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A categorical proof of the Parshin reciprocity laws on algebraic surfaces

Denis Osipov and Xinwen Zhu

We define and study the 2-category of torsors over a Picard groupoid, a central extension of a group by a Picard groupoid, and commutator maps in this central extension. Using this in the context of two-dimensional local fields and two-dimensional adèle theory we obtain the two-dimensional tame symbol and a new proof of Parshin reciprocity laws on an algebraic surface.

1. Introduction

Let C be a projective algebraic curve over a perfect field k. The famous Weil reciprocity law states that

$$\prod_{p\in C} \mathrm{Nm}_{k(p)/k}\{f, g\}_p = 1, \tag{1-1}$$

where $f, g \in k(C)^\times$,

$$\{f, g\}_p = (-1)^{\nu_p(f)\nu_p(g)} \frac{f^{\nu_p(g)}}{g^{\nu_p(f)}}(p)$$

is the one-dimensional tame symbol, and $k(p)$ is the residue field of the point p. The product (1-1) contains only finitely many terms not equal to 1.

There is a proof of this law (and the analogous reciprocity law for residues of rational differential forms: sum of residues equals to zero) by reduction to the case of $\mathbf{P}^1_k$ using the connection between tame symbols (and residues of differentials) in extensions of local fields; see, for example, [Serre 1988, Chapters 2 and 3].

On the other hand, Tate [1968] gave a definition of the local residue of a differential form as the trace of a certain infinite-dimensional matrix. Starting from this definition he gave an intrinsic proof of the residue formula on a projective algebraic curve C using the fact that $\dim_k H^i(C, \mathcal{O}_C) < \infty$, for $i = 0, 1$.

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The multiplicative analog of Tate's approach, i.e., the case of the tame symbol and the proof of Weil reciprocity law, was done later by Arbarello, De Concini and Kac [1988]. They used the central extension of the infinite-dimensional group $\mathrm{GL}(K)$ of continuous automorphisms of K, where $K = k((t))$, by the group $k^\times$, and obtained the tame symbol up to sign as the commutator of the lifting of two elements from $K^\times \subset \mathrm{GL}(K)$ to this central extension. Hence, as in Tate's proof mentioned above, they obtained an intrinsic proof of the Weil reciprocity law on an algebraic curve. However, in this proof the exterior algebra of finite-dimensional k-vector spaces was used. Therefore difficult sign conventions were used in this paper to obtain the reciprocity law. To avoid these difficulties, Beilinson, Bloch and Esnault [2002] used the category of graded lines instead of the category of lines. The category of graded lines has nontrivial commutativity constraints multipliers $(-1)^{mn}$, where $m, n \in \mathbb{Z}$ are corresponding gradings. In other words, they used the Picard groupoid of graded lines which is a nonstrictly commutative instead of strictly commutative Picard groupoid. It was the first application of this notion of nonstrictly commutative Picard groupoid.

Now let X be an algebraic surface over a perfect field k. For any pair $x \in C$, where $C \subset X$ is a curve that $x \in C$ is a closed point, it is possible to define the ring $K_{x,C}$ such that $K_{x,C}$ is isomorphic to the two-dimensional local field

$$k(x)((t))((s))$$

when x is a smooth point on C and X. If x is not a smooth point, then $K_{x,C}$ is a finite direct sum of two-dimensional local fields (see Section 5B of this paper). For any two-dimensional local field $k'((t))((s))$ one can define the two-dimensional tame symbol of 3 variables with values in $k'^\times$, see Section 4A and [Parshin 1975, 1984, §3]. Parshin formulated and proved the reciprocity laws for two-dimensional tame symbols, but his proof was never published. Contrary to the one-dimensional case, there are a lot of reciprocity laws for two-dimensional tame symbols, which belong to two types. For the first type we fix a point on the surface and will vary irreducible curves containing this point. For the second type we fix a projective irreducible curve on the surface and will vary points on this curve. Parshin's idea for the proof, for example, of more unexpected first type of reciprocity laws, was to use the chain of successive blowups of points on algebraic surfaces. Later, Kato [1986, Proposition 1] generalized the reciprocity laws for excellent schemes by using the reduction to the reciprocity law of Bass and Tate for Milnor K-groups of some field $L(t)$. He used them to construct an analog of the Gersten–Quillen complex for Milnor K-theory.

In this paper, we give a generalization of Tate's proof of the reciprocity law on an algebraic curve to the case of two-dimensional tame symbols and obtain an intrinsic proof of Parshin reciprocity laws for two-dimensional tame symbols on an algebraic surface.

To fulfill this goal, we first generalize the notion of a central extension of a group by a commutative group and of the commutator map associated to the central extension. More precisely, we define and study in some detail the properties of the category of central extensions of a group G by a (nonstrictly commutative) Picard groupoid $\mathcal{P}$. Roughly speaking, an object in this category is a rule to assign every $g \in G$ a $\mathcal{P}$-torsor, satisfying certain properties. For such a central extension $\mathcal{L}$ we define a map $C_3^{\mathcal{L}}$ which is an analog of the commutator map. In this case when G is abelian, this commutator map is an antisymmetric and trimultiplicative map from G^3 to the group $\pi_1(\mathcal{P})$. Let us remark that to obtain some of these properties, we used the results of Breen [1999] on group-like monoidal 2-groupoids. We hope these constructions will be of some independent interest.

We then apply this formalism to $\mathcal{P} = \mathcal{P}ic^{\mathbb{Z}}$, where $\mathcal{P}ic^{\mathbb{Z}}$ stands for the Picard groupoid of graded lines. The key ingredient here is Kapranov's [2001] graded-determinantal theory, which associates a $\mathcal{P}ic^{\mathbb{Z}}$-torsor to every 1-Tate vector space (a locally linearly compact vector space). This allows one to construct the central extension $\mathcal{D}et$ of $\mathrm{GL}(\mathbb{K})$ by $\mathcal{P}ic^{\mathbb{Z}}$, where $\mathbb{K}$ is a two-dimensional local field (or more generally, a 2-Tate vector space). It turns out that the two-dimensional tame symbol coincides with the commutator map $C_3^{\mathcal{D}et}$. Finally, using "semilocal" adèle complexes on an algebraic surface we obtain that the corresponding central extension constructed by semilocal fields on the surface is the trivial one. This leads us to a new proof of Parshin's reciprocity laws on an algebraic surface, which is distinct from both Parshin's original approach as well as Kato's.

Our approach to the reciprocity laws on the algebraic surfaces has the following features. First, we use the nonstrictly commutative Picard groupoid, which can be regarded as another application of this notion after [Beilinson et al. 2002]. However, unlike the one-dimensional case where one can just plays with the usual Picard groupoid of lines (though complicated, as done in [Arbarello et al. 1988]), the use of $\mathcal{P}ic^{\mathbb{Z}}$ is essential here. This indicates that the nonstrictly commutative Picard groupoid is an important and fruitful mathematical object that deserves further attention. Also, in order to apply this notion, we develop certain constructions in higher categories (e.g., the commutator map $C_3^{\mathcal{L}}$), which could be potentially useful elsewhere. Second, as in the one-dimensional case, our approach uses a local-to-global (in other words, factorization) principle. Since the local-to-global (factorization) principle in the one-dimensional story is very important in the Langlands program and conformal field theory, we hope our approach is just a shadow of a whole fascinating realm of mathematics yet to be explored. Finally, our approach can be generalized by replacing the ground field k by an Artinian ring A (and even more general rings) and we can obtain reciprocity laws for two-dimensional

Contou–Carrère symbols. By choosing A appropriately, this specializes to residue formulas for algebraic surfaces.[1] We will carefully discuss this in a future paper.

The paper is organized as follows. In Section 2 we describe some categorical constructions, which we need further on. In Section 2A we recall the definition of a Picard groupoid. In Section 2B we discuss the difference between strictly commutative and nonstrictly commutative Picard groupoids. In Section 2C we describe the 2-category of $\mathcal{P}$-torsors, where $\mathcal{P}$ is a Picard groupoid. In Section 2D we study the Picard groupoid of homomorphisms from a group G to a Picard groupoid $\mathcal{P}$ and describe the "commutator" of two commuting elements from G with values in $\pi_1(\mathcal{P})$. In Section 2E we define and study the Picard 2-groupoid of central extensions of a group G by a Picard groupoid $\mathcal{P}$. We define and study properties of the commutator category of such a central extension, and finally study the "commutator" of three commuting elements form G with values in $\pi_1(\mathcal{P})$. This section may be of independent interest.

In Section 3 we recall the theory of graded-determinantal theories on Tate vector spaces. We recall the definition and basic properties of the category of n-Tate vector spaces in Section 3A. In Section 3B we recall the definition of determinant functor from the exact category (Tate_0, isom) to the Picard groupoid $\mathcal{P}ic^{\mathbb{Z}}$ of graded lines and the definition of graded-determinantal theory on the exact category Tate_1 of 1-Tate vector spaces.

In Section 4 we apply the constructions given above to one-dimensional and two-dimensional local fields. In Section 4A we review one-dimensional and two-dimensional tame symbols. In Section 4B we obtain a description of the one-dimensional (usual) tame symbol as some commutator. In Section 4C we obtain the two-dimensional tame symbol as commutator of 3 elements in some central extension of the group $K^\times = k((t))((s))^\times$ by the Picard groupoid $\mathcal{P}ic^{\mathbb{Z}}$.

In Section 5 we obtain the reciprocity laws. In Section 5A we give the proof of Weil reciprocity law using the constructions given above and adèle complexes on a curve. In Section 5B we apply the previous results in order to obtain a proof of Parshin's reciprocity laws on an algebraic surface using "semilocal" adèle complexes on an algebraic surface.

2. General nonsense

2A. *Picard groupoid.* Let $\mathcal{P}$ be a Picard groupoid, i.e., a symmetric monoidal group-like groupoid. Let us recall that this means that $\mathcal{P}$ is a groupoid, together with a bifunctor

$$+ : \mathcal{P} \times \mathcal{P} \to \mathcal{P}$$

[1]The generalization of Tate's approach to the n-dimensional residue of differential forms was done in [Beilinson 1980], but that note contains no proofs.

and natural (functorial) isomorphisms

$$a_{x,y,z} : (x+y)+z \simeq x+(y+z),$$

called the associativity constraints, and natural (functorial) isomorphisms

$$c_{x,y} : x+y \simeq y+x,$$

called the commutativity constraints, such that:

(i) For each $x \in \mathcal{P}$, the functor $y \mapsto x+y$ is an equivalence.

(ii) The pentagon axiom holds, i.e., the following diagram is commutative:

$$\begin{tikzcd}
& (x+y)+(z+w) \arrow[dl] \arrow[dr] & \\
x+(y+(z+w)) \arrow[d] & & ((x+y)+z)+w \arrow[d] \\
x+((y+z)+w) \arrow[rr] & & (x+(y+z))+w
\end{tikzcd} \tag{2-1}$$

(iii) The hexagon axiom holds, i.e., the following diagram is commutative:

$$\begin{tikzcd}
& (x+y)+z \arrow[dl] \arrow[dr] & \\
(y+x)+z \arrow[d] & & x+(y+z) \arrow[d] \\
y+(x+z) \arrow[dr] & & x+(z+y) \arrow[dl] \\
& (x+z)+y &
\end{tikzcd} \tag{2-2}$$

(iv) For any $x, y \in \mathcal{P}$, $c_{y,x}c_{x,y} = id_{x+y}$.

A unit (e, φ) of $\mathcal{P}$ is an object $e \in \mathcal{P}$ together with an isomorphism $\varphi : e+e \simeq e$. It is an exercise to show that (e, φ) exists and is unique up to a unique isomorphism. For any $x \in \mathcal{P}$, there is a unique isomorphism $e+x \simeq x$ such that the following diagram is commutative:

$$\begin{tikzcd}
(e+e)+x \arrow[rr] \arrow[dr] & & e+(e+x) \arrow[dl] \\
& e+x &
\end{tikzcd}$$

and therefore $x+e \simeq e+x \simeq x$. For any $x \in \mathcal{P}$, we choose an object, denoted by $-x$, together with an isomorphism $\phi_x : x+(-x) \simeq e$. The pair $(-x, \phi_x)$ is called an inverse of x, and it is unique up to a unique isomorphism. We choose for each x its inverse $(-x, \phi_x)$, then we have a canonical isomorphism

$$\begin{aligned} -(-x) &\simeq e+(-(-x)) \simeq (x+(-x))+(-(-x)) \\ &\simeq x+((-x)+(-(-x))) \simeq x+e \simeq x, \end{aligned} \tag{2-3}$$

and therefore a canonical isomorphism

$$(-x)+x \simeq (-x)+(-(-x)) \simeq e. \tag{2-4}$$

Observe that we have another isomorphism $(-x)+x \simeq x+(-x) \simeq e$ using the commutativity constraint. When the Picard groupoid $\mathcal{P}$ is strictly commutative (Section 2B), these two isomorphisms are the same [Zhu 2009, Lemma 1.6], but in general they are different.

If $\mathcal{P}_1, \mathcal{P}_2$ are two Picard groupoids, then $\mathrm{Hom}(\mathcal{P}_1, \mathcal{P}_2)$ is defined as follows. Objects are 1-homomorphisms, i.e., functors $F : \mathcal{P}_1 \to \mathcal{P}_2$ together with isomorphisms $F(x+y) \simeq F(x)+F(y)$ such that the following diagrams are commutative:

$$\begin{tikzcd} F((x+y)+z) \arrow[r] \arrow[d] & (F(x)+F(y))+F(z) \arrow[d] \\ F(x+(y+z)) \arrow[r] & F(x)+(F(y)+F(z)), \end{tikzcd}$$

$$\begin{tikzcd} F(x+y) \arrow[r] \arrow[d] & F(x)+F(y) \arrow[d] \\ F(y+x) \arrow[r] & F(y)+F(x). \end{tikzcd} \tag{2-5}$$

Morphisms in $\mathrm{Hom}(\mathcal{P}_1, \mathcal{P}_2)$ are 2-isomorphisms, i.e., natural transformations

$$\theta : F_1 \to F_2$$

such that the following diagram is commutative:

$$\begin{tikzcd} F_1(x+y) \arrow[r] \arrow[d, "\theta"'] & F_1(x)+F_1(y) \arrow[d, "\theta"] \\ F_2(x+y) \arrow[r] & F_2(x)+F_2(y). \end{tikzcd}$$

It is clear that $\mathrm{Hom}(\mathcal{P}_1, \mathcal{P}_2)$ has a natural structure as a Picard groupoid. Namely,

$$(F_1+F_2)(x) := F_1(x)+F_2(x),$$

and the isomorphism $(F_1+F_2)(x+y) \simeq (F_1+F_2)(x)+(F_1+F_2)(y)$ is the unique one such that the following diagram is commutative:

$$\begin{tikzcd} F_1(x+y)+F_2(x+y) \arrow[r] \arrow[d] & (F_1(x)+F_1(y))+(F_2(x)+F_2(y)) \arrow[dl] \\ (F_1(x)+F_2(x))+(F_1(y)+F_2(y)) & \end{tikzcd}$$

The associativity constraints and the commutativity constraints for $\mathrm{Hom}(\mathcal{P}_1, \mathcal{P}_2)$ are clear. If $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$ are three Picard groupoids, then $\mathrm{Hom}(\mathcal{P}_1, \mathcal{P}_2; \mathcal{P}_3)$ is defined as $\mathrm{Hom}(\mathcal{P}_1, \mathrm{Hom}(\mathcal{P}_2, \mathcal{P}_3))$, called the Picard groupoid of bilinear homomorphisms from $\mathcal{P}_1 \times \mathcal{P}_2$ to $\mathcal{P}_3$. The Picard groupoid of trilinear homomorphisms from $\mathcal{P}_1 \times \mathcal{P}_2 \times \mathcal{P}_3$ to $\mathcal{P}_4$ is defined similarly.

For a (small) monoidal group-like groupoid (or gr-category) C we denote by $\pi_0(C)$ the group[2] of isomorphism classes of objects. We denote by $\pi_1(C)$ the group $\mathrm{Aut}_C(e)$, where e is the unit object of C. It follows that $\pi_1(C)$ is an abelian group. If C is a Picard groupoid, then $\pi_0(C)$ is also an abelian group.

2B. ***Strictly commutative vs. nonstrictly commutative Picard groupoids.*** If the commutativity constraints c further satisfy $c_{x,x} = \mathrm{id}$, then the Picard groupoid $\mathcal{P}$ is called strictly commutative. It is a theorem of Deligne's [1973] that the 2-category of strictly commutative Picard groupoids is 2-equivalent to the 2-category of 2-term complexes of abelian groups concentrated on degree -1 and 0, whose terms of degree -1 are injective abelian groups.[3]

Example 2.1. The most famous example is $\mathcal{P} = BA$, where A is an abelian group, and BA is the category of A-torsors. The tensor products of A-torsors make BA a strictly commutative Picard groupoid. The 2-term complex of abelian groups that represents BA under Deligne's theorem is any injective resolution of $A[1]$. If $A = k^\times$ is the group of invertible elements in a field k, then BA is also denoted by $\mathcal{P}ic$, which is the symmetric monoidal category of one-dimensional k-vector spaces.

However, it is also important for us to consider nonstrictly commutative Picard groupoids. The following example of a nonstrictly commutative Picard groupoid is crucial.

Example 2.2. Let $\mathcal{P}ic^{\mathbb{Z}}$ denote the category of graded lines (one-dimensional k-vector spaces with gradings) over a base field k. An object in $\mathcal{P}ic^{\mathbb{Z}}$ is a pair (ℓ, n), where ℓ is a one-dimensional k-vector space, and n is an integer. The morphism set $\mathrm{Hom}_{\mathcal{P}ic^{\mathbb{Z}}}((\ell_1, n_1), (\ell_2, n_2)$ is empty unless $n_1 = n_2$, and in this case, it is just

[2] The group structure on $\pi_0(C)$ is induced by the monoidal structure of C.

[3] In fact, Deligne's theorem holds in any topos.

