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**Reflexivity and rigidity for complexes, II  
Schemes**

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# Reflexivity and rigidity for complexes, II Schemes

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We prove basic facts about reflexivity in derived categories over noetherian schemes, and about related notions such as semidualizing complexes, invertible complexes, and Gorenstein-perfect maps. Also, we study a notion of rigidity with respect to semidualizing complexes, in particular, relative dualizing complexes for Gorenstein-perfect maps. Our results include theorems of Yekutieli and Zhang concerning rigid dualizing complexes on schemes. This work is a continuation of part I (*Algebra and Number Theory* **4**:1 (2010), 47–86), which dealt with commutative rings.

Introduction	380
1. Derived reflexivity over schemes	385
1.1. Standard homomorphisms	385
1.2. Derived multiplication by global functions	387
1.3. Derived reflexivity	388
1.4. Perfect complexes	390
1.5. Invertible complexes	391
2. Gorenstein-type properties of scheme-maps	394
2.1. Perfect maps	394
2.2. Ascent and descent	398
2.3. Gorenstein-perfect maps	401
2.4. Quasi-Gorenstein maps	404
2.5. Composition, decomposition, and base change	405
3. Rigidity over schemes	408
3.1. Rigid complexes	409
3.2. Morphisms of rigid complexes	412
3.3. Relative rigidity	415
Appendices: Background	420
References	428

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## Introduction

This paper is concerned with properties of complexes over noetherian schemes, that play important roles in duality theory. Some such properties, like (derived) reflexivity, have been an integral part of the theory since its inception; others, like rigidity, appeared only recently. Our main results reveal new aspects of such concepts and establish novel links between them.

Similar questions over commutative rings were examined in [Avramov et al. 2010a]. Additional topics treated there are semidualizing complexes, complexes of finite Gorenstein dimension, perfect complexes, invertible complexes, and rigidity with respect to semidualizing complexes, as well as versions of these notions relative to essentially-finite-type ring-homomorphisms that have finite flat dimension or, more generally, finite Gorenstein dimension. In this sequel we globalize such considerations, that is, extend them to the context of schemes.

This work is a substantial application of Grothendieck duality theory, seen as the study of a twisted inverse image pseudofunctor  $(-)^!$  defined on appropriate categories of schemes. Duality theory provides interpretations of the local facts, a technology to globalize them, and suggestions for further directions of development.

To place our work in context, we review two methods for proving existence of  $(-)^!$  for noetherian schemes and separated scheme-maps of finite type. The original approach of Grothendieck involves the construction of a “coherent family” of dualizing complexes; details are presented in [Hartshorne 1966] and revised in [Conrad 2000]. An alternative method, based on Nagata compactifications and sketched in [Deligne 1966] and [Verdier 1969], is developed in [Lipman 2009]. Recent extensions of these approaches to maps essentially of finite type provide a principal object of this study — the concept of rigidity — and one of our main tools.

Indeed, rigid dualizing complexes over rings, introduced by van den Bergh [1997] in the context of noncommutative algebraic geometry, are used by Yekutieli and Zhang [2008; 2009] in an ongoing project aiming to simplify Grothendieck’s construction of  $(-)^!$ , and extend it to schemes essentially of finite type over a regular ring of finite Krull dimension. On the other hand, Nayak [2009] proved an analog of Nagata’s compactification theorem and extended the pseudofunctor  $(-)^!$  to the category of all noetherian schemes and their separated maps essentially of finite type. We work in this category.

Next we describe in some detail the notions and results of the paper. Comparison with earlier work is postponed until the end of this Introduction.

All schemes are assumed to be noetherian; all scheme-maps are assumed to be essentially of finite type and separated. Let  $X$  be a scheme,  $D(X)$  the derived category of the category of  $\mathcal{O}_X$ -modules, and  $D_c^b(X) \subset D(X)$  the full subcategory whose objects are the complexes with coherent homology that vanishes in all but finitely many degrees.

For  $F$  and  $A$  in  $D(X)$ , we say that  $F$  is *derived  $A$ -reflexive* if both  $F$  and  $R\mathcal{H}om_X(F, A)$  are in  $D_c^b(X)$ , and if the canonical  $D(X)$ -map is an isomorphism

$$F \xrightarrow{\sim} R\mathcal{H}om_X(R\mathcal{H}om_X(F, A), A).$$

When  $\mathcal{O}_X$  itself is derived  $A$ -reflexive the complex  $A$  is said to be *semidualizing*. (The classical notion of *dualizing complex* includes the additional requirement that  $A$  be isomorphic, in  $D(X)$ , to a bounded complex of injective sheaves.)

In Chapter 1 we prove basic results about semidualizing complexes in  $D(X)$ , and examine their interplay with perfect complexes, that is, complexes  $F \in D_c^b(X)$  such that for every  $x \in X$  the stalk  $F_x$  is isomorphic in  $D(\mathcal{O}_{X,x})$  to a bounded complex of flat  $\mathcal{O}_{X,x}$ -modules (or equivalently, such that  $F$  is isomorphic in  $D(X)$  to a bounded complex of flat  $\mathcal{O}_X$ -modules).

In Chapter 2 we explore conditions on a scheme-map  $f : X \rightarrow Y$  that allow for the transfer of properties, like reflexivity, along standard functors  $D(Y) \rightarrow D(X)$ . These basic global notions turn out to be local, not only in the Zariski topology but also in the flat topology; that is, we find that they behave rather well under faithfully flat maps. (This opens the way to examination of more general sites, not undertaken here.)

One such condition involves the notion of *perfection relative to  $f$* , defined for  $F$  in  $D_c^b(X)$  by replacing  $\mathcal{O}_{X,x}$  with  $\mathcal{O}_{Y,f(x)}$  in the definition of perfection. If this condition holds with  $F = \mathcal{O}_X$ , then  $f$  is said to be perfect (aka finite flat dimension, or finite tor-dimension). Flat maps are classical examples.

As a sample of results concerning ascent and descent along perfect maps, we quote from Theorem 2.2.5 and Corollary 2.2.6:

**Theorem 1.** *Let  $f : X \rightarrow Y$  be a perfect map and  $B$  a complex in  $D_c^+(Y)$ .*

*If  $M \in D(Y)$  is derived  $B$ -reflexive, then the complex  $Lf^*M$  in  $D(X)$  is both derived  $Lf^*B$ -reflexive and derived  $f^!B$ -reflexive. For  $M = \mathcal{O}_Y$  this says that if  $B$  is semidualizing then so are  $Lf^*B$  and  $f^!B$ .*

*For each of these four statements, the converse holds if  $M$  and  $B$  are in  $D_c^b(Y)$ , and  $f$  is faithfully flat, or  $f$  is perfect, proper and surjective.*

The perfection of  $f$  can be recognized by its *relative dualizing complex*,  $f^!\mathcal{O}_Y$ . Indeed,  $f$  is perfect if and only if  $f^!\mathcal{O}_Y$  is relatively perfect. Furthermore, if  $f$  is perfect, then every perfect complex in  $D(X)$  is derived  $f^!\mathcal{O}_Y$ -reflexive.

In particular, when  $f$  is perfect the complex  $f^!\mathcal{O}_Y$  is semidualizing. We take this condition as the definition of  $G$ -perfect maps. (Here  $G$  stands for Gorenstein.) In the affine case, these are just maps of finite  $G$ -dimension (see Example 2.3.6). They form a class significantly larger than that of perfect maps. For instance, when the scheme  $Y$  is Gorenstein every scheme map  $X \rightarrow Y$  is  $G$ -perfect. In Section 2.3, we prove some basic properties of such maps, and, more generally, of  $\mathcal{O}_X$ -complexes that are derived  $f^!\mathcal{O}_Y$ -reflexive. For such complexes, called  $G$ -perfect relative to  $f$ , there exist nice dualities with respect to the relative dualizing complex (see Corollary 2.3.12).

*Quasi-Gorenstein* maps are defined by the condition that  $f^!\mathcal{O}_Y$  is perfect. A very special case has been extensively studied: a *flat* map is quasi-Gorenstein if and only if all its fibers are Gorenstein schemes. On the other hand, every map of Gorenstein schemes is quasi-Gorenstein. Every quasi-Gorenstein map is  $G$ -perfect.

The conditions of relative perfection and  $G$ -perfection interact in many pleasing ways with composition and base change of scheme-maps, as explicated mainly in Section 2.5. Included there are a number of additional results about ascent and descent along perfect maps. Application to the case of structure sheaves produces facts, such as the following — all taken from Section 2.5, about the behavior of perfect, of  $G$ -perfect and of quasi-Gorenstein maps.

**Theorem 2.** *Let  $Z \xrightarrow{g} X \xrightarrow{f} Y$  be scheme-maps, with  $g$  perfect.*

(i) *If  $f$  is perfect (resp.  $G$ -perfect) then so is  $fg$ . The converse holds if  $g$  is faithfully flat.*

(ii) *Suppose that  $fg$  is quasi-Gorenstein. Then so is  $g$ ; and if  $g$  is faithfully flat, then also  $f$  is quasi-Gorenstein.*

**Theorem 3.** *Let  $Z \xrightarrow{g} X \xrightarrow{f} Y$  be scheme-maps, with  $f$  quasi-Gorenstein.*

*The composition  $fg$  is  $G$ -perfect if and only if so is  $g$ .*

*Also, if  $g$  is quasi-Gorenstein then so is  $fg$ .*

**Theorem 4.** *Let there be given a fiber square, with  $u$  flat:*

$$\begin{array}{ccc} \bullet & \xrightarrow{v} & \bullet \\ h \downarrow & & \downarrow f \\ \bullet & \xrightarrow{u} & \bullet \end{array}$$

(i) *If  $f$  is  $G$ -perfect then so is  $h$ . The converse holds if  $u$  is faithfully flat.*

(ii) *If  $f$  is quasi-Gorenstein then so is  $h$ . The converse holds if  $u$  is faithfully flat.*

In Chapter 3 we define *rigidity* with respect to an arbitrary semidualizing complex  $A \in D(X)$ . An  $A$ -rigid structure on  $F$  in  $D_c^b(X)$  is a  $D(X)$ -isomorphism

$$\rho : F \xrightarrow{\sim} R\mathcal{H}om_X(R\mathcal{H}om_X(F, A), F).$$

We say that  $(F, \rho)$  is an *A-rigid pair*;  $F \in D_c^b(X)$  is an *A-rigid complex* if such an isomorphism  $\rho$  exists. Morphisms of rigid pairs are defined in the obvious way.

In Theorem 3.1.7 we establish the basic fact about rigid pairs:

**Theorem 5.** *Let  $A$  be a semidualizing complex in  $D(X)$ .*

*For each quasicohherent  $\mathcal{O}_X$ -ideal  $I$  such that  $I^2 = I$ , there exists a canonical A-rigid structure on  $IA$ ; and every A-rigid pair is uniquely isomorphic in  $D(X)$  to such an  $IA$  along with its canonical structure.*

The theorem validates the term “rigid”, as it implies that the only automorphism of a rigid pair is the identity. It also shows that isomorphism classes of A-rigid complexes correspond bijectively to the open-and-closed subsets of  $X$ . A more precise description — in terms of those subsets — of the skeleton of the category of rigid pairs appears in Theorem 3.2.6.

In the derived category, gluing over open coverings is usually not possible; but it is for idempotent ideals (Proposition C.8). Consequently the uniqueness expressed by Theorem 5 leads to gluing for rigid pairs, in the following strong sense:

**Theorem 6.** *For any open cover  $(U_\alpha)$  of  $X$  and family  $(F_\alpha, \rho_\alpha)$  of  $A|_{U_\alpha}$ -rigid pairs such that for all  $\alpha, \alpha'$  the restrictions of  $(F_\alpha, \rho_\alpha)$  and  $(F_{\alpha'}, \rho_{\alpha'})$  to  $U_\alpha \cap U_{\alpha'}$  are isomorphic, there is a unique (up to unique isomorphism) A-rigid pair  $(F, \rho)$ , such that for each  $\alpha$ ,  $(F, \rho)|_{U_\alpha} \simeq (F_\alpha, \rho_\alpha)$ .*

This gluing property holds even under the flat topology, see Theorem 3.2.9.

In Section 3.3 we study complexes that are *relatively rigid*, that is, rigid with respect to the relative dualizing complex  $f^! \mathcal{O}_Y$  of a G-perfect map  $f: X \rightarrow Y$  (a complex that is, by the definition of such maps, semidualizing). As a consequence of gluing for rigid complexes under the flat topology, gluing for relatively rigid complexes holds under the étale topology, see Proposition 3.3.1.

Relative rigidity behaves naturally with respect to (G-)perfect maps, in the sense that certain canonical isomorphisms from duality theory, involving relative dualizing complexes, respect the additional rigid structure. In Corollary 3.3.5 we show that, when  $g$  is perfect, the twisted inverse image functor  $g^!$  preserves relative rigidity; and also, for a composition  $Z \xrightarrow{g} X \xrightarrow{f} Y$  where  $f$  is G-perfect, we demonstrate the interaction of rigidity with the canonical isomorphism

$$g^! \mathcal{O}_X \otimes_Z^L Lg^* f^! \mathcal{O}_Y \xrightarrow{\sim} (fg)^! \mathcal{O}_Y.$$

In Corollary 3.3.7 we do the same with respect to flat base change. Such results are obtained as applications of simple necessary and sufficient condition for additive functors of rigid complexes to be liftable to rigid pairs, detailed in Theorem 3.3.2.

The results above can be applied to complete some work started in [Avramov et al. 2010b]. In that paper, we associated a relative dualizing complex to each essentially-finite-type homomorphism of commutative rings, but did not touch

upon the functoriality properties of that complex. This aspect of the construction can now be supplied by using the fact that the sheafification of the complex in [Avramov et al. 2010b] is a relative dualizing complex for the corresponding map of spectra; see Example 2.3.2. One can then use the results in Section 3.3, discussed above, to enrich the reduction isomorphism [Avramov et al. 2010b, 4.1] to a functorial one. For such applications, it is crucial to work with scheme-maps that are *essentially* of finite type; this is one of our reasons for choosing this category in the setup for this paper.

Notions and notation related to scheme-maps, as well as pertinent material from Grothendieck duality theory, as used in this paper, are surveyed in the Appendices.

We finish the introduction by reviewing connections to earlier work.

The results in Chapter 1 are, for the most part, extensions to the global situation of results proved over commutative rings in [Avramov et al. 2010a]; the transfer is fairly straightforward.

Homomorphisms of commutative noetherian rings that track Gorenstein-type properties were introduced and studied in [Avramov and Foxby 1992; Avramov and Foxby 1997; Iyengar and Sather-Wagstaff 2004], without finiteness hypotheses. Those papers are based on Auslander and Bridger's [1969] theory of Gorenstein dimension, which is defined in terms of resolutions by finite modules or projective modules, and so does not globalize. The scheme-maps defined and studied in Chapter 2 are based on a different description of finite Gorenstein dimension for ring-homomorphisms essentially of finite type, obtained in [Avramov et al. 2010a, 2.2].

The developments in Chapter 3 are largely motivated and inspired by work of Yekutieli and Zhang [2004; 2008; 2009] (see also [Yekutieli 2010]). One of their goals was to construct a new foundation for Grothendieck duality theory. Making extensive use of differential graded algebras (DGAs), Yekutieli and Zhang [2008; 2009] extended van den Bergh's construction [1997] of rigid *dualizing* complexes to schemes *essentially of finite type over a regular ring of finite Krull dimension*, and analyzed the behavior of such complexes under some types of perfect maps. Theirs is a novel approach, especially with regard to the introduction of DGAs into the subject. However, it remains to be seen whether, once all the details are fully exposed, it will prove to be simpler than the much more generally applicable theory presented, for example, in [Lipman 2009].

We come to rigidity from the opposite direction, presupposing duality theory and making no use of DGAs. The concept obtained in this way applies to *semidualizing* complexes over arbitrary schemes, and behaves well under *all* perfect scheme-maps. In the setup of [Yekutieli and Zhang 2009], the regularity of the base

ring implies that relative dualizing complexes are actually dualizing. To compare results, one also needs to know that, when both apply, our concept of rigidity coincides with Yekutieli and Zhang’s. This follows from the Reduction Theorem [Avramov et al. 2010b, 4.1]; see [Avramov et al. 2010a, 8.5.5].

### 1. Derived reflexivity over schemes

*Rings are assumed to be commutative, and both rings and schemes are assumed to be noetherian.*

**1.1. Standard homomorphisms.** Let  $(X, \mathbb{O}_X)$  be a scheme and  $D(X)$  the derived category of the category of sheaves of  $\mathbb{O}_X$ -modules.

Let  $D^+(X)$ , resp.  $D^-(X)$ , be the full subcategory of  $D(X)$  having as objects those complexes whose cohomology vanishes in all but finitely many negative, resp. positive, degrees; set  $D^b(X) := D^+(X) \cap D^-(X)$ . For  $\bullet = +, -$  or  $b$ , let  $D_\bullet^\circ(X)$ , resp.  $D_{qc}^\bullet(X)$ , be the full subcategory of  $D(X)$  with objects those complexes all of whose cohomology sheaves are *coherent*, resp. *quasicoherent*.

To lie in  $D_\bullet^*(X)$  ( $\bullet = c$  or  $qc$ , and  $\bullet = +, -$  or  $b$ ) is a *local condition*: if  $(U_\alpha)$  is an open cover of  $X$ , then  $F \in D(X)$  lies in  $D_\bullet^*(X)$  if and only if for all  $\alpha$  the restriction  $F|_{U_\alpha}$  lies in  $D_\bullet^*(U_\alpha)$ .

A number of canonical homomorphisms play a fundamental role in this paper.

**Remark 1.1.1.** There is a standard trifunctorial isomorphism, relating the derived tensor and sheaf-homomorphism functors (see e.g., [Lipman 2009, §2.6]):

$$R\mathcal{H}om_X(E \otimes_X^L F, G) \xrightarrow{\sim} R\mathcal{H}om_X(E, R\mathcal{H}om_X(F, G)) \quad (E, F, G \in D(X)) \tag{1.1.1.1}$$

from which one gets, by application of the composite functor  $H^0R\Gamma(X, -)$ ,

$$\mathrm{Hom}_{D(X)}(E \otimes_X^L F, G) \xrightarrow{\sim} \mathrm{Hom}_{D(X)}(E, R\mathcal{H}om_X(F, G)). \tag{1.1.1.2}$$

The map corresponding via (1.1.1.2) to the identity map of  $R\mathcal{H}om_X(F, G)$

$$\varepsilon = \varepsilon_G^F : R\mathcal{H}om_X(F, G) \otimes_X^L F \rightarrow G \quad (F, G \in D(X)) \tag{1.1.1.3}$$

is called *evaluation*. When  $F$  is a flat complex in  $D^-(X)$  (or more generally, any  $q$ -flat complex in  $D(X)$ , see [Lipman 2009, §2.5]), and  $G$  is an injective complex in  $D^+(X)$  (or more generally, any  $q$ -injective complex in  $D(X)$ , see [Lipman 2009, §2.3]), one verifies that  $\varepsilon$  is induced by the family of maps of complexes

$$\varepsilon(U) : \mathrm{Hom}_{\mathbb{O}_X(U)}(F(U), G(U)) \otimes_{\mathbb{O}_X(U)} F(U) \rightarrow G(U) \quad (U \subseteq X \text{ open})$$

where, for homogeneous  $\alpha \in \mathrm{Hom}_{\mathbb{O}_X(U)}(F(U), G(U))$  and  $b \in F(U)$ ,

$$\varepsilon(U)(\alpha \otimes b) = \alpha(b).$$



Basic properties of supports of complexes are recalled for further reference.

**Remark 1.1.2.** For any  $F \in D(X)$ , the *support of  $F$*  is the set

$$\text{Supp}_X F := \{ x \in X \mid H^n(F_x) \neq 0 \text{ for some } n \}. \tag{1.1.2.1}$$

If  $F \in D_c^b(X)$ , then  $\text{Supp}_X F$  is a *closed subset* of  $X$ . Also, for all  $F$  and  $G$  in  $D_c^-(X)$ , it follows from, e.g., [Avramov et al. 2010a, A.6] that

$$\text{Supp}_X(F \otimes_X^L G) = \text{Supp}_X F \cap \text{Supp}_X G. \tag{1.1.2.2}$$

Note that  $\text{Supp}_X F = \emptyset$  if and only if  $F = 0$  in  $D(X)$ .

The following example opens the door to applications of the results in [Avramov et al. 2010a].

**Example 1.1.3.** Let  $R$  be a ring. Let  $D(R)$  be the derived category of the category of  $R$ -modules, and define, as above, its full subcategories  $D^\bullet(R)$  for  $\bullet = +, -$  or  $b$ . Let  $D_f^\bullet(R)$  be the full subcategory of  $D^\bullet(R)$  having as objects those complexes whose cohomology modules are all *finite*, i.e., finitely generated, over  $R$ .

For the affine scheme  $X = \text{Spec } R$ , the functor that associates to each complex  $M \in D(R)$  its sheafification  $M^\sim$  is an *equivalence of categories*  $D_f^\bullet(R) \xrightarrow{\sim} D_c^\bullet(X)$ , see [Bökstedt and Neeman 1993, 5.5]; when  $\bullet = +$  or  $b$ , see also [Hartshorne 1966, p. 133, 7.19].

There is a natural bifunctorial isomorphism

$$(M \otimes_R^L N)^\sim \xrightarrow{\sim} M^\sim \otimes_X^L N^\sim \quad (M, N \in D(R)); \tag{1.1.3.1}$$

to define it one may assume that  $M$  and  $N$  are suitable flat complexes, so that  $\otimes^L$  becomes ordinary  $\otimes$ , see [Lipman 2009, §2.5 and (2.6.5)].

There is also a natural bifunctorial map

$$\text{RHom}_R(M, N)^\sim \longrightarrow \text{R}\mathcal{H}om_X(M^\sim, N^\sim), \tag{1.1.3.2}$$

defined to be the one that corresponds via (1.1.1.2) to the composite map

$$\text{RHom}_R(M, N)^\sim \otimes_X^L M^\sim \xrightarrow{\sim} (\text{RHom}_R(M, N) \otimes_R^L M)^\sim \xrightarrow{\varepsilon^\sim} N^\sim,$$

where the isomorphism comes from (1.1.3.1), and the *evaluation map*  $\varepsilon$  corresponds to the identity map of  $\text{RHom}_R(M, N)$  via the analog of (1.1.1.2) over  $D(R)$ .

The map (1.1.3.2) is an *isomorphism* if  $M \in D_f^-(R)$  and  $N \in D^+(R)$ . To show this for variable  $M$  and fixed  $N$  one can use the “way-out” Lemma [Hartshorne 1966, p. 68, 7.1], with  $A$  the opposite of the category of  $R$ -modules and  $P$  the family  $(R^n)_{n>0}$ , to reduce to the case  $M = R$ , where, one checks, the map is the obvious isomorphism.

