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On ramification filtrations and p-adic differential modules I: the equal characteristic case

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Let k be a complete discretely valued field of equal characteristic $p > 0$ with possibly imperfect residue field, and let G_k be its Galois group. We prove that the conductors computed by the arithmetic ramification filtrations on G_k defined by Abbes and Saito (*Amer. J. Math* **124**:5, 879–920) coincide with the differential Artin conductors and Swan conductors of Galois representations of G_k defined by Kedlaya (*Algebra Number Theory* **1**:3, 269–300). As a consequence, we obtain a Hasse–Arf theorem for arithmetic ramification filtrations in this case. As applications, we obtain a Hasse–Arf theorem for finite flat group schemes; we also give a comparison theorem between the differential Artin conductors and Borger's conductors (*Math. Ann.* **329**:1, 1–30).

Introduction

Let k be a complete discretely valued field and let G_k be the Galois group of a fixed separable closure k^{sep} over k. When the residue field κ_k of k is perfect, classical ramification theory gives Artin conductors and Swan conductors, which measure the ramification of representations of G_k of finite local monodromy (i.e., the image of the inertia group being finite). A fundamental result, the Hasse–Arf theorem, states that Artin and Swan conductors are nonnegative integers. However, when the residue field κ_k is not perfect, classical ramification theory is no longer applicable.

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For one thing, the transition functions ϕ and ψ in [Serre 1979, §IV.3] fail the basic properties; for another, the extension of the rings of integers may not be generated by a single element (compare [Serre 1979, §III.6, Proposition 12]).

Kato [1989] defined Swan conductors for one-dimensional representations when the residue field is not perfect. Later, Abbes and Saito [2002; 2003] defined an arithmetic (nonlogarithmic) filtration and a logarithmic variant on G_k by counting geometric connected components of certain rigid spaces $as^a_{l/k}$ and $as^a_{l/k,\log}$ over k, which we refer to as Abbes–Saito spaces. The filtrations give the arithmetic Artin conductors and Swan conductors naturally.

Abbes and Saito [2009] showed that their definition of Swan conductors coincides with Kato's when k is of equal characteristic $p > 0$. Moreover, they proved that the subquotients of both filtrations are abelian groups [Abbes and Saito 2003]. (See also Saito's proof [2009] that the subquotients of the logarithmic filtration on wild inertia are elementary abelian p-groups.) However, they were not able to establish an integrality result analogous to the classical Hasse–Arf theorem.

Through a completely different path, when k is of equal characteristic $p > 0$ and has perfect residue field, Christol, Cook, Matsuda, Mebkhout, and Tsuzuki (see [Matsuda 2002]) gave a completely new interpretation of the classical Swan conductors using the theory of p-adic differential modules. Given a p-adic Galois representation of finite local monodromy, they associated a p-adic differential module over the Robba ring and proved that the Swan conductor of the representation can be retrieved from the irregularity of the differential module, or equivalently, the spectral norms of the differential operator.

Partly inspired by [Matsuda 2004], Kedlaya generalized this framework to the case when the residue field κ_k is not perfect. In [Kedlaya 2007], he adopted the same construction and counted in the effects of other differential operators corresponding to elements in a p-basis of κ_k. He defined the differential Swan conductor to be, vaguely speaking, the maximum of the numbers computed by each of the differential operators, under certain normalization; he was aware of a definition for differential Artin conductors using a slightly different normalization. Most importantly, he was able to prove a Hasse–Arf theorem for differential Swan conductors [Kedlaya 2007, Theorem 3.5.8]; his argument can easily be adapted to prove a Hasse–Arf theorem for differential Artin conductors. For a precise statement, see Theorem 2.4.1.

Kedlaya [2007] asked, as Matsuda suggested, whether the differential conductors are the same as the arithmetic ones, in which case the Hasse–Arf theorem for the arithmetic filtrations in the equal characteristic case would follow from that for the differential conductors. Chiarellotto and Pulita [2009] gave an affirmative answer to this question when the representations are one-dimensional, using the setting of Kato's conductors [Kato 1989].

There is a third story of defining conductors. Borger [2004] introduced the notation of generic perfection of a complete discretely valued field and defined the Artin conductors to be the ones obtained by base change to the generic residual perfection of k, which is a complete discretely valued field with perfect residue field satisfying certain universal properties. The Hasse–Arf theorem of these conductors will follow immediately from that of the classical ones. Kedlaya [2007, p. 297] asked if this also coincides with the two definitions above.

This paper answers these questions in the affirmative for all representations of finite local monodromy. Our precise result is the following.

Theorem. *Let k be a complete discretely valued field of equal characteristic $p > 0$ and let G_k be its absolute Galois group.*

(1) (Hasse–Arf Theorem) *Let $\rho : G_k \to \mathrm{GL}(V_\rho)$ be a p-adic representation of finite local monodromy. Then the arithmetic Artin conductor* $\mathrm{Art}_{\mathrm{ar}}(\rho)$, *the differential Artin conductor* $\mathrm{Art}_{\mathrm{dif}}(\rho)$, *and the Borger's conductor* $\mathrm{Art}_B(\rho)$ *are the same. Similarly, the arithmetic Swan conductor* $\mathrm{Swan}_{\mathrm{ar}}(\rho)$ *is the same as the differential Swan conductor* $\mathrm{Swan}_{\mathrm{dif}}(\rho)$. *As a consequence, they are all nonnegative integers.*

(2) *The subquotients* $\mathrm{Fil}^a G_k / \mathrm{Fil}^{a+} G_k$ *of the arithmetic ramification filtrations are trivial if $a \notin \mathbb{Q}$ and are elementary p-abelian groups if $a \in \mathbb{Q}_{>1}$; the subquotients* $\mathrm{Fil}^a_{\log} G_k / \mathrm{Fil}^{a+}_{\log} G_k$ *of the arithmetic logarithmic ramification filtrations are trivial if $a \notin \mathbb{Q}$ and are elementary p-abelian groups if $a \in \mathbb{Q}_{>0}$.*

This theorem consists of Theorems 4.4.1 and 5.4.3 and Corollary 4.4.3.

We now explain the main idea of the proof, which shows that arithmetic conductors and differential conductors coincide in a natural way. (We will use the comparison of Artin conductors as an example; that of Swan conductors is proved similarly.)

Let k be a complete discretely valued field of equal characteristic p, with residue field κ_k. Let l be a finite Galois extension of k with residue field κ_l. One immediately reduces the comparison to proving that the arithmetic highest ramification break of l/k is the same as the differential one. There are three main ingredients.

(a) A useful way of visualizing spectral norms is to consider the convergence loci or radii at a generic point; see for example [Kedlaya 2005, §5]. However, the convergence loci cannot be defined on the rigid annulus because one cannot separate $m + 1$ differential operators on a one-dimensional space. Matsuda [2004] made a pioneering attempt to obtain an $(m + 1)$-dimensional space on which we may discuss convergence loci. Our approach, which is independently developed and looks very similar to Matsuda's work, uses a thickening technique. (Alas, we do not know how to relate the two methods.) If the field k can be realized as the field of rational functions on a smooth variety over certain perfect field, the thickening space is just a subspace of the generic fiber of the tube corresponding to the diagonal embedding in a formal lifting (see Section 3.1). This thickening space, after a certain base change, "looks the same" as the Abbes–Saito space $as^a_{l/k}$, whose geometric connected components give the ramification information. However, we have the following technical issue.

(b) The thickening space is a rigid space over K, the fraction field of a Cohen ring of κ_k, which in particular is a field of characteristic zero. In contrast, the Abbes–Saito space $as^a_{l/k}$ is a rigid space over k, which is of characteristic p. In order to relate the two spaces, we need a lifting technique (see Section 1) to lift the Abbes–Saito space to characteristic zero and compare the geometric connected components before and after the lifting process. A similar idea is also alluded to as a conjecture in [Matsuda 2004]. (Again, we do not know whether our result answers Matsuda's conjecture.)

(c) The lifted Abbes–Saito space is isomorphic to the thickening space after a certain base change (Theorem 4.3.6), but not in the naïve way. Very vaguely speaking, if the extension l/k is generated by a series of equations, then the Abbes–Saito space consists of the points which are close to the solutions to those equations; in contrast, the (base change of the) thickening space consists of points which are solutions to some equations whose coefficients are close to the original equations. These two types of points coincide when l/k is totally and wildly ramified.

Combining these three ingredients, we can prove the comparison between the arithmetic conductors and the differential ones. The following diagram may be helpful to illustrate the process:

$$\begin{tikzcd}
Y = A^1_L[\eta_0^{1/e}, 1) \arrow[d] & TS^a \times_{\tilde{\pi}, Z} Y \arrow[l] \arrow[d] \arrow[r, squiggly, dash, "(c)"] & AS^a_{l/k} & as^a_{l/k} \arrow[l, squiggly, "(b)"'] \\
Z = A^1_K[\eta_0, 1) & TS^a = \text{``}\bigcup_{\eta \in [\eta_0, 1)} A^1_K[\eta, 1) \times A^{m+1}_K[0, \eta^a).\text{''} \arrow[l, "\tilde{\pi}", "(a)"']
\end{tikzcd}$$

Here K and L are the fraction fields of Cohen rings of κ_k and κ_l, respectively; $A^1_K[\eta_0, 1)$ is the half-open annulus over K (centered at the origin) with inner radius η_0 and outer radius 1, for some $\eta_0 \in (0, 1)$; $A^{m+1}_K[0, \eta^a)$ is the open polydisc (centered at the origin) of dimension $m+1$ and radius η^a for some $a \in \mathbb{Q}_{>1}$ (for the quotation marks on $\bigcup_{\eta \in [\eta_0, 1)} A^1_K[\eta, 1) \times A^{m+1}_K[0, \eta^a)$, see Caution 3.2.4); TS^a denotes the space obtained by the thickening process (a); $as^a_{l/k}$ is the rigid analytic space over k defined by Abbes and Saito with respect to a set of distinguished generators; and $AS^a_{l/k}$ is the lifting space given by lifting process (b); the argument in (c) links the two spaces as shown in the diagram.

Part (a) is carried out throughout Section 3 (see Theorem 3.4.12). Part (b) is developed in Section 1 (see Corollary 1.2.12 and Example 1.3.4). Part (c) occupies Section 4 (see Theorem 4.3.6). We finally wrap up the proof in Theorem 4.4.1.

We also obtain a comparison theorem between Borger's Artin conductors and the differential Artin conductors, or equivalently the arithmetic Artin conductors. The key is to show that the differential Artin conductors are invariant under the operation of adding generic p^∞-th roots (see Definition 5.2.2). This fact follows easily from the study of differential operators.

Plan of the paper. In Section 1, we make a construction to lift a rigid space over k to a rigid space over an annulus over K. We prove that the connected components of the original rigid space are in one-to-one correspondence with the connected components of the lifting space, when the annulus is "thin" enough. This part is written in a relatively independent and self-contained manner, since we feel that it is interesting on its own.

In Section 2, we discuss how to associate a differential module $\mathscr{E}_\rho$ on the Robba ring over K with a representation ρ of G_k of finite local monodromy. Then we review the definition of differential Swan conductors following [Kedlaya 2007]. We also introduce differential Artin conductors and discuss their properties.

Section 3 introduces a thickening construction. In Section 3.1, as an intuitive example, we construct the thickening space when k can be realized geometrically. In Section 3.2, we define thickening spaces for general k and discuss spectral properties of the differential module obtained by pulling back $\mathscr{E}_\rho$ to the thickening spaces. In Sections 3.3 and 3.4, we link the (highest) differential breaks and spectral norms with the connected components of a certain base change of the thickening spaces.

In Section 4, we first quickly review the definition of arithmetic ramification filtrations, following [Abbes and Saito 2002]. Then, in Section 4.2, we define the standard Abbes–Saito spaces $as^a_{l/k}$ and their lifts $AS^a_{l/k}$. Next, we prove in Section 4.3 that the lifted Abbes–Saito spaces and (the base change of) the thickening spaces are isomorphic (Theorem 4.3.6). From this, in Section 4.4, we deduce our main result, Theorem 4.4.1: differential conductors coincide with arithmetic conductors.

Section 5 gives two applications. In Section 5.1 we deduce a Hasse–Arf theorem for finite flat group schemes; in Sections 5.2–5.4 we compare the arithmetic and differential Artin conductors with Borger's Artin conductors [Borger 2004].

1. Lifting rigid spaces

In this section, which is largely self-contained, we introduce a construction to lift a rigid space over a field of characteristic $p > 0$ to a rigid space over an annulus over a field of characteristic zero. The notation will not be carried over to later sections unless explicitly noted.[1]

Remark 1.0.1. For most of this paper, we implicitly use rigid analytic spaces in the sense of Berkovich spaces [1990] by allowing discs or annuli with irrational radii. This is mostly for notational convenience. Only in two places (see Remarks 1.2.13 and 4.2.5) will we have to shift back to the classical rigid analytic setting to talk about connected components by assuming some rationality on the radii of discs or annuli.

1.1. ***A Gröbner basis argument.*** In this subsection, we introduce a division algorithm using a Gröbner basis, which enables us to find a representative in the quotient ring achieving the quotient norm.

Notation 1.1.1. Let K be a complete discretely valued field of mixed characteristic $(0, p)$, with ring of integers $\mathcal{O}_K$ and residue field κ. Fix a uniformizer π_K and normalize the valuation $v_K(\cdot)$ on K so that $v_K(\pi_K) = 1$. We also normalize the norm on K so that $|p| = p^{-1}$.

Notation 1.1.2. For a nonarchimedean ring R, we use $R\langle u_1, \dots, u_n\rangle$ to denote the Tate algebra, consisting of formal power series $\sum_{i_1,\dots,i_n\in\mathbb{Z}_{\geq 0}} f_{i_1,\dots,i_n} u_1^{i_1}\cdots u_n^{i_n}$ with $f_{i_1,\dots,i_n} \in R$ and $|f_{i_1,\dots,i_n}| \to 0$ as $i_1+\cdots+i_n \to +\infty$. For $\eta_1, \dots, \eta_n \in (0, 1]$, the ring admits a $(\eta_1, \dots, \eta_n)$-*Gauss norm* given by

$$\Big|\sum_{i_1,\dots,i_n\in\mathbb{Z}_{\geq 0}} f_{i_1,\dots,i_n} u_1^{i_1}\cdots u_n^{i_n}\Big|_{\eta_1,\dots,\eta_n} = \max_{i_1,\dots,i_n} \{|f_{i_1,\dots,i_n}|\eta_1^{i_1}\dots\eta_n^{i_n}\}.$$

Notation 1.1.3. Fix a positive integer n, and put

$$R^{\mathrm{int}} = \mathcal{O}_K\langle u_1, \dots, u_n\rangle((S)), \qquad R = R^{\mathrm{int}} \otimes_{\mathcal{O}_K} K,$$
$$R_\kappa = R^{\mathrm{int}} \otimes_{\mathcal{O}_K} \kappa \cong \kappa[u_1, \dots, u_n]((S)) = \kappa((S))\langle u_1, \dots, u_n\rangle.$$

For $\eta \in (0, 1]$, let $|\cdot|_\eta$ (for short) denote the $(1, \dots, 1, \eta)$-Gauss norm on R.

Notation 1.1.4. The *lexicographic* order on $\mathbb{Z}^n$ is: for $(i_1, \dots, i_n)$ and $(i'_1, \dots, i'_n)$ both in $\mathbb{Z}^n$, we have $(i_1, \dots, i_n) \succ (i'_1, \dots, i'_n)$ if there exists some $j \in \{1, \dots, n\}$ such that $i_1 = i'_1, \dots, i_{j-1} = i'_{j-1}$ and $i_j > i'_j$.

Definition 1.1.5. We equip R_κ with the lexicographic term ordering induced by the correspondence $u_1^{i_1}\cdots u_n^{i_n}S^j \mapsto (-j, i_1, \dots, i_n)$, i.e., we write $\bar\alpha u_1^{i_1}\cdots u_n^{i_n}S^j \succeq \bar\beta u_1^{i'_1}\cdots u_n^{i'_n}S^{j'}$ if $(-j, i_1, \dots, i_n) \succeq (-j', i'_1, \dots, i'_n)$ under the lexicographic order, where $\bar\alpha, \bar\beta \in \kappa^\times$.

Using this ordering, we define the *leading term* $\mathrm{lead}(\bar f)$ of a nonzero element $\bar f \in R_\kappa$ to be its largest term under the ordering. In particular, for $\bar f, \bar g \in R_\kappa\backslash\{0\}$, $\mathrm{lead}(\bar f\bar g) = \mathrm{lead}(\bar f)\,\mathrm{lead}(\bar g)$.

[1] Most of the proofs in this section should be credited to Kedlaya, to whom I am thankful for allowing their inclusion.

For an ideal I_κ of R_κ, a *Gröbner basis* of I_κ is a finite subset $\{\bar{r}_1, \dots, \bar{r}_m\} \subset I_\kappa$ such that no leading term of an $\bar{r}_i$ has exponents in S and such that the ideal consisting of the leading terms of all elements of I_κ is generated by $\operatorname{lead}(\bar{r}_1), \dots, \operatorname{lead}(\bar{r}_m)$. Such a basis exists because R_κ is Noetherian. By [Eisenbud 1995, Lemma 15.5], $\bar{r}_1, \dots, \bar{r}_m$ also generate I_κ.

Proposition 1.1.6. *For any $\bar{f} \in R_\kappa$, there exists $\bar{g}_1, \dots, \bar{g}_m, \bar{f}' \in R_\kappa$ such that*

$$\bar{f} = \bar{g}_1\bar{r}_1 + \cdots + \bar{g}_m\bar{r}_m + \bar{f}', \tag{1.1.7}$$

where any term of $\bar{f}'$ is not divisible by any $\operatorname{lead}(\bar{r}_h)$, *and* $\operatorname{lead}(\bar{f}) \succeq \operatorname{lead}(\bar{g}_h\bar{r}_h)$ *for all h.*

Proof. Let j be the exponent of S in $\operatorname{lead}(\bar{f})$ and let $S^j\bar{f}_{(j)}$ be the sum of terms in $\bar{f}$ for which the exponents of S are j. Applying [Eisenbud 1995, Proposition-Definition 15.6] to $\bar{f}_{(j)}$, we can write

$$\bar{f}_{(j)} \equiv \bar{g}_{1,(j)}\bar{r}_1 + \cdots + \bar{g}_{m,(j)}\bar{r}_m + \bar{f}'_{(j)} \pmod{S \cdot \kappa[u_1, \dots, u_m][\![S]\!]},$$

where $\bar{g}_{h,(j)} \in \kappa[u_1, \dots, u_m]$ and $\operatorname{lead}(\bar{g}_{h,(j)}\bar{r}_h) \preceq \operatorname{lead}(\bar{f}_{(j)})$ for $h = 1, \dots, m$ and any term in $\bar{f}'_{(j)} \in \kappa[u_1, \dots, u_m]$ is not divisible by any $\operatorname{lead}(\bar{r}_h)$.

If we repeat the above argument for $\bar{f}_{(j)} - S^j(\bar{g}_{1,(j)}\bar{r}_1 + \cdots + \bar{g}_{m,(j)}\bar{r}_m + \bar{f}'_{(j)}) \in S^{j+1} \cdot \kappa[u_1, \dots, u_m][\![S]\!]$ in place of $\bar{f}$, we will obtain $\bar{f}'_{(j')}$ and $\bar{g}_{h,(j')}$ for $h = 1, \dots, m$ and for some $j' \geq j+1$. We can then iterate this process.

For $h = 1, \dots, m$, put $\bar{g}_h = S^j\bar{g}_{h,(j)} + S^{j+1}\bar{g}_{h,(j+1)} + \cdots$ and $\bar{f}' = S^j\bar{f}'_{(j)} + S^{j+1}\bar{f}'_{(j+1)} + \cdots$; the power series converge to the elements in R_κ we seek. □

Definition 1.1.8. For $f \in R$, write

$$f = \sum_{i_1,\dots,i_n,j} f_{i_1,\dots,i_n,j}u_1^{i_1}\cdots u_n^{i_n}S^j. \tag{1.1.9}$$

Of the monomials for which $|f_{i_1,\dots,i_n,j}| = |f|_1$, there must be one which is lexicographically largest; we call the corresponding term $f_{i_1,\dots,i_n,j}u_1^{i_1}\cdots u_n^{i_n}S^j$ the 1-*leading term* of f, denoted by $\operatorname{Lead}(f)$.

Hypothesis 1.1.10. Let I^{int} be an ideal of R^{int} such that $R^{\text{int}}/I^{\text{int}}$ is *flat* over $\mathfrak{o}_K$.

Notation 1.1.11. Define $I = I^{\text{int}} \otimes_{\mathcal{O}_K} K$ and $I_\kappa = I^{\text{int}} \otimes_{\mathcal{O}_K} \kappa$; the latter is an ideal in R_κ by the flatness hypothesis above. Choose $r_1, \dots, r_m \in I^{\text{int}}$ which project to elements of a Gröbner basis $\bar{r}_1, \dots, \bar{r}_m$ of I_κ.

For $f \in R$, let $\boldsymbol{j}_f$ denote the minimal exponents of S in the expression (1.1.9) of f. Set $\boldsymbol{j}_I = \min\{\boldsymbol{j}_{r_h}; h = 1, \dots, m\}$; it is a nonpositive integer.

Notation 1.1.12. In this subsection, fix $\eta_0 \in (|\pi_K|^{-1/\boldsymbol{j}_I}, 1)$. We have $|\pi_K|\,\eta_0^{\boldsymbol{j}_I} < 1$.

Notation 1.1.13. Let $\mathscr{R}_{\eta_0}$ be the Fréchet completion of R for $|\cdot|_\eta$ for $\eta \in [\eta_0, 1)$. Let $R^{\mathrm{int}}_{\eta_0}$ denote $\{f \in \mathscr{R}_{\eta_0} \mid |f|_1 \leq 1\}$ and put $R_{\eta_0} = R^{\mathrm{int}}_{\eta_0} \otimes_{\mathscr{O}_K} K$ and $I_{\eta_0} = I \otimes_R R_{\eta_0}$.

Notation 1.1.14. For an element $f \in \mathscr{R}_{\eta_0}$ written as in (1.1.9) and $l \in \mathbb{Z}$, let $\pi_K^l f_{(l)}$ be the sum of all terms $f_{i_1,\dots,i_n,j} u_1^{i_1} \cdots u_n^{i_n} S^j$ for which $v_K(f_{i_1,\dots,i_n,j}) = l$. Thus, $f_{(l)} \in R^{\mathrm{int}}_{\eta_0}$; we use $\bar{f}_{(l)}$ denote its reduction in R_κ.

Lemma 1.1.15. *For $h = 1, \dots, m$ and $\eta \in [\eta_0, 1]$,*

$$|r_h|_\eta = 1, \qquad |r_{h,(l)}|_\eta \leq \eta^{\boldsymbol{j}_I} \quad \text{for } l \in \mathbb{Z}_{\geq 0}.$$

Proof. The equality follows from the choice of η_0 in Notation 1.1.12. The rest follows from the definition of $\boldsymbol{j}_I$ in Notation 1.1.11. □

Construction 1.1.16. For $f \in R_{\eta_0}$ with $|f|_1 = |\pi_K|^{l_0}$, the *division algorithm* is the following procedure. Put $f_{l_0} = f$. Given f_l for $l \geq l_0$, we apply Proposition 1.1.6 to write

$$\bar{f}_{l,(l)} = \bar{g}_{l,1}\bar{r}_1 + \cdots + \bar{g}_{l,m}\bar{r}_m + \bar{f}'_{l,(l)},$$

where $\bar{g}_{l,h} \in R_\kappa$ and $\mathrm{lead}(\bar{g}_{l,h}\bar{r}_h) \preceq \mathrm{lead}(\bar{f}_{l,(l)})$ for $h = 1, \dots, m$ and any term of $\bar{f}'_{l,(l)} \in R_\kappa$ is not divisible by any $\mathrm{lead}(\bar{r}_h)$. For each h, pick lifts $g_{l,h}$ of $\bar{g}_{l,h}$ in R^{int} so that $g_{l,h} = g_{l,h,(0)}$, that is, we only lift nonzero terms. Put

$$f_{l+1} = f_l - \pi_K^l (g_{l,1}r_1 + \cdots + g_{l,m}r_m).$$

Remark 1.1.17. The division algorithm depends on many choices, but we prove in Proposition 1.1.21 that the outcome $\lim_{l \to +\infty} f_l$ is uniquely determined by f.

Lemma 1.1.18. *At each step of the division algorithm, for $\eta \in [\eta_0, 1]$, $h = 1, \dots, m$, we have*

$$|g_{l,h}|_\eta \leq |f_{l,(l)}|_\eta, \quad |f_{l+1,(l')} - f_{l,(l')}|_\eta \begin{cases} \leq \eta^{\boldsymbol{j}_I} |f_{l,(l)}|_\eta & \text{if } l' > l, \\ \leq |f_{l,(l)}|_\eta & \text{if } l' = l, \\ = 0 & \text{if } l' < l. \end{cases} \tag{1.1.19}$$

Proof. The inequality on the left holds because $\mathrm{lead}(\bar{g}_{l,h}\bar{r}_h) \preceq \mathrm{lead}(\bar{f}_{l,(l)})$. The rest follows using Lemma 1.1.15. □

Corollary 1.1.20. *For $h = 1, \dots, m$, the series $g_h = \pi_K^{l_0} g_{l_0,h} + \pi_K^{l_0+1} g_{l_0+1,h} + \cdots$ converges under $|\cdot|_\eta$ for $\eta \in [\eta_0, 1)$. Consequently, $g_h \in R_{\eta_0}$ for $h = 1, \dots, m$.*

Proof. By Lemma 1.1.18,

$$\begin{aligned} |\pi_K^l g_{l,h}|_\eta &\leq |\pi_K^l f_{l,(l)}|_\eta \leq |\pi_K|^l \max\{\eta^{\boldsymbol{j}_I} |f_{l-1,(l-1)}|_\eta, |f_{l-1,(l)}|_\eta\} \\ &\leq |\pi_K|^l \max\{\eta^{2\boldsymbol{j}_I} |f_{l-2,(l-2)}|_\eta, \eta^{\boldsymbol{j}_I} |f_{l-2,(l-1)}|_\eta, \eta^{\boldsymbol{j}_I} |f_{l-2,(l-2)}|_\eta, |f_{l-2,(l)}|_\eta\} \\ &\leq \cdots \\ &\leq |\pi_K|^l \max_{l' < l} \{\eta^{(l-l')\boldsymbol{j}_I} |f_{(l')}|_\eta\} \leq \max_{l' < l} \{(|\pi_K| \eta_0^{\boldsymbol{j}_I})^{l-l'} |\pi_K^{l'} f_{(l')}|_\eta\}. \end{aligned}$$

This goes to zero as $l \to +\infty$. □

Proposition 1.1.21. *Keep the notation as above. The quantity $f - g_1 r_1 - \cdots - g_m r_m$ is the unique element of $f + I_{\eta_0}$ for which none of its terms is divisible by any* $\mathrm{Lead}(r_h)$.

Proof. It follows from the definition of $g_1, \ldots, g_m$ that no term of $f - \sum_{i=1}^m g_i r_i$ is divisible by any $\mathrm{Lead}(r_h)$.

Assume that $f \in R_{\eta_0}$ does not contain any term divisible by any of $\mathrm{Lead}(r_h)$, then we need to show that for any nonzero $g \in I_{\eta_0}$, there is a term in $f + g$ divisible by some of $\mathrm{Lead}(r_h)$. Assume the contrary. Let $n = \log_{|\pi_K|} |g|_1$. Then $\bar{g}_{(n)} \in I_\kappa$ does not contain any term which divides any of $\mathrm{lead}(\bar{r}_h)$. This forces $\bar{g}_{(n)} = 0$ because the leading term of any nonzero element in I_κ is divisible by some $\mathrm{lead}(\bar{r}_h)$. This is a contradiction. The lemma follows. ☐

Lemma 1.1.22. *For $\eta \in [\eta_0, 1]$, $\left| f - \sum_{i=1}^m g_i r_i \right|_\eta$ equals the minimum η-norm of any element of $f + I_{\eta_0}$. Moreover, this continues to hold if we pass from R_{η_0} to its completion $R_{\eta_0}^{\wedge,\eta}$ under $|\cdot|_\eta$.*

Proof. For $\eta \in [\eta_0, 1]$, by Lemma 1.1.18, $|f_{l+1}|_\eta \le |f_l|_\eta$, so $\left| f - \sum_{i=1}^m g_i r_i \right|_\eta \le |f|_\eta$. By Proposition 1.1.21, starting with any element in $f + I_{\eta_0}$, the division algorithm will eventually lead to a unique element $f - \sum_{i=1}^m g_i r_i$; hence the first statement follows.

