ \gamma = (a,\bar{c})$;

(3) $\alpha = (a,c),\ \beta = (b,\bar{c}),\ \gamma = (a,\bar{b})$; (4) $\alpha = (b,c),\ \beta = (a,\bar{c}),\ \gamma = (a,\bar{b})$;

(5) $\alpha = (a,b),\ \beta = (b,\bar{b}),\ \gamma = (a,\bar{a})$; (6) $\alpha = (a,\bar{a}),\ \beta = (b,\bar{b}),\ \gamma = (a,\bar{b})$.

Case (1) is the same as in type A. Cases (2)–(4) each have the three subcases $k \geq c$, $b \leq k < c$, and $a \leq k < b$; while cases (5) and (6) each have the two subcases $k \geq b$ and $a \leq k < b$. For instance, if $b \leq k < c$ in case (3), the subsequence of $\Gamma(k)$ consisting of α, β, γ is $((a, \overline{b}), (a, c), (a, \overline{b}), (b, \bar{c}))$. □

Hence, we can construct a λ-chain as a concatenation $\Gamma := \Gamma^{\lambda_1} \ldots \Gamma^1$, where

$$\Gamma^i = \Gamma(\lambda_i') = \Gamma'_{i2} \ldots \Gamma'_{i,\lambda_i'} \Gamma_{i1} \ldots \Gamma_{i,\lambda_i'} \quad \text{and} \quad \Gamma_{ij} = \Gamma_j(\lambda_i'), \;\; \Gamma'_{ij} = \Gamma'_j. \tag{4-1}$$

This λ-chain is fixed for the remainder of this paper. Thus, we can replace the notation $\mathscr{A}(\Gamma)$ with $\mathscr{A}(\lambda)$.

Example 4.2. Consider $n = 3$ and $\lambda = (3, 2, 1)$, for which we have the λ-chain below. The factorization of Γ into subchains is indicated with vertical bars, while the double vertical bars separate the subchains corresponding to different columns. The underlined pairs are only relevant in Example 4.3 below.

$$\begin{aligned}
\Gamma &= \Gamma_{31} \,\|\, \Gamma'_{22}\Gamma_{21}\Gamma_{22} \,\|\, \Gamma'_{12}\Gamma'_{13}\Gamma_{11}\Gamma_{12}\Gamma_{13} \\
&= \big((1, \overline{2}), \underline{(1, \overline{3})}, (1, \overline{1}), (1, 3), (1, 2) \,\| \\
&\qquad \underline{(1, \overline{2})} \mid (1, \overline{3}), (1, \overline{1}), (1, 3) \mid (1, \overline{2}), (2, \overline{3}), \underline{(2, \overline{2})}, \underline{(2, 3)} \,\| \\
&\qquad (1, \overline{2}) \mid (1, \overline{3}), (2, \overline{3}) \mid (1, \overline{1}) \mid (1, \overline{2}), (2, \overline{2}) \mid (1, \overline{3}), (2, \overline{3}), (3, \overline{3})\big).
\end{aligned} \tag{4-2}$$

We represent the Young diagram of λ inside a broken 3×2 rectangle, as below. In this way, a reflection in Γ can be viewed as swapping entries and/or changing signs in the two parts of each column, or only the top part.

$$\begin{array}{|c|c|c|}
\hline
1 & 1 & 1 \\
\hline
\multicolumn{1}{c|}{} & 2 & 2 \\
\cline{2-3}
\multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 3 \\
\cline{3-3}
\end{array}$$

$$\begin{array}{|c|c}
\hline
2 & \\
\hline
3 & \multicolumn{1}{c|}{3} \\
\hline
\end{array}$$

Given the λ-chain Γ above, in Section 3.2 we considered subsets

$$J = \{j_1 < \cdots < j_s\}$$

of $[m]$, where m is the length of Γ. Instead of J, it is now convenient to use the subsequence T of the roots in Γ whose positions are in J. This is viewed as a concatenation with distinguished factors T_{ij} and T'_{ik} induced by the factorization (4-1) of Γ.

All the notions defined in terms of J are now redefined in terms of T. As such, from now on we will write $\phi(T)$, $\mu(T)$, and $|T|$, the latter being the size of T; see (3-3). If J is a w-admissible subset for some w in B_n, we will also call the corresponding T a *w-admissible sequence*, and (w, T) an admissible pair. We will use the notation $\mathscr{A}(\Gamma)$ and $\mathscr{A}(\lambda)$ accordingly. We denote by $wT_{\lambda_1,1}\dots T_{ij}$ and $wT_{\lambda_1,1}\dots T'_{ik}$ the permutations obtained from w via right multiplication by the transpositions in $T_{\lambda_1,1},\dots,T_{ij}$ and $T_{\lambda_1,1},\dots,T'_{ik}$, considered from left to right. This agrees with the convention of using pairs to denote both roots and the corresponding reflections. As such, $\phi(T)$ can now be written simply as T.

Example 4.3. We continue Example 4.2, by picking the admissible pair (w, J) with $w = \bar{1}\,\bar{2}\,\bar{3} \in B_3$ and $J = \{2, 6, 12, 13\}$ (see the underlined positions in (4-2)). Thus, we have

$$T = T_{31} \parallel T'_{22}T_{21}T_{22} \parallel T'_{12}T'_{13}T_{11}T_{12}T_{13} = ((1,\bar{3}) \parallel (1,\bar{2}) \mid \ \mid (2,\bar{2}), (2,3) \parallel \ \mid \ \mid \ \mid \ \mid \ ).$$

The corresponding decreasing chain in Bruhat order is the following, where the swapped entries are shown in bold (we represent permutations as broken columns starting with w, as discussed in Example 4.2, and we display the splitting of the chain into subchains induced by the splitting of T just given):

$$\begin{array}{c}\boxed{\bar{\mathbf{1}}}\\ \\ \boxed{\bar{2}}\\ \boxed{\bar{\mathbf{3}}}\end{array} > \begin{array}{c}\boxed{3}\\ \\ \boxed{\bar{2}}\\ \boxed{1}\end{array} \parallel \begin{array}{c}\boxed{\mathbf{3}}\\ \boxed{\bar{\mathbf{2}}}\\ \\ \boxed{1}\end{array} > \begin{array}{c}\boxed{2}\\ \boxed{\bar{3}}\\ \\ \boxed{1}\end{array} \mid \begin{array}{c}\boxed{2}\\ \boxed{\bar{3}}\\ \\ \boxed{1}\end{array} \mid \begin{array}{c}\boxed{2}\\ \boxed{\bar{\mathbf{3}}}\\ \\ \boxed{1}\end{array} > \begin{array}{c}\boxed{2}\\ \boxed{\mathbf{3}}\\ \\ \boxed{\mathbf{1}}\end{array} > \begin{array}{c}\boxed{2}\\ \boxed{1}\\ \\ \boxed{3}\end{array} \parallel \begin{array}{c}\boxed{2}\\ \boxed{1}\\ \boxed{3}\end{array} \mid \begin{array}{c}\boxed{2}\\ \boxed{1}\\ \boxed{3}\end{array} \mid \begin{array}{c}\boxed{2}\\ \boxed{1}\\ \boxed{3}\end{array} \mid \begin{array}{c}\boxed{2}\\ \boxed{1}\\ \boxed{3}\end{array} \mid \begin{array}{c}\boxed{2}\\ \boxed{1}\\ \boxed{3}\end{array}.$$

Given a (not necessarily admissible) pair (w, T), with T split into factors T_{ij} and T'_{ik} as above, we consider the permutations

$$\pi_{ij} = \pi_{ij}(w,T) := wT_{\lambda_1,1}\dots T_{i,j-1}, \quad \pi'_{ik} = \pi'_{ik}(w,T) := wT_{\lambda_1,1}\dots T'_{i,k-1};$$

when undefined, $T_{i,j-1}$ and $T'_{i,k-1}$ are given by conventions similar to (2-3), based on the corresponding factorization (4-1) of the λ-chain Γ. In particular, $\pi_{\lambda_1,1} = w$.

