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of the variety of principal minors
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The variety of principal minors of $n \times n$ symmetric matrices, denoted Z_n, is invariant under the action of a group $G \subset \mathrm{GL}(2^n)$ isomorphic to $\mathrm{SL}(2)^{\times n} \ltimes \mathfrak{S}_n$. We describe an irreducible G-module of degree-four polynomials constructed from Cayley's $2 \times 2 \times 2$ hyperdeterminant and show that it cuts out Z_n set-theoretically. This solves the set-theoretic version of a conjecture of Holtz and Sturmfels. Standard techniques from representation theory and geometry are explored and developed for the proof of the conjecture and may be of use for studying similar G-varieties.

1. Introduction

The problem of finding the relations among principal minors of a matrix of indeterminates goes back at least to [Nanson 1897], where relations among the principal minors of an arbitrary 4×4 matrix are given. An expression for the determinant of a matrix in terms of a subset of its principal minors is given in [Stouffer 1928], but the number of independent principal minors was essentially known to that author even earlier [1924], as pointed out in [Griffin and Tsatsomeros 2006]. (In fact, in his 1928 paper Stouffer says that the result was already known to MacMahon in 1893 and later to Muir.)

Subsequently, interest in the subject seems to have diminished, but much more recently, there has been renewed interest in the relations among principal minors and their application to matrix theory, probability, statistical physics and spectral graph theory.

In response to questions about principal minors of *symmetric* matrices, Holtz and Sturmfels [2007] introduced the algebraic variety of principal minors of symmetric $n \times n$ matrices (denoted Z_n herein — see Section 3B for the precise definition) and asked for generators of its ideal. In the first nontrivial case, those authors showed that Z_3 is an irreducible hypersurface in $\mathbb{P}^7$ cut out by a special degree-four polynomial, namely Cayley's hyperdeterminant of format $2 \times 2 \times 2$. In the next case they showed (with the aid of a computer calculation) that the ideal of Z_4 is minimally generated by 20 degree-four polynomials, but only 8 of these polynomials are copies of the hyperdeterminant constructed by natural substitutions. The other 12 polynomials were separated into classes based on their multidegrees. This was done in a first draft of [Holtz and Sturmfels 2007], and at that point, the geometric meaning of the remaining polynomials and their connection to the hyperdeterminant was still somewhat mysterious. Because of the symmetry of the hyperdeterminant, Landsberg suggested to Holtz and Sturmfels the following:

Theorem 1.1 [Holtz and Sturmfels 2007, Theorem 12]. *The variety Z_n is invariant under the action of*

$$\mathrm{SL}(2)^{\times n} \ltimes \mathfrak{S}_n.$$

It should be noted that Borodin and Rains [2005] had found a similar result for two other cases: when the matrix is not necessarily symmetric and for a Pfaffian analog. In [Oeding 2009], we showed that Z_n is a linear projection of the well-known Lagrangian Grassmannian, which can also be interpreted as the variety of all minors of a symmetric matrix. We used this projection to prove Theorem 1.1 geometrically.

In [Holtz and Sturmfels 2007], the span of the $(\mathrm{SL}(2)^{\times n} \ltimes \mathfrak{S}_n)$-orbit of the $2 \times 2 \times 2$ hyperdeterminant is named *the hyperdeterminantal module* (denoted *HD* herein — see Section 3). It was then understood — and included in the final version of the paper — that the 20 degree-four polynomials are a basis of the hyperdeterminantal module when $n = 4$. This interpretation led to the following:

Conjecture 1.2 [Holtz and Sturmfels 2007, Conjecture 14]. *The prime ideal of the variety of principal minors of symmetric matrices is generated in degree 4 by the hyperdeterminantal module for all $n \geq 3$.*

While the first two cases of the conjecture ($n = 3, 4$) were proved using a computer, the dimension of the hyperdeterminantal module and the number of variables both grow exponentially with n, making computational methods ineffective already in the case $n = 5$, for which the hyperdeterminantal module has a basis of 250 degree-four polynomials in 32 variables. Our point of departure is the use of the symmetry of Z_n via tools from representation theory and the geometry of G-varieties.

Our main purpose is to solve the set-theoretic version of the Holtz–Sturmfels conjecture (see Example 3.3 for the representation-theoretic description of the hyperdeterminantal module in terms of Schur modules used in this statement):

Theorem 1.3 (Main Theorem). *The variety of principal minors of symmetric $n \times n$ matrices, Z_n, is cut out set-theoretically by the hyperdeterminantal module, which is the irreducible* $(\mathrm{SL}(2)^{\times n} \ltimes \mathfrak{S}_n)$*-module of degree-four polynomials*

$$HD = S_{(2,2)} S_{(2,2)} S_{(2,2)} S_{(4)} \ldots S_{(4)}.$$

The set-theoretic result is sufficient for many applications related to principal minors of symmetric matrices. In particular, set-theoretic defining equations of Z_n are necessary and sufficient conditions for a given vector of length 2^n to be expressed as the principal minors of a symmetric matrix. We state this practical membership test as follows:

Corollary 1.4. *Suppose $w = [w_{[i_1,\ldots,i_n]}] \in \mathbb{C}^{2^n}$ with $i_j \in \{0,1\}$. Then w represents the principal minors of a symmetric $n \times n$ matrix if and only if w and all images under changes of coordinates by* $\mathrm{SL}(2)^{\times n} \ltimes \mathfrak{S}_n$ *are zeros of Cayley's $2 \times 2 \times 2$ hyperdeterminant*

$$(w_{I_{[0,0,0]}})^2 (w_{I_{[1,1,1]}})^2 + (w_{I_{[1,0,0]}})^2 (w_{I_{[0,1,1]}})^2 + (w_{I_{[0,1,0]}})^2 (w_{I_{[1,0,1]}})^2$$
$$+ (w_{I_{[0,0,1]}})^2 (w_{I_{[1,1,0]}})^2 - 2 w_{I_{[0,0,0]}} w_{I_{[1,0,0]}} w_{I_{[0,1,1]}} w_{I_{[1,1,1]}} - 2 w_{I_{[0,0,0]}} w_{I_{[0,1,0]}} w_{I_{[1,0,1]}} w_{I_{[1,1,1]}}$$
$$- 2 w_{I_{[0,0,0]}} w_{I_{[0,0,1]}} w_{I_{[1,1,0]}} w_{I_{[1,1,1]}} - 2 w_{I_{[1,0,0]}} w_{I_{[0,1,0]}} w_{I_{[0,1,1]}} w_{I_{[1,0,1]}}$$
$$- 2 w_{I_{[1,0,0]}} w_{I_{[0,0,1]}} w_{I_{[0,1,1]}} w_{I_{[1,1,0]}} - 2 w_{I_{[0,1,0]}} w_{I_{[0,0,1]}} w_{I_{[1,0,1]}} w_{I_{[1,1,0]}}$$
$$+ 4 w_{I_{[0,0,0]}} w_{I_{[0,1,1]}} w_{I_{[1,0,1]}} w_{I_{[1,1,0]}} + 4 w_{I_{[0,0,1]}} w_{I_{[0,1,0]}} w_{I_{[1,0,0]}} w_{I_{[1,1,1]}},$$

where $I_{[i_1,i_2,i_3]} = [i_1, i_2, i_3, 0, \ldots, 0]$ for $i_j \in \{0,1\}$.

A second, unifying purpose of this work is to study Z_n as a prototypical (non-homogeneous) G-variety. We aim to show the usefulness of standard constructions in representation theory and geometry, and to further develop general tools for studying geometric and algebraic properties of such varieties. We anticipate these techniques will be applicable to other G-varieties in spaces of tensors, such as those that arise naturally in computational complexity [Landsberg 2008; Bürgisser et al. 1997], signal processing [Comon et al. 2008; Comon and Rajih 2006; de Lathauwer and de Baynast 2008], and algebraic statistics [Pachter and Sturmfels 2005; Allman and Rhodes 2008] (see also [Landsberg $\geq$ 2011] for a unified presentation of the use of geometry and representation theory in these areas), and especially to the case of principal minors of arbitrary matrices studied by Lin and Sturmfels [2009] and Borodin and Rains [2005]. In fact, we use techniques similar to those found here as well as Theorem 1.3 in the sequel [Oeding 2011], which investigates a connection between principal minors of symmetric matrices and the tangential variety

to the Segre product of projective spaces, and solves the set-theoretic version of a conjecture of Landsberg and Weyman [2007].

1A. *Outline.* This paper is organized as follows. Section 2 discusses applications of Theorem 1.3 to statistics, physics and graph theory. In Section 3 we recall basic notions concerning tensors, representations and G-varieties. We point out many standard facts from representation theory that we will use to study the ideal of Z_n and the hyperdeterminantal module. In particular, we recall a method used by Landsberg and Manivel to study G-modules of polynomials via Schur modules. We also show how to use weights and lowering operators to describe and identify Schur modules. We use these concepts to prove Lemma 7.2, a stepping-stone to Proposition 7.1; the latter, in turn, is crucial to our proof of Theorem 1.3.

Sections 3B, 4, 5, and 6 deal with geometric aspects of the variety of principal minors of symmetric matrices and the zero set of the hyperdeterminantal module. In Section 3B we set up notation and give a precise definition of the variety. We also recall two useful facts; a symmetric matrix is determined up to the signs of its off-diagonal terms by its 1×1 and 2×2 principal minors, and the dimension of Z_n is $\binom{n+1}{2}$. In Section 4 we describe the nested structure of Z_n. In particular, in Proposition 4.2 we show that Z_n contains all possible Segre products of Z_p and Z_q where $p + q = n$. We use this interpretation in Proposition 7.1.

In Section 5 we study properties of the hyperdeterminantal module. In particular, we point out that it has dimension $\binom{n}{3}5^{n-3}$. In Proposition 5.2 we show that it actually is an irreducible $(\mathrm{SL}(2)^{\times n} \ltimes \mathfrak{S}_n)$-module of polynomials that occurs with multiplicity 1 in the space of degree-four homogeneous polynomials. This is a consequence of a more general fact about modules with structure similar to that of the hyperdeterminantal module, which we record in Lemma 5.4. In Proposition 5.5 we record the fact (originally proved in [Holtz and Sturmfels 2007]) that the hyperdeterminantal module is in the ideal of Z_n. Then in Proposition 5.6 we generalize the idea to other varieties that have similar structure.

In Section 6 we extract a general property of the hyperdeterminantal module that we call augmentation. We explore properties of augmented modules by means of polarization of tensors, a technique from classical invariant theory used, for example, in the study of secant varieties. Of particular interest is the augmentation lemma (Lemma 6.4), in which we give a geometric description of the zero set of a general augmented module. We apply that lemma to obtain a geometric characterization of the zero set of the hyperdeterminantal module in Lemma 6.6. We use Lemma 6.6 in the proof of Theorem 1.3. Proposition 6.8 is another application of the Augmentation Lemma to polynomials that define Segre products of projective spaces. We use a slightly more complicated version of Proposition 6.8 in the proof of Lemma 7.2.

In Sections 7 and 8 we pull together all of the ideas from the previous parts to prove Theorem 1.3. In particular, we show that any point in the zero set of the hyperdeterminantal module has a symmetric matrix that maps to it under the principal minor map.

In Section 7 we work to understand the case when all principal minors of a symmetric matrix agree with a given vector except possibly the determinant. Of particular importance is Proposition 7.1, which essentially says that for $n \geq 4$, if z is a vector in the zero set of the hyperdeterminantal module, then a specific subset of the coordinates of z determine the rest of its coordinates.

In order to prove Proposition 7.1, we use practically all of the tools from representation theory that we have introduced and developed earlier in the paper. With the aid of Proposition 7.1, we prove Theorem 1.3 in Section 8.

2. Applications of Theorem 1.3

We conclude this introduction by describing how Theorem 1.3 answers questions in other areas via three examples: in statistics and the study of negatively correlated random variables, in physics and the study of determinantal point processes, and in spectral graph theory and the study of graph invariants [Holtz and Schneider 2002; Holtz and Sturmfels 2007; Griffin and Tsatsomeros 2006; Borcea et al. 2009; Holtz 1999; Wagner 2008; Mikkonen 2007].

2A. *Application to covariance of random variables.* Consider a nonsingular real symmetric $n \times n$ matrix A. The principal minors of A can be interpreted as values of a function $\omega : \mathcal{P}(\{1, \ldots, n\}) \to [0, \infty)$, where $\mathcal{P}$ is the power set. This function ω, under various restrictions, is of interest to statisticians. In this setting, the off-diagonal entries of the matrix A^{-1} are associated to covariances of random variables. Wagner [2008] asked:

Question 2.1. *When is it possible to prescribe the principal minors of the matrix A as well as the off-diagonal entries of A^{-1}?*

In [Holtz and Sturmfels 2007, Theorem 6] this question is answered using the hyperdeterminantal equations in degree 4, another set of degree-10 equations, and the strict Hadamard–Fischer inequalities.

Our main result provides an answer to the first part of the question: it is possible to prescribe the principal minors of a symmetric matrix if and only if the candidate principal minors satisfy all the relations given by the hyperdeterminantal module.

For the second part of the question we can give a partial answer. It is not hard to see that the off-diagonal entries of A^{-1} are determined up to sign by the 0×0, 1×1 and 2×2 principal minors, and the rest of the principal minors further restrict the freedom in the choices of signs.

Another useful fact is that if A is invertible, then

$$A^{-1} = \frac{\mathrm{adj}(A)}{\det(A)},$$

where $\mathrm{adj}(A)_{i,j} = ((-1)^{i+j} \det(A_i^j))$ is the adjugate matrix. So up to scale, the vector of principal minors of A^{-1} is the vector of principal minors of A in reverse order. Therefore the determinant and the $(n-1) \times (n-1)$ and $(n-2) \times (n-2)$ principal minors of A determine the off-diagonal entries of A^{-1} up to $\binom{n}{2}$ choices in combinations of signs, and the rest of the principal minors further restrict the choices of combinations of signs.

2B. *Application to determinantal point processes.* Determinantal point processes were introduced by Macchi in 1975, and subsequently have received significant attention in many areas. A nonzero point $p_S \in \mathbb{C}^{2^n}$ is called determinantal if there is an integer m and an $(n+m) \times (n+m)$ matrix K such that for $S \subset \{1, 2, \ldots, n\}$,

$$p_S = \det_{S \cup \{n+1, \ldots, n+m\}} (K).$$

Borodin and Rains [2005, Theorem 4.6] were able to completely classify all such points for the case $n = 4$ by giving a nice geometric characterization. Lin and Sturmfels [2009] studied the geometric and algebraic properties of the algebraic variety of determinantal points and independently arrived at the same result as Borodin and Rains. Moreover, Lin and Sturmfels gave a complete proof of the claim of [Borodin and Rains 2005] that the ideal of the variety is generated in degree 12 by 718 polynomials.