The second statement follows from the fact that any element in $f + I_{\eta_0} R_{\eta_0}^{\wedge,\eta}$ is a limit of elements in $f + I_{\eta_0}$. ☐

Proposition 1.1.23. *Let f be a rigid analytic function on the space*

$$\boldsymbol{X}_{\eta_0} = \left\{ (u_1, \ldots, u_n, S) \in \mathbb{A}_K^{n+1} \,\middle|\, \eta_0 \le |S| < 1;\ |u_1|, \ldots, |u_n| \le 1;\ r_1, \ldots, r_m = 0 \right\}.$$

Then the following are equivalent:

(a) *f is induced by an element of $R_{\eta_0}^{\mathrm{int}}$.*

(b) *There exists a function $r : [\eta_0, 1) \to \mathbb{R}$ with $\lim_{\eta \to 1^-} r(\eta) \le 1$, such that for each $\eta \in [\eta_0, 1)$, f lifts to an element of the $|\cdot|_\eta$-completion of R_{η_0} having η-norm less than or equal to $r(\eta)$.*

Proof. It is clear that (a) implies (b), so assume (b). We can write f as a Fréchet limit of the projections of some sequence of elements $f_1, f_2, \ldots$ of R, under the quotient norms associated with $|\cdot|_\eta$ for $\eta \in [\eta_0, 1)$. Use the division algorithm to write $f_l = g_{l,1} r_1 + \cdots + g_{l,m} r_m + h_l$ with $g_{l,1}, \ldots, g_{l,m}, h_l \in R_{\eta_0}$. Moreover, as $f_l - f_{l+1}$ tends to zero under the Fréchet topology, so does $h_l - h_{l+1}$ since it can be obtained from the division algorithm of $f_l - f_{l+1}$ and Lemma 1.1.18 ensures that $|f_l - f_{l+1}|_\eta \ge |h_l - h_{l+1}|_\eta$. Hence, the h_l form a Fréchet convergent sequence; denote the limit by h, which is a lift of f. Note that for a fixed η, $|h_l|_\eta$ equals the η-quotient norm of f_l, which in turn equals the η-quotient norm of f when l is large enough. Thus, $|h|_\eta \le r(\eta)$ for all $\eta \in [\eta_0, 1)$. Hence it lies in $R_{\eta_0}^{\mathrm{int}}$. ☐

Notation 1.1.24. Define

$$A^{\mathrm{int}} = R^{\mathrm{int}}/I^{\mathrm{int}}, \quad A = R/I, \quad A_{\eta_0} = R_{\eta_0}/I_{\eta_0}, \quad A_\kappa = A^{\mathrm{int}} \otimes_{\mathfrak{o}_K} \kappa \cong R_\kappa/I_\kappa.$$

We may view A_κ as an affinoid algebra over $\kappa((S))$, whose corresponding rigid analytic space is denoted by X.

1.2. *Quotient norms versus spectral norms.* In this subsection, we compare spectral norms with the quotient norms discussed in previous section. As an application, we deduce that the connected components of $\boldsymbol{X}_{\eta_0}$ when $\eta_0 \to 1^-$ as a rigid space over K are the same as the connected components of X as a rigid space over $\kappa((S))$.

Hypothesis 1.2.1. In this subsection, we assume that A_κ is reduced.

Notation 1.2.2. Let $|\cdot|_{\kappa,\mathrm{quot}}$ denote the quotient norm on A_κ induced by the Gauss norm on R_κ. Let $|\cdot|_{\kappa,\mathrm{sp}} = \lim_{n\to+\infty} |\cdot^n|_{\kappa,\mathrm{quot}}^{1/n}$ be the spectral norm; it is a norm because A_κ is reduced. By [Bosch et al. 1984, Theorem 6.2.4/1], there exists $c > 0$ such that $|\cdot|_{\kappa,\mathrm{sp}} \leq |\cdot|_{\kappa,\mathrm{quot}} \leq |S|_\kappa^{-c} |\cdot|_{\kappa,\mathrm{sp}}$, where $|S|_\kappa$ is the norm of S in $\kappa((S))$.

Notation 1.2.3. In this subsection, we fix η_0 in the interval $(|\pi_K|^{1/(-\boldsymbol{j}_I+pc)}, 1)$. In particular, $|\pi_K|\eta_0^{\boldsymbol{j}_I} < \eta_0^{pc}$ and $\eta_0 > p^{-1/pc}$.

Notation 1.2.4. For $\eta \in [\eta_0, 1]$, let $|\cdot|_{\eta,\mathrm{quot}}$ denote the quotient norm on A_{η_0} or A induced by the η-Gauss norm on R_{η_0} or R. Similarly, we have the η-spectral seminorm $|\cdot|_{\eta,\mathrm{sp}} = \lim_{n\to+\infty} |\cdot^n|_{\eta,\mathrm{quot}}^{1/n}$; we will see in Lemma 1.2.6 that it is a norm.

Proposition 1.2.5. *The quotient norm $|\cdot|_{1,\mathrm{quot}}$ on A is the same as the spectral (semi)norm $|\cdot|_{1,\mathrm{sp}}$. As a consequence, the map $A^{\mathrm{int}} \to A_\kappa$ induces an isomorphism $A^\circ/A^{\circ\circ} \cong A_\kappa$, where $A^\circ = \{f \in A \mid |f|_{1,\mathrm{sp}} \leq 1\}$ and $A^{\circ\circ} = \{f \in A \mid |f|_{1,\mathrm{sp}} < 1\}$.*

Proof. Since $A^{\mathrm{int}}/\mathfrak{m}_K A^{\mathrm{int}} = A_\kappa$ is reduced, by [Bosch et al. 1984, 6.2.1/4(iii)], the quotient norm on A is equal to the spectral seminorm, $A^\circ = A^{\mathrm{int}}$, and $A^{\circ\circ} = \mathfrak{m}_K A^{\mathrm{int}}$. This proves the claim. ∎

Lemma 1.2.6. *For $\eta \in [\eta_0, 1)$, we have $|\cdot|_{\eta,\mathrm{sp}} \leq |\cdot|_{\eta,\mathrm{quot}} \leq \eta^{-pc/(p-1)} |\cdot|_{\eta,\mathrm{sp}}$ on A_{η_0}. The same is true when extending both norms to the completion of A_{η_0} with respect to $|\cdot|_{\eta,\mathrm{quot}}$ (which is the same as the completion with respect to the spectral norm). In particular, this shows that $|\cdot|_{\eta,\mathrm{sp}}$ is a norm on A_{η_0}.*

Proof. It suffices to show that for any $f \in A_{\eta_0}$, $|f^p|_{\eta,\mathrm{quot}} \geq \eta^{pc}|f|_{\eta,\mathrm{quot}}^p$; then it would follow that $|f^{p^n}|_{\eta,\mathrm{quot}} \geq \eta^{(p^n-1)pc/(p-1)}|f|_{\eta,\mathrm{quot}}^{p^n}$ for all $n \in \mathbb{N}$ by iteration, and hence the statement follows by taking the limit.

Pick a representative $\tilde{f}$ of f in R_{η_0} containing no terms divisible by any $\mathrm{Lead}(r_h)$ (hence by Proposition 1.1.21, $|\tilde{f}|_\eta = |f|_{\eta,\mathrm{quot}}$). Fixing $\eta \in [\eta_0, 1)$, we will show that

$$|\tilde{f}^p|_{\eta,\mathrm{quot}} = \Big|\sum_l (\pi_K^l \tilde{f}_{(l)})^p\Big|_{\eta,\mathrm{quot}} \geq \eta^{pc}|\tilde{f}|_\eta^p = \eta^{pc}|f|_{\eta,\mathrm{quot}}^p. \tag{1.2.7}$$

First, we remark that, given the middle inequality, the former equality follows; this is because $\tilde{f}^p - \sum_l (\pi_K^l \tilde{f}_{(l)})^p$ consists of products of $\pi_K^l \tilde{f}_{(l)}$ with an extra factor p from the multinomial coefficients. Then

$$\Big|\tilde{f}^p - \sum_l (\pi_K^l \tilde{f}_{(l)})^p\Big|_{\eta,\mathrm{quot}} \leq \Big|\tilde{f}^p - \sum_l (\pi_K^l \tilde{f}_{(l)})^p\Big|_{\eta} \leq p^{-1}|\tilde{f}|_\eta^p < \eta^{pc}|\tilde{f}|_\eta^p,$$

for $\eta \in [\eta_0, 1)$. So it suffices to prove the middle inequality in (1.2.7). For any l, we have

$$|(\bar{\tilde{f}}_{(l)})^p|_{\kappa,\mathrm{quot}} \geq |(\bar{\tilde{f}}_{(l)})^p|_{\kappa,\mathrm{sp}} = |\bar{\tilde{f}}_{(l)}|_{\kappa,\mathrm{sp}}^p \geq |S|_\kappa^{pc} \cdot |\bar{\tilde{f}}_{(l)}|_{\kappa,\mathrm{quot}}^p.$$

Let $(\tilde{f}_{(l)})^p = g_{l,1}r_1 + \cdots + g_{l,m}r_m + h_l$ be the result of the first step of applying the division algorithm to $(\tilde{f}_{(l)})^p$. Then $\log_\eta |h_{l,(0)}|_\eta = \log_{|S|_\kappa} |(\bar{\tilde{f}}_{(l)})^p|_{\kappa,\mathrm{quot}}$ and hence $|h_{l,(0)}|_\eta \geq \eta^{pc}|\tilde{f}_{(l)}|_\eta^p$. Moreover, by Lemma 1.1.18, $|h_l - h_{l,(0)}|_\eta \leq \eta^{j_I}|\pi_K||\tilde{f}_{(l)}|_\eta^p < \eta^{pc}|\pi_K|^{-pl}|\tilde{f}|_\eta^p$; this implies that $|h_{l,(0)}|_{\eta,\mathrm{quot}} = |h_{l,(0)}|_\eta$.

Now, we can write

$$\sum_l (\pi_K^l \tilde{f}_{(l)})^p = \sum_l \pi_K^{pl} h_{l,(0)} + \sum_l \pi_K^{pl}(h_l - h_{l,(0)}) \tag{1.2.8}$$

in the quotient ring. The first term on the right-hand side of (1.2.8) has (quotient) norm at least $\eta^{pc}|\tilde{f}|_\eta^p$ because none of the summands is divisible by any $\mathrm{Lead}(r_h)$. In contrast, the latter term on the right-hand side of (1.2.8) has norm strictly less than $\eta^{pc}|\tilde{f}|_\eta^p$. Thus, the inequality in (1.2.7) holds. □

Remark 1.2.9. It is attractive to think that $|\cdot|_{\eta,\mathrm{sp}} \leq |\cdot|_{\eta,\mathrm{quot}} \leq \eta^{-c}|\cdot|_{\eta,\mathrm{sp}}$ when $\eta \to 1^-$. However, the best we know is that for any $c' > c$, we have an ϵ depending on c', for which $|\cdot|_{\eta,\mathrm{sp}} \leq |\cdot|_{\eta,\mathrm{quot}} \leq \eta^{-c'}|\cdot|_{\eta,\mathrm{sp}}$ for all $\eta \in [\epsilon, 1)$.

Corollary 1.2.10. *For a rigid analytic function f on $\boldsymbol{X}_{\eta_0}$, the following are equivalent.*

(a) *f is an element in $A_{\eta_0}^{\mathrm{int}}$.*

(b) *There exists a function $r : [\eta_0, 1) \to \mathbb{R}$ with $\lim_{\eta\to 1} r(\eta) \leq 1$, such that for each $\eta \in [\eta_0, 1)$, $|f|_{\eta,\mathrm{sp}} \leq r(\eta)$.*

Proof. It follows from combining Lemma 1.2.6 with Proposition 1.1.23. □

Theorem 1.2.11. *There are one-to-one correspondences among the following sets*: (a) *the idempotent elements of A_κ*; (b) *the idempotent elements of $A_{\eta_0}^{\mathrm{int}}$*; (c) *the idempotent elements of A_{η_0}*; *and* (d) *the idempotent elements on $\boldsymbol{X}_{\eta_0}$*.

Proof. By Corollary 1.2.10, the sets (b), (c), and (d) are the same because idempotent elements have spectral norms 1. It suffices to match up (a) and (b). We have a map from the set of idempotent elements of $A_{\eta_0}^{\mathrm{int}}$ to the set of idempotent elements of A_κ by reducing modulo π_K. We first show the injectivity. Let $f, g \in R_{\eta_0}^{\mathrm{int}}$ be idempotents whose reductions modulo π_K are the same, i.e., $\bar{f} = \bar{g} \in A_\kappa$. This implies that $\bar{f}^{p-1} + \bar{f}^{p-2}\bar{g} + \cdots + \bar{g}^{p-1} = 0$ in A_κ. Since $f - g = f^p - g^p = (f-g)(f^{p-1} + f^{p-2}g + \cdots + g^{p-1})$, we have

$$\begin{aligned}|f-g|_{1,\mathrm{quot}} &= |(f-g)(f^{p-1}+f^{p-2}g+\cdots+g^{p-1})|_{1,\mathrm{quot}} \\ &\leq |f-g|_{1,\mathrm{quot}}\, |f^{p-1}+f^{p-2}g+\cdots+g^{p-1}|_{1,\mathrm{quot}} \leq |f-g|_{1,\mathrm{quot}}\, |\pi_K|.\end{aligned}$$

This forces $|f-g|_{1,\mathrm{quot}} = 0$ and hence $f = g$.

To prove surjectivity, we start with an idempotent $\bar{f} \in A_\kappa$, viewed as an element in R_κ with none of its terms divisible by any of $\mathrm{Lead}(\bar{r}_h)$; pick a lift $\tilde{f}_0 \in R^{\mathrm{int}}$ of $\bar{f}$ which only contains terms present in $\bar{f}$, and let $f_0 \in A^{\mathrm{int}}$ denote its image in A^{int}. If we set $\tilde{h}_0$ to the result of applying the division algorithm to $\tilde{f}_0^2 - \tilde{f}_0$ and $h_0 = f_0^2 - f_0$, then $|h_0|_{1,\mathrm{quot}} = |\tilde{h}_0|_{1,\mathrm{quot}} \leq |\pi_K|$ and $|h_0|_{\eta,\mathrm{quot}} = |\tilde{h}_0|_{\eta,\mathrm{quot}} \leq p^{-1}\eta^{-2c} < 1$ for all $\eta \in [\eta_0, 1)$, where the latter inequality holds because all terms in $\tilde{f}_0$ come from terms in $\bar{f}$ having norms at most $|\bar{f}|_{\kappa,\mathrm{quot}} \leq |S|_\kappa^{-c} |\bar{f}|_{\kappa,\mathrm{sp}} = |S|_\kappa^{-c}$. As in the proof of Hensel's lemma, we iteratively modify f_0 as follows. For $\alpha \geq 0$, we set $f_{\alpha+1} = f_\alpha + h_\alpha - 2h_\alpha f_\alpha$ and

$$h_{\alpha+1} := f_{\alpha+1}^2 - f_{\alpha+1} = (f_\alpha + h_\alpha - 2h_\alpha f_\alpha)^2 - (f_\alpha + h_\alpha - 2h_\alpha f_\alpha) = h_\alpha^2(4h_\alpha - 3).$$

Hence, $|h_{\alpha+1}|_{\eta,\mathrm{quot}} \leq |h_\alpha|_{\eta,\mathrm{quot}}^2$ for all $\eta \in [\eta_0, 1]$. Thus $|h_\alpha|_{\eta,\mathrm{quot}} \to 0$ as $\alpha \to +\infty$; hence f_α converges to an element $f \in A_{\eta_0}^{\mathrm{int}}$ which is idempotent. It is clear from the construction that the reduction of f modulo π_K is the same as $\bar{f}$. This proves the surjectivity. □

Corollary 1.2.12. *When $\eta_0 \in p^{\mathbb{Q}}$, there is a one-to-one correspondence between the connected components of X and those of $\boldsymbol{X}_{\eta_0}$.*

Remark 1.2.13. This is the first place where we need the rationality of $\log_p \eta_0$ to ensure that we are in the classical rigid analytic space setting to talk about connected components [Bosch et al. 1984, 9.1.4/8].

1.3. *Lifting construction.*

In order to apply the results from the previous two subsections later in the paper, we, reversing the picture, start with a rigid analytic space X and try to construct $\boldsymbol{X}_{\eta_0}$ from it.

Let κ and K be as before.

Definition 1.3.1. Let X be a *reduced* affinoid rigid space over $\kappa((S))$ with ring of analytic functions $A_\kappa = R_\kappa / I_\kappa$ where $R_\kappa = \kappa((S))\langle u_1, \ldots, u_n\rangle$ and I_κ is some ideal. The *lifting construction* refers to the following.

(1) Find an ideal I^{int} in $R^{\mathrm{int}} = K\langle u_1, \dots, u_n\rangle((S))$ so that $R^{\mathrm{int}}/I^{\mathrm{int}}$ is *flat* over $\mathbb{O}_K$ and $I^{\mathrm{int}} \otimes_{\mathbb{O}_K} \kappa = I_\kappa$.

(2) Choose a Gröbner basis of I_κ, lift its elements to $r_1, \dots, r_m \in I^{\mathrm{int}}$ as in Notation 1.1.11, and define η_0 as in Notation 1.2.3.

(3) We call the rigid analytic space

$$\boldsymbol{X}_{\eta_0} = \left\{(u_1, \dots, u_n, S) \in \mathbb{A}_K^{n+1} \,\middle|\, \eta_0 \le |S| < 1;\ |u_1|, \dots, |u_n| \le 1;\ r_1, \dots, r_m = 0\right\}$$

the *lifting space* of X; it depends only on the choice of I^{int} and η_0.

Remark 1.3.2. We do not know if such a lifting space exists in general. The only obstruction is finding an ideal I^{int} lifting I_κ such that $R^{\mathrm{int}}/I^{\mathrm{int}}$ is flat over $\mathbb{O}_K$.

Question 1.3.3. It would be interesting to know if this lifting construction can be globalized for arbitrary rigid spaces over $\kappa((S))$. In particular, given a morphism between two rigid spaces over $\kappa((S))$, can we lift the morphism (noncanonically) to a morphism between (some strict neighborhood of) their lifting spaces? Can we "glue" the lifting spaces up to homotopy? This situation is very similar to Berthelot's construction [1996] of rigid cohomology.

For an affinoid subdomain of a polydisc, we explicate this lifting process.

Example 1.3.4. Let $p_1, \dots, p_m \in \kappa[\![S]\!][u_1, \dots, u_n]$ be polynomials and take $a_1, \dots, a_m \in \mathbb{N}$. Consider the following affinoid subdomain of the unit polydisc:

$$X = \left\{(u_1, \dots, u_n) \in \mathbb{A}^n_{\kappa((S))} \,\middle|\, |u_1|, \dots, |u_n| \le 1;\ |p_1| \le |S|^{a_1}, \dots, |p_m| \le |S|^{a_m}\right\}.$$

The ring of analytic functions on X is

$$\kappa((S))\langle u_1, \dots, u_n, v_1, \dots, v_m\rangle \big/ (v_1 S^{a_1} - p_1, \dots, v_m S^{a_m} - p_m).$$

For each i, let P_i be a lift of p_i in $\mathbb{O}_K[\![S]\!][u_1, \dots, u_n]$ (here we allow P_i to have new terms other than the terms of p_i). We claim that the ring

$$\mathbb{O}_K\langle u_1, \dots, u_n, v_1, \dots, v_m\rangle((S)) \big/ (v_1 S^{a_1} - P_1, \dots, v_m S^{a_m} - P_m) \tag{1.3.5}$$

is flat over $\mathbb{O}_K$. This is because the ring

$$\mathbb{O}_K((S))[u_1, \dots, u_n, v_1, \dots, v_m] \big/ (v_1 S^{a_1} - P_1, \dots, v_m S^{a_m} - P_m),$$

being isomorphic to $\mathbb{O}_K((S))[u_1, \dots, u_n]$, is flat and hence torsion-free over $\mathbb{O}_K$, and (1.3.5) is its completion with respect to the topology induced by the various $(p, S)^r \mathbb{O}_K[\![S]\!][u_1, \dots, u_n, v_1, \dots, v_m]$, for $r \in \mathbb{N}$. Therefore, by Definition 1.3.1,

$$\begin{aligned}\boldsymbol{X}_{\eta_0} = \Big\{(u_1, \dots, u_n, S) \in \mathbb{A}_K^{n+1} \,\Big|\, \eta_0 \le |S| < 1,\ |u_1|, \dots, |u_n| \le 1,\\ |P_1| \le |S|^{a_1}, \dots, |P_m| \le |S|^{a_m}\Big\}\end{aligned}$$

is a lifting space for X, for some $\eta_0 \in (0, 1)$.

2. Differential conductors

In this section, we recall the definition of differential Swan conductors following [Kedlaya 2007]. Along the way, we define the differential Artin conductors using a slightly different normalization.

2.1. ***Setup.*** Recall that we do not use any notation from the previous section.

Convention 2.1.1. Let J be an index set. We write e_J for a tuple $(e_j)_{j\in J}$. For an element x, we use x^{e_J} to denote $(x^{e_j})_{j\in J}$. For another tuple b_J, we set $b_J^{e_J} = \prod_{j\in J} b_j^{e_j}$ if only a finite number of e_j are nonzero. We also use $\sum_{e_J=0}^{n}$ to mean the sum over $e_j \in \{0, 1, \dots, n\}$ for each $j \in J$, only allowing finitely many summands to be nonzero.

Definition 2.1.2. For a finite field extension l/k of characteristic $p > 0$, a *p-basis* of l over k is a set $(c_j)_{j\in J} \subset l$ such that $c_J^{e_J}$, where $e_j \in \{0, 1, \dots, p-1\}$ for all $j \in J$ and $e_j = 0$ for all but a finite number of j, form a basis of the vector space l over kl^p. By a *p-basis* of l we mean a p-basis of l over l^p; it is an empty set if and only if l is perfect. (For more details, see [Eisenbud 1995, p. 565] or [Grothendieck 1964, Ch. 0, §21].)

Remark 2.1.3. For a p-basis $c_J \subset l$, the dc_J form a basis for the differentials Ω^1_l as an l-vector space.

Convention 2.1.4. Throughout this paper, all differentials are p-adically continuous. In other words, for a continuous homomorphism $A \to B$ of p-adic rings, $\Omega^1_{B/A}$ is the relative p-adically continuous differentials. Sometimes, we may use in the notation the corresponding geometric objects, such as the rigid space $\mathrm{Max}(B)$, instead of A or B. When $A = \mathbb{Z}_p$, we may suppress it from the notation, writing simply Ω^1_B.

For a homomorphism $A \to B$ between rings, a *∇-module* or a *differential module* over B *relative to* A is a finite projective B-module M equipped with an integrable connection $\nabla : M \to M \otimes \Omega^1_{B/A}$. Sometimes, we may use the corresponding geometric objects instead of A or B in the notation. When $A = \mathbb{Z}_p$, we may omit the reference to the base ring.

Notation 2.1.5. Let k be a complete discretely valued field of equal characteristic $p > 0$. Denote its ring of integers, maximal ideal, and residue field by $\mathcal{O}_k$, $\mathfrak{m}_k$, and κ_k, respectively. Fix a uniformizer s and a noncanonical isomorphism

$$\kappa_k((s)) \simeq k. \tag{2.1.6}$$

Let $v_k(\cdot)$ denote the valuation, normalized so that $v_k(s) = 1$. Let $(\bar{b}_j)_{j\in J}$ be a p-basis of κ_k, where J is an index set. Let b_j be the image of $\bar{b}_j$ in k under the isomorphism (2.1.6). Hence, $(db_j)_{j\in J}$ and ds form a basis of $\Omega^1_{\mathcal{O}_k/\mathbb{F}_p}$. We set $\kappa_0 = \bigcap_{n>0} k^{p^n} \cong \bigcap_{n>0} \kappa_k^{p^n}$; it is a perfect field.

Notation 2.1.7. Let $\mathbb{O}_K$ denote the Cohen ring of κ_k with respect to $(\bar{b}_j)_{j\in J}$ and let $(B_j)_{j\in J} \subset \mathbb{O}_K$ be the canonical lifts of the p-basis. (For more about Cohen rings, see [Kedlaya 2007, §3.1] or [Whitney 2002].) Let $K = \operatorname{Frac} \mathbb{O}_K$. We use $\mathbb{O}_{K_0}$ to denote the ring of Witt vectors $W(\kappa_0)$ of κ_0, as a subring of $\mathbb{O}_K$. Let $K_0 = \operatorname{Frac} \mathbb{O}_{K_0}$.

We insert here a proposition discussing the functoriality of Cohen rings. For a more detailed study of functoriality of Cohen rings, see [Whitney 2002].

Proposition 2.1.8. *Keep the notation as above and let R be a complete Noetherian local ring with the maximal ideal $\mathfrak{m}$ containing p. Assume that we have a homomorphism $\overline{\psi} : \kappa_k \hookrightarrow R/\mathfrak{m}$. Then, for any $B'_J \subseteq R$ lifting $\overline{\psi}(\bar{b}_J)$, there exists a unique continuous homomorphism $\psi : \mathbb{O}_K \to R$ lifting $\overline{\psi}$ and sending B_j to B'_j for all $j \in J$.*

Proof. For any $n \in \mathbb{N}$, a *level-n expression* of an element $g \in \mathbb{O}_K$ is a (noncanonical) way of writing g as

$$g = \sum_{i,i'\geq 0} \sum_{e_J=0}^{p^n-1} p^i A_{i,i',e_J}^{p^n} B_J^{e_J} \tag{2.1.9}$$

for some $A_{i,i',e_J} \in \mathbb{O}_K$ and for a fixed i, $A_{i,i',e_J} = 0$ when $i' \gg 0$ for all e_J. Then we set

$$\psi_n(g) = \sum_{i,i'\geq 0} \sum_{e_J=0}^{p^n-1} p^i \tilde{A}_{i,i',e_J}^{p^n} B_J'^{e_J}$$

where $\tilde{A}_{i,i',e_J}$ is some lift of $\overline{\psi}(a_{i,i',e_J})$ in R with a_{i,i',e_J} being the reduction of A_{i,i',e_J} in κ_k. Different choices of lifts $\tilde{A}_{i,i',e_J}$ may change the definition of $\psi_n(g)$ by an element in $\mathfrak{m}^n$; a different level-n expression as in (2.1.9) may also vary $\psi_n(g)$ by some element in $\mathfrak{m}^n$. If $n \geq 1$, we can rewrite a level-n expression of g as in (2.1.9) in the form

$$g = \sum_{i,i'\geq 0} \sum_{e'_J=0}^{p-1} \sum_{e_J=0}^{p^{n-1}-1} p^i (A_{i,i',e_J+p^{n-1}e'_J}^{p} B_J^{e'_J})^{p^{n-1}} B_J^{e_J},$$

lowering the level by 1. From this we conclude that $\psi_n(g) \equiv \psi_{n-1}(g) \bmod \mathfrak{m}^{n-1}$. Taking $n \to \infty$, we get our map $\psi(g) = \lim_{n\to\infty} \psi_n(g)$. It is not hard to check that ψ is actually a homomorphism; this is because for $g, h \in \mathbb{O}_K$, the formal sum and product of level-n expressions of g and h are level-n expressions of $g + h$ and gh, respectively.

To prove the uniqueness, take another continuous homomorphism $\psi' : \mathbb{O}_K \to R$ satisfying all the conditions. Then, for a level-n expression of g as in (2.1.9),

$$\psi'\Big(\sum_{i,i'\geq 0} \sum_{e_J=0}^{p^n-1} p^i A_{i,i',e_J}^{p^n} B_J^{e_J}\Big) = \sum_{i,i'\geq 0} \sum_{e_J=0}^{p^n-1} p^i \psi'(A_{i,i',e_J})^{p^n} B_J'^{e_J}$$

is exactly one possible definition for ψ_n. As we proved above, $\psi'(g) \equiv \psi_n(g) \equiv \psi(g) \bmod \mathfrak{m}^n$. Let $n \to \infty$ and we have $\psi = \psi'$. $\square$

Corollary 2.1.10. *Suppose $J = \{1, \dots, m\}$. There exists a unique continuous homomorphism $\psi : \mathbb{O}_K \to \mathbb{O}_K[\![\delta_1, \dots, \delta_m]\!]$ such that for all $j \in J$, $\psi(B_j) = B_j + \delta_j$ and for any $g \in \mathbb{O}_K$, $\psi(g) - g$ lies in the ideal generated by $\delta_1, \dots, \delta_m$. Moreover, ψ is an $\mathbb{O}_{K_0}$-homomorphism.*

Proof. The first statement follows from previous proposition. By the functoriality of Witt vectors, ψ has to be identity when restricted to $\mathbb{O}_{K_0}$ because κ_0 is perfect. Hence, ψ is an $\mathbb{O}_{K_0}$-homomorphism. $\square$

Corollary 2.1.11. *Assume that κ_k has a finite p-basis b_J. Fix $j \in J$ and let $b'_j \in \mathbb{O}_k$ be an element such that $b'_j \equiv b_j \pmod{\mathfrak{m}_k}$. Then there exists an automorphism $g^* : k \to k$ such that $g^*(s) = s$, $g^*(b_j) = b'_j$, and $g^*(b_{J \setminus j}) = b_{J \setminus j}$.*

Proof. Applying Proposition 2.1.8 to $R = \kappa_k[\![s]\!]$ and $\mathfrak{m} = (s)$ gives us a homomorphism $g^* : \mathbb{O}_K/(p) = \kappa_k \to k[\![s]\!]$ such that $g^*(b_j) = b'_j$, and $g^*(b_{J \setminus j}) = b_{J \setminus j}$. One can extend this to an automorphism $g^* : k \to k$ by setting $g^*(s) = s$. $\square$

2.2. *Construction of differential modules.* In this subsection we review Tsuzuki's construction [2002] of differential modules over the Robba ring associated with p-adic Galois representations. For a systematic treatment, one may consult, for example, [Kedlaya 2007, §3].

Notation 2.2.1. Keep the notation as in the previous subsection. Fix a separable closure k^{sep} of k and let $G_k = \mathrm{Gal}(k^{\mathrm{sep}}/k)$ be the absolute Galois group of k.

For a (not necessarily algebraic) separable extension l/k of complete discretely valued fields, the *naïve ramification degree e* is the index of the valuation group of k in that of l; note that this might not be the same as the usual ramification degree because the inseparable part of the residue field extension κ_l/κ_k is not counted in. We say l/k is *tamely ramified* if $p \nmid e$ and the residue field extension is algebraic and separable. Moreover, if $e = 1$, we say that l/k is *unramified*.

Notation 2.2.2. By a *representation* of G_k, we mean a continuous homomorphism $\rho : G_k \to \mathrm{GL}(V_\rho)$, where V_ρ is a vector space over a (topological) field F of characteristic zero. We say that ρ is a *p-adic representation* if F is a finite extension of $\mathbb{Q}_p$.