$\mathrm{Hom}_k(\ell_1, \ell_2) \setminus 0$. Observe that as a groupoid, $\mathcal{P}ic^{\mathbb{Z}}$ is not connected. In fact $\pi_0(\mathcal{P}) \simeq \mathbb{Z}$. The tensor product $\mathcal{P}ic^{\mathbb{Z}} \times \mathcal{P}ic^{\mathbb{Z}} \to \mathcal{P}ic^{\mathbb{Z}}$ is given as

$$(\ell_1, n_1) \otimes (\ell_2, n_2) \mapsto (\ell_1 \otimes \ell_2, n_1 + n_2).$$

There is a natural associativity constraint that makes $\mathcal{P}ic^{\mathbb{Z}}$ a monoidal groupoid.

Convention. For the Picard groupoids $\mathcal{P}ic$ and $\mathcal{P}ic^{\mathbb{Z}}$, we will often use in this article the usual notation $\otimes$ for monoidal structures in these categories, although for a general Picard groupoid we denoted it as $+$.

We note that the commutativity constraint in category $\mathcal{P}ic^{\mathbb{Z}}$ is the interesting one. Namely,

$$c_{\ell_1,\ell_2} : (\ell_1 \otimes \ell_2, n_1 + n_2) \simeq (\ell_2 \otimes \ell_1, n_2 + n_1), \quad c_{\ell_1,\ell_2}(v \otimes w) = (-1)^{n_1 n_2} w \otimes v.$$

Of course, there is another commutativity constraint on the category of graded lines given by $c(v \otimes w) = w \otimes v$. Then as a Picard groupoid with this naive commutativity constraints, it is just the strictly commutative Picard groupoid $\mathcal{P}ic \times \mathbb{Z}$. There is a natural monoidal equivalence $\mathcal{P}ic^{\mathbb{Z}} \simeq \mathcal{P}ic \times \mathbb{Z}$, but this equivalence is *not* symmetric monoidal (that is, it is not a 1-homomorphism of Picard groupoids). We denote by

$$F_{\mathcal{P}ic} : \mathcal{P}ic^{\mathbb{Z}} \to \mathcal{P}ic$$

the natural monoidal functor.

The importance of $\mathcal{P}ic^{\mathbb{Z}}$ lies in the following observation. Let us make the following convention.

Convention. For any category $\mathcal{C}$ we denote by $(\mathcal{C}, \mathrm{isom})$ a category with the same objects as in the category $\mathcal{C}$, and morphisms in the category $(\mathcal{C}, \mathrm{isom})$ are the isomorphisms in the category $\mathcal{C}$.

Now let Tate_0 be the category of finite dimensional vector spaces over a field k. The categories Tate_0 and $(\mathrm{Tate}_0, \mathrm{isom})$ are symmetric monoidal categories under the direct sum. The commutativity constraints in the categories Tate_0 and $(\mathrm{Tate}_0, \mathrm{isom})$ are defined in the natural way. Namely, the map $c_{V,W} : V \oplus W \to W \oplus V$ is given by $c_{V,W}(v, w) = (w, v)$. Then there is a natural symmetric monoidal functor

$$\det : (\mathrm{Tate}_0, \mathrm{isom}) \to \mathcal{P}ic^{\mathbb{Z}}, \tag{2-6}$$

which assigns to every V its top exterior power and the grading $\dim V$, the dimension of the vector space V over the field k. Observe, however, that the functor $F_{\mathcal{P}ic} \circ \det : (\mathrm{Tate}_0, \mathrm{isom}) \to \mathcal{P}ic$ is *not* symmetric monoidal.

It is a folklore theorem that the category of Picard groupoids (not necessarily strictly commutative) is equivalent to the category of spectra whose only nonvanishing homotopy groups are π_0 and π_1[4]. For example, $\mathcal{P}ic^{\mathbb{Z}}$ should correspond to the truncation $\tau_{\leq 1}\mathcal{K}$, where $\mathcal{K}$ is the spectra of algebraic K-theory of k.

2C. ***$\mathcal{P}$-torsors.*** Let $\mathcal{P}$ be a Picard groupoid. Recall (see also [Beilinson et al. 2002, Appendix A6] and [Drinfeld 2006, §5.1]) that a $\mathcal{P}$-torsor $\mathcal{L}$ is a module category over $\mathcal{P}$, i.e., there is a bifunctor

$$+ : \mathcal{P} \times \mathcal{L} \to \mathcal{L}$$

together with natural isomorphisms

$$a_{x,y,v} : (x+y)+v \simeq x+(y+v), \quad x, y \in \mathcal{P}, v \in \mathcal{L},$$

satisfying

(i) the pentagon axiom, i.e., a diagram similar to (2-1) holds;

(ii) for any $x \in \mathcal{P}$, the functor from $\mathcal{L}$ to $\mathcal{L}$ given by $v \mapsto x+v$ is an equivalence;

(iii) for any $v \in \mathcal{L}$, the functor from $\mathcal{P}$ to $\mathcal{L}$ given by $x \mapsto x+v$ is an equivalence of categories.

It is clear that we can verify the condition (ii) of this definition only for the unit object e of $\mathcal{P}$.

For any $v \in \mathcal{L}$, there is a unique isomorphism $e+v \simeq v$ such that the following diagram is commutative:

$$\begin{tikzcd} (e+e)+v \arrow[rr] \arrow[dr] & & e+(e+v) \arrow[dl] \\ & e+v & \end{tikzcd} .$$

If $\mathcal{L}_1, \mathcal{L}_2$ are $\mathcal{P}$-torsors, then $\mathrm{Hom}_{\mathcal{P}}(\mathcal{L}_1, \mathcal{L}_2)$ is the category defined as follows. Objects are 1-isomorphisms, i.e., equivalences $F : \mathcal{L}_1 \to \mathcal{L}_2$ together with isomorphisms $\lambda : F(x+v) \simeq x + F(v)$ such that the following diagram is commutative:

$$\begin{tikzcd} F((x+y)+v) \arrow[r] \arrow[d] & (x+y)+F(v) \arrow[d] \\ F(x+(y+v)) \arrow[r] & x+(y+F(v)) \end{tikzcd} \tag{2-7}$$

[4]Indeed, consider the geometrization of the nerve of $\mathcal{P}$. Then the Picard structure of $\mathcal{P}$ puts an E_∞-structure on this space.

Morphisms are natural transformations $\theta : F_1 \to F_2$ such that the following diagram is commutative:

$$\begin{tikzcd} F_1(x+v) \arrow[r] \arrow[d, "\theta"'] & x+F_1(v) \arrow[d, "\theta"] \\ F_2(x+v) \arrow[r] & x+F_2(v) \end{tikzcd}$$

From these discussions it follows that all $\mathscr{P}$-torsors form a 2-category, denoted by $B\mathscr{P}$. We will choose, once and for all, for any $\mathscr{P}$-torsors $\mathscr{L}_1, \mathscr{L}_2$ and any F in $\mathrm{Hom}_{\mathscr{P}}(\mathscr{L}_1, \mathscr{L}_2)$, a quasi-inverse F^{-1} of F together with an isomorphism $F^{-1}F \simeq \mathrm{id}$.

Moreover, $B\mathscr{P}$ is a category enriched over itself. That is, for any $\mathscr{P}$-torsors $\mathscr{L}_1, \mathscr{L}_2$ the category $\mathrm{Hom}_{\mathscr{P}}(\mathscr{L}_1, \mathscr{L}_2)$ is again a $\mathscr{P}$-torsor, where an action of $\mathscr{P}$ on $\mathrm{Hom}_{\mathscr{P}}(\mathscr{L}_1, \mathscr{L}_2)$ is defined as follows: for any $z \in \mathscr{P}$, $v \in \mathscr{L}_1$, $F \in \mathrm{Hom}_{\mathscr{P}}(\mathscr{L}_1, \mathscr{L}_2)$ we put $z+F \in \mathrm{Hom}_{\mathscr{P}}(\mathscr{L}_1, \mathscr{L}_2)$ as $(z+F)(v) := z+F(v)$. Now the isomorphism λ for the equivalence $z+F$ is defined by means of the braiding maps c in $\mathscr{P}$ (commutativity constraints from Section 2A). Then the diagram (2-7) for the equivalence $z+F$ follows from hexagon diagram (2-2). It is clear that this definition is extended to the definition of a bifunctor

$$+ : \mathscr{P} \times \mathrm{Hom}_{\mathscr{P}}(\mathscr{L}_1, \mathscr{L}_2) \to \mathrm{Hom}_{\mathscr{P}}(\mathscr{L}_1, \mathscr{L}_2) \tag{2-8}$$

such that the axioms of $\mathscr{P}$-torsor are satisfied (see the beginning of this section).

We note that to prove that the category $B\mathscr{P}$ is enriched over itself we used the commutativity constraints in $\mathscr{P}$. The commutativity constraints will be important also below to define the sum of two $\mathscr{P}$-torsors.

The category $B\mathscr{P}$ furthermore forms a Picard 2-groupoid. We will not make the definition of Picard 2-groupoids precise. (However, one refers to [Kapranov and Voevodsky 1994; Breen 1994] for details). We will only describe the Picard structure on $B\mathscr{P}$ in the way we need.

First, if $\mathscr{L}_1, \mathscr{L}_2$ are two $\mathscr{P}$-torsors, then $\mathscr{L}_1 + \mathscr{L}_2$ is defined to be the category whose objects are pairs (v, w), where $v \in \mathscr{L}_1$ and $w \in \mathscr{L}_2$. The morphisms from (v, w) to (v', w') are defined as the equivalence classes of triples $(x, \varphi_1, \varphi_2)$, where $x \in \mathscr{P}$, $\varphi_1 \in \mathrm{Hom}_{\mathscr{L}_1}(v, x+v')$ and $\varphi_2 \in \mathrm{Hom}_{\mathscr{L}_2}(x+w, w')$, and $(x, \varphi_1, \varphi_2) \sim (y, \phi_1, \phi_2)$ if there exists a map $f : x \to y$ such that $\phi_1 = f(\varphi_1)$ and $\varphi_2 = f(\phi_2)$. The identity in $\mathrm{Hom}_{\mathscr{L}_1+\mathscr{L}_2}((v, w), (v, w))$ and the composition

$$\mathrm{Hom}_{\mathscr{L}_1+\mathscr{L}_2}((v, w), (v', w')) \times \mathrm{Hom}_{\mathscr{L}_1+\mathscr{L}_2}((v', w'), (v'', w'')) \\ \to \mathrm{Hom}_{\mathscr{L}_1+\mathscr{L}_2}((v, w), (v'', w''))$$

are clear. (To define the composition we have to use the commutativity constraints in $B\mathscr{P}$.) So $\mathscr{L}_1 + \mathscr{L}_2$ is a category. Define the action of $\mathscr{P}$ on $\mathscr{L}_1 + \mathscr{L}_2$ by

$$x + (v, w) := (x + v, w).$$

The natural isomorphism $(x + y) + (v, w) \simeq x + (y + (v, w))$ is the obvious one. It is easy to check that $\mathcal{L}_1 + \mathcal{L}_2$ is a $\mathcal{P}$-torsor.

There is an obvious 1-isomorphism of $\mathcal{P}$-torsors

$$A : (\mathcal{L}_1 + \mathcal{L}_2) + \mathcal{L}_3 \simeq \mathcal{L}_1 + (\mathcal{L}_2 + \mathcal{L}_3),$$

which is the associativity constraint. Namely, objects in $(\mathcal{L}_1 + \mathcal{L}_2) + \mathcal{L}_3$ and in $\mathcal{L}_1 + (\mathcal{L}_2 + \mathcal{L}_3)$ are both canonically bijective to triples (v_1, v_2, v_3) where $v_i \in \mathcal{L}_i$. Then A is identity on objects. A morphism from (v_1, v_2, v_3) to (w_1, w_2, w_3) in $(\mathcal{L}_1 + \mathcal{L}_2) + \mathcal{L}_3$ is of the form $(x, (y, \varphi_1, \varphi_2), \varphi_3)$, where $x, y \in \mathcal{P}$, $\varphi_1 : v_1 \to y + (x + w_1)$, $\varphi_2 : y + v_2 \to w_2$, $\varphi_3 : x + v_3 \to w_3$. Then A maps $(x, (y, \varphi_1, \varphi_2), \varphi_3)$ to $(x + y, \varphi_1', (x, \varphi_2', \varphi_3'))$, where $\varphi_1' : v_1 \to (x + y) + w_1$ comes from

$$v_1 \xrightarrow{\varphi_1} y + (x + w_1) \simeq (y + x) + w_1 \simeq (x + y) + w_1,$$

$\varphi_2' : (x + y) + v_2 \to x + w_2$ comes from

$$(x + y) + v_2 \simeq x + (y + v_2) \xrightarrow{x+\varphi_2} x + w_2,$$

and $\varphi_3' : x + v_3 \to w_3$ is the same as φ_3.

To complete the definition of A, we should specify for every $x \in \mathcal{P}$, $(v_1, v_2, v_3) \in (\mathcal{L}_1 + \mathcal{L}_2) + \mathcal{L}_3$, an isomorphism $\lambda : A(x + (v_1, v_2, v_3)) \simeq x + A(v_1, v_2, v_3)$ such that the diagram (2-7) is commutative for $F = A$. It is clear that $\lambda = \mathrm{id} : (x + v_1, v_2, v_3) = (x + v_1, v_2, v_3)$ will suffice for this purpose.

It is clear from definition of A that we can similarly construct a 1-morphism A^{-1} of $\mathcal{P}$-torsors such that the following equalities are satisfied:

$$A^{-1}A = AA^{-1} = \mathrm{id}.$$

From above construction of the associativity constraints (1-morphisms A and A^{-1}) it follows that for any $\mathcal{P}$-torsors $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \mathcal{L}_4$ the following diagram of 1-morphisms (pentagon diagram) is commutative:

$$\begin{tikzcd}[column sep=small]
 & (\mathcal{L}_1 + \mathcal{L}_2) + (\mathcal{L}_3 + \mathcal{L}_4) \arrow[dl] \arrow[dr] & \\
\mathcal{L}_1 + (\mathcal{L}_2 + (\mathcal{L}_3 + \mathcal{L}_4)) \arrow[d] & & ((\mathcal{L}_1 + \mathcal{L}_2) + \mathcal{L}_3) + \mathcal{L}_4 \arrow[d] \\
\mathcal{L}_1 + ((\mathcal{L}_2 + \mathcal{L}_3) + \mathcal{L}_4) \arrow[rr] & & (\mathcal{L}_1 + (\mathcal{L}_2 + \mathcal{L}_3)) + \mathcal{L}_4
\end{tikzcd} \tag{2-9}$$

(To prove this diagram we note that this diagram is evident for objects from category $(\mathcal{L}_1 + \mathcal{L}_2) + (\mathcal{L}_3 + \mathcal{L}_4)$. To verify this diagram for morphisms from this category one needs to make some routine calculations. The analogous reasonings are also applied to the diagram (2-13) below.)

The following axioms are satisfied in the category $B\mathscr{P}$ and describe the functoriality of the associativity constraints. Let $\mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3, \mathscr{L}_1'$ be any $\mathscr{P}$-torsors, and $\mathscr{L}_1 \to \mathscr{L}_1'$ be any 1-morphism of $\mathscr{P}$-torsors, then the following diagram of 1-morphisms is commutative:

$$\begin{tikzcd}
(\mathscr{L}_1+\mathscr{L}_2)+\mathscr{L}_3 \arrow[r] \arrow[d] & (\mathscr{L}_1'+\mathscr{L}_2)+\mathscr{L}_3 \arrow[d] \\
\mathscr{L}_1+(\mathscr{L}_2+\mathscr{L}_3) \arrow[r] & \mathscr{L}_1'+(\mathscr{L}_2+\mathscr{L}_3).
\end{tikzcd} \tag{2-10}$$

Let $\mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3, \mathscr{L}_2'$ be any $\mathscr{P}$-torsors, and $\mathscr{L}_2 \to \mathscr{L}_2'$ be any 1-morphism of $\mathscr{P}$-torsors, then the following diagram of 1-morphisms is commutative:

$$\begin{tikzcd}
(\mathscr{L}_1+\mathscr{L}_2)+\mathscr{L}_3 \arrow[r] \arrow[d] & (\mathscr{L}_1+\mathscr{L}_2')+\mathscr{L}_3 \arrow[d] \\
\mathscr{L}_1+(\mathscr{L}_2+\mathscr{L}_3) \arrow[r] & \mathscr{L}_1+(\mathscr{L}_2'+\mathscr{L}_3).
\end{tikzcd} \tag{2-11}$$

Let $\mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3, \mathscr{L}_3'$ be any $\mathscr{P}$-torsors, and $\mathscr{L}_3 \to \mathscr{L}_3'$ be any 1-morphism of $\mathscr{P}$-torsors, then the following diagram of 1-morphisms is commutative:

$$\begin{tikzcd}
(\mathscr{L}_1+\mathscr{L}_2)+\mathscr{L}_3 \arrow[r] \arrow[d] & (\mathscr{L}_1+\mathscr{L}_2)+\mathscr{L}_3' \arrow[d] \\
\mathscr{L}_1+(\mathscr{L}_2+\mathscr{L}_3) \arrow[r] & \mathscr{L}_1+(\mathscr{L}_2+\mathscr{L}_3').
\end{tikzcd} \tag{2-12}$$

(In diagrams (2-10)–(2-12) the vertical arrows are the associativity constraints.)

Next we define the commutativity constraints. Recall that we have chosen for each $x \in \mathscr{P}$ its inverse $(-x, \phi_x)$, and then obtained the isomorphism (2-4). This gives an obvious 1-isomorphism

$$C : \mathscr{L}_1+\mathscr{L}_2 \simeq \mathscr{L}_2+\mathscr{L}_1.$$

Namely, C will map the object (v_1, v_2) to (v_2, v_1), and $(x, \varphi_1, \varphi_2) : (v_1, v_2) \to (w_1, w_2)$ to $(-x, \varphi_1', \varphi_2') : (v_2, v_1) \to (w_2, w_1)$, where

$$\varphi_1' : v_2 \simeq e+v_2 \simeq (-x+x)+v_2 \simeq -x+(x+v_2) \overset{-x+\varphi_2}{\to} -x+w_2,$$

$$\varphi_2' : -x+v_1 \overset{-x+\varphi_1}{\to} -x+(x+w_1) \simeq (-x+x)+w_1 \simeq e+w_1 \simeq w_1.$$

We also define for each $x \in \mathscr{P}$, $(v_1, v_2) \in \mathscr{L}_1+\mathscr{L}_2$, the isomorphism

$$\lambda : C(x+(v_1, v_2)) = (v_2, x+v_1) \to x+C(v_1, v_2) = (x+v_2, v_1)$$

by $\lambda = (-x, \varphi_1, \varphi_2)$, where $\varphi_1 : v_2 \simeq (-x+x)+v_2 \simeq -x+(x+v_2)$ and $\varphi_2 : -x+(x+v_1) \simeq (-x+x)+v_1 \simeq v_1$.