**1.2. Derived multiplication by global functions.** Let  $(X, \mathbb{O}_X)$  be a scheme. Here we discuss some technicalities about the natural action of  $H^0(X, \mathbb{O}_X)$  on  $D(X)$ .

We identify  $H^0(X, \mathbb{O}_X)$  with  $\text{Hom}_{D(X)}(\mathbb{O}_X, \mathbb{O}_X)$  via the ring isomorphism that takes  $\alpha \in H^0(X, \mathbb{O}_X)$  to multiplication by  $\alpha$ . For  $\alpha \in H^0(X, \mathbb{O}_X)$  and  $F \in D(X)$ , let  $\mu_F(\alpha)$  (“multiplication by  $\alpha$  in  $F$ ”) be the natural composite  $D(X)$ -map

$$F \simeq \mathbb{O}_X \otimes_X^L F \xrightarrow{\alpha \otimes_X^L 1} \mathbb{O}_X \otimes_X^L F \simeq F,$$

or equivalently,

$$F \simeq F \otimes_X^L \mathbb{O}_X \xrightarrow{1 \otimes_X^L \alpha} F \otimes_X^L \mathbb{O}_X \simeq F.$$

Clearly, for any  $D(X)$ -map  $\phi: F \rightarrow C$ ,

$$\phi\alpha := \phi \circ \mu_F(\alpha) = \mu_C(\alpha) \circ \phi =: \alpha\phi.$$

Furthermore, using the obvious isomorphism  $(\mathbb{O}_X \otimes_X^L F)[1] \xrightarrow{\sim} \mathbb{O}_X \otimes_X^L F[1]$  one sees that  $\mu_F(\alpha)$  commutes with translation, that is,  $\mu_F(\alpha)[1] = \mu_{F[1]}(\alpha)$ .

Thus, the family  $(\mu_F)_{F \in D(X)}$  maps  $H^0(X, \mathbb{O}_X)$  into the ring  $C_X$  consisting of endomorphisms of the identity functor of  $D(X)$  that commute with translation — the *center* of  $D(X)$ . It is straightforward to verify that this map is an injective ring homomorphism onto the subring of *tensor-compatible* members of  $C_X$ , that is, those  $\eta \in C_X$  such that for all  $F, G \in D(X)$ ,

$$\eta(F \otimes_X^L G) = \eta(F) \otimes_X^L \text{id}^G = \text{id}^F \otimes_X^L \eta(G).$$

The category  $D(X)$  is  $C_X$ -linear: for all  $F, G \in D(X)$ ,  $\text{Hom}_{D(X)}(F, G)$  has a natural structure of  $C_X$ -module, and composition of maps is  $C_X$ -bilinear. So  $D(X)$  is also  $H^0(X, \mathbb{O}_X)$ -linear, via  $\mu$ .

**Lemma 1.2.1.** *For any  $F, G \in D(X)$  and  $D(X)$ -homomorphism  $\alpha: \mathbb{O}_X \rightarrow \mathbb{O}_X$ , and  $\mu_\bullet(\alpha)$  as above, there are equalities*

$$R\mathcal{H}om_X(\mu_F(\alpha), G) = \mu_{R\mathcal{H}om_X(F, G)}(\alpha) = R\mathcal{H}om_X(F, \mu_G(\alpha)).$$

*Proof.* Consider, for any  $E \in D(X)$ , the natural trifunctorial isomorphism

$$\tau: \text{Hom}_{D(X)}(E \otimes_X^L F, G) \xrightarrow{\sim} \text{Hom}_{D(X)}(E, R\mathcal{H}om_X(F, G)).$$

From tensor-compatibility in the image of  $\mu$ , and  $H^0(X, \mathbb{O}_X)$ -linearity of  $D(X)$ , it follows that for any  $\alpha \in H^0(X, \mathbb{O}_X)$ , the map  $\mu_E(\alpha)$  induces multiplication by  $\alpha$  in both the source and target of  $\tau$ . Functoriality shows then that  $\tau$  is an isomorphism of  $H^0(X, \mathbb{O}_X)$ -modules.

Again, tensor-compatibility implies that  $\mu_F(\alpha)$  induces multiplication by  $\alpha$  in the source of the  $H^0(X, \mathbb{O}_X)$ -linear map  $\tau$ , hence also in the target. Thus, by functoriality,  $R\mathcal{H}om_X(\mu_F(\alpha), G)$  induces multiplication by  $\alpha$  in the target of  $\tau$ . For  $E = R\mathcal{H}om_X(F, G)$ , this gives  $R\mathcal{H}om_X(\mu_F(\alpha), G) = \mu_{R\mathcal{H}om_X(F, G)}(\alpha)$ . One shows similarly that  $R\mathcal{H}om_X(F, \mu_G(\alpha)) = \mu_{R\mathcal{H}om_X(F, G)}(\alpha)$ .  $\square$

**1.3. Derived reflexivity.** Let  $(X, \mathbb{O}_X)$  be a scheme.

One has, for all  $A$  and  $F$  in  $D(X)$ , a *biduality morphism*

$$\delta_F^A : F \rightarrow R\mathcal{H}om_X(R\mathcal{H}om_X(F, A), A),$$

corresponding via (1.1.1.2) to the natural composition

$$F \otimes_X^L R\mathcal{H}om_X(F, A) \xrightarrow{\sim} R\mathcal{H}om_X(F, A) \otimes_X^L F \xrightarrow{\varepsilon_A^F} A.$$

The map  $\delta_F^A$  “commutes” with restriction to open subsets (use [Lipman 2009, 2.4.5.2]).

When  $A$  is a  $q$ -injective complex in  $D(X)$ ,  $\delta_F^A$  is induced by the family

$$\delta(U) : F(U) \rightarrow \text{Hom}_{\mathbb{O}_X(U)}(\text{Hom}_{\mathbb{O}_X(U)}(F(U), A(U)), A(U)) \quad (U \subseteq X \text{ open})$$

of maps of complexes, where, for each  $n \in F(U)$  of degree  $b$ , the map  $\delta(U)(n)$  is

$$\alpha \mapsto (-1)^{ab} \alpha(n),$$

for  $\alpha \in \text{Hom}_{\mathbb{O}_X(U)}(F(U), A(U))$  homogeneous of degree  $a$ .

**Definition 1.3.1.** Given  $A$  and  $F$  in  $D(X)$ , we say that  $F$  is *derived  $A$ -reflexive* if both  $F$  and  $R\mathcal{H}om_R(F, A)$  are in  $D_c^b(X)$  and  $\delta_F^A$  is an isomorphism.

This is a *local condition*: for any open cover  $(U_\alpha)$  of  $X$ ,  $F$  is derived  $A$ -reflexive if and only if the same is true over every  $U_\alpha$  for the restrictions of  $F$  and  $A$ . Also, as indicated below, if  $U$  is affine, say  $U := \text{Spec } R$ , and  $C, M \in D_f^b(R)$ , then

$$M^\sim \text{ is derived } C^\sim\text{-reflexive in } D(U) \iff M \text{ is derived } C\text{-reflexive in } D(R).$$

**Example 1.3.2.** When  $X = \text{Spec } R$  and  $M, C \in D(R)$ , it follows that with  $\delta_M^C$  as in [Avramov et al. 2010a, (2.0.1)], the map  $\delta_M^C$  factors naturally as

$$M^\sim \xrightarrow{(\delta_M^C)^\sim} (\text{RHom}_R(\text{RHom}_R(M, C), C))^\sim \xrightarrow{s} R\mathcal{H}om_X(R\mathcal{H}om_X(M^\sim, C^\sim), C^\sim),$$

where, as in (1.1.3.2), the map  $s$  is an isomorphism if  $M \in D_f^-(R)$ ,  $C \in D^+(R)$  and  $\text{RHom}_R(M, C) \in D_f^b(R)$ . Thus, derived reflexivity globalizes the notion in [Avramov et al. 2010a, §2].

From [Avramov et al. 2010a, 2.1 and 2.3] one now gets:

**Proposition 1.3.3.** *Let  $X$  be a noetherian scheme, and let  $A, F \in D_c^b(X)$ . Then the following conditions are equivalent.*

- (i)  $F$  is derived  $A$ -reflexive.
- (ii)  $R\mathcal{H}om_X(F, A) \in D^-(X)$  and there exists an isomorphism in  $D(X)$

$$F \xrightarrow{\sim} R\mathcal{H}om_X(R\mathcal{H}om_X(F, A), A).$$

- (iii)  $R\mathcal{H}om_X(F, A)$  is derived  $A$ -reflexive and  $\text{Supp}_X F \subseteq \text{Supp}_X A$ . □

**Remark 1.3.4.** For  $A = \mathbb{O}_X$  the theorem above shows that  $F \in D_c^b(X)$  is derived  $\mathbb{O}_X$ -reflexive if and only if so is  $R\mathcal{H}om_X(F, \mathbb{O}_X)$ .

In the affine case,  $X = \text{Spec } R$ , for any  $M \in D_f^b(R)$ , the derived  $\mathbb{O}_X$ -reflexivity of  $M^\sim$  is equivalent to finiteness of the Gorenstein dimension of  $M$ , as defined by Auslander and Bridger [1969].

**Definition 1.3.5.** An  $\mathbb{O}_X$ -complex  $A$  is *semidualizing* if  $\mathbb{O}_X$  is derived  $A$ -reflexive. In other words,  $A \in D_c^b(X)$  and the map  $\chi^A : \mathbb{O}_X \rightarrow R\mathcal{H}om_X(A, A)$  corresponding via (1.1.1.2) to the natural map  $\mathbb{O}_X \otimes_X^L A \rightarrow A$  is an isomorphism.

As above, this condition is local on  $X$ . When  $X = \text{Spec } R$ , a complex  $C \in D_f^b(R)$  is semidualizing in the commutative-algebra sense (that is,  $R$  is derived  $C$ -reflexive; see, e.g., [Avramov et al. 2010a, §3]) if and only if  $C^\sim$  is semidualizing in the present sense.

**Lemma 1.3.6.** *If  $A \in D(X)$  is semidualizing then each  $D(X)$ -endomorphism of  $A$  is multiplication by a uniquely determined  $\alpha \in H^0(X, \mathbb{O}_X)$ .*

*Proof.* With  $\chi^A : \mathbb{O}_X \rightarrow R\mathcal{H}om_X(A, A)$  as in Definition 1.3.5, the map  $\mu_A$  is easily seen to factor as follows:

$$\begin{aligned} \text{Hom}_{D(X)}(\mathbb{O}_X, \mathbb{O}_X) &\xrightarrow{\text{via } \chi^A} \text{Hom}_{D(X)}(\mathbb{O}_X, R\mathcal{H}om_X(A, A)) \\ &\cong \text{Hom}_{D(X)}(\mathbb{O}_X \otimes_X^L A, A) \\ &\cong \text{Hom}_{D(X)}(A, A). \end{aligned}$$

The assertion results. □

**Lemma 1.3.7.** *Let  $X$  be a noetherian scheme. If  $A$  is a semidualizing  $\mathbb{O}_X$ -complex, then  $\text{Supp}_X A = X$ . Furthermore, if there is an isomorphism  $A \simeq A_1 \oplus A_2$  then  $\text{Supp}_X A_1 \cap \text{Supp}_X A_2 = \emptyset$ .*

*Proof.* The  $\mathbb{O}_X$ -complex  $R\mathcal{H}om_X(A, A)$ , which is isomorphic in  $D(X)$  to  $\mathbb{O}_X$ , is acyclic over the open set  $X \setminus \text{Supp}_X A$ . This implies  $\text{Supp}_X A = X$ .

As to the second assertion, taking stalks at arbitrary  $x \in X$  reduces the problem to showing that if  $R$  is a local ring, and  $M_1$  and  $M_2$  in  $D(R)$  are such that the natural map

$$R \rightarrow R\text{Hom}_R(M_1 \oplus M_2, M_1 \oplus M_2) = \bigoplus_{i,j=1}^2 R\text{Hom}_R(M_i, M_j)$$

is an isomorphism, then either  $M_1 = 0$  or  $M_2 = 0$ .

But clearly,  $R$  being local, at most one of the direct summands  $R\text{Hom}_R(M_i, M_j)$  can be nonzero, so for  $i = 1$  or  $i = 2$  the identity map of  $M_i$  is 0, whence the conclusion. □

**1.4. Perfect complexes.** Again,  $(X, \mathbb{O}_X)$  is a scheme.

**Definition 1.4.1.** An  $\mathbb{O}_X$ -complex  $P$  is *perfect* if  $X$  is a union of open subsets  $U$  such that the restriction  $P|_U$  is  $D(U)$ -isomorphic to a bounded complex of finite-rank locally free  $\mathbb{O}_U$ -modules.

From [Illusie 1971, p. 115, 3.5 and p. 135, 5.8.1], one gets:

**Remark 1.4.2.** The complex  $P$  is perfect if and only if  $P \in D_c(X)$  and  $P$  is isomorphic in  $D(X)$  to a bounded complex of flat  $\mathbb{O}_X$ -modules.

Perfection is a local condition. If  $X = \text{Spec } R$  and  $M \in D(R)$  then  $M^\sim$  is perfect if and only if  $N$  is isomorphic in  $D(R)$  to a bounded complex of finite projective  $R$ -modules; cf. [Avramov et al. 2010a, §4]. The next result is contained in [Christensen 2000, 2.1.10]; see also [Avramov et al. 2010a, 4.1].

**Theorem 1.4.3.**  $P \in D_c^b(X)$  is perfect if and only if so is  $R\mathcal{H}om_X(P, \mathbb{O}_X)$ .  $\square$

**Proposition 1.4.4.** Let  $A$  and  $P$  be in  $D(X)$ , with  $P$  perfect.

If  $F \in D(X)$  is derived  $A$ -reflexive then so is  $P \otimes_X^L F$ ; in particular,  $P$  is derived  $\mathbb{O}_X$ -reflexive. If  $A$  is semidualizing then  $P$  is derived  $A$ -reflexive.

*Proof.* The assertion being local, we may assume that  $P$  is a bounded complex of finite-rank free  $\mathbb{O}_X$ -modules. If two vertices of a triangle are derived  $A$ -reflexive then so is the third, whence an easy induction on the number of degrees in which  $P$  is nonzero shows that if  $F$  is  $A$ -reflexive then so is  $P \otimes_X^L F$ . To show that  $P$  is derived  $\mathbb{O}_X$ -reflexive, take  $A = \mathbb{O}_X = F$ .

For the final assertion, take  $F = \mathbb{O}_X$ .  $\square$

A partial converse is given by the next result:

**Theorem 1.4.5.** Let  $F \in D_c(X)$ , let  $A \in D_c^+(X)$ , and let  $P$  be a perfect  $\mathbb{O}_X$ -complex with  $\text{Supp}_X P \supseteq \text{Supp}_X F$ . If  $P \otimes_X^L F$  is in  $D_c^b(X)$ , or  $P \otimes_X^L F$  is perfect, or  $P \otimes_X^L F$  is derived  $A$ -reflexive, then the corresponding property holds for  $F$ .

*Proof.* The assertions are all local, and the local statements are proved in [Avramov et al. 2010a, 4.3, 4.4, and 4.5], modulo sheafification; see Example 1.1.3.  $\square$

We'll need the following isomorphisms, for which cf. [Illusie 1971, pp. 152–153, 7.6 and 7.7].

Let  $E, F$  and  $G$  be complexes in  $D(X)$ , and consider the map

$$R\mathcal{H}om_X(E, F) \otimes_X^L G \rightarrow R\mathcal{H}om_X(E, F \otimes_X^L G), \quad (1.4.5.1)$$

corresponding via (1.1.1.2) to the natural composition

$$(R\mathcal{H}om_X(E, F) \otimes_X^L G) \otimes_X^L E \xrightarrow{\sim} (R\mathcal{H}om_X(E, F) \otimes_X^L E) \otimes_X^L G \xrightarrow{\varepsilon \otimes_X^L 1} F \otimes_X^L G.$$

where  $\varepsilon$  is the evaluation map from (1.1.1.3).

**Lemma 1.4.6.** *Let  $E, F$  and  $G$  be complexes in  $D(X)$ .*

(1) *When either  $E$  or  $G$  is perfect, the map (1.4.5.1) is an isomorphism*

$$\mathbb{R}\mathcal{H}om_X(E, F) \otimes_X^L G \xrightarrow{\sim} \mathbb{R}\mathcal{H}om_X(E, F \otimes_X^L G).$$

(2) *When  $G$  is perfect, there is a natural isomorphism*

$$\mathbb{R}\mathcal{H}om_X(E \otimes_X^L G, F) \xrightarrow{\sim} \mathbb{R}\mathcal{H}om_X(E, F) \otimes_X^L \mathbb{R}\mathcal{H}om_X(G, \mathbb{O}_X).$$

*Proof.* (1). Whether the map (1.4.5.1) is an isomorphism is a local question, so if  $E$  is perfect then one may assume that  $E$  is a bounded complex of finite-rank free  $\mathbb{O}_X$ -modules. The affirmative answer is then given by a simple induction on the number of degrees in which  $E$  is nonzero.

A similar argument applies when  $G$  is perfect.

(2). Setting  $\check{G} := \mathbb{R}\mathcal{H}om_X(G, \mathbb{O}_X)$ , we get from (1), with  $(E, F, G)$  changed to  $(G, \mathbb{O}_X, F)$ , an isomorphism

$$F \otimes_X^L \check{G} \simeq \check{G} \otimes_X^L F \xrightarrow{\sim} \mathbb{R}\mathcal{H}om_X(G, F).$$

This induces the second isomorphism below:

$$\begin{aligned} \mathbb{R}\mathcal{H}om_X(E \otimes_X^L G, F) &\xrightarrow{\sim} \mathbb{R}\mathcal{H}om_X(E, \mathbb{R}\mathcal{H}om_X(G, F)) \\ &\xrightarrow{\sim} \mathbb{R}\mathcal{H}om_X(E, F \otimes_X^L \check{G}) \\ &\xrightarrow{\sim} \mathbb{R}\mathcal{H}om_X(E, F) \otimes_X^L \check{G}; \end{aligned}$$

the first isomorphism comes from (1.1.1.1) and the third from (1), since  $\check{G}$  is also perfect, by Theorem 1.4.3. The desired map is the composite isomorphism.  $\square$

**1.5. Invertible complexes.** Again,  $(X, \mathbb{O}_X)$  is a scheme.

**Definition 1.5.1.** A complex in  $D(X)$  is *invertible* if it is semidualizing and perfect.

This condition is local. If  $X = \text{Spec } R$  and  $M \in D(R)$ , then  $M$  is invertible in the sense of [Avramov et al. 2010a, §5] if and only if  $M^\sim$  is invertible in the present sense.

Recall that  $\Sigma$  denotes the usual translation (suspension) operator on complexes.

**Theorem 1.5.2.** *For  $L \in D_c^b(X)$  the following conditions are equivalent.*

- (i)  *$L$  is invertible.*
- (ii)  *$L^{-1} := \mathbb{R}\mathcal{H}om_X(L, \mathbb{O}_X)$  is invertible.*
- (iii) *Each  $x \in X$  has an open neighborhood  $U_x$  such that for some integer  $r_x$ , there is a  $D(U_x)$ -isomorphism  $L|_{U_x} \simeq \Sigma^{r_x} \mathbb{O}_{U_x}$ .*
- (iii') *For each connected component  $U$  of  $X$  there is an integer  $r$ , a locally free rank-one  $\mathbb{O}_U$ -module  $\mathcal{L}$ , and a  $D(U)$ -isomorphism  $L|_U \simeq \Sigma^r \mathcal{L}$ .*

(iv) For some  $F \in D_c(X)$  there is an isomorphism  $F \otimes_X^L L \simeq \mathbb{C}_X$ .

(v) For all  $G \in D(X)$  the evaluation map  $\varepsilon$  from (1.1.1.3) is an isomorphism

$$\mathcal{R}\mathcal{H}om_X(L, G) \otimes_X^L L \xrightarrow{\sim} G.$$

(v') For all  $G$  and  $G'$  in  $D(X)$ , the natural composite map (see (1.1.1.1))

$$\begin{aligned} \mathcal{R}\mathcal{H}om_X(G' \otimes_X^L L, G) \otimes_X^L L &\xrightarrow{\sim} \mathcal{R}\mathcal{H}om_X(L \otimes_X^L G', G) \otimes_X^L L \\ &\xrightarrow{\sim} \mathcal{R}\mathcal{H}om_X(L, \mathcal{R}\mathcal{H}om_X(G', G)) \otimes_X^L L \\ &\xrightarrow[\varepsilon]{} \mathcal{R}\mathcal{H}om_X(G', G) \end{aligned}$$

is an isomorphism.

*Proof.* When (i) holds, Lemma 1.4.6(2), with  $E = \mathbb{C}_X$  and  $G = L = F$ , yields:

$$\mathbb{C}_X \xrightarrow{\sim} \mathcal{R}\mathcal{H}om_X(L, L) \xrightarrow{\sim} L \otimes_X^L L^{-1}. \quad (1.5.2.1)$$

(i)  $\Leftrightarrow$  (ii). By Theorem 1.4.3, the  $\mathbb{C}_X$ -complex  $L$  is perfect if and only if so is  $L^{-1}$ . If (i) holds, then (1.5.2.1), Proposition 1.4.4 (with  $A = \mathbb{C}_X = F$ ,  $P = L$ ), and Lemma 1.4.6(1) give isomorphisms

$$\mathbb{C}_X \xrightarrow{\sim} L \otimes_X^L L^{-1} \xrightarrow{\sim} \mathcal{R}\mathcal{H}om_X(L^{-1}, \mathbb{C}_X) \otimes_X^L L^{-1} \xrightarrow{\sim} \mathcal{R}\mathcal{H}om_X(L^{-1}, L^{-1}),$$

so that by Proposition 1.3.3(ii) (with  $F = \mathbb{C}_X$  and  $A = L^{-1}$ ), the  $\mathbb{C}_X$ -module  $L^{-1}$  is semidualizing; since it also perfect (ii) holds.

The same argument with  $L$  and  $L^{-1}$  interchanged establishes that (ii)  $\Rightarrow$  (i).

(i)  $\Rightarrow$  (iii). One may assume here that  $X$  is affine. Then, since  $L$  is invertible, §5.1 of [Avramov et al. 2010a] gives that the stalk at  $x$  of the cohomology of  $L$  vanishes in all but one degree, where it is isomorphic to  $\mathbb{C}_{X,x}$ . The cohomology of  $L$  is bounded and coherent, therefore there is an open neighborhood  $U_x$  of  $x$  over which the cohomology of  $L$  vanishes in all but one degree, where it is isomorphic to  $\mathbb{C}_{U_x}$ , i.e., (iii) holds.

(iii)  $\Rightarrow$  (iv). If (iii) holds then the evaluation map (1.1.1.3) (with  $A = L$  and  $G = \mathbb{C}_X$ ) is an isomorphism  $L^{-1} \otimes_X^L L \xrightarrow{\sim} \mathbb{C}_X$ .