Let F be a finite extension of $\mathbb{Q}_p$. Let $\mathbb{O}$ and $\mathbb{F}_q$ denote its ring of integers and residue field, respectively, where q is a power of p. Write $\mathbb{Z}_q$ for the Witt vectors $W(\mathbb{F}_q)$ and $\mathbb{Q}_q$ for its fraction field. By an *$\mathbb{O}$-representation* of G_k, we mean a continuous homomorphism $\rho : G_k \to \mathrm{GL}(V_\rho)$ with V_ρ a finite free $\mathbb{O}$-module.

We always assume that $\mathbb{F}_q \subseteq \kappa_0$ (see Remark 2.4.2). Let $K' = KF$. Since $F/\mathbb{Q}_q$ is totally ramified, we have the ring of integers $\mathbb{O}_{K'} \cong \mathbb{O}_K \otimes_{\mathbb{Z}_q} \mathbb{O}$. Let $v_{K'}$ denote the valuation on K' normalized so that $v_{K'}(p) = 1$.

Notation 2.2.3. Let C_k be the Cohen ring of k relative to the p-basis $\{(b_j)_{j\in J}, s\}$. By the functoriality of Cohen rings (Proposition 2.1.8), C_k has a natural structure as an $\mathcal{O}_K$-algebra, via the isomorphism (2.1.6). In particular, the (canonical) lifts of $(b_j)_{j\in J}$ in C_k are $(B_j)_{j\in J}$. We denote the canonical lift of s in C_k by S.

Put $\Gamma = C_k \otimes_{\mathbb{Z}_q} \mathcal{O}$; it is a complete discrete valuation ring since $\mathcal{O}$ is totally ramified over $\mathbb{Z}_q$. It carries a Frobenius structure ϕ lifting the q-th power Frobenius on k which acts trivially on $\mathcal{O}$.

Definition 2.2.4. Let $\sigma : R \to R$ be an endomorphism. A (σ, ∇)-*module* over R is a ∇-module over R (relative to $\mathbb{Z}_p$) equipped with an isomorphism $\sigma^* M \to M$ of ∇-modules.

Definition 2.2.5. For every $\mathcal{O}$-representation $\rho : G_k \to \mathrm{GL}(V_\rho)$, define its associated (ϕ, ∇)-module over Γ by

$$D(\rho) = (V_\rho \otimes_{\mathcal{O}} \widehat{\Gamma^{\mathrm{unr}}})^{G_k},$$

where $\widehat{\Gamma^{\mathrm{unr}}}$ is the p-adic completion of the maximal unramified extension of Γ. All ∇-modules we encounter in this section are relative to $\mathbb{Z}_p$, so we omit the reference to the base ring $\mathbb{Z}_p$ in the notation.

Proposition 2.2.6. *For any Frobenius lift ϕ on Γ, the functor D from $\mathcal{O}$-representations of G_k to (ϕ, ∇)-modules over Γ is an equivalence of categories.*

Proof. For the convenience of the reader, we briefly describe the functor here; for more details, one may consult [Kedlaya 2007, Propositions 3.2.7 and 3.2.8]. It is well-known that D establishes an equivalence between the category of representations and the category of ϕ-modules over Γ (finite free Γ-modules with semilinear ϕ-actions), with $V(M) = (M \otimes_\Gamma \widehat{\Gamma^{\mathrm{unr}}})^{\phi=1}$ as the inverse. The nontrivial part is that every ϕ-module over Γ admits a unique structure of (ϕ, ∇)-module; this involves a standard approximation argument. $\square$

Definition 2.2.7. Let $I_k = \mathrm{Gal}(k^{\mathrm{sep}}/k^{\mathrm{unr}})$ be the inertia subgroup of G_k, where k^{unr} is the maximal unramified extension of k in k^{sep}. We say that an ($\mathcal{O}$-)representation ρ has *finite local monodromy* if the image $\rho(I_k)$ is finite.

For an $\mathcal{O}$-representation ρ of finite monodromy, one can refine the (ϕ, ∇)-module associated with ρ as follows.

Construction 2.2.8. Since C_k has an $\mathcal{O}_K$-algebra structure, any element $x \in \Gamma$ can be uniquely written in the form of $\sum_{i\in\mathbb{Z}} x_i S^i$ for $x_i \in \mathcal{O}_K \otimes_{\mathbb{Z}_q} \mathcal{O} = \mathcal{O}_{K'}$ such that the indices i for which $v_{K'}(x_i) \le n$ are bounded below.

For $r > 0$, put $\Gamma^r = \{x \in \Gamma \mid \lim_{n\to -\infty} v_{K'}(x_n) + rn = \infty\}$ and $\Gamma^\dagger = \bigcup_{r>0} \Gamma^r$; the latter is commonly known as the *integral Robba ring* over K'. It is not hard to show that the Frobenius ϕ preserves $\Gamma^\dagger$ and that $\Omega^1_{\Gamma^\dagger/\mathcal{O}} = \bigoplus_{j\in J} \Gamma^\dagger dB_j \oplus \Gamma^\dagger dS$. Also, $\Gamma^\dagger$ is a Henselian discrete valuation ring as cited in Lemma 2.2.10.

Since $\mathcal{O}_{K'} \hookrightarrow \Gamma^\dagger$, we can identify $\mathcal{O}_{K'}^{\mathrm{unr}} \hookrightarrow (\Gamma^\dagger)^{\mathrm{unr}}$, where the superscript unr means taking the maximal unramified extensions of discrete valuation rings. Put

$$\tilde{\Gamma}^\dagger = \widehat{\mathcal{O}_K^{\mathrm{unr}}} \otimes_{\mathcal{O}_K^{\mathrm{unr}}} (\Gamma^\dagger)^{\mathrm{unr}} \subset \widehat{\Gamma^{\mathrm{unr}}},$$

where we take the p-adic completion. For a p-adic representation ρ with finite local monodromy, define

$$D^\dagger(\rho) = D(\rho) \cap (V_\rho \otimes_{\mathcal{O}} \tilde{\Gamma}^\dagger) = (V_\rho \otimes_{\mathcal{O}} \tilde{\Gamma}^\dagger)^{G_k}. \tag{2.2.9}$$

Lemma 2.2.10 [Kedlaya 2005, Proposition 3.20]. *The integral Robba ring $\Gamma^\dagger$ is a henselian discrete valuation ring.*

Theorem 2.2.11 [Kedlaya 2007, Theorem 3.3.6]. *Let ϕ be a Frobenius lift on Γ acting on $\Gamma^\dagger$. Then $D^\dagger$ induces an equivalence between the category of $\mathcal{O}$-representations with finite local monodromy and the category of (ϕ, ∇)-modules over $\Gamma^\dagger$.*

Notation 2.2.12. For $I \subset [0, +\infty)$ an interval, let $A_K^1(I)$ denote the annulus (centered at the origin) with radii in I. We do not impose any rationality condition on the endpoints of I, so this space should be viewed as an analytic space in the sense of [Berkovich 1990]. If $I = [\alpha, \beta]$, we write $A_K^1[\alpha, \beta]$ for $A_K^1([\alpha, \beta])$.

For $0 \leq \alpha \leq \beta < \infty$, let $K\langle \alpha/t, t/\beta\rangle$ denote the ring of analytic functions on $A_K^1[\alpha, \beta]$. (If $\alpha = 0$, we write $K\langle t/\beta\rangle$ instead.) For $\eta \in [\alpha, \beta]\backslash\{0\}$, the ring $K\langle \alpha/t, t/\beta\rangle$ admits an η-Gauss norm: for $f = \sum_{i\in\mathbb{Z}} a_i x^i \in K\langle \alpha/t, t/\beta\rangle$,

$$|f|_\eta = \max_{i\in\mathbb{Z}}\{|a_i|\eta^i\}.$$

Notation 2.2.13. For $\eta_0 \in (0, 1)$, we use $Z_k^{\geq\eta_0}$ as a shorthand for $A_K^1[\eta_0, 1)$. Denote the ring of analytic functions on it by $\mathcal{R}_K^{\eta_0}$. We define the *Robba ring over K* to be $\mathcal{R}_K = \bigcup_{\eta\in[\eta_0,1)} \mathcal{R}_K^\eta$. Also let $\mathcal{R}_{K'}^{\eta_0} = \mathcal{R}_K^{\eta_0} \otimes_{\mathbb{Q}_q} F$ and $\mathcal{R}_{K'} = \mathcal{R}_K \otimes_{\mathbb{Q}_q} F$. We will only be interested in the behavior when η_0 is close to 1.

Remark 2.2.14. We use k in the subscript of $Z_k^{\geq\eta_0}$ because the space is functorial in k but not in K, as we made a noncanonical choice in (2.1.6).

Now, we restrict the (ϕ, ∇)-module $D^\dagger(\rho)$ to the Robba ring over K as follows.

Construction 2.2.15. Consider the natural injection $\Gamma^\dagger \hookrightarrow \mathcal{R}_{K'}$. Note that the Frobenius ϕ extends by continuity to $\mathcal{R}_{K'}$. Thus, from an $\mathcal{O}$-representation ρ with finite local monodromy, we obtain a differential module $\mathcal{E}_\rho = D^\dagger(\rho) \otimes_{\Gamma^\dagger} \mathcal{R}_{K'}$ over $\mathcal{R}_{K'}$.

Moreover, if we start with a p-adic representation $\rho : G_k \to \mathrm{GL}(V_\rho)$ of finite local monodromy, we can choose an $\mathcal{O}$-lattice V_ρ^{int} of V_ρ stable under the action of G_k. Then we associate a differential module $\mathcal{E}_\rho$ with the $\mathcal{O}$-representation given by V_ρ^{int}. It is clear that $\mathcal{E}_\rho$ does not depend on the choice of the lattice V_ρ^{int}. We call $\mathcal{E}_\rho$ the *differential module associated to ρ*.

Proposition 2.2.16 [Kedlaya 2007, Proposition 3.5.1]. *The* (ϕ, ∇)*-module* $\mathscr{E}_\rho$ *over* $\mathscr{R}_{K'}$ *does not depend on the choice of the* p*-basis* (*up to a canonical isomorphism*).

Proposition 2.2.17. *The differential module* $\mathscr{E}_\rho$ *descends to a differential module over* $\mathscr{R}_{K'}^{\eta_0}$ *for some* $\eta_0 \in (0, 1)$.

Proof. Defining a differential module requires only a finite amount of data. So, we can realize it on a certain annulus. See [Kedlaya 2007, Remark 3.4.1]. □

Remark 2.2.18. We will often make η_0 closer to 1^- in proving the main theorems. We will see later that all we care about is the asymptotic behavior of $\mathscr{E}_\rho$ as $\eta_0 \to 1^-$.

Remark 2.2.19. The current construction of associating a differential module with a representation (Constructions 2.2.8 and 2.2.15) is *not* functorial with respect to the base field F of the representation. If F' is a finite extension of F, for a p-adic representation ρ over F of finite local monodromy, one can naturally obtain $\rho \otimes_F F'$ as a p-adic representation over F'. Assume that κ_k contains the residue field $\mathbb{F}_{q'}$ of F'. Then the differential modules associated with ρ and $\rho \otimes_F F'$ are the same if F'/F is unramified and $\mathscr{E}_\rho \otimes_F F' = \mathscr{E}_{\rho \otimes_F F'}$ if F'/F is totally ramified.

There are two reasons for keeping this nonfunctoriality flaw. For one, the differential conductors we define later will be the same if we change ρ to $\rho \otimes_F F'$. For the other, if we define $\mathscr{E}_\rho$ using the tensor over $\mathbb{Z}_p$ instead of $\mathbb{Z}_q$ in Notation 2.2.3, in which case we do have the functoriality, we will get the direct sum of $[\mathbb{F}_q : \mathbb{F}_p]$ copies of $\mathscr{E}_\rho$ as differential modules. When proving the integrality of Swan conductors, we have to come back to study $\mathscr{E}_\rho$ because $K \otimes_{\mathbb{Z}_p} \mathbb{O} \simeq K'^{\oplus[\mathbb{F}_q:\mathbb{F}_p]}$ is not a field if $q > p$.

2.3. ***Differential conductors.*** Given a p-adic representation ρ of finite local monodromy, Kedlaya [2007, §3.5] showed that one can define a differential Swan conductor for ρ, using the p-adic differential module associated with ρ. In this subsection, we review this definition and give an analogous definition for the differential Artin conductor.

Remark 2.3.1. Starting from this subsection, the Frobenius ϕ plays almost no role in our theory; most of the arguments work for solvable differential modules [Kedlaya 2007, Definition 2.5.1], and since all the decompositions for differential modules we encounter are canonical, they automatically respect the Frobenius structure. The only place we need Frobenius is to link back with representations; see Proposition 2.3.22.

Hypothesis 2.3.2. In this subsection, we make an auxiliary hypothesis that k admits a *finite* p-basis $\{b_1, \ldots, b_m, s\}$.

Notation 2.3.3. Let $J = \{1, \ldots, m\}$ for notational convenience. We save the letters j and m for indexing the p-basis, except in Section 4.1 (see Notation 4.1.2). We also use J^+ to denote $J \cup \{0\}$, where 0 refers to the uniformizer s in the p-basis.

Definition 2.3.4. Let E be a differential field of order 1 and characteristic zero, i.e., a field of characteristic zero equipped with a derivation ∂. Assume that E is complete for a nonarchimedean norm $|\cdot|$. Let M be a finite differential module over E, i.e., a finite dimensional E-vector space equipped with an action of ∂ satisfying the Leibniz rule. The *spectral norm of* ∂ *on* M is defined to be

$$|\partial|_{M,\mathrm{sp}} = \lim_{n\to\infty} |\partial^n|_M^{1/n}$$

for any norm $|\cdot|_M$ on M; it does not depend on the choice of $|\cdot|_M$. One can prove that $|\partial|_{M,\mathrm{sp}} \geq |\partial|_{E,\mathrm{sp}}$ [Kedlaya 2010, Lemma 6.2.4].

Remark 2.3.5. For a complete extension E' of E, to which the derivation ∂ extends, $M \otimes_E E'$ can be viewed as a differential module over E' with spectral norm $|\partial|_{M\otimes_E E',\mathrm{sp}} = \max\{|\partial|_{M,\mathrm{sp}}, |\partial|_{E',\mathrm{sp}}\}$.

Notation 2.3.6. Let $\partial_0 = \partial/\partial S$, $\partial_1 = \partial/\partial B_1, \dots, \partial_m = \partial/\partial B_m$ denote the elements of a dual basis of $\Omega^1_{\mathscr{O}_K[\![S]\!]/\mathscr{O}_{K_0}}$ with respect to $dS, dB_1, \dots, dB_m$; they also give a dual basis of $\Omega^1_{\mathscr{R}^{\eta_0}_{K'}/K_0}$ for all $\eta_0 \in (0,1)$. For a (ϕ,∇)-module $\mathscr{E}$ over $\mathscr{R}^{\eta_0}_{K'}$, these differential operators act on $\mathscr{E}$, commuting with each other and commuting with the Frobenius action.

Notation 2.3.7. For $\eta \in [\alpha,\beta] \subset (0,+\infty)$, we denote by F'_η the completion of $\mathrm{Frac}(K'\langle \alpha/t, t/\beta\rangle)$ with respect to the η-Gauss norm; this does not depend on the choices of α and β.

Example 2.3.8. For $\eta \in \mathbb{R}_{>0}$, the operator norms of ∂_{J^+} and spectral norms on F'_η are as follows.

$$|\partial_j|_{F'_\eta} = \begin{cases} \eta^{-1} & \text{for } j=0,\\ 1 & \text{for } j\in J;\end{cases} \qquad |\partial_j|_{F'_\eta,\mathrm{sp}} = \begin{cases} p^{-1/(p-1)}\eta^{-1} & \text{for } j=0,\\ p^{-1/(p-1)} & \text{for } j\in J.\end{cases}$$

Definition 2.3.9. Let $\mathscr{E}$ be a ∇-module over $\mathscr{R}^{\eta_0}_{K'}$. For $\eta \in [\eta_0, 1)$, we set $\mathscr{E}_\eta = \mathscr{E} \otimes_{\mathscr{R}^{\eta_0}_{K'}} F'_\eta$, which inherits differential operators ∂_{J^+}. Define the (*nonlogarithmic*) *generic radius* (*of convergence*) $\boldsymbol{T}(\mathscr{E},\eta)$ of $\mathscr{E}_\eta$ to be

$$\min\left\{\frac{p^{-1/(p-1)}}{|\partial_j|_{\mathscr{E}_\eta,\mathrm{sp}}};\ j\in J^+\right\}. \tag{2.3.10}$$

If $\mathscr{E}_{\eta,i}$ $(i=1,\dots,n)$ are the Jordan–Hölder factors of $\mathscr{E}_\eta$ as a ∇-module over F'_η, we define the (*nonlogarithmic*) *radius multiset* $\boldsymbol{S}(\mathscr{E},\eta)$ to be the set consisting of the generic radius of $\mathscr{E}_{\eta,i}$ with multiplicity $\dim_{F'_\eta}\mathscr{E}_{\eta,i}$ for each i.

We define the *logarithmic generic radius (of convergence)* $\boldsymbol{T}_{\log}(\mathscr{E},\eta)$ to be

$$\min\left\{\frac{p^{-1/(p-1)}\eta^{-1}}{|\partial_0|_{\mathscr{E}_\eta,\mathrm{sp}}},\ \frac{p^{-1/(p-1)}}{|\partial_j|_{\mathscr{E}_\eta,\mathrm{sp}}};\ j\in J\right\}. \tag{2.3.11}$$

Similarly, we define the *logarithmic radius multiset* $\boldsymbol{S}_{\log}(\mathscr{E},\eta)$ of $\mathscr{E}$.

Remark 2.3.12. We have $\boldsymbol{T}(\mathscr{E}, \eta) \leq \eta$; more generally, every element in $\boldsymbol{S}(\mathscr{E}, \eta)$ is less than or equal to η.

Remark 2.3.13. The logarithmic generic radius and logarithmic radius multiset are the same as the notions of the generic radius of convergence and radius multiset in [Kedlaya 2007].

Definition 2.3.14. For $j \in J^+$, we call ∂_j *dominant* for $\mathscr{E}_\eta$ if the minimum of $\boldsymbol{T}(\mathscr{E}, \eta)$ in (2.3.10) is achieved by the term involving the spectral norm of ∂_j. The term *log-dominant* is defined likewise, with reference to $\boldsymbol{T}_{\log}(\mathscr{E}, \eta)$ in (2.3.11).

Lemma 2.3.15. *For a (ϕ, ∇)-module $\mathscr{E}$ over $\mathscr{R}_{K'}^{\eta_0}$ and $j \in J^+$, there exists $\eta_0' \in (0, 1)$ such that one of the following two statements is true*:

- *For all $\eta \in [\eta_0', 1)$, ∂_j is (log-)dominant for $\mathscr{E}_\eta$.*
- *For all $\eta \in [\eta_0', 1)$, ∂_j is not (log-)dominant for $\mathscr{E}_\eta$.*

Proof. The logarithmic case is proved in [Kedlaya 2007, Lemma 2.7.5]. The proof for nonlogarithmic case is very similar. □

Definition 2.3.16. Keep the notation as in previous lemma. For $j \in J^+$, ∂_j is called *eventually (log-)dominant* for $\mathscr{E}$ if it is (log-)dominant for $\mathscr{E}_\eta$ for $\eta \to 1^-$.

Lemma 2.3.17. *Keep the notation as in Lemma 2.3.15. Assume that ∂_0 is* not *eventually dominant and ∂_j is. Consider the rotation $g^* : B_j \mapsto B_j + S$, $B_{J \setminus j} \mapsto B_{J \setminus j}$, and $S \mapsto S$ given by Proposition 2.1.8. Then $\partial_0 = \partial / \partial S$ is eventually dominant in $g^*\mathscr{E}$.*

Proof. Use the fact that the action of ∂_0 on $g^*\mathscr{E}$ is the pull-back of the action of $\partial_0 + \partial_j$ on $\mathscr{E}$. For details, see the proof of [Kedlaya 2007, Lemma 2.7.9]. □

Remark 2.3.18. The rotation g in the lemma corresponds to changing the isomorphism (2.1.6) so that $\bar{b}_j$ is sent to $b_j + s$ instead; such an isomorphism can be obtained by Corollary 2.1.11. In particular, if $\mathscr{E}_\rho$ comes from a p-adic representation ρ of finite local monodromy by Constructions 2.2.8 and 2.2.15, $g^*\mathscr{E}_\rho$ is the differential module associated with the same ρ using the aforementioned alternative isomorphism in place of (2.1.6).

Proposition 2.3.19. *The functions*

$$f(r) = \log \boldsymbol{T}(\mathscr{E}, e^{-r}) \quad \textit{and} \quad f_{\log}(r) = \log \boldsymbol{T}_{\log}(\mathscr{E}, e^{-r})$$

on $(0, -\log \eta_0]$ are piecewise linear, concave functions with slopes in $\dfrac{1}{(\operatorname{rank} \mathscr{E})!}\mathbb{Z}$. They are linear in a neighborhood of 0.

Proof. The logarithmic case is proved in [Kedlaya 2007, §2.5]. For the nonlogarithmic case, the only difference is a factor η^{-1} in the spectral norm of ∂_0, which gives an extra linear term r. □

Definition 2.3.20. As a consequence of the previous proposition, there exists $b_{\mathrm{dif}}(\mathscr{E}) \in \mathbb{Q}_{\geq 0}$ and $\eta_0 \in (0, 1)$ such that $\boldsymbol{T}(\mathscr{E}, \eta) = \eta^{b_{\mathrm{dif}}(\mathscr{E})}$ for all $\eta \in [\eta_0, 1)$. This $b_{\mathrm{dif}}(\mathscr{E})$ is called the (*nonlogarithmic*) *differential ramification break* of $\mathscr{E}$. We say that $\mathscr{E}$ has *uniform slope b* if the radius multiset $\boldsymbol{S}(\mathscr{E}, \eta)$ consists only of η^b when $\eta \to 1$. The notions of *logarithmic differential ramification break* $b_{\mathrm{dif,log}}(\mathscr{E})$ and *uniform log-slope b* are defined likewise, with reference to $\mathbf{T}_{\log}(\mathscr{E}, \eta)$ and $\mathbf{S}_{\log}(\mathscr{E}, \eta)$.

The ramification breaks give rise to the break decomposition.

Theorem 2.3.21. *Let $\mathscr{E}$ be a (ϕ, ∇)-module over $\mathscr{R}_{K'}^{\eta_0}$, for some $\eta_0 \in (0, 1)$. Then after making η_0 sufficiently close to 1^-, there exists a* unique *decomposition of (ϕ, ∇)-modules $\mathscr{E} = \bigoplus_{b \in \mathbb{Q}_{\geq 1}} \mathscr{E}_b$ (resp. $\mathscr{E} = \bigoplus_{b \in \mathbb{Q}_{\geq 0}} \mathscr{E}_{b,\log}$) over $\mathscr{R}_{K'}^{\eta_0}$, where each of $\mathscr{E}_b$ (resp. $\mathscr{E}_{b,\log}$) has uniform slope (resp. log-slope) b.*

Proof. Since the differential operators act trivially on $\mathscr{O}$ and commute with ϕ, It suffices to obtain the decomposition of $\mathscr{E}$ as a ∇-module over $A_{K'}^1[\eta_0, 1)$; the uniqueness of the decomposition of $\mathscr{E}$ follows from the uniqueness of that over F'_η for $\eta \in [\eta_0, 1)$. The logarithmic part of this theorem is proved in [Kedlaya 2007, Theorem 2.7.2]. We will give the proof of the nonlogarithmic decomposition by applying several lemmas from the same paper.

We need to show that if $\mathscr{E}$ does not have uniform slope at least 1, then $\mathscr{E}$ is decomposable when η_0 is taken sufficiently close to 1. (See Remark 2.3.12 for the reason for having 1 instead of 0.) If ∂_0 is eventually dominant, the decomposition theorem of Christol and Mebkhout [Kedlaya 2007, Lemma 2.7.3] gives the decomposition. If ∂_0 is not eventually dominant, Lemma 2.3.15 implies that ∂_j is eventually dominant for some $j \in J$. By Lemma 2.3.17, ∂_0 is eventually dominant for $g^*\mathscr{E}$. Applying the decomposition theorem [Kedlaya 2007, Lemma 2.7.3] to $g^*\mathscr{E}$ and pulling back the decomposition along g^{-1}, we obtain a nontrivial decomposition of $\mathscr{E}$ on $\mathscr{R}_{K'}^{\eta_0}$ for some $\eta_0 \in (0, 1)$. □

Proposition 2.3.22. *In Theorem 2.3.21, if the (ϕ, ∇)-module $\mathscr{E}_\rho$ is associated with a p-adic representation ρ of finite local monodromy, then the decomposition of (ϕ, ∇)-modules induces a direct sum decomposition of the representation ρ so that each direct summand of $\mathscr{E}_\rho$ is the differential module associated with a direct summand of ρ.*

Proof. By slope filtration [Kedlaya 2007, Theorem 3.4.6], the Frobenius action on each direct summand of $\mathscr{E}$ is of unit-root; the decomposition of the representation follows by [Kedlaya 2007, Proposition 3.4.4]. □

Definition 2.3.23. Let $\rho : G_k \to \mathrm{GL}(V_\rho)$ be a p-adic representation with finite local monodromy. Let $\mathscr{E}$ be the differential module corresponding to $V_\rho / V_\rho^{I_k}$ by Constructions 2.2.8 and 2.2.15, where $V_\rho^{I_k}$ is the unramified piece of V_ρ consisting of the elements in V_ρ which are fixed by I_k. By Theorem 2.3.21 above, there exists a multiset $\{a_1, \dots, a_d\}$ such that for all η sufficiently close to 1, $\boldsymbol{S}(\mathscr{E}, \eta) = \{\eta^{a_1}, \dots, \eta^{a_d}\}$. Define the *differential Artin conductor* of $\mathscr{E}$ (or ρ) by

$$\mathrm{Art}_{\mathrm{dif}}(\mathscr{E}) = \mathrm{Art}_{\mathrm{dif}}(\rho) = a_1 + \cdots + a_d.$$

The *differential Swan conductor* of $\mathscr{E}$ (or ρ), denoted by $\mathrm{Swan}_{\mathrm{dif}}(\mathscr{E})$ or $\mathrm{Swan}_{\mathrm{dif}}(\rho)$, is defined similarly, by adding the subscript log everywhere.

Remark 2.3.24. In this definition, we split off the unramified part, because it has both conductors 0. We need to do so because the convergence radius multiset cannot distinguish between the unramified and the tame parts, which give different contributions to the Artin conductor. This does not matter for Swan conductors, and we may define the Swan conductor without first taking out the unramified piece.

Remark 2.3.25. By [Kedlaya 2007, Proposition 2.6.6], the definition of differential Swan conductors does not depend on the choice of a uniformizer s and a p-basis $\{b_1, \dots, b_m, s\}$. We are also free to remove Hypothesis 2.3.2 and define the differential Swan conductors for arbitrary complete discretely valued fields of equal characteristic p [Kedlaya 2007, Corollary 3.5.7]. A similar statement holds for differential Artin conductors; the proof is the same as for Swan conductors.

2.4. *Basic properties.* We do not impose any hypothesis on k.

Theorem 2.4.1. *Differential conductors satisfy the following properties*:

(0) *When the residue field κ_k is perfect, the differential Artin and Swan conductors are the same as the classical ones defined in* [Serre 1979].

(1) *For any representation ρ of finite local monodromy*, $\mathrm{Swan}_{\mathrm{dif}}(\rho) \in \mathbb{Z}_{\geq 0}$ *and* $\mathrm{Art}_{\mathrm{dif}}(\rho) \in \mathbb{Z}_{\geq 0}$.

(2) *Let k'/k be a tamely ramified extension of ramification degree e'. Let ρ be a representation of G_k of finite local monodromy and let ρ' denote the restriction of ρ to $G_{k'}$. Then* $\mathrm{Swan}_{\mathrm{dif}}(\rho') = e' \cdot \mathrm{Swan}_{\mathrm{dif}}(\rho)$. *If $e' = 1$, i.e., k'/k is unramified, then* $\mathrm{Art}_{\mathrm{dif}}(\rho') = \mathrm{Art}_{\mathrm{dif}}(\rho)$.

(3) *Let ρ be a faithful p-adic representation of the Galois group of a Galois extension l/k. If l/k is tamely ramified and not unramified, then $b_{\mathrm{dif}}(\rho) = 1$ and $b_{\mathrm{dif},\log}(\rho) = 0$. If l/k is unramified, $b_{\mathrm{dif}}(\rho) = b_{\mathrm{dif},\log}(\rho) = 0$.*

(4) *Put $G_k^0 = G_k$ and $G_k^a = I_k$ for $a \in (0, 1]$. For $a > 1$, let R_a be the set of finite image representations ρ with differential ramification breaks less than a. Define $G_k^a = \bigcap_{\rho \in R_a} (I_k \cap \ker \rho)$ and write G_k^{a+} for the closure of $\bigcup_{b>a} G_k^b$. This defines a differential filtration on G_k such that for all finite image representations ρ, $\rho(G_k^a)$ is trivial if and only if $\rho \in R_a$.*

Similarly, put $G^0_{k,\log} = G_k$. *For* $a > 0$, *let* $R_{a,\log}$ *be the set of finite image representations* ρ *with logarithmic differential ramification breaks less than* a. *Define* $G^a_{k,\log} = \bigcap_{\rho \in R_{a,\log}} (I_k \cap \ker \rho)$ *and write* $G^{a+}_{k,\log}$ *for the closure of* $\bigcup_{b>a} G^b_{k,\log}$. *This defines a differential logarithmic filtration on* G_k *such that for all finite image representations* ρ, $\rho(G^a_{k,\log})$ *is trivial if and only if* $\rho \in R_{a,\log}$.