Consider the case where we impose the restrictions that the matrix K to be symmetric and the integer m equals 0; we call these restricted determinantal points *symmetric determinantal points*.

Restatement. *The variety of all symmetric determinantal points is cut out set-theoretically by the hyperdeterminantal module.*

This restatement is useful because it provides a complete list of necessary and sufficient conditions for determining which symmetric determinantal points can possibly exist.

2C. *Application to spectral graph theory.* A standard construction in graph theory is the following. To a weighted directed graph Γ one can assign an adjacency matrix $\Delta(\Gamma)$.

The eigenvalues of $\Delta(\Gamma)$ are invariants of the graph. The first example involves the standard graph Laplacian. Kirchoff's well-known matrix-tree theorem states that any $(n-1) \times (n-1)$ principal minor of $\Delta(\Gamma)$ counts the number of spanning trees of Γ.

There are many generalizations of the Matrix-Tree Theorem, such as the Matrix-Forest Theorem, which states that $\Delta(\Gamma)_S^S$, the principal minor of the graph Laplacian formed by omitting rows and columns indexed by the set $S \subset \{1, \dots, n\}$, computes the number of spanning forests of Γ rooted at vertices indexed by S.

The principal minors of the graph Laplacian are graph invariants. The relations among principal minors are then also relations among graph invariants. Relations among graph invariants are central in the study of the theory of unlabeled graphs. In fact, Mikkonen [2007, p. 1] holds that "the most important problem in graph theory of unlabeled graphs is the problem of determining graphic values of arbitrary sets of graph invariants."

Theorem 1.3 gives relations among the graph invariants that come from principal minors, and in particular, since a graph can be reconstructed from a symmetric matrix, Theorem 1.3 implies the following:

Restatement. *There exists an undirected weighted graph Γ with invariants $[v] \in \mathbb{P}^{2^n-1}$ specified by the principal minors of a symmetric matrix $\Delta(\Gamma)$ if and only if $[v]$ is a zero of all the polynomials in the hyperdeterminantal module.*

3. Background on *G*-varieties in spaces of tensors and their ideals as *G*-modules

An $n \times n$ matrix has 2^n principal minors (determinants of submatrices centered on the main diagonal, including the trivial 0×0 minor), so vectors of principal minors may be considered in the space $\mathbb{C}^{2^n}$. However, the natural ambient space for vectors of principal minors from the point of view of symmetry (Theorem 1.1) is the n-fold tensor product $\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2$. With this in mind, in this section we study tensor products of several vector spaces, natural group actions on tensors, representation theory for tensor products, and classical subvarieties in spaces of tensors.

For the sake of the reader not familiar with representation theory, we have chosen to include many definitions and basic concepts that we might have skipped otherwise. For more background, one may consult [Fulton and Harris 1991; Landsberg $\geq$ 2011; Goodman and Wallach 1998; Weyl 1997; Harris 1992; Cox et al. 2007].

If V is a vector space and $G \subset \mathrm{GL}(V)$, a variety $X \subset \mathbb{P}V$ is said to be a *G-variety* or *G-invariant* if it is preserved by the action of G, specifically $g.x \in X$ for every $x \in X$ and $g \in G$. In this article our vector spaces are always assumed to be finite-dimensional. Our study fits into the more general context of arbitrary *G*-varieties for a linearly reductive group G, and we sometimes allude to this setting, but for the sake of efficiency and clarity we often present the necessary representation-theoretic concepts only in the case of tensors. The expert reader might try to envision the basic techniques we use in their more general context.

3A. *Examples of classical G-varieties in spaces of tensors.* Let $V_1, \ldots, V_n$ be complex vector spaces and let $V_1 \otimes \cdots \otimes V_n$ denote their tensor product. We give two classic examples of G-varieties in the space of tensors $\mathbb{P}(V_1 \otimes \cdots \otimes V_n)$ that happen to show up in our study of the variety of principal minors of symmetric matrices. These definitions can be found in many texts on algebraic geometry, such as [Harris 1992].

The space of all rank-1 tensors (also called decomposable tensors) is the *Segre variety*, defined by the embedding

$$\mathrm{Seg} : \mathbb{P}V_1 \times \cdots \times \mathbb{P}V_n \longrightarrow \mathbb{P}(V_1 \otimes \cdots \otimes V_n),$$

$$([v_1], \ldots, [v_n]) \longmapsto [v_1 \otimes \cdots \otimes v_n].$$

$\mathrm{Seg}(\mathbb{P}V_1 \times \cdots \times \mathbb{P}V_n)$ is a G-variety for $G = \mathrm{GL}(V_1) \times \cdots \times \mathrm{GL}(V_n)$, and it is *homogeneous* (the G-orbit of a single point) since $\mathrm{Seg}(\mathbb{P}V_1 \times \cdots \times \mathbb{P}V_n) = G.[v_1 \otimes \cdots \otimes v_n]$. If $X_1 \subset \mathbb{P}V_1, \ldots, X_n \subset \mathbb{P}V_n$ are varieties, let $\mathrm{Seg}(X_1 \times \cdots \times X_n)$ denote their Segre product.

The *rth secant variety* to a variety $X \subset \mathbb{P}V$, denoted $\sigma_r(X)$, is the Zariski closure of all embedded secant $\mathbb{P}^{r-1}$'s to X, that is,

$$\sigma_r(X) = \overline{\bigcup_{x_1, \ldots, x_r \in X} \mathbb{P}(\mathrm{span}\{x_1, \ldots, x_r\})} \subset \mathbb{P}V.$$

Secant varieties inherit the symmetry of the underlying variety. In particular,

$$\sigma_r(\mathrm{Seg}(\mathbb{P}V_1 \times \cdots \times \mathbb{P}V_n))$$

is a G-variety for $G = \mathrm{GL}(V_1) \times \cdots \times \mathrm{GL}(V_n)$. However, homogeneity is not preserved in general.

3B. *The variety of principal minors of symmetric matrices.* Let $I = [i_1, \ldots i_n]$ be a binary multi-index, with $i_k \in \{0, 1\}$ for $k = 1, \ldots, n$, and let $|I| = \sum_{k=1}^{n} i_k$. A natural basis of $(\mathbb{C}^2)^{\otimes n}$ is the set of tensors $X^I := x_1^{i_1} \otimes x_2^{i_2} \otimes \cdots \otimes x_n^{i_n}$ for all length-n binary indices I. We use this basis to introduce coordinates; if $P = [C_I X^I] \in \mathbb{P}(\mathbb{C}^2)^{\otimes n}$, the coefficients C_I are the homogeneous coordinates of the point P. (We use the summation convention that the implied summation is over the index I, which appears as a superscript and a subscript.)

Let $S^2\mathbb{C}^n$ denote the space of symmetric $n \times n$ matrices. If $A \in S^2\mathbb{C}^n$, then let $\Delta_I(A)$ denote the principal minor of A formed by taking the determinant of the principal submatrix of A indexed by I in the sense that the submatrix of A is formed by including the kth row and column of A whenever $i_k = 1$ and striking the kth row and column whenever $i_k = 0$.

The projective variety of principal minors of $n \times n$ symmetric matrices, Z_n, is defined by

$$\varphi : \mathbb{P}(S^2\mathbb{C}^n \oplus \mathbb{C}) \dashrightarrow \mathbb{P}(\mathbb{C}^2)^{\otimes n}, \quad [A, t] \mapsto [t^{n-|I|}\Delta_I(A)\, X^I].$$

The map φ is defined on the open set where $t \neq 0$. Moreover, φ is homogeneous of degree n, so it is a well-defined rational map on projective space. The 1×1 principal minors of a matrix A are the diagonal entries of $A = (a_{i,j})$, and if A is a symmetric matrix, the 1×1 and 2×2 principal minors determine the off-diagonal entries of A up to sign in light of the equation

$$a_{i,i}a_{j,j} - a_{i,j}^2 = \Delta_{[0,\dots,0,1,0,\dots,0,1,\dots,0]}(A),$$

where the 1's in $[0, \dots, 0, 1, 0, \dots, 0, 1, \dots, 0]$ occur in positions i and j. So φ is generically finite-to-one and Z_n is a $\binom{n+1}{2}$-dimensional variety. The affine map (on the set $\{t = 1\}$) defines a closed subset of $\mathbb{C}^{2^n}$ [Holtz and Sturmfels 2007].

3C. *Ideals of G-varieties in spaces of tensors.*

Let V be a finite-dimensional vector space over $\mathbb{C}$. Let V^* denote the dual vector space of linear maps $V \to \mathbb{C}$. Let $S^d V^*$ denote the space of homogeneous degree-d polynomials on V, and let $\mathrm{Sym}(V^*) = \bigoplus_d S^d V^*$ denote the polynomial ring.

If $X \subset \mathbb{P}V$ is a projective algebraic variety, let $\mathcal{I}(X) \subset \mathrm{Sym}(V^*)$ denote the ideal of polynomials vanishing on X, and let $\widehat{X} \subset V$ denote the cone over X. If M is a set of polynomials, let $\mathcal{V}(M)$ denote its zero set. Often algebraic varieties are given via an explicit parameterization by a rational map, but the vanishing ideal may be unknown. A basic question in algebraic geometry is to find generators for the ideal of a given variety. Though there are many known theoretical techniques, this remains a difficult practical problem.

Fact. *X is a G-variety if and only if $\mathcal{I}(X)$ is a G-module.*

This observation, which comes directly from the definitions, is key because it allows us to use the representation theory of G-modules to study $\mathcal{I}(X)$.

By definition, all projective varieties are preserved by the action of $\mathbb{C} \setminus \{0\}$ by rescaling. It is well-known that this action induces a grading by degree on the ideal $\mathcal{I}(X) = \bigoplus_d \mathcal{I}_d(X)$, where $\mathcal{I}_d X := S^d(V^*) \cap \mathcal{I}(X)$. In parallel, when a larger, linearly reductive group G acts on X, we get a finer decomposition of each module $\mathcal{I}_d(X)$ into a direct sum of irreducible G-modules. The irreducible modules in $\mathcal{I}_d(X)$ are a subset of those in $S^d V^*$. This simple observation leads to a useful ideal membership test, which is developed and discussed in [Landsberg and Manivel 2004; Landsberg ≥ 2011].

The group $\mathrm{GL}(V_1) \times \cdots \times \mathrm{GL}(V_n)$ acts on $V_1 \otimes \cdots \otimes V_n$ by change of coordinates in each factor. When V_i are all isomorphic, there is also a natural action of the symmetric group $\mathfrak{S}_n$ on $V_1 \otimes \cdots \otimes V_n$ by permuting the factors. With this convention,

one can define a left action of the semidirect product $\mathrm{GL}(V) \ltimes \mathfrak{S}_n$ on $V^{\otimes n}$.

If V is a vector space and $G \subset \mathrm{GL}(V)$, we say that V is a *G-module* or *a representation of G* if it is preserved by the action of $G \subset \mathrm{GL}(V)$. A *G*-module is said to be *irreducible* if it has no nontrivial *G*-invariant subspaces.

The general linear group $\mathrm{GL}(V)$ has well-understood representation theory. In particular, [Fulton and Harris 1991, Proposition 15.47] says that every $\mathrm{GL}(V)$-module is isomorphic to a Schur module of the form $S_\pi V$, where π is a partition of an integer d. We refer the reader to the book just cited or to [Landsberg ≥ 2011] for general background on Schur modules.

Two common representations (in this language) are the space of symmetric tensors $S^d V = S_{(d)} V$ and the space of skew-symmetric tensors $\bigwedge^d V = S_{(1^d)} V$, where 1^d denotes the partition $(1, \ldots, 1)$ with 1 repeated d times.

We will be interested in representations of $\mathrm{SL}(V)$. In light of the isomorphism $\mathrm{SL}(V) \cong \mathrm{GL}(V)/Z(\mathrm{GL}(V))$, where the center $Z(\mathrm{GL}(V)) = \mathbb{C} \setminus \{0\}$ is isomorphic to scalar multiples of the identity, the representation theory of $\mathrm{GL}(V)$ is essentially the same at that of $\mathrm{SL}(V)$. Specifically, if V is m-dimensional, two representations $S_\pi V$ and $S_\lambda V$ of $\mathrm{GL}(V)$ are isomorphic as $\mathrm{SL}(V)$ modules if $\pi = \lambda + k^m$, for some $k \in \mathbb{Z}$, where k^m is the partition $(k, \ldots, k)$ with k repeated m times. However, since we care about how the modules we are studying are embedded in the space of polynomials, we will not reduce partitions via this equivalence.

We are interested in the case when $X \subset \mathbb{P}(V_1 \otimes \cdots \otimes V_n)$ is a variety in a space of tensors, and X is invariant under the action of $G = \mathrm{GL}(V_1) \times \cdots \times \mathrm{GL}(V_n)$. To study $\mathcal{I}_d(X)$ as a G-module, we need to understand how to decompose the space of homogeneous degree-d polynomials $S^d(V_1^* \otimes \cdots \otimes V_n^*)$ into a direct sum of irreducible G-modules. This is a standard computation in representation theory, made explicit, for example, in this result:

Proposition 3.1 [Landsberg and Manivel 2004, Proposition 4.1]. *Let $V_1, \ldots, V_n$ be vector spaces and let $G = \mathrm{GL}(V_1) \times \cdots \times \mathrm{GL}(V_n)$. Then $S^d(V_1^* \otimes \cdots \otimes V_n^*)$ can be decomposed into a direct sum of irreducible G-modules as*

$$S^d(V_1^* \otimes \cdots \otimes V_n^*) = \bigoplus_{|\pi_1| = \cdots = |\pi_n| = d} ([\pi_1] \otimes \cdots \otimes [\pi_n])^{\mathfrak{S}_d} \otimes S_{\pi_1} V_1^* \otimes \cdots \otimes S_{\pi_n} V_n^*,$$

where the $[\pi_i]$ are representations of the symmetric group $\mathfrak{S}_d$ indexed by partitions π_i of d, and $([\pi_1] \otimes \cdots \otimes [\pi_n])^{\mathfrak{S}_d}$ denotes the space of $\mathfrak{S}_d$-invariants (that is, instances of the trivial representation) in the tensor product.

When the vector spaces V_i^* are all isomorphic to the same vector space V^*, this decomposition specializes to give the equality of $\mathrm{GL}(V) \times \cdots \times \mathrm{GL}(V)$-modules

$$S^d(V^* \otimes \cdots \otimes V^*) = \bigoplus_{|\pi_1| = \cdots = |\pi_n| = d} (S_{\pi_1} V^* \otimes \cdots \otimes S_{\pi_n} V^*)^{\oplus N_{\pi_1, \ldots, \pi_k}}, \qquad (1)$$

(found in [Landsberg and Manivel 2004]), where the multiplicity $N_{\pi_1,\ldots,\pi_k}$ can be computed via characters. The modules $(S_{\pi_1} V^* \otimes \cdots \otimes S_{\pi_n} V^*)^{\oplus N_{\pi_1,\ldots,\pi_k}}$ are called *isotypic components*.