Definition 4.4. The *filling map* is the map f from pairs (w, T), not necessarily admissible, to fillings $\sigma = f(w, T)$ of the shape $\hat{\lambda}$, defined (based on the notation (2-2)) by

$$C_{ij} = \pi_{ij}[1, \lambda'_i], \quad C'_{ik} = \pi'_{ik}[1, \lambda'_i]. \tag{4-3}$$

Example 4.5. Given (w, T) as in Example 4.3, we have

$$f(w,T) = \begin{array}{|c|c|c|c|c|c|c|}\hline \bar{1} & 3 & 2 & 2 & 2 & 2 & 2\\ \hline \multicolumn{1}{c|}{} & \bar{2} & \bar{3} & \bar{3} & 1 & 1 & 1\\ \cline{2-7} \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 3 & 3 & 3\\ \cline{5-7}\end{array}.$$

The following theorem describes the way in which our tableau formula (2-7) is obtained by compressing Ram's version of Schwer's formula (3-5). Recall that λ is a regular weight, so $B_n^\lambda = B_n$, and thus the pairs (w, J) in $\mathscr{A}(\lambda)$ are only subject to the decreasing chain condition in (3-4); this fact is implicitly used in the proof of the theorem.

Theorem 4.6. (i) *We have* $f(\mathcal{A}(\lambda)) = \mathcal{F}(\lambda)$.

(ii) *Given any* $\sigma \in \mathcal{F}(\lambda)$ *and* $(w, T) \in f^{-1}(\sigma)$, *we have* $\operatorname{ct} f(w, T) = w(\mu(T))$.

(iii) *The following compression formula holds*:

$$\sum_{(w,T)\in f^{-1}(\sigma)} t^{\frac{1}{2}(\ell(w)+\ell(wT)-|T|)}(1-t)^{|T|} = t^{N(\sigma)}(1-t)^{\operatorname{des}\sigma}. \tag{4-4}$$

Proof. We start with part (i). That we have $f(\mathcal{A}(\lambda)) \subseteq \mathcal{F}(\lambda)$ is clear from the definition of the set of fillings $\mathcal{F}(\lambda)$ in Section 2 and the construction (4-1) of the fixed λ-chain Γ. Vice versa, given a filling σ in $\mathcal{F}(\lambda)$, it is not hard to construct an admissible pair (w, T) in $f^{-1}(\sigma)$. We will assign to the columns C_{ij} and C'_{ij} signed permutations ρ_{ij} and ρ'_{ij} in B_n recursively, starting with $\rho_{11} := C_{11}$; in parallel, we construct the reverse $\operatorname{rev} T$ of the mentioned chain of roots T, and conclude by letting $w := \rho_{\lambda_1,1}$. Each time we pass to the left neighbor C'_{ik} of a column $C'_{i,k+1} = C'_{ik}(r_1, \overline{k}) \dots (r_p, \overline{k})$, we append to the part of $\operatorname{rev} T$ already constructed the roots $(r_p, \overline{k}), \dots, (r_1, \overline{k})$ and let $\rho'_{ik} := \rho'_{i,k+1}(r_p, \overline{k}) \dots (r_1, \overline{k})$. We proceed similarly when passing to the left neighbor C_{ij} of a column $C_{i,j+1}$, where $C_{i,j+1}$ differs from $D = C_{ij}(r_1, \overline{\jmath}) \dots (r_p, \overline{\jmath})$ only in position j; the only difference is that, in this case, we start by applying to $\rho_{i,j+1}$ and appending to $\operatorname{rev} T$ the reflection that exchanges the entry $C_{i,j+1}(j)$ with $D(j)$, and then we proceed as above.

Parts (ii) and (iii) of the theorem are proved in Sections 6 and 7. □

Remarks 4.7. (i) The Kashiwara–Nakashima tableaux [1994] of shape λ index the basis elements of the irreducible representation of $\mathfrak{sp}_{2n}$ of highest weight λ. It is shown in Proposition 4.8 below that these tableaux correspond precisely to the surviving fillings in our formula (2-7) when we set $t = 0$.

(ii) In (4-4), in general, we cannot replace the filling map f with the map $\overline{f}$, sending (w, T) to the compressed version $\overline{f(w, T)}$ of $f(w, T)$. Indeed, consider $n = 2$, $\lambda = (3, 2)$, and the following filling of $2\lambda = (6, 4)$, which happens to be the "doubled" version of a Kashiwara–Nakashima tableau:

$$\overline{\sigma} = \begin{array}{|c|c|c|c|c|c|}\hline \overline{2} & \overline{2} & \overline{2} & \overline{2} & 1 & 1 \\ \hline \multicolumn{2}{c|}{} & \overline{1} & \overline{1} & 2 & 2 \\ \cline{3-6}\end{array}.$$

If $(w, T) \in \overline{f}^{-1}(\overline{\sigma})$, we must have $w = \overline{21}$ and

$$T \subseteq \Gamma_{21}\Gamma_{22} = \big((1, \overline{1}) | (1, \overline{2}), (2, \overline{2})\big),$$

where the full λ-chain factors as follows:

$$\Gamma = \Gamma_{31} \parallel \Gamma'_{22}\Gamma_{21}\Gamma_{22} \parallel \Gamma'_{12}\Gamma_{11}\Gamma_{12}.$$

There are two elements (w, T^1) and (w, T^2) in $\overline{f}^{-1}(\overline{\sigma})$, namely

$$T^1 = \big((1,\overline{2})\big) \quad \text{and} \quad T^2 = \big((1,\overline{1}), (1,\overline{2}), (2,\overline{2})\big).$$

But we have

$$\sum_{(w,T)\in \overline{f}^{-1}(\overline{\sigma})} t^{(1/2)(\ell(w)+\ell(wT)-|T|)}(1-t)^{|T|} = t(1-t)+(1-t)^3 = (1-t)(1-t+t^2).$$

In general, this sum has several factors not of the form t or $(1-t)$, which are hard to control.

(iii) To measure the compression phenomenon, we define the *compression factor* $c(\lambda)$ as in [Lenart 2011], as the ratio of the number of terms in Ram's version of Schwer's formula for λ and the number of terms in the tableau formula. The compression factor is considerably larger in type C. For instance, for $\lambda = (3, 2, 1, 0)$ in C_4 we have 23,495 terms in the compressed formula, while $c(\lambda) = 44.9$.

