Algebra & Number Theory

Volume 5

2011

No. 3

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mathematical sciences publishers

ALGEBRA AND NUMBER THEORY 5:3(2011)

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Let G be a connected linear algebraic group over a field k. We say that G is toric-friendly if for any field extension K/k and any maximal K-torus T in G the group $G(K)$ acts transitively on $(G/T)(K)$. Our main result is a classification of semisimple (and under certain assumptions on k, of connected) toric-friendly groups.

Introduction

Let k be a field and X be a homogeneous space of a connected linear algebraic group G defined over k. The first question one usually asks about X is whether or not it has a k-point. If the answer is "yes", then one often wants to know whether or not the set $X(k)$ of k-points of X forms a single orbit under the group $G(k)$.

In this paper we shall focus on the case where the geometric stabilizers for the G-action on X are maximal tori of $G_{\overline{k}} := G \times_k \overline{k}$ (here $\overline{k}$ stands for a fixed algebraic closure of k). Such homogeneous spaces arise, in particular, in the study of the adjoint action of a connected reductive group G on its Lie algebra or of the conjugation action of G on itself; see [Colliot-Thélène et al. 2011]. It is shown in Corollary 4.6 of the same reference (see also [Kottwitz 1982, Lemma 2.1]) that every homogeneous space X of this type has a k-point, assuming that G is split and $\mathrm{char}(k) = 0$. Therefore it is natural to ask if this point is unique up to translations by $G(k)$.

Definition 0.1. Let k be a field. We say that a connected linear k-group G is *toric-friendly* if for every field extension K/k the following condition is satisfied:

(∗) For every maximal K-torus T of $G_K := G \times_k K$, the group $G(K)$ has only one orbit in $(G_K/T)(K)$; equivalently, the natural map $\pi : G(K) \to (G_K/T)(K)$ is surjective.

Borovoi was partially supported by the Hermann Minkowski Center for Geometry. Reichstein was partially supported by NSERC Discovery and Accelerator Supplement grants.

MSC2000: primary 20G10; secondary 20G15, 14G05.

Keywords: toric-friendly group, linear algebraic group, semisimple group, maximal torus, rational point, elementary obstruction.

Examining the cohomology exact sequence associated to the K-subgroup T of G_K [Serre 1994, I.5.4, Proposition 36], we see that G is toric-friendly if and only if $\ker[H^1(K,T)\to H^1(K,G)]=1$ for every field extension K/k and every maximal K-torus T of G_K.

Observe that G is toric-friendly if and only if condition $(*)$ of Definition 0.1 is satisfied for all *finitely generated* extensions K/k.

We are interested in classifying toric-friendly groups. In Section 1 we partially reduce this problem to the case where the group is semisimple. The rest of this paper will be devoted to proving the following classification theorem for semisimple toric-friendly groups.

Main Theorem 0.2. *Let k be a field. A connected semisimple k-group G is toric-friendly if and only if G is isomorphic to a direct product $\prod_i R_{F_i/k}G'_i$, where each F_i is a finite separable extension of k and each G'_i is an inner form of PGL_{n_i,F_i} for some integer n_i.*

Notation. Unless otherwise specified, k will denote an arbitrary field. For any field K we denote by K_s a separable closure of K.

By a k-group we mean an affine algebraic group scheme over k, not necessarily smooth or connected. However, when talking of a *reductive* or *semisimple* k-group, we implicitly assume smoothness and connectedness.

Let S be a k-group. We denote by $H^i(k,S)$ the i-th flat cohomology set for $i=0,1$ [Waterhouse 1979, 17.6]. If S is abelian, we denote by $H^i(k,S)$ the i-th flat cohomology group for $i\geq 0$ [Berhuy et al. 2007, Appendix B]. There are exact sequences for flat cohomology similar to those for Galois cohomology, [Waterhouse 1979, 18.1; Berhuy et al. 2007, Appendix B]. When S is smooth, the flat cohomology $H^i(k,S)$ can be identified with Galois cohomology.

1. First reductions

Lemma 1.1. *Let $1\to U\to G\xrightarrow{\varphi} G'\to 1$ be an exact sequence of smooth connected k-groups, where U is unipotent. We assume that U is k-split, that is, has a composition series over k whose successive quotients are isomorphic to $\mathbb{G}_{\mathrm{a},k}$. Then G is toric-friendly if and only if G' is toric-friendly.*

Proof. Choose a field extension K/k and a maximal K-torus $T\subset G_K$. Set $T'=\varphi(T)\subset G'_K$, then T' is a maximal torus of G'_K. The map $\varphi^T: T\to T'$ is an isomorphism, because $T\cap U_K=1$ (as U_K is unipotent). Conversely, let us start from a maximal torus T' of G'_K. The preimage

$$H=\varphi^{-1}(T')\subset G_K$$

of T' is smooth and connected, so any maximal torus T of H maps isomorphically onto T' and therefore it is maximal in G_K.

Now we have a commutative diagram

$$\begin{array}{ccc} H^1(K,T) & \longrightarrow & H^1(K,G) \\ {\scriptstyle \varphi_*^T}\downarrow & & \downarrow{\scriptstyle \varphi_*} \\ H^1(K,T') & \longrightarrow & H^1(K,G') \end{array}$$

Since $\varphi^T : T \to T'$ is an isomorphism of tori, the left vertical arrow φ_*^T is an isomorphism of abelian groups. On the other hand, by [Sansuc 1981, Lemma 1.13], the right vertical arrow φ_* is a bijective map. We see that the top horizontal arrow in the diagram is injective if and only if the bottom horizontal arrow is injective, which proves the lemma. □

Let k be a perfect field and G be a connected k-group. Recall that over a perfect field the unipotent radical of G makes sense; that is, the "geometric" unipotent radical over an algebraic closure is defined over k, by Galois descent. We denote the unipotent radical of G by $R_{\mathrm{u}}(G)$.

Corollary 1.2. *Let k be a perfect field, G be a connected k-group, and $R_{\mathrm{u}}(G)$ be its unipotent radical. Then G is toric-friendly if and only if the associated reductive k-group $G/R_{\mathrm{u}}(G)$ is toric-friendly.*

Proof. Since k is perfect, the smooth connected unipotent k-group $R_{\mathrm{u}}(G)$ is k-split [Borel 1991, Theorem 15.4], and the corollary follows from Lemma 1.1. □

Let k be a field. We recall that a k-group G is called *special* if $H^1(K,G) = 1$ for every field extension K/k. This notion was introduced by J.-P. Serre [1958]. Semisimple special groups over an algebraically closed field were classified by A. Grothendieck [1958]; we shall use his classification later on.

