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Local positivity, multiplier ideals, and syzygies of abelian varieties

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We use the language of multiplier ideals in order to relate the syzygies of an abelian variety in a suitable embedding with the local positivity of the line bundle inducing that embedding. This extends to higher syzygies a result of Hwang and To on projective normality.

Introduction

Hwang and To [2010] observed that there is a relation between local positivity on an abelian variety A and the projective normality of suitable embeddings of A. The purpose of this note is to extend their result to higher syzygies, and to show that the language of multiplier ideals renders the computations extremely quick and transparent.

Turning to details, let A be an abelian variety of dimension g, and let L be an ample line bundle on A. Recall that the *Seshadri constant* $\varepsilon(A, L)$ is a positive real number that measures the local positivity of L at any given point $x \in A$: for example, it can be defined by counting asymptotically the number of jets that the linear series $|kL|$ separates at x as $k \to \infty$. We refer to [Lazarsfeld 2004, Chapter 5] for a general survey of the theory, and in particular to Section 5.3 of that book for a discussion of local positivity on abelian varieties.

Our main result is this:

Theorem A. *Assume that*

$$\varepsilon(A, L) \; > \; (p+2)g.$$

Then L satisfies property (N_p).

The reader may consult for instance [Lazarsfeld 2004, Chapter 1.8.D], [Green and Lazarsfeld 1987] or [Eisenbud 2005] for the definition of property (N_p) and

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further references. Suffice it to say here that (N_0) holds when L defines a projectively normal embedding of A, while (N_1) means that the homogeneous ideal of A in this embedding is generated by quadrics. For $p > 1$ the condition is that the first p modules of syzygies among these quadrics are generated in minimal possible degree. The result of Hwang and To [2010] is essentially the case $p = 0$ of Theorem A.

In general it is difficult to control Seshadri constants. However, it was shown in [Lazarsfeld 1996] that on an abelian variety they are related to a metric invariant introduced in [Buser and Sarnak 1994]. Specifically, write $A = V/\Lambda$, where V is a complex vector space of dimension g and $\Lambda \subseteq V$ is a lattice. Then L determines a hermitian form $h = h_L$ on V, and the Buser–Sarnak invariant is (the square of) the minimal length with respect to h of a nonzero period of Λ:

$$m(A, L) := \min_{0 \neq \ell \in \Lambda} h_L(\ell, \ell).$$

The main result of [Lazarsfeld 1996] is that

$$\varepsilon(A, L) \geq \frac{\pi}{4} \cdot m(A, L).$$

On the other hand, one can estimate $m(A, L)$ for very general (A, L). In fact, suppose that the polarization L has elementary divisors

$$d_1 \mid d_2 \mid \cdots \mid d_g,$$

and put $d = d(L) = d_1 \cdot \cdots \cdot d_g$. By adapting an argument of Buser–Sarnak in the case of principal polarizations, Bauer [1998] showed that if (A, L) is very general, then

$$m(A, L) \geq \frac{2^{1/g}}{\pi} \sqrt[g]{d \cdot g!}.$$

Therefore we obtain:

Corollary B. *Assume that*

$$d(L) > \frac{4^g (p+2)^g g^g}{2g!}.$$

Then (N_p) holds for very general (A, L) of the given type.

The essential interest in statements of this sort occurs when L is primitive (that is, $d_1 = 1$), or at least when d_1 is small: as far as we know, our result is the first to give statements for higher syzygies of primitive line bundles in large dimension. By contrast, if L is a suitable multiple of some ample line bundle, then much stronger statements are known. Most notably, the second author proved in [Pareschi 2000] that (N_p) always holds as soon as $d_1 \geq p + 3$. This was strengthened and systematized in [Pareschi and Popa 2003; 2004], while (for $p = 0$) other statements appear in [Iyer 2003] and [Fuentes García 2005].

