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A ring extension $A \subseteq B$ is said to have depth one if B is isomorphic to a direct summand of A^n as an (A, A)-bimodule, for some positive integer n. We prove group-theoretic characterizations of this property in the case $kH \subseteq kG$, where H is a subgroup of a finite group G and k is a field. We determine when the source algebra of a block of kG with defect group P is a depth-one extension of kP.

Introduction

A *ring extension* is a unitary ring homomorphism $f\colon A \to B$ between two rings A and B. In this situation, the ring B can be viewed as an (A, A)-bimodule using the map f. A ring extension $f\colon A \to B$ is said to be of *depth one* (or *centrally projective* [Kadison 1999]) if B is isomorphic, as an (A, A)-bimodule, to a direct summand of A^n for some positive integer n. We write $B \mid A^n$ for this condition. Whenever A is a unitary subring of B and $f\colon A \to B$ is the inclusion map, we denote the corresponding ring extension by $A \subseteq B$.

To the best of our knowledge, centrally projective ring extensions were first considered by Hirata [1969]. The identification of centrally projective ring extensions with ring extensions of depth one appeared in [Kadison and Szlachányi 2003]. Ring extensions of higher depth were studied in [Kadison 2008], for example.

In this paper we try to answer the question of when a ring extension $kH \subseteq kG$ of group rings has depth one. Here and throughout this paper we denote by k a commutative ring, by G a group and by H a subgroup of finite index in G. In [Boltje and Külshammer 2010] we considered the question of when the ring extension $kH \subseteq kG$ has depth two (that is, when $kG \otimes_{kH} kG \mid (kG)^n$ as (kG, kH)-bimodules, or equivalently as (kH, kG)-bimodules, for some positive integer n). It turns out that this is equivalent to H being normal in G, independently of k. If the ring extension $kH \subseteq kG$ has depth one it also has depth two, since one can apply the functor $kG \otimes_{kH} -$ to the relation $kG \mid (kH)^n$. In particular, H has to be normal in G. The converse is not true in general, as our main results, Theorems 1.7 and 1.9,

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show. In these theorems we have to assume that k is a field (or a complete discrete valuation ring of characteristic 0 and positive residual characteristic p). In both cases the depth-one condition is equivalent to a purely group theoretic condition on the inclusion $H \leqslant G$, namely that $G = HC_G(X)$, for every cyclic subgroup X of H in the characteristic 0 case, and for every p-hypoelementary subgroup X of H in the modular case (see Remark 1.8 for a definition of p-hypoelementary groups). Therefore, the group theoretic depth-one condition does depend on the base ring k. We do not have a group theoretic reformulation in the case $k = \mathbb{Z}$.

At the end we study the depth-one condition for the ring extension $FP \to A$, where P is the defect group of a block of a group algebra over a field F of positive characteristic p, and A is a source algebra of the block.

1. Depth one for group algebra extensions

1.1. Assume that g belongs to $N_G(H)$, the normalizer of H in G. Then

$$c_{g^{-1}} : kH \to kH, \quad a \mapsto g^{-1}ag$$

is a k-algebra automorphism of kH. Restriction along this automorphism defines a functor

$$C_g : {}_{kH}\mathsf{Mod} \to {}_{kH}\mathsf{Mod}, \quad M \mapsto {}^{g}M$$

on the category of left kH-modules. More explicitly, for $M \in {}_{kH}\mathsf{Mod}$, the left kH-module ${}^{g}M$ is defined to be equal to M as an abelian group, and it is endowed with the module structure $a * m := (g^{-1}ag) \cdot m$, where "$\cdot$" denotes the original kH-module structure of M. The functor C_g maps a homomorphism $f : M_1 \to M_2$ in ${}_{kH}\mathsf{Mod}$ to ${}^{g}f := f : {}^{g}M_1 \to {}^{g}M_2$.

If C_g is naturally equivalent to the identity functor on ${}_{kH}\mathsf{Mod}$, we say that g *acts trivially on* ${}_{kH}\mathsf{Mod}$. For this paper, we say that G *acts trivially on* ${}_{kH}\mathsf{Mod}$ if H is normal in G and g acts trivially on ${}_{kH}\mathsf{Mod}$ for every $g \in G$. Note that H acts trivially on ${}_{kH}\mathsf{Mod}$. This is also an immediate consequence of the next proposition.