Recall that a k-torus T is called quasitrivial, if its character group $\mathbb{X}(T)$ is a permutation Galois module. Split tori and, more general, quasitrivial tori are special.

Proposition 1.3. *Let $1 \to C \to G \xrightarrow{\varphi} G' \to 1$ be an exact sequence of k-groups, where G and G' are reductive, and $C \subset G$ is central, hence of multiplicative type (not necessarily connected or smooth).*

(a) *If G is toric-friendly, so is G'.*

(b) *If C is a special k-torus, then G is toric-friendly if and only if G' is toric-friendly.*

Proof. Let K/k be a field extension. The map $T \mapsto T' := \varphi(T)$ is a bijection between the set of maximal K-tori $T \subset G_K$ and the set of maximal K-tori $T' \subset G'_K$

(the inverse map is $T' \mapsto T := \varphi^{-1}(T')$). For such T and $T' = \varphi(T)$ we have commutative diagrams

$$\begin{array}{ccc} G_K & \xrightarrow{\varphi} & G'_K \\ \pi \downarrow & & \downarrow \pi' \\ G_K/T & \xrightarrow[\cong]{\varphi_*} & G'_K/T' \end{array} \qquad \begin{array}{ccc} G(K) & \xrightarrow{\varphi} & G'(K) \\ \pi \downarrow & & \downarrow \pi' \\ (G_K/T)(K) & \xrightarrow[\cong]{\varphi_*} & (G'_K/T')(K) \end{array}$$

where $\varphi_* : G_K/T \xrightarrow{\sim} G'_K/T'$ is an isomorphism of K-varieties, and the induced map on K-points $\varphi_* : (G_K/T)(K) \to (G'_K/T')(K)$ is a bijection. Now, if G is toric-friendly, then the map $\pi : G(K) \to (G_K/T)(K)$ is surjective, and we see from the right-hand diagram that then the map $\pi' : G'(K) \to (G'_K/T')(K)$ is surjective as well. This shows that G' is toric-friendly, thus proving (a).

To prove (b), assume that G' is toric-friendly and C is a special k-torus. Then the map $\pi' : G'(K) \to (G'_K/T')(K)$ is surjective (because G' is toric-friendly) and the map $\varphi : G(K) \to G'(K)$ is surjective (because C is special). We see from the right-hand diagram that the map $\pi : G(K) \to (G_K/T)(K)$ is surjective as well. Hence G is toric-friendly. □

We record the following immediate corollary of Proposition 1.3(b).

Corollary 1.4. *Let G be a reductive k-group. Suppose that the radical* $\mathrm{R}(G)$ *is a special k-torus (in particular, this condition is satisfied if* $\mathrm{R}(G)$ *is a quasitrivial k-torus). Then G is toric-friendly if and only if the semisimple group $G/\mathrm{R}(G)$ is toric-friendly.* □

The next result follows from Corollaries 1.2 and 1.4. It partially reduces the problem of classifying toric-friendly groups G to the case where G is semisimple.

Corollary 1.5. *Let k be a perfect field. Let G be a connected k-group containing a split maximal torus. Then G is toric-friendly if and only if the semisimple group $G/\mathrm{R}(G)$ is toric-friendly.* □

The following two lemmas will be used to reduce the problem of classifying *adjoint* semisimple toric-friendly groups G to the case where G is an absolutely simple adjoint k-group.

Lemma 1.6. *A direct product $G = G' \times_k G''$ of connected k-groups is toric-friendly if and only if both G' and G'' are toric-friendly.*

Proof. Let K/k be a field extension. Let $T' \subset G'_K$ and $T'' \subset G''_K$ be maximal K-tori, then $T := T' \times_K T'' \subset G_K$ is a maximal K-torus, and every maximal K-torus in G_K is of this form. The commutative diagram

$$\begin{array}{ccc} G(K) & = & G'(K)\times G''(K) \\ \downarrow & & \downarrow \\ (G_K/T)(K) & = & (G'_K/T')(K)\times(G''_K/T'')(K) \end{array}$$

shows that every K-point of G_K/T lifts to G if and only if every K-point of G'_K/T' lifts to G' and every K-point of G''_K/T'' lifts to G''. □

Lemma 1.7. *Let l/k be a finite separable field extension, G' a connected l-group, and $G = R_{l/k}G'$. Then G is toric-friendly if and only if G' is toric-friendly.*

Proof. Let K/k be a field extension. Then $l\otimes_k K = L_1\times\cdots\times L_r$, where L_i are finite separable extensions of K. It follows that $G_K = \prod_i R_{L_i/K}G'_{L_i}$. Let $T\subset G_K$ be a maximal K-torus, then $T=\prod_i R_{L_i/K}T'_i$, where T'_i is a maximal L_i-torus of G'_{L_i} for each i. We have

$$G(K)=G_K(K)=\Big(\prod_i R_{L_i/K}G'_{L_i}\Big)(K)=\prod_i G'_{L_i}(L_i)=\prod_i G'(L_i)$$

and similarly $(G_K/T)(K)=\prod_i(G'_{L_i}/T'_i)(L_i)$, yielding a commutative diagram

$$\begin{array}{ccc} G(K) & = & \prod_i G'(L_i) \\ \downarrow & & \downarrow \\ (G_K/T)(K) & = & \prod_i(G'_{L_i}/T'_i)(L_i) \end{array}$$

If G' is toric-friendly, then the right vertical arrow in the diagram is surjective, hence the left vertical arrow is surjective and G is toric-friendly.

Conversely, assume that G is toric-friendly. Let L/l be a field extension and $T'\subset G'_L$ a maximal L-torus. Set $K:=L$ and $T:=T'$ in the diagram above. Then we can identify L with one of L_i in the decomposition $l\otimes_k K = L_1\times\cdots\times L_r$, say with L_1. In this way we identify G'_L with G'_{L_1} and G'_L/T' with G'_{L_1}/T'_1. Since G is toric-friendly, the left vertical arrow in the diagram is surjective, hence the right vertical arrow is also surjective. This means that the map $G'(L_i)\to(G'_{L_i}/T'_i)(L_i)$ is surjective for each i and in particular, for $i=1$. Consequently, the map $G'(L)\to(G'_L/T')(L)$ is surjective, and G' is toric-friendly, as desired. □