For $a > 0$, *the group* G^a_k / G^{a+}_k *is abelian and killed by* p *(and trivial if* $a \notin \mathbb{Q}$*).*

For $a > 1$, *the group* $G^a_{k,\log} / G^{a+}_{k,\log}$ *is abelian and killed by* p *(and trivial if* $a \notin \mathbb{Q}$*).*

Proof. For (0), see [Kedlaya 2005, Theorem 5.23]. For the rest of the statements, the proof for Swan conductors can be found in [Kedlaya 2007, §3.5]; we will only prove the corresponding properties for differential Artin conductors. As in the proof for differential Swan conductors, we may first reduce to the case where Hypothesis 2.3.2 holds.

(1) We can follow the proof of [Kedlaya 2007, Theorem 2.8.2], because of the decomposition Theorem 2.3.21. An alternative proof is to apply Lemma 2.3.17, and reduce to the case where ∂_0 is dominant (see also Remark 2.3.18); then one can forget about $\partial_1, \dots, \partial_m$ and hence reduce to the perfect residue field case, which is statement (0) of the theorem.

(2) Since an unramified extension l/k only changes the field K but not the uniformizer s, we can use the same s as the uniformizer of l. The corresponding differential module $\mathscr{E}_{\rho'}$ of ρ' is just a simple extension of scalars. Since the calculation of spectral norms does not depend on the base field (see Remark 2.3.5), we compute the same result on spectral norms and hence have the same Artin conductor.

(3) is an immediate consequence of the Swan case. Attention: differential ramification breaks cannot distinguish unramified extensions from tamely ramified ones. (See also Remark 2.3.24.)

(4) The proof for the nonlogarithmic differential filtration is much simpler than the logarithmic case because of the different normalization in Definition 2.3.9. By virtue of the proof of [Kedlaya 2007, Theorem 3.5.13], it suffices to show that we can rotate so that ∂_0 becomes dominant; this is the content of Lemma 2.3.17. □

Remark 2.4.2. The invariance of the differential conductors under unramified base changes enables us to assume that κ_0 is algebraically closed. This justifies the assumption we made in Notation 2.2.2.

3. The thickening technique

In this section, we introduce a thickening technique. Loosely speaking, it consists in constructing what can be thought of as a tubular neighborhood of the diagonal embedding of $A^1_K[\eta_0, 1)$ into $A^1_K[\eta_0, 1) \times_{K_0} A^1_K[\eta_0, 1)$, but note that the latter rigid space is not really well-defined.

We start with a geometric interpretation of this construction and then move on to the abstract definition of the thickening space.

We keep Hypothesis 2.3.2 throughout this section.

Notation 3.0.1. For $\alpha \in (0, +\infty)$, denote by $A^m_K[0, \alpha]$ and $A^m_K[0, \alpha)$ the closed and open polydiscs with radius α and center at the origin. Let $K\langle u_1/\alpha, \dots, u_m/\alpha\rangle$ denote the ring of analytic functions on the disc $A^m_K[0, \alpha]$.

Later, we will see many homomorphisms between rings of functions on K-rigid spaces, which are only K_0-linear. It is unfair to say that they induce morphisms of rigid spaces; however, we prefer to keep some geometric flavor of the whole construction. On the other hand, these rigid spaces are all quasi-Stein or affinoid; knowing the ring of analytic functions is equivalent to knowing the rigid spaces.

Notation 3.0.2. For a continuous homomorphism $f^* : A \to B$ between affinoid or Fréchet algebras (not necessarily respecting the ground field K), we write formally $f : \mathrm{Max}(B) \to \mathrm{Max}(A)$, as the *geometric incarnation* of the homomorphism. Pullbacks along maps and Cartesian diagrams are thought of as (completed) tensor products. (In fact, in all cases we encounter, we do not need to take the completion for the tensor products.) In short, whenever such a map is given, strictly speaking, we should view it as a continuous ring homomorphism.

3.1. ***Geometric thickening.*** In this subsection, we describe the thickening technique when the residue field κ_k can be realized as the field of rational functions on a smooth κ_0-variety. The purpose of this subsection is solely to provide some geometric intuition for the thickening construction in the next subsection; the content in this subsection will not be used in the rest of this paper.

Hypothesis 3.1.1. *Only* in this subsection, we assume that the field κ_k is a finite separable extension of $\kappa_0(\bar b_1, \dots, \bar b_m)$.

Construction 3.1.2. Let $\bar X$ be a smooth variety over κ_0 whose field of rational functions is κ_k; such an $\bar X$ exists because we may realize it as an affine scheme étale over $\operatorname{Spec} \kappa_0[\bar b_1, \dots, \bar b_m]$ which induces the extension $\kappa_k/\kappa_0(\bar b_1, \dots, \bar b_m)$. We may further shrink $\bar X$ so that it is the special fiber of an affine smooth formal scheme $\mathfrak{X}$ over $\mathcal{O}_{K_0}$ of topological finite type, i.e., $\mathfrak{X} \times_{\operatorname{Spf} \mathcal{O}_{K_0}} \operatorname{Spec} \kappa_0 = \bar X$. We may further shrink $\mathfrak{X}$ and $\bar X$ so that we have lifts $B_1, \dots, B_m$ of $\bar b_1, \dots, \bar b_m$ on $\mathfrak{X}$ and $dB_1, \dots, dB_m$ form a basis of the sheaf of relative differentials $\Omega^1_{\mathfrak{X}/\mathcal{O}_{K_0}}$. We use $\boldsymbol{X}$ to denote the "generic fiber" of $\mathfrak{X}$ as a rigid space over $\mathrm{Sp}(K_0)$, in the sense of Raynaud; it is affinoid.

Consider the commutative diagram

$$\begin{tikzcd}
\overline{X} \arrow[r] \arrow[d, hook] & \mathfrak{X} \arrow[d, hook] & \boldsymbol{X} \arrow[l] \arrow[d, hook] \\
P = \overline{X} \times_{\kappa_0} \mathbb{A}^1_{\kappa_0} \arrow[r] \arrow[d] & \mathcal{P} = \mathfrak{X} \times_{\mathrm{Spf}\, \mathbb{O}_{K_0}} \widehat{\mathbb{A}}^1_{\mathbb{O}_{K_0}} \arrow[d] & \boldsymbol{P} = \boldsymbol{X} \times_{K_0} A^1_{K_0}[0,1] \arrow[l] \arrow[d] \\
\operatorname{Spec} \kappa_0 \arrow[r] & \operatorname{Spf} \mathbb{O}_{K_0} & \operatorname{Sp}(K_0). \arrow[l]
\end{tikzcd}$$

where the vertical arrows from the first row to the second are all embeddings of zero sections and the coordinates of $\mathbb{A}^1_{\kappa_0}$ and $\widehat{\mathbb{A}}^1_{\mathbb{O}_{k_0}}$ are denoted by s and S, respectively.

The tube of $\overline{X}$ in $\boldsymbol{P}$, denoted by $]\overline{X}[_{\mathcal{P}}$, is isomorphic to $\boldsymbol{X} \times A^1_{K_0}[0, 1)$. Let $\mathbb{O}_{\boldsymbol{X}}$ be the ring of rigid analytic functions on $\boldsymbol{X}$; then K is exactly the p-adic completion of $\mathrm{Frac}\, \mathbb{O}_{\boldsymbol{X}}$. If we base-change the tube $]\overline{X}[_{\mathcal{P}}$ from $\boldsymbol{X}$ over to K, we get $A^1_K[0, 1)$. We are interested in the annulus $A^1_K[\eta_0, 1)$ for some $\eta_0 \in (0, 1)$, which can be obtained by base-changing $\boldsymbol{X} \times A^1_{K_0}[\eta_0, 1)$ from $\boldsymbol{X}$ to K.

Now, we consider the thickening space of this annulus $A^1_K[\eta_0, 1)$.

Construction 3.1.3. Consider the commutative diagram

$$\begin{tikzcd}
\overline{X} \arrow[r, hook] \arrow[dr] & P \arrow[r, hook, "\Delta_P"] \arrow[d] & P \times_{\kappa_0} P \arrow[r, hook] \arrow[dl] & \mathcal{P} \times_{\mathbb{O}_{K_0}} \mathcal{P} \arrow[d] & \boldsymbol{P} \times_{K_0} \boldsymbol{P} \arrow[l] \arrow[d] \\
& \operatorname{Spec} \kappa_0 \arrow[rr] & & \operatorname{Spf} \mathbb{O}_{K_0} & \operatorname{Sp}(K_0) \arrow[l]
\end{tikzcd}$$

where we use $\mathrm{pr}_i : \mathcal{P} \times_{\mathbb{O}_{K_0}} \mathcal{P} \to \mathcal{P}$ to denote the projection to the i-th factor for $i = 1, 2$. Then $\mathcal{P} \times_{\mathbb{O}_{K_0}} \mathcal{P}$ has a set of local parameters given by $B_1 = \mathrm{pr}_1^*(B_1), \dots, B_m = \mathrm{pr}_1^*(B_m)$, $S = \mathrm{pr}_1^*(S)$, $B'_1 = \mathrm{pr}_2^*(B_1), \dots, B'_m = \mathrm{pr}_2^*(B_m)$, and $S' = \mathrm{pr}_2^*(S)$. By Berthelot's fibration theorem [1996, théorème 1.3.2], we have an isomorphism

$$]\overline{X}[_{\mathcal{P} \times_{\mathbb{O}_{K_0}} \mathcal{P}} \simeq]\overline{X}[_{\mathcal{P}} \times_{K_0} A^{m+1}_{K_0}[0, 1),$$

where the factor $]\overline{X}[_{\mathcal{P}}$ respects the projection pr_1 and the coordinates for the open polydisc on the right-hand side are given by $\delta_0 = S - S'$, $\delta_1 = B_1 - B'_1, \dots,$ $\delta_m = B_m - B'_m$. The geometric thickening space is the subspace of $]\overline{X}[_{\mathcal{P} \times_{\mathbb{O}_{K_0}} \mathcal{P}}$ where $|\delta_0| = |S - S'| < |S|$, or, more precisely,

$$\boldsymbol{X} \times_{K_0} \{(S, \delta_0) \in A^2_{K_0}[0, 1) \mid |\delta_0| < |S|\} \times_{K_0} A^m_{K_0}[0, 1).$$

Thus, the *thickening space*, denoted by $TS^{\geq \eta_0}_k$, of $A^1_K[\eta_0, 1)$ is the space obtained by base-changing

$$\boldsymbol{X} \times_{K_0} \{(S, \delta_0) \in A^2_{K_0}[0, 1) \mid |S| \geq \eta_0, |\delta_0| < |S|\} \times_{K_0} A^m_{K_0}[0, 1).$$

from $\boldsymbol{X}$ to K.

The projection $\mathrm{pr}_1 : \boldsymbol{P} \times_{K_0} \boldsymbol{P} \to \boldsymbol{P}$ gives a K-morphism of rigid spaces

$$\pi : TS_k^{\geq \eta_0} \to A_K^1[\eta_0, 1);$$

the projection $\mathrm{pr}_2 : \boldsymbol{P} \times_{K_0} \boldsymbol{P} \to \boldsymbol{P}$ gives a K_0-morphism of rigid spaces

$$\tilde{\pi} : TS_k^{\geq \eta_0} \to A_K^1[\eta_0, 1).$$

The morphism $\tilde{\pi}$ does not respect the K-rigid space structure; one should always think of $\tilde{\pi}$ as the ring homomorphism between the corresponding ring of analytic functions. In our earlier notation, this is just the geometric incarnation of the map on the ring of global sections.

3.2. ***General thickening construction.*** We now introduce thickening spaces and study basic properties of differential modules over them.

We keep Hypothesis 2.3.2 in this subsection. However, Hypothesis 3.1.1 is no longer in force from now on.

Definition 3.2.1. For $\eta \in (0, 1)$, we write $Z_k^\eta = A_K^1[\eta, \eta]$. For $a \in \mathbb{Q}_{>1}$ and $\eta_0 \in (0, 1)$, we define the *thickening space of* $A_K^1[\eta_0, 1)$ *and level* a to be the rigid space over K of the form

$$TS_k^{a, \geq \eta_0} = \bigl\{(S, \delta_{J^+}) \in A_K^{m+2}[0, 1) \bigm| |S| \geq \eta_0;\ |\delta_j| \leq |S|^a \text{ for } j \in J^+\bigr\}. \tag{3.2.2}$$

For $\eta \in [\eta_0, 1)$, we put

$$TS_k^{a,\eta} = A_K^1[\eta, \eta] \times_K A_K^{m+1}[0, \eta^a].$$

Similarly, for $a \in \mathbb{Q}_{>0}$ and $\eta_0 \in (0, 1)$, we define the *log-thickening space of* $A_K^1[\eta_0, 1)$ *and level* a to be

$$TS_{k,\log}^{a, \geq \eta_0} = \bigl\{(S, \delta_{J^+}) \in A_K^{m+2}[0, 1) \bigm| |S| \geq \eta_0;\ |\delta_0| \leq |S|^{a+1};\ |\delta_j| \leq |S|^a \text{ for } j \in J\bigr\}. \tag{3.2.3}$$

For $\eta \in [\eta_0, 1)$, we set

$$TS_{k,\log}^{a,\eta} = A_K^1[\eta, \eta] \times_K A_K^1[0, \eta^{a+1}] \times_K A_K^m[0, \eta^a].$$

We denote by $\mathcal{O}_{TS_k^{a,\geq\eta_0}}$, $\mathcal{O}_{TS_k^{a,\eta}}$, $\mathcal{O}_{TS_{k,\log}^{a,\geq\eta_0}}$, and $\mathcal{O}_{TS_{k,\log}^{a,\eta}}$ the rings of analytic functions on these spaces.

Let $|\cdot|_{Z_k^\eta}$ denote the η-Gauss norm on Z_k^η. For $a \in \mathbb{Q}_{>1}$, let $|\cdot|_{TS_k^{a,\eta}}$ denote the Gauss norm on $TS_k^{a,\eta}$; for $a > 0$, let $|\cdot|_{TS_{k,\log}^{a,\eta}}$ denote the Gauss norm on $TS_{k,\log}^{a,\eta}$.

The union of all $TS_k^{a,\geq\eta_0}$ is the $TS_k^{\geq\eta_0}$ we discussed in Construction 3.1.3.

Caution 3.2.4. One may want to write $TS_k^{a,\geq\eta_0} = \bigcup_{\eta\in[\eta_0,1)} A_K^1[\eta, 1) \times_K A_K^{m+1}[0, \eta^a]$ for simplicity, as in the introduction. However, this will not define the same rigid space as in (3.2.2), because the union does *not* give an admissible cover of $TS_k^{a,\geq\eta_0}$.

A similar expression for log-thickening space is not valid either. Nevertheless, it might be helpful to think the space and picture the geometry this way.

On the other hand, it is true that an element of $K[\![S, \delta_0, \dots, \delta_m]\!]$ lies in $\mathcal{O}_{TS_k^{a,\geq\eta_0}}$ (resp. $\mathcal{O}_{TS_{k,\log}^{a,\geq\eta_0}}$) if and only if it has bounded norms for all $|\cdot|_{TS_k^{a,\eta}}$ (resp. $|\cdot|_{TS_{k,\log}^{a,\eta}}$) for all $\eta \in [\eta_0, 1)$.

Remark 3.2.5. We need $a \in \mathbb{Q}$ in Definition 3.2.1 to make sure that (3.2.2) and (3.2.3) actually define a (Berkovich) rigid analytic space. For individual spaces $TS_k^{a,\eta}$ and $TS_{k,\log}^{a,\eta}$, one can allow $a \in \mathbb{R}$.

Notation 3.2.6. For $a \in \mathbb{Q}_{>1}$ and $\eta_0 \in (0, 1)$, denote by $\Delta : Z_k^{\geq\eta_0} \hookrightarrow TS_k^{a,\geq\eta_0}$ the natural embedding of $Z_k^{\geq\eta_0}$ into the locus where $\delta_j = 0$ for $j \in J^+$. Also, we have the naïve projection $\pi : TS_k^{a,\geq\eta_0} \to Z_k^{\geq\eta_0}$ to the first factor.

For $a \in \mathbb{Q}_{>0}$, we define likewise $\Delta : Z_k^{\geq\eta_0} \hookrightarrow TS_{k,\log}^{a,\geq\eta_0}$ and $\pi : TS_{k,\log}^{a,\geq\eta_0} \to Z_k^{\geq\eta_0}$.

All these morphisms remain compatible under changes in a and η_0, and under the replacement of $\geq \eta_0$ by η for some $\eta \in [\eta_0, 1)$.

To simplify notation, for a and η_0 as above, we identify $\mathcal{O}_{Z_k^{\geq\eta_0}}$ with a subring of $\mathcal{O}_{TS_k^{a,\geq\eta_0}}$ and of $\mathcal{O}_{TS_{k,\log}^{a,\geq\eta_0}}$ via π^*, and likewise for η instead of $\geq \eta_0$. Note that π^* is an isometry; hence the identification will not change any calculation on norms.

Corollary 2.1.10 has this immediate consequence:

Proposition 3.2.7. *There is a unique continuous $\mathcal{O}_{K_0}$-homomorphism*

$$\tilde{\pi}^* : \mathcal{O}_K[\![S]\!] \to \mathcal{O}_K[\![S, \delta_{J^+}]\!]$$

such that $\tilde{\pi}^*(S) = S + \delta_0$ and $\tilde{\pi}^*(B_j) = B_j + \delta_j$ for all $j \in J$. Moreover, for $g \in \mathcal{O}_K$, $\tilde{\pi}^*(g) - g \in (\delta_1, \dots, \delta_m)(g)\mathcal{O}_K[\![\delta_1, \dots, \delta_m]\!]$.

Theorem 3.2.8. *For $a \in \mathbb{Q}_{>1}$ (resp. $a \in \mathbb{Q}_{>0}$) and $\eta_0 \in (0, 1)$, the homomorphism $\tilde{\pi}^*$ induces a K_0-homomorphism $\tilde{\pi}^* : \mathcal{O}_{Z_k^{\geq\eta_0}} = \mathcal{R}_K^{\eta_0} \to \mathcal{O}_{TS_k^{a,\geq\eta_0}}$ (resp. $\tilde{\pi}^* : \mathcal{O}_{Z_k^{\geq\eta_0}} \to \mathcal{O}_{TS_{k,\log}^{a,\geq\eta_0}}$) such that $\Delta^* \circ \tilde{\pi}^* = \mathrm{id}$; the same if replacing $\geq \eta_0$ by η for some $\eta \in [\eta_0, 1)$.*

For any $g \in \mathcal{O}_{Z_k^\eta}$ and for $a > 1$ (resp. $a > 0$),

$$|\tilde{\pi}^*(g) - g|_{TS_k^{a,\eta}} \leq \eta^{a-1} \cdot |g|_{Z_k^\eta} \quad (\textit{resp. } |\tilde{\pi}^*(g) - g|_{TS_{k,\log}^{a,\eta}} \leq \eta^a \cdot |g|_{Z_k^\eta}). \tag{3.2.9}$$

In particular, $|\tilde{\pi}^*(g)|_{TS_k^{a,\eta}} = |\tilde{\pi}^*(g)|_{TS_{k,\log}^{a,\eta}} = |g|_{Z_k^\eta}$. Moreover, we have the following bound for $TS_k^{a,\eta}$: if $g \in \mathcal{O}_{Z_k^\eta} \cap \mathcal{O}_K[\![S]\!]$, then

$$|\tilde{\pi}^*(g) - g|_{TS_k^{a,\eta}} \leq \eta^a. \tag{3.2.10}$$

Proof. We need only to establish the bound on the norms. Take

$$g = \sum_{i \in \mathbb{Z}} a_i S^i \in K[\![S]\!]$$

such that $|g|_{Z_k^\eta} < +\infty$. We have

$$\begin{aligned}\tilde{\pi}^*(g) - g &= \sum_{i\in\mathbb{Z}} \big(\tilde{\pi}^*(a_i)(S+\delta_0)^i - a_i S^i\big) \\ &= \sum_{i\in\mathbb{Z}} \big((\tilde{\pi}^*(a_i) - a_i)(S+\delta_0)^i + a_i((S+\delta_0)^i - S^i)\big).\end{aligned} \tag{3.2.11}$$

Since $\tilde{\pi}^*(a_i) - a_i \in (\delta_1, \dots, \delta_m)(a_i)\mathcal{O}_K[\![\delta_1, \dots, \delta_m]\!]$, we have

$$|\tilde{\pi}(a_i) - a_i|_{TS_k^{a,\eta}} \leq |a_i|\eta^a, \quad |\tilde{\pi}(a_i) - a_i|_{TS_{k,\log}^{a,\eta}} \leq |a_i|\eta^a. \tag{3.2.12}$$

We can bound $(S+\delta_0)^i - S^i$ by

$$|(S+\delta_0)^i - S^i|_{TS_k^{a,\eta}} \leq \eta^{a+i-1}, \quad |(S+\delta_0)^i - S^i|_{TS_{k,\log}^{a,\eta}} \leq \eta^{a+i}. \tag{3.2.13}$$

Plugging the estimates (3.2.12) and (3.2.13) into (3.2.11), we obtain (3.2.9). When $g \in \mathcal{O}_K[\![S]\!]$, (3.2.13) always gives $|(S+\delta_0)^i - S^i|_{TS_k^{a,\eta}} \leq \eta^a$ for $i \geq 0$ (when $i = 0$, we have zero). Equation (3.2.10) follows.

Finally, the equalities $|\tilde{\pi}^*(g)|_{TS_k^{a,\eta}} = |\tilde{\pi}^*(g)|_{TS_{k,\log}^{a,\eta}} = |g|_{Z_k^\eta}$ ensure that we have well-defined continuous homomorphisms $\tilde{\pi}^* : \mathcal{O}_{Z_k^{\geq\eta_0}} \to \mathcal{O}_{TS_k^{a,\geq\eta_0}}$ or $\mathcal{O}_{TS_{k,\log}^{a,\geq\eta_0}}$. □

Notation 3.2.14. We use $\tilde{\pi} : TS_k^{a,\geq\eta_0} \to Z_k^{\geq\eta_0}$ and $\tilde{\pi} : TS_{k,\log}^{a,\geq\eta_0} \to Z_k^{\geq\eta_0}$ to denote the geometric incarnations of the homomorphisms $\tilde{\pi}^*$ constructed in Theorem 3.2.8; the same for η in place of $\geq \eta_0$ when $\eta \in [\eta_0, 1)$. To emphasize again, whenever we refer to $\tilde{\pi}$, strictly speaking, we are referring to the corresponding homomorphism $\tilde{\pi}^*$ on rings.

Remark 3.2.15. For $a > 0$, one can factor the map $\tilde{\pi}$ for a nonlog thickening space as $TS_k^{a+1,\geq\eta_0} \to TS_{k,\log}^{a,\geq\eta_0} \xrightarrow{\tilde{\pi}} Z_k^{\geq\eta_0}$, where the second map is the $\tilde{\pi}$ for the log-thickening space. Again, this should be thought of as factorization for ring homomorphisms.

Notation 3.2.16. For a ∇-module $(\mathcal{E}, \nabla_{\mathcal{E}})$ over $Z_k^{\geq\eta_0}$ relative to K_0, we call $\tilde{\pi}^*\mathcal{E}$ the *thickened differential module* of $\mathcal{E}$, denoted by $\mathcal{F}$. We view $\mathcal{F}$ as a differential module over $TS_k^{a,\geq\eta_0}$ or $TS_{k,\log}^{a,\geq\eta_0}$ relative to $Z_k^{\geq\eta_0}$, with respect to the differential operators $\partial/\partial\delta_0, \dots, \partial/\partial\delta_m$. In precise terms, the connection is given by

$$\begin{aligned}\mathcal{F} = \mathcal{E} \otimes_{\mathcal{R}_K^{\eta_0}, \tilde{\pi}^*} \mathcal{O}_{TS_k^{a,\geq\eta_0}} &\xrightarrow{\nabla_{\mathcal{E}}} \mathcal{E} \otimes \Omega^1_{Z_k^{\geq\eta_0}/K_0} \otimes_{\mathcal{R}_K^{\eta_0}, \tilde{\pi}^*} \mathcal{O}_{TS_k^{a,\geq\eta_0}} \\ &\longrightarrow \mathcal{E} \otimes_{\mathcal{R}_K^{\eta_0}, \tilde{\pi}^*} \Omega^1_{TS_k^{a,\geq\eta_0}/K_0} \\ &\longrightarrow \mathcal{E} \otimes_{\mathcal{R}_K^{\eta_0}, \tilde{\pi}^*} \Omega^1_{TS_k^{a,\geq\eta_0}/Z_k^{\geq\eta_0}}\end{aligned}$$

in the nonlog case. The log case is obtained similarly, with subscript log at the appropriate places. This construction is compatible for different a's and η_0's.

We next link the spectral norms on $\mathscr{E}$ and the spectral norms on its thickening $\mathscr{F}$.

Proposition 3.2.17. *Let $\eta \in [\eta_0, 1)$. The spectral norms of ∂_{J^+} on $\mathscr{E}_\eta$ over Z_k^η and the spectral norms of $\partial/\partial\delta_{J^+}$ on $\mathscr{F}_{a,\eta} = \mathscr{F} \otimes \operatorname{Frac}(\mathbb{O}_{TS_k^{a,\eta}})^\wedge$ and $\mathscr{F}_{a,\eta,\log} = \mathscr{F} \otimes \operatorname{Frac}(\mathbb{O}_{TS_{k,\log}^{a,\eta}})^\wedge$ are related as follows*:

$$|\partial/\partial\delta_j|_{\mathscr{F}_{a,\eta},\mathrm{sp}} = \max\{|\partial_j|_{\mathscr{E}_\eta,\mathrm{sp}}, p^{-1/(p-1)}\eta^{-a}\} \qquad \textit{for } j \in J^+,$$
$$|\partial/\partial\delta_0|_{\mathscr{F}_{a,\eta,\log},\mathrm{sp}} = \max\{|\partial_0|_{\mathscr{E}_\eta,\mathrm{sp}}, p^{-1/(p-1)}\eta^{-a-1}\},$$
$$|\partial/\partial\delta_j|_{\mathscr{F}_{a,\eta,\log},\mathrm{sp}} = \max\{|\partial_j|_{\mathscr{E}_\eta,\mathrm{sp}}, p^{-1/(p-1)}\eta^{-a}\} \qquad \textit{for } j \in J.$$

Proof. Note that $\tilde{\pi}^*(dB_j) = dB_j + d\delta_j$ for $j \in J$ and $\tilde{\pi}^*(dS) = dS + d\delta_0$. The actions of $\partial/\partial\delta_j$, $j \in J$ (resp. $j = 0$), on $\mathscr{F}_{a,\eta}$ and $\mathscr{F}_{a,\eta,\log}$ are the same as the action of $\partial/\partial B_j$ (resp. $\partial/\partial S$) on $\mathscr{E}_\eta$. More precisely, we have $\tilde{\pi}^*(\partial/\partial S(x)) = \partial/\partial\delta_0(\tilde{\pi}^*(x))$ and $\tilde{\pi}^*(\partial/\partial B_j(x)) = \partial/\partial\delta_j(\tilde{\pi}^*(x))$ for any $j \in J$ and $x \in \mathscr{R}_K^\eta$ *or* $\mathscr{E}_\eta$.

The statement follows, because that δ_J are transcendental over $\mathbb{O}_{Z_k^\eta}$ and the homomorphism $\tilde{\pi}^*$ is isometric (by Theorem 3.2.8). □

3.3. ***Good generators of the extension.*** We now show that when l/k is totally and wildly ramified, we can choose nice generators of $\mathbb{O}_l$ as an $\mathbb{O}_k$-algebra, so that the corresponding extension on the Robba rings takes a simple form. Then we give a more explicit construction of the differential module associated with a p-adic representation.

We keep Hypothesis 2.3.2 for this subsection.

Hypothesis 3.3.1. For the rest of this section, we assume that l/k is a finite totally and wildly ramified Galois extension.

Remark 3.3.2. This is a mild hypothesis, since both arithmetic and differential conductors behave well under unramified extensions and the tamely ramified case is well-known: see Theorem 2.4.1(3) and Proposition 4.1.7(6).

Notation 3.3.3. Let l be as above, and let $G_{l/k}$ denote the Galois group of l/k. Denote the ring of integers and the residue field of l by $\mathbb{O}_l$ and κ_l, respectively. Given a uniformizer t of l, we fix a noncanonical isomorphism $\kappa_l((t)) \simeq l$. For a p-basis $\bar{c}_J$ of κ_l, we use c_J to denote the image of $\bar{c}_J$ under this isomorphism; we may use the same index set J because κ_l/κ_k is a finite extension.

Let $\mathbb{O}_L$ be the Cohen ring of κ_l with respect to $\bar{c}_J$ and let C_J be the canonical lifts of $\bar{c}_J$. Set $L = \operatorname{Frac}\mathbb{O}_L$.

Caution 3.3.4. The residue field extension κ_l/κ_k is typically not separable and hence *cannot* be embedded into the extension $l = \kappa_l((t))$ over $k = \kappa_k((s))$.

The reader may skip the next construction and remark on first reading. Their gist is to provide "good" generators and relations of $\mathbb{O}_l$ as an $\mathbb{O}_k$-algebra.

Construction 3.3.5. We temporarily drop the finiteness Hypothesis 2.3.2 on the p-basis for this construction. Let $\boldsymbol{k}_0 = \kappa_k$ with p-basis $(\bar{b}_j)_{j\in J}$. By possibly rearranging the indexing in $\bar{b}_J$, we will inductively construct a "good" p-basis $(\bar{c}_j)_{j\in J}$ of κ_l and $\boldsymbol{k}_j = \kappa_k(\bar{c}_1, \dots, \bar{c}_j)$ with p-basis $\{\bar{c}_1, \dots, \bar{c}_j, \bar{b}_{J\setminus\{1,\dots,j\}}\}$ so that $\boldsymbol{k}_m = \kappa_l$ for m sufficiently large.