In addition, by (2-3), there is an equality of 1-morphisms $C^2 = \mathrm{id}$.

The commutativity constrains together with the associativity constrains satisfy the hexagon diagram; i.e., for any $\mathcal{P}$-torsors $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$ the following diagram of 1-morphisms is commutative:

$$\begin{tikzcd}
 & (\mathcal{L}_1+\mathcal{L}_2)+\mathcal{L}_3 \arrow[dl] \arrow[dr] & \\
(\mathcal{L}_1+\mathcal{L}_2)+\mathcal{L}_3 \arrow[d] & & \mathcal{L}_1+(\mathcal{L}_2+\mathcal{L}_3) \arrow[d] \\
\mathcal{L}_1+(\mathcal{L}_2+\mathcal{L}_3) \arrow[dr] & & \mathcal{L}_1+(\mathcal{L}_2+\mathcal{L}_3) \arrow[dl] \\
 & (\mathcal{L}_1+\mathcal{L}_2)+\mathcal{L}_3 &
\end{tikzcd} \tag{2-13}$$

The following axiom is satisfied in the category $B\mathcal{P}$ and describes the functoriality of the commutativity constraints. Let $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_1'$ be any $\mathcal{P}$-torsors, and $\mathcal{L}_1 \to \mathcal{L}_1'$ be any 1-morphism of $\mathcal{P}$-torsors, then the following diagram of 1-morphisms is commutative:

$$\begin{tikzcd}
\mathcal{L}_1+\mathcal{L}_2 \arrow[r] \arrow[d] & \mathcal{L}_1'+\mathcal{L}_2 \arrow[d] \\
\mathcal{L}_2+\mathcal{L}_1 \arrow[r] & \mathcal{L}_2+\mathcal{L}_1',
\end{tikzcd} \tag{2-14}$$

where the vertical arrows are the commutativity constraints.

By regarding $\mathcal{P}$ as a $\mathcal{P}$-torsor, there is a canonical 1-isomorphism of $\mathcal{P}$-torsors $\mathcal{P}+\mathcal{L} \to \mathcal{L}$, $(x, v) \mapsto x+v$ satisfying the associativity and commutativity constraints. This means that $\mathcal{P}$ is the unit in $B\mathcal{P}$. For each $\mathcal{L} \in B\mathcal{P}$, we have an object

$$-\mathcal{L} := \mathrm{Hom}_{\mathcal{P}}(\mathcal{L}, \mathcal{P}),$$

together with a natural 1-isomorphism of $\mathcal{P}$-torsors $\varphi_{\mathcal{L}} : \mathcal{L}+(-\mathcal{L}) \simeq \mathcal{P}$. This object is called an inverse of $\mathcal{L}$.

For $\mathcal{L}$ a $\mathcal{P}$-torsor, $\mathrm{Hom}_{\mathcal{P}}(\mathcal{L}, \mathcal{L})$ is a natural monoidal groupoid (by composition). The natural homomorphism

$$\mathfrak{L} : \mathcal{P} \to \mathrm{Hom}_{\mathcal{P}}(\mathcal{L}, \mathcal{L}) \tag{2-15}$$

given by $\mathfrak{X}(z) = z + \mathrm{id}$[5] is a 1-isomorphism of monoidal groupoids. We will fix once and for all its inverse, i.e., we choose an 1-isomorphism of monoidal groupoids

$$\mathfrak{X}^{-1} : \mathrm{Hom}_{\mathcal{P}}(\mathcal{L}, \mathcal{L}) \to \mathcal{P} \tag{2-16}$$

together with a 2-isomorphism $\mathfrak{X}^{-1} \circ \mathfrak{X} \simeq \mathrm{id}$.

Remark 2.3. We constructed some "semistrict" version of Picard 2-groupoid, because diagrams (2-9)–(2-14) are true in $B\mathcal{P}$ for 1-morphisms without consideration of additional 2-morphisms which involve higher coherence axioms for braided monoidal 2-categories as in [Kapranov and Voevodsky 1994] and [Baez and Neuchl 1996]. Besides, from the equality $C^2 = \mathrm{id}$ we obtain at one stroke that our 2-category $B\mathcal{P}$ is strongly braided, i.e, the diagram (8.4.6) in [Breen 1994, p. 149] holds. Let us mention that in *loc. cit.*, the commutativity constraint C is denoted by R.

2D. *The case $H^1(BG, \mathcal{P})$*. Let $\mathcal{P}$ be a Picard groupoid, and G be a group. Then we define $H^1(BG, \mathcal{P})$ to be the Picard groupoid of homomorphisms from G to $\mathcal{P}$. That is, the objects are monoidal functors from G to $\mathcal{P}$, where G is regarded as a discrete monoidal category (the monoidal groupoid whose objects are elements of G and whose only morphisms are the unit morphisms of objects), and morphisms between these monoidal functors are monoidal natural transformations. In concrete terms, $f \in H^1(BG, \mathcal{P})$ is a functor $f : G \to \mathcal{P}$, together with isomorphisms

$$f(gg') \simeq f(g) + f(g')$$

which are compatible with the associativity constraints. The monoidal structure on $H^1(BG, \mathcal{P})$ is given by $(f + f')(g) = f(g) + f(g')$. The natural isomorphism $(f + f')(gg') \simeq (f + f')(g) + (f + f')(g')$ is the obvious one. The associativity constraints and the commutativity constraints on $H^1(BG, \mathcal{P})$ are clear. Let (e, φ) be a unit of $\mathcal{P}$, and e is regarded as a discrete Picard groupoid with one object. Then $f : G \to \mathcal{P}$ is called trivial if it is isomorphic to $G \to e \to \mathcal{P}$.

Example 2.4. If $\mathcal{P} = BA$, then $H^1(BG, BA)$ is equivalent to the category of central extensions of G by A as Picard groupoids.

Let $Z_2 \subset G \times G$ be the subset of commuting elements, so that if G itself is an abelian group, then $Z_2 = G \times G$. In general, fix $g \in G$, then $Z_2 \cap (G \times g) \simeq Z_2 \cap (g \times G) \simeq Z_G(g)$, the centralizer of g in G.

Lemma-Definition 2.5. *There is a well defined antisymmetric bimultiplicative map* $\mathrm{Comm}(f) : Z_2 \to \pi_1(\mathcal{P}) = \mathrm{End}_{\mathcal{P}}(e)$.

[5]Recall that we constructed the bifunctor $+ : \mathcal{P} \times \mathrm{Hom}_{\mathcal{P}}(\mathcal{L}, \mathcal{L}) \to \mathrm{Hom}_{\mathcal{P}}(\mathcal{L}, \mathcal{L})$ in (2-8).

Proof. The definition of $\mathrm{Comm}(f)$ is as follows. For $g_1, g_2 \in Z_2$, we have

$$f(g_1 g_2) \simeq f(g_1) + f(g_2) \simeq f(g_2) + f(g_1) \simeq f(g_2 g_1) = f(g_1 g_2),$$

where the first and the third isomorphisms come from the constraints for the homomorphism f, and the second isomorphism comes from the commutativity constraints of the Picard groupoid $\mathcal{P}$. We thus obtain an element

$$\mathrm{Comm}(f)(g_1, g_2) \in \mathrm{Aut}_{\mathcal{P}}(f(g_1 g_2)) \simeq \pi_1(\mathcal{P}).$$

Since $\mathcal{P}$ is Picard, i.e., the commutativity constraints satisfy

$$c_{f(g_1), f(g_2)} = c^{-1}_{f(g_2), f(g_1)},$$

the map Comm is antisymmetric. One checks directly by diagram that $\mathrm{Comm}(f)$ is also bimultiplicative (see the analogous diagram (2-29) below).

Here we will give another proof of bimultiplicativity whose higher categorical analog we will use in the proof of Lemma-Definition 2.13. We construct the following category H_f, where objects of H_f are all possible expressions

$$f(g_1) + \cdots + f(g_k) := (\cdots(f(g_1) + f(g_2)) + f(g_3)) + \cdots) + f(g_k), \ \text{where } g_i \in G,$$

and morphisms in H_f are defined as follows:

$$\begin{aligned} &\mathrm{Hom}_{H_f}(f(g_{i_1}) + \cdots + f(g_{i_k}), f(g_{j_1}) + \cdots + f(g_{j_l})) \\ &= \begin{cases} \varnothing & \text{if } g_{i_1} \ldots g_{i_k} \neq g_{j_1} \ldots g_{j_l}, \\ \mathrm{Hom}_{\mathcal{P}}(f(g_{i_1}) + \cdots + f(g_{i_k}), f(g_{j_1}) + \cdots + f(g_{j_l})) & \text{if } g_{i_1} \ldots g_{i_k} = g_{j_1} \ldots g_{j_l}. \end{cases} \end{aligned}$$

The category H_f is a monoidal group-like groupoid (or gr-category), where the monoidal structure on H_f is given in an obvious way by using the associativity constraints in the category $\mathcal{P}$. We have $\pi_0(H_f) = G$, and H_f is equivalent to the trivial gr-category. We consider $\pi_1(\mathcal{P})$-torsor E over Z_2 which is the commutator of H_f (see [Breen 1999, §3]). The fiber of E over $(g_1, g_2) \in Z_2$ is the set

$$E_{g_1, g_2} = \mathrm{Hom}_{H_f}(f(g_1) + f(g_2), f(g_2) + f(g_1)).$$

The $\pi_1(\mathcal{P})$-torsor E has a natural structure of a weak biextension of Z_2 by $\pi_1(\mathcal{P})$ (see [Breen 1999, Proposition 3.1]), i.e., there are partial composition laws on E which are compatible (see also (2-22)). Now the commutativity constraints $c_{f(g_1), f(g_2)}$ give a section of E over Z_2 which is compatible with partial composition laws on E, i.e., "bimultiplicative". (The compatibility of this section with the composition laws follows at once from the definition of the partial composition laws on E and the hexagon diagram (2-2).) The other section of E which is compatible with partial composition laws on E is obtained as the composition of following two morphisms from definition of f: $f(g_1) + f(g_2) \simeq f(g_1 g_2) = f(g_2 g_1) \simeq f(g_2) + f(g_1)$. (The compatibility of this section with composition laws follows from diagrams

(3.10) and (1.4) of [Breen 1999], because of the compatibility of our homomorphism f with the associativity constraints.) Now the difference between the first section and the second section coincides with $\mathrm{Comm}(f)$, which is, thus, a bimultiplicative function, because both sections are "bimultiplicative". □

Remark 2.6. In [Breen 1999, §2] the notion of a weak biextension was intoduced only for $Z_2 = B \times B$ where B is an abelian group. Here, we generalize this notion by allowing B to be non-commutative and by replacing $B \times B$ by Z_2. But all the axioms for partial composition laws in *loc. cit.* are still applicable in this setting. The same remark applies when we talk about (2, 2)-extensions on page 308.

Remark 2.7. It is clear that if $f \simeq f'$ in $H^1(BG, \mathcal{P})$, then $\mathrm{Comm}(f) = \mathrm{Comm}(f')$.

Remark 2.8. When $\mathcal{P} = BA$, this construction reduces to the usual construction of inverse to the commutator pairing maps for central extensions.

Corollary 2.9. *One has* $\mathrm{Comm}(f + f') = \mathrm{Comm}(f) + \mathrm{Comm}(f')$.

Proof. It can be easily checked directly by diagrams. See, for example, analogous formulas and diagrams in (2-30)–(2-32) below. □

Corollary 2.10. *Assume that G is abelian so that $Z_2 = G \times G$. Then* $\mathrm{Comm}(f)$ *is trivial if and only if the* 1*-homomorphism f is a* 1*-homomorphism of Picard groupoids. In particular, if the homomorphism f is trivial, then* $\mathrm{Comm}(f)$ *is trivial.*

Proof. This follows from diagram (2-5). □

Together, these two corollaries can be rephrased as saying that if G is abelian, then there is an exact sequence of Picard groupoids

$$1 \to \mathrm{Hom}(G, \mathcal{P}) \to H^1(BG, \mathcal{P}) \to \mathrm{Hom}\big(\textstyle\bigwedge^2 G, \pi_1(\mathcal{P})\big).$$

2E. *The case $H^2(BG, \mathcal{P})$*. If $\mathcal{P}'$ is a Picard n-groupoid, and G is a group, one should be able to define $H^1(BG, \mathcal{P}')$ as the Picard n-groupoid of homomorphisms from G to $\mathcal{P}'$. When $n = 1$, this is what we discussed in the previous subsection. The next step for consideration is $n = 2$. Again, instead of discussing general Picard 2-groupoids, we will focus on the case when $\mathcal{P}' = B\mathcal{P}$, where $\mathcal{P}$ is a Picard groupoid. Then one can interpret $H^1(BG, B\mathcal{P})$ as the Picard groupoid[6] of central extensions of the group G by the Picard groupoid $\mathcal{P}$. For this reason, we also denote $H^1(BG, B\mathcal{P})$ by $H^2(BG, \mathcal{P})$.

In concrete terms, an object $\mathcal{L}$ in $H^2(BG, \mathcal{P})$ is a rule to assign to every $g \in G$ a $\mathcal{P}$-torsor $\mathcal{L}_g$, and to every g, g' an equivalence $\mathcal{L}_{gg'} \simeq \mathcal{L}_g + \mathcal{L}_{g'}$ of $\mathcal{P}$-torsors, and

[6]As we just mentioned, it is in fact a Picard 2-groupoid.

to every g, g', g'' an isomorphism between two equivalences

$$\begin{tikzcd}
& \mathscr{L}_{gg'g''} \arrow[dl] \arrow[dr] & \\
\mathscr{L}_{gg'} + \mathscr{L}_{g''} \arrow[d] & \Downarrow & \mathscr{L}_g + \mathscr{L}_{g'g''} \arrow[d] \\
(\mathscr{L}_g + \mathscr{L}_{g'}) + \mathscr{L}_{g''} \arrow[rr] & & \mathscr{L}_g + (\mathscr{L}_{g'} + \mathscr{L}_{g''})
\end{tikzcd} \tag{2-17}$$

such that for every g, g', g'', g''', the natural compatibility condition holds, which we describe below.

Remark 2.11. Our notation for the 2-arrow in diagram (2-17) is symbolic, and is distinct from the traditional notation of 2-arrows in a 2-category, because this 2-arrow is between a pair of 1-arrows from $\mathscr{L}_{gg'g''}$ to $\mathscr{L}_g + (\mathscr{L}'_g + \mathscr{L}''_g)$ and should be written horizontally from left to right rather than vertically. This notation for the 2-arrow will be important for us in diagram (2-28).

We define an isomorphism between two central extensions of G by $\mathscr{P}$. An isomorphism between two central extensions $\mathscr{L}$, $\mathscr{L}'$ is a rule which assigns to any g a $\mathscr{P}$-torsor 1-isomorphism $\mathscr{L}_g \simeq \mathscr{L}'_g$, and to any g, g' the following 2-isomorphism

$$\begin{tikzcd}
\mathscr{L}_{gg'} \arrow[r] \arrow[d] & \mathscr{L}_g + \mathscr{L}_{g'} \arrow[d] \\
\mathscr{L}'_{gg'} \arrow[r] \arrow[ur, phantom, "\Nearrow"] & \mathscr{L}'_g + \mathscr{L}'_{g'}
\end{tikzcd} .$$

In addition, these assignments have to be compatible with diagram (2-17) in an obvious way.