The subset $kgH = gkH = kHg$ of kG is a (kH, kH)-subbimodule of kG. It is isomorphic to ${}^{(g,1)}kH$ if we view (kH, kH)-bimodules M as left $k[H \times H]$-modules via

$$(h_1, h_2) \cdot m := h_1 m h_2^{-1} \quad \text{for } h_1, h_2 \in H \text{ and } m \in M.$$

Assume that R is an arbitrary ring and that α is an automorphism of R. Let R_α denote the (R, R)-bimodule that equals R as abelian group but has the twisted action $axb := \alpha(a)xb$, for $a, x, b \in R$. It is well-known and straightforward to prove that α is an inner automorphism if and only if R_α is isomorphic to R as an (R, R)-bimodule. It is also equivalent to α acting trivially on ${}_{R}\mathsf{Mod}$. The following proposition is a special case, and we leave its proof to the reader.

Proposition 1.2. *For $g \in N_G(H)$, the following are equivalent*:

(i) *$kgH \cong kH$ as (kH, kH)-bimodules.*

(ii) *There exists a unit u of kH such that $gag^{-1} = uau^{-1}$ for all $a \in kH$.*

(iii) *g acts trivially on ${}_{kH}\mathsf{Mod}$.*

For every subset X of H we denote by $C_G(X)$ the centralizer of X in G.

Corollary 1.3. *If $G = HC_G(H)$, then the ring extension $kH \subseteq kG$ has depth one. Conversely, if the ring extension $kH \subseteq kG$ has depth one, then H is normal in G.*

Proof. Suppose first that $G = HC_G(H)$. Then every $g \in G$ satisfies condition (ii). Using condition (i) together with the decomposition $kG = \bigoplus_{gH \in G/H} kgH$ into (kH, kH)-subbimodules, the first assertion follows. The second assertion was already observed in the introduction. □

1.4. If Λ is a k-order (that is, a k-algebra that is finitely generated and projective as a k-module) we say that the *Krull–Schmidt theorem holds* for Λ-lattices if the following two properties hold for every Λ-module M that is finitely generated and projective as a k-module:

- M has a decomposition $M = U_1 \oplus \cdots \oplus U_r$ into indecomposable Λ-submodules, and
- if $M = U_1 \oplus \cdots \oplus U_r = V_1 \oplus \cdots \oplus V_s$ are two decompositions into indecomposable Λ-submodules, then $r = s$ and there exists a permutation σ of $\{1, \ldots, r\}$ such that $U_i \cong V_{\sigma(i)}$ for all $i \in \{1, \ldots, r\}$.

If k is a field or a complete discrete valuation ring, then the Krull–Schmidt theorem holds for every k-order Λ [Curtis and Reiner 1981, Theorem 6.12].

Proposition 1.5. *Assume that G is finite and that the Krull–Schmidt theorem holds for $k[H \times H]$-lattices. The following are equivalent*:

(i) *$kH \subseteq kG$ is a ring extension of depth one.*

(ii) *H is normal in G and $kgH \cong kH$ as (kH, kH)-bimodules for every $g \in G$.*

(iii) *H is normal in G and $kG \cong (kH)^{[G:H]}$ as (kH, kH)-bimodules.*

(iv) *$kG \cong (kH)^{[G:H]}$ as (kH, kH)-bimodules.*

Proof. (i) $\Rightarrow$ (ii): Since the ring extension $kH \subseteq kG$ has depth one, Corollary 1.3 implies that H is normal in G. Therefore, for every $g \in G$, one has $kgH \mid kG \mid (kH)^n$ as (kH, kH)-bimodules for some positive integer n. Since the Krull–Schmidt theorem holds for $k[H \times H]$-lattices, every indecomposable direct summand of the $k[H \times H]$-module kgH is isomorphic to an indecomposable direct summand of the $k[H \times H]$-module kH. But the indecomposable direct summands of kH are the blocks of kH, and they are pairwise nonisomorphic. Since kgH is isomorphic to

${}^{(g,1)}kH$ and since $C_{(g,1)}\colon {}_{k[H\times H]}\mathsf{Mod} \to {}_{k[H\times H]}\mathsf{Mod}$ is a category equivalence, the indecomposable direct summands of kgH are also pairwise nonisomorphic. Thus, we can decompose kgH and kH multiplicity-free into a direct sum of indecomposable $k[H \times H]$-submodules. The number of these summands coincides, since $kgH \cong {}^{(g,1)}kH$. Since every summand of kgH occurs as a summand of kH, we can conclude that $kgH \cong kH$ as (kH, kH)-bimodules.