Proposition 4.8. *The map $\sigma \mapsto \overline{\sigma}$ defined by (2-4) is a bijection between the fillings σ in $\mathcal{F}(\lambda)$ with $N(\sigma) = 0$ and the "doubled" versions of the type C Kashiwara–Nakashima tableaux of shape λ.*

Proof. Adamczak and the author [2009] proved that for each type C Kashiwara–Nakashima tableau of shape λ there is a unique admissible pair (w, T) whose associated chain in Bruhat order is saturated and ends at the identity, such that the compressed version $\overline{\sigma}$ of $\sigma = f(w, T)$ is the "doubled" version of the given tableau. It follows that the term associated to (w, T) in (3-5) is $t^0(1-t)^{|T|}x^{w(\mu(T))}$, and therefore $N(\sigma) = 0$, by (4-4). On the other hand, since $P_\lambda(x; 0)$ is the irreducible character indexed by λ, which is expressed in terms of Kashiwara–Nakashima tableaux, we conclude that all σ in $\mathcal{F}(\lambda)$ with $N(\sigma) = 0$ arise in this way. □

5. The tableau formula in type B

We now restrict ourselves to the root system of type B_n. This can be represented as $\Phi = \{\pm\varepsilon_i \pm \varepsilon_j : 1 \le i < j \le n\} \cup \{\pm\varepsilon_i : 1 \le i \le n\}$. The simple roots are $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$, for $i = 1, \dots, n-1$ and $\alpha_n = \varepsilon_n$. The fundamental weights are $\omega_i = \varepsilon_1 + \cdots + \varepsilon_i$, for $i = 1, \dots, n-1$ and $\omega_n = \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_n)$. A dominant weight $\lambda = \alpha_1\omega_1 + \cdots + \alpha_n\omega_n$, where $\alpha_i \in \mathbb{Z}_{\ge 0}$, is identified with the partition $\mu = (n^{\alpha_n}, \dots, 1^{\alpha_1})$; we let $\ell(\mu) := \alpha_1 + \cdots + \alpha_n$, and write $\mu = (\mu_1, \dots, \mu_{\ell(\mu)})$. A dominant weight is regular if $\alpha_i > 0$ for all i. Let us now fix such a weight λ.

The corresponding Weyl group W is the same group of signed permutations B_n considered above. For simplicity, we again use the same notation for roots and the corresponding reflections; see Section 2. The pairs (i, j) and $(i, \overline{j})$ have the same meaning as in type C, whereas (i) denotes the positive root ε_i. Note that, as a reflection in B_n, (i) is the same as $(i, \overline{\imath})$ in type C.

The canonical ω_k-chains and λ-chains are very similar to those in type C. If $k<n$, let

$$\Gamma(k) := \Gamma'_1 \dots \Gamma'_k \Gamma_1(k) \dots \Gamma_k(k),$$

where

$$\begin{aligned}\Gamma'_j &:= \big((1,\bar{\jmath}),\quad (2,\bar{\jmath}),\quad \dots,(j-1,\bar{\jmath}),(j)\big);\\ \Gamma_j(k) &:= \big((1,\bar{\jmath}),\quad (2,\bar{\jmath}),\quad \dots,(j-1,\bar{\jmath}),\\ &\qquad (j,\overline{k+1}),(j,\overline{k+2}),\ \dots,(j,\bar{n}),\\ &\qquad (j),\\ &\qquad (j,n),\quad (j,n-1),\ \dots,(j,k+1)\big).\end{aligned}$$

On the other hand, we let

$$\Gamma(n) := \Gamma'_1 \dots \Gamma'_n = \Gamma_1(n) \dots \Gamma_n(n).$$

As in the type C case, we can prove that $\Gamma(k)$ is an ω_k-chain for any k. Hence, we can construct a λ-chain as a concatenation $\Gamma := \Gamma^{\ell(\mu)} \dots \Gamma^1$, where $\Gamma^i = \Gamma(\mu_i)$.

The filling map is defined as in Definition 4.4. This gives rise to fillings

$$\sigma = \mathscr{C}^{\ell(\mu)} \dots \mathscr{C}^1,$$

where each $\mathscr{C}^i$ is a concatenation of columns of height μ_i:

$$\mathscr{C}^i := \begin{cases} C'_{i1} \dots C'_{i,\mu_i} C_{i1} \dots C_{i\mu_i} & \text{if } \mu_i < n,\\ C_{i1} \dots C_{i,\mu_i} & \text{if } i \neq 1 \text{ and } \mu_i = n,\\ C_{11} & \text{if } i = 1.\end{cases}$$

The fillings are subject to the same conditions (1)–(3) as in type C in Section 2, where condition (3) is made more precise below. In fact, the λ-chain Γ above specifies the way in which each column is related to its left neighbor. Essentially, everything is similar to type C, except for a small difference in the passage from some column C'_{ik} to $C'_{i,k+1}$. Namely, there exist positions $1 \le r_1 < \cdots < r_p < k$ (possibly $p=0$) such that $C'_{i,k+1} = C'_{ik}(r_1,\bar{k}) \dots (r_p,\bar{k})$, like in type C, or $C'_{i,k+1} = C'_{ik}(r_1,\bar{k}) \dots (r_p,\bar{k})(k)$, in which case we also require $C'_{i,k+1}(k) \le n$.

The weight of a filling and the statistics $N(\sigma)$ and $\operatorname{des}\sigma$ are defined completely similarly to type C. The only minor addition is the definition of $N(D,C)$ and $\operatorname{des}(D,C)$ when $C = D(r_1,\bar{k}) \dots (r_p,\bar{k})(k)$. With the notation in Case 2 of the definition of $N(D,C)$, we set

$$N(D,C) := N(D,D_p) + N(D_p,C), \quad \operatorname{des}(D,C) := p+1.$$

Here $N(D,D_p)$ is defined in Case 2, whereas $N(D_p,C)$ is given by the second formula in (7-1); more precisely,

$$N(D_p,C) := \tfrac{1}{2}\big|(\bar{a},a) \setminus \{\pm D_p(i) : i = 1,\dots,k\}\big|,$$

where $a := D_p(k)$.

Given these constructions, the proof of the following theorem is completely similar to its counterparts in type C, since no new situations arise.

Theorem 5.1. *Theorems 2.2 and 4.6 hold in type B, with the appropriate constructions explained above.*

Remark 5.2. The situation in type D is slightly more complex. In this case, applying the preceding ideas leads to an analog of the compression formula (4-4) that contains factors of the form $1-t^k$ with $k>1$ in the right side. However, these factors are not hard to control, while no extra factors appear.

6. Proof of Theorem 4.6(ii)

Recall the λ-chain Γ in Section 4. Let us write $\Gamma=(\beta_1,\dots,\beta_m)$, as in Section 3.2. As such, we recall the hyperplanes H_{β_k,l_k} and the corresponding affine reflections $\hat{r}_k=s_{\beta_k,l_k}=s_{\beta_k}+l_k\beta_k$.

Now fix a signed permutation w in B_n and a subset $J=\{j_1<\cdots<j_s\}$ of $[m]$ (not necessarily w-admissible). Let Π be the alcove path corresponding to Γ, and define the alcove walk Ω as in Section 3.2, by

$$\Omega:=\phi_{j_1}\dots\phi_{j_s}(w(\Pi)).$$

Given k in $[m]$, let $i=i(k)$ be the largest index in $[s]$ for which $j_i<k$, and let $\gamma_k:=wr_{j_1}\dots r_{j_i}(\beta_k)$. Then the hyperplane containing the face F_k of Ω (see Definition 3.2) is of the form H_{γ_k,m_k}; in other words,

$$H_{\gamma_k,m_k}=w\hat{r}_{j_1}\dots\hat{r}_{j_i}(H_{\beta_k,l_k}).$$

Our first goal is to describe m_k purely in terms of the filling associated to (w,J).