Now we describe the compatibility condition which we need after diagram (2-17). If we don't consider the associativity constraints in category $B\mathscr{P}$, then the 2-arrows induced by the one in (2-17) should satisfy the compatibility condition described by the following cube:

$$\begin{tikzcd}
& \mathscr{L}_g + \mathscr{L}_{g'} + \mathscr{L}_{g''g'''} \arrow[rr] & & \mathscr{L}_g + \mathscr{L}_{g'} + \mathscr{L}_{g''} + \mathscr{L}_{g'''} \\
\mathscr{L}_g + \mathscr{L}_{g'g''g'''} \arrow[ur] \arrow[rr] & & \mathscr{L}_g + \mathscr{L}_{g'g''} + \mathscr{L}_{g'''} \arrow[ur] & \\
& \mathscr{L}_{gg'} + \mathscr{L}_{g''g'''} \arrow[uu, dashed] \arrow[rr, dashed] & & \mathscr{L}_{gg'} + \mathscr{L}_{g''} + \mathscr{L}_{g'''} \arrow[uu] \\
\mathscr{L}_{gg'g''g'''} \arrow[uu] \arrow[ur, dashed] \arrow[rr] & & \mathscr{L}_{gg'g''} + \mathscr{L}_{g'''} \arrow[uu] \arrow[ur] &
\end{tikzcd} \tag{2-18}$$

To obtain the correct compatibility diagram for 2-morphisms, we have to replace in diagram (2-18) the arrow (an edge of cube)

$$\mathcal{L}_{gg'} + \mathcal{L}_{g''} + \mathcal{L}_{g'''} \longrightarrow \mathcal{L}_{g} + \mathcal{L}_{g'} + \mathcal{L}_{g''} + \mathcal{L}_{g'''}$$

by the following commutative diagram of 1-morphisms in the category $B\mathcal{P}$

$$\begin{tikzcd}
(\mathcal{L}_{gg'} + \mathcal{L}_{g''}) + \mathcal{L}_{g'''} \arrow[r] \arrow[d] & ((\mathcal{L}_{g} + \mathcal{L}_{g'}) + \mathcal{L}_{g''}) + \mathcal{L}_{g'''} \arrow[d] \\
\mathcal{L}_{gg'} + (\mathcal{L}_{g''} + \mathcal{L}_{g'''}) \arrow[r] & (\mathcal{L}_{g} + \mathcal{L}_{g'}) + (\mathcal{L}_{g''} + \mathcal{L}_{g'''})
\end{tikzcd} \tag{2-19}$$

(where the vertical arrows are associativity constraints); we have to replace in diagram (2-18) the arrow (an edge of the cube)

$$\mathcal{L}_{g} + \mathcal{L}_{g'g''} + \mathcal{L}_{g'''} \longrightarrow \mathcal{L}_{g} + \mathcal{L}_{g'} + \mathcal{L}_{g''} + \mathcal{L}_{g'''}$$

by the following commutative diagram of 1-morphisms in the category $B\mathcal{P}$:

$$\begin{tikzcd}
(\mathcal{L}_{g} + \mathcal{L}_{g'g''}) + \mathcal{L}_{g'''} \arrow[r] \arrow[d] & (\mathcal{L}_{g} + (\mathcal{L}_{g'} + \mathcal{L}_{g''})) + \mathcal{L}_{g'''} \arrow[d] \\
\mathcal{L}_{g} + (\mathcal{L}_{g'g''} + \mathcal{L}_{g'''}) \arrow[r] & \mathcal{L}_{g} + ((\mathcal{L}_{g'} + \mathcal{L}_{g''}) + \mathcal{L}_{g'''})
\end{tikzcd} \tag{2-20}$$

(where vertical arrows are associativity constraints); we have to replace in diagram (2-18) the arrow (an edge of the cube)

$$\mathcal{L}_{g} + \mathcal{L}_{g'} + \mathcal{L}_{g''g'''} \longrightarrow \mathcal{L}_{g} + \mathcal{L}_{g'} + \mathcal{L}_{g''} + \mathcal{L}_{g'''}$$

by the following commutative diagram of 1-morphisms in the category $B\mathcal{P}$:

$$\begin{tikzcd}
(\mathcal{L}_{g} + \mathcal{L}_{g'}) + \mathcal{L}_{g''g'''} \arrow[r] \arrow[d] & (\mathcal{L}_{g} + \mathcal{L}_{g'}) + (\mathcal{L}_{g''} + \mathcal{L}_{g'''}) \arrow[d] \\
\mathcal{L}_{g} + (\mathcal{L}_{g'} + \mathcal{L}_{g''g'''}) \arrow[r] & \mathcal{L}_{g} + (\mathcal{L}_{g'} + (\mathcal{L}_{g''} + \mathcal{L}_{g'''}))
\end{tikzcd} \tag{2-21}$$

(where vertical arrows are associativity constraints). Besides, instead of the vertex $\mathcal{L}_g + \mathcal{L}_{g'} + \mathcal{L}_{g''} + \mathcal{L}_{g'''}$ in diagram (2-18) we insert the commutative diagram which is the modification of pentagon diagram (2-9) for $\mathcal{L}_g$, $\mathcal{L}_{g'}$, $\mathcal{L}_{g''}$, $\mathcal{L}_{g'''}$, and this diagram is always true in category $B\mathcal{P}$. The correct compatibility diagram for 2-morphisms from diagrams (2-17) has 15 vertices.

We note that diagrams (2-19)–(2-21) are commutative for 1-morphisms; that is, the corresponding 2-isomorphisms equal identity morphisms. These diagrams express the "functoriality" of associativity constraints in $B\mathcal{P}$ and follow from axioms-diagrams (2-10)–(2-12) in category $B\mathcal{P}$.

The trivial central extension of G by $\mathcal{P}$, which we will denote by the same letter $\mathcal{P}$, is the rule that assigns to every $g \in G$ the trivial $\mathcal{P}$-torsor $\mathcal{P}$, to every g, g' the natural 1-isomorphism $\mathcal{P} \simeq \mathcal{P} + \mathcal{P}$,[7] and to every g, g', g'' the corresponding natural 2-isomorphism.

Remark 2.12. A central extension $\mathcal{L}$ of G by $\mathcal{P}$ gives rise to a gr-category, $\tilde{\mathcal{L}}$, together with a short exact sequence of gr-categories in the sense of [Breen 1992, Definition 2.1.2]

$$1 \to \mathcal{P} \xrightarrow{i} \tilde{\mathcal{L}} \xrightarrow{\pi} G \to 1.$$

Namely, as a category, $\tilde{\mathcal{L}} = \bigcup_{g \in G} \mathcal{L}_g$. Then the natural equivalence $\mathcal{L}_{gg'} \simeq \mathcal{L}_g + \mathcal{L}_{g'}$ together with the compatibility conditions endows $\tilde{\mathcal{L}}$ with a gr-category structure. The natural morphism $\pi : \tilde{\mathcal{L}} \to G$ is clearly monoidal, and one can show that $\ker \pi = \mathcal{L}_e$ is 1-isomorphic to $\mathcal{P}$.

As is shown in *loc. cit.*, such a short exact sequence endows every $\tilde{\mathcal{L}}_g := \pi^{-1}(g) = \mathcal{L}_g$ with a $\mathcal{P}$-bitorsor structure. This $\mathcal{P}$-bitorsor structure is nothing but the canonical $\mathcal{P}$-bitorsor structure on $\mathcal{L}_g$ (observe that the morphism $\mathcal{X} : \mathcal{P} \to \mathrm{Hom}_{\mathcal{P}}(\mathcal{L}_g, \mathcal{L}_g)$ as in (2-15) induces a canonical $\mathcal{P}$-bitorsor structure on $\mathcal{L}_g$).

The upshot is that an object $\mathcal{L}$ in $H^2(BG, \mathcal{P})$ gives rise to a categorical generalization of a central extension of a group by an abelian group. This justifies our terminology. Indeed, one can define a central extension of G by $\mathcal{P}$ as a short exact sequence as above such that the induced $\mathcal{P}$-bitorsor structure on each $\tilde{\mathcal{L}}_g$ is the canonical one induced from its left $\mathcal{P}$-torsor structure. Since we do not use this second definition, we will not make it precise.

Finally, let us define the Picard structure on $H^2(BG, \mathcal{P})$. Let $\mathcal{L}$ and $\mathcal{L}'$ be two central extensions of G by $\mathcal{P}$. Then we define the central extension $\mathcal{L} + \mathcal{L}'$ by the following way:

$$(\mathcal{L} + \mathcal{L}')_g := \mathcal{L}_g + \mathcal{L}'_g,$$

and the equivalence $(\mathcal{L} + \mathcal{L}')_{gg'} \simeq (\mathcal{L} + \mathcal{L}')_g + (\mathcal{L} + \mathcal{L}')_{g'}$ as the composition of the equivalences

$$\begin{aligned}(\mathcal{L} + \mathcal{L}')_{gg'} = \mathcal{L}_{gg'} + \mathcal{L}'_{gg'} &\simeq (\mathcal{L}_g + \mathcal{L}_{g'}) + (\mathcal{L}'_g + \mathcal{L}'_{g'}) \\ &\simeq (\mathcal{L}_g + \mathcal{L}'_g) + (\mathcal{L}_{g'} + \mathcal{L}'_{g'}) = (\mathcal{L} + \mathcal{L}')_g + (\mathcal{L} + \mathcal{L}')_{g'}.\end{aligned}$$

The corresponding 2-isomorphism for central extension $\mathcal{L} + \mathcal{L}'$ and any elements g, g', g'' of G follows from diagrams (2-17) for central extensions $\mathcal{L}$ and $\mathcal{L}'$. The further compatibility conditions for these 2-isomorphisms hold as in diagrams (2-18)–(2-21), since they follow at once from the corresponding diagrams for central extensions $\mathcal{L}$ and $\mathcal{L}'$.

[7] Naturality means this 1-isomorphism is the chosen quasi-inverse of the natural 1-isomorphism $\mathcal{P} + \mathcal{P} \to \mathcal{P}$.

Again, let Z_2 denote the subset of $G \times G$ consisting of commuting elements. We will give a categorical analog of Lemma-Definition 2.5. For this purpose, let us first explain some terminology. A 1-morphism $f : Z_2 \to \mathcal{P}$ is called bimultiplicative if for fixed $g \in G$, $(Z_G(g), g) \subset Z_2 \to \mathcal{P}$ and $(g, Z_G(g)) \subset Z_2 \to \mathcal{P}$ are homomorphisms, i.e., monoidal functors from discrete monoidal categories $(Z_G(g), g)$ and $(g, Z_G(g))$ to $\mathcal{P}$. In addition, the following diagram must be commutative (which is the compatibility condition between these two homomorphisms):

$$\begin{tikzcd}
f(g_1g_2, g_3) + f(g_1g_2, g_4) \arrow[r] & (f(g_1, g_3) + f(g_2, g_3)) + (f(g_1, g_4) + f(g_2, g_4)) \arrow[dd, "\simeq"] \\
f(g_1g_2, g_3g_4) \arrow[u, "\simeq"'] \arrow[d, "\simeq"'] & \\
f(g_1, g_3g_4) + f(g_2, g_3g_4) \arrow[r] & (f(g_1, g_3) + f(g_1, g_4)) + (f(g_2, g_3) + f(g_2, g_4))
\end{tikzcd} \tag{2-22}$$

When $\mathcal{P} = BA$, a bimultiplicative 1-morphism from $Z_2 \to BA$ is the same as a weak biextension of Z_2 by A as defined in [Breen 1999, §2] (see also Remark 2.6).

A 1-morphism $f : Z_2 \to \mathcal{P}$ is called antisymmetric if there is a 2-isomorphism $\theta : f \simeq -f \circ \sigma$, where σ is the natural flip on Z_2, such that for any $(g_1, g_2) \in Z_2$, the following diagram is commutative:

$$\begin{tikzcd}
f(g_1, g_2) \arrow[r, "\simeq"] \arrow[d, equal] & -f(g_2, g_1) \arrow[d, "\simeq"] \\
f(g_1, g_2) & -(-f(g_1, g_2)) \arrow[l, "\simeq"']
\end{tikzcd}$$

We need some more terminology. Following [Breen 1999, §7], we *define* a weak (2, 2)-extension of Z_2 by $\mathcal{P}$ as a rule which assigns to every $(g, g') \in Z_2$ a $\mathcal{P}$-torsor $\mathcal{E}_{(g,g')}$ such that its restrictions to $(g, Z_G(g))$ and $Z_G(g), g)$ are central extensions of $Z_G(g)$ by $\mathcal{P}$, and that the corresponding diagram (2-22) is 2-commutative (i.e., commutative modulo some 2-isomorphism), and these 2-isomorphisms satisfy further compatibility conditions (see (7.1), (7.3) in *loc. cit.* where these compatibility conditions are carefully spelt out).

Lemma-Definition 2.13. *There is an antisymmetric bimultiplicative homomorphism $C_2^{\mathcal{L}} : Z_2 \to \mathcal{P}$.*

Proof. As in the proof of 2.5, using the commutativity constraints $C : \mathcal{L}_g + \mathcal{L}_{g'} \simeq \mathcal{L}_{g'} + \mathcal{L}_g$ in the category $B\mathcal{P}$, one constructs the following composition of 1-isomorphisms:

$$\mathcal{L}_{gg'} \simeq \mathcal{L}_g + \mathcal{L}_{g'} \simeq \mathcal{L}_{g'} + \mathcal{L}_g \simeq \mathcal{L}_{g'g} = \mathcal{L}_{gg'},$$

for $(g, g') \in Z_2$. In this way, we obtain a functor $Z_2 \to \mathrm{Hom}_{\mathcal{P}}(\mathcal{L}_{gg'}, \mathcal{L}_{gg'})$. Using $\mathfrak{X}^{-1} : \mathrm{Hom}_{\mathcal{P}}(\mathcal{L}_{gg'}, \mathcal{L}_{gg'}) \to \mathcal{P}$ (see (2-16)), we get a morphism $C_2^{\mathcal{L}} : Z_2 \to \mathcal{P}$.

We need to construct the following canonical isomorphisms

$$C_2^{\mathcal{L}}(gg', g'') \simeq C_2^{\mathcal{L}}(g, g'') + C_2^{\mathcal{L}}(g', g''), \quad C_2^{\mathcal{L}}(g, g'g'') \simeq C_2^{\mathcal{L}}(g, g') + C_2^{\mathcal{L}}(g, g''),$$

satisfying the natural compatibility conditions. We now construct the first isomorphism. The second is similar. Let $\mathfrak{X} : \mathcal{P} \to \mathrm{Hom}_{\mathcal{P}}(\mathcal{L}_{gg'g''}, \mathcal{L}_{gg'g''})$ be the canonical equivalence as monoidal groupoids as in (2-15). It is enough to construct a canonical 2-isomorphism $\mathfrak{X}(C_2^{\mathcal{L}}(gg', g'')) \simeq \mathfrak{X}(C_2^{\mathcal{L}}(g, g'') + C_2^{\mathcal{L}}(g', g''))$.

By the definition of the morphism $C_2^{\mathcal{L}}$, there is a canonical 2-isomorphism from 1-isomorphism $\mathfrak{X}(C_2^{\mathcal{L}}(g, g'') + C_2^{\mathcal{L}}(g', g''))$ to the following composition of 1-isomorphisms:

$$\begin{aligned}\mathcal{L}_{gg'g''} &\simeq \mathcal{L}_g + \mathcal{L}_{g'g''} \simeq \mathcal{L}_g + (\mathcal{L}_{g'} + \mathcal{L}_{g''}) \simeq \mathcal{L}_g + (\mathcal{L}_{g''} + \mathcal{L}_{g'}) \simeq \mathcal{L}_g + \mathcal{L}_{g''g'} \simeq \mathcal{L}_{gg''g'} \\ &\simeq \mathcal{L}_{gg''} + \mathcal{L}_{g'} \simeq (\mathcal{L}_g + \mathcal{L}_{g''}) + \mathcal{L}_{g'} \simeq (\mathcal{L}_{g''} + \mathcal{L}_g) + \mathcal{L}_{g'} \simeq \mathcal{L}_{gg''} + \mathcal{L}_{g'} \simeq \mathcal{L}_{gg'g''}\end{aligned} \tag{2-23}$$

By the definition of the central extension of G by $\mathcal{P}$ (see diagram (2-17)), there is a canonical 2-isomorphism from the above composition of 1-isomorphisms to the following composition of 1-isomorphisms

$$\begin{aligned}\mathcal{L}_{gg'g''} &\simeq \mathcal{L}_g + \mathcal{L}_{g'g''} \simeq \mathcal{L}_g + (\mathcal{L}_{g'} + \mathcal{L}_{g''}) \simeq \mathcal{L}_g + (\mathcal{L}_{g''} + \mathcal{L}_{g'}) \\ &\simeq (\mathcal{L}_g + \mathcal{L}_{g''}) + \mathcal{L}_{g'} \simeq (\mathcal{L}_{g''} + \mathcal{L}_g) + \mathcal{L}_{g'} \simeq \mathcal{L}_{gg''} + \mathcal{L}_{g'} \simeq \mathcal{L}_{gg'g''}.\end{aligned} \tag{2-24}$$

From the hexagon axiom for 1-morphisms in the category $B\mathcal{P}$ (see diagram (2-13)) we have that the 1-isomorphism which is the composition of the 1-isomorphisms above is equal to the 1-isomorphism that is the composition of the 1-isomorphisms

$$\begin{aligned}\mathcal{L}_{gg'g''} &\simeq \mathcal{L}_g + \mathcal{L}_{g'g''} \simeq \mathcal{L}_g + (\mathcal{L}_{g'} + \mathcal{L}_{g''}) \simeq (\mathcal{L}_g + \mathcal{L}_{g'}) + \mathcal{L}_{g''} \\ &\simeq \mathcal{L}_{g''} + (\mathcal{L}_g + \mathcal{L}_{g'}) \simeq (\mathcal{L}_{g''} + \mathcal{L}_g) + \mathcal{L}_{g'} \simeq \mathcal{L}_{gg''} + \mathcal{L}_{g'} \simeq \mathcal{L}_{gg'g''}.\end{aligned} \tag{2-25}$$

By the "functoriality" of the commutativity constraints in the category $B\mathcal{P}$ (see axiom-diagram (2-14)), the 1-isomorphism that is the composition of the 1-isomorphisms above is equal to the one that is the composition of the 1-isomorphisms

$$\begin{aligned}\mathcal{L}_{gg'g''} &\simeq \mathcal{L}_g + \mathcal{L}_{g'g''} \simeq \mathcal{L}_g + (\mathcal{L}_{g'} + \mathcal{L}_{g''}) \simeq (\mathcal{L}_g + \mathcal{L}_{g'}) + \mathcal{L}_{g''} \simeq \mathcal{L}_{gg'} + \mathcal{L}_{g''} \\ &\simeq \mathcal{L}_{g''} + \mathcal{L}_{gg'} \simeq \mathcal{L}_{g''} + (\mathcal{L}_g + \mathcal{L}_{g'}) \simeq (\mathcal{L}_{g''} + \mathcal{L}_g) + \mathcal{L}_{g'} \simeq \mathcal{L}_{gg''} + \mathcal{L}_{g'} \simeq \mathcal{L}_{gg'g''}.\end{aligned} \tag{2-26}$$

Again, by the definition of the central extension of G by $\mathcal{P}$ (see diagram (2-17), which we apply twice now), there is a canonical 2-isomorphism from the composition of 1-isomorphisms above to the composition of 1-isomorphisms

$$\mathcal{L}_{gg'g''} \simeq \mathcal{L}_{gg'} + \mathcal{L}_{g''} \simeq \mathcal{L}_{g''} + \mathcal{L}_{gg'} \simeq \mathcal{L}_{gg'g''}, \tag{2-27}$$

which is canonically isomorphic to $\mathfrak{X}(C_2^{\mathscr{L}}(gg', g''))$.