(ii) $\Rightarrow$ (iii): This follows from the decomposition $kG = \bigoplus_{gH \in G/H} kgH$ into (kH, kH)-subbimodules.

(iii) $\Rightarrow$ (iv): This is trivial.

(iv) $\Rightarrow$ (i): This is immediate from the definition of depth one. □

Remark 1.6. In the proof of Proposition 1.5, the Krull–Schmidt property is only used for permutation $k[H \times H]$-modules. By [Boltje and Glesser 2007, Theorem 1.6], the four conditions in Proposition 1.5 are still equivalent when k is a local domain containing a root of unity of order e, where e is defined as the exponent $\exp(H)$ of H in the case that k has characteristic 0, and as the p'-part of $\exp(H)$ in the case that k has positive characteristic p.

Next we will study the case where k is a field of characteristic 0 and H is a finite group. In this case we will denote by $\bar{k}$ an algebraic closure of k and by $\mathrm{Irr}(H)$ the set of irreducible characters of $\bar{k}H$. Recall that $N_G(H)$ acts from the left on $\mathrm{Irr}(H)$ via $\chi \mapsto {}^g\chi$ for $g \in N_G(H)$, where ${}^g\chi(h) = \chi(g^{-1}hg)$, for $h \in H$. Recall also that $N_G(H)$ acts on the set of conjugacy classes of H via $\mathcal{K} \mapsto {}^g\mathcal{K}$, where ${}^g\mathcal{K} = \{ghg^{-1} \mid h \in \mathcal{K}\}$ for a conjugacy class $\mathcal{K}$ of H.

Theorem 1.7. *Assume that k is a field of characteristic* 0 *and that G is finite. The following are equivalent*:

(i) *$kH \subseteq kG$ is a ring extension of depth one.*

(ii) *H is normal in G and G acts trivially on* $\mathrm{Irr}(H)$.

(iii) *H is normal in G and G acts trivially on the set of conjugacy classes of H.*

(iv) *For every cyclic subgroup X of H one has $G = HC_G(X)$.*

Proof. (i) $\Longleftrightarrow$ (ii): Using Corollary 1.3, we may assume that H is normal in G. Using Propositions 1.2 and 1.5, one sees that it suffices to show that, for every $g \in G$, the (kH, kH)-bimodules kgH and kH are isomorphic if and only if g acts trivially on $\mathrm{Irr}(H)$. By the Deuring–Noether theorem [Nagao and Tsushima 1989, Theorem II.3.1], $kgH \cong kH$ as (kH, kH)-bimodules if and only if $\bar{k}gH \cong \bar{k}H$ as $(\bar{k}H, \bar{k}H)$-bimodules. By Proposition 1.2, the latter is equivalent to g acting trivially on ${}_{\bar{k}H}\mathsf{Mod}$, which implies that g acts trivially on $\mathrm{Irr}(H)$. Conversely, if ${}^g\chi = \chi$ for every $\chi \in \mathrm{Irr}(H)$ then $\bar{k}gH \cong \bar{k}H$ as $\bar{k}[H \times H]$-modules, since the character of $\bar{k}H$ is equal to $\sum_{\chi\in\mathrm{Irr}(H)} \chi \times \bar{\chi}$ and the character of $\bar{k}gH$ is equal to

$$^{(g,1)}\Big(\sum_{\chi\in\mathrm{Irr}(H)} \chi\times\bar{\chi}\Big) = \sum_{\chi\in\mathrm{Irr}(H)} {}^{g}\chi\times\bar{\chi} = \sum_{\chi\in\mathrm{Irr}(H)} \chi\times\bar{\chi}.$$

(ii) $\Longleftrightarrow$ (iii): This follows immediately from Brauer's permutation lemma [Nagao and Tsushima 1989, Lemma III.2.19].

(iii) $\Rightarrow$ (iv): Let $X = \langle x\rangle$ be a cyclic subgroup of H and let g be an element of G. Since gxg^{-1} lies in the same conjugacy class as x, there exists $h \in H$ such that $gxg^{-1} = hxh^{-1}$. This implies $h^{-1}g \in C_G(x)$ and $g \in hC_G(x) \subseteq HC_G(X)$.