The irreducible $SL(V)^{\times n} \ltimes \mathfrak{S}_n$-modules are constructed by taking an irreducible $SL(2)^{\times n}$ module $S_{\pi_1} V \otimes \cdots \otimes S_{\pi_n} V$ and summing over all permutations in $\mathfrak{S}_n$ that yield nonredundant modules. When the vector space is understood, we denote this compactly as

$$S_{\pi_1} S_{\pi_2} \ldots S_{\pi_n} := \sum_{\sigma \in \mathfrak{S}_n} S_{\pi_{\sigma(1)}} V^* \otimes \cdots \otimes S_{\pi_{\sigma(n)}} V^*.$$

The decomposition formula (1) is essential for understanding the structure of the ideals of G-varieties. There is an implementation of (1) in the computer program LiE, and we wrote an implementation in Maple.

The combinatorial description of Schur modules in terms of collections of partitions can be used to construct polynomials in spaces of tensors. We refer the reader to [Landsberg and Manivel 2004; Landsberg ≥ 2011] for a complete explanation. A copy of our implementation of these algorithms may be obtained by contacting the author.

3D. *Weights, raising operators, and highest weight vectors.* The ideas of *weights*, *weight vectors*, *highest weight vectors*, and *raising/lowering operators* are well-known practical tools for studying representations and polynomials in spaces of tensors. Here we recall definitions and concepts that can be found in standard textbooks on representation theory in order to define the terms we use in this paper and to explain our use of these representation-theoretic tools. Many of the concepts in this section are practical reinterpretations of concepts in the previous section.

Choose a basis $\{x_i^0, x_i^1\}$ for each V_i and assign the integer *weight* -1 to x_i^0 and the weight $+1$ to x_i^1. Weights of tensors in the algebra $(V_1 \otimes \cdots \otimes V_n)^{\otimes}$ are length-n integer vectors defined first on monomials and extended by linearity. Specifically,

$$(x_1^0)^{\otimes p_1} \otimes (x_1^1)^{\otimes q_1} \otimes (x_2^0)^{\otimes p_2} \otimes (x_2^1)^{\otimes q_2} \otimes \cdots \otimes (x_n^0)^{\otimes p_n} \otimes (x_n^1)^{\otimes q_n}$$

has weight

$$(q_1 - p_1, q_2 - p_2, \ldots, q_n - p_n).$$

A tensor is called a *weight vector* if all of its monomials have the same weight, and this is the only time it makes sense to assign a weight to a tensor. This is the standard assignment of weights for the connected component containing the identity in $SL(2)^{\times n} \ltimes \mathfrak{S}_n$, and is also known as grading by multidegree.

The Lie algebra of $SL(2)$ is $\mathfrak{sl}(2)$, the algebra of traceless 2×2 matrices acting as derivations. The raising (respectively lowering) operators can be thought of as upper (respectively lower) triangular matrices when, for example, the lowering

operator in $\mathfrak{sl}_2$ acts on $V = \{x_0, x_1\}$ by sending x_0 to a scalar multiple of x_1 and sending x_1 to 0.

The Lie algebra of $SL(2)^{\times n}$ is $\mathfrak{sl}_2^{\oplus n}$, where each $\mathfrak{sl}_2$ acts on a single factor of the tensor product $V_1 \otimes \cdots \otimes V_n$. This action is extended to $S^d(V_1 \otimes \cdots \otimes V_n)$ by noting that the differential operators obey the Leibnitz rule. The raising (lowering) operators fix the degree of a polynomial.

A weight vector in a G-module is called a *highest weight vector* (respectively *lowest weight vector*) if it is in the kernel of all of the raising (respectively lowering) operators. Consider the irreducible module $S_{\pi_1} V_1 \otimes \cdots \otimes S_{\pi_n} V_n$, with each π_i a partition of d. Since $V_i \simeq \mathbb{C}^2$ for every $1 \leq i \leq n$, each π_i is of the form (π_i^1, π_i^2) with $\pi_i^1 + \pi_i^2 = d$. A highest weight vector in $S_{\pi_1} V_1 \otimes \cdots \otimes S_{\pi_n} V_n$ has weight $(\pi_1^2 - \pi_1^1, \pi_2^2 - \pi_2^1, \ldots, \pi_n^2 - \pi_n^1)$. If w is the weight of a nonzero vector in $S_{\pi_1} V_1 \otimes \cdots \otimes S_{\pi_n} V_n$, then $-w$ is also the weight of a nonzero vector, and if w is the weight of a highest weight vector in a module then $-w$ is the weight of a lowest weight vector.

Fact. *Assume G is a linearly reductive connected algebraic group. Each finite-dimensional irreducible G-module is the span of the G-orbit of a highest (or lowest) weight vector.*

Remark 3.2. If T is a nonzero homogeneous polynomial on $V_1 \otimes \cdots \otimes V_n$, and T is a highest (or lowest) weight vector, then the degree d and weight $(w_1, w_2, \ldots, w_n)$ of T are sufficient information to determine (up to isomorphism) a module of the form $S_{\pi_1} V_1 \otimes \cdots \otimes S_{\pi_n} V_n$ in which it occurs. We say that we know in which isotypic component the module lives. Specifically, we have $d = \pi_i^1 + \pi_i^2$ and $w_i = \pi_i^2 - \pi_i^1$, so $\pi_i = \frac{1}{2}(d - w_i, d + w_i)$.

In general, the degree and weight of a highest weight polynomial are not enough to find how the module $S_{\pi_1} V_1^* \otimes \cdots \otimes S_{\pi_n} V_n^*$ is embedded in $S^d(V_1^* \otimes \cdots \otimes V_n^*)$ (that is, how it is embedded in the isotypic component). On the other hand, if the found module occurs with multiplicity one in $S^d(V_1^* \otimes \cdots \otimes V_n^*)$, then the degree and weight of a highest weight vector are sufficient information to identify the module.

Example 3.3. The hyperdeterminant of format $2 \times 2 \times 2$ is invariant under the action of $SL(2) \times SL(2) \times SL(2)$, and therefore it must have weight $(0, 0, 0)$. This, together with the knowledge that it is a degree-4 polynomial annihilated by each raising operator, immediately tells us that it must be in the module $S_{(2,2)}\mathbb{C}^2 \otimes S_{(2,2)}\mathbb{C}^2 \otimes S_{(2,2)}\mathbb{C}^2$, which occurs with multiplicity one in $S^4(\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2)$. Moreover, one can write the $2 \times 2 \times 2$ hyperdeterminant on the variables $X^{[i_1, i_2, i_3, 0, \ldots, 0]}$. The weight of this polynomial is $(0, 0, 0, -4, \ldots, -4)$ and it is a highest weight vector, and therefore the span of its $(SL(2)^{\times n} \ltimes \mathfrak{S}_n)$-orbit is the hyperdeterminantal module

$$HD := S_{(2,2)} S_{(2,2)} S_{(2,2)} S_{(4)} \ldots S_{(4)}.$$

3E. *An algorithm to produce a G-module from a polynomial.* Suppose we can write down a polynomial h in some (unknown) G-module M. (Again we are assuming that our modules are finite-dimensional and the group G is a linearly reductive connected algebraic group, and we are specifically thinking of the example $G = \mathrm{SL}(V)^{\times n}$.) Since M is a G-module, it is also a $\mathfrak{g}$-module, where $\mathfrak{g}$ is the Lie algebra associated to the Lie group G. The following algorithm is a standard idea in representation theory and can be used to find more polynomials in M, and in fact we will find submodules of M. In particular, this procedure is essential in the proof of Lemma 7.2 below.

By successively applying lowering operators, we determine the lowest weight space in which a summand of h can live. The lowest weight vector that we construct will generate a submodule of M.

Input: $h \in M$.

Step 0. Choose an ordered basis of lowering operators $\mathfrak{g}_- = \{\alpha_1, \ldots, \alpha_n\}$.

Step 1. Find the largest integer $k_1 \geq 0$ such that $\alpha_1^{k_1}.h \neq 0$, and let $h^{(1)} = \alpha_1^{k_1}.h$.

Step 2. Find the largest integer $k_2 \geq 0$ such that $\alpha_2^{k_2}.h^{(1)} \neq 0$, and let $h^{(2)} = \alpha_2^{k_2}.h^{(1)}$.

Step n. Find the largest integer $k_n \geq 0$ such that $\alpha_n^{k_n}.h^{(n-1)} \neq 0$, and let $h^{(n)} = \alpha_n^{k_n}.h^{(n-1)}$.

Output: The vector $h^{(n)}$ is a lowest weight vector in M and $\mathrm{span}\{G.h^{(n)}\}$ is a submodule of M containing $h^{(n)}$.

In the case $\mathfrak{g} = \mathfrak{sl}(2)^{\oplus n}$, the natural ordered basis of $(\mathfrak{sl}(2)^{\oplus n})_-$ is $\{\alpha_1, \ldots, \alpha_n\}$, where α_i is the lowering operator acting on the V_i^* factor in $S^d(V_1^* \otimes \cdots \otimes V_n^*)$.

Remark 3.4. In the case that M is irreducible, by the same procedure of applying lowering operators to (this time) a highest weight vector h, we can construct a *weight basis M*, namely a basis of M consisting of weight vectors in M of every possible weight.

4. The nested structure of Z_n via Segre products

Proposition 4.1. *The variety* $\mathrm{Seg}(Z_{(n-1)} \times \mathbb{P}V_n)$ *is a subvariety of* Z_n. *In particular, any point of* $\mathrm{Seg}(Z_{(n-1)} \times \mathbb{P}V_n)$ *is, after a possible change of coordinates, the principal minors of an* $(n-1) \times (n-1)$ *block of an* $n \times n$ *matrix.*

Proof. We prove the second claim first. Let $[\eta \otimes v]$ be a point in $\mathrm{Seg}(Z_{(n-1)} \times \mathbb{P}V_n)$. Then change coordinates in V_n to send $[\eta \otimes v]$ to $[\eta \otimes x_n^0]$. Now $[\eta \otimes x_n^0]$ is in $\mathrm{Seg}(Z_{(n-1)} \times \mathbb{P}\{x_n^0\})$, which is the image under ϕ of matrices of the form

$$\left[\begin{pmatrix} P & 0 \\ 0 & 0 \end{pmatrix}, t\right],$$

where P is a symmetric $(n-1) \times (n-1)$ submatrix of an $n \times n$ symmetric matrix.

The first claim then follows from the $(\mathrm{SL}(2)^{\times n} \ltimes \mathfrak{S}_n)$-invariance of Z_n and the fact that the $\mathrm{SL}(V_n)$-orbit of $\mathrm{Seg}(Z_{(n-1)} \times \mathbb{P}\{x_n^0\})$ is $\mathrm{Seg}(Z_{(n-1)} \times \mathbb{P}V_n)$. $\square$

In fact, Proposition 4.1 generalizes as follows.

Proposition 4.2. *Let $p + q = n$ and $Z_p \subset \mathbb{P}(V_1 \otimes \cdots \otimes V_p)$ and*

$$Z_q \subset \mathbb{P}(V_{p+1} \otimes \cdots \otimes V_n).$$

Then $\mathrm{Seg}(Z_p \times Z_q)$ is a subvariety of Z_n.
Let $U_0 = \{[z] \in \mathbb{P}(V_1 \otimes \cdots \otimes V_n) \mid z = z_I X^I \in V_1 \otimes \cdots \otimes V_n, z_{[0,\dots,0]} \neq 0\}$. Then $\varphi([A, t]) \in \mathrm{Seg}(Z_p \times Z_q) \cap U_0$ if and only if A is of the form

$$\begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix},$$

where $P \in S^2\mathbb{C}^p$ and $Q \in S^2\mathbb{C}^q$.

Proof. Let φ^i denote the principal minor map on $i \times i$ matrices and let J and K be multi-indices of length p and q. Let $[x \otimes y] \in \mathrm{Seg}(Z_p \times Z_q)$ be such that $[x] = \varphi^p([P, r]) = [r^{p-|J|}\Delta_J(P)X^J]$ and $[y] = \varphi^q([Q, s]) = [s^{q-|K|}\Delta_K(Q)X^K]$, with $P \in S^2\mathbb{C}^p$ and $Q \in S^2\mathbb{C}^q$.

Notice that if $r = 0$, then $[x] = [0, \dots, 0, \det(P)] \in \mathrm{Seg}(\mathbb{P}V_1 \times \cdots \times \mathbb{P}V_p)$, and similarly if $s = 0$, then $[y] = [0, \dots, 0, \det(Q)] \in \mathrm{Seg}(\mathbb{P}V_{p+1} \times \cdots \times \mathbb{P}V_{p+q})$. So the cases where $r = 0$ or $s = 0$ are covered by iterations of Proposition 4.1.

Now assume $r \neq 0, s \neq 0$ so we can set $r = s = 1$. Consider a blocked matrix of the form

$$A = \begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix}, \tag{2}$$

where $P \in S^2\mathbb{C}^p$ and $Q \in S^2\mathbb{C}^q$. We claim that $\varphi^{p+q}([A, 1]) = [x \otimes y]$. The determinant of a block diagonal matrix is the product of the determinants of the blocks, and principal submatrices of block diagonal matrices are still block diagonal, so

$$\varphi^n([A, 1]) = [\Delta_J(P)\Delta_K(Q) \, X^{J,K}],$$

where $X^{J,K} = X^J \otimes X^K$. But we can reorder the terms in the product to find

$$[\Delta_J(P)\Delta_K(Q)X^{J,K}] = [(\Delta_J(P)X^J) \otimes (\Delta_K(Q)X^K)] = [x \otimes y].$$

For the second part of the proposition, notice that for $[x \otimes y] \in \mathrm{Seg}(Z_p \times Z_q) \cap U_0$, we have exhibited a matrix A as in (2) such that $\varphi^n([A, 1]) = [x \otimes y]$. But symmetric matrices are determined up to sign by their 1×1 and 2×2 principal minors. Any other matrix must have the same blocked form as the one in (2). $\square$

Remark 4.3. Proposition 4.2 gives a useful tool in finding candidate modules for $I(Z_n)$: we are forced to consider

$$I(Z_n) \subset \bigcap_{\substack{p+q=n \\ p,q \geq 1}} I(\mathrm{Seg}(Z_p \times Z_q)).$$

5. Properties of the hyperdeterminantal module

Because of Theorem 1.1, the defining ideal of Z_n, $\mathscr{I}(Z_n) \subset \mathrm{Sym}(V_1^* \otimes \cdots \otimes V_n^*)$, is a $(\mathrm{SL}(2)^{\times n} \ltimes \mathfrak{S}_n)$-module. We consider the $(\mathrm{SL}(2)^{\times n} \ltimes \mathfrak{S}_n)$-module $HD = S_{(2,2)} S_{(2,2)} S_{(2,2)} S_{(4)} \ldots S_{(4)}$ (called the hyperdeterminantal module in [Holtz and Sturmfels 2007]). In this section we compute the dimension of the hyperdeterminantal module and show that it occurs with multiplicity one in $S^4(V_1^* \otimes \cdots \otimes V_n^*)$. Also, in the course of our observations, we arrive at a practical ideal membership test for a class of varieties that includes the variety of principal minors.