Let us write down a diagram which will represent the above 2-isomorphisms. To simplify the notation, we will denote the 2-commutative diagram (2-17) by

$$\begin{tikzcd} \mathscr{L}_{gg'g''} \arrow[d, Rightarrow] \\ \mathscr{L}_g + \mathscr{L}_{g'} + \mathscr{L}_{g''} \end{tikzcd} \tag{2-28}$$

Then, the 2-isomorphism $\mathfrak{X}(C_2^{\mathscr{L}}(gg', g'')) \simeq \mathfrak{X}(C_2^{\mathscr{L}}(g, g'')) + \mathfrak{X}(C_2^{\mathscr{L}}(g', g''))$ is represented by the diagram

$$\begin{tikzcd}[column sep=small] & & \mathscr{L}_{gg''g'} \arrow[d, Rightarrow] \arrow[ddrr, "\mathfrak{X}(C_2^{\mathscr{L}}(g,g''))"] & & \\ & & \mathscr{L}_g + \mathscr{L}_{g''} + \mathscr{L}_{g'} \arrow[dl] \arrow[dr] & & \\ \mathscr{L}_{gg'g''} \arrow[r, Rightarrow] \arrow[uurr, "\mathfrak{X}(C_2^{\mathscr{L}}(g',g''))"] \arrow[rrrr, bend right, "\mathfrak{X}(C_2^{\mathscr{L}}(gg',g''))"'] & \mathscr{L}_g + \mathscr{L}_{g'} + \mathscr{L}_{g''} \arrow[rr] & & \mathscr{L}_{g''} + \mathscr{L}_g + \mathscr{L}_{g'} & \mathscr{L}_{g''gg'} \arrow[l, Rightarrow] \end{tikzcd} \tag{2-29}$$

To check all the compatibility conditions between these canonical isomorphisms we generalize the proof of Lemma-Definition 2.5. We construct a 2-category $H_{\mathscr{L}}$ whose objects are objects from categories given by all expressions

$$\mathscr{L}_{g_1} + \cdots + \mathscr{L}_{g_k} := (\cdots(\mathscr{L}_{g_1} + \mathscr{L}_{g_2}) + \mathscr{L}_{g_3}) + \cdots) + \mathscr{L}_{g_k}, \text{ where } g_i \in G;$$

the 1-morphisms in $H_{\mathscr{L}}$ are defined as follows:

$$\begin{aligned} &\operatorname{Hom}_{H_{\mathscr{L}}}(\mathscr{L}_{g_{i_1}} + \cdots + \mathscr{L}_{g_{i_k}}, \mathscr{L}_{g_{j_1}} + \cdots + \mathscr{L}_{g_{j_l}}) \\ &\qquad = \begin{cases} \varnothing & \text{if } g_{i_1} \dots g_{i_k} \neq g_{j_1} \dots g_{j_l}, \\ \operatorname{Hom}_{B\mathscr{P}}(\mathscr{L}_{g_{i_1}} + \cdots + \mathscr{L}_{g_{i_k}}, \mathscr{L}_{g_{j_1}} + \cdots + \mathscr{L}_{g_{j_l}}) & \text{if } g_{i_1} \dots g_{i_k} = g_{j_1} \dots g_{j_l}, \end{cases} \end{aligned}$$

and the 2-morphisms in the 2-category $H_{\mathscr{L}}$ come from the 2-morphisms of category $B\mathscr{P}$. The category $H_{\mathscr{L}}$ is a monoidal group-like 2-groupoid (or a 2-gr-category), see [Breen 1994, §8], where the monoidal structure on $H_{\mathscr{L}}$ is given in an obvious way by using the associativity constraints in the category $B\mathscr{P}$ and the pentagon diagram (2-9). We have $\pi_0(H_{\mathscr{L}}) = G$. We consider the $\mathscr{P}$-torsor $\mathscr{E}^{\mathscr{L}}$ on Z_2 which is the commutator of $H_{\mathscr{L}}$. (See [Breen 1999, §8].[8]) The fiber of $\mathscr{E}^{\mathscr{L}}$ over $(g_1, g_2) \in Z_2$

[8] L. Breen assumed for simplicity in *loc. cit.* that the group π_1 of a 2-gr-category is equal to 0. We have $\pi_1(H_{\mathscr{L}}) \neq 0$, but the constructions and its properties which we need remain true in our situation.

is the $\mathcal{P}$-torsor

$$\mathcal{E}^{\mathcal{L}}_{g_1,g_2} = \mathrm{Hom}_{H_{\mathcal{L}}}(\mathcal{L}_{g_1} + \mathcal{L}_{g_2}, \mathcal{L}_{g_2} + \mathcal{L}_{g_1}).$$

The $\mathcal{P}$-torsor $\mathcal{E}^{\mathcal{L}}$ on Z_2 has a natural structure of a weak $(2,2)$-extension (see [Breen 1999, Proposition 8.1]), i.e., there are partial composition (group) laws on $\mathcal{E}^{\mathcal{L}}$ which are compatible (see diagrams (7.1), (7.3) in *loc. cit*). Now the commutativity constraints C from $B\mathcal{P}$ give a trivialization of $\mathcal{P}$-torsor $\mathcal{E}^{\mathcal{L}}$ on Z_2 which is compatible with partial composition laws on $\mathcal{E}^{\mathcal{L}}$, i.e., “bimultiplicative”. (The compatibility of this trivialization with composition laws follows at once from definition of partial composition laws on $\mathcal{E}^{\mathcal{L}}$ and hexagon diagram (2-13). See also the discussion in the end of [Breen 1999, §8] regarding the braiding structure in $H_{\mathcal{L}}$, which gives the “bimultiplicative” trivialization of the $\mathcal{P}$-torsor $\mathcal{E}^{\mathcal{L}}$ on Z_2.) The other trivialization of the $\mathcal{P}$-torsor $\mathcal{E}^{\mathcal{L}}$ on Z_2 which is compatible with partial composition laws on $\mathcal{E}^{\mathcal{L}}$ is obtained as the composition of the following two equivalences from definition of $\mathcal{L}$:

$$S_{\mathcal{L}_{g_1},\mathcal{L}_{g_2}} : \mathcal{L}_{g_1} + \mathcal{L}_{g_2} \simeq \mathcal{L}_{g_1 g_2} = \mathcal{L}_{g_2 g_1} \simeq \mathcal{L}_{g_2} + \mathcal{L}_{g_1}.$$

Now the difference between the first trivialization and the second trivialization of the $\mathcal{P}$-torsor $\mathcal{E}^{\mathcal{L}}$ on Z_2 coincides with $C_2^{\mathcal{L}}$, which is, thus, a bimultiplicative homomorphism, because both trivializations are “bimultiplicative”.

We have shown that $C_2^{\mathcal{L}} : Z_2 \to \mathcal{P}$ is a bimultiplicative 1-morphism. One readily checks from the above constructions that this is antisymmetric from Z_2 to $\mathcal{P}$, since $C^2 = \mathrm{id}$. □

Remark 2.14. If $\mathcal{P} = BA$, then the construction of $C_2^{\mathcal{L}}$ given above is equivalent to the construction of the commutator category of the central extension $-\mathcal{L}$ introduced in [Deligne 1991].

We also have the following categorical analog of Corollary 2.9. First, let us remark that if $f_1, f_2 : Z_2 \to \mathcal{P}$ are two bimultiplicative homomorphisms, one can define $f_1 + f_2$, which is again a bimultiplicative homomorphism, in the same way as defining the Picard structure on $H^1(BG, \mathcal{P})$.

Lemma 2.15. *For any two central extensions $\mathcal{L}$ and $\mathcal{L}'$ of G by $\mathcal{P}$ there is a natural bimultiplicative 2-isomorphism (i.e., it respects the bimultiplicative structure) between bimultiplicative 1-morphisms $C_2^{\mathcal{L}+\mathcal{L}'}$ and $C_2^{\mathcal{L}} + C_2^{\mathcal{L}'}$.*

Proof. Recall that we have the following canonical 1-isomorphism

$$\mathfrak{X} : \mathcal{P} \to \mathrm{Hom}(\mathcal{L}_{gg'} + \mathcal{L}'_{gg'}, \mathcal{L}_{gg'} + \mathcal{L}'_{gg'}).$$

We construct a canonical isomorphism

$$\mathfrak{X}(C_2^{\mathcal{L}+\mathcal{L}'}(g,g')) \simeq \mathfrak{X}(C_2^{\mathcal{L}}(g,g') + C_2^{\mathcal{L}'}(g,g'))$$

for any $(g,g')\in Z_2$ as follows. By definition, $\mathfrak{L}(C_2^{\mathscr{L}+\mathscr{L}'}(g,g'))$ is canonically 2-isomorphic to the composition of 1-morphisms

$$\begin{aligned}(\mathscr{L}+\mathscr{L}')_{gg'}&=\mathscr{L}_{gg'}+\mathscr{L}'_{gg'}\simeq(\mathscr{L}_g+\mathscr{L}_{g'})+(\mathscr{L}'_g+\mathscr{L}'_{g'})\simeq(\mathscr{L}_g+\mathscr{L}'_g)+(\mathscr{L}_{g'}+\mathscr{L}'_{g'})\\&=(\mathscr{L}+\mathscr{L}')_g+(\mathscr{L}+\mathscr{L}')_{g'}\simeq(\mathscr{L}+\mathscr{L}')_{g'}+(\mathscr{L}+\mathscr{L}')_g\simeq(\mathscr{L}_{g'}+\mathscr{L}'_{g'})+(\mathscr{L}_g+\mathscr{L}'_g)\\&\simeq(\mathscr{L}_{g'}+\mathscr{L}_g)+(\mathscr{L}'_{g'}+\mathscr{L}'_g)\simeq\mathscr{L}_{g'g}+\mathscr{L}'_{g'g}=(\mathscr{L}+\mathscr{L}')_{g'g}=(\mathscr{L}+\mathscr{L}')_{gg'}.\end{aligned}\tag{2-30}$$

Using the functoriality of commutativity constraints, i.e., applying diagram (2-14) twice, and using the following commutative diagram (which is written without associativity constraints)

$$\begin{tikzcd}
& \mathscr{L}_g+\mathscr{L}_{g'}+\mathscr{L}'_g+\mathscr{L}'_{g'} \arrow[dl] \arrow[dr] & \\
\mathscr{L}_g+\mathscr{L}_{g'}+\mathscr{L}'_{g'}+\mathscr{L}'_g \arrow[d] \arrow[drr] & & \mathscr{L}_g+\mathscr{L}'_g+\mathscr{L}_{g'}+\mathscr{L}'_{g'} \arrow[ll] \arrow[d] \\
\mathscr{L}_{g'}+\mathscr{L}_g+\mathscr{L}'_{g'}+\mathscr{L}'_g & & \mathscr{L}_{g'}+\mathscr{L}'_{g'}+\mathscr{L}_g+\mathscr{L}'_g \arrow[ll]
\end{tikzcd}\tag{2-31}$$

(to obtain the correct diagram we have to replace every triangle in this diagram by a hexagon coming from (2-13)), we obtain that the composition of 1-morphisms in (2-30) is equal to the composition of 1-morphisms

$$\begin{aligned}\mathscr{L}_{gg'}+\mathscr{L}'_{gg'}&\simeq(\mathscr{L}_g+\mathscr{L}_{g'})+(\mathscr{L}'_g+\mathscr{L}'_{g'})\\&\simeq(\mathscr{L}_{g'}+\mathscr{L}_g)+(\mathscr{L}'_{g'}+\mathscr{L}'_g)\simeq\mathscr{L}_{g'g}+\mathscr{L}'_{g'g}=\mathscr{L}_{gg'}+\mathscr{L}'_{gg'},\end{aligned}\tag{2-32}$$

which is, by definition, 2-isomorphic to $\mathfrak{L}(C_2^{\mathscr{L}}(g,g')+C_2^{\mathscr{L}'}(g,g'))$.

To complete the proof, we need to show that the diagram

$$\begin{tikzcd}
C_2^{\mathscr{L}+\mathscr{L}'}(g,g'')+C_2^{\mathscr{L}+\mathscr{L}'}(g',g'') \arrow[r] & (C_2^{\mathscr{L}}(g,g'')+C_2^{\mathscr{L}'}(g,g''))+(C_2^{\mathscr{L}}(g',g'')+C_2^{\mathscr{L}'}(g',g'')) \arrow[dd] \\
C_2^{\mathscr{L}+\mathscr{L}'}(gg',g'') \arrow[u] \arrow[d] & \\
C_2^{\mathscr{L}}(gg',g'')+C_2^{\mathscr{L}'}(gg',g'') \arrow[r] & (C_2^{\mathscr{L}}(g,g'')+C_2^{\mathscr{L}}(g',g''))+(C_2^{\mathscr{L}'}(g,g'')+C_2^{\mathscr{L}'}(g',g''))
\end{tikzcd}\tag{2-33}$$

and a similar diagram involving $C_2^{\mathscr{L}+\mathscr{L}'}(g,g'g'')$ are commutative. To prove this, let us recall that the 2-isomorphism $C_2^{\mathscr{L}}(gg',g'')\simeq C_2^{\mathscr{L}}(g,g'')+C_2^{\mathscr{L}}(g',g'')$ is the composition of the 2-isomorphisms

$$\mathfrak{L}(C_2^{\mathscr{L}}(g,g'')+C_2^{\mathscr{L}}(g',g''))\to\text{(2-23)}\to\text{(2-24)}\to\cdots\to\text{(2-27)}\to\mathfrak{L}(C_2^{\mathscr{L}}(gg',g'')).$$

Let us denote the 1-isomorphism (2-23) for $\mathcal{L}$ (resp. $\mathcal{L}'$, resp. $\mathcal{L}+\mathcal{L}'$) as $(2\text{-}23)_{\mathcal{L}}$ (resp. $(2\text{-}23)_{\mathcal{L}'}$, resp. $(2\text{-}23)_{\mathcal{L}+\mathcal{L}'}$) and etc. Then it is readily checked that there exists a canonical 2-isomorphism

$$(2\text{-}23)_{\mathcal{L}} + (2\text{-}23)_{\mathcal{L}'} \simeq (2\text{-}23)_{\mathcal{L}+\mathcal{L}'}$$

between corresponding 1-isomorphisms $\mathcal{L}_{gg'g''} + \mathcal{L}'_{gg'g''} \to \mathcal{L}_{gg'g''} + \mathcal{L}_{gg'g''}$, and canonical 2-isomorphisms for (2-24)–(2-27) such that the diagram

$$\begin{tikzcd}
(2\text{-}23)_{\mathcal{L}} + (2\text{-}23)_{\mathcal{L}'} \arrow[r] \arrow[d] & (2\text{-}23)_{\mathcal{L}+\mathcal{L}'} \arrow[d] \\
(2\text{-}24)_{\mathcal{L}} + (2\text{-}24)_{\mathcal{L}'} \arrow[r] & (2\text{-}24)_{\mathcal{L}+\mathcal{L}'}
\end{tikzcd}$$

and similar diagrams for (2-24)–(2-27) commute. In addition, the following diagrams commute:

$$\begin{tikzcd}
\mathfrak{X}(C_2^{\mathcal{L}+\mathcal{L}'}(g,g'')+C_2^{\mathcal{L}+\mathcal{L}'}(g',g'')) \arrow[r] \arrow[d] & \begin{array}{l}\mathfrak{X}\big((C_2^{\mathcal{L}}(g,g'')+C_2^{\mathcal{L}}(g',g'')) \\ \quad +(C_2^{\mathcal{L}'}(g,g'')+C_2^{\mathcal{L}'}(g',g''))\big)\end{array} \arrow[d] \\
(2\text{-}23)_{\mathcal{L}+\mathcal{L}'} \arrow[r] & (2\text{-}23)_{\mathcal{L}} + (2\text{-}23)_{\mathcal{L}'}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathfrak{X}(C_2^{\mathcal{L}+\mathcal{L}'}(gg',g'')) \arrow[r] & \mathfrak{X}(C_2^{\mathcal{L}}(gg',g'')+C_2^{\mathcal{L}'}(gg',g'')) \\
(2\text{-}27)_{\mathcal{L}+\mathcal{L}'} \arrow[r] \arrow[u] & (2\text{-}27)_{\mathcal{L}} + (2\text{-}27)_{\mathcal{L}'} \arrow[u]
\end{tikzcd}$$

These facts together imply the commutativity of diagram (2-33). □

Fix $g \in G$, the induced map $Z_G(g) \to \mathcal{P}$ given by $g' \mapsto C_2^{\mathcal{L}}(g,g')$ is denoted by $C_g^{\mathcal{L}}$. The bimultiplicativity of $C_2^{\mathcal{L}}$ implies that $C_g^{\mathcal{L}}$ is an object in $H^1(BZ_G(g),\mathcal{P})$. It is easy to see from the definition the following lemma:

Lemma 2.16. (i) *If two central extensions $\mathcal{L}$ and $\mathcal{L}'$ of G by $\mathcal{P}$ are isomorphic in $H^2(BG,\mathcal{P})$, then for any g the induced two homomorphisms $C_g^{\mathcal{L}}$ and $C_g^{\mathcal{L}'}$ are isomorphic in $H^1(BZ_G(g),\mathcal{P})$.*

(ii) *$C_g^{\mathcal{P}}$ is the trivial homomorphism for any $g \in G$.*

Let $Z_3 \subset G \times G \times G$ be the subset of pairwise commuting elements.

Proposition 2.17. *The map*

$$C_3^{\mathcal{L}} : Z_3 \to \pi_1(\mathcal{P})$$

defined by

$$C_3^{\mathcal{L}}(g,g',g'') := \mathrm{Comm}(C_g^{\mathcal{L}})(g',g'').$$

is an antisymmetric trimultiplicative homomorphism from Z_3 to $\pi_1(\mathcal{P})$.

Proof. We check the trimultiplicativity of the map $C_3^{\mathscr{L}}$. The multiplicativity of this map with respect to g' or g'' follows from Lemma-Definition 2.5. Multiplicativity with respect to g follows from Lemma-Definition 2.13 and Corollary 2.9.