(iv) $\Rightarrow$ (ii): Condition (iv) implies immediately that H is normal in G. Now let $g \in G$, $\chi \in \mathrm{Irr}(H)$ and $x \in H$. Then there exists $h \in H$ and $c \in C_G(x)$ such that $g = hc$. Hence $\chi(gxg^{-1}) = \chi(hcxc^{-1}h^{-1}) = \chi(hxh^{-1}) = \chi(x)$. Thus ${}^{g^{-1}}\chi = \chi$ and $\chi = {}^{g}\chi$. $\square$

Remark 1.8. Next we study the depth-one condition for the ring extension $kH \subseteq kG$ in the case where k is a field of positive characteristic p, or where k is a complete discrete valuation ring of characteristic 0 and positive residual characteristic p. We will need the theory of species developed by Benson and Parker [1998, Section 5.5]. For this remark assume that G is finite and that k contains a root of unity whose order is equal to $\exp(G)$ if k has characteristic 0, and to the p'-part of $\exp(G)$ if k has characteristic p. Let S and T be finite left G-sets. We denote the corresponding permutation kG-modules by kS and kT. The goal of this remark is to derive a criterion for kS being isomorphic to kT. Denote by $\mathscr{H}_p(G)$ the set of p-hypoelementary subgroups E of G, that is, subgroups E that have a normal (Sylow) p-subgroup P such that E/P is cyclic. We claim that

$$kS \cong kT \text{ as } kG\text{-modules} \iff |S^E| = |T^E| \text{ for all } E \in \mathscr{H}_p(G), \tag{1.8.a}$$

where $|S^E|$ denotes the cardinality of the set S^E of E-fixed points on S. In order to see this equivalence, it suffices to show that

$$s_{E,g}(kS) = |S^E| \tag{1.8.b}$$

for all $E \in \mathscr{H}_p(G)$ and all p'-elements $g \in E$ such that gP generates E/P, where P denotes the Sylow p-subgroup of E. For a definition of $s_{E,g}$ see [Benson 1998, Section 5.5]. Since $s_{E,g}(kS) = s_{E,g}(\mathrm{Res}^G_E(kS))$, we may assume that $G = E$. Moreover, since $s_{E,g}$ is additive, we may assume that S is a transitive E-set, that is, $S = E/D$ for some subgroup D of E. Then $kS \cong \mathrm{Ind}_D^E(k)$. If P is not contained in D, then no indecomposable direct summand of kS has vertex P, and both sides of Equation (1.8.b) are equal to 0. If $P \leqslant D < E$, the Brauer species of $\mathrm{Ind}_D^E(k)$ at g equals 0, since $g \notin D$, and again both sides in Equation (1.8.b) are equal to 0. Finally, if $D = E$, it is immediate that both sides of the equation are equal to 1.

In the next theorem we will apply the criterion in (1.8.a) to the $H \times H$-sets gH and H for $g \in N_G(H)$.

For a subgroup X of H, we set $\Delta X := \{(x, x) \mid x \in X\} \leqslant H \times H$.

Theorem 1.9. *Assume that G is finite and that k is a field of characteristic $p > 0$ or a complete discrete valuation ring of characteristic 0 with residual characteristic $p > 0$. The following are equivalent*:

(i) *The ring extension $kH \subseteq kG$ has depth one.*

(ii) *H is normal in G, and $|(gH)^E| = |H^E|$ for all $g \in G$ and $E \in \mathcal{H}_p(H \times H)$.*

(iii) *H is normal in G, and $|(gH)^{\Delta X}| = |H^{\Delta X}|$ for all $g \in G$ and $X \in \mathcal{H}_p(H)$.*

(iv) *For all $X \in \mathcal{H}_p(H)$ one has $G = C_G(X)H$.*

Proof. (i) $\Longleftrightarrow$ (ii): By Corollary 1.3, we may assume that H is normal in G. Now the equivalence of (i) and (ii) follows immediately from Proposition 1.5 and Remark 1.8 applied to the $k[H \times H]$-modules kgH and kH, for $g \in G$. In fact, by [Benson 1998, Corollary 3.11.4(i)] and the Deuring–Noether theorem [Nagao and Tsushima 1989, Theorem II.3.1], we have $kgH \cong kH$ as $k[H \times H]$-modules if and only if $k'gH \cong k'H$ as $k'[H \times H]$-modules, where k' is obtained from k by adjoining a root of unity whose order is equal to $\exp(H)$ if k has characteristic 0, and to the p'-part of $\exp(H)$ if k has characteristic p.

(ii) $\Rightarrow$ (iii): This is trivial.