Let $\hat{t}_k$ be the affine reflection in the hyperplane H_{γ_k,m_k}. Note that

$$\hat{t}_k=w\hat{r}_{j_1}\dots\hat{r}_{j_i}\hat{r}_k\hat{r}_{j_i}\dots\hat{r}_{j_1}w^{-1}.$$

Thus, we can see that

$$w\hat{r}_{j_1}\dots\hat{r}_{j_i}=\hat{t}_{j_i}\dots\hat{t}_{j_1}w.$$

Let $T=((a_1,b_1),\dots,(a_s,b_s))$ be the subsequence of Γ indexed by the positions in J; see Section 4. Let T^i be the initial segment of T with length i, let $w_i:=wT^i$, and let $\sigma_i:=\overline{f(w,T^i)}$; see (2-4). In particular, σ_0 is the filling with all entries in row i equal to $w(i)$, and $\sigma:=\sigma_s=\overline{f(w,T)}$. The columns of a filling of 2λ are numbered left to right by $2\lambda_1$ to 1. We split each segment Γ^k of Γ into two parts: the head, which is a concatenation of $\Gamma'_{k\cdot}$, and the tail, which is a concatenation of $\Gamma_{k\cdot}$; see (4-1). We say that the head corresponds to column $2k$ of the Young diagram 2λ, whereas the tail corresponds to column $2k-1$ (see the construction of $f(w,T)$ in Section 4 and (2-4)). If $\beta_{j_{i+1}}=(a_{i+1},b_{i+1})=(a,b)$ falls in the segment of Γ corresponding to column p of 2λ, then σ_{i+1} is obtained from σ_i by replacing the entry $w_i(a)$ with $w_i(b)$ in the columns $p-1, \dots, 1$ of σ_i, as well as, possibly, the entry $w_i(\overline{b})$ with $w_i(\overline{a})$ in the same columns.

Now fix a position k, and consider $i = i(k)$ and the roots β_k, $\gamma := \gamma_k$, as above, where γ_k might be negative. Assume that β_k falls in the segment of Γ corresponding to column q of 2λ. Given a filling ϕ, we denote by $\phi^{[p]}$ the part of ϕ consisting of columns $2\lambda_1, 2\lambda_1-1, \dots, p$, and by $\phi^{(p,q]}$ the part consisting of columns $p-1,\ p-2, \dots, q$. We also recall the definition (2-5) and conventions related to the content of a filling; this definition now applies to any filling of some Young diagram.

Proposition 6.1. *With the same notation, we have*

$$m_k = \langle \mathrm{ct}\, \sigma^{[q]}, \gamma^\vee \rangle.$$

Proof. We apply induction on i, which starts at $i = 0$, when the verification is straightforward. We now proceed from $j_1 < \cdots < j_i < k$, where $i = s$ or $k \le j_{i+1}$, to $j_1 < \cdots < j_{i+1} < k$, and we freely use the notation above.

Assume that $\beta_{j_{i+1}}$ falls in the segment of Γ corresponding to column p of 2λ, where $p \ge q$.

We need to compute

$$w\hat{r}_{j_1} \dots \hat{r}_{j_{i+1}}(H_{\beta_k, l_k}) = \hat{t}_{j_{i+1}} \dots \hat{t}_{j_1} w(H_{\beta_k, l_k}) = \hat{t}_{j_{i+1}}(H_{\gamma, m}),$$

where $m = \langle \mathrm{ct}\, \sigma_i^{[q]}, \gamma^\vee \rangle$, by induction. Let $\gamma' := \gamma_{j_{i+1}}$, and $\hat{t}_{j_{i+1}} = s_{\gamma', m'}$, where $m' = \langle \mathrm{ct}\, \sigma_i^{[p]}, (\gamma')^\vee \rangle$, by induction. We use the formula

$$s_{\gamma', m'}(H_{\gamma, m}) = H_{s_{\gamma'}(\gamma), m - m'\langle \gamma', \gamma^\vee \rangle}.$$

Thus, the proof is reduced to showing that

$$m - m'\langle \gamma', \gamma^\vee \rangle = \langle \mathrm{ct}\, \sigma_{i+1}^{[q]}, s_{\gamma'}(\gamma^\vee) \rangle.$$

An easy calculation, based on the information above, shows that the latter equality is nontrivial only if $p > q$, in which case it is equivalent to

$$\langle \mathrm{ct}\, \sigma_{i+1}^{(p,q]} - \mathrm{ct}\, \sigma_i^{(p,q]}, \gamma^\vee \rangle = \langle \gamma', \gamma^\vee \rangle \langle \mathrm{ct}\, \sigma_{i+1}^{(p,q]}, (\gamma')^\vee \rangle.$$

This equality is a consequence of

$$\mathrm{ct}\, \sigma_{i+1}^{(p,q]} = s_{\gamma'}(\mathrm{ct}\, \sigma_i^{(p,q]}),$$

which follows from the construction of σ_{i+1} from σ_i explained above. □

Proof of Theorem 4.6(ii). We apply induction on the size of T, using freely the notation above. We prove the statement for $T = (\beta_{j_1}, \dots, \beta_{j_{s+1}})$, assuming it holds for $T^s = (\beta_{j_1}, \dots, \beta_{j_s})$. We have

$$w(\mu(T)) = w\hat{r}_{j_1} \dots \hat{r}_{j_{s+1}}(\lambda) = \hat{t}_{j_{s+1}} \dots \hat{t}_{j_1} w(\lambda) = \hat{t}_{j_{s+1}}(\mathrm{ct}\,\sigma_s),$$

by induction. We need to show that

$$\hat{t}_{j_{s+1}}(\mathrm{ct}\,\sigma_s) = \mathrm{ct}\,\sigma_{s+1}. \tag{6-1}$$

Let $\gamma := \gamma_{j_{s+1}}$ and assume that $\beta_{j_{s+1}}$ falls in the segment of Γ corresponding to column p of 2λ. Based on Proposition 6.1, (6-1) is rewritten as

$$s_\gamma(\mathrm{ct}\,\sigma_s) + \langle \mathrm{ct}\,\sigma_s^{[p]}, \gamma^\vee \rangle \gamma = \mathrm{ct}\,\sigma_{s+1}. \tag{6-2}$$

Decomposing $\mathrm{ct}\,\sigma_s$ as $\mathrm{ct}\,\sigma_s^{[p]} + \mathrm{ct}\,\sigma_s^{(p,1]}$ (using the notation above), and $\mathrm{ct}\,\sigma_{s+1}$ in a similar way, (6-2) would follow from

$$\begin{aligned} s_\gamma(\mathrm{ct}\,\sigma_s^{[p]}) + \langle \mathrm{ct}\,\sigma_s^{[p]}, \gamma^\vee \rangle \gamma &= \mathrm{ct}\,\sigma_{s+1}^{[p]}, \\ s_\gamma(\mathrm{ct}\,\sigma_s^{(p,1]}) &= \mathrm{ct}\,\sigma_{s+1}^{(p,1]}. \end{aligned}$$