The hard part now is to prove that the map $C_3^{\mathscr{L}}$ is antisymmetric. Let us write C_2 instead of $C_2^{\mathscr{L}}$, and C_3 instead of $C_3^{\mathscr{L}}$ for simplicity. Let $(g, g', g'') \in Z_3$. First of all, let us observe that by definition, there is a canonical isomorphism

$$C_2(g, g'g'') + C_2(g', g'') \simeq C_2(g', g'') + C_2(g, g''g') \tag{2-34}$$

induced by the 2-commutative diagram

$$\begin{tikzcd}
\mathscr{L}_{gg'g''} \arrow[r] \arrow[rrr, bend left, "\mathfrak{X}(C_2(g,g'g''))"] \arrow[ddd, "\mathfrak{X}(C_2(g',g''))"'] & \mathscr{L}_g + \mathscr{L}_{g'g''} \arrow[r] \arrow[d] & \mathscr{L}_{g'g''} + \mathscr{L}_g \arrow[r] \arrow[d] & \mathscr{L}_{gg'g''} \arrow[ddd, "\mathfrak{X}(C_2(g',g''))"] \\
& \mathscr{L}_g + (\mathscr{L}_{g'} + \mathscr{L}_{g''}) \arrow[r] \arrow[d] & (\mathscr{L}_{g'} + \mathscr{L}_{g''}) + \mathscr{L}_g \arrow[d] & \\
& \mathscr{L}_g + (\mathscr{L}_{g''} + \mathscr{L}_{g'}) \arrow[r] \arrow[d] & (\mathscr{L}_{g''} + \mathscr{L}_{g'}) + \mathscr{L}_g \arrow[d] & \\
\mathscr{L}_{gg''g'} \arrow[r] \arrow[rrr, bend right, "\mathfrak{X}(C_2(g,g''g'))"'] & \mathscr{L}_g + \mathscr{L}_{g''g'} \arrow[r] & \mathscr{L}_{g'g''} + \mathscr{L}_g \arrow[r] & \mathscr{L}_{gg''g'}
\end{tikzcd}$$

The following lemma can be checked using the definition of $B\mathscr{P}$.

Lemma 2.18. *The isomorphism (2-34) is the same as the commutativity constraint in $\mathscr{P}$.*

Now, there are two isomorphisms between $(C_2(g, g') + C_2(g, g'')) + C_2(g', g'')$ and $C_2(g', g'') + (C_2(g, g'') + C_2(g, g'))$. Namely, the first isomorphism is obtained by the associativity and commutativity constraints in $\mathscr{P}$. (Recall that such isomorphism is unique by Mac Lane's coherence theorem for Picard category.) The second isomorphism is

$$\begin{aligned}(C_2(g, g') + C_2(g, g'')) + C_2(g', g'') \simeq C_2(g, g'g'') + C_2(g', g'') \overset{(2\text{-}34)}{\simeq} \\ C_2(g', g'') + C_2(g, g''g') \simeq C_2(g', g'') + (C_2(g, g'') + C_2(g, g')).\end{aligned} \tag{2-35}$$

By the lemma, the difference between these two isomorphisms is $C_3(g, g', g'')$. If we recall the definition of $C_2(g, g'g'') \simeq C_2(g, g') + C_2(g, g'')$ by (2-29), we see that the isomorphism (2-35) can be represented by the diagram

$$
\begin{tikzcd}
 & \mathscr{L}_{g'gg''} \arrow[rrr, "\mathfrak{X}(C_2(g,g''))"] \arrow[dr, Rightarrow] & & & \mathscr{L}_{g'g''g} \arrow[dl, Rightarrow] \arrow[ddr, "\mathfrak{X}(C_2(g',g''))"] & \\
 & & \mathscr{L}_{g'}+\mathscr{L}_g+\mathscr{L}_{g''} \arrow[r] & \mathscr{L}_{g'}+\mathscr{L}_{g''}+\mathscr{L}_g \arrow[dr] & & \\
\mathscr{L}_{gg'g''} \arrow[uur, "\mathfrak{X}(C_2(g,g'))"] \arrow[ddr, "\mathfrak{X}(C_2(g',g''))"'] \arrow[r, Rightarrow] & \mathscr{L}_g+\mathscr{L}_{g'}+\mathscr{L}_{g''} \arrow[ur] \arrow[dr] & & & \mathscr{L}_{g''}+\mathscr{L}_{g'}+\mathscr{L}_g & \mathscr{L}_{g''g'g} \arrow[l, Rightarrow] \\
 & & \mathscr{L}_g+\mathscr{L}_{g''}+\mathscr{L}_{g'} \arrow[r] & \mathscr{L}_{g''}+\mathscr{L}_g+\mathscr{L}_{g'} \arrow[ur] & & \\
 & \mathscr{L}_{gg''g'} \arrow[rrr, "\mathfrak{X}(C_2(g,g''))"'] \arrow[ur, Rightarrow] & & & \mathscr{L}_{g''gg'} \arrow[ul, Rightarrow] \arrow[uur, "\mathfrak{X}(C_2(g,g'))"'] &
\end{tikzcd}
$$

This diagram clearly implies that $C_3(g, g', g'') = C_3(g', g'', g)$. This, together with the fact that $C_3(g, g', g'') = -C_3(g, g'', g')$ (because the map $\mathrm{Comm}(C_g^{\mathscr{L}})$ is antisymmetric), implies that C_3 is antisymmetric. □

Corollary 2.19. (i) *If two central extensions $\mathscr{L}$ and $\mathscr{L}'$ of G by $\mathscr{P}$ are isomorphic in $H^2(BG, \mathscr{P})$, then $C_3^{\mathscr{L}} = C_3^{\mathscr{L}'}$.*

(ii) *$C_3^{\mathscr{P}}$ is trivial.*

Corollary 2.20. *For any two central extensions $\mathscr{L}$ and $\mathscr{L}'$ of G by $\mathscr{P}$ we have*

$$C_3^{\mathscr{L}+\mathscr{L}'} = C_3^{\mathscr{L}} + C_3^{\mathscr{L}'}.$$

Proof. This follows from Lemma 2.15, Corollary 2.9 and the definition of C_3. □

Remark 2.21. If $\mathscr{P} = BA$, where A is an abelian group, then a central extension $\mathscr{L}$ of a group G by the Picard groupoid $\mathscr{P}$ is a gr-category such that these gr-categories are classified by the group $H^3(G, A)$ with the trivial G-module A. In this case the map $C_3^{\mathscr{L}}$ coincides with the symmetrization of corresponding 3-cocycle; see [Breen 1999, §4]. (This follows from Remarks 2.14, 2.8 and [Osipov 2003, Proposition 10].)

3. Tate vector spaces

3A. ***The category of Tate vector spaces.*** We first review the definition of Tate vector spaces, following [Osipov 2007; Arkhipov and Kremnizer 2010]. Let us fix a base field k.

Recall that Beilinson [1987] associates to an exact category $\mathscr{C}$ in the sense of [Quillen 1973] another exact category $\underset{\longleftrightarrow}{\lim}\,\mathscr{C}$, which is again an exact category. In nowadays terminology, this is the category of locally compact objects of $\mathscr{C}$.

For an exact category $\mathscr{E}$, let $\hat{\mathscr{E}}$ denote the category of left exact additive contravariant functors from the category $\mathscr{E}$ to the category of abelian groups. This is again an exact category (in fact an abelian category), in which arbitrary small colimits exist. The Yoneda embedding $h : \mathscr{E} \to \hat{\mathscr{E}}$ is exact. Then the category $\mathrm{Ind}(\mathscr{E})$ of (strict) ind-objects of $\mathscr{E}$, is the full subcategory of $\hat{\mathscr{E}}$ consisting of objects of the form $\varinjlim_{i\in I} h(X_i)$, where I is a filtered small category, and $X_i \in \mathscr{E}$, such that for $i \to j$ in I, the map $X_i \to X_j$ is an admissible monomorphism. This category is a natural exact category. Likewise, one can define $\mathrm{Pro}(\mathscr{E})$ as $\mathrm{Ind}(\mathscr{E}^{\mathrm{op}})^{\mathrm{op}}$.

Definition 3.1. Let $\mathscr{E}$ be an exact category. Then $\underleftrightarrow{\lim}\, \mathscr{E}$ is the full subcategory of $\mathrm{Pro}(\mathrm{Ind}(\mathscr{E}))$ consisting of objects that can be represented as

$$\varprojlim_{i\in I^{\mathrm{op}}} \varinjlim_{j\in I} h(X_{ij})$$

such that, for any $i \to i'$, $j \to j'$, the following diagram is cartesian (which automatically makes it cocartesian).

$$\begin{tikzcd} X_{ij} \arrow[r] \arrow[d] & X_{ij'} \arrow[d] \\ X_{i'j} \arrow[r] & X_{i'j'} \end{tikzcd}$$

One can show that $\underleftrightarrow{\lim}\, \mathscr{E}$ is an exact category and the embedding $\underleftrightarrow{\lim}\, \mathscr{E} \to \mathrm{Pro}(\mathrm{Ind}(\mathscr{E}))$ is exact. Further, there is a natural embedding $\underleftrightarrow{\lim}\, \mathscr{E} \to \mathrm{Ind}(\mathrm{Pro}(\mathscr{E}))$ which is again exact. It is clear that the natural embedding $\mathrm{Ind}(\mathscr{E}) \to \mathrm{Pro}(\mathrm{Ind}(\mathscr{E}))$ lands in $\underleftrightarrow{\lim}\, \mathscr{E}$, and similarly the natural embedding $\mathrm{Pro}(\mathscr{E}) \to \mathrm{Ind}(\mathrm{Pro}(\mathscr{E}))$ lands in $\underleftrightarrow{\lim}\, \mathscr{E}$.

Definition 3.2. Define Tate_0 to be the category of finite dimensional vector spaces, together with its canonical exact category structure. Define $\mathrm{Tate}_n = \underleftrightarrow{\lim}\, \mathrm{Tate}_{n-1}$, together with the canonical exact category structure given by Beilinson.

There is a canonical forgetful functor $F_n : \mathrm{Tate}_n \to \mathscr{T}op$, where $\mathscr{T}op$ denotes the category of topological vector spaces. As is shown in [Osipov 2007], the functor is fully faithful when $n = 1$, but this is in general not the case when $n > 1$.

Definition 3.3. Let V be an object of Tate_n. A *lattice* L of V is an object in Tate_n which actually belongs to $\mathrm{Pro}(\mathrm{Tate}_{n-1})$, together with an admissible monomorphism $L \to V$ such that the object V/L belongs to $\mathrm{Ind}(\mathrm{Tate}_{n-1})$. A *colattice* L^c of V is an object in Tate_n which actually belongs to $\mathrm{Ind}(\mathrm{Tate}_{n-1})$, together with an addmissible monomorphism $L^c \to V$ such that the object V/L^c belongs to $\mathrm{Pro}(\mathrm{Tate}_{n-1})$.

It is clear that if L is a lattice of V and L^c is a colattice, then $L \cap L^c$ belongs to Tate_{n-1}.

The main players of this paper are Tate_1 and Tate_2. The category Tate_1 is just the category of locally linearly compact k-vector spaces. A typical object in Tate_1 is the field of formal Laurent series $k((t))$, that is, the field of fractions of the ring $k[[t]]$. The field $k((t))$ is equipped with the standard topology, where the base of neighborhoods of zero consists of integer powers of the maximal ideal of $k[[t]]$. The subspace $k[[t]]$ is a lattice in $k((t))$ and $k[t^{-1}]$ is a colattice. Observe that $k[t] \subset k((t))$ is neither a lattice nor a colattice, because the subspace $k[t]$ is not closed in the topological space $k((t))$. Therefore the embedding $k[t] \hookrightarrow k((t))$ is not an admissible monomorphism, since any admissible (exact) triple in the category Tate_1 is of the form

$$0 \longrightarrow V_1 \longrightarrow V_2 \longrightarrow V_3 \longrightarrow 0,$$

where the locally linearly compact vector space V_1 is a closed vector subspace in a locally linearly compact vector space V_2, and the locally linearly compact vector space V_3 has the quotient topology on the quotient vector space.

A typical object in Tate_2 is $k((t))((s))$, since

$$k((t))((s)) = \varprojlim_{l\in\mathbb{Z}} \varinjlim_{m\leq l} s^m k((t))[[s]] \,/\, s^l k((t))[[s]] = \varinjlim_{m\in\mathbb{Z}} \varprojlim_{l\geq m} s^m k((t))[[s]] \,/\, s^l k((t))[[s]],$$

and $s^m k((t))[[s]] \,/\, s^l k((t))[[s]]$ is a locally linearly compact k-vector space.

The k-space $k((t))[[s]]$ is a lattice, and the k-space $k((t))[s^{-1}]$ is a colattice in the k-space $k((t))((s))$. As just mentioned above, it is not enough to regard them as topological vector spaces. On the other hand $k[[t]]((s))$ is not a lattice in $k((t))((s))$ although the natural map $k[[t]]((s)) \to k((t))((s))$ is an admissible monomorphism.

Remark 3.4. The category Tate_n coincides with the category of complete C_n-spaces from [Osipov 2007].

3B. ***Determinant theories of Tate vector spaces.*** We consider Tate_0 as an exact category. Then $\det : (\mathrm{Tate}_0, \mathrm{isom}) \to \mathcal{P}ic^{\mathbb{Z}}$ (see (2-6)) is a functor satisfying the following additional property: for each injective homomorphism $V_1 \to V$ in the category Tate_0, there is a canonical isomorphism

$$\det(V_1) \otimes \det(V/V_1) \simeq \det(V), \tag{3-1}$$

such that:

(i) for $V_1 = 0$ (resp. $V_1 = V$), equality (3-1) is the same as

$$\ell_0 \otimes \det(V) \simeq \det(V) \tag{3-2}$$

resp.

$$\det(V) \otimes \ell_0 \simeq \det(V), \tag{3-3}$$

where ℓ_0 is the trivial k-line of degree zero.

(ii) For any diagram

$$
\begin{tikzcd}
0 \arrow[r] & U_1 \arrow[r] \arrow[d, "\simeq"] & U \arrow[r] \arrow[d, "\simeq"] & U/U_1 \arrow[r] \arrow[d, "\simeq"] & 0 \\
0 \arrow[r] & V_1 \arrow[r] & V \arrow[r] & V/V_1 \arrow[r] & 0,
\end{tikzcd}
\tag{3-4}
$$

the following diagram is commutative:

$$
\begin{tikzcd}
\det(U_1) \otimes \det(U/U_1) \arrow[r] \arrow[d] & \det(U) \arrow[d] \\
\det(V_1) \otimes \det(V/V_1) \arrow[r] & \det(V).
\end{tikzcd}
\tag{3-5}
$$

(iii) For any diagram

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & U_1 \arrow[r] \arrow[d] & V_1 \arrow[r] \arrow[d] & W_1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & U \arrow[r] \arrow[d] & V \arrow[r] \arrow[d] & W \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & U/U_1 \arrow[r] \arrow[d] & V/V_1 \arrow[r] \arrow[d] & W/W_1 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 & ,
\end{tikzcd}
\tag{3-6}
$$

the following diagram is commutative:

$$
\begin{tikzcd}
(\det(U_1) \otimes \det(U/U_1)) \otimes (\det(W_1) \otimes \det(W/W_1)) \arrow[r] \arrow[d, "\text{ass. and comm. constraints}"] & \det(U) \otimes \det(W) \arrow[dd] \\
(\det(U_1) \otimes \det(W_1)) \otimes (\det(U/U_1) \otimes \det(W/W_1)) \arrow[d] & \\
\det(V_1) \otimes \det(V/V_1) \arrow[r] & \det(V)
\end{tikzcd}
\tag{3-7}
$$

Definition 3.5. Let $\mathcal{P}$ be a Picard groupoid. A determinant functor from the category $(\mathrm{Tate}_0, \mathrm{isom})$ to $\mathcal{P}$ is a functor $D : (\mathrm{Tate}_0, \mathrm{isom}) \to \mathcal{P}$ together with isomorphisms (3-1) satisfying equalities and diagrams (3-2)–(3-7), where we have to change the notation det to D everywhere in these formulas.

The next proposition is obvious.

Proposition 3.6. *Let $D : (\mathrm{Tate}_0, \mathrm{isom}) \to \mathcal{P}$ be a determinant functor. Then there is a 1-homomorphism of Picard groupoids $\tilde{D} : \mathcal{P}ic^{\mathbb{Z}} \to \mathcal{P}$ and a monoidal natural transformation $\varepsilon : \tilde{D} \circ \det \simeq D$. Furthermore, the pair $(\tilde{D}, \varepsilon)$ is unique up to a unique isomorphism.*

Remark 3.7. All the above discussions are valid when one replaces k by a noetherian commutative ring A, and replaces Tate_0 by the category of finitely generated projective A-modules.

Next we turn to Tate_1. The following result is fundamental and is due to Kapranov [2001] (but see also [Drinfeld 2006, §5.1–5.3]).

Proposition 3.8. *There is a natural functor*

$$\mathcal{D}et \, : \, (\mathrm{Tate}_1, \mathrm{isom}) \to B\mathcal{P}ic^{\mathbb{Z}},$$

and for each admissible monomorphism $\mathcal{V}_1 \to \mathcal{V}$ there is a 1-isomorphism

$$\mathcal{D}et(\mathcal{V}_1) + \mathcal{D}et(\mathcal{V}/\mathcal{V}_1) \to \mathcal{D}et(\mathcal{V}) \tag{3-8}$$

that coincides with the canonical 1-isomorphism $\mathcal{P} + \mathcal{D}et(\mathcal{V}) \simeq \mathcal{D}et(\mathcal{V})$ if $\mathcal{V}_1 = 0$ and with the canonical 1-isomorphism $\mathcal{D}et(\mathcal{V}) + \mathcal{P} \simeq \mathcal{D}et(\mathcal{V})$ if $\mathcal{V}_1 = \mathcal{V}$.

For each admissible diagram (3-4) of 1-Tate vector spaces, the corresponding diagram (3-5) is commutative. For each admissible diagram (3-6) of 1-Tate vector spaces, there is a 2-isomorphism for the corresponding diagram (3-7).

Remark 3.9. Under conditions of Proposition 3.8, the 2-isomorphisms which appear from diagram (3-7) satisfy further compatibility conditions.