2. The elementary obstruction

2.1. Let K be a field and X be a smooth geometrically integral K-variety. Write $\mathfrak{g}=\mathrm{Gal}(K_s/K)$, where K_s is a fixed separable closure of K. Recall from [Colliot-Thélène and Sansuc 1987, Definition 2.2.1] that the *elementary obstruction* $\mathrm{ob}(X)$

is the class in $\mathrm{Ext}^1_{\mathfrak{g}}(K_s(X)^*/K_s^*, K_s^*)$ of the extension

$$1 \to K_s^* \to K_s(X)^* \to K_s(X)^*/K_s^* \to 1.$$

In particular, $\mathrm{ob}(X) = 0$ if and only if this extension of $\mathfrak{g}$-modules splits. If X has a K-point, then $\mathrm{ob}(X) = 0$ [Colliot-Thélène and Sansuc 1987, Proposition 2.2.2(a)]. Conversely, if Y is a T-torsor over K for some K-torus T, and $\mathrm{ob}(Y) = 0$, then Y has a K-point, by Lemma 2.1(iv) of [Borovoi et al. 2008]. However, if X is an H-torsor over K for some simply connected semisimple K-group H, then $\mathrm{ob}(X) = 0$ even when X has no K-points; see Lemma 2.2(viii) of that same reference. (The standing assumption in [Borovoi et al. 2008] is that $\mathrm{char}(K) = 0$; however, the proofs of Lemmas 2.1(iv) and 2.2(viii) go through in arbitrary characteristic.)

The following key lemma was suggested to us by J.-L. Colliot-Thélène.

Lemma 2.2. *Let K be a field, T be a K-torus, H be a simply connected semisimple K-group, X be a H-torsor over K and Y be a T-torsor over K. If Y has an F-point over the function field $F = K(X)$ of X, then Y has a K-point.*

Proof. Since H is simply connected, $\mathrm{ob}(X) = 0$; see Section 2.1 above. Suppose Y has an F-point. This means that there exist a K-rational map $X \dashrightarrow Y$. By [Wittenberg 2008, Lemma 3.1.2], if we have a K-rational map $X \dashrightarrow Y$ between smooth geometrically integral K-varieties, then $\mathrm{ob}(X) = 0$ implies $\mathrm{ob}(Y) = 0$. Since T is a K-torus, if $\mathrm{ob}(Y) = 0$, then $Y(K) \neq \varnothing$; see Section 2.1 above. Thus in our situation Y has a K-point, as claimed. □

Lemma 2.3. *Let k be a field. Assume we have a commutative diagram of k-groups*

$$\begin{array}{ccc} S & \longrightarrow & T \\ \downarrow & & \downarrow \\ H & \longrightarrow & G \end{array}$$

where G is a smooth connected k-group, the vertical map $T \to G$ is the inclusion of a maximal k-torus T into G, and H is semisimple and simply connected. If there exists a field extension K/k such that the map

$$H^1(K, S) \to H^1(K, T)$$

is nontrivial, then G is not toric-friendly.

Proof. Choose K and $s \in H^1(K, S)$ such that the image $t \in H^1(K, T)$ of s in $H^1(K, T)$ is nontrivial. Let $h \in H^1(K, H)$ be the image of $s \in H^1(K, S)$ in $H^1(K, H)$, and let $g \in H^1(K, G)$ be the image of t (and of h) in $H^1(K, G)$, as shown in the commutative diagram below:

$$\begin{array}{ccc} H^1(K,S) & \longrightarrow & H^1(K,T) \\ \downarrow & & \downarrow \\ H^1(K,H) & \longrightarrow & H^1(K,G) \end{array} \qquad \begin{array}{ccc} s & \longrightarrow & t \\ \downarrow & & \downarrow \\ h & \longrightarrow & g \end{array}$$

Let X be an H-torsor over K representing h and let $F = K(X)$ be the function field of X. We denote by h_F the image of h in $H^1(F, H)$, and similarly we define s_F, t_F, and g_F. Clearly X has an F-point, hence $h_F = 1$ in $H^1(F, H)$ and therefore $g_F = 1$ in $H^1(F, G)$. On the other hand, by Lemma 2.2, $t_F \neq 1$. We conclude that the kernel of the natural map $H^1(F, T) \to H^1(F, G)$ contains $t_F \neq 1$ and hence, is nontrivial. This implies that G is not toric-friendly. □

2.4. Let G be a reductive k-group. Let G^{ss} be the derived group of G (it is semisimple), and let G^{sc} be the universal cover of G^{ss} (it is semisimple and simply connected). Consider the composed homomorphism $f : G^{sc} \twoheadrightarrow G^{ss} \hookrightarrow G$.

Let K/k be a field extension. There is a canonical bijective correspondence $T \leftrightarrow T^{sc}$ between the set of maximal K-tori $T \subset G_K$ and the set of maximal K-tori $T^{sc} \subset G^{sc}$. Starting from a maximal K-torus $T \subset G_K$, we define a maximal K-torus $T^{sc} := f^{-1}(T) \subset G^{sc}_K$. Conversely, starting from a maximal K-torus $T^{sc} \subset G^{sc}_K$, we define a maximal K-torus $T := f(T^{sc}) \cdot R(G)_K \subset G_K$, where $R(G)$ is the radical of G.

Proposition 2.5. *Let G be a reductive k-group. Let G^{sc} and $f : G^{sc} \to G$ be as in Section 2.4 above. Let K/k be a field extension, $T \subset G_K$ be a maximal K-torus of G_K, and set $T^{sc} = f^{-1}(T) \subset G^{sc}_K$ as above. If the natural map $H^1(K, T^{sc}) \to H^1(K, T)$ is nontrivial, then G is not toric-friendly.*

Proof. Immediate from Lemma 2.3. □

Proposition 2.6. *Let G be a semisimple k-group, $f : G^{sc} \to G$ be the universal covering and $C := \ker(f)$. Then the following conditions are equivalent*:

(a) *G is toric-friendly.*

(b) *The map $H^1(K, T^{sc}) \to H^1(K, T)$ is trivial (identically zero) for every field extension K/k and every maximal K-torus T^{sc} of G^{sc}. Here $T := f(T^{sc})$.*

(c) *The map $H^1(K, C) \to H^1(K, T^{sc})$ is surjective for every field extension K/k and every maximal K-torus T^{sc} of G^{sc}.*

(d) *The connecting homomorphism $\partial_T : H^1(K, T) \to H^2(K, C)$ is injective for every field extension K/k and every maximal K-torus T of G.*

(e) *The natural map $H^1(K, T) \to H^1(K, G)$ is injective for every field extension K/k and every maximal K-torus T of G.*

Proof. (a) $\Rightarrow$ (b) by Proposition 2.5. Examining the cohomology sequence

$$H^1(K,C) \to H^1(K,T^{\mathrm{sc}}) \to H^1(K,T) \to H^2(K,C)$$

associated to the exact sequence $1 \to C \to T^{\mathrm{sc}} \to T \to 1$ of k-groups, we see that (b), (c) and (d) are equivalent.