The first equality is clear since $\sigma_s^{[p]} = \sigma_{s+1}^{[p]}$, while the second one follows from the construction of σ_{s+1} from σ_s explained above. □

7. Proof of Theorem 4.6(iii)

We start by recalling some basic facts about the group B_n and some notation from Section 2. We will use the following notation related to a word $w = w_1 \dots w_l$ with integer letters, which is sometimes identified with its set of letters

$$w[i,j] := w_i \dots w_j, \quad N_{ab}(w) := |(a,b) \cap w|, \quad N_{ab}(\pm w) := N_{ab}(w) + N_{ab}(-w),$$

where $-w := \overline{w_1} \dots \overline{w_l}$. Given words $w^1, \dots, w^p$, we let

$$N_{ab}(w^1, \dots, w^p) := N_{ab}(w^1) + \dots + N_{ab}(w^p).$$

We also let

$$\tau_{ab} := \begin{cases} 1 & \text{if } a, b \le n, \\ 0 & \text{otherwise.} \end{cases}$$

With this notation, given a signed permutation w in B_n and $1 \le i < j \le n$, $a := w(i)$, $b := w(j)$, we have

$$\begin{aligned} \frac{\ell(w(i,j)) - \ell(w) - 1}{2} &= N_{ab}(w[i,j]), \\ \frac{\ell(w(i,\bar{\imath})) - \ell(w) - 1}{2} &= N_{a\bar{a}}(w[i,n]), \\ \frac{\ell(w(i,\bar{\jmath})) - \ell(w) - 1}{2} &= N_{a\bar{b}}\big(w[i,j-1], \pm w[j+1,n]\big) + \tau_{ab}, \end{aligned} \tag{7-1}$$

assuming that the left side is nonnegative (that is, that we go up in Bruhat order upon applying the indicated reflection); these facts are used implicitly throughout.

Given a chain of roots Δ, we define $\mathcal{A}^r(\Delta)$ as in (3-4) except that we impose an increasing chain condition and $w \in W$. The following simple lemma will be useful throughout, for splitting chains into subchains.

Lemma 7.1. *Consider (w, T) with T written as a concatenation $S_1 \dots S_p$. Let $w_i := wS_1 \dots S_i$, so $w_0 = w$. Then*

$$\tfrac{1}{2}(\ell(w)+\ell(wT)-|T|)=\tfrac{1}{2}(\ell(w_{p-1})+\ell(w_p)-|S_p|)+\sum_{i=1}^{p-1}\tfrac{1}{2}(\ell(w_{i-1})-\ell(w_i)-|S_i|).$$

Let Δ be the chain

$$\begin{aligned}\Delta := \big(&(1, p+1), (1, p+2), \ \dots, (1, n),\\ &(1, \overline{1}),\\ &(1, \bar{n}), \quad (1, \overline{n-1}), \dots, (1, \overline{p+1})\big).\end{aligned}$$

Proposition 7.2. *Consider a signed permutation w in B_n with $a := w(1)$, a position $1 \le p \le n$, and a value $b \in \{\pm a\} \cup (\pm w[p+1, n])$ such that $b \ge a$. Then*

$$\sum_{\substack{T:(w,T)\in\mathcal{A}^r(\Delta)\\ wT(1)=b}} t^{\frac{1}{2}(\ell(wT)-\ell(w)-|T|)}(1-t)^{|T|} = t^{N_{ab}(w[2,p])}(1-t)^{1-\delta_{ab}}, \tag{7-2}$$

where δ_{ab} is the Kronecker delta.

Proof. Given $s \in \{\overline{1}, \pm(p+1), \dots, \pm n\}$, we let Δ_s be the subchain of Δ starting with $(1, s)$. We also let

$$S(w, s) := \sum_{\substack{T:(w,T)\in\mathcal{A}^r(\Delta_s)\\ wT(1)=b}} t^{\frac{1}{2}(\ell(wT)-\ell(w)-|T|)}(1-t)^{|T|}.$$

We consider three cases: $b = w(q)$, $b = \overline{w(q)}$, and $b = \bar{a}$. The proof in the first case is identical to that of the analogous result for type A, namely [Lenart 2011, Proposition 5.3].

Case (ii): $b = \overline{w(q)}$. Let $c := w(q) = \overline{b}$ and $p < q \le s$. We start by showing that

$$S(w, \bar{s}) = t^{N_{a\bar{c}}(w[2,q-1], w[q+1,s], \pm w[s+1,n])+\tau_{ac}}(1-t). \tag{7-3}$$

We use induction on s, which starts at $s = q$. For $s > q$, let $w^1 := w[2, q-1]$, $w^2 := w[q+1, s-1]$, $w^3 := w[s+1, n]$, and $d := w(s)$. The sum $S(w, \bar{s})$ splits into two sums: over s such that $(1, \bar{s}) \notin T$ and such that $(1, \bar{s}) \in T$. By induction, the first sum is

$$S(w, \overline{s-1}) = t^{N_{a\bar{c}}(w^1, w^2, \pm d w^3)+\tau_{ac}}(1-t).$$

Again by induction, if $a<\bar{d}<\bar{c}$, then the second sum is

$$t^{N_{a\bar{d}}(w^1cw^2,\pm w^3)+\tau_{ad}}(1-t)S(w(1,\bar{s}),\overline{s-1})$$
$$=t^{N_{a\bar{d}}(w^1cw^2,\pm w^3)+N_{\bar{d}\bar{c}}(w^1,w^2,\pm\bar{a}w^3)+\tau_{ad}+\tau_{\bar{d}c}}(1-t)^2;$$

otherwise, it is empty. Adding up the two sums into which $S(w,\bar{s})$ splits, we obtain

$$t^{N_{a\bar{c}}(w^1,w^2d,\pm w^3)+\tau_{ac}}(1-t).$$

This last claim rests on the easily verified facts that if $a<\bar{d}<\bar{c}$, then

$$\tau_{ad}+\tau_{\bar{d}c}=\tau_{ac},\qquad N_{a\bar{d}}(c)+N_{\bar{d}\bar{c}}(\bar{a})=N_{a\bar{c}}(d).$$

Still assuming that $c=w(q)=\bar{b}$ and $p<q$, we now show that

$$S(w,\bar{1})=t^{N_{a\bar{c}}(w[2,q-1],w[q+1,n])+\tau'_{ac}}(1-t),\tag{7-4}$$

where

$$\tau'_{ac}:=\begin{cases}1 & \text{if } a<c\le n,\\ 0 & \text{otherwise.}\end{cases}$$