Observation 5.1. The module $S_{(2,2)} \mathbb{C}^2$ is 1-dimensional and the module $S_{(4)} \mathbb{C}^2$ is 5-dimensional and therefore the dimension of the hyperdeterminantal module is

$$\dim(S_{(2,2)} S_{(2,2)} S_{(2,2)} S_{(4)} \ldots S_{(4)}) = \binom{n}{3} 5^{n-3}.$$

Proposition 5.2. *The module $HD = S_{(2,2)} S_{(2,2)} S_{(2,2)} S_{(4)} \ldots S_{(4)}$ occurs with multiplicity 1 in $S^4(V_1^* \otimes \cdots \otimes V_n^*)$. Moreover, HD is an irreducible $(\mathrm{SL}(2)^{\times n} \ltimes \mathfrak{S}_n)$-module.*

Remark 5.3. The fact that HD occurs with multiplicity 1 saves us a lot of work because we do not have to worry about which isomorphic copy of the module occurs in the ideal.

Proof of Proposition 5.2. For the "moreover" part, recognize that by definition, $HD = S_{(2,2)} S_{(2,2)} S_{(2,2)} S_{(4)} \ldots S_{(4)}$ is a direct sum over permutations yielding distinct $\mathrm{SL}(2)^{\times n}$-modules. It is a standard fact that each summand is an irreducible $\mathrm{SL}(2)^{\times n}$-module, and this makes HD an irreducible $(\mathrm{SL}(2)^{\times n} \ltimes \mathfrak{S}_n)$-module.

We need to examine the $\mathrm{SL}(2)^{\times n}$-module decomposition of $S^4(V_1^* \otimes \cdots \otimes V_n^*)$. It suffices to prove for any fixed permutation σ that $S_{(2,2)} V_{\sigma(1)}^* \otimes S_{(2,2)} V_{\sigma(2)}^* \otimes S_{(2,2)} V_{\sigma(3)}^* \otimes S_{(4)} V_{\sigma(4)}^* \otimes \cdots \otimes S_{(4)} V_{\sigma(n)}^*$ is an $\mathrm{SL}(2)^{\times n}$-module that occurs with multiplicity 1 in the decomposition of $S^4(V_1^* \otimes \cdots \otimes V_n^*)$.

We follow notation and calculations similar to [Landsberg and Manivel 2004]. For a representation $[\pi]$ of the symmetric group $\mathfrak{S}_d$, let χ_π denote its character. The number of occurrences of $S_{\pi_1} V_1^* \otimes \cdots \otimes S_{\pi_n} V_n^*$ in the decomposition of

$$S^d(V_1^* \otimes \cdots \otimes V_n^*)$$

is computed by $\dim(([\pi_1] \otimes \cdots \otimes [\pi_n])^{\mathfrak{S}_d})$, the dimension of the space of $\mathfrak{S}_d$ invariants. This may be computed by the formula

$$\dim(([\pi_1] \otimes \cdots \otimes [\pi_n])^{\mathfrak{S}_d}) = \frac{1}{d!} \sum_{\sigma \in \mathfrak{S}_d} \chi_{\pi_1}(\sigma) \dots \chi_{\pi_n}(\sigma).$$

In our case, we need to compute

$$\dim\left((([2,2)] \otimes [(2,2)] \otimes [(2,2)] \otimes [(4)] \otimes \cdots \otimes [(4)])^{\mathfrak{S}_4}\right)$$
$$= \frac{1}{4!} \sum_{\sigma \in \mathfrak{S}_4} \chi_{(2,2)}(\sigma) \chi_{(2,2)}(\sigma) \chi_{(2,2)}(\sigma) \chi_{(4)}(\sigma) \dots \chi_{(4)}(\sigma).$$

But $\chi_{(4)}(\sigma) = 1$ for every $\sigma \in \mathfrak{S}_4$. So our computation reduces to

$$\dim\left((([2,2)] \otimes [(2,2)] \otimes [(2,2)] \otimes [(4)] \otimes \cdots \otimes [(4)])^{\mathfrak{S}_n}\right)$$
$$= \frac{1}{4!} \sum_{\sigma \in \mathfrak{S}_4} \chi_{(2,2)}(\sigma) \chi_{(2,2)}(\sigma) \chi_{(2,2)}(\sigma) = 1,$$

where the last equality is found by direct computation. The module $S_{(2,2)} V_1^* \otimes S_{(2,2)} V_2^* \otimes S_{(2,2)} V_3^*$ occurs with multiplicity 1 in $S^4(V_1^* \otimes V_2^* \otimes V_3^*)$. (The full decomposition of $S^4(V_1^* \otimes V_2^* \otimes V_3^*)$ was computed in [Landsberg and Manivel 2004, Proposition 4.3].) Therefore the module $S_{(2,2)} V_{\sigma(1)}^* \otimes S_{(2,2)} V_{\sigma(2)}^* \otimes S_{(2,2)} V_{\sigma(3)}^* \otimes S_{(4)} V_{\sigma(4)}^* \otimes \cdots \otimes S_{(4)} V_{\sigma(n)}^*$ occurs with multiplicity 1 in $S^4(V_1^* \otimes \cdots \otimes V_n^*)$.

We have seen that each summand of HD is an irreducible $\mathrm{SL}(2)^{\times n}$-module that occurs with multiplicity 1 in $S^4(V_1^* \otimes \cdots \otimes V_n^*)$. Therefore HD is an irreducible $(\mathrm{SL}(2)^{\times n} \ltimes \mathfrak{S}_n)$-module, and it occurs with multiplicity 1 in $S^4(V_1^* \otimes \cdots \otimes V_n^*)$. $\square$

This argument generalizes:

Lemma 5.4. *For every collection $\pi_1, \dots, \pi_n$ of partitions of d,*

$$\dim(([\pi_1] \otimes \cdots \otimes [\pi_n])^{\mathfrak{S}_d}) = \dim(([\pi_1] \otimes \cdots \otimes [\pi_n] \otimes [(d)])^{\mathfrak{S}_d}).$$

In particular, if M is any irreducible $\mathrm{SL}(V_1) \times \cdots \times \mathrm{SL}(V_n)$-module that occurs with multiplicity m in $S^d(V_1^* \otimes \cdots \otimes V_n^*)$, then $M \otimes S^d V_{n+1}^*$ is an irreducible $\left(\mathrm{SL}(V_1) \times \cdots \times \mathrm{SL}(V_n) \times \mathrm{SL}(V_{n+1})\right)$-module that occurs with multiplicity m in $S^d(V_1^* \otimes \cdots \otimes V_n^* \otimes V_{n+1}^*)$.

Proof. Use

$$\dim(([\pi_1] \otimes \cdots \otimes [\pi_n])^{\mathfrak{S}_d}) = \frac{1}{d!} \sum_{\sigma \in \mathfrak{S}_d} \chi_{\pi_1}(\sigma) \dots \chi_{\pi_n}(\sigma)$$

and note that $\chi_{(d)}(\sigma) = 1$ for every $\sigma \in \mathfrak{S}_d$. $\square$

Proposition 5.5. *The hyperdeterminantal module is contained in the ideal of the variety of principal minors of symmetric matrices, that is,*

$$HD = S_{(2,2)} S_{(2,2)} S_{(2,2)} S_{(4)} \ldots S_{(4)} \subseteq \mathcal{I}(Z_n),$$

and in particular, $Z_n \subseteq \mathcal{V}(HD)$.

Proof. This statement is proved in [Holtz and Sturmfels 2007]. What follows is a slightly different proof that uses representation theory. Both HD and $\mathcal{I}(Z_n)$ are $(\mathrm{SL}(2)^{\times n} \ltimes \mathfrak{S}_n)$-modules and HD is an irreducible $(\mathrm{SL}(2)^{\times n} \ltimes \mathfrak{S}_n)$-module, so we only need to show that the highest weight vector of HD vanishes on all points of Z_n. The highest weight vector of HD is the hyperdeterminant of format $2 \times 2 \times 2$ on the variables $X^{[i_1, i_2, i_3, 0, \ldots, 0]}$. The set

$$Z_n \cap \mathrm{span}\{X^{[i_1, i_2, i_3, 0, \ldots, 0]} \mid i_1, i_2, i_3 \in \{0, 1\}\}$$

is the set of principal minors of the upper-left 3×3 block of $n \times n$ matrices. The highest weight vector of HD vanishes on these principal minors because of the case $n = 3$, so there is nothing more to show. □

Proposition 5.6. *Let V and W be complex vector spaces with $\dim(V) \geq 2$. Suppose $Y \subset \mathbb{P}W$ and $X \subset \mathbb{P}(V \otimes W)$ are varieties such that $\mathrm{Seg}(Y \times \mathbb{P}W) \subset X$. Suppose $M \subset S^d V^*$ is a space of polynomials. Then $M \otimes S^d W^* \subset \mathcal{I}_d(X)$ only if $M \subset \mathcal{I}_d(Y)$.*

Proof. By Lemma 5.4, it makes sense to think of $M \otimes S^d W^* \subset S^d (V^* \otimes W^*)$ as a space of polynomials.

There exists a basis of $S^d W^*$ of vectors of the form α^d. So $M \otimes S^d W^*$ has a basis of vectors of the form $f \otimes \alpha^d$ with $f \in M$ and $\alpha \in W^*$. It suffices to prove the proposition on this basis.

Suppose $f \otimes \alpha^d$ is a basis vector in $M \otimes S^d(W^*) \subset \mathcal{I}_d(X)$. Then $\mathrm{Seg}(Y \times \mathbb{P}W) \subset X$ implies that $f \otimes \alpha^d \in \mathcal{I}_d(\mathrm{Seg}(Y \times \mathbb{P}W)) \subset S^d(V^* \otimes W^*)$. This means that $f \otimes \alpha^d (y \otimes w) = 0$ for all $y \in Y$ and for all $w \in W$. It is a fact that $\alpha^d(w) = \alpha(w)^d$ (this can be deduced from Lemma 6.3 below, for instance), so we can evaluate

$$f \otimes \alpha^d (y \otimes w) = f(y) \alpha^d(w) = f(y) \alpha(w)^d.$$

Since $\dim(V) \geq 2$, $\mathcal{V}(\alpha)$ is a hyperplane. It is no problem to choose a point that misses a hyperplane, so we can choose a particular $w \in W$ so that $\alpha(w) \neq 0$.

So we have $f(y)\alpha(w)^d = 0$ for all $y \in Y$ and $\alpha(w) \neq 0$, so $f(y) = 0$ for all $y \in Y$ and hence $f \in \mathcal{I}_d(Y)$. We can repeat the argument for any $f \in M$ we choose, so we are done. □

Proposition 5.6 fails to be an if and only if statement. Explicitly, we *cannot* say that every module in the space $\mathcal{I}_d(X)$ occurs as $M \otimes S^d V^*$ for a subset $M \subset I_d(Y)$.

In Section 6 we study the zero sets of modules of the form $I_d(Y) \otimes S^d V^*$, and this sheds light on the failure of the converse of Proposition 5.6.

Remark 5.7. Proposition 4.1 says that $\mathrm{Seg}(Z_n \times \mathbb{P}V_{n+1}) \subset Z_{n+1}$. We can use this proposition to study the variety of principal minors in two ways. First, if M is a module in $\mathscr{I}_d(Z_n)$, then $M \otimes S^d V_{n+1}$ is a module in $\mathscr{I}_d(Z_{n+1})$. The second use is the contrapositive version. It gives an easy test for ideal membership for modules that have at least one $S_{(d)}V_i^*$ factor. Suppose we know $\mathscr{I}_d(Z_n)$ for some n. If we want to test whether $M = S_{\pi_1}V_1^* \otimes \cdots \otimes S_{\pi_{n+1}}V_{n+1}^*$ is in $\mathscr{I}_d(Z_{n+1})$ and we know that M has at least one $\pi_i = (d)$, then we can remove $S_{\pi_i}V_i^*$ and check whether the module we have left is in $\mathscr{I}_d(Z_n)$.

6. A geometric characterization of the zero set
of the hyperdeterminantal module via augmentation

The hyperdeterminantal module has a useful inductive description that we would like to be able to exploit. In particular, for $n \geq 3$, the module is always of the form

$$S_{(2,2)} S_{(2,2)} S_{(2,2)} S_{(4)} \ldots S_{(4)},$$

where the number of $S_{(4)}$ factors is $n - 3$. Then for $n \geq 3$, to construct the $(\mathrm{SL}(2)^{\times n} \ltimes \mathfrak{S}_n)$-module HD in the case $n = k + 1$ from HD in the case $n = k$, we simply append another S^4.

More generally, if M is an $\mathrm{SL}(V)$-module, we will call a $(\mathrm{SL}(V) \times \mathrm{SL}(W))$-module of the form $M \otimes S^d W^*$ an *augmentation* or *augmented module*. So for $n \geq 4$, HD can be considered as the sum of augmented modules.

In this section, we study augmented modules and their zero sets in order to arrive at a geometric description of the zero set of an augmented module (Lemma 6.4). By applying this geometric description to the hyperdeterminantal module, we get a geometric description of its zero set (Lemma 6.6). This description is essential in our proof of Theorem 1.3.

6A. *Polarization and its application to augmented modules.* Augmentation is similar to prolongation, a concept found in the study of the ideals of secant varieties. A difference between the two is that augmentation does not change the degree of the polynomials, whereas prolongation increases the degree.

It is not a surprise that we can get inspiration from the techniques used to study secant varieties when studying augmented modules. In particular, polarization is a tool from classical invariant theory [Weyl 1997, pp. 5–6], and is useful in the study of ideals of secant varieties (see [Landsberg and Manivel 2003; Sidman and Sullivant 2009] for recent examples). In what follows, we use polarization to better understand the polynomials in an augmented module.

Polarization allows for the passage from a homogeneous polynomial to a symmetric multilinear form. Let $x_1, \ldots, x_n$ be a basis of V, and let $v_i = v_{i,1}x_1 + \cdots + v_{i,n}x_n$ for $1 \le i \le d$. Given a homogeneous degree-d polynomial f in the polynomial ring $\mathbb{C}[x_1, \ldots, x_n]$, the *polarization* of f is a symmetric multilinear form $\vec{f} \in S^d V^*$, where we define $\vec{f}(v_1, \ldots, v_d)$ to be the coefficient of $t_1 t_2 \ldots t_d$ in the expansion of

$$f(t_1 v_1 + \cdots + t_d v_d)$$

considered as a polynomial in $t_1, \ldots, t_d$. For example if $f(x_1, x_2) = (x_1)^2 x_2$, one calculates that $\vec{f}(v_1, v_2, v_3) = 2(v_{1,1}v_{2,1}v_{3,2} + v_{1,1}v_{3,1}v_{2,2} + v_{2,1}v_{3,1}v_{1,2})$.