(iv)  $\Rightarrow$  (i). This is a local statement that is established (along with some other unstated equivalences) in [Avramov et al. 2010a, 5.1]; see also [Frankild et al. 2009, 4.7].

(iii)  $\Rightarrow$  (iii'). The function  $x \mapsto r_x$  must be locally constant, so of constant value, say  $r$ , on  $U$ ; and then in  $D(U)$ ,  $L \simeq \Sigma^r H^{-r}(L)$ .

(iii')  $\Rightarrow$  (iii). This implication is clear.

(i)  $\Rightarrow$  (v). The first of the following isomorphisms comes from Lemma 1.4.6(2) (with  $(E, F, G) = (L, G, \mathbb{C}_X)$ ), and the second from (1.5.2.1):

$$\mathcal{R}\mathcal{H}om_X(L, G) \otimes_X^L L \xrightarrow{\sim} L^{-1} \otimes_X^L G \otimes_X^L L \xrightarrow{\sim} G.$$

That this composite isomorphism is  $\varepsilon$  is essentially the definition of the isomorphism

$$L^{-1} \otimes_X^L G = R\mathcal{H}om_X(L, \mathbb{O}_X) \otimes_X^L G \xrightarrow{\sim} R\mathcal{H}om_X(L, G);$$

see the proof of Lemma 1.4.6.

(v)  $\Rightarrow$  (iv). Set  $F := L^{-1}$ , and apply (v) for  $G = \mathbb{O}_X$ .

(v)  $\Leftrightarrow$  (v'). Replace  $G$  in (v) by  $R\mathcal{H}om_X(G', G)$ ; or  $G'$  in (v') by  $\mathbb{O}_X$ . □

**Corollary 1.5.3.** *Let  $L_1$  and  $L_2$  be complexes in  $D_c(X)$ .*

- (1) *If  $L_1$  and  $L_2$  are invertible, then so is  $L_1 \otimes_X^L L_2$ .*
- (2) *If  $L_1$  is in  $D_c^b(X)$  and  $L_1 \otimes_X^L L_2$  is invertible, then  $L_1$  is invertible.*
- (3) *For any scheme-map  $g : Z \rightarrow X$ , if  $L_1$  is invertible then so is  $Lg^*L_1$ .*

*Proof.* For (1), use Theorem 1.5.2(iii'); for (2), Theorem 1.5.2(iv) — noting that the  $F$  there may be taken to be the invertible complex  $L^{-1}$ , and that tensoring with an invertible complex takes  $D_c(X)$  into itself; and for (3), the fact that  $g$  maps any connected component of  $Z$  into a connected component of  $X$ . □

**Corollary 1.5.4.** *Let  $A, L$  and  $F$  be complexes in  $D_c^b(X)$ , with  $L$  invertible.*

- (1)  *$F$  is derived  $A$ -reflexive if and only if it is derived  $L \otimes_X^L A$ -reflexive.*
- (2)  *$F$  is derived  $A$ -reflexive if and only if  $F \otimes_X^L L$  is derived  $A$ -reflexive.*
- (3)  *$A$  is semidualizing if and only if  $L \otimes_X^L A$  is semidualizing.*

*Proof.* From, say, Theorem 1.5.2(iii') and Lemma 1.4.6(1), one gets

$$R\mathcal{H}om_X(F, A) \in D_c^b(X) \iff R\mathcal{H}om_X(F, L \otimes_X^L A) \in D_c^b(X).$$

Since  $L^{-1} \otimes_X^L L \simeq \mathbb{O}_X$ , (1) follows now from Lemma 1.4.6; (2) follows from Theorem 1.5.2(iii); and (3) follows from (1). □

**Remark 1.5.5.** A complex  $A \in D_c^b(X)$  is *pointwise dualizing* if every  $F \in D_c^b(X)$  is derived  $A$ -reflexive (see [Avramov et al. 2010a, 6.2.2]). Such an  $A$  is semidualizing: take  $F = \mathbb{O}_X$ .

It is proved in [Avramov et al. 2010a, 8.3.1] that  $\mathbb{O}_X$  is *pointwise dualizing if and only if  $X$  is a Gorenstein scheme* (i.e., the local ring  $\mathbb{O}_{X,x}$  is Gorenstein for all  $x \in X$ ).

It follows from [Avramov et al. 2010a, 5.7] that invertible complexes can be characterized as those that are semidualizing and derived  $\mathbb{O}_X$ -reflexive. Hence *when  $X$  is Gorenstein,*

$$A \in D_c^b(X) \text{ is semidualizing} \iff A \text{ is pointwise dualizing} \iff A \text{ is invertible.}$$



## 2. Gorenstein-type properties of scheme-maps

All schemes are assumed to be noetherian; all scheme-maps are assumed to be essentially of finite type (see Appendix A) and separated.

**2.1. Perfect maps.** Let  $f : X \rightarrow Y$  be a scheme-map.

Let  $f_0 : X \rightarrow Y$  denote the underlying map of topological spaces, and  $f_0^{-1}$  the left adjoint of the direct image functor  $f_{0*}$  from sheaves of abelian groups on  $X$  to sheaves of abelian groups on  $Y$ . There is then a standard way of making  $f_0^{-1}\mathcal{O}_Y$  into a sheaf of commutative rings on  $X$ , whose stalk at any point  $x \in X$  is  $\mathcal{O}_{Y, f(x)}$ .

**Definition 2.1.1.** An  $\mathcal{O}_X$ -complex  $F$  is *perfect relative to  $f$*  — or, as we will write, *perfect over  $f$*  — if it is in  $D_c^b(X)$ , and in the derived category of the category of  $f_0^{-1}\mathcal{O}_Y$ -modules  $F$  is isomorphic to a bounded complex of flat  $f_0^{-1}\mathcal{O}_Y$ -modules.

The map  $f$  is *perfect* if  $\mathcal{O}_X$  is perfect over  $f$  [Illusie 1971, p. 250, définition 4.1].

Perfection over  $\text{id}^X$  is equivalent to perfection in  $D(X)$ ; see Remark 1.4.2.

**Remark 2.1.2.** Using [Illusie 1971, p. 242, 3.3], one sees that perfection over  $f$  is local on  $X$ , in the sense that  $F$  has this property if and only if every  $x \in X$  has an open neighborhood  $U$  such that  $F|_U$  is perfect over  $f|_U$ .

Since  $f$  is essentially of finite type, there is always such a  $U$  for which  $f|_U$  factors as (essentially smooth)  $\circ$  (closed immersion). If  $X \xrightarrow{i} W \rightarrow Y$  is such a factorization, then  $F$  is perfect over  $f$  if and only if  $i_*F$  is perfect over  $\text{id}^W$ : the proof of [Illusie 1971, pp. 252, 4.4] applies here (see Remark A.3).

**Remark 2.1.3.** Each complex that is perfect over  $f$  is derived  $f^!\mathcal{O}_Y$ -reflexive. In particular, if the map  $f$  is perfect, then  $\mathcal{O}_X$  is derived  $f^!\mathcal{O}_Y$ -reflexive.

This is given by [Illusie 1971, p. 259, 4.9.2], in whose proof “smooth” can be replaced by “essentially smooth”; see [Avramov et al. 2010b, 5.1].

Let  $P(f)$  be the full subcategory of  $D(X)$  whose objects are all the complexes that are perfect over  $f$ ; and let  $P(X) := P(\text{id}^X)$  be the full subcategory of  $D(X)$  whose objects are all the perfect  $\mathcal{O}_X$ -complexes.

**Example 2.1.4.** If the map  $f : X = \text{Spec } S \rightarrow \text{Spec } K = Y$  corresponds to a homomorphism of noetherian rings  $\sigma : K \rightarrow S$ , then  $P(f)$  is equivalent to the full subcategory  $P(\sigma) \subseteq D_f^b(S)$  with objects those complexes  $M$  that are isomorphic in  $D(K)$  to some bounded complex of flat  $K$ -modules; this follows from [Illusie 1971, p. 168, 2.2.2.1 and p. 242, 3.3], in view of the standard equivalence, given by sheafification, between finite  $S$ -modules and coherent  $\mathcal{O}_X$ -modules.

Recall that an exact functor  $F : D(Y) \rightarrow D(X)$  is said to be *bounded below* if there is an integer  $d$  such that for all  $M \in D(Y)$  and  $n \in \mathbb{Z}$  the following holds:

$$H^i(M) = 0 \text{ for all } i < n \implies H^j(F(M)) = 0 \text{ for all } j < n - d,$$

By substituting  $>$  for  $<$  in the preceding definition one obtains the notion of *bounded above*. If  $F$  is bounded below, then clearly  $FD^+(Y) \subseteq D^+(Y)$ ; likewise, if  $F$  is bounded above, then  $FD^-(Y) \subseteq D^-(Y)$ .

**Remark 2.1.5.** For every scheme-map  $f$  the functor  $Lf^*$  is bounded above. It is bounded below if and only if  $f$  is perfect. When  $f$  is perfect, one has

$$Lf^*D_c^b(Y) \subseteq D_c^b(X).$$

For,  $Lf^*$  is bounded above and below, hence, as above,  $Lf^*D^b(Y) \subseteq D^b(X)$ . Also,  $Lf^*D_c(Y) \subseteq D_c(X)$ ; see [Hartshorne 1966, p.99, 4.4], whose proof uses 7.3 on page 73 of the same reference as well as compatibility of  $Lf^*$  with open base change to reduce to the assertion that  $Lf^*\mathcal{O}_Y = \mathcal{O}_X$ .

The following characterization of perfection of  $f$ , in terms of the twisted inverse image functor  $f^!$ , was proved for finite-type  $f$  in [Lipman 2009, 4.9.4] and then extended to the essentially finite-type case in [Nayak 2009, 5.9].

**Remark 2.1.6.** For any scheme-map  $f : X \rightarrow Y$ , and for all  $M, N$  in  $D_{qc}^+(Y)$ , there is defined in [Lipman 2009, §4.9] and [Nayak 2009, 5.7–5.8] a functorial  $D(X)$ -map

$$f^!M \otimes_X^L Lf^*N \rightarrow f^!(M \otimes_Y^L N). \tag{2.1.6.1}$$

The following conditions on  $f$  are equivalent:

- (i) The map  $f$  is perfect.
- (ii) The functor  $f^! : D_{qc}^+(Y) \rightarrow D_{qc}^+(X)$  is bounded above and below.
- (iii) The complex  $f^!\mathcal{O}_Y$  is perfect over  $f$ .
- (iv) When  $M$  is perfect,  $f^!M$  is perfect over  $f$ ; and when  $M \otimes_Y^L N$  is in  $D_{qc}^+(Y)$ , the natural map (2.1.6.1) is an isomorphism

$$f^!M \otimes_X^L Lf^*N \xrightarrow{\sim} f^!(M \otimes_Y^L N). \tag{2.1.6.2}$$

From (ii) one gets, as above,  $f^!D_{qc}^b(Y) \subseteq D^b(X)$ ; and the last paragraph in §5.4 of [Nayak 2005] gives

$$f^!D_c^+(Y) \subseteq D_c^+(X). \tag{2.1.6.3}$$

Thus, for perfect  $f$ , one has

$$f^!D_c^b(Y) \subseteq D_c^b(X). \tag{2.1.6.4}$$

Next we establish some further properties of perfect maps for later use.

**Lemma 2.1.7.** Let  $f : X \rightarrow Y$  be a scheme-map, and  $M, B$  complexes in  $D(Y)$ .

If  $f$  is an open immersion, or if  $f$  is perfect,  $M$  is in  $D_c^-(Y)$  and  $B$  is in  $D_{qc}^+(Y)$ , then there are natural isomorphisms

$$Lf^*R\mathcal{H}om_Y(M, B) \xrightarrow{\sim} R\mathcal{H}om_X(Lf^*M, Lf^*B), \tag{2.1.7.1}$$

$$f^!R\mathcal{H}om_Y(M, B) \xrightarrow{\sim} R\mathcal{H}om_X(Lf^*M, f^!B). \tag{2.1.7.2}$$

*Proof.* As a map in  $D(X)$ , (2.1.7.1) comes from (B.1.5). To show that it is an isomorphism we may assume that  $Y$  is affine, say  $Y = \text{Spec } R$ . Then by [Bökstedt and Neeman 1993, 5.5] and [Hartshorne 1966, p.42, 4.6.1 (dualized)], any  $M \in D_c^-(Y)$  is isomorphic to the sheafification of a complex of finite-rank free  $R$ -modules, vanishing in all large degrees; so [Lipman 2009, p. 181, (4.6.7)] gives the desired assertion.

For (2.1.7.2), use [Lipman 2009, 4.2.3(e)] when  $f$  is proper; and then in the general case, compactify, see Appendix A. □

**Remark 2.1.8.** Let  $f : X \rightarrow Y$  be a perfect and proper scheme map.

One has  $Rf_*(D_c^b(X)) \subseteq D_c^b(Y)$ , by [Illusie 1971, p.237, 2.2.1]. Moreover, if  $F \in D_c^b(X)$  is perfect, then so is  $Rf_*F$ ; see Remark 1.4.2 and [Illusie 1971, p.250, Proposition 3.7.2].

**Remark 2.1.9.** In  $D(X)$  there is a natural map

$$\alpha(E, F, G) : R\mathcal{H}om_X(E, F) \rightarrow R\mathcal{H}om_X(E \otimes_X^L G, F \otimes_X^L G) \quad (E, F, G \in D(X)),$$

corresponding via (1.1.1.2) to

$$(R\mathcal{H}om_X(E, F) \otimes_X^L E) \otimes_X^L G \xrightarrow{\varepsilon \otimes_X^L 1} F \otimes_X^L G$$

where  $\varepsilon$  is evaluation (1.1.1.3).

Assume now that  $f$  is perfect. By Remark 2.1.6 there is a natural isomorphism

$$Lf^*N \otimes_X^L f^! \mathcal{O}_Y \simeq f^!N \quad (N \in D_{qc}^+(Y)). \quad (2.1.9.1)$$

Hence  $\alpha(Lf^*M, Lf^*N, f^! \mathcal{O}_Y)$  gives rise to a natural map, for all  $M, N \in D_{qc}^+(Y)$ ,

$$\beta(M, N, f) : R\mathcal{H}om_X(Lf^*M, Lf^*N) \rightarrow R\mathcal{H}om_X(f^!M, f^!N). \quad (2.1.9.2)$$

**Lemma 2.1.10.** *When  $f : X \rightarrow Y$  is perfect,  $M$  is in  $D_c^b(Y)$ , and  $N$  is in  $D_{qc}^+(Y)$ , the map  $\beta(M, N, f)$  is an isomorphism.*

*Proof.* One checks, using B.3(i) and Lemma 2.1.7, that the question is local on both  $X$  and  $Y$ . Hence, via [Hartshorne 1966, p.133, 7.19], one may assume that  $Y$  is affine, that  $M$  is a bounded-above complex of finite-rank free  $\mathcal{O}_Y$ -modules, and that  $N$  is a quasicohherent complex in  $D^+(X)$ .

By Remarks 2.1.5 and 2.1.6, respectively, the functors  $Lf^*$  and  $f^!$  are bounded (both above and below). Therefore, for every fixed  $N$ , the source and target of  $\beta(M, N, f)$  are bounded-below functors of  $M$ . So one can argue as in the proof of [Hartshorne 1966, p.69, (iv)] to reduce the problem to the case  $M = \mathcal{O}_Y$ . This case can be dealt with as follows (cf. [Lipman 2009, p.239, (c)]).

The question being local on  $X$ , one may assume there is a factorization  $f = pi$  as in Remark 2.1.2 ( $i : X \rightarrow W$  a closed immersion,  $p : W \rightarrow Y$  essentially smooth),

with  $i_*\mathbb{O}_X$  a perfect complex. Since the functor  $i_*$  preserves stalks of sheaves, it suffices then to show that the composite map

$$\begin{aligned} i_*Li^*p^*N &\cong i_*R\mathcal{H}om_X(Lf^*\mathbb{O}_Y, Lf^*N) \\ &\xrightarrow{i_*\beta} i_*R\mathcal{H}om_X(f^!\mathbb{O}_Y, f^!N) && (\beta = \beta(\mathbb{O}_Y, N, f)) \\ &\cong i_*R\mathcal{H}om_X(i^!p^!\mathbb{O}_Y, i^!p^!N) \\ &\cong R\mathcal{H}om_W(i_*i^!p^!\mathbb{O}_Y, p^!N) && \text{(see (B.6.1))} \\ &\cong R\mathcal{H}om_W(R\mathcal{H}om_W(i_*\mathbb{O}_X, p^!\mathbb{O}_Y), p^!N) && \text{(see (B.6.1))} \end{aligned}$$

is an isomorphism in  $D(W)$ .

Since both  $i_*\mathbb{O}_X$  and  $p^!\mathbb{O}_Y$  are perfect complexes (see (B.5.1), therefore the target of  $i_*\beta$  is a bounded functor of  $N$ , that preserves direct sums. (This well-known fact about perfect complexes  $P$  can be shown by an easy induction on the number of degrees in which  $P$  is nonzero.) Also, using (B.1.3), one gets

$$i_*i^*p^*N \cong i_*(\mathbb{O}_X \otimes_X^L i^*p^*N) \cong i_*\mathbb{O}_X \otimes_W^L p^*N;$$

and hence the source of  $i_*\beta$  is a bounded functor of  $N$ , that preserves direct sums.

Every quasicohherent  $\mathbb{O}_Y$ -module being a homomorphic image of a free one, arguing as in [Hartshorne 1966, p.69, (iii) and (iv)(dualized)] reduces the isomorphism question to the case  $N = \mathbb{O}_Y$ . It remains to observe that  $\beta(\mathbb{O}_Y, \mathbb{O}_Y, f)$  is isomorphic to the natural map  $\mathbb{O}_X \rightarrow R\mathcal{H}om_X(f^!\mathbb{O}_Y, f^!\mathbb{O}_Y)$ , a map that, by Remark 2.1.3, is indeed an isomorphism.  $\square$

**Lemma 2.1.11.** *Let  $f : X \rightarrow Y$  be a perfect map.*

*When  $M$  is in  $D_c^-(Y)$  and  $B$  is in  $D_c^+(Y)$ , the complex  $Lf^*M$  is derived  $Lf^*B$ -reflexive if and only if it is derived  $f^!B$ -reflexive.*

*Proof.* We deal first with the boundedness conditions in Definition 1.3.1. The condition  $Lf^*M \in D_c^b(X)$  holds throughout, by assumption.

Assume that  $R\mathcal{H}om_X(Lf^*M, Lf^*B)$  is in  $D_c^b(X)$ . As  $R\mathcal{H}om_Y(M, B) \in D_c^+(Y)$  (see [Hartshorne 1966, p. 92, 3.3]), one gets from Remark 2.1.6 and (2.1.7.1) an isomorphism

$$f^!\mathbb{O}_Y \otimes_X^L R\mathcal{H}om_X(Lf^*M, Lf^*B) \simeq f^!R\mathcal{H}om_Y(M, B). \tag{2.1.11.1}$$

By Remark 2.1.6(iii),  $f^!\mathbb{O}_Y \in D_c^b(X)$ , so it follows that  $f^!R\mathcal{H}om_Y(M, B) \in D_c^-(X)$ . On the other hand, by (2.1.6.3),  $f^!R\mathcal{H}om_Y(M, B) \in D_c^+(X)$ . We conclude that  $f^!R\mathcal{H}om_Y(M, B) \in D_c^b(X)$ , and so by (2.1.7.2), that  $R\mathcal{H}om_X(Lf^*M, f^!B) \in D_c^b(X)$ .

Suppose, conversely, that  $R\mathcal{H}om_X(Lf^*M, f^!B) \in D_c^b(X)$ , so that by (2.1.7.2), there is an integer  $n$  such that

$$H^i(f^!R\mathcal{H}om_Y(M, B)) = 0 \text{ for all } i > n.$$

Using (2.1.7.1) and Remark 2.1.5 one gets

$$\mathcal{R}\mathcal{H}om_X(Lf^*M, Lf^*B) \simeq Lf^*\mathcal{R}\mathcal{H}om_Y(M, B) \in D_c(X).$$

Also,  $f^!\mathcal{O}_Y \in D_c^b(X)$ , by Remark 2.1.6, and it follows from an application of (i)–(iii) in B.3 to a local factorization of  $f$  as (essentially smooth)◦(closed immersion) — or from Proposition 2.3.9 — that  $\text{Supp}_X f^!\mathcal{O}_Y = X$ . So except for the trivial case where  $X$  is empty, there is an integer  $m$  such that

$$H^m f^!\mathcal{O}_Y \neq 0 \quad \text{and} \quad H^j f^!\mathcal{O}_Y = 0 \quad \text{for all } j > m.$$

Hence, by (2.1.11.1), for each  $x$  in  $X$  and for all  $k > n - m$ , [Avramov et al. 2010a, A.4.3] gives  $(H^k \mathcal{R}\mathcal{H}om_X(Lf^*M, Lf^*B))_x = 0$ . It follows that

$$\mathcal{R}\mathcal{H}om_X(Lf^*M, Lf^*B) \in D_c^b(X).$$

The desired assertions now result from the isomorphisms

$$\begin{aligned} \mathcal{R}\mathcal{H}om_X(\mathcal{R}\mathcal{H}om_X(Lf^*M, Lf^*B), Lf^*B) &\xrightarrow{\sim} \mathcal{R}\mathcal{H}om_X(Lf^*\mathcal{R}\mathcal{H}om_X(M, B), Lf^*B) \\ &\xrightarrow{\sim} \mathcal{R}\mathcal{H}om_X(f^!\mathcal{R}\mathcal{H}om_X(M, B), f^!B) \\ &\xrightarrow{\sim} \mathcal{R}\mathcal{H}om_X(\mathcal{R}\mathcal{H}om_X(Lf^*M, f^!B), f^!B), \end{aligned}$$

given by formula (2.1.7.1), Lemma 2.1.10, and formula (2.1.7.2), respectively.  $\square$

**2.2. Ascent and descent.** Let  $f : X \rightarrow Y$  be a scheme-map.

**Remark 2.2.1.** Recall that  $f$  is said to be *faithfully flat* if it is flat and surjective; and that for any flat  $f$ , the canonical map to  $f^*$  from its left-derived functor  $Lf^*$  is an isomorphism — in brief,  $Lf^* = f^*$ .

**Lemma 2.2.2.** *Let  $f : X \rightarrow Y$  be a perfect scheme-map and  $M$  a complex in  $D(Y)$ . If  $M$  is in  $D_c^b(Y)$  then  $Lf^*M$  is in  $D_c^b(X)$ . The converse holds when  $M$  is in  $D_c(Y)$  and  $f$  is faithfully flat, or proper and surjective.*

*Proof.* The forward implication is contained in Remark 2.1.5. For the converse, when  $f$  is faithfully flat there are isomorphisms  $H^n(f^*M) \cong f^*H^n(M)$  ( $n \in \mathbb{Z}$ ); so it suffices that  $f^*H^n(M) = 0$  imply  $H^n(M) = 0$ . This can be seen stalkwise, where we need only recall, for a flat local homomorphism  $R \rightarrow S$  of local rings and any  $R$ -module  $P$ , that  $P \otimes_R S = 0$  implies  $P = 0$ .