Let $w^1:=w[2,q-1]$, as before, and let $w^2:=w[q+1,n]$. The sum $S(w,\bar{1})$ splits into two sums, depending on whether $(1,\bar{1})\notin T$ or and $(1,\bar{1})\in T$. By (7-3), the first sum is

$$S(w,\bar{n})=t^{N_{a\bar{c}}(w^1,w^2)+\tau_{ac}}(1-t).$$

Again by (7-3), if $c<a\le n$, then the second sum is

$$t^{N_{a\bar{a}}(w^1cw^2)}(1-t)S(w(1,\bar{1}),\bar{n})=t^{N_{a\bar{a}}(w^1cw^2)+N_{\bar{a}\bar{c}}(w^1,w^2)+\tau_{\bar{a}c}}(1-t)^2;$$

otherwise, it is empty. Adding up the two sums into which $S(w,\bar{s})$ splits, we obtain

$$t^{N_{a\bar{c}}(w^1,w^2)+\tau'_{ac}}(1-t).$$

Assuming that $c=w(q)=\bar{b}$ and $p<q<s$, we now show that

$$S(w,s)=t^{N_{a\bar{c}}(w[2,q-1],w[q+1,s-1])+\tau'_{ac}}(1-t).\tag{7-5}$$

We use decreasing induction on s. As before, we let $w^1:=w[2,q-1]$, $w^2:=w[q+1,s-1]$, and $d:=w(s)$. The sum $S(w,s)$ splits into two sums, depending on whether $(1,s)\notin T$ or $(1,s)\in T$. By induction, the first sum is

$$S(w,s+1)=t^{N_{a\bar{c}}(w^1,w^2d)+\tau'_{ac}}(1-t).$$

Again by induction, if $a<d<\bar{c}$, then the second sum is

$$t^{N_{ad}(w^1cw^2)}(1-t)S(w(1,s),s+1)=t^{N_{ad}(w^1cw^2)+N_{d\bar{c}}(w^1,w^2a)+\tau'_{dc}}(1-t)^2;$$

otherwise, it is empty. (In both calculations, induction works by substituting $\overline{1}$ for $n+1$ when $s=n$, and by using (7-4) in this case.) Adding up the two sums into which $S(w,s)$ splits, we obtain

$$t^{N_{a\bar{c}}(w^1,w^2)+\tau'_{ac}}(1-t).$$

This last claim rests on the easily verified fact that if $a<d<\bar{c}$, then

$$N_{ad}(c)+\tau'_{dc}=\tau'_{ac}.$$

Case (iii): $b=\bar{a}$. We need to show that

$$S(w,\,p{+}1)=t^{N_{a\bar{a}}(w[2,p])}(1-t). \tag{7-6}$$

We use decreasing induction on p, which starts at $p=n$; in this case Δ only contains the pair $(1,\overline{1})$, so the convention of substituting $\overline{1}$ for $n+1$ works well here too. For $p<n$, we let $d:=w(p+1)$. The sum $S(w,\,p{+}1)$ splits into two sums, depending on whether $(1,\,p{+}1)\notin T$ or $(1,\,p{+}1)\in T$. By induction, the first sum is

$$S(w,\,p{+}2)=t^{N_{a\bar{a}}(w[2,p]d)}(1-t).$$

If $a<d<\bar{a}$, then by (7-5) of case (ii), the second sum is

$$t^{N_{ad}(w[2,p])}(1-t)S\big(w(1,\,p{+}1),\,p{+}2\big)=t^{N_{ad}(w[2,p])+N_{d\bar{a}}(w[2,p])+\tau'_{da}}(1-t)^2;$$

otherwise, it is empty. Adding up the two sums into which $S(w,\,p{+}1)$ splits, we obtain the desired result.

Case (ii) (continued). Assuming that $c=w(q)=\overline{b}$ and $p<q$, we now show that

$$S(w,q)=t^{N_{a\bar{c}}(w[2,q-1])}(1-t). \tag{7-7}$$

The sum $S(w,q)$ splits into two sums, depending on whether $(1,q)\notin T$ or $(1,q)\in T$. By (7-5) of case (ii), the first sum is

$$S(w,\,q{+}1)=t^{N_{a\bar{c}}(w[2,q-1])+\tau'_{ac}}(1-t).$$

If $a<c\le n$, then by (7-6) of case (iii), the second sum is

$$t^{N_{ac}(w[2,q-1])}(1-t)S(w(1,q)),\,q{+}1)=t^{N_{ac}(w[2,q-1])+N_{c\bar{c}}(w[2,q])}(1-t)^2;$$

otherwise, it is empty. Adding up the two sums into which $S(w,q)$ splits, we obtain the desired result.

The final step in case (ii) is to prove that

$$S(w,\,p{+}1)=t^{N_{a\bar{c}}(w[2,p])}(1-t). \tag{7-8}$$

This can be done by decreasing induction on p, starting with $p = q-1$, which is the case proved in (7-7). The procedure is completely similar to the ones above, and, in fact, to the one for type A in [Lenart 2011, Proposition 5.3]. □

Consider the chain

$$\begin{aligned}\Phi := \Gamma_1(n)\ldots\Gamma_n(n) = \big(\,&(1,\bar{1}),\\ &(1,\bar{2}),\ (2,\bar{2}),\\ &\qquad\ldots\\ &(1,\bar{n}),\ (2,\bar{n}),\ \ldots,\ (n-1,\bar{n})\big).\end{aligned} \tag{7-9}$$

We denote by Φ_{ij} the subchain of Φ starting with $(i,\bar{j})$. Given a signed permutation w, recall the definition (2-1) of $\ell_+(w)$ and $\ell_-(w)$. Given (i,j) with $1 \le i \le j \le n$, we also define

$$\begin{aligned}\ell_-^{ij}(w) &:= |\{(k,l) : (k,\bar{l}) \in \Phi\setminus\Phi_{ij}\,,\ w(k) > \overline{w(l)}\}|,\\ \overline{\ell}_-^{ij} &:= \ell_-(w) - \ell_-^{ij}(w).\end{aligned} \tag{7-10}$$

Proposition 7.3. *Fix (i,j) with $1 \le i \le j \le n$ and a signed permutation w in B_n. We have*

$$\sum_{T:(w,T)\in\mathcal{A}(\Phi_{ij})} t^{\frac{1}{2}(\ell(w)+\ell(wT)-|T|)}(1-t)^{|T|} = t^{\ell_+(w)+\ell_-^{ij}(w)}. \tag{7-11}$$

In particular, if this sum is over $(w,T)\in\mathcal{A}(\Phi)$, then the right side is $t^{\ell_+(w)}$.

Proof. Let us denote the sum in the left side of (7-11) by $S(w,i,j)$, and the corresponding sum over $(w,T)\in\mathcal{A}(\Phi_{ij}\setminus\{(i,\bar{j})\})$ by $S'(w,i,j)$. We view the chain Φ as a total order on the pairs $(i,\bar{j})$, with $(1,\bar{1})$ being the smallest pair. With this in mind, we use decreasing induction on pairs $(i,\bar{j})$. Given such a pair, if $w(i) < \overline{w(j)}$, then the induction step is clear, so assume the contrary. We can now split $S(w,i,j)$ into two sums, depending on whether $(i,\bar{j}) \notin T$ or $(i,\bar{j}) \in T$. By induction, the first sum is

$$S'(w,i,j) = t^{1+\ell_+(w)+\ell_-^{ij}(w)}.$$

By induction and Lemma 7.1, the second sum is

$$\begin{aligned}&(1-t)t^{\frac{1}{2}(\ell(w)-\ell(w(i,\bar{j}))-1)}S'(w(i,\bar{j}),i,j)\\ &\qquad\qquad = (1-t)t^{\frac{1}{2}(\ell(w)-\ell(w(i,\bar{j}))-1)+\ell_+(w(i,\bar{j}))+\ell_-^{ij}(w(i,\bar{j}))}.\end{aligned}$$

The induction step is completed once we show that

$$\ell_+(w)+\ell_-^{ij}(w) = \tfrac{1}{2}\,(\ell(w)-\ell(w(i,\bar{j}))-1)+\ell_+(w(i,\bar{j}))+\ell_-^{ij}(w(i,\bar{j})).$$