We conclude this introduction by sketching a proof of the theorem of [Hwang and To 2010] via the approach of the present paper. Following a time-honored device, one considers the diagonal $\Delta \subseteq A \times A$, with ideal sheaf $\mathcal{I}_\Delta$. Writing

$$L \boxtimes L = \mathrm{pr}_1^* L \otimes \mathrm{pr}_2^* L$$

for the exterior product of L with itself, the essential point is to prove

$$H^1\big(A \times A, L \boxtimes L \otimes \mathcal{I}_\Delta\big) \;=\; 0. \tag{*}$$

Hwang and To [2010] achieve this by establishing a somewhat delicate upper bound on the volume of a one-dimensional analytic subvariety of a tubular neighborhood of Δ (or, more generally, of a tubular neighborhood of any subtorus of an abelian variety). This allows them to control the positivity required to apply vanishing theorems on the blow-up of $A \times A$ along Δ. While their calculation is of substantial independent interest, for the task at hand it is considerably quicker to deduce (*) directly from Nadel vanishing.

Specifically, using the hypothesis that $\varepsilon(A, L) > 2g$, a standard argument (see Lemma 1.2) shows that for suitable $0 < c \ll 1$, one can construct an effective $\mathbb{Q}$-divisor

$$E_0 \equiv_{\mathrm{num}} \frac{1-c}{2} L$$

on A whose multiplier ideal vanishes precisely at the origin: $\mathcal{J}(A, E_0) = \mathcal{I}_0$. Now consider the difference map

$$\delta : A \times A \to A, \qquad (x, y) \mapsto x - y,$$

and set $E = \delta^* E_0$. Since forming multiplier ideals commutes with pullback under smooth morphisms, we have on the one hand

$$\mathcal{J}(A \times A, E) = \delta^* \mathcal{J}(A, E_0) = \mathcal{I}_\Delta.$$

On the other hand, one knows that

$$L^2 \boxtimes L^2 = \delta^*(L) \otimes N \tag{**}$$

for a suitable nef line bundle N on $A \times A$. Thanks to our choice of E_0, this implies that $(L \boxtimes L)(-E)$ is ample. Therefore Nadel vanishing gives (*), as required.

The proof of the general case of Theorem A proceeds along similar lines. Following an idea going back to Green [1984], one works on the $(p+2)$-fold product of A, where one has to check a vanishing involving the ideal sheaf of a union of pairwise diagonals.[1] To realize this as a multiplier ideal, we pull back a suitable

[1] The possibility of applying vanishing theorems on a blow-up to verify Green's criterion was noted already in [Bertram et al. 1991, Remark on p. 600]. Nowadays one can invoke the theory of [Li 2009] to control the blow-ups involved: the pairwise diagonals $\Delta_{0,1}, \ldots, \Delta_{0,p+1}$ form a building

divisor under a multisubtraction map: this is carried out in Section 1. The positivity necessary for Nadel vanishing is verified using an analogue of (∗∗) established in Section 2. Finally, Section 3 contains some complements and variants, including a criterion for L to define an embedding in which the homogeneous coordinate ring of A is Koszul.

For applications of Nadel vanishing, one typically has to estimate the positivity of formal twists of line bundles by $\mathbb{Q}$-divisors. To this end, we allow ourselves to be a little sloppy in mixing additive and multiplicative notation. Thus, given a $\mathbb{Q}$-divisor D and a line bundle L, the statement $D \equiv_{\mathrm{num}} bL$ is intended to mean that D is numerically equivalent to $b \cdot c_1(L)$. Similarly, to say that $(bL)(-D)$ is ample indicates that $b \cdot c_1(L) - D$ is an ample numerical class. We trust that no confusion will result.

1. Proof of Theorem A

As in the Introduction, let A be an abelian variety of dimension g, and let L be an ample line bundle on A.

We start by recalling a geometric criterion that guarantees property (N_p) in our setting. Specifically, form the $(p+2)$-fold product $X = A^{\times(p+2)}$ of A with itself, and inside X consider the reduced algebraic set

$$\Sigma = \big\{(x_0, \dots, x_{p+1}) \,|\, x_0 = x_i \text{ for some } 1 \le i \le p+1\big\} = \Delta_{0,1} \cup \Delta_{0,2} \cup \dots \cup \Delta_{0,p+1}$$

arising as the union of the indicated pairwise diagonals. Thus Σ has $p+1$ irreducible components, each of codimension g in X.