(iii) $\Rightarrow$ (iv): Let $g \in G$ and let $X \in \mathcal{H}_p(H)$. Since $1 \in H^{\Delta X}$, the set $(gH)^{\Delta X}$ is nonempty. Let $h \in H$ such that $gh \in (gH)^{\Delta X}$. Then $gh \in C_G(X)$ and $g \in C_G(X)H$.

(iv) $\Rightarrow$ (ii): From (iv) we have immediately that H is normal in G. Now let $g \in G$ and let $E \in \mathcal{H}_p(H)$. The $H \times H$-sets H and gH are transitive. The stabilizer of $1 \in H$ is ΔH and the stabilizer of $g \in gH$ is ${}^{(g,1)}\Delta H$. One has $|H^E| = 0 = |(gH)^E|$ unless E is $H \times H$-conjugate to a subgroup of ΔH or ${}^{(g,1)}\Delta H$. Since the number of fixed points does not change if we replace E by an $H \times H$-conjugate of E, we may assume that $E \leqslant \Delta H$ or $E \leqslant {}^{(g,1)}\Delta H$. We first assume that $E \leqslant \Delta H$. Then $E = \Delta X$ for some $X \in \mathcal{H}_p(H)$. Since $g \in C_G(X)H$, we can write $g = ch$ with $c \in C_G(X)$ and $h \in H$. Then $gH = cH$, and for $h' \in H$ we have

$$h' \in H^{\Delta X} \Longleftrightarrow h' \in C_G(X) \Longleftrightarrow ch' \in C_G(X) \Longleftrightarrow ch' \in (cH)^{\Delta X}.$$

It follows that $|H^{\Delta X}| = |(cH)^{\Delta X}| = |(gH)^{\Delta X}|$. Finally, if $E \leqslant {}^{(g,1)}\Delta H$, then ${}^{(g^{-1},1)}E = \Delta X$ for some $X \in \mathcal{H}_p(H)$. Again we can write $g = ch$ with $c \in C_G(X)$ and $h \in H$. Then $g = h'c$ with $h' = ghg^{-1} \in H$ and $E = {}^{(g,1)}(\Delta X) = {}^{(h'c,1)}(\Delta X) = {}^{(h',1)}(\Delta X)$ is $H \times H$-conjugate to ΔX. By the first case, this implies

$$|(gH)^E| = |(gH)^{\Delta X}| = |H^{\Delta X}| = |H^E|. \qquad \square$$

Remark 1.10. In the case $k = \mathbb{Z}$, we do not know if there is a similar equivalence (i) $\Longleftrightarrow$ (iv) as in Theorem 1.9 with $\mathscr{H}_p(H)$ replaced by some other set $\mathscr{S}(H)$ of subgroups of H. Even if there existed such a set $\mathscr{S}(H)$, we don't have a good guess what it should be.

If $\mathbb{Z}H \subseteq \mathbb{Z}G$ has depth one, then $kH \subseteq kG$ has depth one for every commutative ring k (by scalar extension). In particular, this implies that $G = HC_G(X)$ for every p-hypoelementary subgroup X of H for all primes p. We do not know if the converse holds. On the other hand, if $G = HC_G(H)$, then $\mathbb{Z}H \subseteq \mathbb{Z}G$ has depth one by Corollary 1.3. However, the converse is not true. In fact, by [Hertweck 2001, Theorem A], there exist a finite group H (metabelian of order $2^{25} \cdot 97^2$), a noninner automorphism g of H, and a unit u of $\mathbb{Z}H$ with $g(a) = uau^{-1}$. We set $G := H \rtimes \langle g \rangle$. By Proposition 1.2, we obtain $\mathbb{Z}g^i H \cong \mathbb{Z}H$ as $(\mathbb{Z}H, \mathbb{Z}H)$-bimodules for every integer i. This implies that $\mathbb{Z}G = \bigoplus_{xH \in G/H} \mathbb{Z}xH \cong (\mathbb{Z}H)^{[G:H]}$ and that $\mathbb{Z}H \subseteq \mathbb{Z}G$ has depth one. But $g \notin C_G(H)H$, since g is not an inner automorphism of H. This shows that if, for each finite group H, there exists a set of subgroups $\mathscr{S}(H)$ of H that replaces $\mathscr{H}_p(H)$ in Theorem 1.9(iv) in the case $k = \mathbb{Z}$, then $H \notin \mathscr{S}(H)$ for Hertweck's group H.