This equality can be rewritten as

$$\Delta\ell_+(w) + \Delta\ell_-^{ij}(w) - 1 = \Delta\overline{\ell}_-^{ij}(w),$$

where $\Delta\ell_+(w) := \ell_+(w) - \ell_+(w(i,\overline{j}))$, and similarly for the other two variations. To prove this, we first fix k between i and j, and analyze the contribution to the three variations of the pairs (i,k) and (k,j); see (2-1) and (7-10). For simplicity, let $a := w(i)$, $b := w(k)$, and $c := w(j)$, where $a > \overline{c}$. The nonzero contributions are as follows:

- the pair (i,k) contributes 1 to $\Delta\ell_+(w)$ if $a > b > \overline{c}$;
- the pair (k,j) contributes -1 to $\Delta\ell_+(w)$ if $\overline{a} < b < c$, which is equivalent to $a > \overline{b} > \overline{c}$;
- the pair (i,k) contributes 1 to $\Delta\ell_-^{ij}(w)$ if $a > \overline{b} > \overline{c}$;
- the pair (k,j) contributes 1 to $\Delta\overline{\ell}_-^{ij}(w)$ if $a > b > \overline{c}$.

The second and third contributions cancel out, whereas the first and fourth are equal. The analysis is completely similar if $k < i$ or $k > j$. The pair (i,j) only contributes 1 to $\Delta\ell_-^{ij}(w)$. As far as the pairs (i,i) and (j,j) are concerned, the contribution of the first one to $\Delta\ell_-^{ij}(w)$ and of the second one to $\Delta\overline{\ell}_-^{ij}(w)$ are both equal to σ_{ac}; see (2-6). □

Proof of Theorem 4.6(iii). Fix a filling σ in $\mathcal{F}(\lambda)$ with columns C_{ij} and C'_{ij}, as explained in Section 2. Recall the chain $\Phi := \Gamma_1(n)\dots\Gamma_n(n) = \Gamma_{11}\dots\Gamma_{1n}$ in (7-9). By splitting the λ-chain Γ into the tail Φ and its complement, and by using Lemma 7.1, the sum in the left side of (4-4) can be rewritten as

$$\begin{aligned}\sum_{(w,T)\in f^{-1}(\sigma)} & t^{\frac{1}{2}(\ell(w)+\ell(wT)-|T|)}(1-t)^{|T|} \\ &= \left(\sum_{\substack{(w,T)\in f^{-1}(\sigma)\\ T_{11}=\dots=T_{1n}=\varnothing}} t^{(1/2)(\ell(w)-\ell(wT)-|T|)}(1-t)^{|T|}\right) \\ &\qquad\times\left(\sum_{T:(C_{11},T)\in\mathcal{A}(\Phi)} t^{\frac{1}{2}(\ell(C_{11})+\ell(C_{11}T)-|T|)}(1-t)^{|T|}\right). \qquad (7\text{-}12)\end{aligned}$$

Here the column C_{11}, which has height n, is viewed as a signed permutation in B_n. By Proposition 7.3, the second bracket is $t^{\ell_+(C_{11})}$.

To evaluate the first bracket, we reverse all chains. Let us start by recalling the construction (4-1) of the λ-chain Γ, and in particular the order in which the subchains Γ_{ij} and Γ'_{ij} are concatenated (including the conventions in Section 2 related to $\Gamma_{i,j+1}$ and $\Gamma'_{i,j+1}$). We denote by Γ^r_{ij} and $(\Gamma'_{ij})^r$ the corresponding reverse chains. Also recall that we defined $\mathcal{A}^r(\cdot)$ as in (3-4) except that we imposed an increasing chain condition and $w \in W$. We consider pairs (w_{ij}, S_{ij}) in $\mathscr{A}^r(\Gamma^r_{ij})$ and (w'_{ij}, S'_{ij}) in $\mathscr{A}^r((\Gamma'_{ij})^r)$, where w_{ij} and w'_{ij} are defined by

$$w_{ij} := C_{11} S'_{1,\lambda'_1} \dots S_{i,j+1}, \quad w'_{ij} := C_{11} S'_{1,\lambda'_1} \dots S_{i,j+1},$$

where the concatenation order for S_{ij} and S'_{ij} comes from that for Γ_{ij} and Γ'_{ij}; in particular, $w'_{1,\lambda'_1} = C_{11}$. Given this notation, we define the sum

$$\Sigma_{ij} := \sum_{\substack{S_{ij}:(w_{ij},S_{ij})\in\mathscr{A}^r(\Gamma^r_{ij}) \\ w_{ij}S_{ij}[1,\lambda'_i]=C_{ij}}} t^{\frac{1}{2}(\ell(w_{ij}S_{ij})-\ell(w_{ij})-|S_{ij}|)}(1-t)^{|S_{ij}|},$$

and the similar sum Σ'_{ij}. We can now evaluate the first bracket in the right side of (7-12):

$$\sum_{\substack{(w,T)\in f^{-1}(\sigma) \\ T_{11}=\dots=T_{1n}=\varnothing}} t^{\frac{1}{2}(\ell(w)-\ell(wT)-|T|)}(1-t)^{|T|} = \Sigma_{\lambda_1,1} \dots \Sigma'_{ij} \dots \Sigma_{ij} \dots \Sigma'_{1,\lambda'_1}.$$

In fact, we first write the sum in the left side as an iterated sum, which factors in the way shown above because Σ_{ij} only depends on $w_{ij}[1,\lambda'_i] = C_{i,j+1}[1,\lambda'_i]$ (rather than the whole w_{ij}), by Proposition 7.2.

We conclude the proof by calculating the sum Σ_{ij}, the calculation for Σ'_{ij} being similar but simpler. For simplicity, let $d := \lambda'_i$, $w = w_{ij}$, $C := C_{i,j+1}[1,\lambda'_i]$, and $D := C_{ij}$. Assume that C differs from $D' := D(r_1,\bar{\jmath})\dots(r_p,\bar{\jmath})$ with $1 \le r_1 < \dots < r_p < j$ (possibly $p = 0$) only in position j. Let $\Gamma^r_{ij} = \Delta\Delta'$, where

$$\begin{aligned} \Delta :=\big((j,d{+}1),\ (j,d{+}2),\ &\dots,\ (j,n), \\ (j,\bar{\jmath}),& \\ (j,\bar{n}),\ (j,\overline{n-1}),\ &\dots,\ (j,\overline{d+1})\big), \\ \Delta' :=\big((j{-}1,\bar{\jmath}),\ \dots,\ &(2,\bar{\jmath}),(1,\bar{\jmath})\big). \end{aligned}$$

Correspondingly, the chains S_{ij} split into a head S, which can vary, and a fixed tail

$$S' := ((r_p,\bar{\jmath}),\dots,(r_1,\bar{\jmath})).$$

We have

$$\Sigma_{ij} = t^e(1-t)^p \sum_{\substack{S:(w,S)\in\mathscr{A}^r(\Delta) \\ wS(j)=D'(j)}} t^{\frac{1}{2}(\ell(wS)-\ell(w)-|S|)}(1-t)^{|S|},$$

where $e := \frac{1}{2}(\ell(wSS') - \ell(wS) - p)$. By Proposition 7.2, the sum in the right side is

$$t^{N_{C(j),D'(j)}(D[j+1,d])}(1-t);$$

note that this sum is missing when $D' = C$, which is another possibility. The exponent e is calculated based on (7-1). $\square$

References

[Adamczak and Lenart 2009] W. Adamczak and C. Lenart, "The alcove path model and Young tableaux", preprint, 2009, available at http://www.albany.edu/∼lenart/articles/tableaux-v3.pdf.