Proof. We recall the definition of a graded-determinantal theory Δ on a 1-Tate vector space $\mathcal{V}$. This is a rule that assign to every lattice $L \subset \mathcal{V}$ an object $\Delta(L)$ from $\mathcal{P}ic^{\mathbb{Z}}$ and to every lattices $L_1 \subset L_2 \subset \mathcal{V}$ an isomorphism

$$\Delta_{L_1,L_2} \, : \, \Delta(L_1) \otimes \det(L_2/L_1) \longrightarrow \Delta(L_2)$$

such that for any three lattices $L_1 \subset L_2 \subset L_3 \subset \mathcal{V}$ the following diagram is commutative:

$$\begin{tikzcd}
\Delta(L_1) \otimes \det(L_2/L_1) \otimes \det(L_3/L_2) \arrow[r] \arrow[d] & \Delta(L_1) \otimes \det(L_3/L_1) \arrow[d] \\
\Delta(L_2) \otimes \det(L_3/L_2) \arrow[r] & \Delta(L_3).
\end{tikzcd}$$

Let $\mathcal{D}et(\mathcal{V})$ be the category of graded-determinantal theories on $\mathcal{V}$. This is a $\mathcal{P}ic^{\mathbb{Z}}$-torsor, where for any $x \in \mathcal{P}ic^{\mathbb{Z}}$, $\Delta \in \mathcal{D}et(\mathcal{V})$, we have $(x + \Delta)(L) := x \otimes \Delta(L)$.

Now for an admissible (exact) sequence

$$0 \longrightarrow \mathcal{V}_1 \longrightarrow \mathcal{V} \xrightarrow{\epsilon} \mathcal{V}/\mathcal{V}_1 \longrightarrow 0$$

the 1-isomorphism (3-8) is constructed as

$$\Delta(L) := \Delta_1(L \cap \mathcal{V}_1) \otimes \Delta_2(\epsilon(L)),$$

where L is a lattice in $\mathcal{V}$, $\Delta_1 \in \mathcal{D}et(\mathcal{V}_1)$, $\Delta_2 \in \mathcal{D}et(\mathcal{V}/\mathcal{V}_1)$, $\Delta \in \mathcal{D}et(\mathcal{V})$. (We used that the k-space $L \cap \mathcal{V}_1$ is a lattice in the 1-Tate vector space $\mathcal{V}_1$, and the k-space $\epsilon(L)$ is a lattice in the 1-Tate vector space $\mathcal{V}_3$).

We note that, by construction, $\mathcal{V} \mapsto \mathcal{D}et(\mathcal{V})$ is naturally a contravariant functor from the category $(\mathrm{Tate}_1, \mathrm{isom})$ to the category $B\mathcal{P}ic^{\mathbb{Z}}$. To obtain the covariant functor we have to inverse arrows in the category $(\mathrm{Tate}_1, \mathrm{isom})$. $\square$

4. Applications to the case $G = \mathrm{GL}(k((t)))$ and $\mathrm{GL}(k((t))((s)))$

4A. ***Tame symbols.*** Let us first review the tame symbols. Recall that if K is a field with discrete valuation $\nu : K^\times \to \mathbb{Z}$, and k denote its residue field, then there are so-called boundary maps for any $i \in \mathbb{N}$

$$\partial_i \,:\, K_i^M(K) \longrightarrow K_{i-1}^M(k),$$

where $K_i^M(F)$ denotes the i-th Milnor K-group of a field F. Recall also that for a field F, the i-th Milnor K-group $K_i^M(F)$ is the quotient of the abelian group $F^\times \otimes_{\mathbb{Z}} F^\times \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} F^\times$ modulo the so-called Steinberg relations. Then the tame symbol is defined as the composition of the maps

$$\{\cdot,\cdot\} \,:\, K^\times \otimes_{\mathbb{Z}} K^\times \longrightarrow K_2^M(K) \xrightarrow{\partial_2} K_1^M(k) \simeq k^\times.$$

Explicitly, let $\pi \subset K$ be the maximal ideal. Then

$$\{f, g\} = (-1)^{\nu(f)\nu(g)} \frac{f^{\nu(g)}}{g^{\nu(f)}} \mod \pi \tag{4-1}$$

Now, let $\mathbb{K}$ be a two-dimensional local field, whose residue field is denoted by K, whose residue field is k. Then we define the map

$$\nu_{\mathbb{K}} \,:\, \mathbb{K}^\times \otimes_{\mathbb{Z}} \mathbb{K}^\times \longrightarrow K_2^M(\mathbb{K}) \xrightarrow{\partial_2} K_1^M(K) \xrightarrow{\partial_1} K_0^M(k) \simeq \mathbb{Z},$$

and define the two-dimensional tame symbol as

$$\{\cdot,\cdot,\cdot\} \,:\, \mathbb{K}^\times \otimes_{\mathbb{Z}} \mathbb{K}^\times \otimes_{\mathbb{Z}} \mathbb{K}^\times \longrightarrow K_3^M(\mathbb{K}) \xrightarrow{\partial_3} K_2^M(K) \xrightarrow{\partial_2} K_1^M(k) \simeq k^\times.$$

We have the following explicit formulas for $\nu_{\mathbb{K}}$ and $\{\cdot,\cdot,\cdot\}$ (see [Osipov 2003]). Let $\nu_1 : \mathbb{K} \to \mathbb{Z}$, and $\nu_2 : K \to \mathbb{Z}$ be discrete valuations. Let $\pi_{\mathbb{K}}$ be the maximal ideal of $\mathbb{K}$, π_K be the maximal ideal of K. For an element $f \in \mathcal{O}_{\mathbb{K}}$, let $\bar{f}$ denote its residue class in K. Then

$$\nu_{\mathbb{K}}(f, g) = \nu_2\left(\overline{\frac{f^{\nu_1(g)}}{g^{\nu_1(f)}}}\right) \tag{4-2}$$

and

$$\{f, g, h\} = \operatorname{sgn}(f, g, h) f^{\nu_{\mathbb{K}}(g,h)} g^{\nu_{\mathbb{K}}(h,f)} h^{\nu_{\mathbb{K}}(f,g)} \mod \pi_{\mathbb{K}} \mod \pi_K \tag{4-3}$$

where

$$\operatorname{sgn}(f, g, h) = (-1)^A, \tag{4-4}$$

with

$$A = \nu_{\mathbb{K}}(f, g)\nu_{\mathbb{K}}(f, h) + \nu_{\mathbb{K}}(g, h)\nu_{\mathbb{K}}(g, f) + \nu_{\mathbb{K}}(h, f)\nu_{\mathbb{K}}(h, g) \\ + \nu_{\mathbb{K}}(f, g)\nu_{\mathbb{K}}(g, h)\nu_{\mathbb{K}}(h, f).$$

Remark 4.1. Originally one used another explicit formula for the sign of the two-dimensional tame symbol. This other formula was introduced in [Parshin 1975].

It is easy to see that tame symbols $\{\cdot, \cdot\}$, $\{\cdot, \cdot, \cdot\}$ and the map $\nu_{\mathbb{K}}$ are antisymmetric.

4B. ***The one-dimensional story.*** Let $\mathcal{V}$ be a 1-Tate vector space over k. The group of automorphisms of $\mathcal{V}$ in this category is denoted by $\mathrm{GL}(\mathcal{V})$.

Proposition 4.2. *There is a homomorphism $\mathcal{D}et_{\mathcal{V}} : \mathrm{GL}(\mathcal{V}) \to \mathcal{P}ic^{\mathbb{Z}}$, which is canonical up to a unique isomorphism in $H^1(B\mathrm{GL}(\mathcal{V}), \mathcal{P}ic^{\mathbb{Z}})$.*

Proof. According to Proposition 3.8, we have a homomorphism

$$\mathrm{GL}(\mathcal{V}) \to \mathrm{Hom}_{\mathcal{P}ic^{\mathbb{Z}}}(\mathcal{D}et(\mathcal{V}), \mathcal{D}et(\mathcal{V})) \simeq \mathcal{P}ic^{\mathbb{Z}}$$

via $\mathfrak{X}^{-1}$, where $\mathfrak{X} : \mathcal{P}ic^{\mathbb{Z}} \to \mathrm{Hom}_{\mathcal{P}ic^{\mathbb{Z}}}(\mathcal{D}et(\mathcal{V}), \mathcal{D}et(\mathcal{V}))$ is a natural homomorphism from Section 2C. □

Choose $\mathcal{L} \subset \mathcal{V}$ a lattice. It follows from the proof of Proposition 3.8 that in concrete terms, one has to assign to $\mathcal{D}et_{\mathcal{V}}(g)$ the graded line

$$\det(\mathcal{L} \mid g\mathcal{L}) := \det\left(\frac{g\mathcal{L}}{\mathcal{L} \cap g\mathcal{L}}\right) \otimes \det\left(\frac{\mathcal{L}}{\mathcal{L} \cap g\mathcal{L}}\right)^{-1}, \tag{4-5}$$

where $g \in \mathrm{GL}(\mathcal{V})$. Then, it is well-known that there is a canonical isomorphism

$$\det(\mathcal{L} \mid gg'\mathcal{L}) \simeq \det(\mathcal{L}|g\mathcal{L}) \otimes \det(g\mathcal{L} \mid gg'\mathcal{L}) \simeq \det(\mathcal{L} \mid g\mathcal{L}) \otimes \det(\mathcal{L} \mid g'\mathcal{L}),$$

which is compatible with the associativity constraints in the category $\mathcal{P}ic^{\mathbb{Z}}$ (see, for example, [Frenkel and Zhu 2008, §1]). For different choice of $\mathcal{L}$, the resulting objects in $H^1(BGL(\mathcal{V}), \mathcal{P}ic^{\mathbb{Z}})$ are isomorphic.

We also have the following lemma, which easily follows from the construction of homomorphism $\mathcal{D}et_{\mathcal{V}}$ and the discussion in Section 3B (in particular the diagram (3-7)).

Lemma 4.3. *If* $0 \to \mathcal{V}' \to \mathcal{V} \to \mathcal{V}'' \to 0$ *is a short exact sequence of* 1*-Tate vector spaces* (*recall that* Tate_1 *is an exact category*). *Let* P *be the subgroup of* $\mathrm{GL}(\mathcal{V})$ *that preserves this sequence*, *then there is a canonical* 1*-isomorphism* $\mathcal{D}et_{\mathcal{V}'} + \mathcal{D}et_{\mathcal{V}''} \simeq \mathcal{D}et_{\mathcal{V}}$ *in* $H^1(BP, \mathcal{P}ic^{\mathbb{Z}})$.

Remark 4.4. The 1-homomorphism $F_{\mathcal{P}ic} \circ \mathcal{D}et_{\mathcal{V}} : \mathrm{GL}(\mathcal{V}) \to \mathcal{P}ic$ is essentially constructed in [Arbarello et al. 1988]. However, the above lemma does not hold for this 1-homomorphism. This is the complication of the sign issues in that reference.

Now let k'/k be a finite extension and $K = k'((t))$ be a local field with residue field k'. Then K has a natural structure as a 1-Tate vector space over k. Let $H = K^{\times}$. The multiplication gives a natural embedding $H \subset \mathrm{GL}(K)$. The following proposition is from [Beilinson et al. 2002].

Proposition 4.5. *If* $f, g \in H$, *then*

$$\mathrm{Comm}(\mathcal{D}et_K)(f, g) = \mathrm{Nm}_{k'/k}\{f, g\}^{-1}$$

that is inverse to the tame symbol of f *and* g.

Remark 4.6. Since the natural functor $F_{\mathcal{P}ic}$ is monoidal, the restriction to H of the functor $F_{\mathcal{P}ic} \circ \mathcal{D}et_K$ determines a homomorphism $H \to \mathcal{P}ic$. The commutator pairing $\mathrm{Comm}(f, g)$ constructed by this homomorphism is

$$(-1)^{\mathrm{ord}(f)\,\mathrm{ord}(g)}\,\mathrm{Nm}_{k'/k}\{f, g\}^{-1}.$$

By Definition 3.3, a lattice $\mathcal{L}$ of $\mathcal{V}$ is a linearly compact open k-subspace of $\mathcal{V}$ such that $\mathcal{V}/\mathcal{L}$ is a discrete k-space. A colattice $\mathcal{L}^c$ is a k-subspace of $\mathcal{V}$ such that for any lattice $\mathcal{L}$, both $\mathcal{L}^c \cap \mathcal{L}$ and $\mathcal{V}/(\mathcal{L}^c + \mathcal{L})$ are finite dimensional.

Lemma 4.7. *Let* $P \subset \mathrm{GL}(\mathcal{V})$ *be a subgroup of* $\mathrm{GL}(\mathcal{V})$ *that preserves a lattice* (*or a colattice*) *in* $\mathcal{V}$, *then the homomorphism* $\mathcal{D}et_{\mathcal{V}}$ *is trivial on* P.

Proof. Let $\mathcal{L} \subset \mathcal{V}$ be a lattice such that the group P preserves it. We consider an exact sequence of 1-Tate vector spaces

$$0 \longrightarrow \mathcal{L} \longrightarrow \mathcal{V} \longrightarrow \mathcal{V}/\mathcal{L} \longrightarrow 0.$$

Then the group P preserves this sequence. Therefore by Lemma 4.3, it is enough to prove that the homomorphisms $\mathcal{D}et_{\mathcal{L}}$ and $\mathcal{D}et_{\mathcal{L}/\mathcal{V}}$ are trivial on P. But this is obvious from the proof of Proposition 4.2.

For a colattice $\mathcal{L}^c \subset \mathcal{V}$ we have to use the analogous reasonings. □

4C. *The two-dimensional story.* If $\mathbb{V} \in \mathrm{Tate}_2$, then we denote by $\mathrm{GL}(\mathbb{V})$ the group of automorphisms of $\mathbb{V}$ in this category.

There should be a determinantal functor from $(\mathrm{Tate}_2, \mathrm{isom})$ to $B^2\mathcal{P}ic^{\mathbb{Z}}$, which assigns to every such $\mathbb{V}$ the graded gerbal theory in the sense of [Arkhipov and Kremnizer 2010], satisfying properties generalizing those listed in Proposition 3.8 (and further compatibility conditions). We do not make it precise. But we define the corresponding central extension of $\mathrm{GL}(\mathbb{V})$ as follows. Pick a lattice $\mathbb{L}$ of $\mathbb{V}$. Then one associates with g the $\mathcal{P}ic^{\mathbb{Z}}$-torsor

$$\mathfrak{D}et_{\mathbb{V}}(g) = \mathfrak{D}et(\mathbb{L} \mid g\mathbb{L}) := \mathfrak{D}et\left(\frac{g\mathbb{L}}{\mathbb{L} \cap g\mathbb{L}}\right) - \mathfrak{D}et\left(\frac{\mathbb{L}}{\mathbb{L} \cap g\mathbb{L}}\right). \tag{4-6}$$

This definition is correct because both k-spaces $\frac{g\mathbb{L}}{\mathbb{L}\cap g\mathbb{L}}$ and $\frac{\mathbb{L}}{\mathbb{L}\cap g\mathbb{L}}$ belong to objects of category Tate_1. We define the 1-isomorphism as

$$\mathfrak{D}et(\mathbb{L} \mid gg'\mathbb{L}) \simeq \mathfrak{D}et(\mathbb{L} \mid g\mathbb{L}) + \mathfrak{D}et(g\mathbb{L} \mid gg'\mathbb{L}) \simeq \mathfrak{D}et(\mathbb{L} \mid g\mathbb{L}) + \mathfrak{D}et(\mathbb{L} \mid g'\mathbb{L}). \tag{4-7}$$

One uses Proposition 3.8 to check that this defines a central extension of $\mathrm{GL}(\mathbb{V})$ by $\mathcal{P}ic^{\mathbb{Z}}$. This central extension depends on the chosen lattice $\mathbb{L}$ of $\mathbb{V}$. If we change the lattice, then the central extension constructed by a new lattice will be isomorphic to the previous one.

Remark 4.8. If one replaces $\mathcal{P}ic^{\mathbb{Z}}$ by $\mathcal{P}ic$, such a central extension was constructed in [Osipov 2003; Frenkel and Zhu 2008]. In the first of these references the two-dimensional tame symbol up to sign was obtained as an application of this construction, and the reciprocity laws on algebraic surfaces were proved up to sign.

As generalization of Lemma 4.3 and Lemma 4.7 it is not difficult to prove the following lemmas.

Lemma 4.9. *If* $0 \to \mathbb{V}' \to \mathbb{V} \to \mathbb{V}'' \to 0$ *is a short exact sequence of* 2*-Tate vector spaces* (*recall that* Tate_2 *is an exact category*). *Let* P *be the subgroup of* $\mathrm{GL}(\mathbb{V})$ *that preserves this sequence*, *then there is a canonical* 1*-isomorphism* $\mathfrak{D}et_{\mathbb{V}'} + \mathfrak{D}et_{\mathbb{V}''} \simeq \mathfrak{D}et_{\mathbb{V}}$ *in* $H^2(BP, \mathcal{P}ic^{\mathbb{Z}})$.

Lemma 4.10. *Let* P *be subgroup of* $\mathrm{GL}(\mathbb{V})$ *which preserves a lattice or a colattice in* $\mathbb{V}$, *then the central extension restricted to* P *can be trivialized.*

Let k'/k be a finite field extension, and $\mathbb{K} = k'((t))((s))$ be a two-dimensional local field. Then $\mathbb{K}$ has a natural structure as a 2-Tate vector space over k. The group $H = \mathbb{K}^{\times}$ acts on $\mathbb{K}$ by left multiplications, which gives rise to an embedding $H \to \mathrm{GL}(\mathbb{K})$.

Theorem 4.11. *For* $f, g, h \in H$, *one has*

$$C_3^{\mathfrak{D}et}(f, g, h) = \mathrm{Nm}_{k'/k}\{f, g, h\},$$

where the map $C_3^{\mathcal{D}et}$ is constructed in Proposition 2.17 and $\{\cdot,\cdot,\cdot\}$ is the two-dimensional tame symbol.

In what follows, we will denote the bimultiplicative homomorphism $C_2^{\mathcal{D}et}$ by C_2, the homomorphism $C_g^{\mathcal{D}et}$ by C_g and the map $C_3^{\mathcal{D}et}$ by C_3.

Proof. Since both maps C_3 and $\mathrm{Nm}_{k'/k}\{\cdot,\cdot,\cdot\}$ are antisymmetric and trimultiplicative, we just need to consider the following cases: (i) $f, g, h \in \mathcal{O}_{\mathbb{K}}^{\times}$; (ii) $f, g \in \mathcal{O}_{\mathbb{K}}^{\times}$, $h = s$; (iii) $f \in \mathcal{O}_{\mathbb{K}}^{\times}$, $g = h = s$; (iv) $f = g = h = s$. Here $\mathcal{O}_{\mathbb{K}} = k'((t))[[s]]$ is the ring of integers of the field $\mathbb{K}$, which is also a lattice in $\mathbb{K}$. We will fix $\mathbb{L} = \mathcal{O}_{\mathbb{K}}$.