It was observed by Green [1984, §3] that property (N_p) for L is implied by a vanishing on X involving the ideal sheaf of $\mathcal{I}_\Sigma$, generalizing the condition (*) for projective normality. We refer to [Inamdar 1997] for a statement and careful discussion of the criterion in general.[2] In the present situation, it shows that Theorem A is a consequence of the following:

Proposition 1.1. *Assume that $\varepsilon(A, L) > (p+2)g$. Then*

$$H^i\Big(A^{\times(p+2)}, \overset{p+2}{\boxtimes} L \otimes Q \otimes \mathcal{I}_\Sigma\Big) = 0$$

for any nef line bundle Q on X and all $i > 0$.[3]

set in the sense of [Li 2009] on the $(p+2)$-fold self product of a smooth variety. However, in the case of abelian varieties treated here, elementary properties of multiplier ideals are used to obviate the need for any blow-ups.

[2] The argument appearing in [Green 1984] is somewhat oversimplified.

[3] As explained in [Inamdar 1997] one actually needs the vanishings

$$H^1\big(A^{\times(p'+2)}, L^q \boxtimes L \boxtimes \cdots \boxtimes L \otimes \mathcal{I}_\Sigma\big) = 0$$

The plan is to deduce the proposition from Nadel vanishing. To this end, it suffices to produce an effective $\mathbb{Q}$-divisor E on X having two properties:

$$\mathcal{J}(X, E) = \mathcal{I}_{\Sigma}. \tag{1-1}$$

$$\Big(\overset{p+2}{\boxtimes} L\Big)(-E) \text{ is ample.} \tag{1-2}$$

The rest of this section is devoted to the construction of E and the verification of these requirements.

The first point is quite standard:

Lemma 1.2. *Assuming that $\varepsilon(A, L) > (p+2)g$, there exists an effective $\mathbb{Q}$-divisor F_0 on A having the properties that*

$$F_0 \equiv_{\text{num}} \frac{1-c}{p+2} L$$

for some $0 < c \ll 1$, and

$$\mathcal{J}(A, F_0) = \mathcal{I}_0.$$

Here naturally $\mathcal{I}_0 \subseteq \mathcal{O}_A$ denotes the ideal sheaf of the origin $0 \in A$.

Proof of Lemma 1.2. We claim that for suitable $0 < c \ll 1$ and sufficiently divisible $k \gg 0$, there exists a divisor $D \in |k(1-c)L|$ with

$$\text{mult}_0(D) = (p+2)gk,$$

where, in addition, D has a smooth tangent cone at the origin $0 \in A$ and is nonsingular away from 0. Granting this, it suffices to put $F_0 = (1/(p+2)k)D$. As for the existence of D, let

$$\rho : A' = \text{Bl}_0(A) \to A$$

be the blowing up of A at 0, with exceptional divisor $T \subseteq A'$. Then, by definition of $\varepsilon(A, L)$, the class $(1-c)\rho^*L - (p+2)gT$ is ample on A' for $0 < c \ll 1$. If D' is a general divisor in the linear series corresponding to a large multiple of this class, Bertini's theorem on A' implies that $D = \rho_*(D')$ has the required properties. □

Now form the $(p+1)$-fold product $Y = A^{\times(p+1)}$ of A with itself, and write $\text{pr}_i : Y \to A$ for the i-th projection. Consider the reduced algebraic subset

$$\Lambda = \bigcup_{i=1}^{p+1} \text{pr}_i^{-1}(0) = \big\{(y_1, \dots, y_{p+1}) \mid y_i = 0 \text{ for some } 1 \le i \le p+1\big\}.$$

---

for $0 \le p' \le p$ and $q \ge 1$, but these are all implied by the assertion of Proposition 1.1.