Assume that we have constructed $\boldsymbol{k}_{j-1}$. Let r_j be the unique integer such that $\kappa_l \subseteq \boldsymbol{k}_{j-1}^{p^{-r_j}}$ but $\kappa_l \nsubseteq \boldsymbol{k}_{j-1}^{p^{-r_j+1}}$. If $r_j = 0$, we must have $\boldsymbol{k}_{j-1} = \kappa_l$; in this case, we set $\bar{c}_\alpha = \bar{b}_\alpha$ and $r_\alpha = 0$ for all $\alpha \in J\setminus\{1, \dots, j-1\}$ and stop the induction. Otherwise we assume that $r_j > 0$. Take $\bar{c}_j$ to be any element in $\kappa_l\setminus\boldsymbol{k}_{j-1}^{p^{-r_j+1}}$ and let $\boldsymbol{k}_j = \boldsymbol{k}_{j-1}(\bar{c}_j)$. Then $\bar{c}_j^{p^{r_j}} \in \boldsymbol{k}_{j-1}$ and $[\boldsymbol{k}_j : \boldsymbol{k}_{j-1}] = p^{r_j}$. There must exist one element in $\bar{b}_{J\setminus\{1,\dots,j-1\}}$ such that the rest together with $\bar{c}_1, \dots, \bar{c}_j$ form a p-basis of $\boldsymbol{k}_j$. We assume that this element is $\bar{b}_j$ by reindexing $\bar{b}_{J\setminus\{1,\dots,j-1\}}$. This finishes the induction.

From the induction, one can see that the r_j form a decreasing sequence of nonnegative integers; but we do not need this fact.

Since κ_l/κ_k is finite, the construction ensures that $\bar{c}_j \in \kappa_k^\times$ for $j \in J\setminus J_0$ with $J_0 = \{1, \dots, m\}$ a finite subset. By the functoriality of p-bases (Corollary 2.1.11), we may change the isomorphism $\kappa_l((t)) \simeq l$ so that $\bar{c}_{J\setminus J_0}$ are sent to elements in $\mathbb{O}_k^\times$. Let c_J denote the images of $\bar{c}_J$ under the above isomorphism.

As a consequence, $c_1, \dots, c_m$ and t generate $\mathbb{O}_l$ over $\mathbb{O}_k$. More precisely,

$$\left\{c_J^{e_J} t^i \;\middle|\; i \in \{0, \dots, e-1\};\ e_j \in \{0, \dots, p^{r_j}-1\} \text{ for } j = 1, \dots, m\right\}$$

is a basis of $\mathbb{O}_l$ as a finite free $\mathbb{O}_k$-module. It is also a basis of l as a k-vector space.

Remark 3.3.6. It is attractive to hope that we can find a p-basis $(\bar{b}_j)_{j\in J}$ of κ_k such that $\kappa_l = \kappa_k(\bar{b}_j^{p^{-r_j}})$ for some $r_j \in \mathbb{Z}_{\geq 0}$. But this is false in general, as pointed out to us by Shun Ohkubo; a counterexample is provided by [Sweedler 1968, Example 1.1]. Sweedler called the case where such a basis can be found *modular*.

Let κ_0 be a perfect field of characteristic p and let X, Y, Z be indeterminates. Set $\kappa_k = \kappa_0(X^p, Y^p, Z^{p^2})$ and $\kappa_l = \kappa_k(Z, XY+Z)$. Then $[\kappa_l : \kappa_l \cap \kappa_k^{p^{-1}}] = p^2$ and $[\kappa_l \cap \kappa_k^{p^{-1}} : \kappa_k] = p$. Hence, κ_l/κ_k cannot be modular.

Now we go back to assuming Hypothesis 2.3.2.

Notation 3.3.7. For a nonarchimedean ring R, we use $R\langle u_0, \dots, u_m\rangle$ to denote the completion of $R[u_0, \dots, u_m]$ with respect to the natural topology induced from R. When $R = F$ is a complete nonarchimedean field, $F\langle u_0, \dots, u_m\rangle$ is the ring of analytic functions on the unit polydisc $A_F^{m+1}[0, 1]$.

Notation 3.3.8. Let $\mathbb{O}_k\langle u_0, \dots, u_m\rangle/\mathcal{I} \xrightarrow{\sim} \mathbb{O}_l$ be the homomorphism that sends u_0 to t and u_j to c_j, for each $j \in J$. We choose a set of generators $p_0, \dots, p_m$ of $\mathcal{I}$ as follows: each $c_j^{p^{r_j}}$ or t^e can be written in terms of the basis of $\mathbb{O}_l$ over $\mathbb{O}_k$ listed in

Construction 3.3.5. This gives us an element p_j in $\mathcal{I}$ (the index $j=0$ being used for t^e). Obviously, the p_i generate $\mathcal{I}$. Moreover,

$$p_0 \in u_0^e - \mathfrak{d}s + (u_0 s, s^2)\cdot \mathbb{O}_k[u_0, \dots, u_m],$$
$$p_j \in u_j^{p^{r_j}} - \mathfrak{b}_j + (u_0, s)\cdot \mathbb{O}_k[u_0, \dots, u_m] \quad \text{for } j = 1, \dots, m,$$

where $\mathfrak{b}_j$ is a polynomial in $u_1, \dots, u_{j-1}$ with coefficients in $\mathbb{O}_k$ and with degree on $u_{j'}$ strictly smaller than $p^{r_{j'}}$ for $j' = 1, \dots, j-1$, and $\mathfrak{d} \in \mathbb{O}_k[u_J]$ such that $\mathfrak{d}(c_1, \dots, c_m) \in \mathbb{O}_L^\times$. Let $\bar{\mathfrak{b}}_j$ be the reduction of $\mathfrak{b}_j$ in $\kappa_k[u_1, \dots, u_{j-1}]$.

Remark 3.3.9. The need for introducing $\mathfrak{d}$ was pointed out to us by Shun Ohkubo: in general, one may not be able to find uniformizers s and t of k and l, respectively, such that $t^e \equiv s \bmod t^{e+1}\mathbb{O}_l$. This is shown by the next example, provided by Ohkubo. (We do not know if there is a counterexample for which L/K is Galois.)

Example 3.3.10. Let k be a complete discretely valued field with nonperfect residue field κ_k. Let $b \in \mathbb{O}_k$ be such that $\bar{b} \in \kappa_k \backslash \kappa_k^p$. Choose $\alpha, \beta \in \overline{K}$ as follows: let α be a root of polynomial $X^p + sX + b \in k[X]$ and β a root of polynomial $Y^p + sY + s\alpha \in k(\alpha)[Y]$. Let $l = k(\alpha, \beta)$. Then l/k is a separable extension of degree p^2 with naïve ramification degree p. The rings of integers of $k(\alpha)$ and $k(\alpha, \beta)$ are $\mathbb{O}_k[\alpha]$ and $\mathbb{O}_k[\alpha, \beta]$, respectively. We claim that we cannot choose uniformizers t and s so that $t^p/s \equiv 1 \bmod \mathfrak{m}_l$.

It is clear that β is a uniformizer of l. For any uniformizer t of l,

$$\frac{t^p}{s} = \frac{\beta^p}{s}\left(\frac{t}{\beta}\right)^p \in (-\alpha-\beta)(\mathbb{O}_l^\times)^p \xrightarrow{(\mathrm{mod}\ \mathfrak{m}_l)} (-\alpha)\kappa_l^p \subset \kappa_l.$$

In particular, t^p/s is not congruent to 1 modulo $\mathfrak{m}_l$.

Remark 3.3.11. Generally, the kernel of $\mathbb{O}_k[u_0, \dots, u_m] \to \mathbb{O}_l$ is not generated by $p_0, \dots, p_m$. This will not matter since we take $a > 0$ and $a > 1$ in Definition 3.2.1.

Construction 3.3.12. For each $j \in J$, fix an element in $\mathbb{O}_L[\![T]\!]$ lifting $b_j \in \mathbb{O}_k \subset \kappa_l[\![t]\!]$; also fix an element in $T^e + T^{e+1}\mathbb{O}_L[\![T]\!]$ lifting $s \in \mathbb{O}_k \subset \kappa_l[\![t]\!]$. By Proposition 2.1.8, there exists a continuous homomorphism $f^* : C_k \hookrightarrow C_l$ sending B_J and S to the elements chosen above; it naturally restricts to $f^* : \mathbb{O}_K[\![S]\!] \hookrightarrow \mathbb{O}_L[\![T]\!]$.

The proof of the following lemma is not enlightening. The reader may skip it on a first reading. The upshot is that we can turn the good generators and relations of $\mathbb{O}_l$ as an $\mathbb{O}_k$-algebra into good generators and relations of $\mathcal{R}_L^{\eta_0^{1/e}}$ as an $\mathcal{R}_K^{\eta_0}$-algebra.

Lemma 3.3.13. *Keep the notation as above.*

(1) *The homomorphism f^* is finite, and $C_1, \dots, C_m$ and T generate $\mathbb{O}_L[\![T]\!]$ over $\mathbb{O}_K[\![S]\!]$. Hence, f^* induces a surjective map $\mathbb{O}_K[\![S]\!]\langle U_0, \dots, U_m\rangle \twoheadrightarrow \mathbb{O}_L[\![T]\!]$ sending U_0 to T and U_j to C_j for $j \in J$. Moreover, one can choose generators*

$P_0, \dots, P_m$ of the kernel so that, modulo p, they are exactly p_{J^+} in Notation 3.3.8. In particular,

$$P_0 \in U_0^e - \mathfrak{D}S + (p, U_0S, S^2) \cdot \mathcal{O}_K[\![S]\!]\langle U_0, \dots, U_m\rangle,$$
$$P_j \in U_j^{p^{r_j}} - \mathfrak{B}_j + (p, U_0, S) \cdot \mathcal{O}_K[\![S]\!]\langle U_0, \dots, U_m\rangle,$$

where $\mathfrak{B}_j$ is a polynomial in $U_1, \dots, U_{j-1}$ with coefficients in $\mathcal{O}_K$ and with degree on $U_{j'}$ strictly smaller than $p^{r_{j'}}$ for $j' = 1, \dots, j-1$, and $\mathfrak{D} \in \mathcal{O}_K[U_J]$ lifts $\mathfrak{d}$. Moreover, $\{U_{J^+}^{e_{J^+}} \mid 0 \le e_0 < e; 0 \le e_j < p^{r_j}, j \in J\}$ form a basis of $\mathcal{O}_K[\![S]\!]\langle U_0, \dots, U_m\rangle/(P_{J^+})$ over $\mathcal{O}_K[\![S]\!]$.

(2) *The map f^* extends to a map $f_\eta^* : K\langle \eta/S, S/\eta\rangle \to L\langle \eta^{1/e}/T, T/\eta^{1/e}\rangle$ for $\eta \in [0, 1)$. Thus f^* extends by continuity to a homomorphism $f^* : \mathcal{R}_K^{\eta_0} \to \mathcal{R}_L^{\eta_0^{1/e}}$, or in geometric notation, $f : A_K^1[\eta_0, 1) \to A_L^1[\eta_0^{1/e}, 1)$ for $\eta_0 \in (0, 1)$.*

(3) *Let $\Gamma_K^\dagger$ and $\Gamma_L^\dagger$ be the integral Robba rings over K and L, respectively, similarly constructed as in Construction 2.2.8 but without tensoring with F. Let $\mathcal{R}_L$ be the Robba ring over L as in Notation 2.2.13. Then $\Gamma_L^\dagger$ is a finite étale extension of $\Gamma_K^\dagger$ with Galois group $G_{l/k}$. Moreover, $\mathcal{R}_L \simeq \Gamma_L^\dagger \otimes_{\Gamma_K^\dagger} \mathcal{R}_K$.*

(4) *For some $\eta_0 \in (0, 1)$, $A_L^1[\eta_0^{1/e}, 1)$ is Galois étale over $\eta \in [\eta_0, 1)$ via f^* with Galois group $G_{l/k}$. Hence, $\mathcal{R}_L^{\eta_0^{1/e}}$ becomes a regular $G_{l/k}$-representation over $\mathcal{R}_K^{\eta_0}$ via f^*.*

Proof. (1) is equivalent to its mod p version, which is exactly Construction 3.3.5.

(2) It suffices to prove that f^* is continuous with respect to the norms $|\cdot|_{Z_k^\eta}$ on C_k and $|\cdot|_{Z_l^{\eta^{1/e}}}$ on C_l, for all $\eta \in [\eta_0, 1)$. Since $f^*(\mathcal{O}_K) \in \mathcal{O}_L[\![T]\!]$ and $f^*(S) \in T^e + T^{e+1}\mathcal{O}_L[\![T]\!]$, we have $|g|_{Z_k^\eta} = |f^*(g)|_{Z_l^{\eta^{1/e}}}$ for any $g \in C_k$. Hence the map f^* extends continuously to $f_\eta^* : K\langle \eta/S, S/\eta\rangle \to L\langle \eta^{1/e}/T, T/\eta^{1/e}\rangle$.

(3) The first statement follows from Lemma 2.2.10. The second statement is true because $\Gamma_L^\dagger \otimes_{\Gamma^\dagger} \mathcal{R}_K$ is complete and dense in $\mathcal{R}_L$.

(4) follows from (2) and (3) since $\mathcal{R}_K$ and $\mathcal{R}_L$ are limits of $\mathcal{R}_K^{\eta_0}$ and $\mathcal{R}_L^{\eta_0^{1/e}}$, respectively. □

Remark 3.3.14. The homomorphism f^* does not respect the naïve K-algebra structure on $\mathcal{R}_L^{\eta_0^{1/e}}$; this is precisely because of Caution 3.3.4. But it respects the K-algebra structure on $\mathcal{R}_L^{\eta_0^{1/e}}$ induced by $\mathcal{O}_K \hookrightarrow \mathcal{O}_K[\![S]\!] \xrightarrow{f^*} \mathcal{O}_L[\![T]\!]$. So, it might be better not to view $Z_l^{\ge \eta_0^{1/e}} \to Z_k^{\ge \eta_0}$ as a morphism between rigid spaces, but rather as the geometric incarnation of f^*.

Construction 3.3.15. Keep the notation as in Construction 2.2.15. Let $\rho : G_{l/k} \to \mathrm{GL}(V_\rho)$ be a p-adic representation, where V_ρ is a finite dimensional vector space over F. We have

$$\begin{aligned}\mathscr{E}_\rho &= D^\dagger(\rho) \otimes_{\Gamma_K^\dagger} \mathscr{R}_K = (V_\rho \otimes_{\mathbb{O}} \tilde{\Gamma}^\dagger)^{G_k} \otimes_{\Gamma_K^\dagger} \mathscr{R}_K \\ &= (V_\rho \otimes_{\mathbb{Z}_q} \Gamma_L^\dagger)^{G_{l/k}} \otimes_{\Gamma_K^\dagger} \mathscr{R}_K = (V_\rho \otimes_{\mathbb{Z}_q} \mathscr{R}_L)^{G_{l/k}}.\end{aligned}$$

Hence, for some $\eta_0 \in (0, 1)$, the differential module $\mathscr{E}_\rho$ descends to

$$\mathscr{E}_\rho = \big(V_\rho \otimes_{\mathbb{Z}_q} f_* \mathbb{O}_{Z_l^{\geq \eta_0^{1/e}}}\big)^{G_{l/k}};$$

this is a differential module over $\mathscr{R}_K^{\eta_0} \otimes_{\mathbb{Q}_q} F = \mathscr{R}_{K'}^{\eta_0}$ relative to K_0. This construction respects tensor products, i.e., given another p-adic representation ρ' of $G_{l/k}$ over F, we have

$$\mathscr{E}_{\rho \otimes \rho'} = \mathscr{E}_\rho \otimes_{\mathscr{R}_{K'}^{\eta_0}} \mathscr{E}_{\rho'}.$$

Hypothesis 3.3.16. From now on, we always assume that $\eta_0 \in (0, 1)$ is close enough to 1^- that all statements in Lemma 3.3.13 hold and $\mathscr{E}_\rho$ descends to $\mathscr{R}_{K'}^{\eta_0}$.

3.4. ***Spectral norms and connected components of thickening spaces.*** We now relate the spectral norms of differential operators on $\mathscr{E}$ to the connected components of certain rigid spaces. We keep Hypotheses 2.3.2, 3.3.1, and 3.3.16 in this subsection.

Definition 3.4.1. Let $a \in \mathbb{Q}_{>1}$. We define

$$\begin{aligned}\mathbb{O}_{TS_{l/k}^{a, \geq \eta_0}} &= \mathscr{R}_L^{\eta_0^{1/e}} \otimes_{f^*, \mathscr{R}_K^{\eta_0}, \pi^*} \mathbb{O}_{TS_k^{a, \geq \eta_0}}, \\ \mathbb{O}_{TS_{k\backslash l}^{a, \geq \eta_0}} &= \mathbb{O}_{TS_k^{a, \geq \eta_0}} \otimes_{\tilde{\pi}^*, \mathscr{R}_K^{\eta_0}, f^*} \mathscr{R}_L^{\eta_0^{1/e}}, \\ \mathbb{O}_{TS_{l/k\backslash l}^{a, \geq \eta_0}} &= \mathscr{R}_L^{\eta_0^{1/e}} \otimes_{f^*, \mathscr{R}_K^{\eta_0}, \pi^*} \mathbb{O}_{TS_k^{a, \geq \eta_0}} \otimes_{\tilde{\pi}^*, \mathscr{R}_K^{\eta_0}, f^*} \mathscr{R}_L^{\eta_0^{1/e}}.\end{aligned}$$

Here we do not have to complete the tensor products because f^* is finite. (We intentionally put the tensor products on different sides so that it is easy to distinguish the two base changes by f^* through π^* and $\tilde{\pi}^*$ respectively.) Let $TS_{l/k}^{a, \geq \eta_0}$, $TS_{k\backslash l}^{a, \geq \eta_0}$, and $TS_{l/k\backslash l}^{a, \geq \eta_0}$, respectively, denote the geometric incarnations of these rings. We have formally the following Cartesian diagram:

$$\begin{tikzcd} Z_l^{\geq \eta_0^{1/e}} \arrow[d, "f"] & TS_{l/k}^{a, \geq \eta_0} \arrow[l, "1 \times \pi"'] \arrow[d, "f \times 1"] & TS_{l/k\backslash l}^{a, \geq \eta_0} \arrow[l, "\tilde{f}"'] \arrow[d] \\ Z_k^{\geq \eta_0} & TS_k^{a, \geq \eta_0} \arrow[l, "\pi"'] \arrow[d, "\tilde{\pi}"] & TS_{k\backslash l}^{a, \geq \eta_0} \arrow[l, "1 \times f"'] \arrow[d] \\ & Z_k^{\geq \eta_0} & Z_l^{\geq \eta_0^{1/e}}. \arrow[l, "f"'] \end{tikzcd} \tag{3.4.2}$$

We make similar constructions for the logarithmic version of all spaces if $a \in \mathbb{Q}_{>0}$.

Remark 3.4.3. The morphisms π and $1\times f$ are genuine morphisms between rigid spaces over $Z_k^{\geq\eta_0}$, and $\tilde f$ and $1\times\pi$ are genuine morphisms between rigid spaces over $Z_l^{\geq\eta_0^{1/e}}$. This is because the rigid space structures on thickening spaces are given by the projections π and $1\times\pi$, respectively. In contrast, all the vertical arrows in (3.4.2) should all be thought of as just geometric incarnations of the corresponding ring homomorphisms.

Remark 3.4.4. The naïve base change $f\times 1$ helps to realize geometric connected components as connected components (see Theorem 3.4.12). The base change $\tilde f$ (and also $1\times f$) encodes the ramification information, which is what we are interested in.

Remark 3.4.5. One may want to relate $TS_{l/k\backslash l}^{a,\geq\eta_0}$ to the thickening space of $Z_l^{\geq\eta_0^{1/e}}$. However, it is not clear how to compare the levels or radii of the two spaces. We will not need this result.

Corollary 3.4.6. *The space $TS_{l/k\backslash l}^{a,\geq\eta_0}$ admits an action of $G_{l/k}$ by morphisms between K-rigid spaces, obtained by pulling back the action on $Z_l^{\geq\eta_0^{1/e}}$ over $Z_k^{\geq\eta_0}$ via $\tilde\pi\circ(f\times 1)$. Under this action, $\tilde f_*\mathcal{O}_{TS_{l/k\backslash l}^{a,\geq\eta_0}}$ is a regular representation of $G_{l/k}$ over $\mathcal{O}_{TS_{l/k}^{a,\geq\eta_0}}$. For a p-adic representation ρ of $G_{l/k}$ over F, define*

$$\tilde{\mathscr{F}}_\rho=(V_\rho\otimes_{\mathbb{Q}_q}\tilde f_*\mathcal{O}_{TS_{l/k\backslash l}^{a,\geq\eta_0}})^{G_{l/k}};$$

this is a differential module over $TS_{l/k}^{a,\geq\eta_0}\times_{\mathbb{Q}_q}F$ relative to $Z_l^{\geq\eta_0^{1/e}}\times_{\mathbb{Q}_q}F$. Moreover, $\tilde{\mathscr{F}}_\rho\simeq(f\times 1)^*\tilde\pi^*\mathscr{E}_\rho$.

The same statement also holds for log-space.

Proof. The differential module structure on $\tilde f_*\mathcal{O}_{TS_{l/k\backslash l}^{a,\geq\eta_0}}$ is given by the composition of natural homomorphisms

$$\begin{aligned}\tilde f_*\mathcal{O}_{TS_{l/k\backslash l}^{a,\geq\eta_0}}\longrightarrow \tilde f_*\big(\Omega^1_{TS_{l/k\backslash l}^{a,\geq\eta_0}/Z_l^{\geq\eta_0^{1/e}}}\big)&\simeq\tilde f_*\Big(\tilde f^*\big(\Omega^1_{TS_{l/k}^{a,\geq\eta_0}/Z_l^{\geq\eta_0^{1/e}}}\big)\Big)\\ &\simeq \tilde f_*\mathcal{O}_{TS_{l/k\backslash l}^{a,\geq\eta_0}}\otimes\Omega^1_{TS_{l/k}^{a,\geq\eta_0}/Z_l^{\geq\eta_0^{1/e}}}.\end{aligned}$$

(In fact this construction works for any finite étale morphisms.) The statement of the corollary is an easy consequence of flat base change for the two Cartesian squares on the right in (3.4.2). $\square$

Notation 3.4.7. We may view $1\times\pi: TS_{l/k}^{a,\eta_0}\to Z_l^{\eta_0^{1/e}}$ as bundles, whose fibers are polydiscs (of different radii) with parameters $\delta_0,\dots,\delta_m$; again this morphism is a genuine morphism between rigid spaces. By the *zero section* $\boldsymbol{Z}$, we mean the natural closed subspace of this bundle defined by $\delta_0=0,\dots,\delta_m=0$.

Notation 3.4.8. Let M be a differential module over a differential ring R with derivatives $\partial_1,\dots,\partial_n$. For $x\in M$ and $r_1,\dots,r_n\in R$, we define the *Taylor series*

$$\mathbb{T}(x;\partial_1,\dots,\partial_n;r_1,\dots,r_n)=\sum_{\alpha_1,\dots,\alpha_n\in\mathbb{Z}_{\geq 0}}\frac{r_1^{\alpha_1}\dots r_n^{\alpha_n}\partial_1^{\alpha_1}\dots\partial_n^{\alpha_n}}{n!}(x),$$

if it converges. If $x\in R$, we have $\mathbb{T}(ax;\partial_1,\dots,\partial_n;r_1,\dots,r_n)=\mathbb{T}(a;\partial_1,\dots,\partial_n;r_1,\dots,r_n)\cdot\mathbb{T}(x;\partial_1,\dots,\partial_n;r_1,\dots,r_n)$ if all terms converge.

Notation 3.4.9. Let M be a differential module over a differential ring R with derivatives $\partial_1,\dots,\partial_n$. Let $H^0_\nabla(R,M)=\{x\in M\mid\partial_i(x)=0,i=1,\dots,n\}$ be the set of horizontal sections of M over R. In particular, if $r_1,\dots,r_n\in R$ are elements such that $\partial_i(r_j)=1$ if $i=j$ and 0 otherwise, then an elementary calculation shows that the Taylor series $\mathbb{T}(x;\partial_i,\dots,\partial_n;r_1,\dots,r_n)$ is an element in $H^0_\nabla(R,M)$ for any $x\in M$ such that the Taylor series converges.

We usually use the geometric counterparts in places of R and M in the notation. For example, we write $H^0_\nabla(\mathrm{Max}(R),M)$ if R is an affinoid algebra.

The following lemma will be frequently used in proving the theorem below. It works in greater generality, but we content ourselves with this special case.

Lemma 3.4.10 (Dwork's transfer theorem). *Let $a>1$. Let $\tilde{\mathscr{F}}$ be a differential module over $TS^{a,\geq\eta_0}_{l/k}$ relative to $Z_l^{\geq\eta_0^{1/e}}$. Assume $|\partial/\partial\delta_i|_{\tilde{\mathscr{F}}_{a,\eta}}\leq p^{-1/(p-1)}\eta^{-a}$ for all $j\in J$ and $\eta\in[\eta_0,1)$. Then, for any rational number $c>a$, the natural homomorphism of finite $\mathscr{R}_L^{\eta_0^{1/e}}$-modules*

$$\Theta: H^0_\nabla(TS^{c,\geq\eta_0}_{l/k},\tilde{\mathscr{F}})\xrightarrow{\sim}\Gamma(\mathbf{Z},\tilde{\mathscr{F}}|_{\mathbf{Z}})\tag{3.4.11}$$

is an isomorphism. In particular, $\tilde{\mathscr{F}}$ is a trivial ∇-module relative to $Z_l^{\geq\eta_0^{1/e}}$. The same statement is also true if we base-change everything to F over $\mathbb{Q}_q$. When $\tilde{\mathscr{F}}=f_*\mathscr{O}_{TS^{a,\geq\eta_0}_{l/k\backslash l}}$, Θ induces a ring *homomorphism for any rational number $c>a$:*

$$\Gamma\big(\mathbf{Z},f_*\mathscr{O}_{TS^{c,\geq\eta_0}_{l/k\backslash l}}\big|_{\mathbf{Z}}\big)\xleftarrow[\sim]{\Theta}H^0_\nabla\big(TS^{c,\geq\eta_0}_{l/k},f_*\mathscr{O}_{TS^{c,\geq\eta_0}_{l/k\backslash l}}\big)\hookrightarrow\Gamma\big(TS^{c,\geq\eta_0}_{l/k\backslash l},\mathscr{O}_{TS^{c,\geq\eta_0}_{l/k\backslash l}}\big).$$

The same statements hold for the log version with $a>0$, inserting the subscript log *appropriately.*

Proof. We prove the lemma for the nonlog case over $\mathbb{Q}_q$. The proof for the log case differs only by inserting subscript log appropriately, using the δ_0 coordinate, and increasing the exponents on η by 1. The proof for the tensor F version is also the same, except we need to tensor F everywhere.

We may define an inverse of the map Θ using Taylor series:

$$\Theta^{-1}(x)=\mathbb{T}(\tilde{x};\partial/\partial\delta_0,\dots,\partial/\partial\delta_m;\delta_0,\dots,\delta_m)$$

for $x\in\Gamma(\mathbf{Z},\tilde{\mathscr{F}}|_{\mathbf{Z}})$, where $\tilde{x}$ is a lift of x in $\Gamma(TS^{c,\geq\eta_0}_{l/k},\tilde{\mathscr{F}})$. The Taylor series converges over $TS^{c,\geq\eta_0}_{l/k}$ by the condition $|\partial/\partial\delta_i|_{\tilde{\mathscr{F}}_{a,\eta}}\leq p^{-1/(p-1)}\eta^{-a}<p^{-1/(p-1)}\eta^{-c}$ for all $j\in J$ and $\eta\in[\eta_0,1)$. Moreover, the Taylor series converges to a horizontal section in $H^0_\nabla(TS^{c,\geq\eta_0}_{l/k},\tilde{\mathscr{F}})$.

When $\tilde{\mathscr{F}} = f_*\mathcal{O}_{TS^{a,\geq\eta_0}_{l/k\backslash l}}$, Θ is a homomorphism, which can also be seen from the fact that the Taylor series gives a ring homomorphism (see Notation 3.4.8). □

The following theorem is one of the key steps of the proof of the Hasse–Arf theorem. This is the main ingredient (a) described in the introduction. It allows us to compare the differential ramification breaks with the geometric connected components of the thickening spaces; we will later identify the thickening spaces with the lifts of the Abbes–Saito spaces (Theorem 4.3.6).

Theorem 3.4.12. *Let $\rho : G_{l/k} \to \mathrm{GL}(V_\rho)$ be a* faithful *p-adic representation over F with l/k satisfying Hypotheses 2.3.2 and 3.3.1. Then, for $b > 1$, the following conditions are equivalent*:

(a) *ρ has differential ramification break $\leq b$.*

(b) *For any rational number $c > b$, when $\eta_0 \to 1^-$, $\tilde{\mathscr{F}} = \tilde{\mathscr{F}}_\rho$ is a trivial ∇-module over $TS^{c,\geq\eta_0}_{l/k} \times_{\mathbb{Q}_q} F$ relative to $Z_l^{\geq\eta_0^{1/e}} \times_{\mathbb{Q}_q} F$.*

(c) *For any rational number $c > b$, when $\eta_0 \to 1^-$, $TS^{c,\geq\eta_0}_{l/k\backslash l}$ has exactly $[l:k]$ connected components.*

(d) *For any rational number $c > b$, when $\eta_0 \to 1^-$, $Z_{l'}^{\geq\eta_0^{1/e'}} \times_{Z_l^{\geq\eta_0^{1/e}},\pi} TS^{c,\geq\eta_0}_{l/k\backslash l}$ has exactly $[l:k]$ connected components for some finite extension l'/l, where e' is the naïve ramification degree of l'/k.*

For $b > 0$, the corresponding conditions for logarithmic spaces are equivalent.

Proof. We prove the statement for nonlogarithmic spaces; in the logarithmic case we just need to add the subscript log and change the scales on ∂_0 and $\partial/\partial\delta_0$ from η^b to η^{b+1} and η^c to η^{c+1}.