The following useful characterization is straightforward from the definition, and while it is a consequence of classical facts [Weyl 1997], we found it stated in [Sidman and Sullivant 2009, Lemma 2.5(1)].

Lemma 6.1. *If F is a homogeneous degree-d polynomial in $x_1, \ldots, x_n$, let $\vec{F}$ be its polarization. Let $v = t_1 x_1 + \cdots + t_k x_k$. Then*

$$F(v) = \vec{F}(v, \ldots, v) = \sum_{\beta} \frac{1}{\beta!} t^\beta \vec{F}(x^\beta), \qquad (3)$$

where $\beta = (\beta_1, \ldots, \beta_k)$ is a (nonnegative) partition of d, $\beta! = \beta_1! \ldots \beta_k!$, $t^\beta = t_1^{\beta_1} \ldots t_k^{\beta_k}$, and $\vec{F}(x^\beta) = \vec{F}(x_1^{\beta_1}, \ldots, x_k^{\beta_k})$, and $x_i^{\beta_i}$ is to be interpreted as x_i repeated i times.

Here is an example of the utility of this lemma that we will need later.

Lemma 6.2. *A linear space $L = \operatorname{span}\{x_1, \ldots, x_k\}$ is a subset of $\mathcal{V}(f)$ if and only if $\vec{f}(x^\beta) = 0$ for every partition β of d.*

Proof. A linear space $L = \operatorname{span}\{x_1, \ldots, x_k\}$ is in the zero set of f if and only if $f(t_1 x_1 + \cdots + t_k x_k) = 0$ for all choices of $t_i \in \mathbb{C}$. Formula (3) says that

$$f(t_1 x_1 + \cdots + t_k x_k) = \sum_{\beta} \frac{1}{\beta!} t^\beta \vec{f}(x^\beta), \qquad (4)$$

and thus implies that if $\vec{f}(x^\beta) = 0$ for all β, then $f(t_1 x_1 + \cdots + t_k x_k) = 0$ for all $t_i \in \mathbb{C}$.

For the other direction, suppose $f(t_1 x_1 + \cdots + t_k x_k) = 0$ for all $t_i \in \mathbb{C}$. Consider a fixed partition β' and take the derivative $\partial/\partial t^{\beta'}$ of (4) to get

$$0 = \vec{f}(x^{\beta'}) + \sum_{\beta > \beta'} \frac{1}{(\beta - \beta')!} t^{\beta - \beta'} \vec{f}(x^\beta).$$

Then take limits as $t_i \to 0$ to find that $0 = \vec{f}(x^{\beta'})$. We do this for each β' to conclude. $\qquad \square$

In general, the polarization of the tensor product of two polynomials is not likely to be the product of the polarized polynomials; however, there is something we can say in the following special case:

Lemma 6.3. *Let $F \in S^d(W^*)$ and let $\vec{F}$ denote its polarization. Then for $\gamma \in V^*$ we have*

$$\overrightarrow{F \otimes (\gamma)^d} = \vec{F} \otimes \overrightarrow{(\gamma)^d} = \vec{F} \otimes (\gamma)^d.$$

Proof. A standard fact about the polarization is that $\vec{F}$ is a symmetric multilinear form. It is obvious that

$$\overrightarrow{(\gamma)^d} = (\gamma)^d,$$

because $(\gamma)^d$ is already symmetric and multilinear.

So it remains to prove the first equality in the lemma, which we do by induction on the number of terms in F. Suppose F is a monomial, $F = \boldsymbol{w}^\alpha = w_1^{\alpha_1} \circ \cdots \circ w_n^{\alpha_n}$. Then use the isomorphism $W^{\otimes d} \otimes V^{\otimes d} \simeq (W \otimes V)^{\otimes d}$, and write $\boldsymbol{w}^\alpha \otimes \gamma^d = (w_1^{\alpha_1} \otimes \gamma^{\alpha_1}) \circ \cdots \circ (w_n^{\alpha_n} \otimes \gamma^{\alpha_n}) = (w_1 \otimes \gamma)^{\alpha_1} \circ \cdots \circ (w_n \otimes \gamma)^{\alpha_n} = (\boldsymbol{w} \otimes \gamma)^\alpha$.

If F is not a monomial, suppose $F = F_1 + F_2$ with F_i nonzero polynomials for $i = 1, 2$, each having strictly fewer monomials than F. It is clear that

$$\overrightarrow{F_1 + F_2} = \vec{F}_1 + \vec{F}_2.$$

Also, the operation $\otimes \gamma^d$ is distributive. So $\overrightarrow{F \otimes (\gamma)^d} = \overrightarrow{F_1 \otimes \gamma^d} + \overrightarrow{F_2 \otimes \gamma^d}$. By the induction hypothesis, we know that

$$\overrightarrow{F_i \otimes \gamma^d} = \vec{F}_i \otimes \gamma^d$$

for $i = 1, 2$. We conclude that $\overrightarrow{F_1 \otimes \gamma^d} + \overrightarrow{F_2 \otimes \gamma^d} = (\vec{F}_1 + \vec{F}_2) \otimes \gamma^d = \vec{F} \otimes \gamma^d$. $\square$

The following lemma was inspired by methods found in [Landsberg and Manivel 2003]. It is a geometric description of the zero set of an augmented module.

Lemma 6.4 (augmentation lemma). *Let W and V be complex vector spaces with $\dim(V) \geq 2$. Let $X \subset \mathbb{P}W$ be a variety and let $\mathscr{I}_d(X) = \mathscr{I}(X) \cap S^d W^*$ be the vector space of degree-d polynomials in the ideal $\mathscr{I}(X)$. Then*

$$\mathscr{V}(\mathscr{I}_d(X) \otimes S^d V^*) = \mathrm{Seg}(\mathscr{V}(\mathscr{I}_d(X)) \times \mathbb{P}V) \cup \bigcup_{L \subset \mathscr{V}(\mathscr{I}_d(X))} \mathbb{P}(L \otimes V), \qquad (5)$$

where $L \subset \mathscr{V}(\mathscr{I}_d(X))$ are linear subspaces.

Since the linear spaces L can be one-dimensional, we do have

$$\mathrm{Seg}(\mathscr{V}(\mathscr{I}_d(X)) \times \mathbb{P}V) \subset \bigcup_{L \subset \mathscr{V}(\mathscr{I}_d(X))} \mathbb{P}(L \otimes V),$$

and we use Lemma 6.4 with the two terms on the right side of (5) combined, but we keep the two parts separate for emphasis here.

Remark 6.5. Note that if $\mathcal{I}(X)$ is generated in degree no larger than d, then one can replace $\mathcal{V}(\mathcal{I}_d(X))$ with X in the statement of Lemma 6.4. We use the result of Lemma 6.4 with the induction hypothesis that $\mathcal{V}(HD) = Z_n$ and obtain a description of the zero set $\mathcal{V}(HD \otimes S^4 V_{n+1})$ in terms of the geometry of Z_n.

Proof of Lemma 6.4. First we prove "$\supseteq$". Suppose $\dim(V) = n \geq 2$. Recall that we can choose a basis of $S^d V^*$ consisting of dth powers of linear forms, $\{(\gamma_1)^d, \ldots, (\gamma_r)^d\}$, where $r = \binom{n+d-1}{d}$ and the γ_i are in general linear position. It suffices to work on a basis of the vector space $\mathcal{I}_d(X) \otimes S^d V^*$. We choose a basis consisting of polynomials of the form $f \otimes \gamma^d$, with $f \in \mathcal{I}_d(X)$ and $\gamma \in V^*$.

Suppose $[x \otimes a] \in \text{Seg}(\mathcal{V}(\mathcal{I}_d(X)) \times \mathbb{P}V)$ and evaluate $(f \otimes \gamma^d)(x \otimes a) = f(x)\gamma^d(a)$. But $x \in \mathcal{V}(\mathcal{I}_d(X))$, so $f(x) = 0$ for every $f \in \mathcal{I}_d(X)$, and in particular, $[x \otimes a] \in \mathcal{V}(\mathcal{I}_d(X) \otimes S^d V^*)$. So we have established that $\mathcal{V}(\mathcal{I}_d(X) \otimes S^d V^*) \supset \text{Seg}(\mathcal{V}(\mathcal{I}_d(X)) \times \mathbb{P}V)$.

Now suppose $[v] \in \mathbb{P}(L \otimes V)$ for some linear subspace $L = \text{span}\{x_1, \ldots, x_l\} \subset \mathcal{V}(\mathcal{I}_d(X))$. By expanding an expression of $[v]$ in bases and collecting the coefficients of the x_i, we can write $[v] = [x_1 \otimes a_1 + \cdots + x_l \otimes a_l]$ for $a_i \in V$ not all zero. Consider $f \otimes \gamma^d \in \mathcal{I}_d(X) \otimes S^d V$. By Lemma 6.3,

$$\overrightarrow{f \otimes \gamma^d} = \vec{f} \otimes \gamma^d,$$

and using the polarization formula (3), we write

$$(f \otimes \gamma^d)(v) = (\vec{f} \otimes \gamma^d)(v, \ldots, v) = \sum_\beta \frac{1}{\beta!} \vec{f}(x^\beta)\gamma^d(a^\beta).$$

The choice of $L \subset \mathcal{V}(\mathcal{I}_d(X))$ means that $L \subset \mathcal{V}(f)$, so by Lemma 6.2, $\vec{f}(x^\beta) = 0$ for all β. Every term of $(f \otimes \gamma^d)(v)$ vanishes, so $(f \otimes \gamma^d)(v) = 0$, and hence $[v] \in \mathcal{V}(\mathcal{I}_d(X) \otimes S^d V^*)$. So we have established that $\mathcal{V}(\mathcal{I}_d(X) \otimes S^d V^*) \supset \mathbb{P}(L \otimes V)$ for all linear subspaces $L \subset \mathcal{V}(\mathcal{I}_d(X))$.

Now we prove "$\subseteq$". Consider any $[v] \in \mathbb{P}(W \otimes V)$. Choose a basis $\{a_1, \ldots, a_k\}$ of V (by assumption $k \geq 2$). Then expand the expression of v in bases and collect the coefficients of each a_i to find $[v] = [x_1 \otimes a_1 + \cdots + x_k \otimes a_k]$ with $x_1, \ldots, x_k \in W$ and not all x_i zero.

We need to show that $[v] \in \mathbb{P}(L \otimes V)$ for a linear space $L \subset \mathcal{V}(\mathcal{I}_d(X))$. The natural linear space to consider is $L = \text{span}\{x_1, \ldots, x_k\}$. Since we already have an expression $[v] = [x_1 \otimes a_1 + \cdots + x_k \otimes a_k]$, if we can show that $L = \text{span}\{x_1, \ldots, x_k\} \subset \mathcal{V}(\mathcal{I}_d(X))$, we will be done.

For any $f \otimes \gamma^d \in \mathcal{I}_d(X) \otimes S^d V^*$ we can write

$$0 = (f \otimes \gamma^d)(v) = \sum_\beta \frac{1}{\beta!} \vec{f}(x^\beta)\gamma^d(a^\beta). \tag{6}$$

Let $\{\check{a}_1, \ldots, \check{a}_k\}$ be a basis of V^* dual to $\{a_1, \ldots, a_k\}$. Then let γ vary continuously

in V^* by writing it as

$$\gamma = t_1 \check{a}_1 + \cdots + t_k \check{a}_k,$$

where the parameters $t_i \in \mathbb{C}$ vary. The polynomial γ^d is simple enough that we can expand it as follows:

$$\gamma^d(\boldsymbol{a}^\beta) = \gamma^d(a_1^{\beta_1}, \ldots, a_k^{\beta_k}) = \gamma(a_1)^{\beta_1} \ldots \gamma(a_k)^{\beta_k}.$$

But our choices have made it so that $\gamma(a_i) = t_i$, and therefore $\gamma^d(\boldsymbol{a}^\beta) = \boldsymbol{t}^\beta$. So (6) becomes

$$0 = (f \otimes \gamma^d)(v) = \sum_\beta \frac{1}{\beta!} \vec{f}(\boldsymbol{x}^\beta) \boldsymbol{t}^\beta = f(t_1 x_1 + \cdots + t_k x_k),$$

where we have used Lemma 6.3. So $f(t_1 x_1 + \cdots + t_k x_k) = 0$ for all $t_i \in \mathbb{C}$, and this is an equivalent condition that $L = \mathrm{span}\{x_1, \ldots, x_k\}$ is a subspace of $\mathcal{V}(f)$. Since this was done for arbitrary $f \in \mathcal{I}_d(X)$, we conclude that $L \subset \mathcal{V}(\mathcal{I}_d(X))$. $\qquad \square$

Now we can apply this geometric characterization of augmentation to the hyperdeterminantal module. To do this we need to set up more notation.

Assume $n \geq 4$. Let HD_i be the image of the hyperdeterminantal module at stage $n-1$ under the reindexing isomorphism

$$S^4(V_1^* \otimes \cdots \otimes V_{n-1}^*) \longrightarrow S^4(V_1^* \otimes \cdots \otimes V_{i-1}^* \otimes V_{i+1}^* \otimes \cdots \otimes V_n^*),$$

where we still have $n-1$ vector spaces $V_i \simeq \mathbb{C}^2$, but we have shifted the index on the last $n-i$ terms. Then the hyperdeterminantal module at stage n can be expressed as a sum of augmented modules as follows:

$$HD = \sum_{i=1}^n (HD_i \otimes S^4 V_i^*).$$

Finally note that if $\dim(V) = k$, then $\sigma_s(\mathbb{P}W \times \mathbb{P}V) = \mathbb{P}(W \otimes V)$ for all $s \geq k$. In the case $V_i \simeq \mathbb{C}^2$, we have $\mathbb{P}(L \otimes V_i) = \sigma_2(\mathbb{P}L \times \mathbb{P}V_i)$. Certainly

$$\mathrm{Seg}(\mathcal{V}(M_i) \times \mathbb{P}V_i) \subset \bigcup_{L \subset V(M_i)} \mathbb{P}(L \otimes V_i),$$

for any modules of polynomials M_i. If $L \subset \mathcal{V}(\mathcal{I}_d(X))$, then $\sigma_s(\mathbb{P}L \times \mathbb{P}V) \subseteq \sigma_s(\mathcal{V}(\mathcal{I}_d(X)) \times \mathbb{P}V)$. If A, B, C are vector spaces of polynomials and $C = A + B$, then $\mathcal{V}(C) = \mathcal{V}(A) \cap \mathcal{V}(B)$. Collecting these ideas, we apply the Augmentation Lemma 6.4 to the hyperdeterminantal module to yield the following:

Lemma 6.6 (characterization lemma). *Consider*

$$\sum_{i=1}^n HD_i \otimes S^d V_i^* \subset S^d(V_1^* \otimes \cdots \otimes V_n^*).$$

Then

$$\mathcal{V}\left(\sum_{i=1}^{n} HD_i \otimes S^d V_i^*\right) = \bigcap_{i=1}^{n}\left(\bigcup_{L \subset V(HD_i)} \mathbb{P}(L \otimes V_i)\right) \subseteq \bigcap_{i=1}^{n}(\sigma_2(\mathcal{V}(HD_i) \times \mathbb{P}V_i)).$$

Remark 6.7. A consequence of the characterization lemma is the following test for nonmembership in the zero set of *HD*. Suppose $[z] = [\zeta^1 \otimes x_i^1 + \zeta^2 \otimes x_i^2] \in \mathbb{P}^{2^n-1}$. If either $[\zeta^1]$ or $[\zeta^2]$ is not a vector of principal minors of an $(n-1) \times (n-1)$ symmetric matrix, then $[z]$ is not a zero of the hyperdeterminantal module *HD* and hence not a vector of principal minors of a symmetric matrix, since $\mathcal{V}(HD) \supset Z_n$. This observation can be iterated, and each iteration cuts the size of the vector in question in half until one need only check honest hyperdeterminants of format $2 \times 2 \times 2$. This test, while relatively cheap and accessible, is necessary but not sufficient, as is pointed out in [Holtz and Sturmfels 2007].