When  $f$  is proper then by Remark 2.1.8,  $Rf_*(D_c^b(X)) \subseteq D_c^b(Y)$  and  $Rf_*\mathcal{O}_X$  is perfect. Furthermore, surjectivity of  $f$  implies that

$$\text{Supp}_Y Rf_*\mathcal{O}_X \supseteq \text{Supp}_Y H^0 Rf_*\mathcal{O}_X = \text{Supp}_Y f_*\mathcal{O}_X = Y.$$

In view of the projection isomorphism

$$Rf_*Lf^*M \simeq Rf_*\mathcal{O}_X \otimes_Y^L M,$$

see (B.1.4), the desired converse follows from Theorem 1.4.5.  $\square$

**Proposition 2.2.3.** *Let  $f : X \rightarrow Y$  be a scheme-map and  $M \in D_c^-(Y)$ .*

*If  $M$  is perfect, then  $Lf^*M$  is perfect. The converse holds if  $f$  is faithfully flat, or if  $f$  is perfect, proper and surjective.*

*Proof.* Suppose  $M$  is perfect in  $D(Y)$ . One may assume, after passing to a suitable open cover, that  $M$  is a bounded complex of finite-rank free  $\mathcal{O}_Y$ -modules. Then  $Lf^*M = f^*M$  is a bounded complex of finite-rank free  $\mathcal{O}_X$ -modules. Thus if  $M$  is perfect then so is  $Lf^*M$ .

For the *converse*, when  $f$  is faithfully flat we use the following characterization of perfection ([Illusie 1971, p. 135, 5.8.1]):  *$M \in D(Y)$  is perfect if and only if  $M \in D_c^b(Y)$  and there are integers  $m \leq n$  such that for all  $\mathcal{O}_Y$ -modules  $E$  and all  $i$  outside the interval  $[m, n]$ ,  $H^i(E \otimes_Y^L M) = 0$ .*

Writing  $f^*$  in place of  $Lf^*$  (see Remark 2.2.1) we have, as in the proof of Lemma 2.2.2, that for any  $i$ , the vanishing of

$$H^i(f^*E \otimes_X^L f^*M) = H^i(f^*(E \otimes_Y^L M)) \cong f^*H^i(E \otimes_Y^L M)$$

implies that of  $H^i(E \otimes_Y^L M)$ . Hence the converse holds.

When  $f$  is perfect, proper and surjective, one can argue as in the last part of the proof of Lemma 2.2.2 to show that if  $Lf^*M$  is perfect then  $M$  is perfect. □

**Proposition 2.2.4.** *Let  $f : X \rightarrow Y$  be a proper scheme-map and  $B \in D_{qc}^+(Y)$ .*

*If  $F \in D(X)$  is derived  $f^!B$ -reflexive then  $Rf_*F$  is derived  $B$ -reflexive.*

*Proof.* Since  $F$  and  $R\mathcal{H}om_X(F, f^!B)$  are in  $D_c^b(X)$ , it follows from Remark 2.1.8 that  $Rf_*F$  is in  $D_c^b(Y)$ , and (via (B.6.1)) that

$$R\mathcal{H}om_Y(Rf_*F, B) \simeq Rf_*R\mathcal{H}om_X(F, f^!B) \in D_c^b(Y).$$

Now apply the functor  $Rf_*$  to the assumed isomorphism

$$\delta_F^{f^!B} : F \xrightarrow{\sim} R\mathcal{H}om_X(R\mathcal{H}om_X(F, f^!B), f^!B),$$

and use the duality isomorphism (B.6.1) twice, to get the isomorphisms

$$\begin{aligned} Rf_*F &\xrightarrow{\sim} Rf_*R\mathcal{H}om_X(R\mathcal{H}om_X(F, f^!B), f^!B) \\ &\xrightarrow{\sim} R\mathcal{H}om_Y(Rf_*R\mathcal{H}om_X(F, f^!B), B) \\ &\xrightarrow{\sim} R\mathcal{H}om_Y(R\mathcal{H}om_Y(Rf_*F, B), B). \end{aligned}$$

Their composition is actually  $\delta_{Rf_*F}^B$ , though that doesn't seem so easy to show. Fortunately, owing to Proposition 1.3.3(ii) we needn't do so to conclude that  $Rf_*F$  is derived  $B$ -reflexive. □

**Theorem 2.2.5.** *Let  $f : X \rightarrow Y$  be a perfect scheme-map,  $M$  a complex in  $D_c^-(Y)$ , and  $B$  a complex in  $D_c^+(Y)$ .*

If  $M$  is derived  $B$ -reflexive, then  $\mathbb{L}f^*M$  is derived  $\mathbb{L}f^*B$ -reflexive and derived  $f^!B$ -reflexive. If  $f$  is faithfully flat, or proper and surjective, and  $\mathbb{L}f^*M$  is derived  $\mathbb{L}f^*B$ -reflexive or derived  $f^!B$ -reflexive, then  $M$  is derived  $B$ -reflexive.

*Proof.* Suppose first that  $M$  is derived  $B$ -reflexive, so that, by definition, both  $M$  and  $\mathbb{R}\mathcal{H}om_Y(M, B)$  are in  $D_c^b(Y)$ . Then (2.1.7.1) and Remark 2.1.5 show that  $\mathbb{L}f^*M$  and  $\mathbb{R}\mathcal{H}om_X(\mathbb{L}f^*M, \mathbb{L}f^*B)$  are in  $D_c^b(X)$ . Moreover, application of the functor  $\mathbb{L}f^*$  to the  $D(Y)$ -isomorphism  $M \simeq \mathbb{R}\mathcal{H}om_Y(\mathbb{R}\mathcal{H}om_Y(M, B), B)$  yields provides a  $D(X)$ -isomorphism

$$\mathbb{L}f^*M \simeq \mathbb{R}\mathcal{H}om_X(\mathbb{R}\mathcal{H}om_X(\mathbb{L}f^*M, \mathbb{L}f^*B), \mathbb{L}f^*B).$$

Proposition 1.3.3(ii) then gives that  $\mathbb{L}f^*M$  is derived  $\mathbb{L}f^*B$ -reflexive. When  $B$  is in  $D_c^+(Y)$ , Lemma 2.1.11 yields that  $\mathbb{L}f^*M$  is derived  $f^!B$ -reflexive.

Suppose, conversely, that  $\mathbb{L}f^*M$  is derived  $\mathbb{L}f^*B$ -reflexive, or equivalently, that  $\mathbb{L}f^*M$  is derived  $f^!B$ -reflexive (see Lemma 2.1.11). Then, first,  $\mathbb{L}f^*M \in D_c^b(X)$  and, by (2.1.7.1),  $\mathbb{L}f^*\mathbb{R}\mathcal{H}om_Y(M, B) \in D_c^b(X)$ . Lemma 2.2.2 then gives  $M \in D_c^b(Y)$ ; similarly, since  $\mathbb{R}\mathcal{H}om_Y(M, B) \in D_c(Y)$  (see [Hartshorne 1966, p. 92, 3.3]), we obtain  $\mathbb{R}\mathcal{H}om_Y(M, B) \in D_c^b(Y)$ .

Next, when  $f$  is faithfully flat (so that  $\mathbb{L}f^* = f^*$ , see Remark 2.2.1), one checks, with moderate effort, that if

$$\delta := \delta_M^B : M \rightarrow \mathbb{R}\mathcal{H}om_Y(\mathbb{R}\mathcal{H}om_Y(M, B), B)$$

is the canonical  $D(Y)$ -map, then  $f^*\delta$  is identified, via (2.1.7.1), with the canonical  $D(X)$ -map  $\delta_{f^*M}^{f^*B}$ . The latter being an isomorphism, therefore so are all the maps  $H^n(f^*\delta) = f^*H^n(\delta)$ . Verifying that a sheaf-map is an isomorphism can be done stalkwise, and so,  $f$  being faithfully flat, local considerations show that the maps  $H^n(\delta)$  are isomorphisms. Therefore,  $\delta$  is an isomorphism.

Finally, when  $f$  is proper and surjective and  $\mathbb{L}f^*M$  is derived  $f^!B$ -reflexive, whence, by Proposition 2.2.4,  $\mathbb{R}f_*M$  is derived  $B$ -reflexive, one argues as in the last part of the proof of Lemma 2.2.2 to deduce that  $M$  is derived  $B$ -reflexive.  $\square$

Taking  $M = \mathbb{O}_Y$  one gets:

**Corollary 2.2.6.** *Let  $f : X \rightarrow Y$  be a perfect scheme-map and  $B \in D_c^+(Y)$ .*

*If  $B$  is semidualizing, then so are  $\mathbb{L}f^*B$  and  $f^!B$ . Conversely, if  $f$  is faithfully flat, or proper and surjective, and  $\mathbb{L}f^*B$  or  $f^!B$  is semidualizing, then so is  $B$ .  $\square$*

**Corollary 2.2.7.** *Let  $f : X \rightarrow Y$  be a perfect scheme-map and  $M$  a complex in  $D_c^-(Y)$ . Consider the following properties:*

- (a)  $M$  is semidualizing.
- (b)  $M$  is derived  $\mathbb{O}_Y$ -reflexive.
- (c)  $M$  is invertible.

Each of these properties implies the corresponding property for  $Lf^*M$  in  $D(X)$ . The converse holds when  $f$  is faithfully flat, or proper and surjective.

*Proof.* Note that, given Lemma 2.2.2, we may assume that  $M$  is in  $D_c^b(Y)$ . The assertions about properties (a) and (b) are the special cases  $(M, B) = (\mathbb{O}_Y, M)$  and  $(M, B) = (M, \mathbb{O}_Y)$ , respectively, of Theorem 2.2.5. The assertion about (c) follows from the assertion about (a) together with Proposition 2.2.3.  $\square$

**2.3. Gorenstein-perfect maps.** Let  $f : X \rightarrow Y$  be a scheme-map.

**Definition 2.3.1.** A relative dualizing complex for  $f$  is any  $\mathbb{O}_X$ -complex isomorphic in  $D(X)$  to  $f^!\mathbb{O}_Y$ .

Any relative dualizing complex is in  $D_c^+(X)$ . Indeed, § B.3(i) and B.4 reduce the assertion to the case of maps between affine schemes, where the desired assertion follows from the following example.

**Example 2.3.2.** For a homomorphism  $\tau : K \rightarrow P$  of commutative rings denote the  $P$ -module of relative differentials by  $\Omega_\tau$ , and set

$$\Omega_\tau^n = \bigwedge_P^n \Omega_\tau \quad \text{for each } 0 \leq n \in \mathbb{Z}.$$

Let  $\sigma : K \rightarrow S$  be a homomorphism of rings that is essentially of finite type; thus, there exists a factorization

$$K \xrightarrow{\dot{\sigma}} P \xrightarrow{\sigma'} S \tag{2.3.2.1}$$

where  $\dot{\sigma}$  is essentially smooth of relative dimension  $d$  and  $\sigma'$  is finite, see A.1. As in [Avramov et al. 2010a, (8.0.2)], we set

$$D^\sigma := \Sigma^d \text{RHom}_P(S, \Omega_{\dot{\sigma}}^d) \in D(S). \tag{2.3.2.2}$$

With  $f : X = \text{Spec } S \rightarrow \text{Spec } K = Y$  the scheme-map corresponding to  $\sigma$ , the complex of  $\mathbb{O}_X$ -modules  $(D^\sigma)^\sim$  is a relative dualizing complex for  $f$ ; in particular, up to isomorphism,  $D^\sigma$  depends only on  $\sigma$ , and not on the factorization (2.3.2.1).

Indeed, there is a  $D_{qc}(X)$ -isomorphism

$$f^!\mathbb{O}_Y \simeq (D^\sigma)^\sim; \tag{2.3.2.3}$$

for, if  $f = \dot{f}f'$  is the factorization corresponding to (2.3.2.1) then

$$f^!\mathbb{O}_Y \simeq \dot{f}'^! \dot{f}^!\mathbb{O}_Y \simeq \dot{f}'^!(\Sigma^d \Omega_{\dot{\sigma}}^d)^\sim \simeq \Sigma^d \text{RHom}_P(S, \Omega_{\dot{\sigma}}^d)^\sim = (D^\sigma)^\sim,$$

the second isomorphism coming from B.5, and the third from (B.6.2).

**Definition 2.3.3.** A complex  $F$  in  $D(X)$  is said to be *G-perfect* (for Gorenstein-perfect) relative to  $f$  if  $F$  is derived  $f^!\mathbb{O}_Y$ -reflexive. The full subcategory of  $D_c^b(X)$ , whose objects are the complexes that are G-perfect relative to  $f$  is denoted  $G(f)$ .



In particular,  $F$  is in  $G(\text{id}^X)$  if and only if  $F$  is derived  $\mathbb{O}_X$ -reflexive. We set

$$G(X) := G(\text{id}^X).$$

In view of (2.3.2.3), in the affine case G-perfection can be expressed in terms of *finite G-dimension* in the sense of [Auslander and Bridger 1969]; see [Avramov et al. 2010a, §6.3 and 8.2.1].

As is the case for perfection (Remark 2.1.2), G-perfection can be tested locally.

**Remark 2.3.4.** A complex  $F$  in  $D(X)$  is in  $G(f)$  if and only if every  $x \in X$  has an open neighborhood  $U$  such that  $F|_U$  is in  $G(f|_U)$ .

If  $f$  factors as

$$X \xrightarrow{i} W \xrightarrow{h} Y$$

with  $i$  a closed immersion and  $h$  essentially smooth, then  $F$  is in  $G(f)$  if and only if  $i_*F$  is in  $G(W)$ . It suffices to show this locally; and then this is [Avramov et al. 2010a, 8.2.1], in view of the equivalence of categories in Example 1.1.3.

**Definition 2.3.5.** The map  $f : X \rightarrow Y$  is said to be *G-perfect* (for *Gorenstein perfect*) if  $f^!\mathbb{O}_Y$  is semidualizing, that is, if  $\mathbb{O}_X$  is in  $G(f)$ .

A local theory of such maps already exists:

**Example 2.3.6.** If  $X = \text{Spec } S$  and  $Y = \text{Spec } K$ , where  $K$  and  $S$  are noetherian rings, and  $\sigma : K \rightarrow S$  is the ring-homomorphism corresponding to  $f$ , then  $f$  is *G-perfect* if and only if  $\sigma$  is of *finite G-dimension* in the sense of [Avramov and Foxby 1997]; see [Avramov et al. 2010a, 8.4.1].

Recall from Remark 2.1.6 that  $f$  is perfect if and only if  $f^!\mathbb{O}_Y$  is in  $P(f)$ , the full subcategory of  $D(X)$  whose objects are all the complexes that are perfect with respect to  $f$ . There is a similar description of G-perfection:

**Remark 2.3.7.** The map  $f$  is G-perfect if and only if  $f^!\mathbb{O}_Y \in G(f)$ . This follows from Proposition 1.3.3, since for all  $x \in X$ , the stalk at  $x$  satisfies  $(f^!\mathbb{O}_Y)_x \not\cong 0$ ; see (2.3.2.3).

**Remark 2.3.8.** When  $Y$  is Gorenstein, every map  $f : X \rightarrow Y$  is G-perfect, since [Avramov et al. 2010a, 8.3.1] and (2.3.2.3) together imply that  $G(f) = D_c^b(X)$ .

Via (2.3.2.3), a slight generalization of [Illusie 1971, p. 258, 4.9ff] globalizes [Avramov et al. 2010b, 1.2]:

**Proposition 2.3.9.** *Let  $f : X \rightarrow Y$  be a scheme-map.*

*The following inclusion holds:  $P(f) \subseteq G(f)$ .*

*If  $M \in P(Y)$  then the functor  $R\mathcal{H}om_X(-, f^!M)$  takes  $P(f)$  (resp.  $G(f)$ ) into itself; and if  $M \in G(Y)$  then  $R\mathcal{H}om_X(-, f^!M)$  takes  $P(f)$  into  $G(f)$ .*

*Proof.* The first assertion is a restatement of Remark 2.1.3.

The second assertion is local on  $X$ , so one may suppose  $f$  factors as  $X \xrightarrow{i} W \xrightarrow{h} Y$  with  $i$  a closed immersion and  $h$  essentially smooth. For any  $F \in D_c^b(X)$  and  $M \in D_{qc}^+(Y)$  one has, using formula (B.6.1), B.5 and Lemma 1.4.6,

$$i_* \mathcal{R}\mathcal{H}om_X(F, i^! h^! M) \simeq \mathcal{R}\mathcal{H}om_W(i_* F, h^! M) \simeq \mathcal{R}\mathcal{H}om_W(i_* F, h^* M) \otimes_W^L h^! \mathcal{O}_Y,$$

where  $h^! \mathcal{O}_Y$  is invertible. Consequently, by Remark 2.1.2,

$$\begin{aligned} \mathcal{R}\mathcal{H}om_X(F, f^! M) \in \mathcal{P}(f) &\iff i_* \mathcal{R}\mathcal{H}om_X(F, f^! M) \in \mathcal{P}(W) \\ &\iff \mathcal{R}\mathcal{H}om_W(i_* F, h^* M) \in \mathcal{P}(W). \end{aligned}$$

Similarly, by Remark 2.3.4 and Corollary 1.5.4(2),

$$\begin{aligned} \mathcal{R}\mathcal{H}om_X(F, f^! M) \in \mathcal{G}(f) &\iff i_* \mathcal{R}\mathcal{H}om_X(F, f^! M) \in \mathcal{G}(h) \\ &\iff \mathcal{R}\mathcal{H}om_W(i_* F, h^* M) \text{ is derived } \mathcal{O}_W\text{-reflexive.} \end{aligned}$$

If  $F \in \mathcal{P}(f)$  then  $i_* F$  is a perfect  $\mathcal{O}_W$ -complex, and by Lemma 1.4.6(2),

$$\mathcal{R}\mathcal{H}om_W(i_* F, h^* M) \simeq h^* M \otimes_W^L \mathcal{R}\mathcal{H}om_W(i_* F, \mathcal{O}_W), \quad (2.3.9.1)$$

where  $\mathcal{R}\mathcal{H}om_W(i_* F, \mathcal{O}_W)$  is perfect (see Theorem 1.4.3).

If  $M \in \mathcal{P}(Y)$  then by Proposition 2.2.3,  $h^* M \in \mathcal{P}(W)$ , and then (2.3.9.1) shows that  $\mathcal{R}\mathcal{H}om_W(i_* F, h^* M) \in \mathcal{P}(W)$ . Thus  $\mathcal{R}\mathcal{H}om_X(F, f^! M) \in \mathcal{P}(f)$ .

If  $M \in \mathcal{G}(Y)$ , then  $h^* M$  is derived  $\mathcal{O}_W$ -reflexive, hence so is  $\mathcal{R}\mathcal{H}om_W(i_* F, h^* M)$ ; see Theorem 2.2.5, (2.3.9.1) and Proposition 1.4.4. So  $\mathcal{R}\mathcal{H}om_X(F, f^! M) \in \mathcal{G}(f)$ .

If  $F \in \mathcal{G}(f)$  and  $M \in \mathcal{P}(Y)$  then  $i_* F \in \mathcal{G}(h)$  is  $\mathcal{O}_W$ -reflexive and  $h^* M$  is perfect; so by Lemma 1.4.6(1), (2.3.9.1) still holds, so  $\mathcal{R}\mathcal{H}om_W(i_* F, h^* M)$  is  $\mathcal{O}_W$ -reflexive, by Remark 1.3.4 and Proposition 1.4.4. So again,  $\mathcal{R}\mathcal{H}om_X(F, f^! M) \in \mathcal{G}(f)$ .  $\square$

From Proposition 2.3.9 one gets the following result. It can also be seen as the special case  $g = \text{id}^Y$  of Proposition 2.5.2 below.

**Corollary 2.3.10.** *Any perfect map is  $\mathcal{G}$ -perfect.*  $\square$

Applying Proposition 2.3.9 to  $\mathcal{R}\mathcal{H}om_X(\mathcal{O}_X, f^! F)$ , one gets:

**Corollary 2.3.11.** *If  $f : X \rightarrow Y$  is perfect then  $f^! \mathcal{P}(Y) \subseteq \mathcal{P}(f)$  and furthermore  $f^! \mathcal{G}(Y) \subseteq \mathcal{G}(f)$ . If  $f$  is  $\mathcal{G}$ -perfect then  $f^! \mathcal{P}(Y) \subseteq \mathcal{G}(f)$ .*  $\square$

Also, in view of Proposition 1.3.3(iii):

**Corollary 2.3.12.** *For any scheme-map  $f : X \rightarrow Y$ , the relative dualizing functor  $\mathcal{R}\mathcal{H}om_X(-, f^! \mathcal{O}_Y)$  induces a commutative diagram of categories, where horizontal arrows represent equivalences:*

$$\begin{array}{ccc}
G(f)^{\text{op}} & \xleftarrow{\cong} & G(f) \\
\cup & & \cup \\
P(f)^{\text{op}} & \xleftarrow{\cong} & P(f)
\end{array}$$

These equivalences are dualities, in the sense of [Avramov et al. 2010a, §6].  $\square$

**2.4. Quasi-Gorenstein maps.** For the following notion of quasi-Gorenstein map, cf. [Avramov and Iyengar 2008, 2.2] and [Avramov et al. 2010a, §8.6.1]. For the case when  $f$  is flat, see also [Hartshorne 1966, p. 298, Exercise 9.7], which can be done, e.g., along the lines of the proof of [Lipman 1979, Lemma 1].)

**Definition 2.4.1.** A map  $f : X \rightarrow Y$  is *quasi-Gorenstein* if  $f^!\mathcal{O}_Y$  is invertible. If, in addition,  $f$  is perfect, then  $f$  is said to be a *Gorenstein map*.

If  $f : X \rightarrow Y$  is quasi-Gorenstein then, clearly,  $\mathcal{O}_X \in G(f)$ , i.e.,  $f$  is *G-perfect*. More generally, Corollary 1.5.4 shows that  $G(f) = G(X)$ .

**Example 2.4.2.** Let  $f : X \rightarrow Y$  be a scheme map. If  $X$  is Gorenstein and  $f$  is G-perfect, then  $f$  is quasi-Gorenstein; see Remark 1.5.5. Remark 2.3.8 shows then that when  $X$  and  $Y$  are both Gorenstein  $f$  is quasi-Gorenstein.