(d) $\Rightarrow$ (e): The diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & C & \longrightarrow & T^{\mathrm{sc}} & \longrightarrow & T & \longrightarrow & 1 \\ & & \| & & \big\downarrow & & \big\downarrow & & \\ 1 & \longrightarrow & C & \longrightarrow & G^{\mathrm{sc}} & \longrightarrow & G & \longrightarrow & 1 \end{array}$$

of K-groups induces compatible connecting morphisms

$$\begin{array}{ccc} H^1(K,T) & & \\ \big\downarrow & \overset{\partial_T}{\searrow} & \\ & & H^2(K,C) \\ & \overset{\partial_G}{\nearrow} & \\ H^1(K,G) & & \end{array}$$

Suppose $\alpha, \beta \in H^1(K,T)$ map to the same element in $H^1(K,G)$. Then the diagram above shows that $\partial_T(\alpha) = \partial_T(\beta)$ in $H^2(K,C)$. Part (d) now tells us that $\alpha = \beta$.

(e) $\Rightarrow$ (a) is obvious, since (a) is equivalent to the assertion that $H^1(K,T) \to H^1(K,G)$ has trivial kernel for every K and T; see Definition 0.1. □

Corollary 2.7. *With the assumptions and notation of Proposition 2.6, if G is toric-friendly and quasisplit, then*

(a) *the map* $H^1(K,G^{\mathrm{sc}}) \to H^1(K,G)$ *is trivial for every* K/k,

(b) *the map* $H^1(K,C) \to H^1(K,G^{\mathrm{sc}})$ *is surjective for every* K/k,

(c) *the connecting map* $\partial_G : H^1(K,G) \to H^2(K,C)$ *has trivial kernel for every* K/k.

Proof. Examining the cohomology sequence

$$H^1(K,C) \to H^1(K,G^{\mathrm{sc}}) \to H^1(K,G) \to H^2(K,C)$$

associated to the exact sequence $1 \to C \to G^{\mathrm{sc}} \to G \to 1$, we see that (a), (b) and (c) are equivalent.

To prove (a), recall that since G_K is quasisplit, by a theorem of Steinberg [1965, Theorem 1.8] every $x^{\mathrm{sc}} \in H^1(K,G^{\mathrm{sc}})$ lies in the image of the map $H^1(K,T^{\mathrm{sc}}) \to$

$H^1(K, G^{\mathrm{sc}})$ for some maximal K-torus T^{sc} of G^{sc}_K. Since G is toric-friendly, by Proposition 2.6 the map $H^1(K, T^{\mathrm{sc}}) \to H^1(K, T)$ is trivial. The commutative diagram

$$\begin{array}{ccc} H^1(K, T^{\mathrm{sc}}) & \longrightarrow & H^1(K, T) \\ \downarrow & & \downarrow \\ H^1(K, G^{\mathrm{sc}}) & \longrightarrow & H^1(K, G) \end{array}$$

now shows that the image of x^{sc} in $H^1(K, G)$ is 1. Thus the map $H^1(K, G^{\mathrm{sc}}) \to H^1(K, G)$ is trivial. □

Theorem 2.8. *Let G be a split semisimple k-group and $f : G^{\mathrm{sc}} \to G$ be its universal covering map. If G is toric-friendly, then G^{sc} is special.*

Proof. Let T^{sc} be a split maximal torus of G^{sc}. Recall that T^{sc} is special (as is any split torus). Set $C = \ker f$, then $C \subset T^{\mathrm{sc}}$. For any field extension K/k, the map $H^1(K, C) \to H^1(K, G^{\mathrm{sc}})$ factors through $H^1(K, T^{\mathrm{sc}}) = 1$ and hence is trivial. By Corollary 2.7(b) this map is also surjective. This shows that $H^1(K, G^{\mathrm{sc}}) = 1$ for every K/k, that is, G^{sc} is special. □

Remark 2.9. Our proof of Theorem 2.8 goes through for any (not necessarily split) semisimple k-group G, as long as G^{sc} contains a special maximal k-torus T^{sc}. In particular, Theorem 2.8 remains valid for any quasisplit semisimple k-group G, in view of Lemma 2.10 below. This lemma is a special case of [Colliot-Thélène et al. 2004, Lemma 5.6]; however, for the sake of completeness we supply a short self-contained proof.

Lemma 2.10. *Let G be a semisimple, simply connected, quasisplit k-group over a field k. Let $B \subset G$ be a Borel subgroup defined over k, and let $T \subset B \subset G$ be a maximal k-torus of G contained in B. Then T is a quasitrivial k-torus.*

Proof. We write $\bar{k}$ for a fixed algebraic closure of k. Let $\mathbb{X}^\vee(T)$ denote the group of cocharacters of T. Let $R^\vee = R^\vee(G_{\bar{k}}, T_{\bar{k}}) \subset \mathbb{X}^\vee(T)$ denote the coroot system of $G_{\bar{k}}$ with respect to $T_{\bar{k}}$, and let $\Pi^\vee \subset R^\vee$ denote the basis of $R^\vee$ corresponding to B. The Galois group $\mathrm{Gal}(k_s/k)$ acts on $\mathbb{X}^\vee(T)$. Since T, G, and B are defined over k, the subsets $R^\vee$ and $\Pi^\vee$ of $\mathbb{X}^\vee(T)$ are invariant under this action. Since G is simply connected, $\Pi^\vee$ is a $\mathbb{Z}$-basis of $\mathbb{X}^\vee(T)$. Thus $\mathrm{Gal}(k_s/k)$ permutes the $\mathbb{Z}$-basis $\Pi^\vee$ of $\mathbb{X}^\vee(T)$; in other words, T is a quasitrivial torus. □

Remark 2.11. A similar assertion for *adjoint* quasisplit groups was proved by G. Prasad [1989, Proof of Lemma 2.0].

3. Examples in type A

Let k be a field and A a central simple k-algebra of dimension n^2. We write $\mathrm{GL}_{1,A}$ for the k-group with $\mathrm{GL}_{1,A}(R) = (A \otimes_k R)^*$ for any unital commutative k-algebra

R (here $(\ )^*$ denotes the group of invertible elements). The k-group $\mathrm{GL}_{1,A}$ is an inner form of $\mathrm{GL}_{n,k}$.