2. Depth one for source algebras of blocks

2.1. Let G be a finite group, let p be a prime, and let (K, R, F) be a p-modular system. Thus, R is a complete discrete valuation ring of characteristic zero, K is the field of fractions of R, and F, the residue field of R, has characteristic p. We assume that R contains a root of unity of order $\exp(G)$ and that F is algebraically closed. Then K and F are splitting fields for KG and FG, respectively. For an R-order A, we denote by $\bar{A}$ the finite-dimensional F-algebra $F \otimes_R A$. In the following, let $k \in \{R, F\}$.

In this section, we will consider the depth-one condition for blocks and source algebras. For general background, we refer to the books [Thévenaz 1995] and [Külshammer 1991]. For the convenience of the reader, we recall some of the basic concepts.

An *interior G-algebra* over k consists of a k-order A and a group homomorphism $i : G \to A^\times$, where $A^\times$ denotes the group of units of A. In this case, we will consider the k-linear extension $kG \to A$ of i as a ring extension. Two interior G-algebras A_1 and A_2 are called *isomorphic* if there exists an isomorphism $f : A_1 \to A_2$ commuting with the structural maps $i_1 : G \to A_1^\times$ and $i_2 : G \to A_2^\times$.

If A is an interior G-algebra, then a *point* of a subgroup H of G on A is an $(A^H)^\times$-conjugacy class β of primitive idempotents in the subalgebra

$$A^H := \{a \in A \mid ha = ah \text{ for all } h \in H\}$$

of A. In this case the pair $(H, \beta) =: H_\beta$ is called a *pointed group* on A.

The point β of H on A is called *local* if $\beta \not\subseteq \mathrm{Tr}_L^H(A^L)$ for every proper subgroup L of H; here $\mathrm{Tr}_L^H\colon A^L \to A^H$, $a \mapsto \sum_{hL\in H/L} hah^{-1}$ is the *relative trace map*. If β is a local point of H on A then H_β is called a *local pointed group* on A. One can show that in this case H has to be a p-group.

Let H_β and L_γ be pointed groups on A. We write $L_\gamma \leq H_\beta$ if $L \leqslant H$ and $jAj \subseteq iAi$ for suitable idempotents $i \in \beta$, $j \in \gamma$. This defines a partial order on the set of pointed groups on A. The group G acts by conjugation on the set of all pointed groups H_β on A, and this action is compatible with the partial order relation. We denote by $N_G(H_\beta)$ the stabilizer of H_β in G. Thus, $N_G(H_\beta)$ is a subgroup of $N_G(H)$.

A *block* of kG is an indecomposable direct summand B of kG, considered as a (kG, kG)-bimodule. In this case B is a k-order in its own right. We consider B as an interior G-algebra via the group homomorphism $G \to B^\times$, $g \mapsto g1_B = 1_B g$. Then $\alpha := \{1_B\}$ is a point of G on B and we consider G_α as a pointed group on B.

The maximal local pointed groups $P_\gamma \leqslant G_\alpha$ are called *defect pointed groups* of G_α (and of B). They are unique up to conjugation in G. If P_γ is a defect pointed group on B, then P is also called a *defect group* of B. For $i \in \gamma$, the k-order $B_\gamma = iBi = ikGi$ is called a *source algebra* of B. One can show that $BiB = B$, so that B and iBi are Morita equivalent k-orders via multiplication with i. The source algebra iBi will always be considered as an interior P-algebra via the map $P \to (iBi)^\times$, $x \mapsto ix = xi$.

The block B is called *nilpotent* if $N_G(Q_\delta)/C_G(Q)$ is a p-group for every local pointed group $Q_\delta \leq G_\alpha$ on B. (Note that indeed $C_G(Q) \subseteq N_G(Q_\delta)$ here.) Puig [1988] determined the structure of the source algebra of a nilpotent block. It is a consequence of his results that every nilpotent block has a unique simple module in characteristic p, up to isomorphism. We will make use of Puig's results in the following theorem.