We wish to realize $\mathscr{I}_\Lambda$ as a multiplier ideal, to which end we simply consider the *exterior sum* of the divisors F_0 just constructed. Specifically, put

$$E_0 = \sum_{i=1}^{p+1} \mathrm{pr}_i^*(F_0).$$

Thanks to [Lazarsfeld 2004, 9.5.22], one has

$$\mathscr{J}(Y, E_0) = \prod_{i=1}^{p+1} \mathrm{pr}_i^* \mathscr{J}(A, F_0) = \prod_{i=1}^{p+1} \mathrm{pr}_i^* \mathscr{I}_0,$$

that is, $\mathscr{J}(Y, E_0) = \mathscr{I}_\Lambda$, as desired.

Next, consider the map

$$\begin{aligned} \delta = \delta_{p+1} : A^{\times(p+2)} &\to A^{\times(p+1)}, \\ (x_0, x_1, \ldots, x_{p+1}) &\mapsto (x_0 - x_1, \ldots, x_0 - x_{p+1}), \end{aligned} \tag{1-3}$$

and note that $\Sigma = \delta^{-1}\Lambda$ (scheme-theoretically). Set

$$E = \delta^*(E_0).$$

Forming multiplier ideals commutes with pulling back under smooth morphisms [Lazarsfeld 2004, 9.5.45]; hence

$$\mathscr{J}(X, E) = \delta^* \mathscr{J}(Y, E_0) = \delta^* \mathscr{I}_\Lambda = \mathscr{I}_\Sigma,$$

and thus (1-1) is satisfied.

In order to verify (1-2), we use the following assertion, which will be established in the next section.

Proposition 1.3. *There is a nef line bundle N on $X = A^{\times(p+2)}$ such that*

$$\delta^*\Big(\overset{p+1}{\boxtimes} L\Big) \otimes N = \overset{p+2}{\boxtimes} L^{p+2}. \tag{1-4}$$

Granting this, the property (1-2) — and with it, Proposition 1.1 — follows easily. Indeed,

$$E \equiv_{\mathrm{num}} \frac{1-c}{p+2} \cdot \Big(\delta^*\Big(\overset{p+1}{\boxtimes} L\Big)\Big).$$

Therefore (1-4) implies that

$$\Big(\overset{p+2}{\boxtimes} L\Big)(-E) \equiv_{\mathrm{num}} c \cdot \Big(\overset{p+2}{\boxtimes} L\Big) + \frac{1-c}{p+2} \cdot N,$$

which is ample. This completes the proof of Theorem A.

2. Proof of Proposition 1.3

Let A be an abelian variety and p a nonnegative integer. Define the maps

$$b : A^{\times(p+2)} \to A, \qquad (x_0, x_1, \ldots, x_{p+1}) \mapsto x_0 + x_1 + \cdots + x_{p+1},$$

and for any $0 \le i < j \le p+1$,

$$d_{ij} : A^{\times(p+2)} \to A, \qquad (x_0, x_1, \ldots, x_{p+1}) \mapsto x_i - x_j.$$

Recall the map δ from the previous section:

$$\delta : A^{\times(p+2)} \to A^{\times(p+1)}, \qquad (x_0, x_1, \ldots, x_{p+1}) \mapsto (x_0 - x_1, \ldots, x_0 - x_{p+1}).$$

Proposition 1.3 follows from the following more precise statement.[4]

Proposition 2.1. *For any ample line bundle L on A, we have*

$$\delta^*\Big(\mathop{\boxtimes}^{p+1} L\Big) \otimes (b^*L) \otimes \Big(\bigotimes_{1\le i<j} d_{ij}^* L\Big) = \mathop{\boxtimes}_{k=0}^{p+1} \big(L^{p+2-k} \otimes (-1)^*L^k\big).$$

Let

$$a : A \times A \to A \qquad \text{and} \qquad d : A \times A \to A$$

be the addition and subtraction maps, $\mathcal{P}$ be a normalized Poincaré line bundle on $A \times \widehat{A}$, and $\phi_L : A \to \widehat{A}$ be the isogeny induced by L. We use the notation

$$P = (1 \times \phi_L)^*\mathcal{P} \qquad \text{and} \qquad P_{ij} = \mathrm{pr}_{ij}^* P,$$

where $\mathrm{pr}_{ij} : A^{\times(p+2)} \to A \times A$ is the projection on the (i, j)-factor. We will use repeatedly the following standard facts.