It is well-known that the ideal of the Segre product of an arbitrary number of projective spaces is generated in degree 2 by the 2×2 minors of flattenings. In essence, this is saying that all of the polynomials in the ideal come from the Segre products of just two projective spaces. The following is a weaker, strictly set-theoretic result in the same spirit. It is another application of the Augmentation Lemma 6.4, and its proof is mimicked in the proof of Lemma 7.11 below.

Proposition 6.8. *For $1 \leq i \leq n$, let V_i be complex vector spaces each with dimension ≥ 2 and assume $n \geq 2$. If for each i, $B_i \subset S^d(V_1^* \otimes \ldots V_{i-1}^* \otimes V_{i+1}^* \otimes \cdots \otimes V_n^*)$ is a set of polynomials with the property*

$$\mathcal{V}(M^i) = \text{Seg}(\mathbb{P}V_1 \times \cdots \times \mathbb{P}V_{i-1} \times \mathbb{P}V_{i+1} \times \cdots \times \mathbb{P}V_n),$$

then

$$\mathcal{V}\left(\bigoplus_i (M_i \otimes S^d V_i^*)\right) = \text{Seg}(\mathbb{P}V_1 \times \cdots \times \mathbb{P}V_n).$$

Proof. Work by induction and use Lemma 6.4. It is clear that

$$\mathcal{V}\left(\bigoplus_i (M_i \otimes S^d V_i^*)\right) \supset \text{Seg}(\mathbb{P}V_1 \times \cdots \times \mathbb{P}V_n).$$

All the linear spaces on $\text{Seg}(\mathbb{P}V_1 \times \cdots \times \mathbb{P}V_n)$ are (up to permutation) of the form $V_1 \otimes \widehat{a_2} \otimes \cdots \otimes \widehat{a_n}$, where $a_i \in V_i$ are nonzero and $\widehat{a_i}$ denotes the line through a_i. Then compute the intersection, $\bigcup_{L^i} \bigcap_{i=1}^{n} \mathbb{P}(L^i \otimes V_i)$, and notice that in the intersection of just 3 factors, all of the resulting linear spaces must live in $\text{Seg}(\mathbb{P}V_1 \times \cdots \times \mathbb{P}V_n)$. □

7. Understanding the case when two zeros of the hyperdeterminantal module disagree in precisely one coordinate

In the proof of Theorem 1.3 below, we work to construct a matrix whose principal minors are a given point in the zero set of the hyperdeterminantal module. The main difficulty is the following. Suppose we have a point $[z] \in \mathcal{V}(HD)$ and a candidate matrix A that satisfies $\Delta_I(A) = z_I$ for all $I \neq [1, \ldots, 1]$. In other words, all of the principal minors of A except possibly for the determinant agree with the entries of z. What can we say about z?

To answer this question, we must study the points in $\widehat{\mathcal{V}(HD)}$ that have all of their coordinates except one equal. Geometrically, we need to understand the points for which a line in the coordinate direction $X^{[1,\ldots,1]}$ above the point z intersects $\widehat{\mathcal{V}(HD)}$ in at least two points. We answer this question in Lemma 7.2 below. Using that lemma, we find the following.

Proposition 7.1. *Let $n \geq 4$. Let $z = z_I X^I$ and $w = w_I X^I$ be points in $\widehat{\mathcal{V}(HD)}$. If $z_I = w_I$ for all $I \neq [1, \ldots, 1]$ and $z_{[0,\ldots,0]} \neq 0$, then $z = w$.*

For the rest of this section we use the following notation. If $K = \{k_1, \ldots, k_s\} \subset \{1, \ldots, n\}$ and $1 \leq k_j \leq n$ for all j, then let $V_K \simeq V_{k_1} \otimes \cdots \otimes V_{k_s}$ for $s \leq n$. We assume $V_k \simeq \mathbb{C}^2$ for all k, so that $V_K \simeq (\mathbb{C}^2)^{\otimes s}$. Let $\mathcal{P}^2(\{n_1, \ldots, n_n\})$ denote the collection of all partitions of $\{n_1, \ldots, n_n\}$ into mutually disjoint subsets of cardinality 2 or less, that is, $\mathcal{P}^2(\{1, \ldots, n\})$ consists of the sets $\{K_1, \ldots, K_m\}$ such that $K_p \subset \{1, \ldots, n\}$ and $|K_p| \leq 2$ for every $1 \leq p \leq m$, $K_p \cap K_q = \varnothing$ whenever $p \neq q$, and $\cup_{p=1}^m K_p = \{1, \ldots, n\}$.

Lemma 7.2. *Let $n \geq 4$. Let $z = z_I X^I$ and $w = w_I X^I$ be points in $\widehat{\mathcal{V}(HD)}$. If $z_I = w_I$ for all $I \neq [1, \ldots, 1]$ but $z_{[1,\ldots,1]} \neq w_{[1,\ldots,1]}$, then*

$$[z], [w] \in \bigcup_{\{K_1,\ldots,K_m\} \in \mathcal{P}^2(\{1,\ldots,n\})} \mathrm{Seg}(\mathbb{P}V_{K_1} \times \cdots \times \mathbb{P}V_{K_m}) \subset Z_n.$$

Note that the notationally dense Segre product is just a product of $\mathbb{P}^3$'s and $\mathbb{P}^1$'s.

Proof of Proposition 7.1. Assume Lemma 7.2. Let $z = z_I X^I$ and $w = w_I X^I$ be points in $\widehat{\mathcal{V}(HD)} \cap \{z \mid z_{[0,\ldots,0]} \neq 0\}$. Suppose that $z_I = w_I$ for all $I \neq [1, \ldots, 1]$, and suppose for contradiction that $z_{[1,\ldots,1]} \neq w_{[1,\ldots,1]}$. Lemma 7.2 implies that $[z], [w]$ are in a Segre product of $\mathbb{P}^1$'s and $\mathbb{P}^3$'s.

$Z_1 \simeq \mathbb{P}^1$ and $Z_2 \simeq \mathbb{P}^3$, and Proposition 4.2 implies that a point $[A, t]$ with $t \neq 0$ mapping to $\mathrm{Seg}(\mathbb{P}V_{K_1} \times \cdots \times \mathbb{P}V_{K_m})$ with $\{K_1, \ldots, K_m\} \in \mathcal{P}^2(\{1, \ldots, n\})$ is permutation equivalent to a block diagonal matrix consisting of 1×1 and 2×2 blocks. Such a block diagonal matrix is a special case of a symmetric tridiagonal matrix, and therefore none of its principal minors depends on the sign of the off-diagonal terms. So fixing the 0×0, 1×1 and 2×2 principal minors fixes the rest

of the principal minors in such a matrix. If we take $z_{[0,\ldots,0]} = w_{[0,\ldots,0]} = 1$ and assume the 1×1 and 2×2 principal minors agree, then the rest of the principal minors must agree, including the determinants; thus the contradiction.

The assumption $z_{[0,\ldots,0]} \neq 0$ is necessary. If $z_{[0,\ldots,0]} = 0$, then consider the image of any two matrices A, B with different nonzero determinants under the principal minor map with $t = 0$. Then $\varphi([A, 0]) = [0, \ldots, 0, \det(A)] \neq \varphi([B, 0]) = [0, \ldots, 0, \det(B)]$. $\square$

Remark 7.3. A key point is that we are not making the claim in Proposition 7.1 for $n = 3$. In this case any two zeros of the hyperdeterminant are principal minors of 3×3 matrices that differ up to sign of the off-diagonal terms. Altering the sign of the off-diagonal terms of a 3×3 symmetric matrix can change the determinant without changing the other principal minors and without forcing the matrix to be blocked as a 2×2 block and a 1×1 block.

Remark 7.4. To see that the analog of Proposition 7.1 holds for Z_n with $n \geq 4$ and $t \neq 0$ requires much less work than the case of $\mathcal{V}(HD)$. We used Maple to construct a generic symmetric 4×4 matrix and computed its principal minors. Then we changed the signs of the off-diagonal terms in every possible combination and compared the number of principal minors that agreed with the principal minors of the original matrix. The result was that the two vectors of principal minors could agree in precisely 11, 13 or 16 entries, but not 15. (Though tedious, the 4×4 case can also be proved without a computer by analyzing the parity of the various products of the off-diagonal terms in the matrix.) We repeated the experiment in the 5×5 case and found that the two vectors could agree in precisely 16, 19, 20, 21, 23, 25 or 32 positions, but never 31 positions.

The general case follows from the 4×4 case by the following. Suppose $n \geq 4$ and $2^n - 1$ of the principal minors of an $n \times n$ symmetric matrix agree with the principal minors of another $n \times n$ symmetric matrix. Then we may assume that the 0×0, 1×1 and 2×2 principal minors of both matrices agree and hence the matrices must agree up to the signs of the off-diagonal terms. Then use the group to move the one position where the principal minors don't agree to be a 4×4 determinant and use the 4×4 result for the contradiction.

To prove Lemma 7.2, we will show that if $w_I = z_I$ for all $I \neq [1, \ldots, 1]$ and $z_{[1,\ldots,1]} \neq w_{[1,\ldots,1]}$, then z is a zero of an auxiliary set of polynomials denoted B. We then show that the zero set $\mathcal{V}(B)$ is contained in the union of Segre varieties. Finally, Proposition 4.2 provides the inclusion into Z_n.

7A. *Reduction to one variable.* Let $n \geq 4$. Suppose $z = z_I X^I$ and $w = w_I X^I$ are points in $\widehat{\mathcal{V}(HD)}$ such that $z_I = w_I$ for all $I \neq [1, \ldots, 1]$. Both points are zeros of every polynomial in HD, but the only coordinate in which they can differ

is $[1, \ldots, 1]$. Now consider the coordinates z_I $(= w_I)$ as fixed constants for all $I \neq [1, \ldots, 1]$, and for $f \in HD$ define f_z by the substitution

$$f(X^{[0,\ldots,0]}, \ldots, X^{[1,\ldots,1]}) \mapsto f(z_{[0,\ldots,0]}, \ldots, z_{[0,1,\ldots,1]}, X^{[1,\ldots,1]}) =: f_z(X^{[1,\ldots,1]}).$$

Let $HD_{[1,\ldots,1]}(z) = \{f_z \mid f \in HD\}$ denote the resulting set of univariate polynomials. Then $z_{[1,\ldots,1]}$ and $w_{[1,\ldots,1]}$ are two (possibly different) roots of each univariate polynomial $f_z \in HD_{[1,\ldots,1]}(z)$.

Lemma 7.5. *If $f \in HD$, then the corresponding polynomial $f_z \in HD_{[1,\ldots,1]}(z)$ is either degree 0, 1, or 2 in $X^{[1,\ldots,1]}$.*

Proof. It suffices to prove the statement for $f \in S_{(2,2)}V_1^* \otimes S_{(2,2)}V_2^* \otimes S_{(2,2)}V_3^* \otimes S_{(4)}V_4^* \otimes \cdots \otimes S_{(4)}V_n^*$. Suppose for contradiction that f has a monomial of the form $(X^{[1,\ldots,1]})^3 X^{[i_1,\ldots,i_n]}$. Its possible weights are $(2+2i_1, \ldots, 2+2i_n)$ with $i_j \in \{0, 1\}$. However, the weight of every polynomial in $S_{(2,2)}V_1^* \otimes S_{(2,2)}V_2^* \otimes S_{(2,2)}V_3^* \otimes S_{(4)}V_4^* \otimes \cdots \otimes S_{(4)}V_n^*$ is of the form $(0, 0, 0, w_4, \ldots, w_n)$, where w_i are even integers with $|w_i| \leq 4$ for $4 \leq i \leq n$, a contradiction since obviously $0 \neq 2+2i_1$ for any $i_1 \in \{0, 1\}$. Therefore the degree of f is less than 3 in $X^{[1,\ldots,1]}$. $\qquad\square$

Now we know that $w_{[1,\ldots,1]}$ and $z_{[1,\ldots,1]}$ are both common zeros of univariate polynomials, all with degree 2 or less. The fact that $w_{[1,\ldots,1]}$ and $z_{[1,\ldots,1]}$ are both common zeros of more than one univariate polynomial comes from the fact that we have required $n \geq 4$; otherwise there is only one polynomial and what we are about to do would be trivial.

A quadratic (not identically zero) in one variable has at most two solutions, and a linear polynomial (not identically zero) has at most one solution. The only way then for us to have $w \neq z$ and $[w], [z] \in \mathcal{V}(HD)$ is if *all* of the linear polynomials were identically zero and if *all* of the quadratics were scalar multiples of each other.

Therefore, we need to study the points $[z] \in \mathcal{V}(HD)$ for which $HD_{[1,\ldots,1]}(z)$ has dimension 1 or less. Define polynomials a_f, b_f, and c_f (which necessarily do not depend on $X^{[1,\ldots,1]}$) for each $f_z \in HD_{[1,\ldots,1]}(z)$ by

$$f_z = a_f(z)(X^{[1,\ldots,1]})^2 + b_f(z)(X^{[1,\ldots,1]}) + c_f(z).$$

The requirement that $HD_{[1,\ldots,1]}(z)$ have dimension 1 or less implies the weaker (but still sufficient) condition that z be a root of the polynomials

$$B' := \mathrm{span}\{a_f b_g - a_g b_f \mid f, g \in HD\}.$$

The polynomials in B' have the property that if $h(z) \neq 0$ for a nonzero $h \in B'$, that is, $[z] \notin \mathcal{V}(B')$, then there is a nontrivial pair of polynomials in $HD_{[1,\ldots,1]}(z)$ that are not scalar multiples of each other, and thus the zero set of $HD_{[1,\ldots,1]}(z)$ is a single point. In this case we must have $w_{[1,\ldots,1]} = z_{[1,\ldots,1]}$. If, however, $h(z) = 0$ for

all $h \in B'$, that is, $z \in \mathcal{V}(B')$, then it is possible that the polynomials in $HD_{[1,\dots,1]}(z)$ have 2 common roots.