[Ardonne and Kedem 2007] E. Ardonne and R. Kedem, "Fusion products of Kirillov–Reshetikhin modules and fermionic multiplicity formulas", *J. Algebra* **308**:1 (2007), 270–294. MR 2008a:17028 Zbl 1122.17014

[Assaf 2010] S. Assaf, "Dual equivalence graphs, I: A combinatorial proof of LLT and Macdonald positivity", preprint, 2010. arXiv 1005.3759

[Gaussent and Littelmann 2005] S. Gaussent and P. Littelmann, "LS galleries, the path model, and MV cycles", *Duke Math. J.* **127**:1 (2005), 35–88. MR 2006c:20092 Zbl 1078.22007

[Grojnowski and Haiman 2007] I. Grojnowski and M. Haiman, "Affine Hecke algebras and positivity of LLT and Macdonald polynomials", preprint, 2007, available at http://math.berkeley.edu/∼mhaiman/ftp/llt-positivity/new-version.pdf.

[Haglund et al. 2005] J. Haglund, M. Haiman, and N. Loehr, "A combinatorial formula for Macdonald polynomials", *J. Amer. Math. Soc.* **18**:3 (2005), 735–761. MR 2006g:05223a Zbl 1061.05101

[Kashiwara and Nakashima 1994] M. Kashiwara and T. Nakashima, "Crystal graphs for representations of the q-analogue of classical Lie algebras", *J. Algebra* **165**:2 (1994), 295–345. MR 95c:17025 Zbl 0808.17005

[Kato 1982] S.-i. Kato, "Spherical functions and a q-analogue of Kostant's weight multiplicity formula", *Invent. Math.* **66**:3 (1982), 461–468. MR 84b:22030 Zbl 0498.17005

[Lascoux and Schützenberger 1979] A. Lascoux and M.-P. Schützenberger, "Croissance des polynômes de Foulkes–Green", *C. R. Acad. Sci. Paris Sér. A-B* **288**:2 (1979), A95–A98. MR 80d:20052 Zbl 0398.20052

[Lecouvey and Shimozono 2007] C. Lecouvey and M. Shimozono, "Lusztig's q-analogue of weight multiplicity and one-dimensional sums for affine root systems", *Adv. Math.* **208**:1 (2007), 438–466. MR 2008d:17022 Zbl 0498.17005

[Lenart 2009] C. Lenart, "On combinatorial formulas for Macdonald polynomials", *Adv. Math.* **220**:1 (2009), 324–340. MR 2009h:05211 Zbl 05376877

[Lenart 2011] C. Lenart, "Hall–Littlewood polynomials, alcove walks, and fillings of Young diagrams", *Discrete Math.* **311**:4 (2011), 258–275.

[Lenart and Postnikov 2007] C. Lenart and A. Postnikov, "Affine Weyl groups in K-theory and representation theory", *Int. Math. Res. Not.* **2007**:12 (2007), Art. ID rnm038, 65. MR 2008j:14105 Zbl 1137.14037

[Lenart and Postnikov 2008] C. Lenart and A. Postnikov, "A combinatorial model for crystals of Kac–Moody algebras", *Trans. Amer. Math. Soc.* **360**:8 (2008), 4349–4381. MR 2010a:17037 Zbl 05308768

[Littelmann 1994] P. Littelmann, "A Littlewood–Richardson rule for symmetrizable Kac–Moody algebras", *Invent. Math.* **116**:1-3 (1994), 329–346. MR 95f:17023 Zbl 0805.17019

[Littelmann 1995] P. Littelmann, "Paths and root operators in representation theory", *Ann. of Math.* (2) **142**:3 (1995), 499–525. MR 96m:17011 Zbl 0858.17023

[Littlewood 1961] D. E. Littlewood, "On certain symmetric functions", *Proc. London Math. Soc.* (3) **11** (1961), 485–498. MR 24 #A173 Zbl 0099.25102

[Lusztig 1983] G. Lusztig, "Singularities, character formulas, and a q-analog of weight multiplicities", pp. 208–229 in *Analysis and topology on singular spaces, II, III* (Luminy, 1981), Astérisque **101**, Soc. Math. France, Paris, 1983. MR 85m:17005 Zbl 0561.22013

[Macdonald 1971] I. G. Macdonald, *Spherical functions on a group of p-adic type*, Publications of the Ramanujan Institute **2**, Ramanujan Institute, 1971. MR 55 #8261 Zbl 0302.43018

[Macdonald 1992] I. G. Macdonald, "Schur functions: theme and variations", pp. 5–39 in *Séminaire Lotharingien de Combinatoire* (Saint-Nabor, 1992), edited by J. Zeng, Publ. Inst. Rech. Math. Av. **498**, Univ. Louis Pasteur, Strasbourg, 1992. MR 95m:05245 Zbl 0889.05073

[Macdonald 2000] I. G. Macdonald, "Orthogonal polynomials associated with root systems", Séminaire Lotharingien de Combinatoire B45a, 2000, available at http://www.emis.de/journals/SLC/wpapers/s45macdonald.pdf. MR 2002a:33021 Zbl 1032.33010

[Nelsen and Ram 2003] K. Nelsen and A. Ram, "Kostka–Foulkes polynomials and Macdonald spherical functions", pp. 325–370 in *Surveys in combinatorics* (Bangor, 2003), edited by C. D. Wensley, London Math. Soc. Lecture Note Ser. **307**, Cambridge Univ. Press, Cambridge, 2003. MR 2004h:05126 Zbl 1036.05049

[Ram 2006] A. Ram, "Alcove walks, Hecke algebras, spherical functions, crystals and column strict tableaux", *Pure Appl. Math. Q.* **2**:4 (2006), 963–1013. MR 2008a:20013 Zbl 1127.20005

[Ram and Yip 2011] A. Ram and M. Yip, "A combinatorial formula for Macdonald polynomials", *Adv. Math.* **226**:1 (2011), 309–331.

[Schwer 2006] C. Schwer, "Galleries, Hall–Littlewood polynomials, and structure constants of the spherical Hecke algebra", *Int. Math. Res. Not.* **2006** (2006), Art. ID 75395, 31. MR 2007m:22018 Zbl 1121.05121

[Stembridge 2005] J. Stembridge, "Kostka–Foulkes polynomials of general type", Lecture notes for the Generalized Kostka Polynomials Workshop at the American Institute of Mathematics, Palo Alto, CA, 2005, available at http://www.math.lsa.umich.edu/~jrs/papers/kostka.ps.gz.

Communicated by Georgia Benkart
Received 2009-07-16 Revised 2010-07-11 Accepted 2010-10-13

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Algebra & Number Theory (ISSN 1937-0652) at Mathematical Sciences Publishers, Department of Mathematics, University of California, Berkeley, CA 94720-3840 is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

ANT peer review and production are managed by EditFLOW™ from Mathematical Sciences Publishers.

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Algebra & Number Theory

Volume 4 No. 7 2010

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