Lemma 2.2. *The following identities hold*:

(i) $a^*L \cong (L \boxtimes L) \otimes P$;

(ii) $d^*L \cong (L \boxtimes (-1)^*L) \otimes P^{-1}$;

(iii) $\mathrm{pr}_{13}^* P \otimes \mathrm{pr}_{23}^* P \cong (a \times 1)^* P$ *on the triple product* $A \times A \times A$.

Proof. Identity (i) is well known (see for example [Mumford 1970, p. 78]) and follows from the seesaw principle. Identity (ii) can then be deduced similarly using the seesaw principle, or from (i) by noting that $d = a \circ (1, -1)$. This gives

$$\begin{aligned} d^*L &\cong (1 \times (-1))^*\big((L \boxtimes L) \otimes (1 \times \phi_L)^*\mathcal{P}\big) \\ &\cong (L \boxtimes (-1)^*L) \otimes (1 \times ((-1) \circ \phi_L))^*\mathcal{P} \\ &\cong (L \boxtimes (-1)^*L) \otimes (1 \times \phi_{(-1)^*L})^*(1, -1)^*\mathcal{P} \\ &\cong (L \boxtimes (-1)^*L) \otimes (1 \times \phi_L)^*\mathcal{P}^{-1}, \end{aligned}$$

[4] Note that L and $(-1)^*L$ differ by a topologically trivial line bundle.

where the last isomorphism follows from the well-known identity

$$((-1)\times 1)^*\mathcal{P} \cong (1\times(-1))^*\mathcal{P} \cong \mathcal{P}^{-1}.$$

Identity (iii) follows from the formula

$$\mathrm{pr}_{13}^*\mathcal{P}\otimes \mathrm{pr}_{23}^*\mathcal{P} \cong (a,1)^*\mathcal{P}$$

on $A\times A\times \widehat{A}$, which in turn is easily verified using the seesaw principle (see, for example, the proof of Mukai's inversion theorem [1981, Theorem 2.2]). □

Proposition 2.1 follows by putting together the formulas in the next Lemma.

Lemma 2.3. *If L is an ample line bundle on A, the following identities hold*:

(i) $b^*L \;\cong\; \Big(\overset{p+2}{\boxtimes} L\Big)\otimes\Big(\bigotimes_{i<j} P_{ij}\Big)$;

(ii) $d_{ij}^*L \;\cong\; \Big(\mathcal{O}_A\boxtimes\cdots\boxtimes \underset{i}{L}\boxtimes\cdots\boxtimes \underset{j}{(-1)^*L}\boxtimes\cdots\boxtimes\mathcal{O}_A\Big)\otimes P_{ij}^{-1}$ *for all* $i<j$;

(iii) $\delta^*\Big(\overset{p+1}{\boxtimes} L\Big) \;\cong\; \big(L^{p+1}\boxtimes(-1)^*L\boxtimes\cdots\boxtimes(-1)^*L\big)\otimes P_{01}^{-1}\otimes\cdots\otimes P_{0,p+1}^{-1}$.

Proof. (i) If $p=0$ this is Lemma 2.2(i). We can inductively obtain the formula for some $p>0$ from that for $p-1$ by noting that b $(=b_{p+2}) = (a,\mathrm{id})\circ b_{p+1}$, where b_k denotes the addition map for k factors, a is the addition map on the first two factors, and id is the identity on the last p factors. Therefore, inductively we have

$$b^*L \cong (a,\mathrm{id})^*\Big(\Big(\overset{p+1}{\boxtimes} L\Big)\otimes\Big(\bigotimes_{i<j} P_{ij}\Big)\Big).$$

The formula follows then by using Lemma 2.2(i) for the addition map a on the first two factors, and Lemma 2.2(iii) for the combination of the first two factors with any of the other p factors.