B' is not $(\mathrm{SL}(2)^{\times n} \ltimes \mathfrak{S}_n)$-invariant. Let $B := \mathrm{span}\{(\mathrm{SL}(2)^{\times n} \ltimes \mathfrak{S}_n).B'\}$ denote the corresponding $(\mathrm{SL}(2)^{\times n} \ltimes \mathfrak{S}_n)$-module. If $g.[z] \notin \mathcal{V}(B')$, then by our remarks above, $g.[z] \in Z_n$, and in particular, $[z] \in Z_n$ (because Z_n is a G-variety). The following lemma allows us to compare G-orbits of points and the zero sets of arbitrary sets of polynomials (not necessarily G-modules).

Lemma 7.6. *Let $z \in \mathbb{P}V$, let $G \subset \mathrm{GL}(V)$ be a group, and let $M \subset \mathrm{Sym}(V^*)$ be a collection of polynomials (M is not necessarily a G-module). Then*

$$G.z \subset \mathcal{V}(M) \text{ if and only if } z \in \mathcal{V}(\mathrm{span}\{G.M\}).$$

Proof. $G.z \subset \mathcal{V}(M)$ if and only if $f(g.z) = 0$ for all $g \in G$ and for all $f \in M$. But from the definition of the G-action on the dual space, $f(g.z) = (g^{-1}.f)(z)$, so $f(g.z) = 0$ for all $g \in G$ and for every $f \in M$. This happens if and only if $(g.f)(z) = 0$ for all $g \in G$ and for all $f \in M$, but this is the condition that $z \in \mathcal{V}(\mathrm{span}\{G.M\})$. $\qquad\square$

We apply Lemma 7.6 to our setting; if $(\mathrm{SL}(2)^{\times n} \ltimes \mathfrak{S}_n).[z] \subset \mathcal{V}(B')$, then $[z] \in \mathcal{V}(B)$ (recall $B := \mathrm{span}\{(\mathrm{SL}(2)^{\times n} \ltimes \mathfrak{S}_n).B'\}$). So, we need to look at the variety $\mathcal{V}(B)$. We conclude that our construction satisfies the property that if $[z] \in \mathcal{V}(HD)$ but $[z] \notin \mathcal{V}(B)$, then $[z] \in Z_n$.

We need to understand the types of points that can be in $\mathcal{V}(B)$, and the following proposition gives sufficient information about $\mathcal{V}(B)$.

Proposition 7.7. *Let $n \geq 4$ and let B be the module of polynomials constructed above. Let $\mathscr{P}^2(\{1, \dots, n\})$ be the collection of all partitions of $\{1, \dots, n\}$ into mutually disjoint subsets of cardinality 2 or less. Then*

$$\mathcal{V}(B) \subset \bigcup_{\{K_1, \dots, K_m\} \in \mathscr{P}^2(\{1,\dots,n\})} \mathrm{Seg}(\mathbb{P}V_{K_1} \times \cdots \times \mathbb{P}V_{K_m}) \subset Z_n.$$

Proof. Proposition 7.7 is proved in several parts. In Lemma 7.8 we find the module $S_{(4,1)}S_{(4,1)}S_{(4,1)}S_{(5)}\dots S_{(5)}$ as a submodule of B using the algorithm in Section 3E. In Lemma 7.12 we identify the zero set of this new module. In particular, we show

$$\mathcal{V}(S_{(4,1)}S_{(4,1)}S_{(4,1)}S_{(5)}\dots S_{(5)}) = \bigcup_{\{K_1, \dots, K_m\} \in \mathscr{P}^2(\{1,\dots,n\})} \mathrm{Seg}(\mathbb{P}V_{K_1} \times \cdots \times \mathbb{P}V_{K_m}).$$

We prove this statement by induction on n, where we prove the base case $n = 3$ in Lemma 7.9 and the induction step in Lemma 7.11. Finally, each $\mathbb{P}V_{K_i}$ is either a copy of $\mathbb{P}^1 \cong Z_1$ or $\mathbb{P}^3 \cong Z_2$, so we can apply Proposition 4.2 to verify the inclusion

$$\mathrm{Seg}(\mathbb{P}V_{K_1} \times \cdots \times \mathbb{P}V_{K_m}) \subset Z_n. \qquad\square$$

Lemma 7.8. *Suppose $n \geq 4$ and let B be constructed as above. Then*

$$S_{(4,1)}S_{(4,1)}S_{(4,1)}S_{(5)}\dots S_{(5)} \subset B.$$

Proof. Here we have a subset of polynomials in B in an explicit form, and we would like to identify $(\mathrm{SL}(2)^{\times n} \ltimes \mathfrak{S}_n)$-modules in B from this information. To do this we use the ideas presented in Section 3D and particularly the algorithm presented in Section 3E. It suffices to work first with $\mathrm{SL}(2)^{\times n}$-modules and later consider the permutations.

Suppose $f_{k_1,k_2,k_3} \in S_{(2,2)}V_{k_1}^* \otimes S_{(2,2)}V_{k_2}^* \otimes S_{(2,2)}V_{k_3}^* \otimes S_{(4)}V_{k_4}^* \otimes \dots \otimes S_{(4)}V_{k_n}^*$ is a lowest weight vector. Define a_{k_1,k_2,k_3}, b_{k_1,k_2,k_3}, c_{k_1,k_2,k_3} by the equation $f_{k_1,k_2,k_3} = a_{k_1,k_2,k_3}(X^{[1,\dots,1]})^2 + b_{k_1,k_2,k_3}(X^{[1,\dots,1]}) + c_{k_1,k_2,k_3}$.

For this proof, we introduce new notation. If k_1, k_2, k_3 are fixed, let $X^{I_{p,q,r}}$ denote the coordinate vector with $k_1 = p$, $k_2 = q$, $k_3 = r$ and $k_s = 0$ for $s \geq 4$.

Since f_{k_1,k_2,k_3} is a hyperdeterminant of format $2 \times 2 \times 2$, we find

$a_{k_1,k_2,k_3} = (X^{I_{0,0,0}})^2.$
$b_{k_1,k_2,k_3} = -2X^{I_{0,0,0}}(X^{I_{1,0,0}}X^{I_{0,1,1}} + X^{I_{0,1,0}}X^{I_{1,0,1}} + X^{I_{0,1,0}}X^{I_{1,1,0}})$
$$+ 4X^{I_{1,0,0}}X^{I_{0,1,0}}X^{I_{0,0,1}}.$$

The weight of a_{k_1,k_2,k_3} is (up to permutation) $(-2,-2,-2,2,\dots,2)$, where the -2's actually occur at $\{k_1, k_2, k_3\}$. The weight of b_{k_1,k_2,k_3} is (up to permutation) $(-1,-1,-1,3,\dots,3)$, where the -1's actually occur at $\{k_1, k_2, k_3\}$. Now consider

$$h_{k_1,k_2,k_3,j_1,j_2,j_3} = a_{k_1,k_2,k_3}b_{j_1,j_2,j_3} - a_{j_1,j_2,j_3}b_{k_1,k_2,k_3} \in B.$$

We notice that $h_{k_1,k_2,k_3,j_1,j_2,j_3}$ cannot have k_1, k_2, k_3 and j_1, j_2, j_3 all equal (this is the zero polynomial). So either two, one or zero pairs of i's and j's match in the indices k_1, k_2, k_3 and j_1, j_2, j_3. Therefore $h_{k_1,k_2,k_3,j_1,j_2,j_3}$ can have 3 different weights (up to permutation), depending on how k_1, k_2, k_3 and j_1, j_2, j_3 match up. The three possible weights of $h_{k_1,k_2,k_3,j_1,j_2,j_3}$ are (up to permutation): $(-3,-3,1,1,5,\dots,5)$, $(-3,1,1,1,1,5,\dots,5)$, or $(1,1,1,1,1,1,5,\dots,5)$.

In each case, apply the algorithm in Section 3E and lower $h_{k_1,k_2,k_3,j_1,j_2,j_3}$ to a nonzero vector with the lowest possible weight. We did this calculation in Maple. The output in each case is a vector of weight (up to permutation) $(3,3,3,5,\dots,5)$. Use Remark 3.2 to identify the module with lowest weight $(3,3,3,5,\dots,5)$ as

$$S_{(4,1)}S_{(4,1)}S_{(4,1)}S_{(5)}\dots S_{(5)},$$

and this must be a submodule of B. $\square$

Lemma 7.9. *As sets in $\mathbb{P}(V_1 \otimes V_2 \otimes V_3)$,*

$\mathcal{V}(S_{(4,1)}V_1^* \otimes S_{(4,1)}V_2^* \otimes S_{(4,1)}V_3^*)$
$\quad = \mathrm{Seg}(\mathbb{P}(V_1 \otimes V_2) \times \mathbb{P}V_3) \cup \mathrm{Seg}(\mathbb{P}(V_1 \otimes V_3) \times \mathbb{P}V_2) \cup \mathrm{Seg}(\mathbb{P}(V_1) \times (V_1 \otimes V_2)).$

Proof. The space $V_1^* \otimes V_2^* \otimes V_3^*$ has seven orbits under the action of $\mathrm{SL}(2)^{\times 3}$ [Gelfand et al. 1994, Example 4.5]. This gives rise to a list of normal forms, which we record below together with the respective $(\mathrm{SL}(2)^{\times n} \ltimes \mathfrak{S}_n)$-orbit closures to which they belong.

- The trivial orbit, $\varnothing$.

- $\mathrm{Seg}(\mathbb{P}V_1 \times \mathbb{P}V_2 \times \mathbb{P}V_3)$: Normal form $[x] = [a \otimes b \otimes c]$.

- $\tau(\mathrm{Seg}(\mathbb{P}V_1 \times \mathbb{P}V_2 \times \mathbb{P}V_3))_{\mathrm{sing}} = \mathfrak{S}_3.\, \mathrm{Seg}(\mathbb{P}(V_1 \otimes V_2) \times \mathbb{P}V_3)$: Normal form (up to permutation) $[x] = [a \otimes b \otimes c + a' \otimes b' \otimes c]$. This union of 3 irreducible varieties is the singular set of the next orbit.

- $\tau(\mathrm{Seg}(\mathbb{P}V_1 \times \mathbb{P}V_2 \times \mathbb{P}V_3))$: Normal form

$$[x] = [a \otimes b \otimes c + a' \otimes b \otimes c + a \otimes b' \otimes c + a \otimes b \otimes c'].$$

- $\sigma(\mathrm{Seg}(\mathbb{P}V_1 \times \mathbb{P}V_2 \times \mathbb{P}V_3))$: Normal form $[x] = [a \otimes b \otimes c + a' \otimes b' \otimes c']$.

The orbit closures are nested:

$$\varnothing \subset \mathrm{Seg}(\mathbb{P}V_1 \times \mathbb{P}V_2 \times \mathbb{P}V_3) \subset \tau(\mathrm{Seg}(\mathbb{P}V_1 \times \mathbb{P}V_2 \times \mathbb{P}V_3))_{\mathrm{sing}}$$
$$\subset \tau(\mathrm{Seg}(\mathbb{P}V_1 \times \mathbb{P}V_2 \times \mathbb{P}V_3)) \subset \sigma(\mathrm{Seg}(\mathbb{P}V_1 \times \mathbb{P}V_2 \times \mathbb{P}V_3)) = \mathbb{P}^7.$$

The lowest weight vector for $S_{(4,1)} S_{(4,1)} S_{(4,1)}$ is

$$
\begin{aligned}
f_{(4,1),(4,1),(4,1)} = {}& (X^{[1,1,1]})^2 \big(X^{[0,0,0]} (X^{[1,1,1]})^2 + 2 X^{[1,0,1]} X^{[0,1,1]} X^{[1,1,0]} \\
& - X^{[1,1,1]} (X^{[0,1,1]} X^{[1,0,0]} + X^{[1,0,1]} X^{[0,1,0]} + X^{[1,1,0]} X^{[0,0,1]}) \big).
\end{aligned}
$$

We took a generic point $x \in \tau(\mathrm{Seg}(\mathbb{P}V_1 \times \mathbb{P}V_2 \times \mathbb{P}V_3))_{\mathrm{sing}}$ and evaluated

$$f_{(4,1),(4,1),(4,1)}(x) = 0.$$

So $\tau(\mathrm{Seg}(\mathbb{P}V_1 \times \mathbb{P}V_2 \times \mathbb{P}V_3))_{\mathrm{sing}} \subset \mathcal{V}(S_{(4,1)} S_{(4,1)} S_{(4,1)})$. We could also see this without calculation by noticing that any point of the form $[a \otimes b \otimes c + a' \otimes b' \otimes c]$ lives in $\mathbb{P}(V_1 \otimes V_2 \otimes \widehat{c})$, where $\widehat{c}$ is the line through c. But every point in this space is a zero of $S_{(4,1)} S_{(4,1)} S_{(4,1)})$ because $S_{(4,1)}(\widehat{c})^* = 0$.

Next we show that the other two varieties are not in $\mathcal{V}(S_{(4,1)} S_{(4,1)} S_{(4,1)})$. The varieties are nested, so take the point $[x] = \big[X^{[1,1,1]} + X^{[0,1,1]} + X^{[1,0,1]} + X^{[1,1,0]} \big] \in \tau(\mathrm{Seg}(\mathbb{P}V_1 \times \mathbb{P}V_2 \times \mathbb{P}V_3))$. But $f_{(4,1),(4,1),(4,1)}(x) = 2 \neq 0$, so the other two varieties are not in $\mathcal{V}(S_{(4,1)} S_{(4,1)} S_{(4,1)})$. Since we have considered all possible normal forms, we are done. $\qquad\square$

Observation 7.10. All the linear spaces on $\mathrm{Seg}(\mathbb{P}V_{K_1} \times \cdots \times \mathbb{P}V_{K_m})$ are (up to permutation) contained in one of the form $V_{K_1} \otimes \widehat{v_{K_2}} \otimes \cdots \otimes \widehat{v_{K_m}}$, where $\widehat{v_K}$ denotes the line through $v_{k_1} \otimes \cdots \otimes v_{k_s}$ in V_K.

Let $\mathscr{P}_{p,q}(\{n_1, \ldots, n_{p+q}\})$ denote the set of partitions of $\{n_1, \ldots, n_{p+q}\}$ into two disjoint sets of cardinality p and q.