Further, Proposition 3.2.17 is unchanged if we replace $\mathscr{F}$ by $\tilde{\mathscr{F}}$, since the spectral norms are invariant under scalar extensions.

We first that (a) implies (b). Assume ρ has differential ramification break at most b. By Definition 2.3.20, for η_0 sufficiently close to 1^-, the generic radius of $\mathscr{E}_\rho$ satisfies $\boldsymbol{T}(\mathscr{E}_\rho, \eta) \geq \eta^b$ for $\eta \in [\eta_0, 1)$, or equivalently $|\partial_j|_{\mathscr{E}_{\rho,\eta},\mathrm{sp}} \leq p^{-1/(p-1)}\eta^{-b}$ for any $j \in J^+$ and $\eta \in [\eta_0, 1)$. Then Proposition 3.2.17 and Remark 2.3.5 imply that for all $\eta \in [\eta_0, 1)$, $|\partial/\partial\delta_j|_{\tilde{\mathscr{F}}_{b,\eta},\mathrm{sp}} \leq p^{-1/(p-1)}\eta^{-b}$, and hence $\tilde{\mathscr{F}}_\rho$ is a trivial differential module over $TS^{c,\geq\eta_0}_{l/k} \times_{\mathbb{Q}_q} F$ relative to $Z_l^{\geq\eta_0^{1/e}} \times_{\mathbb{Q}_q} F$ for any rational number $c > b$ by Dwork's transfer theorem (Lemma 3.4.10). This proves (b).

Now assume (b), i.e., $\tilde{\mathscr{F}}_\rho$ is trivial over $TS^{c,\geq\eta_0}_{l/k} \times_{\mathbb{Q}_q} F$ relative to $Z_l^{\geq\eta_0^{1/e}} \times_{\mathbb{Q}_q} F$ for any rational number $c > b$ and some $\eta_0 \in (0, 1)$. It follows that

$$|\partial/\partial\delta_j|_{\tilde{\mathscr{F}}_{c,\eta},\mathrm{sp}} = |\partial/\partial\delta_j|_{\mathrm{Frac}(\mathcal{O}_{TS^{c,\eta}_{l/k}})^\wedge,\mathrm{sp}} = p^{-1/(p-1)}\eta^{-c}.$$

By Proposition 3.2.17, $|\partial_j|_{\mathrm{sp},\mathscr{E}_\eta} \leq p^{-1/(p-1)}\eta^{-c}$, for any $j \in J^+$, $\eta \in [\eta_0, 1)$, and $c \in \mathbb{Q}_{>b}$. By Definition 2.3.20, this implies that the differential ramification break is at most b, since the rationals are dense in the real numbers.

Obviously, (c) implies (b). To see the converse, note first claim that if $c > b$ is rational, $f_*\mathcal{O}_{TS^{c,\geq\eta_0}_{l/k\backslash l}}$ is a trivial differential module over $TS^{c,\geq\eta_0}_{l/k}$ relative to $Z_l^{\geq\eta_0^{1/e}}$. Indeed, for a rational number $c' \in (b, c)$, we know that $\tilde{\mathcal{F}}_\rho$ is a trivial differential module over $TS^{c',\geq\eta_0}_{l/k} \times_{\mathbb{Q}_q} F$ relative to $Z_l^{\geq\eta_0^{1/e}} \times_{\mathbb{Q}_q} F$, then for any $n \in \mathbb{N}$, $\tilde{\mathcal{F}}_\rho^{\otimes n}$ is also a trivial differential module (relative to $Z_l^{\geq\eta_0^{1/e}} \times_{\mathbb{Q}_q} F$), which corresponds to $V_\rho^{\otimes n}$ by functoriality (Construction 3.3.15). By Lemma 3.4.16 below from the theory of representations of finite groups (or standard Tannakian arguments), the differential module

$$\big(F[G_{l/k}] \otimes_{\mathbb{Q}_q} f_*\mathcal{O}_{TS^{c',\geq\eta_0}_{l/k\backslash l}}\big)^{G_{l/k}} \xrightarrow{\simeq} F \otimes_{\mathbb{Q}_q} f_*\mathcal{O}_{TS^{c',\geq\eta_0}_{l/k\backslash l}} \tag{3.4.13}$$

corresponding to the regular representation is a direct summand of a direct sum of some $\tilde{\mathcal{F}}_\rho^{\otimes n}$'s and hence is a trivial differential module (relative to $Z_l^{\geq\eta_0^{1/e}} \times_{\mathbb{Q}_q} F$). To make it perfectly rigorous, here the isomorphism (3.4.13) of differential modules is given by $\sum_{g\in G_{l/k}} \boldsymbol{f} g \otimes g\boldsymbol{v} \mapsto \boldsymbol{f} \cdot \boldsymbol{v}$, where $\boldsymbol{f} \in F$ and $\boldsymbol{v} \in f_*\mathcal{O}_{TS^{c,\geq\eta_0}_{l/k\backslash l}}$; this map does *not* respect the $F[G_{l/k}]$-module structures.

We have finished the proof of the claim in the case $F = \mathbb{Q}_p$. If $F \neq \mathbb{Q}_p$, we know that, for all $j \in J^+$, the spectral norms of $\partial/\partial\delta_j$ at radius η on the right-hand side of (3.4.13) are $p^{-1/(p-1)}\eta^{-c'}$, which equal the spectral norms of $\partial/\partial\delta_j$ on $f_*\mathcal{O}_{TS^{c',\geq\eta_0}_{l/k\backslash l}}$ at radius η. By Dwork's transfer theorem (Lemma 3.4.10), the claim is proved.

We now apply the second part of Lemma 3.4.10 and obtain, for any rational numbers $c' > c$, a *ring* homomorphism

$$\Gamma\big(\boldsymbol{Z}, f_*\mathcal{O}_{TS^{c',\geq\eta_0}_{l/k\backslash l}}|_{\boldsymbol{Z}}\big) \overset{\Theta}{\underset{\sim}{\longleftarrow}} H^0_\nabla\big(TS^{c',\geq\eta_0}_{l/k}, f_*\mathcal{O}_{TS^{c',\geq\eta_0}_{l/k\backslash l}}\big) \hookrightarrow \Gamma\big(TS^{c',\geq\eta_0}_{l/k\backslash l}, \mathcal{O}_{TS^{c',\geq\eta_0}_{l/k\backslash l}}\big). \tag{3.4.14}$$

The key is that the left-hand side of (3.4.14) is isomorphic to the ring functions on $Z_l^{\geq\eta_0^{1/e}} \times_{Z_k^{\geq\eta_0}} Z_l^{\geq\eta_0^{1/e}}$ because the restrictions of $\tilde{\pi}$ and π to $\boldsymbol{Z}$ are both the same as f. Moreover, since $Z_l^{\geq\eta_0^{1/e}}$ is finite étale Galois over $Z_k^{\geq\eta_0}$ (Lemma 3.3.13), $Z_l^{\geq\eta_0^{1/e}} \times_{Z_k^{\geq\eta_0}} Z_l^{\geq\eta_0^{1/e}} = \coprod_{g\in G_{l/k}} Z_l^{\geq\eta_0^{1/e}}$. In particular, we have fundamental idempotent elements in $\Gamma(\boldsymbol{Z}, f_*\mathcal{O}_{TS^{c',\geq\eta_0}_{l/k\backslash l}}|_{\boldsymbol{Z}})$ corresponding to each connected component. Via the composition of the homomorphisms in (3.4.14), we can "lift" the idempotent elements on $Z_l^{\geq\eta_0^{1/e}} \times_{Z_k^{\geq\eta_0}} Z_l^{\geq\eta_0^{1/e}}$ to idempotent elements in $\mathcal{O}_{TS^{c',\geq\eta_0}_{l/k\backslash l}}$. This shows that $TS^{c',\geq\eta_0}_{l/k\backslash l}$ has at least $[l : k]$ connected components. But this space is finite and flat of degree $[l : k]$ over an irreducible rigid space $TS^{c',\geq\eta_0}_{l/k}$; it can have at most $[l : k]$ connected components. Therefore, (c) holds.

The equivalence between (b) and (d) can be proved similarly, using a version of Lemma 3.4.10 over $Z_{l'}^{\eta_0^{1/e'}}$. The upshot here is that we need a base change to at least $Z_l^{\geq\eta_0^{1/e}}$ in (c) so that we can split the fiber over $\boldsymbol{Z}$; this is why we did not state the theorem for $TS^{c,\geq\eta_0}_{k\backslash l}$ and $\tilde{\mathcal{F}}$ themselves. □

Remark 3.4.15. The faithfulness condition on ρ in the theorem is harmless: we will very easily reduce to this case later in the proof of Theorem 4.4.1.

Lemma 3.4.16. *Let G be a finite group and F be a field of characteristic 0. Let $\rho : G \to \mathrm{GL}(V_\rho)$ be a faithful representation over F. Then the regular representation $F[G]$ is a direct summand of a direct sum of some self-tensor products of V_ρ.*

This is an easy exercise of finite group representations but we do not know a good reference. The author thanks Xuhua He for providing the following proof.

Proof. Let χ be the character of V_ρ and let d be the dimension of V_ρ. Since the representation is injective, $\chi(1) = d$ and $\chi(g) \neq d$ for all $g \in G$ nontrivial. (This is because all the eigenvalues of $\rho(g)$ are roots of unity and cannot all be 1.)

Therefore, for each $g \neq 1$ there exists a polynomial P_g in χ with integer coefficients such that $P_g(\chi(g)) = 0$ but $P_g(d) \neq 0$. Let $P = \prod_{1 \neq g \in G} P_g$; then $P(d) \neq 0$ but $P(\chi(g)) = 0$ for all $g \neq 1$. Multiplying by a constant, we may assume that $\#G$ divides $P(d)$ and $P(d) > 0$. If $P(X) = a_n X^n + \cdots + a_0 \in \mathbb{Z}[X]$, then $(V^{\otimes n})^{\oplus a_n} \oplus \cdots \oplus V^{\oplus a_1} \oplus \mathbf{1}_F^{\oplus a_0} = F[G]^{P(d)/\#G}$ in the Grothendieck group of the representations of G, where $\mathbf{1}_F$ denotes the trivial representation. Consequently, if we take the direct sum of the terms on the left-hand side with positive a_i, the regular representation will be a natural direct summand of it. $\square$

4. Arithmetic ramification filtrations

4.1. ***Review of Abbes and Saito's definition.*** We briefly review the definition of arithmetic ramification filtrations on the Galois group of a complete discretely valued field k. For more details, consult [Abbes and Saito 2002; 2003]. The filtrations can be defined for a k of mixed characteristic; however, for the purpose of this paper, we focus on the case where k is of equal characteristic $p > 0$.

In this subsection, we do *not* make any of the hypothesis we have been using in previous sections.

Notation 4.1.1. Keep the notation as in previous sections. Fix uniformizers s and t for k and l, respectively. Let $v_l(\cdot)$ be the valuation on l normalized so that $v_l(t) = 1$. Let $\theta = |s|$.

Notation 4.1.2. In this subsection, we temporarily free j and J from the restraint introduced in Notation 2.3.3. But in later applications, we will specialize to the case in which j and J actually index p-bases.

Definition 4.1.3. Take $Z = (z_j)_{j \in J} \subset \mathcal{O}_l$ to be a finite set of elements generating $\mathcal{O}_l$ as an $\mathcal{O}_k$-algebra, i.e., $\mathcal{O}_k[(u_j)_{j\in J}]/\mathcal{I} \xrightarrow{\sim} \mathcal{O}_l$ mapping u_j to z_j for $j \in J = \{1, \dots, m\}$ and for some appropriate ideal $\mathcal{I}$. Let $(f_i)_{i=1,\dots,n}$ be a finite set of generators of $\mathcal{I}$. For $a \in \mathbb{Q}_{>0}$, define the (*nonlogarithmic*) *Abbes–Saito space* to be

$$as^a_{l/k,Z} = \left\{ (u_1, \dots, u_m) \in A^m_k[0,1] \,\middle|\, |f_i(u_J)| \leq \theta^a, 1 \leq i \leq n \right\}. \tag{4.1.4}$$

The *geometric* connected components (see [Bosch et al. 1984, 9.1.4/8] for the definition) of $as^a_{l/k,Z}$ are $\pi_0^{\mathrm{geom}}(as^a_{l/k,Z})$. The *arithmetic ramification break* $b_{\mathrm{ar}}(l/k)$ is defined as the minimal number b such that $\#\pi_0^{\mathrm{geom}}(as^a_{l/k,Z}) = [l:k]$ for any $a > b$.

Definition 4.1.5. Keep the notation as above. We single out a subset $P \subset Z$ and assume that P and hence Z contain the uniformizer t. For each $j \in J$, let $e_j = v_l(z_j)$. Take a lift $g_j \in \mathbb{O}_k[(u_j)_{j\in J}]$ of z_j^e/s^{e_j} for each $z_j \in P$, and take a lift $h_{i,j} \in \mathbb{O}_k[(u_j)_{j\in J}]$ of $z_j^{e_i}/z_i^{e_j}$ for each pair $(z_i, z_j) \in P \times P$. For $a \in \mathbb{Q}_{>0}$, define the *logarithmic Abbes–Saito space* to be

$$as^a_{l/k,\log,Z,P} = \left\{ (u_J) \in A^m_k[0,1] \;\middle|\; \begin{array}{ll} |f_i(u_J)| \leq \theta^a, & 1 \leq i \leq n, \\ |u_j^e - s^{e_j} g_j| \leq \theta^{a+e_j}, & \text{for all } z_j \in P, \\ |u_j^{e_i} - u_i^{e_j} h_{i,j}| \leq \theta^{a+e_ie_j/e}, & \text{for all } (z_i, z_j) \in P\times P. \end{array} \right\}$$

Similarly, the *logarithmic arithmetic ramification break* $b_{ar,\log}(l/k)$ is defined to be the minimal number b such that for any $a > b$, $\#\pi_0^{\mathrm{geom}}(as^a_{l/k,\log,Z,P}) = [l:k]$.

Remark 4.1.6. To ease the readers who are not familiar with Abbes and Saito's definition, we give an intuitive way to understand the definition following [Abbes and Saito 2002].

First, if $a \to \infty$, the conditions on $f_1, \dots, f_n$ in (4.1.4) basically restrict the possible u_J to be very close to z_J or other solutions to the equations $f_1 = 0, \dots, f_n = 0$, which are exactly Galois conjugates of z_J. Thus, one may believe that $as^a_{l/k,Z}$ has exactly $[l:k]$ geometric connected components, each of which looks like a small polydisc centered at one of the solutions. In contrast, if $a \to 0^+$, the conditions on $f_1, \dots, f_n$ are almost vacuum and $as^a_{l/k,Z}$ is almost the whole unit polydisc. In particular, the space is likely to be geometrically connected. From the two extreme cases, we know that, when we increase a, the Abbes–Saito space shrinks from a whole unit polydisc to smaller polydiscs and, at some a, a bigger polydisc breaks apart into several smaller polydiscs. The arithmetic ramification break captures the last break point.

We reproduce several statements from [Abbes and Saito 2002; 2003].

Proposition 4.1.7. *Abbes–Saito spaces have the following properties.*

(1) *For $a > 0$, the spaces $as^a_{l/k,Z}$ and $as^a_{l/k,\log,Z,P}$ do not depend on the choice of generators $(f_i)_{i=1,\dots,n}$ of $\mathcal{I}$ and lifts g_j and $h_{i,j}$ for $i, j \in P$* [Abbes and Saito 2002, §3].

(1′) *If, in the definition of both spaces, we choose polynomials $(f_i)_{i=1,\dots,n}$ as a set of generators of $\mathrm{Ker}(\mathbb{O}_k\langle(u_j)_{j\in J}\rangle \to \mathbb{O}_l)$ instead of $\mathrm{Ker}(\mathbb{O}_k[(u_j)_{j\in J}] \to \mathbb{O}_l)$, the spaces will not change.*

(2) *If we substitute in another pair of generating sets Z and P satisfying the same properties, then we have a canonical bijection on the sets of the geometric connected components $\pi_0^{\mathrm{geom}}(as^a_{l/k,Z})$ and $\pi_0^{\mathrm{geom}}(as^a_{l/k,\log,Z,P})$ for different generating sets, where $a > 0$. In particular, both highest arithmetic ramification breaks are well-defined* [Abbes and Saito 2002, §3].

(3) *The highest arithmetic ramification break (resp. highest logarithmic arithmetic ramification break) gives rise to a filtration on the Galois group G_k consisting of normal subgroups* $\mathrm{Fil}^a G_k$ *(resp.* $\mathrm{Fil}^a_{\log} G_k$*) for $a > 0$ such that* $b_{\mathrm{ar}}(l/k) = \inf\{a \mid \mathrm{Fil}^a G_k \subseteq G_l\}$ *(resp.* $b_{ar,\log}(l/k) = \inf\{a \mid \mathrm{Fil}^a_{\log} G_k \subseteq G_l\}$*)* [Abbes and Saito 2002, Theorems 3.3 and 3.11]. *Moreover, for l/k a finite Galois extension, both arithmetic ramification breaks are rational numbers* [Abbes and Saito 2002, Theorems 3.8 and 3.16].

(4) *Let k'/k be an algebraic extension of complete discretely valued fields or the completion of such an extension. If k'/k is unramified, then* $\mathrm{Fil}^a G_{k'} = \mathrm{Fil}^a G_k$ *for $a > 0$* [Abbes and Saito 2002, Proposition 3.7]. *If k'/k is tamely ramified with ramification index e', then* $\mathrm{Fil}^{e'a}_{\log} G_{k'} = \mathrm{Fil}^a_{\log} G_k$ *for $a > 0$* [Abbes and Saito 2002, Proposition 3.15]. *More generally, for a (not necessarily algebraic) extension k'/k of complete discretely valued fields with the* same *valued group and linearly independent from l/k such that $\mathbb{O}_{lk'} = \mathbb{O}_{k'} \otimes_{\mathbb{O}_k} \mathbb{O}_l$, we have* $b_{\mathrm{ar}}(lk'/k') = b_{\mathrm{ar}}(l/k)$ *and* $b_{ar,\log}(lk'/k') = b_{ar,\log}(l/k)$ [Abbes and Mokrane 2004, lemme 2.1.5].

(5) *For $a > 0$, define* $\mathrm{Fil}^{a+} G_k = \overline{\bigcup_{b>a} \mathrm{Fil}^b G_k}$ *and* $\mathrm{Fil}^{a+}_{\log} G_k = \overline{\bigcup_{b>a} \mathrm{Fil}^b_{\log} G_k}$. *Then, the subquotients* $\mathrm{Fil}^a G_k / \mathrm{Fil}^{a+} G_k$ *are abelian p-groups if $a \in \mathbb{Q}_{>1}$ and are* 0 *if $a \notin \mathbb{Q}$* ([Abbes and Saito 2002, Theorem 3.8] *and* [Abbes and Saito 2003, Theorem 1]); *the subquotients* $\mathrm{Fil}^a_{\log} G_k / \mathrm{Fil}^{a+}_{\log} G_k$ *are elementary abelian p-groups if $a \in \mathbb{Q}_{>0}$ and are* 0 *if $a \notin \mathbb{Q}$* ([Abbes and Saito 2002, Theorem 3.16] *and* [Saito 2009, Theorem 1.3.3]).

(6) *The inertia subgroup is* $\mathrm{Fil}^a G_k$ *if $a \in (0, 1]$ and the wild inertia subgroup is* $\mathrm{Fil}^{1+} G_k = \mathrm{Fil}^{0+}_{\log} G_k$ [Abbes and Saito 2002, Theorems 3.7 and 3.15].

(7) *When the residue field κ_k is perfect, the arithmetic ramification filtrations agree with the classical upper numbered filtrations in the following way:* $\mathrm{Fil}^a G_k = \mathrm{Fil}^{a-1}_{\log} G_k = \mathrm{Gal}^{a-1}_k$ *for $a \geq 1$* [Abbes and Saito 2002, §6.1], *where* Gal^a_k *is the classical upper numbered filtration on G_k.*

Proof. For the convenience of readers, we point out some ingredients of the proof. For details, one can consult the original papers.

(1) is straightforward by matching up points.

(1′) is not in the literature. However, it can be proved identically to (1).

(2) One can show that if we add a new (dummy) generator in Z or P, the new Abbes–Saito space admits a fibration over the original Abbes–Saito space whose fibers are closed discs of radius θ^a.

(3) The first statement is just abstract nonsense. The second is true essentially because Abbes–Saito spaces are defined over k and the geometric connect components can be detected over the algebraic closure k^{alg}, which has valued group $|k^\times|^{\mathbb{Q}}$. However, realizing this principle requires formal models of rigid spaces. As we will reprove this result in Theorem 4.4.1, we refer to the original paper for the formal model proof.

(4) When $\mathbb{O}_{lk'} \simeq \mathbb{O}_l \otimes_{\mathbb{O}_k} \mathbb{O}_{k'}$, one can match up the nonlogarithmic Abbes–Saito space for lk'/k' and the extension of the scalar of that for lk'/k' in a natural way. Actually, the logarithmic ramification break is not considered in [Abbes and Mokrane 2004, lemme 2.1.5], but the proof carries over similarly. In the tamely ramified and logarithmic cases, one can also identify two logarithmic Abbes–Saito spaces [2002, Proposition 9.8]; this is slightly more complicated.

(5) The proof used the formal models of the Abbes–Saito spaces and their stable reductions, which is in an orthogonal direction from the present paper. One may consult [Abbes and Saito 2003; Saito 2009] for a complete treatment.

(6) is an easy fact.

(7) follows from an explicit calculation in the monogenic case. □

Remark 4.1.8. In fact, in the proof of the main theorem (Theorem 4.4.1), we do not need (5) or the second statement of (3) on the rationality of the breaks in the proposition above. Therefore, we can obtain these properties from the properties of differential conductors in Theorem 2.4.1 via the comparison in Theorem 4.4.1.

Definition 4.1.9. Let $\rho : G_k \to \mathrm{GL}(V_\rho)$ be a representation of finite local monodromy. Define the *arithmetic Artin and Swan conductors* as

$$\mathrm{Art}_{\mathrm{ar}}(\rho) \stackrel{\mathrm{def}}{=} \sum_{a \in \mathbb{Q}_{\geq 0}} a \cdot \dim\big(V_\rho^{\mathrm{Fil}^{a+} G_k} / V_\rho^{\mathrm{Fil}^{a} G_k}\big), \tag{4.1.10}$$

$$\mathrm{Swan}_{\mathrm{ar}}(\rho) \stackrel{\mathrm{def}}{=} \sum_{a \in \mathbb{Q}_{\geq 0}} a \cdot \dim\big(V_\rho^{\mathrm{Fil}^{a+}_{\log} G_k} / V_\rho^{\mathrm{Fil}^{a}_{\log} G_k}\big). \tag{4.1.11}$$

They are actually finite sums.

Conjecture 4.1.12 (Hasse–Arf Theorem). *Let k be a complete discretely valued field of equal characteristic p. For any representation ρ of G_k of finite local monodromy, the arithmetic conductors are nonnegative integers, namely,* $\mathrm{Art}_{\mathrm{ar}}(\rho) \in \mathbb{Z}_{\geq 0}$ *and* $\mathrm{Swan}_{\mathrm{ar}}(\rho) \in \mathbb{Z}_{\geq 0}$.

Proposition 4.1.13. *Conjecture 4.1.12 is true if the residue field κ_k is perfect.*

Proof. By Proposition 4.1.7(7), we are reduced to the classical Hasse–Arf theorem [Serre 1979, §VI.2, Theorem 1′ and §IV.2, Corollary 3]. Note that in this case, $\mathrm{Swan}_{\mathrm{ar}}(\rho) = \mathrm{Art}_{\mathrm{ar}}(\rho) - \dim V_\rho / V_\rho^{I_k}$. □

We will prove Conjecture 4.1.12 in Corollary 4.4.3.

4.2. ***Standard Abbes–Saito spaces and their lifts.*** In practice, we will only study Abbes–Saito spaces that are given by some particular generators. We explicitly write down spaces and their lifts in the sense of Section 1.

In this subsection, we retrieve Hypotheses 2.3.2 and 3.3.1, assuming that k has finite p-basis and the extension l/k is totally and wildly ramified. Also, we retrieve Notation 2.3.3 on indexing p-basis.

Construction 4.2.1. We take $Z = \{c_1, \dots, c_m, t\}$ to be the set of generators of $\mathbb{O}_l/\mathbb{O}_k$ given by Construction 3.3.5. (Maybe some of them are already in the field k, but we still keep those.) We take $P = \{t\}$. By Proposition 4.1.7(1′), we can take the relations to be $p_0, \dots, p_m$ from Notation 3.3.8. For $a \in \mathbb{Q}_{>0}$, we define the *standard Abbes–Saito spaces* as

$$as^a_{l/k} = \big\{(u_0, \dots, u_m) \in A_k^{m+1}[0, 1] \,\big|\, |p_0(u_{J^+})| \le \theta^a, \dots, |p_m(u_{J^+})| \le \theta^a\big\},$$

$$as^a_{l/k,\log} = \big\{(u_0, \dots, u_m) \in A_k^{m+1}[0, 1] \,\big|\, |p_0(u_{J^+})| \le \theta^{a+1}, |p_1(u_{J^+})| \le \theta^a, \dots, |p_m(u_{J^+})| \le \theta^a\big\}.$$

Let P_{J^+} be the lifts of p_{J^+} as in Lemma 3.3.13. For $a \in \mathbb{Q}_{>0}$ and $\eta_0 \in (0, 1)$, we define the *lifting Abbes–Saito spaces* to be

$$AS^{a,\ge\eta_0}_{l/k} = \left\{ (U_{J^+}, S) \in A_K^{m+2}[0, 1] \,\middle|\, \begin{array}{l} \eta_0 \le |S| < 1, \\ |P_0(U_{J^+}, S)| \le |S|^a, \dots, |P_m(U_{J^+}, S)| \le |S|^a \end{array} \right\},$$

$$AS^{a,\ge\eta_0}_{l/k,\log} = \left\{ (U_{J^+}, S) \in A_K^{m+2}[0, 1] \,\middle|\, \begin{array}{l} \eta_0 \le |S| < 1, \quad |P_0(U_{J^+}, S)| \le |S|^{a+1}, \\ |P_1(U_{J^+}, S)| \le |S|^a, \dots, |P_m(U_{J^+}, S)| \le |S|^a \end{array} \right\};$$

they are viewed as rigid spaces over $Z_k^{\ge\eta_0}$.

Lemma 4.2.2. *Let k'/k be a finite Galois extension of naïve ramification degree e'. If we identify C_k as a subring of $C_{k'}$ as in Construction 3.3.12, we may view P_{J^+} as polynomials in U_{J^+} with coefficients in $\mathbb{O}_{K'}[\![S']\!]$, where K' is the fraction field of the Cohen ring of $\kappa_{k'}$ and S' is a lift of the uniformizer s' in k'. Then, for $\eta_0 \in (0, 1)$ and $a \in \mathbb{Q}_{>0}$, we have*

$$Z_{k'}^{\ge\eta_0^{1/e'}} \times_{Z_k^{\ge\eta_0}} AS^{a,\ge\eta_0}_{l/k} \cong \left\{ (U_{J^+}, S') \in A_{K'}^{m+2}[0, 1] \,\middle|\, \begin{array}{l} \eta_0^{1/e'} \le |S'| < 1, \\ |P_0| \le |S'|^{e'a}, \dots, |P_m| \le |S'|^{e'a} \end{array} \right\},$$

$$Z_{k'}^{\ge\eta_0^{1/e'}} \times_{Z_k^{\ge\eta_0}} AS^{a,\ge\eta_0}_{l/k,\log} \cong \left\{ (U_{J^+}, S') \in A_{K'}^{m+2}[0, 1] \,\middle|\, \begin{array}{l} \eta_0^{1/e'} \le |S'| < 1, \; |P_0| \le |S'|^{e'(a+1)}, \\ |P_1| \le |S'|^{e'a}, \dots, |P_m| \le |S'|^{e'a} \end{array} \right\};$$

Proof. The only thing not obvious is that we replace $|P_j| \leq |S|^{a(+1)}$ by $|P_j| \leq |S'|^{e'a(+e')}$; this is because $|S| = |S'|^{e'}$ as proved in Lemma 3.3.13(2). □

Remark 4.2.3. Note that $Z_k^{\geq \eta_0} \to Z_{k'}^{\eta_0^{1/e'}}$ is not a morphism between rigid spaces for the reason explained in Remark 3.3.14. So, strictly speaking, $Z_{k'}^{\geq \eta_0^{1/e'}} \times_{Z_k^{\geq \eta_0}} AS_{l/k}^{a, \geq \eta_0}$ and the log counterpart should be thought of as the geometric incarnations of the tensor products of the corresponding ring of analytic functions. The new spaces are, however, well-defined rigid analytic spaces over $Z_{k'}^{\geq \eta_0^{1/e'}}$.

Theorem 4.2.4. *For $a \in \mathbb{Q}_{>0}$, there is a one-to-one correspondence between the geometric connected components of $as_{l/k,(\log)}^a$ and the following limit of connected components*:
$$\varprojlim_{k'/k} \lim_{\eta_0 \to 1^-} \pi_0^{\mathrm{geom}}\big(Z_{k'}^{\geq \eta_0^{1/e'}} \times_{Z_k^{\geq \eta_0}} AS_{l/k(,\log)}^{a, \geq \eta_0}\big),$$
where e' is the naïve ramification degree of k'/k and the second limit only takes $\eta_0 \in p^{\mathbb{Q}} \cap (0, 1)$.

Proof. By Lemma 4.2.2 and Example 1.3.4, when $e'a \in \mathbb{Z}$, $Z_{k'}^{\geq \eta_0^{1/e'}} \times_{Z_k^{\geq \eta_0}} AS_{l/k(,\log)}^{a, \geq \eta_0}$ is a lifting space of $as_{l/k(,\log)}^a$. The theorem then follows from Corollary 1.2.12. □

Remark 4.2.5. Here, we need $\eta_0 \in p^{\mathbb{Q}} \cap (0, 1)$ since Corollary 1.2.12 requires it.