Let K be a field. Recall that an n-dimensional commutative étale K-algebra is a finite product $E=\prod_i L_i$, where each L_i is a finite separable field extension of K and $\sum_i [L_i:K]=n$. For such $E=\prod_i L_i$ we define a K-torus $R_{E/K}\mathbb{G}_{m,E}:=\prod_i R_{L_i/K}\mathbb{G}_{m,L_i}$, then $(R_{E/K}\mathbb{G}_{m,E})(K)=E^*$. Clearly the K-torus $R_{E/K}\mathbb{G}_{m,E}$ is quasitrivial.

Proposition 3.1. *Let k be a field, and let A/k be a central simple k-algebra of dimension n^2.*

(a) *The k-group $G=\mathrm{GL}_{1,A}$ is toric-friendly.*

(b) *The k-group $\mathrm{PGL}_{1,A}:=\mathrm{GL}_{1,A}/\mathbb{G}_{m,k}$ is toric-friendly.*

(c) *In particular, $\mathrm{GL}_{n,k}$ and $\mathrm{PGL}_{n,k}$ are toric-friendly.*

Proof. (a) Let K/k be a field extension and let

$$T\subset G_K=\mathrm{GL}_{1,A\otimes_k K}$$

be a maximal K-torus. Let E be the centralizer of T in $A\otimes_k K$. An easy calculation over a separable closure K_s of K shows that E is an n-dimensional commutative étale K-subalgebra of $A\otimes_k K$ and that $T=R_{E/K}\mathbb{G}_{m,E}$. It follows that T is quasitrivial, hence special. Since all maximal K-tori $T\subset G_K$ are special, G is toric-friendly.

(b) follows from (a) and Corollary 1.4. To deduce (c) from (a) and (b), set $A=M_n(k)$ (the matrix algebra). □

We now come to the main result of this section, which asserts that a toric-friendly semisimple groups of type A is necessarily an adjoint group.

Proposition 3.2. *Let k be a field. Consider a k-group $G=(\mathrm{SL}_{n_1}\times\cdots\times\mathrm{SL}_{n_r})/C$, where $C\subset\mu:=\mu_{n_1}\times\cdots\times\mu_{n_r}$ is a central subgroup of $G^{\mathrm{sc}}=\mathrm{SL}_{n_1}\times\cdots\times\mathrm{SL}_{n_r}$, not necessarily smooth. If $C\neq\mu$, then G is not toric-friendly.*

Before proceeding with the proof, we fix some notation. Let L/K be a finite separable field extension of degree n. Set

$$R^1_{L/K}(\mathbb{G}_m):=\ker[N_{L/K}:R_{L/K}\mathbb{G}_{m,L}\to\mathbb{G}_{m,K}],$$

where $N_{L/K}$ is the norm map. Clearly $R^1_{L/K}(\mathbb{G}_m)$ can be embedded into $\mathrm{SL}_{n,K}$ as a maximal K-torus. The embedding $K\hookrightarrow L$ induces an embedding $\mu_{n,K}\hookrightarrow R^1_{L/K}\mathbb{G}_m$, where $n=[L:K]$.

The following two lemmas are undoubtedly known. We include short proofs below because we have not been able to find appropriate references.

Lemma 3.3. *There is a commutative diagram*

$$\begin{array}{ccc} K^*/K^{*n} & \xrightarrow{\cong} & H^1(K,\mu_n) \\ \downarrow & & \downarrow \\ K^*/N_{L/K}(L^*) & \xrightarrow{\cong} & H^1(K, R^1_{L/K}\mathbb{G}_m) \end{array} \tag{1}$$

where the horizontal arrows are canonical isomorphisms, the right vertical arrow is induced by the embedding $\mu_n \hookrightarrow R^1_{L/K}\mathbb{G}_m$, and the left vertical arrow is the natural projection.

Proof. Apply the flat cohomology functor to the commutative diagram of commutative K-groups

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mu_{n,K} & \longrightarrow & \mathbb{G}_{m,K} & \xrightarrow{n} & \mathbb{G}_{m,K} & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow \text{id} & & \\ 1 & \longrightarrow & R^1_{L/K}\mathbb{G}_m & \longrightarrow & R_{L/K}\mathbb{G}_m & \xrightarrow{N_{L/K}} & \mathbb{G}_{m,K} & \longrightarrow & 1 \end{array}$$

and use Hilbert's Theorem 90. □

Lemma 3.4. *Suppose $r \mid n$. Then there is a commutative diagram*

$$\begin{array}{ccc} K^*/K^{*n} & \xrightarrow{\cong} & H^1(K,\mu_n) \\ \downarrow & & \downarrow (n/r)_* \\ K^*/K^{*r} & \xrightarrow{\cong} & H^1(K,\mu_r)\,, \end{array}$$

where the horizontal arrows are canonical isomorphisms, the right vertical arrow is induced by the homomorphism $\mu_n \xrightarrow{n/r} \mu_r$ given by $x \mapsto x^{n/r}$, and the left vertical arrow is the natural projection.

Proof. Similar to that of Lemma 3.3, using the commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mu_n & \longrightarrow & \mathbb{G}_m & \xrightarrow{n} & \mathbb{G}_m & \longrightarrow & 1 \\ & & \downarrow n/r & & \downarrow n/r & & \downarrow \text{id} & & \\ 1 & \longrightarrow & \mu_r & \longrightarrow & \mathbb{G}_m & \xrightarrow{r} & \mathbb{G}_m & \longrightarrow & 1 \end{array}$$

□

Example 3.5. The group $G = \mathrm{SL}_{n,k}$ ($n \geq 2$) is not toric-friendly.

Proof. Since SL_n is special, it suffices to construct an extension K/k and a maximal K-torus $T := R^1_{L/K}(\mathbb{G}_m)$ such that $H^1(K,T) \neq 1$. In view of Lemma 3.3 it suffices to show that $N_{L/K}(L^*) \neq K^*$ for some field extension K/k and some finite separable field extension L/K of degree n. This is well known; see for example the proof of [Rowen 1980, Proposition 3.1.46]. We include a short proof below as a way of motivating a related but more complicated argument at the end of the proof of Proposition 3.2.