Theorem 2.2. *Let B be a block of RG with defect pointed group P_γ, and let B_γ be a corresponding source algebra. Then the following assertions are equivalent*:

(i) *The ring extension $FP \to \overline{B_\gamma}$ defined by the canonical map $P \to \overline{B_\gamma}^\times$ has depth one.*

(ii) *B_γ and RP are isomorphic as interior P-algebras.*

(iii) *B is a nilpotent block, and the unique simple $\overline{B}$-module M has a trivial source.*

Proof. (i) $\Rightarrow$ (ii): Suppose that the ring extension $FP \to \overline{B_\gamma}$ has depth one. Then $\overline{B_\gamma} \mid (FP)^n$ as an (FP, FP)-bimodule, for some positive integer n. Thus every indecomposable direct summand of the (FP, FP)-bimodule $\overline{B_\gamma}$ is isomorphic to FP. Hence [Thévenaz 1995, Theorem 44.3] implies that $N_G(P_\gamma) = PC_G(P)$ and $\overline{B_\gamma} \cong FP$, as an (FP, FP)-bimodule; in particular, we have $\mathrm{rk}_R(B_\gamma) = \dim_F \overline{B_\gamma} = |P|$. The same theorem now implies that $B_\gamma \cong RP$ as interior P-algebras.

(ii) $\Rightarrow$ (iii): Suppose that B_γ and RP are isomorphic interior P-algebras. Then a result by Puig [1988, Theorem 1.6] implies that the block B is nilpotent [Thévenaz 1995, Remark 50.10]. We write $\overline{B_\gamma} = iFGi$, where i is a primitive idempotent in $(FG)^P$. Since every block has at least one simple module whose vertices are defect groups of the block, P is a vertex of the unique simple $\overline{B}$-module M. By [Thévenaz 1995, Proposition 38.3], M has an FP-source V such that $V \mid iM$, as an FP-module. Since $\overline{B}$ and $\overline{B_\gamma}$ are Morita equivalent via multiplication with i, the $\overline{B_\gamma}$-module iM is simple. Since $\overline{B_\gamma} \cong FP$, iM is trivial as an FP-module, and so is V.

(iii) $\Rightarrow$ (i): Suppose that B is nilpotent and that the unique simple $\overline{B}$-module M has a trivial source. Then M has vertex P, as above, and a result by Puig [Thévenaz 1995, Theorem 50.6] implies that $\overline{B_\gamma} \cong S \otimes_F FP$ as interior P-algebras, where S is an interior P-algebra that is simple as an F-algebra. (The tensor product of two interior P-algebras is again an interior P-algebra via the diagonal map.) As above, we write $\overline{B_\gamma} = iFGi$, where i is a primitive idempotent in $(FG)^P$. Since $\overline{B}$ and $\overline{B_\gamma}$ are Morita equivalent via multiplication with i, the module iM is the unique simple $\overline{B_\gamma}$-module, up to isomorphism. Thus, S and $\mathrm{End}_F(iM)$ are isomorphic interior P-algebras; in particular, $S^P \cong \mathrm{End}_{FP}(iM)$ as F-algebras. But S^P is a local ring (since $\overline{B_\gamma}^P$ is), so iM is indecomposable as an FP-module. On the other hand, [Thévenaz 1995, Proposition 38.3] implies that iM has a direct summand, as an FP-module, which is a source of M. Thus $\dim_F iM = 1$. Hence $\dim_F S = 1$, so $S \cong F$ and $\overline{B_\gamma} \cong FP$. In particular, the ring extension $FP \to \overline{B_\gamma}$ has depth one. □

2.3. It would be interesting to have a similar description of the depth-two condition for source algebras of blocks. The goal of this subsection is to show that $RP \to B_\gamma$ (and also $FP \to \overline{B_\gamma}$) is a *symmetric Frobenius extension*, so that the left and right depth-two conditions are equivalent [Kadison and Szlachányi 2003, Proposition 6.4].

Recall from [Kadison 1999, Theorem I.1.2] that a ring extension $f\colon \Gamma \to \Delta$ is called a *Frobenius extension* if there exist a (Γ, Γ)-bimodule homomorphism $E\colon \Delta \to \Gamma$ and elements $x_j, y_j \in \Delta$, $j = 1, \dots, n$, such that

$$\sum_{j=1}^{n} x_j E(y_j a) = a = \sum_{j=1}^{n} E(a x_j) y_j \tag{2.3.a}$$

for all $a \in \Delta$. If in addition

$$E(ca) = E(ac) \tag{2.3.b}$$

holds for all $a \in \Delta$ and $c \in C_\Delta(\Gamma)$, then one calls the extension $f\colon \Gamma \to \Delta$ a *symmetric Frobenius extension*.