One has the following globalization of the *flat* case of [Avramov et al. 2010a, 8.6.2], see also [Avramov and Iyengar 2008, 2.4]:

**Proposition 2.4.3.** *If  $f : X \rightarrow Y$  is a flat Gorenstein map, with diagonal map  $\delta : X \rightarrow X \times_Y X$ , then there are natural isomorphisms*

$$W_f := \mathcal{H}om_X(\delta^!(\mathcal{O}_{X \times_Y X}), \mathcal{O}_X) \xrightarrow[\nu]{\sim} R\mathcal{H}om_X(\delta^!(\mathcal{O}_{X \times_Y X}), \mathcal{O}_X) \xrightarrow{\sim} f^!\mathcal{O}_Y.$$

If furthermore  $g : Z \rightarrow X$  is finite, then (B.6.1) gives a natural isomorphism

$$g_*(fg)^!\mathcal{O}_Y \cong Rg_*g^!f^!\mathcal{O}_Y \xrightarrow{\sim} R\mathcal{H}om_X(g_*\mathcal{O}_Z, W_f).$$

*Proof.* For any flat scheme-map  $f : X \rightarrow Y$  there is a natural isomorphism

$$\delta^!(\mathcal{O}_{X \times_Y X}) \xrightarrow{\sim} R\mathcal{H}om_X(f^!\mathcal{O}_Y, \mathcal{O}_X)$$

(see Corollary 6.5 in [Avramov et al. 2010b], with  $M = \mathcal{O}_X = N$ ).

It follows, when  $f^!\mathcal{O}_Y$  is invertible, that the complex  $\delta^!(\mathcal{O}_{X \times_Y X})$  is invertible, and that there is a natural  $D(X)$ -isomorphism

$$f^!\mathcal{O}_Y \xrightarrow{\sim} R\mathcal{H}om_X(\delta^!(\mathcal{O}_{X \times_Y X}), \mathcal{O}_X).$$

That the natural map  $\nu$  is an isomorphism holds true with any perfect complex in place of  $\delta^!(\mathcal{O}_{X \times_Y X})$ : the assertion is local, hence reduces to the corresponding (obvious) assertion for rings.

For the final assertion, note that the natural map is an isomorphism

$$g_*(fg)^!\mathcal{O}_Y \xrightarrow{\sim} \mathrm{R}g_*(fg)^!\mathcal{O}_Y$$

because the equivalence of categories given in [Hartshorne 1966, p.133, 7.19] allows us to work exclusively with quasicoherent sheaves, on which the functor  $g_*$  is exact.  $\square$

**2.5. Composition, decomposition, and base change.** We turn now to the behavior of relative perfection and G-perfection, especially vis-à-vis the derived direct- and inverse-image functors and the twisted inverse image functor, when several maps are involved.

Generalizing Proposition 2.2.3 (which is the special case  $f = \mathrm{id}^X$ ), one has:

**Proposition 2.5.1** (cf. [Illusie 1971, pp.253–254, 4.5.1]). *Let  $Z \xrightarrow{g} X$  and  $X \xrightarrow{f} Y$  be scheme-maps, with  $g$  perfect.*

*Then  $\mathrm{L}g^*\mathrm{P}(f) \subseteq \mathrm{P}(fg)$ . In particular, if  $f$  is perfect then so is  $fg$ .*

*Conversely, if  $g$  is faithfully flat, or if  $g$  is proper and surjective and  $F \in \mathrm{D}_c(X)$ , then  $\mathrm{L}g^*F \in \mathrm{P}(fg) \implies F \in \mathrm{P}(f)$ . In particular, if  $fg$  is perfect then so is  $f$ .*

*Proof.* Let  $F \in \mathrm{P}(f)$ . By Lemma 2.2.2,  $\mathrm{L}g^*F \in \mathrm{D}_c^b(Z)$ . Hence by [Illusie 1971, p.242, 3.3, p.251, 4.3 and p.115, 3.5(b)] (whose proofs are easily made to apply to essentially finite-type maps of noetherian schemes), for  $\mathrm{L}g^*F$  to be in  $\mathrm{P}(fg)$  it suffices that there be integers  $m \leq n$  such that for any  $\mathcal{O}_Y$ -module  $M$  and any integer  $j \notin [m, n]$ ,

$$0 = H^j(\mathrm{L}g^*F \otimes_Z^{\mathbb{L}} \mathrm{L}(fg)^*M) \cong H^j(\mathrm{L}g^*(F \otimes_X^{\mathbb{L}} \mathrm{L}f^*M)).$$

But by *loc. cit.* this holds because  $F$  is in  $\mathrm{P}(g)$  and  $\mathrm{L}g^*$  is bounded.

Taking  $M = \mathcal{O}_Y$  one gets that if  $f$  is perfect then  $fg$  is perfect.

For the converse, if  $g$  is faithfully flat (so that  $\mathrm{L}g^* = g^*$ ) then for any  $\mathcal{O}_X$ -module  $F$  and any  $j \in \mathbb{Z}$ , one sees stalkwise that

$$H^j(g^*F) \cong g^*H^j(F) = 0 \iff H^j(F) = 0.$$

Hence if  $F \in \mathrm{D}_c(X)$  and  $g^*F \in \mathrm{P}(fg) \subseteq \mathrm{D}_c^b(Z)$  — whence  $F \in \mathrm{D}_c^b(X)$  — then by an argument like that above,  $F \in \mathrm{P}(f)$ .

In the remaining case one argues as in the proof of Proposition 2.2.3. (It should be noted that the relevant part of Theorem 1.4.5 is proved via the above criterion for relative perfection, so it applies not only to perfection but more generally to relative perfection.)  $\square$

Analogously, for  $A := f^!\mathcal{O}_Y$  one has  $(fg)^!\mathcal{O}_Y \simeq g^!A$ , so Theorem 2.2.5 gives

**Proposition 2.5.2** (cf. [Avramov and Foxby 1997, 4.7]). *Let  $Z \xrightarrow{g} X \xrightarrow{f} Y$  be scheme-maps, with  $g$  perfect.*

Then  $\mathrm{L}g^*G(f) \subseteq G(fg)$ . In particular, if  $f$  is  $G$ -perfect then so is  $fg$ .

Conversely, if  $g$  is faithfully flat and  $F \in D_c^-(X)$ , or if  $g$  is proper and surjective and  $F \in D_c(X)$ , then  $\mathrm{L}g^*F \in G(fg)$  implies  $F \in G(f)$ .  $\square$

The next proposition generalizes parts of Proposition 2.3.9. The proof is quite similar, and so is omitted.

**Proposition 2.5.3.** *Let  $Z \xrightarrow{g} X \xrightarrow{f} Y$  be scheme-maps,  $P \in \mathcal{P}(g)$ ,  $F, A \in D(X)$ .*

*If  $F \in \mathcal{P}(f)$  then  $\mathrm{R}\mathcal{H}om_Z(P, g^1F) \in \mathcal{P}(fg)$ . (Cf. [Illusie 1971, p. 258, 4.9].) In other words, the functor  $\mathrm{R}\mathcal{H}om_Z(-, g^1F)$  takes  $\mathcal{P}(g)$  to  $\mathcal{P}(fg)$ .*

*If  $F$  is  $A$ -reflexive then  $\mathrm{R}\mathcal{H}om_Z(P, g^1F)$  is  $g^1A$ -reflexive. For  $A = f^1\mathcal{O}_Y$  this gives that  $\mathrm{R}\mathcal{H}om_X(-, g^1F)$  takes  $\mathcal{P}(g)$  to  $G(fg)$ .  $\square$*

**Proposition 2.5.4.** *Let  $Z \xrightarrow{g} X \xrightarrow{f} Y$  be scheme-maps, with  $g$  perfect.*

*Then  $g^1\mathcal{P}(f) \subseteq \mathcal{P}(fg)$  and  $g^1G(f) \subseteq G(fg)$ .*

*Conversely, if  $g$  is proper and surjective,  $F$  is in  $D_c^+(X)$ , and  $g^1F$  is in  $\mathcal{P}(fg)$  (resp.  $G(fg)$ ) then  $F$  is in  $\mathcal{P}(f)$  (resp.  $G(f)$ ).*

*Proof.* The direct assertions are obtained from Proposition 2.5.3 by taking  $P = \mathcal{O}_Z$ .

If  $g$  is perfect then  $g^1\mathcal{O}_X \in \mathcal{P}(g)$  and

$$\mathrm{R}g_*g^1F \simeq \mathrm{R}g_*(g^1\mathcal{O}_X \otimes_Z^{\mathbb{L}} \mathrm{L}g^*F) \simeq \mathrm{R}g_*g^1\mathcal{O}_X \otimes_X^{\mathbb{L}} F;$$

see Remark 2.1.6. If  $g$  is also proper then  $\mathrm{R}g_*g^1\mathcal{O}_X$  is perfect [Illusie 1971, p. 257, 4.8(a)]. One can then argue as at the end of the proof of Proposition 2.5.1.  $\square$

**Proposition 2.5.5.** *Let  $Z \xrightarrow{g} X \xrightarrow{f} Y$  be scheme-maps, with  $g$  proper.*

*Then  $\mathrm{R}g_*\mathcal{P}(fg) \subseteq \mathcal{P}(f)$  and  $\mathrm{R}g_*G(fg) \subseteq G(f)$ .*

*Proof.* For  $\mathcal{P}$  one can proceed as in [Illusie 1971, p. 257, 4.8]. (This ultimately uses the projection isomorphism (B.1.3).)

For  $G$  apply Proposition 2.2.4 with  $B = f^1\mathcal{O}_Y$ .  $\square$

**Proposition 2.5.6** (cf. [Iyengar and Sather-Wagstaff 2004, 5.2]). *Let  $Z \xrightarrow{g} X \xrightarrow{f} Y$  be scheme-maps, with  $f$  quasi-Gorenstein.*

*Then  $G(fg) = G(g)$ . In particular,  $fg$  is  $G$ -perfect if and only if so is  $g$ .*

*Also, if  $g$  is quasi-Gorenstein then so is  $fg$ .*

*Proof.* For any invertible  $F \in D(X)$  the natural map (see (2.1.6.1))

$$g^1\mathcal{O}_X \otimes_Z^{\mathbb{L}} \mathrm{L}g^*F \rightarrow g^1F$$

is an *isomorphism*: the question being local (see B.4), one reduces via 1.5.2(iii') to the simple case  $F = \mathcal{O}_X$ .

When  $F$  is the invertible complex  $f^1\mathcal{O}_Y$ , there results an isomorphism

$$g^1\mathcal{O}_X \otimes_Z^{\mathbb{L}} \mathrm{L}g^*f^1\mathcal{O}_Y \rightarrow g^1f^1\mathcal{O}_Y \simeq (fg)^1\mathcal{O}_Y.$$

The first assertion follows from Corollary 1.5.4(1) (with  $A = g^! \mathcal{O}_X$ ,  $L = \mathbb{L}g^* f^! \mathcal{O}_Y$ ); and the last holds because if  $g^! \mathcal{O}_X$  is invertible then by Corollary 1.5.3,  $(fg)^! \mathcal{O}_Y$  is invertible as well.  $\square$

The last assertion of Proposition 2.5.6 expresses a composition property of quasi-Gorenstein homomorphisms. Here is a decomposition property:

**Proposition 2.5.7** (cf. [Avramov and Foxby 1992, 4.6], [Iyengar and Sather-Wagstaff 2004, 5.5]). *Let  $Z \xrightarrow{g} X \xrightarrow{f} Y$  be scheme-maps, with  $g$  perfect.*

*If  $fg$  is quasi-Gorenstein then  $g$  is Gorenstein.*

*Suppose  $g$  is faithfully flat, or proper and surjective. If  $fg$  is quasi-Gorenstein (resp. Gorenstein) then so is  $f$ .*

*Proof.* By Remark 2.1.6, one has  $g^! \mathcal{O}_X \in D_c^b(Z)$  and an isomorphism

$$g^! \mathcal{O}_X \otimes_Z^L \mathbb{L}g^* f^! \mathcal{O}_Y \rightarrow g^! f^! \mathcal{O}_Y \simeq (fg)^! \mathcal{O}_Y.$$

Also, the paragraph immediately before §5.5 in [Nayak 2009] yields  $f^! \mathcal{O}_Y \in D_c(X)$ , whence  $\mathbb{L}g^* f^! \mathcal{O}_Y \in D_c(Z)$ . Now Corollary 1.5.3(2) gives the first assertion. It also shows that  $\mathbb{L}g^* f^! \mathcal{O}_Y$  is invertible, whence so is  $f^! \mathcal{O}_Y$  if  $g$  is faithfully flat, or proper and surjective (see Corollary 2.2.7), giving the quasi-Gorenstein part of the second assertion. The last assertion in Proposition 2.5.2 now gives the Gorenstein part.  $\square$

From Propositions 2.5.2, 2.5.4 and 2.5.6 one gets:

**Corollary 2.5.8.** *Let there be given a commutative diagram*

$$\begin{array}{ccc} X' & \xrightarrow{v} & X \\ h \downarrow & & \downarrow f \\ Y' & \xrightarrow{u} & Y \end{array}$$

*with  $u$  quasi-Gorenstein and  $v$  perfect.*

*Then  $\mathbb{L}v^* G(f) \subseteq G(h)$  and  $v^! G(f) \subseteq G(h)$ . Thus, when  $f$  is  $G$ -perfect so is  $h$ .*  $\square$

It is shown in [Illusie 1971, p. 245, 3.5.2] that relative perfection is preserved under tor-independent base change. Here is an analog (and more) for relative  $G$ -perfection.

**Proposition 2.5.9.** *Let there be given a tor-independent fiber square (see §B.2)*

$$\begin{array}{ccc} X' & \xrightarrow{v} & X \\ h \downarrow & & \downarrow f \\ Y' & \xrightarrow{u} & Y \end{array}$$

If the map  $u$  is Gorenstein, or flat, or if  $u$  is perfect and  $f$  is proper, then  $Lv^*G(f) \subseteq G(h)$ . In particular, if  $f$  is  $G$ -perfect then so is  $h$ .

Conversely, suppose that  $u$  is faithfully flat, or that  $u$  is perfect, proper, and surjective and  $f$  is proper. If  $F \in D_c^b(X)$  and  $Lv^*F \in G(h)$ , then  $F \in G(f)$ .

*Proof.* In all cases,  $u$  is perfect, whence so is  $v$  [Illusie 1971, p. 245, 3.5.2].

If  $u$  is Gorenstein, the assertion is contained in Corollary 2.5.8.

By Lemma 2.2.2, if  $F$  is  $f^!\mathcal{O}_Y$ -reflexive then  $Lv^*F$  is  $Lv^*f^!\mathcal{O}_Y$ -reflexive.

If  $u$  (hence  $v$ ) is flat then by B.4, one has

$$Lv^*f^!\mathcal{O}_Y \cong h^!Lu^*\mathcal{O}_Y = h^!\mathcal{O}_{Y'}. \quad (2.5.9.1)$$

Thus  $v^*F$  is  $h^!\mathcal{O}_{Y'}$ -reflexive, i.e.,  $v^*F \in G(h)$ .

The case when  $u$  is perfect and  $f$  is proper is treated similarly through the tor-independent base-change theorem [Lipman 2009, 4.4.3].

For the converse, the assumption is, in view of the isomorphism (2.5.9.1), that  $Lv^*F$  is derived  $Lv^*f^!\mathcal{O}_Y$ -reflexive. Formula (2.1.6.3) gives that  $f^!\mathcal{O}_Y \in D_c^+(X)$ . So since  $v$  satisfies all the same hypotheses as  $u$  does, Theorem 2.2.5 yields that  $F$  is  $f^!\mathcal{O}_Y$ -reflexive, as asserted.  $\square$

**Proposition 2.5.10.** *Let there be given a tor-independent fiber square (see B.2)*

$$\begin{array}{ccc} X' & \xrightarrow{v} & X \\ h \downarrow & & \downarrow f \\ Y' & \xrightarrow{u} & Y \end{array}$$

with either  $u$  flat, or  $u$  perfect and  $f$  proper.

If the map  $f$  is quasi-Gorenstein (resp. Gorenstein) then so is  $h$ .

The converse holds if  $u$  (hence  $v$ ) is faithfully flat, or if  $u$  (hence  $v$ ) is perfect, proper and surjective and  $f$  is proper.

*Proof.* As in the proof of Proposition 2.5.9, one has the isomorphism (2.5.9.1). Hence if  $f^!\mathcal{O}_Y$  is invertible then so is  $h^!\mathcal{O}_{Y'}$  (see Corollary 1.5.3(3)), whence the first quasi-Gorenstein assertion, whose converse follows from Corollary 2.2.7(c). Also, by [Illusie 1971, p. 245, 3.5.2], if  $f$  is perfect then so is  $h$ , whence the first Gorenstein assertion, whose converse follows from the preceding converse and Proposition 2.5.1 (since  $u$  perfect and  $h$  perfect implies  $hu = fv$  perfect).  $\square$

### 3. Rigidity over schemes

As in previous sections, schemes are assumed to be noetherian, and scheme-maps to be essentially of finite type, and separated.

**3.1. Rigid complexes.** Fix a scheme  $X$  and a semidualizing  $\mathbb{O}_X$ -complex  $A$ , and for any  $F \in D(X)$  set

$$F^\dagger := R\mathcal{H}om_X(F, A).$$

**Definition 3.1.1.** An  $A$ -rigid pair  $(F, \rho)$  is one where  $F \in D_c^b(X)$  and  $\rho$  is a  $D(X)$ -isomorphism

$$\rho : F \xrightarrow{\sim} R\mathcal{H}om_X(F^\dagger, F).$$

An  $\mathbb{O}_X$ -complex  $F$  is  $A$ -rigid if there exists a  $\rho$  such that  $(F, \rho)$  is an  $A$ -rigid pair. Such a  $\rho$  is called an  $A$ -rigidifying isomorphism for  $F$ .

A morphism of  $A$ -rigid pairs  $(F, \rho) \rightarrow (G, \sigma)$  is a  $D(X)$ -map  $\phi : F \rightarrow G$  such that the following diagram, with  $\tilde{\phi} : R\mathcal{H}om_X(F^\dagger, F) \rightarrow R\mathcal{H}om_X(G^\dagger, G)$  the map induced by  $\phi$ , commutes:

$$\begin{array}{ccc} F & \xrightarrow{\rho} & R\mathcal{H}om_X(F^\dagger, F) \\ \phi \downarrow & & \downarrow \tilde{\phi} \\ G & \xrightarrow{\sigma} & R\mathcal{H}om_X(G^\dagger, G) \end{array}$$

The terminology “rigid” is motivated by the fact, contained in Theorem 3.2.1, that *the only automorphism of an  $A$ -rigid pair is the identity.*

**Example 3.1.2.** If  $R$  is a ring,  $X = \text{Spec } R$ , and  $M, C \in D_f^b(R)$  are such that  $R\text{Hom}_R(M, C) \in D_f^b(R)$ , then by Example 1.1.3,  $M$  is  $C$ -rigid in the sense of [Avramov et al. 2010a, §7] if and only if  $M^\sim$  is  $C^\sim$ -rigid in the present sense.

Since  $R\mathcal{H}om$  commutes with restriction to open subsets, an  $A$ -rigid pair restricts over any open  $U \subseteq X$  to an  $A|_U$ -rigid pair. However, rigidity is *not* a local condition: any invertible sheaf  $F$  is  $F$ -rigid, but  $\mathbb{O}_X$  is not  $F$ -rigid unless  $F \cong \mathbb{O}_X$ .

On the other hand, *rigid pairs glue*, in the sense explained in Theorem 6 of the Introduction, and generalized in Theorem 3.2.9 below.

The central result of this section, Theorem 3.1.7, a globalization of [Avramov et al. 2010a, 7.2], is that any  $A$ -rigid  $F$  is isomorphic in  $D(X)$  to  $i_*i^*A$ , with  $i$  the inclusion into  $X$  of some open-and-closed subscheme—necessarily the support of  $F$ , see (1.1.2.1); or equivalently,  $F \simeq IA$  for some idempotent  $\mathbb{O}_X$ -ideal  $I$ , uniquely determined by  $F$  (see Appendix C); or equivalently,  $F$  is, in  $D(X)$ , a direct summand of  $A$ .

**Example 3.1.3.** The pair  $(A, \rho^A)$  with  $\rho^A$  the natural composite isomorphism

$$\rho^A : A \xrightarrow{\sim} R\mathcal{H}om_X(\mathbb{O}_X, A) \xrightarrow{\sim} R\mathcal{H}om_X(R\mathcal{H}om_X(A, A), A),$$

is  $A$ -rigid.

Extending this example a little leads to:



**Example 3.1.4.** Let  $U \subseteq X$  be an open-and-closed subset, and  $i : U \hookrightarrow X$  the inclusion. Recall that the  $\mathbb{O}_U$ -module  $i^*A$  is semidualizing; see Corollary 2.2.6. If  $F \in \mathbf{D}(U)$  is  $i^*A$ -rigid then  $i_*F$  is  $A$ -rigid.

Indeed, if  $\rho$  is an  $i^*A$ -rigidifying isomorphism for  $F$ , then one has isomorphisms

$$\begin{aligned} i_*F &\xrightarrow[i_*\rho]{\sim} i_*\mathbf{R}\mathcal{H}om_U(\mathbf{R}\mathcal{H}om_U(F, i^*A), F) \\ &\xrightarrow{\sim} i_*\mathbf{R}\mathcal{H}om_U(i^*\mathbf{R}\mathcal{H}om_X(i_*F, A), F) \\ &\xrightarrow{\sim} \mathbf{R}\mathcal{H}om_X(\mathbf{R}\mathcal{H}om_X(i_*F, A), i_*F), \end{aligned}$$

where the second comes from (B.1.5) (since  $i^*i_*F = F$ ), and the third is a special case of [Lipman 2009, p. 98, (3.2.3.2)] (or see [Lipman 2009, §3.5.4], or just reason directly, using that  $i_*F$  vanishes outside  $U$ ).

The composition of these isomorphisms is  $A$ -rigidifying for  $i_*F$ .

**Definition 3.1.5.** The  $U$ -canonical  $A$ -rigid pair  $(i_*i^*A, \rho^{i_*i^*A})$  is the one constructed in Example 3.1.4 out of the  $i^*A$ -rigid pair  $(i^*A, \rho^{i^*A})$  in Example 3.1.3.

It is well known that *any* monomorphism (resp. epimorphism) in  $\mathbf{D}(X)$  is split, i.e., has a left (resp. right) inverse (see e.g., [Lipman 2009, 1.4.2.1]). Thus, when we speak of mono- or epimorphisms, the adjective “split” will usually be omitted.