(ii) This follows simply by noting that $d_{ij} = d\circ p_{ij}$, where p_{ij} is the projection on the (i,j) factors and d is the difference map. We then apply Lemma 2.2(ii).

(iii) Note that $\delta = (d_{01},\ldots,d_{0,p+1})$. Therefore

$$\delta^*\Big(\overset{p+1}{\boxtimes} L\Big) \cong d_{01}^*L\otimes\cdots\otimes d_{0,p+1}^*L.$$

One then applies the formula in (ii). □

In order to discuss the Koszul property in the next section, we will need a variant of these results. Specifically, fix $k\geq 2$ and consider the mapping

$$\gamma : A^{\times k}\to A^{\times(k-1)},\qquad (x_0,x_1,\ldots,x_k)\mapsto(x_0-x_1,x_1-x_2,\ldots,x_{k-1}-x_k).$$

Consider also for any $0 \le i < j \le k$ the maps

$$a_{ij} : A^{\times k} \to A, \qquad (x_0, x_1, \dots, x_k) \mapsto x_i + x_j.$$

Variant 2.4. *For any ample line bundle L on A we have*

$$\gamma^*\Big(\overset{k-1}{\boxtimes} L\Big) \otimes \Big(\bigotimes_{0\le i\le k-1} a^*_{i,i+1}L\Big) = L^2 \boxtimes \big(L^2 \otimes (-1)^*L\big) \boxtimes \dots \boxtimes \big(L^2 \otimes (-1)^*L\big) \boxtimes \big(L \otimes (-1)^*L\big).$$

Proof. Noting that $a_{ij} = a \circ \mathrm{pr}_{ij}$, where pr_{ij} is the projection on the (i, j) factors and a is the difference map, and using Lemma 2.2(i), we have

$$a^*_{ij}L \cong \Big(\mathcal{O}_A \boxtimes \cdots \boxtimes \underset{i}{L} \boxtimes \cdots \boxtimes \underset{j}{L} \boxtimes \cdots \boxtimes \mathcal{O}_A\Big) \otimes P_{ij}.$$

On the other hand, $\gamma = (d_{01}, d_{12}, \dots, d_{k-1,k})$ and using Lemma 2.3(ii) for each of the factors, we have

$$\gamma^*\Big(\overset{k-1}{\boxtimes} L\Big) \cong \big(L \boxtimes (L \otimes (-1)^*L) \boxtimes \dots \boxtimes (L \otimes (-1)^*L) \boxtimes (-1)^*L\big) \otimes P_{01}^{-1} \otimes \dots \otimes P_{k-1,k}^{-1}. \qquad \square$$

Corollary 2.5. *There is a nef line bundle N on $A^{\times k}$ such that*

$$\gamma^*\Big(\overset{k-1}{\boxtimes} L\Big) \otimes N = \overset{k}{\boxtimes} L^3.$$

3. Complements

This section contains a couple of additional results that are established along the same lines as those above. As before, A is an abelian variety of dimension g, and L is an ample line bundle on A.

We start with a criterion for L to define an embedding in which A satisfies the Koszul property (for a definition and discussion of this property see for instance [Brion and Kumar 2005, §1.5]).