If $\Gamma \subseteq \Delta$ is a symmetric Frobenius extension and e is an idempotent in $C_\Delta(\Gamma)$, then $e\Gamma e \subseteq e\Delta e$ is a symmetric Frobenius extension. In fact, if $E\colon \Delta \to \Gamma$ satisfies

(2.3.a) and (2.3.b), then it is easy to verify that $\widetilde{E}\colon e\Delta e \to e\Gamma e$, $a \mapsto eE(a)e$ satisfies

$$\sum_{j=1}^{n} ex_j e\widetilde{E}(ey_j ea) = a = \sum_{j=1}^{n} \widetilde{E}(aex_j e)ey_j e$$

for all $a \in e\Delta e$. Moreover, Equation (2.3.b) implies $\widetilde{E}(ca) = \widetilde{E}(ac)$ for all $a \in e\Delta e$ and $c \in C_{e\Delta e}(e\Gamma e) = eC_\Delta(\Gamma)e$.

If H is a subgroup of G, then $kH \subseteq kG$ is a symmetric Frobenius extension. In fact, one can choose for $E\colon kG \to kH$ the canonical projection, and for x_j and y_j, coset representatives of G/H and their inverses. Thus, if e is an idempotent in $(kG)^H$, then also $ekHe \to ekGe$ is a symmetric Frobenius extension. This holds even over arbitrary commutative rings k.

Now our claim follows by specializing to $H = P$ and $e = 1_{B_\gamma}$ (or $e = 1_{\overline{B_\gamma}}$), and noting that $kP \to ekPe$, $a \mapsto eae = ea = ae$ is an isomorphism of k-algebras.

By the preceding discussion, we do not need to distinguish between the left and the right depth-two condition in the following proposition.

Proposition 2.4. *Let B be the principal block of RG, and let P_γ be a maximal local pointed group on B (so that P is a Sylow p-subgroup of G). Set $E := N_G(P_\gamma)/PC_G(P)$. Let B_γ be a source algebra of B. Then the following assertions are equivalent*:

(i) *The ring extension $FP \to \overline{B_\gamma}$ defined by the structural map $P \to \overline{B_\gamma}^\times$ has depth two.*

(ii) *B_γ is isomorphic to a twisted group algebra $R_\sharp[P \rtimes E]$ of the semidirect product $P \rtimes E$, as an interior P-algebra.*

Proof. (i) $\Rightarrow$ (ii): Suppose that the ring extension $FP \to \overline{B_\gamma}$ has depth two, and write $A := \overline{B_\gamma} = iFGi$, where i is a primitive idempotent in $(FG)^P$. Then there exists a positive integer n such that

$$\mathrm{Res}^A_{FP}\mathrm{Ind}^A_{FP}\mathrm{Res}^A_{FP}(iM) \mid \mathrm{Res}^A_{FP}(iM)^n$$

for every B-module M. Taking for M the trivial FG-module F, we obtain

$$A \otimes_{FP} iF \mid (iF)^n$$

in ${}_{FP}\mathsf{Mod}$. Thus, P acts trivially on $A \otimes_{FP} iF$. On the other hand, A is a direct sum of (FP, FP)-bimodules of the form $F[PgP]$, for suitable $g \in G$. It is easy to see that $F[PgP] \otimes_{FP} iF \cong \mathrm{Ind}^P_{P\cap gPg^{-1}}(F)$ in ${}_{FP}\mathsf{Mod}$. And if P acts trivially on $\mathrm{Ind}^P_{P\cap gPg^{-1}}(F)$, then $g \in N_G(P)$. Thus A is in fact a direct sum of (FP, FP)-bimodules of the form $F[PgP]$, for suitable $g \in N_G(P)$. Hence [Thévenaz 1995, Theorem 44.3], a result by Puig, implies that $\mathrm{rk}_R B_\gamma = \dim_F \overline{B_\gamma} = |P| \cdot |E|$. Thus [Thévenaz 1995, Theorem 45.11], another result by Puig, implies (ii).

(ii) $\Rightarrow$ (i): Suppose that (ii) holds. Since $R_\sharp[P \rtimes E]$ is a strongly E-graded ring with 1-component $R_\sharp P \cong RP$, [Boltje and Külshammer 2010, Proposition 1.5] shows that the ring extension $RP \to R_\sharp[P \rtimes E]$ has depth two. Tensoring with F, we obtain (i). $\square$

Remark 2.5. The implication (ii) $\Rightarrow$ (i) is valid for arbitrary blocks B of RG. Also, if (ii) holds, one can show that every simple $\overline{B}$-module M has trivial source by noting that P acts trivially on iM.

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