**Lemma 3.1.6.** *Let  $\theta : F \hookrightarrow A$  be a monomorphism in  $\mathbf{D}(X)$ . Let  $(A, \rho^A)$  be the canonical  $A$ -rigid pair in Example 3.1.3. There exists a unique  $A$ -rigidifying isomorphism  $\rho$  for  $F$  such that  $\theta$  is a morphism of rigid pairs  $(F, \rho) \rightarrow (A, \rho^A)$ .*

*Proof.* It suffices to deal with the situation separately over each connected component of  $X$ ; so we may assume that  $X$  is connected. Then, by Lemma 1.3.7, either  $F = 0$  or  $\theta$  is an isomorphism. In either case the assertion is obvious.  $\square$

**Theorem 3.1.7.** *For any  $F \in \mathbf{D}(X)$ , the following conditions are equivalent.*

- (i)  $F$  is  $A$ -rigid.
- (ii) In  $\mathbf{D}(X)$ ,  $F \simeq I \otimes_X^\mathbb{L} A \simeq IA \simeq I^\dagger$  for some idempotent  $\mathbb{O}_X$ -ideal  $I$ .
- (iii) There is an open-and-closed  $U \subseteq X$  such that,  $i : U \hookrightarrow X$  being the inclusion,  $F \simeq i_*i^*A$  in  $\mathbf{D}(X)$ , whence  $U = \text{Supp}_X F$ .
- (iv) There is, in  $\mathbf{D}(X)$ , a monomorphism  $F \rightarrow A$ .

When they hold, there is a unique ideal  $I$  satisfying condition (ii).

*Proof.* (iii)  $\Rightarrow$  (i). In view of Example 3.1.3, this is contained in Example 3.1.4.

(i)  $\Rightarrow$  (iii). For the last assertion in (iii), since  $i_*i^*A$  vanishes outside  $U$ , and since for all  $x \in U$  one has, in  $\mathbf{D}(\mathbb{O}_{U,x})$ ,

$$0 \neq \mathbb{O}_{U,x} \simeq (\mathbf{R}\mathcal{H}om_U(i^*A, i^*A))_x \simeq \mathbf{R}\mathcal{H}om_{\mathbb{O}_{U,x}}((i_*i^*A)_x, (i_*i^*A)_x)$$

therefore  $U = \text{Supp}_X(i_*i^*A)$ .

Now let  $F$  be  $A$ -rigid. Then  $U := \text{Supp}_X F$  is an open-and-closed subset of  $X$ . For,  $X$  is covered by open subsets of the form  $V = \text{Spec } R$ ; and with  $j : V \hookrightarrow X$  the inclusion, the  $j^*A$ -rigid complex  $j^*F$  (resp. its homology) is the sheafification of  $F_V := R\Gamma(V, F)$  (resp. its homology), so  $(\text{Supp}_X F) \cap V = \text{Supp}_R F_V$ . But  $F_V$  is  $R\Gamma(V, A)$ -rigid (since  $(F_V)^\sim \cong j^*F$  is  $j^*A$ -rigid), so by [Avramov et al. 2010a, 7.2],  $\text{Supp}_R F_V = U \cap V$  is an open-and-closed subset of  $V$ . That  $U$  is open-and-closed follows.

Hence, the natural map  $F \rightarrow i_*i^*F$  is a  $D(X)$ -isomorphism; so to prove the theorem we can replace  $(X, A, F)$  by  $(U, i^*A, i^*F)$ , i.e., we may assume  $\text{Supp}_X F = X$ .

In  $D(X)$ , the complex  $L := F^\dagger$  is isomorphic to  $H^0L$ , which is an invertible sheaf: this assertion need only be checked locally, i.e., for affine  $X$ , where it is given by [Avramov et al. 2010a, 4.9]. (The assumptions of that theorem are satisfied because  $F$  and  $A$  are both in  $D_c^b(X)$ .) The invertible complex  $L$  is derived  $A$ -reflexive (take  $F = \mathbb{O}_X$  in 1.5.4(2)); similarly, so is  $L \otimes_X^L L$ . Since  $\text{Supp}_X A = X$ , by Lemma 1.3.7, therefore Proposition 1.3.3(iii) yields that  $F$  is derived  $A$ -reflexive. So  $L^\dagger \simeq F$ , and

$$L^\dagger \simeq R\mathcal{H}om_X(L, L^\dagger) \simeq (L \otimes_X^L L)^\dagger \quad (\text{see (1.1.1.1)}).$$

Applying the functor  $^\dagger$  to these isomorphisms we get  $L \otimes_X^L L \simeq L$ . Tensoring with  $L^{-1}$  shows then that  $L \simeq \mathbb{O}_X$ . Thus  $F \simeq L^\dagger \simeq A$ .

(iii)  $\Rightarrow$  (ii). Associated to any open-and closed  $U \subseteq X$  is the unique idempotent  $\mathbb{O}_X$ -ideal  $I$  that is isomorphic to  $i_*\mathbb{O}_U$  (Corollary C.3). For this  $I$  we have natural isomorphisms, the second from (B.1.3) and the last two from Corollary C.4:

$$i_*i^*A \simeq i_*(\mathbb{O}_U \otimes_U^L i^*A) \simeq i_*\mathbb{O}_U \otimes_X^L A \simeq I \otimes_X^L A \simeq IA \simeq I^\dagger.$$

(ii)  $\Rightarrow$  (iii). Given  $I$  as in (ii), let  $U = \text{Supp}_X I$ , with inclusion  $i : U \hookrightarrow X$ , and use the preceding isomorphisms (see Corollary C.3).

(iii)  $\Rightarrow$  (iv). If  $i$  is as in (iii), then  $i_*i^*A$  is a direct summand of  $A$ .

(iv)  $\Rightarrow$  (i). See Lemma 3.1.6.

It remains to note that the uniqueness of  $I$  in (ii) results from

$$\text{Supp}_X IA = \text{Supp}_X (I \otimes_X^L A) = \text{Supp}_X I \cap \text{Supp}_X A = \text{Supp}_X I \cap X = \text{Supp}_X I,$$

see (1.1.2.2). The proof of Theorem 3.1.7 is now completed. □

Define a *direct decomposition* of  $F \in D(X)$  to be a  $D(X)$ -isomorphism

$$F \simeq F_1 \oplus F_2 \oplus \cdots \oplus F_n \tag{3.1.7.1}$$

such that no  $F_i$  vanishes; call  $F$  *indecomposable* if  $F \neq 0$  and in any direct decomposition of  $F$ , one has  $n = 1$ . Say that (3.1.7.1) is an *orthogonal decomposition* of  $F$  if, in addition,  $F_i \otimes_X^L F_j = 0$  for all  $i \neq j$ .

**Corollary 3.1.8.** *Let  $F \neq 0$  be an  $A$ -rigid complex. Let  $\text{Supp}_X F = \bigsqcup_{s=1}^n U_s$  be a decomposition into disjoint nonempty connected closed subsets, and  $i_s : U_s \hookrightarrow X$  ( $1 \leq s \leq n$ ) the canonical inclusions.*

*The  $U_s$  are then connected components of  $X$ , and there is an orthogonal decomposition into indecomposable  $A$ -rigid complexes:*

$$F \simeq \bigoplus_{s=1}^n (i_s)_*(i_s)^*A.$$

*If  $F \simeq F_1 \oplus \cdots \oplus F_r$  is a direct decomposition with each  $F_t$  indecomposable, then  $r = n$  and (after renumbering) there is for each  $s$  an isomorphism  $F_s \simeq (i_s)_*(i_s)^*A$ .*

*Proof.* Since by Theorem 3.1.7(iii),  $\text{Supp}_X F$  is open and closed in  $X$ , therefore each  $U_s$  is a connected component of  $X$ . Moreover, if  $i : \text{Supp}_X F \hookrightarrow X$  is the inclusion, then  $i^*A$  is semidualizing (Corollary 2.2.6), and compatibility of  $\mathcal{R}\text{Hom}$  with open immersions (to see which, use [Lipman 2009, 2.4.5.2]) implies that  $i^*F$  is  $i^*A$ -rigid. It follows then from Theorem 3.1.7(iii) that we may assume  $F = A$ .

The decomposition  $X = \bigsqcup_{s=1}^n U_s$  now yields a decomposition of  $F \in \mathcal{D}(X)$ :

$$F \simeq \bigoplus_{s=1}^n (i_s)_*(i_s)^*F = \bigoplus_{s=1}^n (i_s)_*(i_s)^*A.$$

As before,  $(i_s)^*A$  is a semidualizing complex of  $\mathbb{C}_{U_s}$ -modules, so its support is  $U_s$ , and it is indecomposable; see Lemma 1.3.7. Hence  $(i_s)_*(i_s)^*A$  is indecomposable, and has support  $U_s$ . It then follows from (1.1.2.2) that the decomposition above is orthogonal. Moreover, the complexes  $(i_s)_*(i_s)^*A$  are  $A$ -rigid; see Definition 3.1.5.

Let  $F \simeq F_1 \oplus \cdots \oplus F_r$  be a direct decomposition. It results from Lemma 1.3.7 that this decomposition is orthogonal. Hence  $X = \text{Supp}_X F = \bigsqcup_{t=1}^r V_t$ . Furthermore,  $F \in \mathcal{D}_c^b(X) \implies F_t \in \mathcal{D}_c^b(X)$  for all  $t$ . Hence  $V_t = \text{Supp}_X F_t$  is open and closed; and since  $F_t$  is indecomposable,  $V_t$  is connected. Thus the  $V_t$  are the connected components of  $X$ . In particular,  $r = n$ , and, after renumbering, one may assume  $V_t = U_t$  for each  $t$ . It remains to observe that  $F_s \simeq (i_s)_*(i_s)^*F \simeq (i_s)_*(i_s)^*A$ .  $\square$

**3.2. Morphisms of rigid complexes.** We present elaborations of Theorem 3.1.7, leading to a simple description of the skeleton of the category of rigid pairs; see Theorem 3.2.6 and Remark 3.2.7.

The result below involves the  $H^0(X, \mathbb{C}_X)$  action on  $\mathcal{D}(X)$  described in 1.2.

**Theorem 3.2.1.** *If  $(F, \rho)$ ,  $(F', \rho')$  are  $A$ -rigid pairs with  $\text{Supp}_X F = \text{Supp}_X F'$  then there exists a unique isomorphism  $(F, \rho) \xrightarrow{\sim} (F', \rho')$ . In particular, any  $A$ -rigid pair  $(F, \rho)$  admits a unique isomorphism into a  $U$ -canonical one, for some open-and-closed  $U \subseteq X$ , necessarily the support of  $F$ .*

Moreover, if  $F' = F$  then with  $U_F := \text{Supp}_X F$ , there is a unique unit  $u$  in the ring  $H^0(U_F, \mathbb{O}_{U_F})$  such that  $\rho' = \rho \bar{u}$ , where  $\bar{u} \in H^0(X, \mathbb{O}_X)$  is  $u$  extended by 0, and the unique isomorphism  $(F, \rho) \xrightarrow{\sim} (F, \rho')$  is multiplication in  $F$  by  $\bar{u}$ .

For any endomorphism  $\phi$  of the  $A$ -rigid pair  $(F, \rho)$  there is a uniquely determined idempotent  $u \in H^0(U_F, \mathbb{O}_{U_F})$  such that  $\phi$  is multiplication by  $\bar{u}$ .

*Proof.* Modulo Theorem 3.1.7, the proof is basically that of [Avramov et al. 2010a, 7.3]. Indeed, Theorem 3.1.7(iii) implies that we may assume  $F = F'$ , and that furthermore, we may replace  $X$  by  $U$ , i.e., assume  $F = A$  (so that  $\bar{u} = u$ ).

Each endomorphism of  $F$  is multiplication by a unique element  $u$  in  $H^0(X, \mathbb{O}_X)$ . From Lemma 1.2.1 it follows that multiplication by  $u$  induces multiplication by  $u$  on  $F^\dagger$  and multiplication by  $u^2$  on  $R\mathcal{H}om_X(F^\dagger, F)$ . With  $u_F$ , resp.  $u_H$ , multiplication by  $u$  on  $F$ , resp. on  $R\mathcal{H}om_X(F^\dagger, F)$ , we have then that  $u_H \rho = \rho u_F$ , see 1.2, so that  $u_H^2 \rho = u_H \rho u_F = \rho u_F^2$ .

In view of this identity, one gets that  $u_F$  is an isomorphism from the rigid pair  $(F, \rho)$  to the rigid pair  $(F, \rho') \iff \rho' u_F = u_H^2 \rho \iff \rho' u_F = \rho u_F^2 \iff \rho' = \rho u_F$ . Thus the sought-after  $u$  is the unique one such that  $u_F$  is the automorphism  $\rho^{-1} \rho'$ .

In the same vein, when  $u_F$  induces an endomorphism of the rigid pair  $(F, \rho)$  one gets a relation  $\rho u = \rho u^2$ , whence,  $\rho$  being an isomorphism,  $u^2 = u$ .  $\square$

**Corollary 3.2.2.** *For any  $A$ -rigid complex  $F$ , the group of automorphisms of  $F$  acts faithfully and transitively on the set of rigidifying isomorphisms  $\rho$  of  $F$ .*

$\square$

**Corollary 3.2.3.** *If  $X$  is connected then every nonzero morphism of  $A$ -rigid pairs is an isomorphism.*

$\square$

**Definition 3.2.4.** For any  $D(X)$ -map  $\phi : F \rightarrow F'$  of  $A$ -rigid pairs,  $\text{Supp}_X \phi$  is the union of those connected components of  $X$  to which the restriction of  $\phi$  is nonzero.

By Corollary 3.2.3, if  $X$  is connected then nonzero maps of  $A$ -rigid pairs are isomorphisms. So for a composable pair  $(\phi, \psi)$  of maps of  $A$ -rigid pairs,

$$\text{Supp}_X(\phi\psi) = \text{Supp}_X \phi \cap \text{Supp}_X \psi. \tag{3.2.4.1}$$

**Corollary 3.2.5.** *Let  $(F, \rho)$  and  $(F', \rho')$  be  $A$ -rigid pairs.*

- (1) *Suppose that  $\text{Supp}_X F \subseteq \text{Supp}_X F'$ . Then there is a unique monomorphism  $(F, \rho) \hookrightarrow (F', \rho')$  and a unique epimorphism  $(F', \rho') \twoheadrightarrow (F, \rho)$ .*
- (2) *For any morphism  $\phi : (F, \rho) \rightarrow (F', \rho')$ , if  $(G, \sigma)$  is an  $A$ -rigid pair with  $\text{Supp}_X G = \text{Supp}_X \phi$  then  $\phi$  factors uniquely as*

$$(F, \rho) \xrightarrow{\phi'} (G, \sigma) \xrightarrow{\phi''} (F', \rho')$$

*with  $\phi'$  an epimorphism and  $\phi''$  a monomorphism.*

*Thus  $\phi$  is uniquely determined by its source, target and support.*

*Proof.* Looking at connected components separately, one reduces to where  $X$  is connected; the assertions then follow from Corollary 3.2.3 and Theorem 3.2.1.  $\square$

Here is a *structure theorem* for the category  $\mathrm{Rp}_A(X)$  of  $A$ -rigid pairs.

**Theorem 3.2.6.** *Let  $\mathrm{OC}(X)$  be the category whose objects are the open-and-closed subsets of  $X$ , and whose maps  $U \rightarrow V$  are the open-and-closed subsets of  $U \cap V$ , the composition of  $S \subseteq U \cap V$  and  $T \subseteq V \cap W$  being  $S \cap T \subseteq U \cap W$ .*

*Let  $\Psi : \mathrm{Rp}_A(X) \rightarrow \mathrm{OC}(X)$  be the functor taking  $(F, \rho) \in \mathrm{Rp}_A(X)$  to  $\mathrm{Supp}_X F$ , and taking a morphism  $\phi \in \mathrm{Rp}_A(X)$  to  $\mathrm{Supp}_X \phi$  (see (3.2.4.1)).*

*This  $\Psi$  is an equivalence of categories.*

*Proof.* Let  $(F, \rho)$  and  $(F', \rho')$  be  $A$ -rigid pairs,  $U := \mathrm{Supp}_X F$ ,  $V := \mathrm{Supp}_X F'$ , and  $S$  an open-and-closed subset of  $U \cap V$ . It follows from Corollary 3.2.5, with  $(G, \sigma)$  the  $S$ -canonical pair, that there is a unique map of  $A$ -rigid pairs  $\phi : (F, \rho) \rightarrow (F', \rho')$  such that  $\mathrm{Supp}_X \phi = S$ , whence the conclusion.  $\square$

**Remark 3.2.7.** A quasi-inverse  $\Phi$  of  $\Psi$  can be constructed as follows:

$\Phi : \mathrm{OC}(X) \rightarrow \mathrm{Rp}_A(X)$  takes an open-and-closed  $U \subseteq X$  to an arbitrarily chosen rigid pair  $(F, \rho)$  with  $\mathrm{Supp}_X F = U$ ; and then, for any  $\mathrm{OC}(X)$ -map  $S \subseteq U \cap V$ ,  $\Phi(S)$  is the unique epimorphism  $\Phi U \twoheadrightarrow \Phi S$  followed by the unique monomorphism  $\Phi S \hookrightarrow \Phi V$  (see Corollary 3.2.5).

That this describes a functor is, modulo (3.2.4.1), straightforward to see.

Taking into account that the map  $S \subseteq U \cap V$  factors as a split epimorphism (namely  $S \subseteq U \cap S$ ) followed by a split monomorphism (namely  $S \subseteq S \cap V$ ), and that any functor respects left and right inverses, one sees that in fact *all* quasi-inverses of  $\Psi$  have the preceding form.

In particular, there is a canonical  $\Phi$ , associating to each  $U$  the  $U$ -canonical pair. Thus  $\mathrm{OC}(X)$  is *canonically isomorphic* to the category of canonical  $A$ -rigid pairs.

The next result is in preparation for establishing a gluing property for rigid pairs.

**Lemma 3.2.8.** *If  $g : Z \rightarrow X$  is a perfect map and  $F$  is an  $A$ -rigid complex in  $\mathrm{D}(X)$ , then  $\mathrm{Lg}^*A \in \mathrm{D}_c^b(Z)$  is semidualizing and  $\mathrm{Lg}^*F$  is  $(\mathrm{Lg}^*A)$ -rigid.*

*Proof.* That  $\mathrm{Lg}^*A$  is semidualizing is given by Corollary 2.2.6.

If  $\rho$  is an  $A$ -rigidifying isomorphism for  $F \in \mathrm{D}(X)$ , then, abusing notation, we let  $\mathrm{Lg}^*\rho$  be the composed isomorphism

$$\begin{aligned} \mathrm{Lg}^*F &\xrightarrow{\sim} \mathrm{Lg}^*\mathrm{R}\mathcal{H}om_X(F^\dagger, F) \\ &\xrightarrow{\sim} \mathrm{R}\mathcal{H}om_Z(\mathrm{Lg}^*F^\dagger, \mathrm{Lg}^*F) \\ &\xrightarrow{\sim} \mathrm{R}\mathcal{H}om_Z(\mathrm{R}\mathcal{H}om_Z(\mathrm{Lg}^*F, \mathrm{Lg}^*A), \mathrm{Lg}^*F), \end{aligned}$$

where the first isomorphism is the result of applying the functor  $\mathrm{Lg}^*$  to  $\rho$ , and the other two come from (2.1.7.1). Thus  $\mathrm{Lg}^*\rho$  is  $(\mathrm{Lg}^*A)$ -rigidifying for  $\mathrm{Lg}^*F$ .  $\square$

**Theorem 3.2.9.** *Let  $g : Z \rightarrow X$  be a faithfully flat scheme-map,  $W := Z \times_X Z$ ,  $\pi_1 : W \rightarrow Z$  and  $\pi_2 : W \rightarrow Z$  the canonical projections.*

*Let  $A \in \mathcal{D}(X)$  be semidualizing. If  $(G, \sigma)$  is a  $(g^*A)$ -rigid pair such that there exists an isomorphism  $\pi_1^*G \simeq \pi_2^*G$ , then there is, up to unique isomorphism, a unique  $A$ -rigid pair  $(F, \rho)$  such that  $(G, \sigma) \simeq (g^*F, g^*\rho)$ .*

*Proof.* (Uniqueness.) If  $g^*F \simeq g^*F'$  then, since

$$g^{-1} \operatorname{Supp}_X F = \operatorname{Supp}_Z g^*F = \operatorname{Supp}_Z g^*F' = g^{-1} \operatorname{Supp}_X F',$$

and  $g$  is surjective, therefore  $\operatorname{Supp}_X F = \operatorname{Supp}_X F'$ ; and so by Theorem 3.2.1, there is a unique isomorphism  $(F, \rho) \xrightarrow{\sim} (F', \rho')$ .

(Existence.) In view of Theorem 3.1.7, we may assume that  $G = Jg^*A$  for some idempotent  $\mathbb{O}_Z$ -ideal  $J$ . Then, for  $i = 1, 2$ , Corollaries C.4 and C.7 yield

$$\begin{aligned} \operatorname{Supp}_W \pi_i^*G &= \operatorname{Supp}_W (\pi_i^*J \otimes_W^L \pi_i^*g^*A) \\ &= \operatorname{Supp}_W \pi_i^*J \cap \operatorname{Supp}_W \pi_i^*g^*A \\ &= \operatorname{Supp}_W \pi_i^*J. \end{aligned}$$

So  $\pi_1^*J$  and  $\pi_2^*J$ , being isomorphic to idempotent ideals with the same support, must be isomorphic. Hence by Proposition C.8, there is a unique idempotent  $\mathbb{O}_X$ -ideal  $I$  such that  $J = I\mathbb{O}_Z$ . If  $F = IA$  then  $G \simeq g^*F$ .

Let  $\rho$  be a rigidifying isomorphism for  $F$ , so that  $(g^*F, g^*\rho)$  is a  $(g^*A)$ -rigid pair. By Theorem 3.2.1, there is a unique isomorphism  $(g^*F, g^*\rho) \xrightarrow{\sim} (G, \sigma)$ .  $\square$

**Remark 3.2.10.** In view of 3.2.1 and 3.2.8, the hypothesis  $\pi_1^*G \simeq \pi_2^*G$  in 3.2.9 means simply that  $\operatorname{Supp}_W \pi_1^*G = \operatorname{Supp}_W \pi_2^*G$ .

**3.3. Relative rigidity.** With reference to a  $G$ -perfect map  $f : X \rightarrow Y$ , we take particular interest in those complexes that are  $f^!\mathbb{O}_Y$ -rigid—complexes we will simply call  *$f$ -rigid*.

For  $g$  any essentially étale map (so that, by Proposition 2.5.2,  $fg$  is  $G$ -perfect), there is a natural isomorphism of functors  $(fg)^! \simeq g^*f^!$  (see B.3). By Lemma 3.2.8, if  $P$  is  $f$ -rigid then  $g^*P$  is  $(fg)$ -rigid.

The following *étale gluing* result (where for simplicity we omit mention of rigidifying isomorphisms) is an immediate consequence of Theorem 3.2.9.