Let $L := k(x_1, \dots, x_n)$, where $x_1, \dots, x_n$ are independent variables, and $K := L^\Gamma$, where Γ is the cyclic group of order n that acts on L by cyclically permuting $x_1, \dots, x_n$. For $0 \neq a \in k[x_1, \dots, x_n]$, let $\deg(a) \in \mathbb{N}$ denote the degree of a as a polynomial in $x_1, \dots, x_n$. If $a \in k(x_1, \dots, x_n)$ is of the form $a = b/c$ with nonzero $b, c \in k[x_1, \dots, x_n]$, then we define $\deg(a) = \deg(b) - \deg(c)$. This yields the usual degree homomorphism $\deg : L^* \to \mathbb{Z}$. Since $\mathrm{N}_{L/K}(a) = \prod_{\gamma \in \Gamma} \gamma(a)$, we see that $\deg(\mathrm{N}_{L/K}(a)) = n \deg(a)$ is divisible by n, for every $a \in L^*$. On the other hand, $s_1 = x_1 + \dots + x_n \in K$ has degree 1. This shows that $N_{L/K}(L^*) \neq K^*$, as claimed. □

3.6. *Proof of Proposition 3.2.* Let K/k be a field extension. For each $i = 1, \dots, r$, let L_i be a separable field extension of degree n_i over K, and let $T = T_1 \times \dots \times T_r$ be a maximal K-torus of G^{sc}, where $T_i := R^1_{L_i/K}(\mathbb{G}_m)$. By Proposition 2.6 it suffices to show that the composition

$$H^1(K, C) \to H^1(K, \mu) \to H^1(K, T) \tag{2}$$

is not surjective for some choice of extensions K/k and L_i/K_i. Since $C \subsetneq \mu$, there exist a prime p and a nontrivial character $\chi : \mu \to \mu_p$ such that $\chi(C) = 1$. By Proposition 1.3(a) we may assume that $C = \ker(\chi)$. For notational simplicity, let us suppose that $n_1, \dots, n_s$ are divisible by p and $n_{s+1}, \dots, n_r$ are not, for some $0 \le s \le r$. Then it is easy to see that χ is of the form

$$\chi(c_1, \dots, c_r) = c_1^{d_1 n_1/p} \cdots c_s^{d_s n_s/p}$$

for some integers $d_1, \dots, d_s$. Since χ is nontrivial on μ, we have $s \ge 1$ and d_i is not divisible by p for some $i = 1, \dots, s$, say for $i = 1$. That is, we may assume that d_1 is not divisible by p.

Lemma 3.3 gives a concrete description of the second map in (2). To determine the image of the map $H^1(K, C) \to H^1(K, \mu)$, we examine the cohomology exact sequence

$$\begin{array}{ccccc} H^1(K, C) & \longrightarrow & H^1(K, \mu) & \xrightarrow{\chi_*} & H^1(K, \mu_p) \\ & & \| & & \| \\ & & \prod_{i=1}^r K^*/K^{*n_i} & \xrightarrow{\chi_*} & K/K^{*p} \end{array}$$

induced by the exact sequence $1 \to C \to \mu \xrightarrow{\chi} \mu_p \to 1$. The image of $H^1(K, C)$ in $H^1(K, \mu)$ is the kernel of χ_*. By Lemma 3.4, χ_* maps the class of $(a_1, \dots, a_r)$

in $H^1(K,\mu)=\prod_{i=1}^r K^*/K^{*n_i}$ to the class of $a_1^{d_1}\cdots a_s^{d_s}$ in $H^1(K,\mu_p)=K/K^{*p}$. In other words, the image of $H^1(K,C)$ in $H^1(K,\mu)$ is the subgroup of classes of r-tuples $(a_1,\dots,a_r)$ in $H^1(K,\mu)=\prod_{i=1}^r K^*/K^{*n_i}$ such that $a_1^{d_1}\dots a_s^{d_s}\in K^{*p}$. Hence, the image of $H^1(K,C)$ in $H^1(K,T)=\prod_{i=1}^r K^*/N_{L_i/K}(L_i^*)$ consists of classes of r-tuples $(a_1,\dots,a_r)$ such that $a_1^{d_1}\dots a_s^{d_s}\in K^{*p}$.

It remains to construct a field extension K/k, separable field extensions L_i/K of degree n_i for $i=1,\dots,r$, and an element $\alpha\in H^1(K,T)=\prod_{i=1}^r K^*/N_{L_i/K}(L_i^*)$, which cannot be represented by $(a_1,\dots,a_r)\in(K^*)^r$ such that $a_1^{d_1}\cdots a_s^{d_s}\in K^{*p}$. This will show that the map $H^1(K,C)\to H^1(K,T)$ is not surjective, as claimed.

Set $L:=k(x_1,\dots,x_n)$, where $n=n_1+\cdots+n_r$ and $x_1,\dots,x_n$ are independent variables. The symmetric group S_n acts on L by permuting these variables; we embed $S_{n_1}\times\cdots\times S_{n_r}$ into S_n in the natural way, by letting S_{n_1} permute the first n_1 variables, S_{n_2} permute the next n_2 variables, etc. Set $K:=L^{S_{n_1}\times\cdots\times S_{n_r}}$, $s_1:=x_1+\cdots+x_n\in K$ and

$$L_1:=K(x_1),\ \ L_2:=K(x_{n_1+1}),\ \ \dots\ \ L_r:=K(x_{n_1+\cdots+n_{r-1}+1})\,.$$

Clearly $[L_i:K]=n_i$. We claim the class of $(s_1,1,\dots,1)$ in $\prod_{i=1}^r K^*/N_{L_i/K}(L_i^*)$ cannot be represented by any $(a_1,\dots,a_r)\in(K^*)^r$ with $a_1^{d_1}\cdots a_s^{d_s}\in K^{*p}$.

Let $\deg: L^*\to\mathbb{Z}$ be the degree map, as in Example 3.5. Arguing as we did there, we see that $\deg(N_{L_i/K}(a))$ is divisible by n_i for every $i=1,\dots,r$ and every $a\in L_i^*$. In particular, $(a_1,\dots,a_r)\mapsto\deg(a_i)+n_i\mathbb{Z}$ is a well-defined function $\prod_{i=1}^r K^*/N_{L_i/K}(L_i^*)\to\mathbb{Z}/n_i\mathbb{Z}$, and consequently,

$$f(a_1,\dots,a_n):=d_1\deg(a_1)+\cdots+d_s\deg(a_s)+p\mathbb{Z}$$

is a well-defined function $H^1(K,T)\to\mathbb{Z}/p\mathbb{Z}$. We have

$$f(a_1,\dots,a_n)=\deg(a_1^{d_1}\cdots a_s^{d_s}).$$

If $a_1^{d_1}\cdots a_s^{d_s}\in K^{*p}$, then $f(a_1,\dots,a_r)=0$ in $\mathbb{Z}/p\mathbb{Z}$. On the other hand, since $\deg(1)=0$, $\deg(s_1)=1$ and d_1 is not divisible by p, we conclude that $f(s_1,1,\dots,1)$ is nonzero in $\mathbb{Z}/p\mathbb{Z}$. This proves the claim and the proposition. □