Proposition 3.1. *Assume that $\varepsilon(A, L) > 3g$. Then under the embedding defined by L, the homogeneous coordinate ring of A is a Koszul algebra.*

Sketch of Proof. Fix $k \ge 2$, and consider the k-fold self product $A^{\times k}$ of A. By analogy to Green's criterion, it is known that the Koszul property is implied by the vanishings (for all $k \ge 2$)

$$H^1\Big(A^{\times k}, \overset{k}{\boxtimes} L \otimes Q \otimes \mathcal{I}_\Gamma\Big) = 0, \tag{3-1}$$

where Q is a nef bundle on $A^{\times k}$, and Γ is the reduced algebraic set

$$\Gamma = \Delta_{1,2} \cup \Delta_{2,3} \cup \cdots \cup \Delta_{k-1,k}$$

(see [Inamdar and Mehta 1994, Proposition 1.9]). As above, this is established by realizing Γ as a multiplier ideal and applying Nadel vanishing. For the first point, one constructs (as in the case $p=2$ of Theorem A) a divisor $F_0 \equiv_{\text{num}} ((1-c)/3)L$ on A, takes its exterior sum on $A^{\times(k-1)}$, and then pulls back under the map $\gamma : A^{\times k} \to A^{\times(k-1)}$ appearing at the end of the last section. The required positivity follows from Corollary 2.5. □

We record an analogue of the result of Hwang and To for Wahl [1992] maps.

Proposition 3.2. *Let L be an ample line bundle on A, and assume that $\varepsilon(A, L) > 2(g+m)$ for some integer $m \geq 0$. Then*

$$h^1\big(A \times A, L \boxtimes L \otimes \mathcal{I}_\Delta^{m+1}\big) = 0.$$

In particular, the m-th Wahl (or Gaussian) map

$$\gamma_L^m : h^0\big(A \times A, L \boxtimes L \otimes \mathcal{I}_\Delta^m\big) \\ \to h^0\big(A \times A, L \boxtimes L \otimes \mathcal{I}_\Delta^m \otimes \mathcal{O}_\Delta\big) \cong h^0\big(A, L^2 \otimes S^m \Omega_A^1\big)$$

is surjective.

Sketch of Proof. One proceeds as in the proof outlined in the Introduction, except that the stronger numerical hypothesis on $\varepsilon(A, L)$ allows one to take $E_0 \equiv_{\text{num}} ((1-c)/2)L$ with $\mathcal{J}(A, E_0) = \mathcal{I}_0^{m+1}$. For the rest one argues as before. □

Remark 3.3. Proposition 3.2, combined with Bauer's result mentioned in the Introduction and with [Colombo et al. 2011, Theorem B], implies the surjectivity of the first Wahl map of curves of genus g sitting on very general abelian surfaces for all $g > 145$. This provides a "nondegenerational" proof — in the range $g > 145$ — of the surjectivity of the map $\gamma_{K_C}^1$ for general curves of genus g, which holds for all $g \geq 12$ and $g = 10$ [Ciliberto et al. 1988].

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References

[Bauer 1998] T. Bauer, "Seshadri constants and periods of polarized abelian varieties", *Math. Ann.* **312**:4 (1998), 607–623. MR 2000a:14054 Zbl 0933.14025

[Bertram et al. 1991] A. Bertram, L. Ein, and R. Lazarsfeld, "Vanishing theorems, a theorem of Severi, and the equations defining projective varieties", *J. Amer. Math. Soc.* **4**:3 (1991), 587–602. MR 92g:14014 Zbl 0762.14012

[Brion and Kumar 2005] M. Brion and S. Kumar, *Frobenius splitting methods in geometry and representation theory*, Progress in Mathematics **231**, Birkhäuser, Boston, 2005. MR 2005k:14104 Zbl 1072.14066

[Buser and Sarnak 1994] P. Buser and P. Sarnak, "On the period matrix of a Riemann surface of large genus", *Invent. Math.* **117**:1 (1994), 27–56. MR 95i:22018 Zbl 0814.14033

[Ciliberto et al. 1988] C. Ciliberto, J. Harris, and R. Miranda, "On the surjectivity of the Wahl map", *Duke Math. J.* **57**:3 (1988), 829–858. MR 89m:14010 Zbl 0684.14009

[Colombo et al. 2011] E. Colombo, P. Frediani, and G. Pareschi, "Hyperplane sections of abelian surfaces", 2011. To appear in *J. Alg. Geom.* arXiv 0903.2781