**Proposition 3.3.1.** *Let  $Z \xrightarrow{g} X \xrightarrow{f} Y$  be scheme-maps, where  $f$  is  $G$ -perfect and  $g$  is essentially étale and surjective. Let  $W := Z \times_X Z$ , with canonical projections  $\pi_1 : W \rightarrow Z$  and  $\pi_2 : W \rightarrow Z$ . If  $P$  is an  $(fg)$ -rigid complex such that there exists an isomorphism  $\pi_1^*P \simeq \pi_2^*P$ , then there exists, up to isomorphism, a unique  $f$ -rigid complex  $F$  with  $g^*F \simeq P$ .  $\square$*

Fix a semidualizing complex  $A$  on a scheme  $X$ . The main result in this section, Theorem 3.3.2, is that for any additive functor from  $A$ -rigid complexes to the derived category of some scheme, that takes  $A$  to a semidualizing complex  $C$  — and hence, by Theorem 3.1.7(iv), takes  $A$ -rigid complexes to  $C$ -rigid complexes — there is a unique lifting to the category of  $A$ -rigid pairs that takes the canonical pair  $(A, \rho^A)$  to  $(C, \rho^C)$ , provided that the functor “respects intersection of supports”.

From Theorem 3.3.2 we will derive the behavior of relatively rigid complexes with respect to perfect maps (Corollaries 3.3.4 and 3.3.5). These results generalize, and were inspired by, results in [Yekutieli and Zhang 2004, Sections 3 and 6].

Let  $\text{Rc}_A(X) \subseteq \text{D}(X)$  be the full subcategory of  $A$ -rigid complexes, and let  $\text{Rp}_A(X)$  be the category of  $A$ -rigid pairs. Let  $\varphi_X : \text{Rp}_A(X) \rightarrow \text{D}(X)$  be the functor taking  $(F, \rho)$  to  $F \in \text{Rc}_A(X)$ . The rigid pair  $(A, \rho^A)$  is defined in Example 3.1.3.

**Theorem 3.3.2.** *Let  $X$  and  $Z$  be schemes, let  $A \in \text{D}(X)$  be semidualizing, and let  $F : \text{Rc}_A(X) \rightarrow \text{D}(Z)$  be an additive functor such that  $FA$  is semidualizing.*

*There exists at most one functor  $\bar{F} : \text{Rp}_A(X) \rightarrow \text{Rp}_{FA}(Z)$ , such that*

$$\varphi_Z \bar{F} = F \varphi_X \quad \text{and} \quad \bar{F}(A, \rho^A) = (FA, \rho^{FA}).$$

*For such an  $\bar{F}$  to exist it is necessary that for any idempotent  $\mathbb{O}_X$ -ideals  $I, J$ ,*

$$\text{Supp}_Z F(IJA) = \text{Supp}_Z F(IA) \cap \text{Supp}_Z F(JA), \quad (3.3.2.1)$$

*and it is sufficient that (3.3.2.1) hold whenever  $IJ = 0$ .*

**Remark 3.3.3.** Let  $a, b \in H^0(X, \mathbb{O}_X)$  be the idempotents such that  $I = a\mathbb{O}_X$  and  $J = b\mathbb{O}_X$ . Since  $IA$  admits a monomorphism into  $A$ , therefore  $F(IA)$  admits a monomorphism into  $FA$ , and it follows from Theorem 3.1.7 that there is a unique idempotent  $f(a) \in H^0(Z, \mathbb{O}_Z)$  with  $F(IA) \simeq f(a)FA$ . By (1.1.2.2), Corollary C.4, and the fact that a semidualizing complex on a scheme is supported at every point of the underlying space, see Lemma 1.3.7, condition (3.3.2.1) amounts then to  $f(ab) = f(a)f(b)$ .

Before proving Theorem 3.3.2, we gather together some examples. Part (1) of the next corollary elaborates Lemma 3.2.8.

Recall that if  $g : Z \rightarrow X$  is perfect then both  $\text{L}g^*B$  and  $g^!A$  are semidualizing; see Corollary 2.2.6. If  $L \in \text{D}(X)$  is invertible then  $L \otimes_X^L A$  is semidualizing, by Corollary 1.5.4(3); and if  $F \in \text{D}_{\text{qc}}^+(X)$ , then there is as in (2.1.6.2) a natural isomorphism  $g^!L \otimes_Z^L \text{L}g^*F \xrightarrow{\sim} g^!(L \otimes_X^L F)$ .

**Corollary 3.3.4.** *Let  $g : Z \rightarrow X$  be a perfect map, and  $A \in \text{D}_c^b(X)$  semidualizing.*

(1) *There is a unique functor  $g^{**} : \text{Rp}_A(X) \rightarrow \text{Rp}_{\text{L}g^*A}(Z)$  such that*

$$\varphi_Z g^{**} = \text{L}g^* \quad \text{and} \quad g^{**}(A, \rho^A) = (\text{L}g^*A, \rho^{\text{L}g^*A}).$$

(2) *There is a unique functor  $g^{!!} : \text{Rp}_A(X) \rightarrow \text{Rp}_{g^!A}(Z)$  such that*

$$\varphi_Z g^{!!} = g^! \quad \text{and} \quad g^{!!}(A, \rho^A) = (g^!A, \rho^{g^!A}).$$

(3) *For each invertible  $L \in \text{D}(X)$  there is a unique bifunctor*

$$g^\otimes : \text{Rp}_{g^!L}(Z) \times \text{Rp}_A(X) \rightarrow \text{Rp}_{g^!(L \otimes_X^L A)}(Z)$$

*such that*

$$\varphi_Z g^\otimes(P, F) = P \otimes_Z^L Lg^*F$$

*and*

$$g^\otimes((g^!L, \rho^{g^!L}), (A, \rho^A)) = (g^!(L \otimes_X^L A), \rho^{g^!(L \otimes_X^L A)}).$$

*Proof.* Corollary C.7 implies that for either functor, one has in Remark 3.3.3 that  $f(a)$  is the image of  $a$  under the natural map  $\text{H}^0(X, \mathbb{O}_X) \rightarrow \text{H}^0(Z, \mathbb{O}_Z)$ . Thus  $f(ab) = f(a)f(b)$  holds, and so (1) and (2) result from Theorem 3.3.2.

For (3) replace  $X$  in Theorem 3.3.2 by the disjoint union  $Z \sqcup X$ . For  $P \in \text{D}(Z)$  and  $F \in \text{D}(X)$ , let  $(P, F) \in \text{D}(Z \sqcup X)$  be the complex whose restriction to  $Z$  is  $P$  and to  $X$  is  $F$ . There is an obvious functor  $F : \text{D}(Z \sqcup X) \rightarrow \text{D}(Z)$  taking  $(P, F)$  to  $P \otimes_Z^L Lg^*F$ . This takes the semidualizing complex  $(g^!L, A)$  to the semidualizing complex  $g^!L \otimes_Z^L Lg^*A \simeq g^!(L \otimes_X^L A)$ . Using (1.1.2.2) and Remark 3.3.3, one verifies that (3.3.2.1) holds; and so (3) results.  $\square$

Recall that if  $Z \xrightarrow{g} X \xrightarrow{f} Y$  are maps such that  $g$  is perfect and  $f$  is  $G$ -perfect then  $fg$  is  $G$ -perfect (Proposition 2.5.2). Taking  $A = f^!\mathbb{O}_Y$  and  $L = \mathbb{O}_X$  in (2) and (3) of Corollary 3.3.4 one gets:

**Corollary 3.3.5.** *Let  $g : Z \rightarrow X$  be perfect, and  $f : X \rightarrow Y$   $G$ -perfect.*

(1) *If  $F$  is  $f$ -rigid then  $g^!F$  is  $fg$ -rigid.*

(2) *If  $P$  is  $g$ -rigid and  $F$  is  $f$ -rigid then  $P \otimes_Z^L Lg^*F$  is  $fg$ -rigid.*  $\square$

**Corollary 3.3.6.** *Let  $g : Z \rightarrow X$  be a proper map such that the natural map is an isomorphism  $\mathbb{O}_X \xrightarrow{\sim} \text{Rg}_*\mathbb{O}_Z$ . Let  $A \in \text{D}_{\text{qc}}^+(X)$  be such that  $g^!A$  is semidualizing.*

*Then  $A$  is semidualizing, the canonical map is an isomorphism  $\text{Rg}_*g^!A \xrightarrow{\sim} A$ , and there is a unique functor  $g_{**} : \text{Rp}_{g^!A}(Z) \rightarrow \text{Rp}_A(X)$  such that*

$$\varphi_X g_{**} = \text{Rg}_*\varphi_Z \quad \text{and} \quad g_{**}(g^!A, \rho^{g^!A}) = (\text{Rg}_*g^!A, \rho^{\text{Rg}_*g^!A}).$$

*Hence, if  $f : X \rightarrow Y$  is such that  $fg$  is  $G$ -perfect then  $f$  is  $G$ -perfect, and if  $P$  is  $fg$ -rigid then  $\text{Rg}_*P$  is  $f$ -rigid.*



*Proof.* That  $A$  is semidualizing is given by Proposition 2.2.4.

There are, for  $E \in D_{qc}(X)$ , natural isomorphisms, the second from B.3(ii), and the third from (B.1.3),

$$\begin{aligned} \mathrm{Hom}_{D(X)}(E, \mathrm{R}g_*g^!A) &\cong \mathrm{Hom}_{D(Z)}(\mathrm{L}g^*E, g^!A) \\ &\cong \mathrm{Hom}_{D(X)}(\mathrm{R}g_*(\mathbb{C}_Z \otimes_Z^{\mathrm{L}} \mathrm{L}g^*E), A) \\ &\cong \mathrm{Hom}_{D(X)}(\mathrm{R}g_*\mathbb{C}_Z \otimes_X^{\mathrm{L}} E, A) \cong \mathrm{Hom}_{D(X)}(E, A). \end{aligned}$$

It follows, via [Lipman 2009, 3.4.7(ii)], that the canonical map is an isomorphism

$$\mathrm{R}g_*g^!A \xrightarrow{\sim} A.$$

By assumption, one has the natural isomorphism  $H^0(X, \mathbb{C}_X) \xrightarrow{\sim} H^0(Z, \mathbb{C}_Z)$ . So there is a bijection between the idempotents in these two rings; and also,  $g$  is surjective. Hence  $g^{-1}$  gives a bijection from the open-and-closed subsets of  $X$  to the open-and-closed subsets of  $Z$ . Furthermore, for any  $P \in D_c^b(Z)$ , the support  $\mathrm{Supp}_Z P$  is closed, whence,  $g$  being proper,  $U := X \setminus g(\mathrm{Supp}_Z P)$  is open; and the restriction of  $P$  to  $g^{-1}U$  is acyclic. Thus  $\mathrm{Supp}_X \mathrm{R}g_*P \subseteq g(\mathrm{Supp}_Z P)$ . One now easily checks (3.3.2.1), with  $F = \mathrm{R}g_*$  and  $A$  replaced by  $g^!A$ , when  $IJ = 0$ —so that  $\mathrm{Supp}_Z(Ig^!A)$  and  $\mathrm{Supp}_Z(Jg^!A)$  are disjoint open-and-closed subsets of  $Z$ . The existence and uniqueness of  $g_{**}$  follows then from Theorem 3.3.2.

For the last assertion, take  $A = f^!\mathbb{C}_Y$ . □

**Corollary 3.3.7.** *Let there be given a tor-independent fiber square (see B.2)*

$$\begin{array}{ccc} X' & \xrightarrow{v} & X \\ h \downarrow & & \downarrow f \\ Y' & \xrightarrow{u} & Y \end{array}$$

in which  $f$  is  $G$ -perfect.

*If  $u$  is flat, or if  $u$  is perfect and  $f$  is proper, then  $h$  is  $G$ -perfect and for any  $f$ -rigid  $\mathbb{C}_X$ -complex  $F$ ,  $\mathrm{L}v^*F$  is  $h$ -rigid.*

*Proof.* Proposition 2.5.9 and [Illusie 1971, p. 245, 3.5.2] imply  $h$  is  $G$ -perfect and  $v$  is perfect. By Corollary 3.3.4(i),  $\mathrm{L}v^*F$  is  $\mathrm{L}v^*f^!\mathbb{C}_Y$ -rigid, i.e.,  $h^!\mathbb{C}_{Y'}$ -rigid; see (2.5.9.1). □

*Proof of Theorem 3.3.2. (Uniqueness.)* Let  $(G, \sigma)$  be an  $A$ -rigid pair.

Set  $(FG, \tau) := \bar{F}(G, \sigma)$ . Let  $\phi_G$  be the unique (split) monomorphism from  $(G, \sigma)$  to the canonical pair  $(A, \rho^A)$ , so that  $\bar{F}(\phi_G)$  is a (split) monomorphism, necessarily the unique one from  $(FG, \tau)$  to  $(FA, \sigma^{FA})$ , see Corollary 3.2.5. It follows then from Lemma 3.1.6 that  $\tau$  depends only on  $F$  and  $(G, \sigma)$ .

Also, for any morphism  $\phi$  of  $A$ -rigid pairs,  $\phi_Z \bar{F} = F$  implies  $\bar{F}\phi = F\phi$ .

(Necessity of (3.3.2.1)). Let  $\Psi_Z : \text{Rp}_{FA}(Z) \rightarrow \text{OC}(Z)$  be as in Theorem 3.2.6. Let  $\Phi : \text{OC}(X) \rightarrow \text{Rp}_A(X)$  be as in Remark 3.2.7, sending an open-and-closed  $U \subseteq X$  to  $I_U A$ , where  $I_U$  is the idempotent  $\mathbb{O}_X$ -ideal that is  $\mathbb{O}_U$  over  $U$  and  $(0)$  elsewhere. Then  $\Psi_Z \bar{F} \Phi : \text{OC}(X) \rightarrow \text{OC}(Z)$  respects composition of maps, i.e., (3.3.2.1) holds.

(Existence.) Since any functor preserves a map's property of being split—mono or epi—Theorem 3.1.7(iv) shows that  $F$  takes  $A$ -rigid complexes to  $FA$ -rigid complexes; and the preceding uniqueness argument shows how  $\bar{F}(G, \sigma)$  must be defined. It remains to prove that for any morphism  $\phi : (G, \sigma) \rightarrow (G', \sigma')$  of  $A$ -rigid pairs,  $F\phi$  is a morphism of  $FA$ -rigid pairs.

Let  $U_1, \dots, U_n$  be the connected components of  $X$ . For each  $j$ , let  $V_j$  be the support of the  $FA$ -rigid complex  $F(I_{U_j} A)$  (see above). The condition (3.3.2.1), for  $IJ = 0$ , guarantees that if  $j \neq k$  then the open-and-closed subsets  $V_j$  and  $V_k$  are disjoint. So we need only show that

(\*) *the restriction of  $F\phi$  over each  $V_j$  is a morphism of  $FA|_{V_j}$ -rigid pairs.*

Corollary 3.1.8 shows that  $\phi = \sum_{j=1}^n \phi_j$  where for each  $j$ , the source and target of  $\phi_j$  each have support that, if not empty, is  $U_j$ . Then, since  $F$  is additive,  $F\phi = \sum_{j=1}^n F\phi_j$ ; and the source and target of  $F\phi_j$  each have support contained in  $V_j$  (see the first assertion in Theorem 3.2.1). Hence the restriction of  $F\phi$  over  $V_j$  is  $F\phi_j$ . Proving (\*) is thus reduced to the case where  $X$  is connected, so that by Corollary 3.2.3,  $\phi$  is either 0 or an isomorphism.

If  $\phi = 0$ , (\*) is obvious. If  $\phi$  (hence  $F\phi$ ) is an isomorphism consider the diagram, where  $(FG, \tau) := \bar{F}(G, \sigma)$ ,  $(FG', \tau') := \bar{F}(G', \sigma')$ , where  $\phi_{G'}$  is as above, and where the maps on the right are induced by those on the left:

$$\begin{array}{ccc}
 FG & \xrightarrow{\tau} & R\mathcal{H}om_Z(R\mathcal{H}om_Z(FG, FA), FG) \\
 F\phi \downarrow & & \downarrow \xi \\
 FG' & \xrightarrow{\tau'} & R\mathcal{H}om_Z(R\mathcal{H}om_Z(FG', FA), FG') \\
 F\phi_{G'} \downarrow & & \downarrow \xi' \\
 FA & \xrightarrow[\sigma_{FA}]{} & R\mathcal{H}om_Z(R\mathcal{H}om_Z(FA, FA), FA)
 \end{array}$$

By the above-indicated definition of  $\tau$  and  $\tau'$ , the bottom square commutes, as does the square obtained by erasing  $\tau'$ . Since  $\xi'$  is a monomorphism, therefore the top square commutes too. Thus  $F\phi$  is a map of  $FA$ -rigid pairs. □

**Remark 3.3.8.** One would naturally like more concrete definitions of the functors in Corollary 3.3.4.

One does find in [Yekutieli and Zhang 2004, §3] some explicitly formulated — in DGA terms — versions of special cases of these functors. (Indeed, that’s what suggested Corollary 3.3.4.) But getting from here to there does not appear to be a simple matter. One might well have to go via the Reduction Theorem [Avramov et al. 2010b, 4.1], the main result of that paper, cf. [Avramov et al. 2010a, 8.5.5]); and, say for smooth maps, make use of nontrivial formal properties of Verdier’s isomorphism (B.5).

In Duality Land the well-cultivated concrete and abstract plains are not presently known to be connected other than by forbidding mountain passes, that can only be traversed by hard slogging.

### Appendices: Background

We review background concepts and basic facts having to do with scheme-maps, insofar as needed in the main text. Of special import is the *twisted inverse-image pseudofunctor*, a fundamental object in Grothendieck duality theory.

*Rings and schemes are assumed throughout to be noetherian.*

#### A. Essentially finite-type maps

**A.1.** A homomorphism  $\sigma : K \rightarrow S$  of commutative rings is *essentially of finite type* if  $\sigma$  can be factored as a composition of ring-homomorphisms

$$K \hookrightarrow K[x_1, \dots, x_d] \rightarrow V^{-1}K[x_1, \dots, x_d] \twoheadrightarrow S,$$

where  $x_1, \dots, x_d$  are indeterminates,  $V \subseteq K[x_1, \dots, x_d]$  is a multiplicatively closed set, the first two maps are canonical and the third is surjective. The map  $\sigma$  is *of finite type* if one can choose  $V = \{1\}$ ; the map  $\sigma$  is *finite* if it turns  $S$  into a finite (that is, finitely generated)  $R$ -module.

A homomorphism  $\hat{\sigma} : K \rightarrow P$  is (*essentially*) *smooth* if it is flat and (essentially) of finite type, and if for each homomorphism of rings  $K \rightarrow k$ , where  $k$  is a field, the ring  $k \otimes_K P$  is regular. By [Grothendieck and Dieudonné 1967, 17.5.1], this notion of smoothness is equivalent to the one defined in terms of lifting of homomorphisms.

When  $\hat{\sigma}$  is essentially smooth the  $P$ -module  $\Omega_{\hat{\sigma}}$  of relative Kähler differentials is finite projective; we say  $\hat{\sigma}$  has *relative dimension*  $d$  if for every  $p \in \text{Spec } S$ , the free  $S_p$ -module  $(\Omega_{\hat{\sigma}})_p$  has rank  $d$ .

**A.2.** A scheme-map  $f : X \rightarrow Y$  is *essentially of finite type* if every  $y \in Y$  has an affine open neighborhood  $V = \text{Spec}(A)$  such that  $f^{-1}V$  can be covered by finitely many affine open sets  $U_i = \text{Spec}(C_i)$  so that the corresponding ring homomorphisms  $A \rightarrow C_i$  are essentially of finite type.

If, moreover, there exists for each  $i$  a multiplicatively closed subset  $V_i \subseteq A$  such that  $A \rightarrow C_i$  factors as  $A \rightarrow V_i^{-1}A \xrightarrow{\sim} C_i$  where the first map is canonical and the second is an isomorphism (in other words,  $A \rightarrow C_i$  is a *localization* of  $A$ ), then we say that  $f$  is *localizing*. If the scheme-map  $f$  is localizing and also set-theoretically injective, then we say that  $f$  is a *localizing immersion*.

The map  $f$  is *essentially smooth* (of relative dimension  $d$ ) if it is essentially of finite type and the above data  $A \rightarrow C_i$  can be chosen to be essentially smooth ring homomorphisms (of relative dimension  $d$ ). The map  $f$  is *essentially étale* if it is essentially smooth of relative dimension 0. Equivalently,  $f$  is essentially smooth (resp. étale) if it is essentially of finite type and formally smooth (resp. étale); see [Grothendieck and Dieudonné 1967, §17.1]. For example, any localizing map is essentially étale.

**Remark A.3.** We will refer a few times to proofs in [Illusie 1971] that make use of the fact that the diagonal of a smooth map is a quasiregular immersion. To ensure that those proofs apply here, we note that the same property for *essentially smooth* maps is given by [Grothendieck and Dieudonné 1967, 16.10.2 and 16.9.4].

Nayak [2009, 4.1], extending a compactification theorem of Nagata, shows that *every essentially-finite-type separated map  $f$  of noetherian schemes factors as  $f = \bar{f}u$  with  $\bar{f}$  proper and  $u$  a localizing immersion*.

**Example A.4.** (Local compactification.) A map  $f : X = \text{Spec } S \rightarrow \text{Spec } K = Y$  coming from an essentially finite-type homomorphism of rings  $K \rightarrow S$  factors as

$$X \xrightarrow{j} W \xrightarrow{i} \bar{W} \xrightarrow{\pi} Y,$$

where  $W$  is the Spec of a finitely generated  $K$ -algebra  $T$  of which  $S$  is a localization,  $j$  being the corresponding map, where  $i$  is an open immersion, and where  $\pi$  is a projective map, so that  $\pi$  is proper and  $ij$  is a localizing immersion.

### B. Review of global duality theory

*All scheme-maps are assumed to be essentially of finite type and separated.*

We recall some global duality theory, referring to [Lipman 2009] and [Nayak 2009] for details.