4. Groups of type C_n and outer forms of A_n

Proposition 4.1. *No absolutely simple k-group of type C_n ($n\geq 2$) is toric-friendly.*

Proof. Clearly we may assume that k is algebraically closed. We may also assume that G is adjoint, see Proposition 1.3(a). We see that $G=\mathrm{PSp}_{2n}$ and $G^{\mathrm{sc}}=\mathrm{Sp}_{2n}$. By Example 3.5, SL_2 is not toric-friendly. This means that there exist a field extension K/k, a maximal K-torus $S\subset\mathrm{SL}_{2,K}$, and a cohomology class $a_S\in H^1(K,S)$ such that $a_S \neq 1$. We consider the standard embedding

$$(\mathrm{SL}_2)^n = (\mathrm{Sp}_2)^n \hookrightarrow \mathrm{Sp}_{2n}, \quad n \geq 2.$$

Set $T^{\mathrm{sc}} = S^n \subset (\mathrm{Sp}_2)^n \subset \mathrm{Sp}_{2n} = G^{\mathrm{sc}}$. Let $\iota : S \hookrightarrow T^{\mathrm{sc}} = S^n$ be the embedding as the first factor. Set $a^{\mathrm{sc}} = \iota_*(a_S) \in H^1(K, T^{\mathrm{sc}})$. Let T be the image of T^{sc} in $G = \mathrm{PSp}_{2n}$, and let a be the image of a^{sc} in $H^1(K, T)$.

Now observe that the homomorphism

$$\chi : T^{\mathrm{sc}} = S^n \to S, \quad (x_1, \dots, x_n) \mapsto x_1 x_2^{-1},$$

factors through T (recall that $n \geq 2$). Since $\chi \circ \iota = \mathrm{id}_S$, we see that $a \neq 1$. On the other hand, the image of a^{sc} in $H^1(K, G^{\mathrm{sc}})$ is 1 (because $G^{\mathrm{sc}} = \mathrm{Sp}_{2n}$ is special), hence $a \in \ker[H^1(K, T) \to H^1(K, G)]$, and we see that $G = \mathrm{PSp}_{2n}$ is not toric-friendly. □

Proposition 4.2. *No absolutely simple k-group of outer type A_n $(n \geq 2)$ is toric-friendly.*

Lemma 4.3. *Let k be a field, K/k a separable quadratic extension, and D/K a central division algebra of dimension r^2 over K with an involution σ of the second kind (i.e., σ acts nontrivially on K and trivially on k). Then there exists a finite separable field extension F/k such that $K_F := K \otimes_k F$ is a field and $D \otimes_K K_F$ is split, that is, K_F-isomorphic to the matrix algebra $M_r(K_F)$.*

Proof of the lemma. Since there are no nontrivial central division algebras over finite fields, we may assume that k and K are infinite. Let

$$H = \{x \in D \mid x^\sigma = x\}$$

denote the k-space of Hermitian elements of D. Consider the embedding $D \hookrightarrow M_r(K_s)$ induced by an isomorphism $D \otimes_K K_s \cong M_r(K_s)$, where K_s is a separable closure of K. An element x of D is called semisimple regular if its image in $D \otimes_K K_s \cong M_r(K_s)$ is a semisimple matrix with r distinct eigenvalues. A standard argument using an isomorphism $D \otimes_k K_s \cong M_r(K_s) \times M_r(K_s)$ shows that there is a dense open subvariety H_{reg} in the space H, consisting of semisimple regular elements. Clearly H_{reg} is defined over k and contains k-points.

Let $x \in H_{\mathrm{reg}}(k) \subset D$ be a semisimple regular Hermitian element. Let L be the centralizer of x in D. Since x is Hermitian (σ-invariant), the k-algebra L is σ-invariant. Since x is semisimple and regular, the algebra L is a commutative étale K-subalgebra of D of dimension r over K, as is easily seen by passing to K_s. Clearly L is a field, $[L : K] = r$, and L is separable over k. Since $L \subset D$ and $[L : K] = r$, the field L is a splitting field for D; see, for example, [Pierce 1982, Corollary 13.3].

Since $L \supset K$, we see that σ acts nontrivially on L. Let $F = L^{\langle\sigma\rangle}$ denote the subfield of L consisting of elements fixed by σ. Then $[L:F]=2$ and $[F:k]=r$. Clearly F is separable over k. Since $F\cap K = k$ and $FK = L$, we conclude that $L = K\otimes_k F := K_F$. This completes the proof of the lemma. □

4.4. ***Proof of Proposition 4.2.*** By Proposition 1.3(a) we may assume that G is adjoint. By Lemma 4.3 there is a finite separable field extension F/k such that $G_F \cong \mathrm{PSU}(L^{n+1}, h)$, where L/F is a separable quadratic extension and h is a Hermitian form on L^{n+1}. It suffices to prove that $G_F = \mathrm{PSU}(L^{n+1}, h)$ is not toric-friendly.

Set $S = R^1_{L/F}\mathbb{G}_m$. We set $G_F^{\mathrm{sc}} = \mathrm{SU}(L^{n+1}, h)$. We may assume that h is a diagonal form [Knus 1991, Proposition 6.2.4(1); Scharlau 1985, Theorem 7.6.3]. Consider the diagonal torus $S^{n+1}\subset \mathrm{U}(L^{n+1}, h)$ and set $T^{\mathrm{sc}} = S^{n+1}\cap \mathrm{SU}(L^{n+1}, h)$.

We claim that there exists a field extension K/F such that $H^1(K, S)\neq 1$. Indeed, take $K = F((t))$, the field of formal Laurent series over F. Then by [Serre 1968, Proposition V.2.3(c)], $H^1(K, S)\cong H^1(F, S)\times \mathbb{Z}/2\mathbb{Z}\neq 1$.

Now let $a_S\in H^1(K, S)$, $a_S\neq 1$, and consider the embedding

$$\iota : S \hookrightarrow T^{\mathrm{sc}} \subset S^{n+1}, \quad x\mapsto (x, x^{-1}, 1, \dots, 1).$$

Set $a^{\mathrm{sc}} = \iota_*(a_S)\in H^1(K, T^{\mathrm{sc}})$. Let T be the image of T^{sc} in $G_F = \mathrm{PSU}(L^{n+1}, h)$ and a be the image of a^{sc} in $H^1(K, T)$.