[Eisenbud 2005] D. Eisenbud, *The geometry of syzygies*, Graduate Texts in Mathematics **229**, Springer, New York, 2005. MR 2005h:13021 Zbl 1066.14001

[Fuentes García 2005] L. Fuentes García, "Some results about the projective normality of abelian varieties", *Arch. Math.* (*Basel*) **85**:5 (2005), 409–418. MR 2006j:14062 Zbl 1082.14046

[Green 1984] M. L. Green, "Koszul cohomology and the geometry of projective varieties, II", *J. Differential Geom.* **20**:1 (1984), 279–289. MR 86j:14011 Zbl 0559.14009

[Green and Lazarsfeld 1987] M. Green and R. Lazarsfeld, "Deformation theory, generic vanishing theorems, and some conjectures of Enriques, Catanese and Beauville", *Invent. Math.* **90**:2 (1987), 389–407. MR 89b:32025 Zbl 0659.14007

[Hwang and To 2010] J.-M. Hwang and W.-K. To, "Buser–Sarnak invariant and projective normality of abelian varieties", preprint, 2010. arXiv 1003.0742

[Inamdar 1997] S. P. Inamdar, "On syzygies of projective varieties", *Pacific J. Math.* **177**:1 (1997), 71–76. MR 98a:14010 Zbl 0898.14015

[Inamdar and Mehta 1994] S. P. Inamdar and V. B. Mehta, "Frobenius splitting of Schubert varieties and linear syzygies", *Amer. J. Math.* **116**:6 (1994), 1569–1586. MR 96a:14054 Zbl 0817.14032

[Iyer 2003] J. N. Iyer, "Projective normality of abelian varieties", *Trans. Amer. Math. Soc.* **355**:8 (2003), 3209–3216. MR 2004e:14070 Zbl 1016.14024

[Lazarsfeld 1996] R. Lazarsfeld, "Lengths of periods and Seshadri constants of abelian varieties", *Math. Res. Lett.* **3**:4 (1996), 439–447. MR 98e:14044 Zbl 0890.14025

[Lazarsfeld 2004] R. Lazarsfeld, *Positivity in algebraic geometry, I*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) **48**, Springer, Berlin, 2004. MR 2005k:14001a Zbl 1093.14501

[Li 2009] L. Li, "Wonderful compactification of an arrangement of subvarieties", *Michigan Math. J.* **58**:2 (2009), 535–563. MR 2011f:14086 Zbl 1187.14060

[Mukai 1981] S. Mukai, "Duality between $D(X)$ and $D(\hat{X})$ with its application to Picard sheaves", *Nagoya Math. J.* **81** (1981), 153–175. MR 82f:14036 Zbl 0417.14036

[Mumford 1970] D. Mumford, *Abelian varieties*, Tata Institute of Fundamental Research Studies in Mathematics **5**, Oxford University Press, London, 1970. MR 44 #219 Zbl 0223.14022

[Pareschi 2000] G. Pareschi, "Syzygies of abelian varieties", *J. Amer. Math. Soc.* **13**:3 (2000), 651–664. MR 2001f:14086 Zbl 0956.14035

[Pareschi and Popa 2003] G. Pareschi and M. Popa, "Regularity on abelian varieties, I", *J. Amer. Math. Soc.* **16**:2 (2003), 285–302. MR 2004c:14086 Zbl 1022.14012

[Pareschi and Popa 2004] G. Pareschi and M. Popa, "Regularity on abelian varieties, II: Basic results on linear series and defining equations", *J. Algebraic Geom.* **13**:1 (2004), 167–193. MR2005a:14059 Zbl 1073.14061

[Wahl 1992] J. Wahl, "Introduction to Gaussian maps on an algebraic curve", pp. 304–323 in *Complex projective geometry* (Trieste/Bergen, 1989), edited by G. Ellingsrud et al., London Math. Soc. Lecture Note Ser. **179**, Cambridge Univ. Press, London, 1992. MR 93m:14029 Zbl 0790.14014

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