**B.1.** To any scheme-map  $f : X \rightarrow Y$  one associates the right-derived direct-image functor  $Rf_* : D_{qc}(X) \rightarrow D_{qc}(Y)$  and its left adjoint, the left-derived inverse-image functor  $Lf^* : D_{qc}(Y) \rightarrow D_{qc}(X)$  [Lipman 2009, 3.2.2, 3.9.1, 3.9.2]. These functors interact with the left-derived tensor product  $\otimes^L$  via a natural isomorphism

$$Lf^*(M \otimes_Y^L N) \xrightarrow{\sim} Lf^*M \otimes_X^L Lf^*N \quad (M, N \in D(Y)), \quad (\text{B.1.1})$$

see [Lipman 2009, 3.2.4]; via the functorial map

$$Rf_*F \otimes_Y^L Rf_*G \rightarrow Rf_*(F \otimes_X^L G) \quad (F, G \in D(X)) \quad (\text{B.1.2})$$

adjoint to the natural composite map

$$Lf^*(Rf_*F \otimes_Y^L Rf_*G) \xrightarrow{\sim} Lf^*Rf_*F \otimes_X^L Lf^*Rf_*G \longrightarrow F \otimes_X^L G;$$

and via the *projection isomorphism*

$$Rf_*F \otimes_Y^L M \xrightarrow{\sim} Rf_*(F \otimes_X^L Lf^*M) \quad (F \in D_{qc}(X), M \in D_{qc}(Y)), \quad (\text{B.1.3})$$

defined qua map to be the natural composition

$$Rf_*F \otimes_Y^L M \rightarrow Rf_*F \otimes_Y^L Rf_*Lf^*M \rightarrow Rf_*(F \otimes_X^L Lf^*M).$$

see [Lipman 2009, 3.9.4]). The projection isomorphism yields a natural isomorphism

$$Rf_*Lf^*M \simeq Rf_*(\mathbb{C}_X \otimes_X^L Lf^*M) \simeq Rf_*\mathbb{C}_X \otimes_Y^L M. \quad (\text{B.1.4})$$

Interactions with the derived (sheaf-)homomorphism functor  $R\mathcal{H}om$  occur via natural bifunctorial maps:

$$Lf^*R\mathcal{H}om_Y(M, N) \rightarrow R\mathcal{H}om_X(Lf^*M, Lf^*N) \quad (M, N \in D(Y)), \quad (\text{B.1.5})$$

(see [Lipman 2009, 3.5.6(a)]) which is an *isomorphism* if  $f$  is an open immersion [Lipman 2009, p. 190, end of §4.6]; and

$$Rf_*R\mathcal{H}om_X(F, G) \rightarrow R\mathcal{H}om_Y(Rf_*F, Rf_*G) \quad (F, G \in D(X)), \quad (\text{B.1.6})$$

the latter corresponding via (1.1.1.2) to the natural composition

$$Rf_*R\mathcal{H}om_X(F, G) \otimes_Y^L Rf_*F \rightarrow Rf_*(R\mathcal{H}om_X(F, G) \otimes_X^L F) \xrightarrow{Rf_*\varepsilon} Rf_*G,$$

where the first map comes from (B.1.2), and  $\varepsilon$  is the evaluation map (1.1.1.3).

**B.2.** For any commutative square of scheme-maps

$$\begin{array}{ccc} X' & \xrightarrow{v} & X \\ \downarrow h & \Xi & \downarrow f \\ Y' & \xrightarrow{u} & Y \end{array} \quad (\text{B.2.1})$$

one has the map  $\theta_\Xi : Lu^*Rf_* \rightarrow Rh_*Lv^*$  adjoint to the natural composite map

$$Rf_* \longrightarrow Rf_*Rv_*Lv^* \xrightarrow{\sim} Ru_*Rh_*Lv^*.$$

When  $\Xi$  is a *fiber square* (which means that the map associated to  $\Xi$  is an isomorphism  $X' \xrightarrow{\sim} X \times_Y Y'$ ), and  $u$  is *flat*, then  $\theta_\Xi$  is an *isomorphism*. In fact, for

any fiber square  $\Xi$ ,  $\theta_\Xi$  is an isomorphism  $\iff \Xi$  is tor-independent [Lipman 2009, 3.10.3].

**B.3.** Duality theory focuses on the *twisted inverse-image pseudofunctor*

$$f^! : D_{qc}^+(Y) \rightarrow D_{qc}^+(X),$$

where “pseudofunctoriality” (also known as “2-functoriality”) entails, in addition to functoriality, a family of functorial isomorphisms  $c_{g,f}^! : (fg)^! \xrightarrow{\sim} g^! f^!$ , one for each composable pair  $Z \xrightarrow{g} X \xrightarrow{f} Y$ , satisfying a natural “associativity” property vis-à-vis any composable triple; see, e.g., [Lipman 2009, 3.6.5].

This pseudofunctor is uniquely determined up to isomorphism by the following three properties:

- (i) If  $f$  is essentially étale then  $f^!$  is the usual restriction functor  $f^*$ .
- (ii) If  $f$  is proper then  $f^!$  is right-adjoint to  $Rf_*$ .
- (iii) If in a fiber square  $\Xi$  as in (B.2.1) the map  $f$  (and hence  $h$ ) is proper and  $u$  is essentially étale, then the functorial *base-change map*

$$\beta_\Xi(M) : v^* f^! M \rightarrow h^! u^* M \quad (M \in D_{qc}^+(Y)), \tag{B.3.1}$$

defined to be adjoint to the natural composition

$$Rh_* v^* f^! M \xrightarrow[\theta_\Xi^{-1}]{\sim} u^* Rf_* f^! M \longrightarrow u^* M,$$

is identical with the natural composite *isomorphism*

$$v^* f^! M = v^! f^! M \xrightarrow{\sim} (fv)^! M = (uh)^! M \xrightarrow{\sim} h^! u^! M = h^! u^* M.$$

For the existence of such a pseudofunctor, see [Nayak 2009, section 5.2 ].

**B.4.** Nayak’s theorem [2009, 5.3] (as elaborated in [Nayak 2005, 7.1.6]) shows that one can associate, in a unique way, to *every* fiber square  $\Xi$  as in (B.2.1) with  $u$  (and hence  $v$ ) flat, a functorial isomorphism

$$\beta_\Xi(M) : v^* f^! M \xrightarrow{\sim} h^! u^* M \quad (M \in D_{qc}^+(Y)),$$

equal to (B.3.1) when  $f$  is proper, and to the natural isomorphism  $v^* f^* \xrightarrow{\sim} h^* u^*$  when  $f$  is essentially étale.

**B.5.** Generalizing (i) in B.3, let  $f : X \rightarrow Y$  be essentially smooth, so that by [Grothendieck and Dieudonné 1967, 16.10.2] the relative differential sheaf  $\Omega_f$  is locally free over  $\mathbb{C}_X$ . On any connected component  $W$  of  $X$ , the rank of  $\Omega_f$  is a constant, denoted  $d(W)$ .

There is a *functorial isomorphism*

$$f^! M \xrightarrow{\sim} \Sigma^d \Omega_f^d \otimes_{\mathbb{C}_X} f^* M \quad (M \in D_{qc}(Y)), \tag{B.5.1}$$

with  $\Sigma^d \Omega_f^d$  the complex whose restriction to any  $W$  is  $\Sigma^{d(W)} \wedge_{\mathbb{O}_W}^{d(W)} (\Omega_f|_W)$ .

( $\Sigma$  is the usual translation automorphism of  $D(X)$ ; and  $\wedge$  denotes exterior power.)

To prove this, one may assume that  $X$  itself is connected, and set  $d := d(X)$ . Noting that the diagonal  $\Delta : X \rightarrow X \times_Y X$  is defined locally by a regular sequence of length  $d$  (see Remark A.3), so that  $\Delta^! \mathbb{O}_{X \times_Y X} \otimes^L L\Delta^* G \cong \Delta^! G$  for all  $G \in D_{\text{qc}}(X \times_Y X)$  [Hartshorne 1966, p. 180, 7.3], one can imitate the proof of [Verdier 1969, p. 397, Thm. 3], where, in view of (a) above, one can drop the properness condition and take  $U = X$ , and where finiteness of Krull dimension is superfluous.

**B.6.** The fact that  $\beta_{\Xi}(M)$  in (B.3.1) is an isomorphism for all  $M$  whenever  $u$  is an open immersion and  $f$  is proper, is shown in [Lipman 2009, §4.6, part V] to be equivalent to *sheafified duality*, which is that for any proper  $f : X \rightarrow Y$ , and any  $F \in D_{\text{qc}}(X)$ ,  $M \in D_{\text{qc}}^+(Y)$ , the natural composition, in which the first map comes from (B.1.6),

$$Rf_* \mathcal{H}om_X(F, f^! M) \rightarrow R\mathcal{H}om_Y(Rf_* F, Rf_* f^! M) \rightarrow R\mathcal{H}om_Y(Rf_* F, M), \quad (\text{B.6.1})$$

is an isomorphism.

Moreover, if the proper map  $f$  has finite flat dimension, then sheafified duality holds for all  $M \in D_{\text{qc}}(Y)$ , see [Lipman 2009, 4.7.4].

If  $f$  is a *finite* map, then (B.6.1) with  $F = \mathbb{O}_X$  determines the functor  $f^!$ . (See also [Conrad 2007, §2.2].) In particular, if  $f : \text{Spec } B \rightarrow \text{Spec } A$  corresponds to a finite ring homomorphism  $A \rightarrow B$ , and  $\sim$  is the standard sheafification functor, then for an  $A$ -complex  $N$ ,  $f^!(N^\sim)$  is the  $B$ -complex

$$f^!(N^\sim) = \text{RHom}_A(B, N)^\sim, \quad (\text{B.6.2})$$

where  $\text{RHom}_A(B, -)$  denotes the right-derived functor of the functor  $\text{Hom}_A(B, -)$  from  $A$ -modules to  $B$ -modules.

### C. Idempotent ideal sheaves

**Definition C.1.** Let  $(X, \mathbb{O}_X)$  be a local-ringed space, i.e.,  $X$  is a topological space and  $\mathbb{O}_X$  is a sheaf of commutative rings whose stalk at each point is a local ring (not necessarily noetherian). An  $\mathbb{O}_X$ -ideal is *idempotent* if it is of finite type (i.e., locally finitely generated) and satisfies the equivalent conditions in the next proposition.

**Proposition C.2.** Let  $(X, \mathbb{O}_X)$  be a local-ringed space. Consider the following conditions on an  $\mathbb{O}_X$ -ideal  $I$ .

- (i) There is an  $a \in H^0(X, \mathbb{O}_X)$  such that  $a^2 = a$  and  $I = a\mathbb{O}_X$ .
- (i') The identity map of  $I$  extends to an  $\mathbb{O}_X$ -homomorphism  $\pi : \mathbb{O}_X \rightarrow I$ .

- (ii) *There is an open and closed  $U \subseteq X$ , with inclusion, say,  $i : U \hookrightarrow X$ , and an  $\mathbb{O}_X$ -isomorphism  $i_*\mathbb{O}_U \simeq I$ .*
- (iii) *The  $\mathbb{O}_X$ -module  $\mathbb{O}_X/I$  is flat.*
- (iv) *For all  $\mathbb{O}_X$ -modules  $F$ , the natural map is an isomorphism  $I \otimes_X F \xrightarrow{\sim} IF$ .*
- (v) *For all  $\mathbb{O}_X$ -ideals  $J$ ,  $IJ = I \cap J$ .*
- (vi)  $I^2 = I$ .

*One has the implications*

$$(i) \iff (i') \iff (ii) \implies (iii) \iff (iv) \iff (v) \implies (vi);$$

*and if  $I$  is of finite type then  $(vi) \implies (i)$ .*

*Proof.* (i)  $\Leftrightarrow$  (i'). If (i) holds, let  $\pi$  be the map taking  $1 \in H^0(X, \mathbb{O}_X)$  to  $a$ . Conversely, given (i'), let  $a = \pi(1)$ .

(ii)  $\Rightarrow$  (i). Let  $a$  be the global section that is 1 over  $U$  and 0 over  $X \setminus U$ .

(i)  $\Rightarrow$  (vi). Trivial.

(vi)  $\Rightarrow$  (ii) when  $I$  is of finite type (whence (i)  $\Rightarrow$  (ii) always). The support of  $I$ ,  $U := \{x \in X \mid I_x \neq 0\}$ , is closed when  $I$  is of finite type. For any  $x \in U$ , since  $I_x$  is a finitely generated  $\mathbb{O}_{X,x}$ -ideal such that  $I_x = I_x^2$ , therefore Nakayama's lemma shows that  $I_x = \mathbb{O}_{X,x}$ . So  $X \setminus U = \{x \in X \mid \mathbb{O}_{X,x}/I_x \neq 0\}$  is closed, and thus  $U$  is open as well as closed. Clearly,  $I|_U = \mathbb{O}_U$  and  $I|_{X \setminus U} = 0$ , whence  $I \simeq i_*\mathbb{O}_U$ .

(i)  $\Rightarrow$  (iii). If (i) holds then the germ of  $a$  at any  $x \in X$  is 1 or 0, so  $(\mathbb{O}/I)_x$  is either (0) or  $\mathbb{O}_{X,x}$ , both of which are flat over  $\mathbb{O}_{X,x}$ .

The remaining implications can be tested stalkwise, and so reduce to the corresponding well-known implications for ideals  $I, J$  in a local ring  $R$ , and  $R$ -modules  $F$ :

(iii)  $\Rightarrow$  (iv). The surjection  $I \otimes_R F \rightarrow IF \subseteq R \otimes_R F$  has kernel  $\text{Tor}_1^R(R/I, F) = 0$ .

(iv)  $\Rightarrow$  (v).  $(I \cap J)/IJ$  is the kernel of the natural injective (by (iv)) map

$$R/IJ \cong I \otimes_R R/J \rightarrow R \otimes_R (R/J) = R/J.$$

(v)  $\Rightarrow$  (iii). Flatness of  $R/I$  is implied by injectivity, for all  $R$ -ideals  $J$ , of the natural map  $J/IJ \cong J \otimes_R (R/I) \rightarrow R \otimes_R (R/I) = R/I$ , with kernel  $(I \cap J)/IJ$ .

(v)  $\Rightarrow$  (vi). Take  $J = I$ . □

**Corollary C.3.** (1) *Taking  $a$  to  $a\mathbb{O}_X$  gives a bijection from the set of idempotent elements of  $H^0(X, \mathbb{O}_X)$  to the set of idempotent  $\mathbb{O}_X$ -ideals.*

(2) *There is a bijection that associates to each idempotent  $\mathbb{O}_X$ -ideal its support — an open-and-closed subset of  $X$  — and to each open-and-closed  $U \subseteq X$ , with inclusion map  $i$ , the unique idempotent  $\mathbb{O}_X$ -ideal isomorphic to  $i_*\mathbb{O}_U$ , that is, the ideal whose restriction to  $U$  is  $\mathbb{O}_U$  and to  $X \setminus U$  is (0).* □



**Corollary C.4.** *A finite-type  $\mathbb{O}_X$ -ideal  $I$  is idempotent if and only if for each  $G \in \mathcal{D}(X)$  there exist  $\mathcal{D}(X)$ -isomorphisms, functorial in  $G$ ,*

$$\mathcal{R}\mathcal{H}om_X(I, G) \simeq I \otimes_X^L G \simeq IG.$$

*Proof.* If  $I$  is idempotent then over the open set  $U := \text{Supp}_X I$  one has  $I = \mathbb{O}_U$ , and over the disjoint open set  $X \setminus U$ ,  $I \simeq 0$ , so the asserted isomorphisms obviously exist over  $X = U \sqcup (X \setminus U)$ .

Conversely, if these isomorphisms hold for all members of the natural triangle

$$I \rightarrow \mathbb{O}_X \rightarrow \mathbb{O}_X/I \xrightarrow{+}$$

then, since  $I(\mathbb{O}_X/I) = 0$ , application of the functor  $\mathcal{R}\mathcal{H}om(I, -)$  yields that the natural map is an isomorphism  $I \simeq I^2$  in  $\mathcal{D}(X)$ , hence in  $\mathbb{O}_X$ , i.e.,  $I = I^2$ .  $\square$

**Corollary C.5.** *Let  $X$  be a locally noetherian scheme. For a complex  $L \in \mathcal{D}(X)$  the following conditions are equivalent.*

- (i)  $L$  is isomorphic in  $\mathcal{D}(X)$  to an idempotent  $\mathbb{O}_X$ -ideal.
- (ii)  $L \in \mathcal{D}_c^b(X)$  and there exists a  $\mathcal{D}(X)$ -isomorphism  $L \otimes_X^L L \xrightarrow{\sim} L$ .

*Proof.* If (i) holds then  $L \in \mathcal{D}_c^b(X)$  is clear; and taking  $G = I$  in C.4, one gets (ii).

When (ii) holds, (i) follows easily from [Avramov et al. 2010a, 4.9].  $\square$

**Proposition C.6.** *Let  $g : Z \rightarrow X$  be a morphism of local ringed spaces (so that for each  $z \in Z$  the associated stalk homomorphism  $\mathbb{O}_{X,gz} \rightarrow \mathbb{O}_{Z,z}$  is a local homomorphism of local rings). Let  $I$  be an  $\mathbb{O}_X$ -ideal. If  $I$  is idempotent then so is  $I\mathbb{O}_Z \cong g^*I \simeq \mathcal{L}g^*I$ . The converse holds if  $g$  is flat and surjective.*

*Proof.* If  $I = I^2$  then  $I\mathbb{O}_Z = (I\mathbb{O}_Z)^2$ . Flatness of  $\mathbb{O}_X/I$  implies that  $I$  is flat and that the natural map  $g^*I \rightarrow g^*\mathbb{O}_X = \mathbb{O}_Z$  is injective, and thus  $\mathcal{L}g^*I \simeq g^*I \cong I\mathbb{O}_Z$ .

If  $g$  is flat and surjective then for each  $x \in X$  there is a  $z \in Z$  such that  $g(z) = x$ , and then there is a flat local homomorphism  $\mathbb{O}_{X,x} \rightarrow \mathbb{O}_{Z,z}$ . Hence if  $I\mathbb{O}_Z = (I\mathbb{O}_Z)^2$  then  $I_x\mathbb{O}_{Z,z} = I_x^2\mathbb{O}_{Z,z}$ , i.e.,  $I_x = I_x^2$ . As this holds for all  $x$ , therefore  $I = I^2$ .  $\square$

**Corollary C.7.** *Let  $g : Z \rightarrow X$  be a morphism of local ringed spaces, and  $I$  an idempotent  $\mathbb{O}_X$ -ideal.*

- (1) *For any  $E \in \mathcal{D}(X)$ , there is a unique isomorphism*

$$\mathcal{L}g^*(IE) \simeq I\mathcal{L}g^*E$$

*whose composition with the natural map  $I\mathcal{L}g^*E \rightarrow \mathcal{L}g^*E$  is the map obtained by applying  $\mathcal{L}g^*$  to the natural map  $IE \rightarrow E$ .*

- (2) *If  $g$  is a perfect scheme-map then for any  $E \in \mathcal{D}_{\text{qc}}^+(X)$ , there exists a unique isomorphism  $g^!(IE) \simeq Ig^!E$  whose composition with the natural map  $Ig^!E \rightarrow g^!E$  is the map obtained by applying  $g^!$  to the natural map  $IE \rightarrow E$ .*

*Proof.* Uniqueness holds because,  $I\mathbb{O}_Z$  being idempotent,  $ILg^*E \simeq I\mathbb{O}_Z \otimes_Z Lg^*E$  is a direct summand of  $\mathbb{O}_Z \otimes_Z Lg^*E \simeq Lg^*E$  (Proposition C.2, (iv) and (i')).

Since both  $I$  and  $\mathbb{O}_X/I$  are flat over  $\mathbb{O}_X$ , there are for all  $F \in D(X)$  natural isomorphisms  $Lg^*I \otimes_Z^L F \simeq g^*I \otimes_Z F \cong IF$ . So for all  $E \in D(X)$ ,

$$Lg^*(IE) \simeq Lg^*(I \otimes_X^L E) \simeq Lg^*I \otimes_Z^L Lg^*E \simeq ILg^*E.$$

The composition of these isomorphisms has the property asserted in (1).

Similarly, if  $g$  is a perfect scheme-map then, using Remark 2.1.6, one gets natural isomorphisms for all  $E \in D_{qc}^+(X)$ ,

$$g^!(IE) \simeq g^!(I \otimes_X^L E) \simeq Lg^*I \otimes_Z^L Lg^*E \otimes_Z^L g^!\mathbb{O}_X \simeq Lg^*I \otimes_Z^L g^!E \simeq Ig^!E,$$

that compose to the isomorphism needed for (2). □

The next result is to the effect that *idempotence satisfies faithfully flat descent* (without any ‘‘cocycle condition’’).

**Proposition C.8.** *Let  $g : Z \rightarrow X$  be a faithfully flat map, and let  $\pi_1 : Z \times_X Z \rightarrow Z$  and  $\pi_2 : Z \times_X Z \rightarrow Z$  be the canonical projections. If  $J$  is an idempotent  $\mathbb{O}_Z$ -ideal such that there exists an isomorphism  $\pi_1^*J \cong \pi_2^*J$  then there is a unique idempotent  $\mathbb{O}_X$ -ideal such that  $J = I\mathbb{O}_Z$ .*

*Proof.* (Uniqueness.) If  $J = I\mathbb{O}_Z = I'\mathbb{O}_Z$  where  $I$  and  $I'$  are idempotent  $\mathbb{O}_X$ -ideals with respective supports  $U$  and  $U'$ , then  $g^{-1}U = g^{-1}U'$  (both being the support of  $J$ ), and since  $g$  is surjective, therefore  $U = U'$ , so  $I = I'$ .

(Existence.) Let  $V$  be the support of  $J$ . The support of  $\pi_1^*J$  is  $\pi_1^{-1}V = V \times_X Z$ , and similarly that of  $\pi_2^*J$  is  $Z \times_X V$ . Hence, since  $\pi_1^*J \cong \pi_2^*J$ , the following subsets of  $Z \times_X Z$  are all the same:

$$V \times_X Z = Z \times_X V = (V \times_X Z) \cap (Z \times_X V) = V \times_X V.$$

If  $v \in V$  and  $w \in Z$  are such that  $g(v) = g(w)$ , then there is a field  $K$  and a map  $\gamma : \text{Spec } K \rightarrow V \times_X Z = V \times_X V$  such that the set-theoretic images of  $\pi_1\gamma$  and  $\pi_2\gamma$  are  $v$  and  $w$  respectively, so  $w \in V$ . Thus  $V = g^{-1}g(V)$ .

We claim that  $g(V)$  is open and closed in  $X$ . For this it suffices to show that for each connected component  $X' \subseteq X$ ,  $g(V \cap g^{-1}X') = X'$ . Without loss of generality, then, we may assume that  $X$  is connected, so  $X' = X$ .

Since  $g$  is flat, if  $y \in g(V)$  then the generic point  $x_1$  of any irreducible component  $X_1$  of  $X$  containing  $y$  is also in  $g(V)$ . In fact  $X_1 \subseteq g(V)$ , else the preceding argument applied to  $\bar{V} := Z \setminus V$  would show that  $x_1 \in g(\bar{V}) = X \setminus g(V)$ . It results that some open neighborhood of  $y$  is in  $g(V)$ ; and thus  $g(V)$  is open. Similarly,  $g(\bar{V}) = X \setminus g(V)$  is open, so  $g(V)$  is closed.

The conclusion follows, with  $I$  the idempotent  $\mathbb{O}_X$ -ideal corresponding to the open-and-closed set  $g(V) \subseteq X$ . □

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
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Volume 5    No. 3    2011

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A categorical proof of the Parshin reciprocity laws on algebraic surfaces DENIS OSIPOV and XINWEN ZHU	289
Quantum differentiation and chain maps of bimodule complexes ANNE V. SHEPLER and SARAH WITHERSPOON	339
Toric-friendly groups MIKHAIL BOROVOI and ZINOVY REICHSTEIN	361
Reflexivity and rigidity for complexes, II Schemes LUCHEZAR AVRAMOV, SRIKANTH B. IYENGAR and JOSEPH LIPMAN	379



1937-0652(2011)5:3;1-E