Consider the $(\mathrm{SL}(2)^{\times n} \ltimes \mathfrak{S}_n)$-module $\tilde{B} = S_{(4,1)}S_{(4,1)}S_{(4,1)}S_{(5)} \ldots S_{(5)}$ that has $n-3$ copies of $S_{(5)}$. We write in $\tilde{B}$ in full detail as

$$\tilde{B} = \bigoplus_{\substack{\{\{k_1,k_2,k_3\},\{k_4,\ldots,k_n\}\} \\ \in \mathscr{P}_{3,n-3}(\{1,\ldots,n\})}} S_{(4,1)}V_{k_1}^* \otimes S_{(4,1)}V_{k_2}^* \otimes S_{(4,1)}V_{k_3}^* \otimes S_{(5)}V_{k_4}^* \otimes \cdots \otimes S_{(5)}V_{k_n}^*.$$

Let $\tilde{B}_k$ denote the $\mathrm{SL}(2)^{n-1} \ltimes \mathfrak{S}_{n-1}$ module

$$\tilde{B}_k = \bigoplus_{\substack{\{\{k_1,k_2,k_3\},\{k_4,\ldots,k_{n-1}\}\} \\ \in \mathscr{P}_{3,n-4}(\{1,\ldots,n\}\setminus\{k\})}} S_{(4,1)}V_{k_1}^* \otimes S_{(4,1)}V_{k_2}^* \otimes S_{(4,1)}V_{k_3}^* \otimes S_{(5)}V_{k_4}^* \otimes \cdots \otimes S_{(5)}V_{k_{n-1}}^*.$$

$\tilde{B} \sum_{i=1}^{n} \tilde{B}_i \otimes S_{(5)}V_i^*$, that is, the $(\mathrm{SL}(2)^{\times n} \ltimes \mathfrak{S}_n)$-module $\tilde{B}$ is constructed as the nonredundant sum over permutations of augmented $\mathrm{SL}(2)^{\times n-1}$-modules.

We want to understand the zero set of this module $\tilde{B}$, and we do this in the next two lemmas by mimicking what we did for Proposition 6.8. We also point out that while notationally more complicated, the resulting Lemma 7.12 is essentially the same idea as Proposition 6.8.

Lemma 7.11. *Suppose $n \geq 4$ and let $\tilde{B}$ and $\tilde{B}_k$ be as above. If*

$$\mathscr{V}(\tilde{B}_k) = \bigcup_{\{K_1,\ldots,K_m\} \in \mathscr{P}^2(\{1,\ldots,n\}\setminus\{k\})} \mathrm{Seg}(\mathbb{P}V_{K_1} \times \mathbb{P}V_{K_2} \times \cdots \times \mathbb{P}V_{K_m}),$$

then

$$\mathscr{V}(\tilde{B}_k \otimes S_{(5)}V_k^*) = \bigcup_{\{K_1,\ldots,K_m\} \in \mathscr{P}^2(\{1,\ldots,n\}\setminus\{k\})} \mathrm{Seg}(\mathbb{P}V_{K_1 \cup \{k\}} \times \mathbb{P}V_{K_2} \times \cdots \times \mathbb{P}V_{K_m}). \quad (7)$$

Proof. Apply Lemma 6.4 to the left side of (7). It remains to check that

$$\bigcup_{L \subset \mathscr{V}(\tilde{B}_k)} \mathbb{P}(L \otimes V_k) = \bigcup_{\{K_1,\ldots,K_m\} \in \mathscr{P}^2(\{1,\ldots,n\}\setminus\{k\})} \mathrm{Seg}(\mathbb{P}V_{K_1 \cup \{k\}} \times \mathbb{P}V_{K_2} \times \cdots \times \mathbb{P}V_{K_m}),$$

where $L \subset \mathscr{V}(\tilde{B}_k)$ are linear spaces. Because of symmetry and our hypothesis, there is only one type of linear space to consider, $V_{I_1} \otimes \widehat{v_{I_2}} \otimes \cdots \otimes \widehat{v_{I_m}} \otimes V_k = V_{I_1 \cup \{k\}} \otimes \widehat{v_{I_2}} \otimes \cdots \otimes \widehat{v_{I_m}}$. It is clear that each of these linear spaces is on one of the Segre varieties on the right side of (7), and moreover every point on the right side of (7) is on one of these linear spaces. $\qquad \square$

Lemma 7.12. *Let $n \geq 4$ and let $\mathcal{P}^2(\{n_1, \ldots, n_n\})$ denote the collection of all partitions of $\{n_1, \ldots, n_n\}$ into mutually disjoint subsets of cardinality 2 or less. Then*

$$\mathcal{V}(S_{(4,1)} S_{(4,1)} S_{(4,1)} S_{(5)} \ldots S_{(5)}) = \bigcup_{\{K_1, \ldots, K_m\} \in \mathcal{P}^2(\{1, \ldots, n\})} \mathrm{Seg}(\mathbb{P}V_{K_1} \times \cdots \times \mathbb{P}V_{K_m}).$$

Proof by induction. The base case is Lemma 7.9. For the induction step, use Lemma 7.11. We need to show that

$$\bigcap_{k=1}^{n} \left(\bigcup_{\{K_1, \ldots, K_m\} \in \mathcal{P}^2(\{1, \ldots, n\} \setminus \{k\})} \mathrm{Seg}(\mathbb{P}V_{K_1 \cup \{k\}} \times \mathbb{P}V_{K_2} \times \cdots \times \mathbb{P}V_{K_m}) \right)$$

$$= \bigcup_{\{K_1, \ldots, K_m\} \in \mathcal{P}^2(\{1, \ldots, n\})} \mathrm{Seg}(\mathbb{P}V_{K_1} \times \cdots \times \mathbb{P}V_{K_m}).$$

It suffices to check that

$$\mathrm{Seg}(\mathbb{P}V_{K_1 \cup \{k\}} \times \mathbb{P}V_{K_2} \times \mathbb{P}V_{K_3} \times \cdots \times \mathbb{P}V_{K_m})$$
$$\cap \, \mathrm{Seg}(\mathbb{P}V_{K_1} \times \mathbb{P}V_{K_2 \cup \{k\}} \times \mathbb{P}V_{K_3} \times \cdots \times \mathbb{P}V_{K_m})$$
$$= \mathrm{Seg}(\mathbb{P}V_{K_1} \times \mathbb{P}V_{K_2} \times \mathbb{P}V_k \times \mathbb{P}V_{K_3} \times \cdots \times \mathbb{P}V_{K_m}).$$

This is equivalent to checking that for any vector spaces V_1, V_2, V_3,

$$\mathrm{Seg}(\mathbb{P}(V_1 \otimes V_2) \times \mathbb{P}V_3) \cap \mathrm{Seg}(\mathbb{P}V_1 \times \mathbb{P}(V_2 \otimes V_3)) = \mathrm{Seg}(\mathbb{P}V_1 \times \mathbb{P}V_2 \times \mathbb{P}V_3)).$$

In this case, let $[T] \in \mathrm{Seg}(\mathbb{P}(V_1 \otimes V_2) \times \mathbb{P}V_3) \cap \mathrm{Seg}(\mathbb{P}V_1 \times \mathbb{P}(V_2 \otimes V_3))$. Then, viewed as a map $T : (V_1 \otimes V_2)^* \to V_3$, the image of T must be one-dimensional, and thus $[T] \in \mathbb{P}(V_1 \otimes V_2 \otimes V_3')$, where $V_3' \subset V_3$ is a one-dimensional subspace. By the same argument using the other Segre variety in the intersection, $[T] \in \mathbb{P}(V_1 \otimes V_2' \otimes V_3)$, where $V_2' \subset V_2$ is a one-dimensional subspace. So $T \in \mathbb{P}(V_1 \otimes V_2' \otimes V_3')$, but this is a linear space on $\mathrm{Seg}(\mathbb{P}V_1 \times \mathbb{P}V_2 \times \mathbb{P}V_3))$, so we are done. $\square$

We have established all of the ingredients for the proof of Lemma 7.2.

8. Proof of Theorem 1.3

The outline of the proof is the following. Proposition 5.5 says that $Z_n \subseteq \mathcal{V}(HD)$. To show the opposite inclusion, we work by induction. In the cases of $n = 3, 4$, the (stronger) ideal-theoretic version of Theorem 1.3 was proved with the aid of a computer in [Holtz and Sturmfels 2007]. Since the theorem is already proved for the cases $n = 3, 4$ we will assume $n \geq 5$. The induction hypothesis is that $\mathcal{V}(HD_i) \simeq Z_{(n-1)}$. We need to show that given a point $[z] \in \mathcal{V}(HD)$, we have $[z] \in Z_n$, that is, that there exists a matrix A such that $\varphi([A, t]) = [z]$. The key tools we use in this proof are Proposition 7.1 and Lemma 6.6.

We work on a preferred open set $U_0 = \{[z] = [z_I X^I] \in \mathbb{P}(V_1 \otimes \cdots \otimes V_n) \mid z_{[0,\ldots,0]} \neq 0\}$. Working on this open set causes no loss of generality because of the following:

Lemma 8.1. *Let* $U_0 = \{[z] = [z_I X^I] \in \mathbb{P}(V_1 \otimes \cdots \otimes V_n) \mid z_{[0,\ldots,0]} \neq 0\}$. *Then* $\mathcal{V}(HD) \cap U_0 \subset Z_n$ *implies that* $\mathcal{V}(HD) \subset Z_n$.

Proof. The result follows from the fact that Z_n and $\mathcal{V}(HD)$ are $(\mathrm{SL}(2)^{\times n} \ltimes \mathfrak{S}_n)$-invariant, and $(\mathrm{SL}(2)^{\times n} \ltimes \mathfrak{S}_n).U_0 = \mathbb{P}(V_1 \otimes \cdots \otimes V_n)$.   □

It suffices to work on the section $\{z = z_I Z^I \in V_1 \otimes \cdots \otimes V_n \mid z_{[0,\ldots,0]} = 1\}$ of the cone over projective space, because afterwards we can rescale everything to get the result on the whole open set U_0.

Suppose we take a point in the zero set (as described by Lemma 6.6)

$$[z] \in \mathcal{V}(HD) = \bigcap_{i=1}^{n} \bigcup_{L^i \subset \mathcal{V}(HD_i)} \mathbb{P}(L^i \otimes V_i).$$

Since $[z]$ is fixed, we can also fix a single L^i for each i so that $[z] \in \bigcap_{i=1}^{n} \mathbb{P}(L^i \otimes V_i)$. Work in our preferred section of the cone over projective space and write n different expressions for the point z (one for each i):

$$z = z_I X^I = \eta^i \otimes x_i^0 + v^i \otimes x_i^1,$$

where $[\eta^i], [v^i] \in L^i \subset \mathcal{V}(HD_i)$. (These expressions are possible because each V_i is 2-dimensional.) Choosing $z_{[0,\ldots,0]} = 1$ also implies that $\eta^i_{[0,\ldots,0]} = 1$. The induction hypothesis says that $Z_{(n-1)} \simeq \mathcal{V}(HD_i)$ for $1 \leq i \leq n$. So each η^i satisfies $\varphi([A^{(i)}, 1]) = \eta^i$ for a symmetric matrix $A^{(i)} \in S^2\mathbb{C}^{n-1}$. For each $0 \leq j \leq n$ denote by $\mathscr{A}^j$ the subset of matrices

$$\mathscr{A}^j = \{A \in S^2\mathbb{C}^n \mid \Delta_I(A) = z_I \text{ for all } I = [i_1, \ldots, i_n] \text{ with } i_j = 0\}.$$

Each matrix $A \in \mathscr{A}^j$ has the property that the principal submatrix of A formed by deleting the jth row and column maps to η^j under the principal minor map. Thus each $A \in \mathscr{A}^j$ is a candidate matrix that might satisfy $\varphi([A, 1]) = [z]$, but we don't know if such a matrix will have a submatrix that maps to the other η^i's. We claim that there is at least one matrix that satisfies all of these conditions.

Lemma 8.2. $\bigcap_{i=1}^{n} \mathscr{A}^i$ *is nonempty.*

Proof. By the induction hypothesis, each $\mathscr{A}^i$ is nonempty. Assume $\bigcap_{i=2}^{n} \mathscr{A}^i$ is nonempty. We show that if $A \in \bigcap_{i=2}^{n} \mathscr{A}^i$, then $A \in \mathscr{A}^1$. The same argument also proves that if $A \in \bigcap_{i=3}^{n} \mathscr{A}^i$, then $A \in \mathscr{A}^1$, and so on, so it suffices to check the last, most restrictive case. Also because of the $\mathfrak{S}_n$ action, we don't have to repeat the proof for every permutation.

If $A \in \bigcap_{i=2}^{n} \mathscr{A}^i$, then $\Delta_I A = z_I$ for all $I \neq [0, i_2, \ldots, i_n]$ with $|I| \leq n - 2$. The only possible exception we could have is that $\Delta_{[0,1,\ldots,1]}$ might not be equal to $z_{[0,1,\ldots,1]}$. Let A' denote the principal submatrix of A formed by deleting the first row and column of A. Now since $n \geq 5$, $|I| \geq 3$, A' is at least as large as 4×4, and we have determined that all of the principal minors of A' except possibly the determinant agree with a fixed point $\eta^1 \in \mathscr{V}(HD_1)$ (in other words $\Delta(A')_I = \eta_I^i$ for all $I \neq [1, \ldots, 1]$), so we can apply Proposition 7.1 to conclude that the determinant of A' also agrees with η^1 (that is, $\Delta_{[1,\ldots,1]}(A') = \eta_{[1,\ldots,1]}^1$). Therefore any such A must have $\Delta_{[0,1,\ldots,1]}(A) = z_{[0,1,\ldots,1]}$, and we have shown $A \in \mathscr{A}^1$. $\qquad\square$

Lemma 8.2 proves the existence of a symmetric matrix A such that $\Delta_I(A) = z_I$ for all $I \neq [1, \ldots, 1]$. Then since both $z_I X^I$ and $\Delta_I(A) X^I$ are points in $\mathscr{V}(HD)$, Proposition 7.1 implies that $\Delta_{[1,\ldots,1]}(A) = z_{[1,\ldots,1]}$, and this finishes the proof of the main theorem.

Remark 8.3 (building a matrix). Note that when $n \geq 4$, the proof we gave can be used also to construct a symmetric matrix whose principal minors are prescribed by a point $z \in \widehat{\mathscr{V}(HD)} \cap \{z \mid z_{[0,\ldots,0]} \neq 0\}$. The entries of z corresponding to 1×1 and 2×2 principal minors determine a large finite set $\mathscr{A}$ of candidate matrices that could map to z. Restrict the set $\mathscr{A}$ to only those matrices whose 3×3 principal minors agree with the corresponding entries of z, that is, keep only the matrices A such that $\Delta(A)_I = z_I$ for all $|I| \leq 3$. We claim that the remaining matrices all map to z under the principal minor map. If A is such that all of the 3×3 principal minors agree with z, then Proposition 7.1 implies that each 4×4 principal minor of A must agree with z also. Iterate this argument to imply that all of the principal minors of A must agree with z.

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