Note that the homomorphism

$$\chi : T^{\mathrm{sc}} \to S, \quad (x_1, \dots, x_n, x_{n+1})\mapsto x_1x_3^{-1},$$

factors through T (recall that $n\geq 2$). Since $\chi\circ\iota = \mathrm{id}_S$, we see that $a\neq 1$. Now by Proposition 2.5, G_F and hence G are not toric-friendly. □

5. Classification of semisimple toric-friendly groups

Lemma 5.1. *Let k be an algebraically closed field. If a semisimple k-group G is toric-friendly, then it is adjoint of type A, that is, $G\cong\prod_i \mathrm{PGL}_{n_i}$ for some integers $n_i\geq 2$.*

Proof. First assume that G is simple. By Theorem 2.8 the simply connected cover G^{sc} of G is special. By a theorem of Grothendieck [1958, Theorem 3], G^{sc} is special if and only if G is of type A_n, $n\geq 1$ or C_n, $n\geq 2$. Proposition 4.1 rules out the second possibility. Thus G is of type A.

Now let G be semisimple. By Proposition 1.3(a), G^{ad} is toric-friendly. Write $G^{\mathrm{ad}} = \prod_i G_i$, where each G_i is an adjoint simple group, then by Lemma 1.6 each G_i is toric-friendly. As we have seen, this implies that each G_i is of type A, that is, isomorphic to PGL_{n_i} for some n_i. By Proposition 3.2, G is adjoint, that is, $G = G^{\mathrm{ad}} = \prod_i \mathrm{PGL}_{n_i}$. □

5.2. ***Proof of the Main Theorem 0.2.*** If G is toric-friendly, then clearly $G_{\bar{k}}$ is toric-friendly, where $\bar{k}$ is an algebraic closure of k. By Lemma 5.1, G is adjoint of type A. Write $G = \prod_i R_{F_i/k} G'_i$, where each F_i/k is a finite separable extension and G'_i is a form of PGL_{n_i,F_i}. By Lemmas 1.6 and 1.7, each G'_i is toric-friendly, and by Proposition 4.2, G'_i is an *inner* form of PGL_{n_i,F_i}.

Conversely, by Proposition 3.1 an inner form G'_i of PGL_{n_i,F_i} is toric-friendly. By Lemmas 1.6 and 1.7, the product $G = \prod_i R_{F_i/k} G'_i$ is toric-friendly. □

Corollary 5.3. *Let G be a nontrivial semisimple k-group. Then there exist a field extension K/k and a maximal K-torus $T \subset G$ that is not special. Equivalently, there exist a field extension K/k and a maximal K-torus T of G such that $H^1(K,T) \neq 1$.*

Proof. Assume the contrary, that is, that for any field extension K/k, any maximal K-torus $T \subset G_K$ is special. We may and shall assume that G is split. Recall that for a (quasi)split group, by [Steinberg 1965, Theorem 11.1], every element of $H^1(K,G)$ lies in the image of the map $H^1(K,T) \to H^1(K,G)$ for some maximal K-torus T of G. Thus, under our assumption we have $H^1(K,G) = 1$ for every field extension K/k, that is, G is special. By [Grothendieck 1958, Theorem 3], this is only possible if G is simply connected and has components only of types A and C. On the other hand, G is clearly toric-friendly (see Definition 0.1), and by the Main Theorem 0.2 no nontrivial simply connected semisimple group can be toric-friendly, a contradiction. □

The next result follows immediately from the Main Theorem 0.2 and Corollary 1.4.

Corollary 5.4. *Let G be a split reductive k-group. The group G is toric-friendly if and only if it satisfies these two conditions*:

(a) *the center $Z(G)$ of G is a k-torus, and*

(b) *the adjoint group $G^{\mathrm{ad}} := G/Z(G)$ is a direct product of simple adjoint groups of type A.* □

Note that in condition (a) we allow the trivial k-torus $\{1\}$.

By Corollary 1.4 if G is a reductive k-group such that $G/R(G)$ is toric-friendly and $R(G)$ is special, then G is toric-friendly. The example below shows that when $G/R(G)$ is toric-friendly but $R(G)$ is not special, G need not be toric-friendly.

Example 5.5. Let $k = \mathbb{R}$, $G = \mathrm{U}_2$, the unitary group in two complex variables. Then $Z(G)$ is the group of scalar matrices in G, it is connected, hence $R(G) = Z(G)$ and $G/R(G) = G^{\mathrm{ad}} = \mathrm{PSU}_2$. Since PSU_2 is an inner form of $\mathrm{PGL}_{2,\mathbb{R}}$, by the Main Theorem 0.2 it is toric-friendly. However, the group $G = \mathrm{U}_2$ is not toric-friendly. This does not contradict Corollary 1.4, because $R(G) = Z(G)$ is not special: $H^1(\mathbb{R}, Z(G)) = \mathbb{R}^*/N_{\mathbb{C}/\mathbb{R}}(\mathbb{C}^*) \cong \mathbb{Z}/2\mathbb{Z}$.

To show that $G = \mathrm{U}_2$ is not toric-friendly, set $S = R^1_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m$. Let T be the diagonal maximal $\mathbb{R}$-torus of U_2. Set $G^{\mathrm{sc}} = \mathrm{SU}_2$, $T^{\mathrm{sc}} = T \cap \mathrm{SU}_2$, then $T^{\mathrm{sc}} \cong S$.

Let $a^{\mathrm{sc}} \in H^1(\mathbb{R}, T^{\mathrm{sc}})$ be the cohomology class of the cocycle given by the element $-1 \in T^{\mathrm{sc}}(\mathbb{R})$ of order 2. Let $a \in H^1(\mathbb{R}, T)$ be the image of a^{sc} in $H^1(\mathbb{R}, T)$. Clearly $a \neq 1$. By Proposition 2.5, G is not toric-friendly. □

Acknowledgements

We are grateful to Jean-Louis Colliot-Thélène for helpful comments and suggestions. In particular, he contributed Lemma 2.2 and the idea of Proposition 2.5, which simplified our earlier arguments. We thank the anonymous referee for a quick and thorough review and an anonymous editor of ANT for helpful comments. We also thank Brian Conrad, Philippe Gille, Boris Kunyavskiĭ, and James S. Milne for stimulating discussions.

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Communicated by Jean-Louis Colliot-Thélène
Received 2010-04-03 Revised 2010-10-17 Accepted 2010-10-17

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Algebra & Number Theory (ISSN 1937-0652) at Mathematical Sciences Publishers, Department of Mathematics, University of California, Berkeley, CA 94720-3840 is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

ANT peer review and production are managed by EditFLOW™ from Mathematical Sciences Publishers.

PUBLISHED BY
mathematical sciences publishers
http://msp.org/
A NON-PROFIT CORPORATION
Typeset in LaTeX

Algebra & Number Theory

Volume 5 No. 3 2011