Remark 4.2.6. Introducing this ramified extension k'/k to make $e'a \in \mathbb{Z}$ may not be essential, but it eases the proof.

4.3. *Comparison of rigid spaces.* In this subsection, we will prove that the lifting Abbes–Saito spaces are isomorphic to some thickening spaces we constructed in Section 3.4. In this subsection, we continue to assume Hypotheses 2.3.2 and 3.3.1.

Before proving the comparison theorem, we need to analyze Construction 3.3.5 closely and give a new view of $\tilde{\pi}^*$ using differentials. However, the proofs of the following two lemmas are not so enlightening in this generality; the reader may skip them when reading the paper for the first time, but see Remark 4.3.5.

Lemma 4.3.1. *Modulo p, the homomorphism $\tilde{\pi}^*$ gives a continuous homomorphism $\bar{\tilde{\pi}}^* : \kappa_k \to \kappa_k[\![\delta_J]\!]$. For $\bar{g} \in \kappa_k$, we can write $d\bar{g} = \bar{g}_1 d\bar{b}_1 + \cdots + \bar{g}_m d\bar{b}_m$ in $\Omega^1_{\kappa_k/\mathbb{F}_p}$. Then $\bar{\tilde{\pi}}^*(\bar{g}) \equiv \bar{g} + \bar{g}_1\delta_1 + \cdots + \bar{g}_m\delta_m$ modulo $(\delta_J)^2 \cdot \kappa_K[\![\delta_J]\!]$.*

Proof. Use the p-basis to express $\bar{g}$ (uniquely) as $\bar{g} = \sum_{e_J=0}^{p-1} \bar{a}_{e_J}^p \bar{b}_J^{e_J}$ for some $\bar{a}_{e_J} \in \kappa_k$. Thus, $d\bar{g} = \sum_{e_J=0}^{p-1} \bar{a}_{e_J}^p d(\bar{b}_J^{e_J})$. On the other hand, we have
$$\bar{\tilde{\pi}}^*(\bar{g}) \equiv \sum_{e_J=0}^{p-1} \bar{a}_{e_J}^p (\bar{b}_J + \delta_J)^{e_J}$$
modulo $(\delta_J)^p \cdot \kappa_K[\![\delta_J]\!]$. The statement follows by comparing the two formulas. □

Remark 4.3.2. The analogous result for $\tilde{\pi}^*$ is true. Actually, $\tilde{\pi}^*$ is the Taylor expansion homomorphism [Kedlaya 2007, Definition 2.2.2].

Lemma 4.3.3. *Keep the notation as in Section 3.3. We have*

$$\det\left(\frac{\partial(\tilde{\pi}^*(P_i)-P_i)}{\partial\delta_j}\right)_{i,j\in J^+}\Bigg|_{\delta_{J^+}=0} \in \left(\mathcal{O}_K[\![S]\!]\langle U_{J^+}\rangle/(P_{J^+})\right)^\times = (\mathcal{O}_L[\![T]\!])^\times.$$

In particular, the corresponding matrix is invertible.

Proof. It is enough to prove that the matrix is of full rank modulo (p, T). First, modulo (p, T), the first row will be all zero except the first element which is $\mathfrak{d}(\bar{c}_1, \dots, \bar{c}_m) \in \kappa_l^\times$. Hence, we need only to look at

$$\left(\frac{\partial(\tilde{\pi}^*(P_i)-P_i)}{\partial\delta_j}\right)_{i,j\in J} \bmod (p, T, \delta_{J^+}) = \left(\frac{\partial(\bar{\tilde{\pi}}^*(\bar{\mathfrak{b}}_i)-\bar{\mathfrak{b}}_i)}{\partial\delta_j}\right)_{i,j\in J} \bmod (t, \delta_{J^+}). \quad (4.3.4)$$

Let $\bar{\alpha}_{ij} \in \kappa_l$ denote the entries in the matrix on the right-hand side of (4.3.4), where we identify $\mathcal{O}_k\langle u_{J^+}\rangle/(p_{J^+}, u_0) \xrightarrow{\sim} \kappa_l$. Under this identification, $\bar{\mathfrak{b}}_i$ will become $\bar{c}_i^{p^{r_i}}$ for all $i \in J$. It suffices to show that the i-th row is κ_l-linearly independent from the first $i-1$ rows for all i. Write

$$\bar{\mathfrak{b}}_i = \sum_{e_1=0}^{p^{r_0}-1} \cdots \sum_{e_{i-1}=0}^{p^{r_{i-1}}-1} \bar{\lambda}_{e_1,\dots,e_{i-1}} u_1^{e_1} \cdots u_{i-1}^{e_{i-1}},$$

where $\bar{\lambda}_{e_1,\dots,e_{i-1}} \in \kappa_k$ for which $d\bar{\lambda}_{e_1,\dots,e_{i-1}} = \bar{\mu}_{e_1,\dots,e_{i-1},1} d\bar{b}_1 + \cdots + \bar{\mu}_{e_1,\dots,e_{i-1},m} d\bar{b}_m$. Then, by Lemma 4.3.1, we can write

$$\begin{aligned}
&\bar{\alpha}_{i1} d\bar{b}_1 + \cdots + \bar{\alpha}_{im} d\bar{b}_m \\
&\qquad = \sum_{e_1=0}^{p^{r_0}-1} \cdots \sum_{e_{i-1}=0}^{p^{r_{i-1}}-1} \bar{c}_1^{e_1} \dots \bar{c}_{i-1}^{e_{i-1}} (\bar{\mu}_{e_1,\dots,e_{i-1},1} d\bar{b}_1 + \cdots + \bar{\mu}_{e_1,\dots,e_{i-1},m} d\bar{b}_m) \\
&\qquad \equiv d(\bar{c}_i^{p^{r_i}}) \quad \text{modulo } (d\bar{c}_1, \dots, d\bar{c}_{i-1})
\end{aligned}$$

in $\Omega^1_{\boldsymbol{k}_{i-1}/\mathbb{F}_p}$; it is in fact nontrivial because $d\bar{c}_1, \dots, d\bar{c}_m$ form a basis of $\Omega^1_{\kappa_L/\mathbb{F}_p}$ and hence there should not be any auxiliary relation among $d\bar{c}_1, \dots, d\bar{c}_i$ in $\Omega^1_{\boldsymbol{k}_i/\mathbb{F}_p}$. But we know that the sums $\bar{\alpha}_{i'1} d\bar{b}_1 + \cdots + \bar{\alpha}_{i'm} db_m$ for $i' < i$ all lie in the submodule of $\Omega^1_{\boldsymbol{k}_{i-1}/\mathbb{F}_p}$ generated by $d\bar{c}_1, \dots, d\bar{c}_{i-1}$. Hence the i-th row of the matrix in (4.3.4) is $\boldsymbol{k}_{i-1}$-linearly independent from the first $i-1$ rows. The lemma follows. □

Remark 4.3.5. When κ_l/κ_k is modular in the sense of [Sweedler 1968], we can choose the p-basis of κ_k so that $\bar{c}_j^{p^{r_j}} = \bar{b}_j$; in that case, the above lemma is much easier to prove because the matrix, modulo (p, T), is lower triangular with 1 on the diagonal. However, this may not be the case in general; see also Remark 3.3.6.

Theorem 4.3.6. *There exists* $\eta_0' \in (0, 1)$ *such that for any* $a \in \mathbb{Q}_{>1}$ *and any* $\eta_0 \in (\max\{p^{-1/a}, \eta_0'\}, 1)$, *there exists an isomorphism of rigid spaces over* $Z_k^{\geq \eta_0}$:

$$TS_{k\backslash l}^{a, \geq \eta_0} \simeq AS_{l/k}^{a, \geq \eta_0}. \tag{4.3.7}$$

Similarly, There exists $\eta_0' \in (0, 1)$ *such that for any* $a \in \mathbb{Q}_{>0}$ *and any* $\eta_0 \in (\max\{p^{-1/a}, \eta_0'\}, 1)$, *there exists an isomorphism of rigid spaces over* $Z_k^{\geq \eta_0}$:

$$TS_{k\backslash l, \log}^{a, \geq \eta_0} \simeq AS_{l/k, \log}^{a, \geq \eta_0}. \tag{4.3.8}$$

Proof. We give the proof for the case of log-spaces and indicate the changes needed for the nonlog case. The only significant difference between the two is that when constructing the morphism χ_2, we have slightly different approximations. We will match up the ring of functions on the two rigid spaces in (4.3.8) in the log case and (4.3.7) in the nonlog case.

Fix an $\eta_0 \in (p^{-1/a}, 1)$ satisfying Hypothesis 3.3.16.

Recall that $\mathcal{O}_{TS_{k,\log}^{a,\geq\eta_0}} = \mathcal{R}_K^{\eta_0}\langle S^{-a-1}\delta_0, S^{-a}\delta_J\rangle$ (resp. $\mathcal{O}_{TS_k^{a,\geq\eta_0}} = \mathcal{R}_K^{\eta_0}\langle S^{-a}\delta_{J^+}\rangle$). For each $j \in J^+$, $\tilde{\pi}^*(P_j)$ is the polynomial P_j with coefficients replaced by their pull-backs to $\mathcal{O}_{TS_{k,\log}^{a,\geq\eta_0}}$ (resp. $\mathcal{O}_{TS_k^{a,\geq\eta_0}}$) via $\tilde{\pi}^*$. So the rings of functions on $TS_{k\backslash l,\log}^{a,\geq\eta_0}$ and $TS_{k\backslash l}^{a,\geq\eta_0}$ are, respectively,

$$\begin{aligned} \mathcal{R}_{1,\log} &= \mathcal{R}_K^{\eta_0}\langle S^{-a-1}\delta_0, S^{-a}\delta_J\rangle\langle U_{J^+}\rangle/\tilde{\pi}^*(P_{J^+}), \\ \mathcal{R}_1 &= \mathcal{R}_K^{\eta_0}\langle S^{-a}\delta_{J^+}\rangle\langle U_{J^+}\rangle/\tilde{\pi}^*(P_{J^+}). \end{aligned} \tag{4.3.9}$$

By Lemma 3.3.13(1),

$$\begin{aligned} &\tilde{\pi}^*(P_j) \in U_j^{p^{r_j}} - \tilde{\pi}^*(\mathfrak{B}_j) + (p, U_0, S, \delta_0)\cdot \mathcal{O}_K[\![\delta_{J^+}, S]\!][U_{J^+}], \\ &\tilde{\pi}^*(P_0) \in U_0^e - \tilde{\pi}^*(\mathfrak{D})S - \delta_0 + (p, U_0S, S^2, U_0\delta_0, S\delta_0, \delta_0^2)\cdot \mathcal{O}_K[\![\delta_{J^+}, S]\!][U_{J^+}]. \end{aligned}$$

Thus, we can view $\mathcal{R}_{1,\log}$ and $\mathcal{R}_1$ as finite free modules over $\mathcal{O}_{TS_{k,\log}^{a,\geq\eta_0}}$ and $\mathcal{O}_{TS_k^{a,\geq\eta_0}}$, respectively, with basis $\{U_{J^+}^{e_{J^+}} \mid 0 \leq e_0 < e;\ 0 \leq e_j < p^{r_j}, j \in J\}$. For each $\eta \in [\eta_0, 1)$, we norm $\mathcal{R}_{1,\log}$ and $\mathcal{R}_1$ as follows: for $g = \sum \lambda_{e_{J^+}} U_{J^+}^{e_{J^+}}$ with $\lambda_{e_{J^+}} \in \mathcal{O}_{TS_{k,\log}^{a,\geq\eta_0}}$ or $\lambda_{e_{J^+}} \in \mathcal{O}_{TS_k^{a,\geq\eta_0}}$, summed over $e_0 = 0, \dots, e-1$ and $e_j = 0, \dots, p^{r_j}-1$ for $j \in J$, we define

$$|g|_{\mathcal{R}_{1,\log},\eta} = \max_{e_{J^+}}\{|\lambda_{e_{J^+}}|_{TS_{k,\log}^{a,\eta}} \cdot \eta^{e_0/e}\} \quad \text{and} \quad |g|_{\mathcal{R}_1,\eta} = \max_{e_{J^+}}\{|\lambda_{e_{J^+}}|_{TS_k^{a,\eta}} \cdot \eta^{e_0/e}\}.$$

It is clear that $\mathcal{R}_{1,\log}$ and $\mathcal{R}_1$ are the Fréchet completions for the norms $|\cdot|_{\mathcal{R}_{1,\log},\eta}$ and $|\cdot|_{\mathcal{R}_1,\eta}$, for all $\eta \in [\eta_0, 1)$.

On the other hand, by the definition of $AS_{l/k,\log}^{a,\geq\eta_0}$ and $AS_{l/k}^{a,\geq\eta_0}$, their respective rings of functions are

$$\begin{aligned} \mathcal{R}_{2,\log} &= \mathcal{R}_K^{\eta_0}\langle S^{-a-1}V_0, S^{-a}V_J\rangle\langle U_{J^+}\rangle/(P_{J^+} - V_{J^+}), \\ \mathcal{R}_2 &= \mathcal{R}_K^{\eta_0}\langle S^{-a}V_{J^+}\rangle\langle U_{J^+}\rangle/(P_{J^+} - V_{J^+}), \end{aligned}$$

which are clearly finite free modules over $W_{\log} = \mathcal{R}_K^{\eta_0}\langle V_0/\eta^{a+1}, V_J/\eta^a\rangle$ and $W = \mathcal{R}_K^{\eta_0}\langle V_{J^+}/\eta^a\rangle$, respectively, with basis $\{U_{J^+}^{e_{J^+}} \mid 0 \le e_0 < e;\ 0 \le e_j < p^{r_j},\ j \in J\}$.

Similarly, for $\eta \in [\eta_0, 1)$, we norm $\mathcal{R}_{2,\log}$ and $\mathcal{R}_2$ as follows: for $g = \sum \lambda_{e_{J^+}} U_{J^+}^{e_{J^+}}$ with $\lambda_{e_{J^+}} \in W_{\log}$ or $\lambda_{e_{J^+}} \in W$, summed over $e_0 = 0, \dots, e-1$ and $e_j = 0, \dots, p^{r_j}-1$ for $j \in J$, we define

$$|g|_{\mathcal{R}_{2,\log},\eta} = \max_{e_{J^+}}\{|\lambda_{e_{J^+}}|_{W_{\log}} \cdot \eta^{e_0/e}\} \quad \text{and} \quad |g|_{\mathcal{R}_2,\eta} = \max_{e_{J^+}}\{|\lambda_{e_{J^+}}|_W \cdot \eta^{e_0/e}\}.$$

It is clear that $\mathcal{R}_{2,\log}$ and $\mathcal{R}_2$ are the Fréchet completions for the norms $|\cdot|_{\mathcal{R}_{2,\log},\eta}$ and $|\cdot|_{\mathcal{R}_2,\eta}$, for all $\eta \in [\eta_0, 1)$.

We will identify the U_{J^+} in different rings, but V_{J^+} will not be same as δ_{J^+}. Be cautioned that the two norms will not be the same under the identification; but they will give the same topology.

Now, we define a continuous K-homomorphism $\chi_1 : \mathcal{R}_{2,\log} \to \mathcal{R}_{1,\log}$ (resp. $\chi_1 : \mathcal{R}_2 \to \mathcal{R}_1$) so that $\chi_1(S) = S$, $\chi_1(U_j) = U_j$, $\chi_1(V_j) = P_j(U_{J^+})$ for all $j \in J^+$. We need only to check that for any $\eta \in [\eta_0, 1)$,

$$\begin{gathered}|\chi_1(V_j)|_{\mathcal{R}_{1,\log},\eta} \le \begin{cases}\eta^{a+1} & \text{if } j = 0,\\ \eta^a & \text{if } j \in J,\end{cases}\\ |\chi_1(V_j)|_{\mathcal{R}_1,\eta} \le \eta^a \ \text{ for all } j \in J^+.\end{gathered} \tag{4.3.10}$$

Here we need separate arguments for the logarithmic and nonlogarithmic cases. In the former case, inequality (3.2.9) says that

$$|P_j - \tilde{\pi}^*(P_j)|_{\mathcal{R}_{1,\log},\eta} \le \eta^a |P_j|_{\mathcal{R}_{2,\log},\eta}$$

for $j \in J^+$, which gives exactly the bound in (4.3.10) because $|P_0|_{\mathcal{R}_{2,\log},\eta} \le \eta$ and $|P_j|_{\mathcal{R}_{2,\log},\eta} \le 1$ for $j \in J$ by Lemma 3.3.13(1).

In the nonlogarithmic case, combining Lemma 3.3.13(1) and inequality (3.2.10), one has $|P_j - \tilde{\pi}^*(P_j)|_{\mathcal{R}_1,\eta} \le \eta^a$ for $j \in J^+$; inequality (4.3.10) follows.

Conversely, we will define a continuous K-homomorphism $\chi_2 : \mathcal{R}_{1,\log} \to \mathcal{R}_{2,\log}$ or $\chi_2 : \mathcal{R}_1 \to \mathcal{R}_2$ as the inverse to χ_1. Obviously, we need $\chi_2(S) = S$, $\chi_2(U_j) = U_j$ for all $j \in J^+$. The only thing not clear is $\chi_2(\delta_j)$ for all $j \in J^+$.

By Lemma 4.3.3, let

$$\begin{aligned}A := \big(\partial(\tilde{\pi}^*(P_i) - P_i)/\partial\delta_j\big)_{i,j\in J^+}\big|_{\delta_{J^+}=0} &\in \mathrm{GL}_{m+1}(\mathcal{O}_L[\![T]\!])\\ &\cong \mathrm{GL}_{m+1}\big(\mathcal{O}_K[\![S]\!]\langle U_{J^+}\rangle/P_{J^+}\big).\end{aligned}$$

Let B be the $(m+1)\times(m+1)$ matrix whose entries are in the free $\mathcal{O}_K[\![S]\!]$-module generated by the basis in Lemma 3.3.13(1) and which has image A^{-1} in $M_{m+1}(\mathcal{O}_K[\![S]\!]\langle U_{J^+}\rangle/(P_{J^+}))$. Then, if I denotes the $(m+1)\times(m+1)$ identity matrix, we have

$$BA - I \in \mathrm{Mat}_{m+1}\big((\delta_{J^+})\cdot\mathcal{O}_K[\![S]\!]\langle U_{J^+}\rangle\big), \tag{4.3.11}$$

Now, we write

$$\begin{pmatrix}\delta_0\\ \vdots\\ \delta_m\end{pmatrix} = (I-BA)\begin{pmatrix}\delta_0\\ \vdots\\ \delta_m\end{pmatrix} - B\left(\begin{pmatrix}\tilde{\pi}^*(P_0)-P_0\\ \vdots\\ \tilde{\pi}^*(P_m)-P_m\end{pmatrix} - A\begin{pmatrix}\delta_0\\ \vdots\\ \delta_m\end{pmatrix}\right) - B\begin{pmatrix}P_0\\ \vdots\\ P_m\end{pmatrix}, \tag{4.3.12}$$

the last term being just $-B\chi_1(V_{J^+})$. We need to bound the first two terms.

By (4.3.11), $I - BA$ has norm $\leq \eta^a$. Hence, in the nonlogarithmic case, the first term in (4.3.12) has norm $\leq \eta^{2a}$; in the logarithmic case the first term in (4.3.12) has norm $\leq \eta^{2a}$, except for the first row, which has norm $\leq \eta^{2a+1}$. By the definition of A and Theorem 3.2.8, the second term in (4.3.12) has norm $\leq \eta^{2a}$ in the nonlogarithmic case; it has norm $\leq \eta^{2a}$ in the logarithmic case, except for the first row, which has norm $\leq \eta^{2a+1}$.

Since we want χ_2 to be the inverse of χ_1, we define recursively

$$\chi_2\begin{pmatrix}\delta_0\\ \vdots\\ \delta_m\end{pmatrix} = -B\begin{pmatrix}V_0\\ \vdots\\ V_m\end{pmatrix} + \chi_2\begin{pmatrix}\Lambda_0\\ \vdots\\ \Lambda_m\end{pmatrix}, \tag{4.3.13}$$

where Λ_{J^+} denotes the sum of the first two terms in (4.3.12). Since Λ_{J^+} has strictly smaller norm than δ_{J^+} and is in the ideal (δ_{J^+}), one can plug the image of $\chi_2(\delta_{J^+})$ back into $\chi_2(\Lambda_{J^+})$ and iterate this substitution. This iteration will converge to the value of $\chi(\delta_{J^+})$ as an element in $\mathcal{R}_{2,\log}$ or $\mathcal{R}_2$. Moreover, from the construction, one can see that

$$\begin{aligned} &|\chi_2(\delta_j)|_{\mathcal{R}_{1,\eta}} \leq \eta^a && \text{for all } \eta\in[\eta_0,1) \text{ and } j\in J^+,\\ &\left.\begin{aligned}|\chi_2(\delta_0)|_{\mathcal{R}_{1,\log},\eta} \leq \eta^{a+1}\\ |\chi_2(\delta_j)|_{\mathcal{R}_{1,\log},\eta} \leq \eta^{a}\end{aligned}\right\} && \text{for all } \eta\in[\eta_0,1) \text{ and } j\in J.\end{aligned}$$

Hence, if we define

$$\chi_2 : \mathcal{R}_K^{\eta_0}\langle S^{-a-1}\delta_0, S^{-a}\delta_J\rangle\langle U_{J^+}\rangle \to \mathcal{R}_{2,\log} \quad \text{and} \quad \chi_2 : \mathcal{R}_K^{\eta_0}\langle S^{-a}\delta_{J^+}\rangle\langle U_{J^+}\rangle \to \mathcal{R}_2$$

by $\chi_2(u_{J^+}) = u_{J^+}$, then $\chi_2(\delta_{J^+})$ is the limit we obtained above; this gives a continuous homomorphism. We will check that this homomorphism factors through $\mathcal{R}_{1,\log}$ or $\mathcal{R}_1$. Indeed, by the recursive formula (4.3.13), which is (4.3.12) after applying χ_2, we see that

$$-B\chi_2\begin{pmatrix}\tilde{\pi}^*(P_0)-P_0\\ \vdots\\ \tilde{\pi}^*(P_m)-P_m\end{pmatrix} - B\begin{pmatrix}V_0\\ \vdots\\ V_m\end{pmatrix} = \begin{pmatrix}0\\ \vdots\\ 0\end{pmatrix}.$$

We know that B has an invertible image in $\mathrm{GL}_{m+1}(\mathcal{O}_K[\![S]\!]\langle U_{J^+}/(P_{J^+})\rangle)$, and so is invertible over $\mathcal{R}_{1,\log}$ or $\mathcal{R}_1$. We must have $0 = \chi_2(\tilde{\pi}^*(P_j) - P_j) + V_j =$

$\chi_2(\tilde{\pi}^*(P_j)) + V_j - P_j = \chi_2(\tilde{\pi}^*(P_j))$ for all $j \in J^+$. This proves that χ_2 factors through $\mathcal{R}_{1,\log}$ or $\mathcal{R}_1$.

Finally, we claim that χ_2 and χ_1 are inverse to each other. One may check this from the definition directly. Alternatively, we observe that, by our definition, they are inverse to one another on a dense subset $K[S, u_{J^+}]$, the polynomial ring inside the Fréchet algebras; therefore, they have to be inverse to one another and give an isomorphism between the ring of functions on Abbes–Saito space and the ring of functions on thickening space. □

Remark 4.3.14. The isomorphisms constructed in Theorem 4.3.6 are canonical in the sense that they match up U_{J^+} on both sides. However, slight perturbations of the isomorphisms will continue to be isomorphic. This point will be important when studying the mixed characteristic case.

4.4. ***Comparison of conductors.*** In this subsection, we will prove the comparison between the arithmetic conductors and the differential conductors. As a reminder, we do not impose Hypotheses 2.3.2 and 3.3.1 in this subsection.

Theorem 4.4.1. *Let k be a complete discretely valued field of equal characteristic $p > 0$ and let G_k be its absolute Galois group. For a p-adic representation $\rho : G_k \to \mathrm{GL}(V_\rho)$ of finite local monodromy, the arithmetic Artin conductor* $\mathrm{Art}_{\mathrm{ar}}(\rho)$ *of ρ coincides with the differential Artin conductor* $\mathrm{Art}_{\mathrm{dif}}(\rho)$; *the arithmetic Swan conductor* $\mathrm{Swan}_{\mathrm{ar}}(\rho)$ *coincides with the differential Swan conductor* $\mathrm{Swan}_{\mathrm{dif}}(\rho)$.

Proof. It suffices to prove this for irreducible representations, as all the conductors are additive. All the conductors remain the same if we pass to the completion of the unramified closure of k, by Proposition 4.1.7(4) and Theorem 2.4.1(2). Thus we may assume that the residue field κ_k is separably closed; hence ρ factors through the Galois group of a finite totally ramified extension l/k as $\rho : G_k \twoheadrightarrow \mathrm{Gal}(l/k) \hookrightarrow \mathrm{GL}(V_\rho)$ with the second map injective. Moreover, we may assume that l/k is wildly ramified because the theorem is known when l/k is tamely ramified, by Proposition 4.1.7(6) and Theorem 2.4.1(3). To sum up, we may assume Hypothesis 3.3.1. In particular, $b_{\mathrm{ar}}(l/k) > 1$ and $b_{ar,\log}(l/k) > 0$.

Next, we want to reduce to the case when the p-basis of k is finite. In view of Construction 3.3.5, one can choose a p-basis of l so that all but a finite number of elements are actually in k. Let $(c_i)_{i\in I}$ be a subset of those elements in the p-basis which lie in k. Set $\tilde{k} = k(c_i^{1/p^n} \mid i \in I, n \in \mathbb{N})^\wedge$ and $\tilde{l} = l\tilde{k}$. We claim that $\mathcal{O}_{\tilde{l}} = \mathcal{O}_l \otimes_{\mathcal{O}_k} \mathcal{O}_{\tilde{k}}$. Indeed, after base change to $\tilde{k}$, the valued groups do not change: $|\tilde{k}^\times| = |k^\times|$. Thus, $[|\tilde{l}^\times| : |\tilde{k}^\times|] \geq [|l^\times| : |k^\times|]$. On the other hand, the residue field extension of $\tilde{l}/\tilde{k}$ has degree at least the same as κ_l/κ_k because $\bar{c}_{J\setminus I}$ are not in the residue field of $\tilde{k}$. But we know that the degree of the extension does not increase. Therefore, we have equality on both naïve ramification degrees and degrees of residue field extension. It is then clear that $\mathbb{O}_{\tilde{l}} = \mathbb{O}_l \otimes_{\mathbb{O}_k} \mathbb{O}_{\tilde{k}}$, as the right-hand side contains the uniformizer of the left-hand side and both sides are isomorphic modulo that uniformizer. Therefore, by Proposition 4.1.7(4), $b_{\mathrm{ar}}(\tilde{l}/\tilde{k}) = b_{\mathrm{ar}}(l/k)$.

On the differential conductors side, [Kedlaya 2007, Lemma 3.5.4] shows for the log case (the nonlog case follows by a similar argument) that we can consider only a finite number of elements in the p-basis and the differential conductors are unchanged after taking an inseparable field extension with respect to other elements in the p-basis.

To sum up, we can make an inseparable extension so that all conductors do not change, and we are reduced to the case where Hypothesis 2.3.2 holds.

Now, we will prove the comparison theorem for the Swan conductors and the proof for the Artin conductors follows verbatim, except replacing Swan by Art, replacing $a > 0$ by $a > 1$, and dropping all the logs in the subscripts.

Since ρ is irreducible, $\mathrm{Swan}_{\mathrm{ar}}(\rho) = b_{ar,\log}(l/k) \cdot \dim V_\rho$. Recall from Section 2.3, we can associate with ρ a differential module $\mathscr{E}_\rho$ over $\mathscr{R}_K^{\eta_0} \otimes_{\mathbb{Q}_q} F$ for some $\eta_0 \in (0, 1)$. As the representation ρ is irreducible, $\mathscr{E}_\rho$ has a unique ramification break $b_{\mathrm{dif},\log}(\mathscr{E}_\rho)$. So the differential Swan conductor of ρ is $\mathrm{Swan}_{\mathrm{dif}}(\rho) = b_{\mathrm{dif},\log}(\mathscr{E}_\rho) \cdot \dim V_\rho$. Therefore, to conclude, it suffices to show that $b_{ar,\log}(l/k) = b_{\mathrm{dif},\log}(\mathscr{E}_\rho)$.

We do this by means of a chain of equivalences. By the equivalence (a) $\Longleftrightarrow$ (d) in Theorem 3.4.12, the inequality $a > b_{\mathrm{dif},\log}(\mathscr{E}_\rho)$ is equivalent to this condition:

For any (or some) extension l'/l with naïve ramification degree e',

$$\pi_0^{\mathrm{geom}}\big(Z_{l'}^{\geq \eta_0^{1/ee'}} \times_{Z_l^{\geq \eta_0^{1/e}}} TS^{a,\geq \eta_0}_{l/k\backslash l,\log}\big) = [l:k], \text{ when } \eta_0 \to 1^-. \qquad (*)$$

By Theorem 4.2.4, the condition $(*)$ is equivalent to $\pi_0^{\mathrm{geom}}(as^a_{l/k,\log}) = [l:k]$, where a is a rational number. But this is the same as $a > b_{ar,\log}(l/k)$. □

Remark 4.4.2. In an early version of this paper, Theorem 4.4.1 is stated for representations with finite image. Andrea Pulita pointed out that this could be extended to the finite local monodromy case by a standard argument as in the proof.

Corollary 4.4.3. (a) (Hasse–Arf Theorem) *Let k be a complete discretely valued field of equal characteristic $p > 0$, let G_k be its absolute Galois group, and let $\rho : G_k \to \mathrm{GL}(V_\rho)$ be a p-adic representation of finite local monodromy. Then the arithmetic Artin conductor $\mathrm{Art}_{\mathrm{ar}}(\rho)$ and the arithmetic Swan conductor $\mathrm{Swan}_{\mathrm{ar}}(\rho)$ are integers.*

(b) *Let k be a complete discretely valued field of equal characteristic $p > 0$. Then the subquotients $\mathrm{Fil}^a G_k/\mathrm{Fil}^{a+} G_k$ (resp. $\mathrm{Fil}^a_{\log} G_k/\mathrm{Fil}^{a+}_{\log} G_k$) of the arithmetic ramification filtrations are elementary p-abelian groups if $a \in \mathbb{Q}_{>1}$ (resp. $a \in \mathbb{Q}_{>0}$) and are trivial if $a \notin \mathbb{Q}$.*

Proof. This follows from Theorems 2.4.1 and 4.4.1. □

5. Applications

In this section, we give two applications of the comparison Theorem 4.4.1. The first is to deduce an integrality result concerning the ramification filtration of finite flat group schemes, introduced in [Abbes and Mokrane 2004]. The other is to compare the arithmetic and differential Artin conductors to the Artin conductor defined by Borger [2004].

Remark 5.0.4. All applications in this section can be carried over to the mixed characteristic case if there is a good theory of differential conductors. For the application to finite flat group schemes, one needs the Hasse–Arf theorem of arithmetic Artin conductors; for the comparison with Borger's Artin conductor, one needs a mixed characteristic version of Proposition 5.4.1. In the absence of these statements, we only focus on the equal characteristic p case throughout this section.

5.1. ***Hasse–Arf theorem for finite flat group schemes.*** We first recall some definitions and basic properties from [Abbes and Mokrane 2004; Hattori 2008]. Then, we use a theorem by Raynaud [Berthelot et al. 1982, théorème 3.1.1] to reduce the integrality result to the case of finite Galois extension of complete discretely valued fields.

Keep the notation as in previous sections. We do not assume any hypothesis on k (and there will be no l in this subsection).

Convention 5.1.1. All finite flat groups schemes are commutative.

The construction of the canonical filtration on a generically étale finite flat group scheme is similar to that of the arithmetic ramification filtration.

Definition 5.1.2. Let A be a finite flat $\mathbb{O}_k$-algebra. Write $A = \mathbb{O}_k[x_1, \dots, x_n]/\mathcal{I}$ with $\mathcal{I}$ an ideal generated by $f_1, \dots, f_r$. For $a \in \mathbb{Q}_{>0}$, define the rigid space

$$X^a = \bigl\{(x_1, \dots, x_n) \in A^n_K[0, 1] \bigm| |f_\alpha(x_1, \dots, x_n)| \leq \theta^a, \alpha = 1, \dots, r\bigr\},$$

where $\theta = |s|$ as in Notation 4.1.1. The *highest break* $b(A/\mathbb{O}_k)$ is the smallest number such that $\pi_0^{\mathrm{geom}}(X^a) = \operatorname{rank}_{\mathbb{O}_k} A$ for all $a > b(A/\mathbb{O}_k)$. This is the same as Definition 4.1.3 if $A = \mathbb{O}_l$, except here we use the ring of integers instead of the fields in the notation.

Notation 5.1.3. A finite flat group scheme $G = \operatorname{Spec} A$ is *generically étale* if $G \times_{\mathbb{O}_k} k$ is étale over k; it is *generically trivial* if $G \times_{\mathbb{O}_k} k$ is a disjoint union of copies of $\operatorname{Spec} k$.

Definition 5.1.4. For a geometrically étale finite flat group scheme $G = \operatorname{Spec} A$, we have a natural map of points $G(k^{\mathrm{alg}}) \hookrightarrow X^a(k^{\mathrm{alg}})$; further composing with the map for geometric connected components, we obtain a map

$$\sigma^a : G(k^{\mathrm{alg}}) \hookrightarrow X^a(k^{\mathrm{alg}}) \to \pi_0^{\mathrm{geom}}(X^a).$$

Define G^a to be the closure of $\ker \sigma^a$. We use $b(G/\mathbb{O}_k)$ to denote the highest break $b(A/\mathbb{O}_k)$; then for $a > b(G/\mathbb{O}_k)$, $G^a = \operatorname{Spec} \mathbb{O}_k$.

Proposition 5.1.5 [Abbes and Mokrane 2004, lemme 2.3.2]. *Let*

$$0 \to G' \to G \to G'' \to 0$$

be an exact sequence of finite flat group schemes. For $a > 0$,

$$0 \to G'^a \to G^a \to G''^a \to 0$$

is exact.

Caution 5.1.6. For a subgroup scheme $H \subset G$ and $a \geq 0$, we do not know how to link H^a with $H \times_G G^a$.

The following question was first raised in [Hattori 2008], and the result is essentially due to Hattori. The author thanks him for clarifying this and for permission to include the proof here.

Theorem 5.1.7. *Let $\mathbb{O}_k$ be a complete discrete valuation ring of equal characteristic p. For any generically trivial finite flat groups scheme G over $\mathbb{O}_k$, $b(G/\mathbb{O}_k)$ is a nonnegative integer.*

Proof. We may assume that G is connected by taking the connected component of the identity. By a theorem of Raynaud [Berthelot et al. 1982, théorème 3.1.1], we may realize G as the kernel of an isogeny $f : \mathfrak{B} \to \mathfrak{A}$ of two abelian schemes over $\operatorname{Spec} \mathbb{O}_k$. Let α and β be generic points of the special fibers of $\mathfrak{A}$ and $\mathfrak{B}$, respectively. Then by [Abbes and Mokrane 2004, lemme 2.1.6], $b(\mathbb{O}^\wedge_{\mathfrak{B},\beta}/\mathbb{O}^\wedge_{\mathfrak{A},\alpha}) = b(G/\mathbb{O}_k)$.

Since the generic fiber of G is a disjoint union of copies of $\operatorname{Spec} k$, we know that $\mathbb{O}^\wedge_{\mathfrak{B},\beta}/\mathbb{O}^\wedge_{\mathfrak{A},\alpha}$ is a generically étale finite Galois extension of complete discrete valuation rings, with Galois group $G(k)$; in particular, all irreducible representations of this Galois group over an algebraically closed field are one-dimensional. By Hasse–Arf Theorem 4.4.1, $b(\mathbb{O}^\wedge_{\mathfrak{B},\beta}/\mathbb{O}^\wedge_{\mathfrak{A},\alpha}) = b(G/\mathbb{O}_k)$ is an integer. □

5.2. *Generic p^∞-th roots.* In this subsection, we introduce the notation of generic p^∞-th roots. This idea was first introduced in [Borger 2004] as a key ingredient of Borger's Artin conductor.

Keep the notation as in previous sections. We assume Hypothesis 2.3.2, that k has a finite p-basis b_J.

Notation 5.2.1. Let $x_1, \dots, x_m$ be transcendental over k. Define k' to be the completion of $k(x_1, \dots, x_m)$ with respect to the $(1, \dots, 1)$-Gauss norm. Set $l' = k'l$. Clearly, l' is the completion of $l(x_1, \dots, x_m)$ with respect to the $(1, \dots, 1)$-Gauss norm. We call $x_1, \dots, x_m$ *dummy variables*.

Definition 5.2.2. We use *adding generic p^∞-th roots* to refer to the following procedure. Consider

$$k \hookrightarrow \tilde{k} = k'\big((b_j + x_j s)^{1/p^n};\ j \in J, n \in \mathbb{N}\big)^\wedge,$$

instead of k; namely, put all p-power roots of $b_j + x_j s$ for all $j \in J$ into k' and then take the completion. We provide $\tilde{k}$ with the p-basis x_J, i.e., replacing b_j by x_j for all $j \in J$. For a finite field extension l/k, we replace it by the extension of the composite $\tilde{l} = l\tilde{k}/\tilde{k}$. Note that $\mathrm{Gal}(\tilde{l}/\tilde{k}) = \mathrm{Gal}(l/k)$ as $\tilde{k}$ is linearly independent from l.

The proof of the following proposition is essentially the same as [Kedlaya 2007, Lemma 3.5.4]. It is also implicitly contained in Borger's construction of Artin conductors (Section 5.3).

Proposition 5.2.3. *Let l/k be a finite Galois extension of complete discretely valued fields of equal characteristic p and with finite p-basis. Then, after a finite number of operations of adding generic p^∞-th roots, the field extension has separable residue field extension.*

Proof. First, the tamely ramified part is always preserved under these operations. So, we can assume that l/k is totally wildly ramified and hence the Galois group $G_{l/k}$ is a p-group. We can filter the extension l/k as $k = k_0 \subset \cdots \subset k_n = l$, where k_i/k_{i-1} is a (wildly ramified) $\mathbb{Z}/p\mathbb{Z}$-Galois extension and k_i/k is Galois for each $i = 1, \dots, n$. Each of these subextensions

(a) either has inseparable residue field extension (and so has naïve ramification degree 1), or

(b) has separable residue field extension (and so has naïve ramification degree p).

Let i_0 be the maximal number such that k_i/k_{i-1} has separable residual extension for $i = 1, \dots, i_0$. Obviously adding generic p^∞-th roots does not decrease i_0 because after adding generic p^∞-th roots, the naïve ramification degree of $\tilde{k}_{i_0}/\tilde{k}$ still equals the degree p^{i_0}. It then suffices to show that after a finite number of operations of adding generic p^∞-th roots, k_{i_0+1}/k_{i_0} has separable residue field extension. Suppose the contrary.

Let $g \in G_{k_{i_0+1}/k_{i_0}} \simeq \mathbb{Z}/p\mathbb{Z}$ be a generator. We claim that

$$\gamma = \min_{w \in \mathbb{O}_{k_{i_0+1}}} \big(v_{k_{i_0+1}}(g(w) - w)\big)$$

decreases by at least 1 after adding p^∞-th roots. This would conclude the proposition, as γ is always a nonnegative integer, which would lead to a contradiction.

Let z be a generator of $\mathbb{O}_{k_{i_0+1}}$ as an $\mathbb{O}_{k_{i_0}}$-algebra. It satisfies an equation

$$z^p + a_1 z^{p-1} + \cdots + a_p = 0, \tag{5.2.4}$$

where $a_1, \dots, a_{p-1} \in \mathfrak{m}_{k_{i_0}}$ and $a_p \in \mathbb{O}_{k_{i_0}}^\times$ with $\bar{a}_p \in \kappa_{k_{i_0}}^\times \backslash (\kappa_{k_{i_0}}^\times)^p = \kappa_k^\times \backslash (\kappa_k^\times)^p$. It is easy to see that $\gamma = v_{k_{i_0}}(g(z) - z)$.

Adding generic p^∞-th roots to k gives us the field $\tilde{k}$. Now, the field extension $\tilde{k}k_{i_0+1}/\tilde{k}k_{i_0}$ is also generated by z as above. But we can write $a_p = \alpha^p + \beta$ for $\alpha \in \mathbb{O}_{\tilde{k}k_{i_0}}$ and $\beta \in \mathfrak{m}_{\tilde{k}k_{i_0}}$. Hence if we substitute $z' = z + \alpha$ into (5.2.4), we get $z'^p + a_1' z'^{p-1} + \cdots + a_p' = 0$, with $a_1', \dots, a_p' \in \mathfrak{m}_{\tilde{k}k_{i_0}}$. Hence, $v_{\tilde{k}k_{i_0+1}}(z') > 0$. By assumption that the extension $\tilde{k}k_{i_0+1}/\tilde{k}k_{i_0}$ has naïve ramification degree 1, a uniformizer $\pi_{k_{i_0}}$ of k_{i_0} is also a uniformizer for $\tilde{k}k_{i_0+1}$ and hence $z'/\pi_{k_{i_0}}$ lies in $\mathbb{O}_{\tilde{k}k_{i_0+1}}$. Thus,

$$\begin{aligned}\gamma' &= \min_{w \in \mathbb{O}_{\tilde{k}k_{i_0+1}}} \big(v_{\tilde{k}k_{i_0+1}}(g(w) - w) \big) \\ &\le v_{\tilde{k}k_{i_0+1}} \big(g(z'/\pi_{k_{i_0}}) - z'/\pi_{k_{i_0}} \big) = v_{k_{i_0+1}}(g(z) - z) - 1 = \gamma - 1.\end{aligned}$$

This proves the claim and hence the proposition. □

5.3. *Borger's Artin conductors.* We start with reviewing Borger's definition of Artin conductors following [Borger 2004]. Then, we prove the comparison theorem linking this to arithmetic and differential conductors.

Keep the notation as above. Let k be a complete discretely valued field of equal characteristic p, with no further hypothesis added. In fact, Borger's construction works in the mixed characteristic case, but we only focus on the equal characteristic case (see Remark 5.0.4).

Definition 5.3.1. An $\mathbb{F}_p$-algebra R is called *perfect* if $F : x \mapsto x^p$ is an isomorphism. For an $\mathbb{F}_p$-algebra R, we use $R^{\mathrm{pf}} = \bigcup_{n \in \mathbb{N}} R^{1/p^n}$ to denote its *perfection*. Let $\mathtt{CRP}_{\mathbb{O}_k}$ be the subcategory of the category of $\mathbb{O}_k$-algebras consisting of flat $\mathbb{O}_k$-algebras A, complete with respect to the $\mathfrak{m}_k$-adic topology and for which $A/\mathfrak{m}_k A$ is perfect.

Proposition 5.3.2 [Borger 2004, Theorem 1.4]. *This category* $\mathtt{CRP}_{\mathbb{O}_k}$ *has an initial object* $\mathbb{O}_k^u$, *the* universal residual perfection *of* $\mathbb{O}_k$. *We have an equivalence of categories*

$$\mathtt{CRP}_{\mathbb{O}_k} \xrightarrow{\sim} \mathtt{PerfAlg}_{\overline{\mathbb{O}_k^u}}, \quad A \mapsto A/\mathfrak{m}_k A, \tag{5.3.3}$$

where $\mathtt{PerfAlg}_{\overline{\mathbb{O}_k^u}}$ *is the category of perfect* $\mathbb{O}_k^u/\mathfrak{m}_k\mathbb{O}_k^u$*-algebras.*

Definition 5.3.4. Let $\mathbb{O}_k^g$ be the inverse image of $\mathrm{Frac}(\mathbb{O}_k^u/\mathfrak{m}_k\mathbb{O}_k^u)$ under (5.3.3), called the *generic residual perfection* of $\mathbb{O}_k$. Let

$$k^g = \mathrm{Frac}(\mathbb{O}_k^g).$$

By Proposition 5.3.2, $\mathbb{O}_k^g$ is a complete discrete valuation ring with perfect residue field.

We have a homomorphism of Galois groups $G_{k^g} \to G_k$. Given a representation ρ of G_k with finite image, we define the *Borger's Artin conductor* $\mathrm{Art}_B(\rho)$ to be $\mathrm{Art}(\rho_{G_{k^g}})$, where the latter term is as in the classical definition [Serre 1979].

Remark 5.3.5. Borger [2004] only defined Artin conductors for representations of finite image. We expect his definition can be extended to representations of finite local monodromy. However, this additional freedom is not essential, so we stick to the finite image case to ease the argument.

Obviously, Borger's Artin conductors have a Hasse–Arf property naturally inherited from that of k^g, a complete discretely valued field with perfect residue field.

Proposition 5.3.6 [Borger 2004, Theorem A]. *Borger's Artin conductor* $\mathrm{Art}_B(\rho)$ *is a nonnegative integer and it coincides with the classical definition when the residue field κ_k is perfect.*

[Borger 2004, Proposition 2.3] *Furthermore*, $\mathrm{Art}_B(\rho)$ *is unchanged after a finite unramified extension of k.*

Moreover, Borger proved that his definition coincides with a variant of arithmetic Artin conductor Art_K for characters using the definition of Kato [1989]. (As we will not use Kato's definition, we just mention the following proposition as a fact.)

Proposition 5.3.7 [Borger 2004, Theorem B]. *Let χ be a class in $H^1(G_k, \mathbb{Q}/\mathbb{Z})$ and χ' its image in $H^1(G_{k^g}, \mathbb{Q}/\mathbb{Z})$. Then* $\mathrm{Art}_K(\chi) = \mathrm{Art}_K(\chi')$. *In particular, for a rank-one representation ρ of G_k with finite image*, $\mathrm{Art}_K(\rho) = \mathrm{Art}_B(\rho)$.

Borger gave the following explicit descriptions of k^u and k^g.

Proposition 5.3.8. *We have $k^u = (\kappa_k[v_{i,j} \mid j \in J, i \in \mathbb{N}])^{\mathrm{pf}}((\pi_{k^u}))$. The homomorphism $k \hookrightarrow k^u$ is determined by $s \mapsto \pi_{k^u}$ and $b_j \mapsto b_j + \sum_{i>0} v_{i,j}\pi_{k^u}^i$. Also, $k^g = \mathrm{Frac}(\kappa_k[v_{i,j};\, j \in J, i \in \mathbb{N}]^{\mathrm{pf}})((\pi_{k^u}))$ and the homomorphism $k \to k^g$ is given by composing $k \hookrightarrow k^u$ with the natural morphism $k^u \hookrightarrow k^g$.*

5.4. ***Comparison with Borger's conductors.*** The key to proving the comparison between Borger's Artin conductors and the arithmetic Artin conductors is to study how the arithmetic Artin conductors behave under the operations of adding generic p^∞-th roots.

In this subsection, we do not impose any hypothesis on k.

Proposition 5.4.1. *Assume Hypothesis 2.3.2. For representations of finite image, the differential Artin conductor for a representation of finite image is unchanged after adding generic p^∞-th roots.*

Proof. Since the operation of adding p^∞-th roots does not change the Galois group of the finite Galois extension, we may assume that the representation is irreducible and totally and wildly ramified. Hence it suffices to consider the differential ramification break of a totally and wildly ramified finite Galois extension l/k.

Recall that we have a differential module $\mathscr{E}$ over $Z_k^{\geq\eta_0} = A_K^1[\eta_0, 1)$ for some $\eta_0 \in (0, 1)$ with differential operators ∂_{B_J} and ∂_S, associated with the regular representation of $\mathrm{Gal}(l/k)$ over $\mathbb{Q}_p$. The base change $k \hookrightarrow k' = k(x_J)^\wedge$ is translated into the base change of $\mathscr{E}$ into $\mathscr{E}'$, from $Z_k^{\geq\eta_0}$ to $Z_{k'}^{\geq\eta_0} = A^1_{K(X_J)^\wedge}[\eta_0, 1)$, where $K' = K(X_J)^\wedge$ is the completion of $K(X_J)$ with respect to the $(1, \dots, 1)$-Gauss norm; $\mathscr{E}'$ has differential operators ∂_{B_J}, ∂_{X_J}, and ∂_S.

Consider the rotation $f : Z_{k'}^{\geq\eta_0} \to Z_{k'}^{\geq\eta_0}$ by $f^*(B_J) = B_J + X_J S$, $f^*(X_J) = X_J$, and $f^*(S) = S$; write ∂'_{B_J}, ∂'_{X_J}, and ∂'_S for the action of differential operators on $f^*\mathscr{E}'$. Then

$$\partial'_{B_J} = \partial_{B_J}, \qquad \partial'_{X_J} = S \cdot \partial_{B_J} + \partial_{X_J}, \qquad \partial'_S = \sum_{j \in J} X_j \cdot \partial_{B_j} + \partial_S.$$

Since X_J are transcendental over K, we have

$$\begin{aligned} \max\{|\partial_{B_J}|_{\mathscr{E}_\eta,\mathrm{sp}}, |\partial_S|_{\mathscr{E}_\eta,\mathrm{sp}}, |\partial_{X_J}|_{\mathscr{E}_\eta,\mathrm{sp}}\} &= |\partial'_S|_{\mathscr{E}'_\eta,\mathrm{sp}} \\ &= \max\{|\partial'_{X_J}|_{\mathscr{E}'_\eta,\mathrm{sp}}, |\partial'_S|_{\mathscr{E}'_\eta,\mathrm{sp}}\} \end{aligned} \tag{5.4.2}$$

for all $\eta \in [\eta_0, 1)$. Note that adding generic p^∞-th roots to k corresponds exactly to replacing $\mathscr{E}$ by $f^*\mathscr{E}'$ and *forgetting* the differential operators B_J. By (5.4.2), the differential nonlogarithmic ramification break of $\tilde{l}/\tilde{k}$ is the same as that of l/k. $\square$

Theorem 5.4.3. *For a complete discretely valued field k of equal characteristic p and a representation ρ of its Galois group G_k with finite image, the arithmetic Artin conductors* $\mathrm{Art}_{\mathrm{ar}}(\rho)$ *as well as the differential Artin conductors* $\mathrm{Art}_{\mathrm{dif}}(\rho)$ *are the same as Borger's Artin conductors* $\mathrm{Art}_B(\rho)$.

Proof. First we may assume that ρ is irreducible and it factors exactly through the Galois group $G_{l/k}$ of a totally ramified Galois extension l/k because all conductors are additive and remain the same under a (finite) unramified extension (Theorem 2.4.1(c) and Propositions 4.1.7(d) and 5.3.6). As k^g has a perfect residue field, $\mathrm{Art}_B(\rho) = \mathrm{Art}_B(\rho|_{G_{k^g}}) = \mathrm{Art}_{\mathrm{dif}}(\rho|_{G_{k^g}})$ are the same as in the classical definition. It suffices to show $\mathrm{Art}_{\mathrm{dif}}(\rho) = \mathrm{Art}_{\mathrm{dif}}(\rho|_{G_{k^g}})$.

Similarly to the proof of Theorem 4.4.1, one may add the p^∞-th roots of all but a finite number of elements of the p-basis into k without changing the differential Artin conductors. In other words, there exists $k \hookrightarrow k_1 = k(b_j^{p^{-n}} \mid j \in J \backslash J_0, n \in \mathbb{N})^\wedge$ for some finite set $J_0 \subset J$, such that $\mathrm{Art}_{\mathrm{dif}}(\rho) = \mathrm{Art}_{\mathrm{dif}}(\rho|_{\mathrm{Gal}_{k_1}})$. Since the residue field of k^g is perfect, there exists $k_1 \hookrightarrow k^g$ extending $k \hookrightarrow k^g$. Hence, we may assume Hypothesis 2.3.2, i.e., k has a finite p-basis.

By Proposition 5.2.3, we can do a finite number of operations of adding generic p^∞-th roots and make the resulting field extension $k_2 l/k_2$ not fiercely ramified and $\mathrm{Art}_{\mathrm{dif}}(\rho|_{G_{k_1}}) = \mathrm{Art}_{\mathrm{dif}}(\rho|_{G_{k_2}})$. In order to link k_2 with k^g, we need to show that we have a homomorphism $k_2 \hookrightarrow k^g$ extending $k_1 \hookrightarrow k^g$, for which we return to the proof of Proposition 5.2.3 and construct the homomorphism step by step.

The r-th ($1 \le r \le r_0$) step of adding generic p^∞-th roots is to construct

$$k_1^{(r)} = \left(k_1^{(r-1)}(x_{r,J})\big((x_{r-1,j} + x_{r,j}\pi_k)^{1/p^n};\ j \in J, n \in \mathbb{N}\big)\right)^\wedge,$$

where $x_{0,j} = b_j$ for $j \in J$ and $k_1^{(0)} = k_1$. One checks that the map given by

$$x_{r,j} \mapsto \sum_{r' \ge r} v_{r',j}\pi_{k^g}^{r'-r}$$

for all $j \in J$ and $r = 1, \ldots, r_0$, gives the desired homomorphism $k_2 \hookrightarrow k^g$.

Now, $k_2 l/k_2$ has naïve ramification degree $[k_2 l : k_2]$, so $\mathbb{O}_{k^g l} = \mathbb{O}_{k^g} \otimes_{\mathbb{O}_{k_2}} \mathbb{O}_{k_2 l}$. Hence we have

$$\mathrm{Art}_{\mathrm{dif}}(\rho|_{G_{k_2}}) = \mathrm{Art}_{\mathrm{ar}}(\rho|_{G_{k_2}}) = \mathrm{Art}_{\mathrm{ar}}(\rho|_{G_{k^g}}) = \mathrm{Art}_{\mathrm{dif}}(\rho|_{G_{k^g}})$$

via Theorem 4.4.1 and Proposition 4.1.7(d). □

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References

[Abbes and Mokrane 2004] A. Abbes and A. Mokrane, "Sous-groupes canoniques et cycles évanescents p-adiques pour les variétés abéliennes", *Publ. Math. Inst. Hautes Études Sci.* 99 (2004), 117–162. MR 2005f:14090 Zbl 1062.14057

[Abbes and Saito 2002] A. Abbes and T. Saito, "Ramification of local fields with imperfect residue fields", *Amer. J. Math.* **124**:5 (2002), 879–920. MR 2003m:11196 Zbl 1084.11064

[Abbes and Saito 2003] A. Abbes and T. Saito, "Ramification of local fields with imperfect residue fields, II", *Doc. Math.* Extra Vol. (2003), 5–72. MR 2005g:11231 Zbl 1127.11349

[Abbes and Saito 2009] A. Abbes and T. Saito, "Analyse micro-locale l-adique en caractéristique $p > 0$: Le cas d'un trait", *Publ. Res. Inst. Math. Sci.* **45**:1 (2009), 25–74. MR 2009m:11197 Zbl 05551347

[Berkovich 1990] V. G. Berkovich, *Spectral theory and analytic geometry over non-Archimedean fields*, Math. Surveys and Monographs **33**, Amer. Math. Soc., Providence, RI, 1990. MR 91k:32038 Zbl 0715.14013

[Berthelot 1996] P. Berthelot, "Cohomologie rigide et cohomologie rigide à support propre", preprint, IRMAR Université de Rennes, 1996, Available at http://tinyurl.com/2fr9nsm.

[Berthelot et al. 1982] P. Berthelot, L. Breen, and W. Messing, *Théorie de Dieudonné cristalline, II*, Lecture Notes in Math. **930**, Springer, Berlin, 1982. MR 85k:14023 Zbl 0516.14015

[Borger 2004] J. M. Borger, "Conductors and the moduli of residual perfection", *Math. Ann.* **329**:1 (2004), 1–30. MR 2005c:11146 Zbl 1096.11040

[Bosch et al. 1984] S. Bosch, U. Güntzer, and R. Remmert, *Non-Archimedean analysis: A systematic approach to rigid analytic geometry*, Grundlehren der Mathematischen Wissenschaften **261**, Springer, Berlin, 1984. MR 86b:32031 Zbl 0539.14017

[Chiarellotto and Pulita 2009] B. Chiarellotto and A. Pulita, "Arithmetic and differential Swan conductors of rank one representations with finite local monodromy", *Amer. J. Math.* **131**:6 (2009), 1743–1794. MR 2567506 Zbl 05656521

[Eisenbud 1995] D. Eisenbud, *Commutative algebra: With a view toward algebraic geometry*, Graduate Texts in Math. **150**, Springer, New York, 1995. MR 97a:13001 Zbl 0819.13001

[Grothendieck 1964] A. Grothendieck, "Éléments de géométrie algébrique, IV: Étude locale des schémas et des morphismes de schémas, I", *Inst. Hautes Études Sci. Publ. Math.* 20 (1964), 259. MR 30 #3885

[Hattori 2008] S. Hattori, "Tame characters and ramification of finite flat group schemes", *J. Number Theory* **128**:5 (2008), 1091–1108. MR 2009a:14060 Zbl 05268132

[Kato 1989] K. Kato, "Swan conductors for characters of degree one in the imperfect residue field case", pp. 101–131 in *Algebraic K-theory and algebraic number theory* (Honolulu, 1987), edited by M. R. Stein and R. K. Dennis, Contemp. Math. **83**, Amer. Math. Soc., Providence, RI, 1989. MR 90g:11164 Zbl 0716.12006

[Kedlaya 2005] K. S. Kedlaya, "Local monodromy of p-adic differential equations: An overview", *Int. J. Number Theory* **1**:1 (2005), 109–154. MR 2006g:12013 Zbl 1107.12005

[Kedlaya 2007] K. S. Kedlaya, "Swan conductors for p-adic differential modules, I: A local construction", *Algebra Number Theory* **1**:3 (2007), 269–300. MR 2009b:11205

[Kedlaya 2010] K. S. Kedlaya, *p-adic differential equations*, Cambridge Studies in Advanced Mathematics **125**, Cambridge University Press, New York, 2010. Zbl 05765055

[Matsuda 2002] S. Matsuda, "Katz correspondence for quasi-unipotent overconvergent isocrystals", *Compositio Math.* **134**:1 (2002), 1–34. MR 2003j:12007 Zbl 1101.14021

[Matsuda 2004] S. Matsuda, "Conjecture on Abbes–Saito filtration and Christol–Mebkhout filtration", pp. 845–856 in *Geometric aspects of Dwork theory*, vol. 2, edited by A. Adolphson et al., de Gruyter, Berlin, 2004. MR 2005g:11233 Zbl 1174.11406

[Saito 2009] T. Saito, "Wild ramification and the characteristic cycle of an l-adic sheaf", *J. Inst. Math. Jussieu* **8**:4 (2009), 769–829. MR 2540880 Zbl 1177.14044

[Serre 1979] J.-P. Serre, *Local fields*, Graduate Texts in Mathematics **67**, Springer, New York, 1979. MR 82e:12016 Zbl 0423.12016

[Sweedler 1968] M. E. Sweedler, "Structure of inseparable extensions", *Ann. of Math.* (2) **87** (1968), 401–410. MR 36 #6391 Zbl 0168.29203

[Tsuzuki 2002] N. Tsuzuki, "Morphisms of F-isocrystals and the finite monodromy theorem for unit-root F-isocrystals", *Duke Math. J.* **111**:3 (2002), 385–418. MR 2003b:14026 Zbl 1055.14022

[Whitney 2002] W. A. Whitney, *Functorial Cohen rings*, Ph.D. thesis, University of California, Berkeley, 2002, Available at http://tinyurl.com/2ugguq6. MR 2707613

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