In Case (i), both C_3 and $\mathrm{Nm}_{k'/k}\{\cdot,\cdot,\cdot\}$ are trivial (to see that C_3 is trivial, one uses Lemma 4.10).

Case (ii). According to formulas (4-2)–(4-4), this case amounts to proving that

$$C_3(f, g, s) = \mathrm{Nm}_{k'/k}\{\bar{f}, \bar{g}\},$$

where $\bar{f}, \bar{g}$ are the image of elements f, g under the map $\mathcal{O}_{\mathbb{K}}^{\times} \to K^{\times}$.

Let us consider a little more general situation. Let $f, g \in \mathrm{GL}(\mathbb{K})$ that leave the lattice $\mathcal{O}_{\mathbb{K}}$ invariant, and let $h \in \mathrm{GL}(\mathbb{K})$ such that $h\mathcal{O}_{\mathbb{K}} \subset \mathcal{O}_{\mathbb{K}}$. Let $\mathcal{V} = \mathcal{O}_{\mathbb{K}}/h\mathcal{O}_{\mathbb{K}}$, which is a 1-Tate vector space over the field k. We assume that f, g, h mutually commute with each other. Then $f, g : \mathcal{O}_{\mathbb{K}} \to \mathcal{O}_{\mathbb{K}}$ induce automorphisms $\pi_h(f), \pi_h(g) : \mathcal{V} \to \mathcal{V}$. Let $\mathcal{D}et$ be the central extension of $\mathrm{GL}(\mathbb{K})$ by $\mathcal{P}ic^{\mathbb{Z}}$ defined by the lattice $\mathbb{L} = \mathcal{O}_{\mathbb{K}}$. By definition, under the isomorphism

$$\mathfrak{X} : \mathcal{P}ic^{\mathbb{Z}} \to \mathrm{Hom}_{\mathcal{P}ic^{\mathbb{Z}}}(\mathcal{D}et(\mathcal{O}_{\mathbb{K}}|hg\mathcal{O}_{\mathbb{K}}), \mathcal{D}et(\mathcal{O}_{\mathbb{K}}|hg\mathcal{O}_{\mathbb{K}})),$$

the 1-isomorphism $C_2(h, g)$ corresponds to the composition of 1-isomorphisms of $\mathcal{P}ic^{\mathbb{Z}}$-torsors:

$$\begin{aligned}\mathcal{D}et(\mathcal{O}_{\mathbb{K}}|hg\mathcal{O}_{\mathbb{K}}) &\to \mathcal{D}et(\mathcal{O}_{\mathbb{K}}|h\mathcal{O}_{\mathbb{K}}) + \mathcal{D}et(h\mathcal{O}_{\mathbb{K}}|hg\mathcal{O}_{\mathbb{K}})\\ &\to \mathcal{D}et(\mathcal{O}_{\mathbb{K}}|h\mathcal{O}_{\mathbb{K}}) + \mathcal{D}et(\mathcal{O}_{\mathbb{K}}|g\mathcal{O}_{\mathbb{K}})\\ &\to \mathcal{D}et(\mathcal{O}_{\mathbb{K}}|g\mathcal{O}_{\mathbb{K}}) + \mathcal{D}et(\mathcal{O}_{\mathbb{K}}|h\mathcal{O}_{\mathbb{K}})\\ &\to \mathcal{D}et(\mathcal{O}_{\mathbb{K}}|g\mathcal{O}_{\mathbb{K}}) + \mathcal{D}et(g\mathcal{O}_{\mathbb{K}}|gh\mathcal{O}_{\mathbb{K}}) \to \mathcal{D}et(\mathcal{O}_{\mathbb{K}}|gh\mathcal{O}_{\mathbb{K}}).\end{aligned}$$

Using the fact that $g\mathcal{O}_{\mathbb{K}} = \mathcal{O}_{\mathbb{K}}$ and Proposition 3.8, this 1-isomorphism is canonically 2-isomorphic to the 1-isomorphism

$$\mathcal{D}et(\mathcal{O}_{\mathbb{K}} \mid hg\mathcal{O}_{\mathbb{K}}) \xrightarrow{\mathfrak{X}(\mathcal{D}et_{\mathcal{V}}(\pi(g)))} \mathcal{D}et(\mathcal{O}_{\mathbb{K}} \mid hg\mathcal{O}_{\mathbb{K}}).$$

Therefore, there is a canonical 2-isomorphism $C_2(h, g) \simeq -\mathcal{D}et_{\mathcal{V}}(\pi_h(g))$, because, by definition (see formula (4-6)), $\mathcal{D}et(\mathcal{O}_{\mathbb{K}} \mid h\mathcal{O}_{\mathbb{K}}) \simeq -\mathcal{D}et(\mathcal{V})$. One readily checks by the construction of Lemma-Definition 2.13, that these 2-isomorphisms fit into the commutative diagrams

$$\begin{tikzcd} C_2(h,\, fg) \arrow[r] \arrow[d] & C_2(h,\, f) + C_2(h,\, g) \arrow[d] \\ -\mathcal{D}et_{\mathcal{V}}(\pi_h(fg)) \arrow[r] & -\mathcal{D}et_{\mathcal{V}}(\pi_h(f)) - \mathcal{D}et_{\mathcal{V}}(\pi_h(g)) \end{tikzcd} \tag{4-8}$$

where the natural isomorphism $\mathcal{D}et_{\mathcal{V}}(\pi_h(fg)) \to \mathcal{D}et_{\mathcal{V}}(\pi_h(f)) + \mathcal{D}et_{\mathcal{V}}(\pi_h(g))$ comes from Proposition 4.2. (We have to use that $\mathcal{D}et((0))$ is canonically isomorphic to $\mathcal{P}ic^{\mathbb{Z}}$, and $\mathcal{O}_{\mathbb{K}}/g\mathcal{O}_{\mathbb{K}} = (0)$, where (0) is the zero-space.)

We now return to our proof of case (ii). Let P_s be the subgroup of $\mathrm{GL}(\mathbb{K})$ consisting of elements that preserve the lattice $\mathcal{O}_{\mathbb{K}}$ and commute with the element s. Then the elements in the group P_s also preserve the lattice $s\mathcal{O}_{\mathbb{K}}$, and therefore induce a group homomorphism

$$\pi_s : P_s \to \mathrm{GL}(K),$$

because $\mathbb{K} = \mathcal{O}_{\mathbb{K}}/s\mathcal{O}_{\mathbb{K}}$. Then the commutative diagram (4-8) amounts to the following lemma.

Lemma 4.12. *The homomorphism $C_s : P_s \to \mathcal{P}ic^{\mathbb{Z}}$ is isomorphic to the minus* (*or the inverse*) *of the homomorphism*

$$\mathcal{D}et_K \circ \pi_s \;:\; P_s \to \mathrm{GL}(K) \to \mathcal{P}ic^{\mathbb{Z}}.$$

By Proposition 4.5, we thus obtain that

$$C_3(f,g,s) = C_3(s,\,f,\,g) = \mathrm{Comm}(C_s)(f,g) = \mathrm{Nm}_{k'/k}\{\bar{f},\bar{g}\}$$

for $f, g \in \mathcal{O}_{\mathbb{K}}^{\times} \subset \mathrm{GL}(\mathbb{K})$. Case (ii) follows.

Case (iii). According to formulas (4-2)–(4-4), one needs to show

$$C_3(f,s,s) = C_f(s,s) = \mathrm{Nm}_{k'/k}(-1)^{\nu_2(\bar{f})} = (-1)^{(\nu_2(\bar{f})[k':k])} = (-1)^{(\nu_2(\bar{f})[k':k])^2}.$$

We have the following exact sequence of 1-Tate vector spaces

$$0 \longrightarrow \frac{s\mathcal{O}_{\mathbb{K}}}{s^2\mathcal{O}_{\mathbb{K}}} \longrightarrow \frac{\mathcal{O}_{\mathbb{K}}}{s^2\mathcal{O}_{\mathbb{K}}} \longrightarrow \frac{\mathcal{O}_{\mathbb{K}}}{s\mathcal{O}_{\mathbb{K}}} \longrightarrow 0.$$

and therefore by Lemma 4.3, for any element $p \in P_s$, there is a canonical isomorphism in $\mathcal{P}ic^{\mathbb{Z}}$

$$\mathcal{D}et_{\frac{\mathcal{O}_{\mathbb{K}}}{s^2\mathcal{O}_{\mathbb{K}}}}(\pi_{s^2}(p)) \simeq \mathcal{D}et_{\frac{s\mathcal{O}_{\mathbb{K}}}{s^2\mathcal{O}_{\mathbb{K}}}}(\pi_{s^2}(p)) + \mathcal{D}et_{\frac{\mathcal{O}_{\mathbb{K}}}{s\mathcal{O}_{\mathbb{K}}}}(\pi_{s^2}(p)). \tag{4-9}$$

On the other hand, we have already shown that there are canonical isomorphisms

$$\begin{aligned} C_2(s,p) &\simeq -\mathcal{D}et_{\frac{\mathbb{O}_{\mathbb{K}}}{s\mathbb{O}_{\mathbb{K}}}}(\pi_s(p)) = -\mathcal{D}et_{\frac{\mathbb{O}_{\mathbb{K}}}{s\mathbb{O}_{\mathbb{K}}}}(\pi_{s^2}(p)), \\ C_2(s^2,p) &\simeq -\mathcal{D}et_{\frac{\mathbb{O}_{\mathbb{K}}}{s^2\mathbb{O}_{\mathbb{K}}}}(\pi_{s^2}(g)). \end{aligned} \tag{4-10}$$

Again, by checking the construction as in Lemma-Definition 2.13, one obtains that under the isomorphisms (4-10), the canonical isomorphism

$$C_2(s^2,p) \simeq C_2(s,p) + C_2(s,p)$$

corresponds to (4-9).

Now let $p = f$ as in Case (iii). We know that $\mathcal{D}et_{\mathbb{O}_{\mathbb{K}}/(s\mathbb{O}_{\mathbb{K}})}(\pi(f))$ is a graded line of degree $\nu_2(\bar{f})[k':k]$. Therefore, using $C_2(a,b) \simeq -C_2(b,a)$ for any commuting elements $a, b \in \mathrm{GL}(\mathbb{K})$, we obtain that Case (iii) follows from the definition of the commutativity constraints in $\mathcal{P}ic^{\mathbb{Z}}$.

Case (iv). One needs to show that $C_s(s,s) = 1$. One can easily show that there are canonical isomorphisms $C_2(s,s) \simeq \ell_0$, $C_2(s^2,s) \simeq \ell_0$, and the canonical isomorphism $C_2(s^2,s) \simeq C_2(s,s) + C_2(s,s)$ corresponds to $\ell_0 \simeq \ell_0 + \ell_0$. (We used that for the k'-space $M = k'[\![t]\!]((s))$ we have $sM = M$, and the k'-space M induce a lattice in every 1-Tate vector space $s^n\mathbb{O}_{\mathbb{K}}/s^{n+l}\mathbb{O}_{\mathbb{K}}$, $n \in \mathbb{Z}$, $l \in \mathbb{N}$.) This case also follows. □

5. Reciprocity laws

We will use the adèle theory on schemes. Adèles on algebraic surfaces were introduced in [Parshin 1976]. On arbitrary noetherian schemes they were considered in [Beilinson 1980]. See the proof of part of results of this latter reference in [Huber 1991]. A survey of adèles can be found in [Osipov 2008].

We fix a perfect field k.

5A. ***Weil reciprocity law.*** To fix the idea, let us first revisit the Weil reciprocity law. Let C be an irreducible projective curve over a field k. Let $k(C)$ be the field of rational functions on the curve C. For a closed point $p \in C$ let $\hat{\mathbb{O}}_p$ be the completion by maximal ideal m_p of the local ring $\mathbb{O}_p$ of point $p \in C$. Let a ring K_p be the localization of the ring $\hat{\mathbb{O}}_p$ with respect to the multiplicative system $\mathbb{O}_p \setminus 0$. (If p is a smooth point, then $K_p = k(C)_p$ is the fraction field of the ring $\hat{\mathbb{O}}_p$, and $K_p = k(p)((t_p))$, $\hat{\mathbb{O}}_p = k(p)[\![t_p]\!]$, where $k(p)$ is the residue field of the point p, t_p is a local parameter at p. For a nonsmooth point $p \in C$, the ring K_p is a finite direct product of one-dimensional local fields.)

We have that K_p is a 1-Tate vector space over k, and $\hat{\mathbb{O}}_p$ is a lattice in K_p for any point $p \in C$.

For any coherent subsheaf $\mathcal{F}$ of the constant sheaf $k(C)$ on the curve C we consider the following adèle complex $\mathcal{A}_C(\mathcal{F})$:

$$\mathbb{A}_{C,0}(\mathcal{F}) \oplus \mathbb{A}_{C,1}(\mathcal{F}) \longrightarrow \mathbb{A}_{C,01}(\mathcal{F})$$

whose cohomology groups coincide with the cohomology groups $H^*(C, \mathcal{F})$. We recall that

$$\mathbb{A}_{C,0}(\mathcal{F}) = k(C) \otimes_{\mathcal{O}_C} \mathcal{F}, \qquad \mathbb{A}_{C,1}(\mathcal{F}) = \prod_{p \in C} \hat{\mathcal{O}}_p \otimes_{\mathcal{O}_C} \mathcal{F},$$
$$\mathbb{A}_{C,01}(\mathcal{F}) = \mathbb{A}_C = {\prod_{p \in C}}' K_p \otimes_{\mathcal{O}_C} \mathcal{F},$$

where $\prod'$ denotes the restricted (adèle) product with respect to $\prod_{p \in C} \hat{\mathcal{O}}_p$. Observe that since $\mathcal{F}$ is a subsheaf of $k(C)$, we have

$$k(C) \otimes_{\mathcal{O}_C} \mathcal{F} = k(C), \qquad K_p \otimes_{\mathcal{O}_C} \mathcal{F} = K_p.$$

The adèle ring $\mathbb{A}_C$ is a 1-Tate vector space over k. This is because

$$\mathbb{A}_C = \varprojlim_{\mathcal{G} \subset k(C)} \varinjlim_{\mathcal{H} \subset k(C)} \mathbb{A}_{C,1}(\mathcal{H}) / \mathbb{A}_{C,1}(\mathcal{G}),$$

and $\dim_k \mathbb{A}_{C,1}(\mathcal{H}) / \mathbb{A}_{C,1}(\mathcal{G}) < \infty$ for coherent subsheaves $0 \neq \mathcal{G} \subset \mathcal{H}$ of $k(C)$. (We used that $\mathbb{A}_{C,1}(\mathcal{H}) / \mathbb{A}_{C,1}(\mathcal{G}) = \mathbb{A}_{C,1} = \bigoplus_{p \in C} \hat{\mathcal{O}}_p \otimes_{\mathcal{O}_C} (\mathcal{H}/\mathcal{G})$). For any coherent subsheaf $\mathcal{F}$ of $k(C)$ the space $\mathbb{A}_{C,1}(\mathcal{F})$ is a lattice in the space $\mathbb{A}_C$. Hence, the k-space $k(C)$ is a colattice in $\mathbb{A}_C$, since from the adelic complex $\mathcal{A}(\mathcal{F})$ it follows that

$$\dim_k k(C) \cap \mathbb{A}_{C,1}(\mathcal{F}) = \dim_k H^0(C, \mathcal{F}) < \infty,$$
$$\dim_k \mathbb{A}_C / (k(C) + \mathbb{A}_{C,1}(\mathcal{F})) = \dim_k H^1(C, \mathcal{F}) < \infty.$$

Let a p be a point of C and f, g a pair of elements of $K_p^\times$. If $K_p = k(p)((t_p))$, then we denote by $\{f, g\}_p$ the element from $k(p)^\times$ which is the corresponding tame symbol. If the ring K_p is isomorphic to the finite product of fields isomorphic to $k(p)((t))$, then we denote by $\{f, g\}_p$ the element from $k(p)^\times$ which is the same finite product of the corresponding tame symbols. Recall that there is the diagonal embedding $k(C) \hookrightarrow \mathbb{A}_C$.

Proposition 5.1 (Weil reciprocity law). *For any elements $f, g \in k(C)^\times$ the following product contains only finitely many nonequal to* 1 *terms and*

$$\prod_{p \in C} \mathrm{Nm}_{k(p)/k} \{f, g\}_p = 1. \tag{5-1}$$

Proof. By Proposition 4.5, we can change $\mathrm{Nm}_{k(p)/k}\{f, g\}_p$ to $\mathrm{Comm}(\mathcal{D}et_{K_p})(f, g)$ for all $p \in C$ in (5-1). There are points $p_1, \ldots, p_l \in C$ such that if $p \in C$ and

$p \neq p_i$ $(1 \le i \le l)$, then $f\hat{\mathbb{O}}_p = \hat{\mathbb{O}}_p$, $g\hat{\mathbb{O}}_p = \hat{\mathbb{O}}_p$, and hence, by Lemma 4.7, $\mathrm{Comm}(\mathcal{D}et_{K_p})(f, g) = 1$ for points $p \neq p_i$ $(1 \le i \le l)$.

We define the group H as the subgroup of the group $k(C)^\times$ generated by the elements f and g. We apply Lemma 4.3 to the 1-Tate k-vector spaces

$$\mathcal{V} = \mathbb{A}_C, \quad \mathcal{V}' = \mathbb{A}_{C \setminus \{p_1, \dots, p_l\}}, \quad \mathcal{V}'' = \prod_{1 \le i \le l} K_{p_i}.$$

The group H preserves the lattice $\prod_{p \in C \setminus \{p_1, \dots, p_l\}} \hat{\mathbb{O}}_p$ in the space $\mathcal{V}'$. Therefore, by Lemma 4.7, the homomorphism $\mathcal{D}et_{\mathbb{A}_C}$ is isomorphic to the homomorphism $\mathcal{D}et_{\mathcal{V}''}$, which is (again by Lemma 4.3) isomorphic to the sum of homomorphisms $\mathcal{D}et_{K_{p_1}}, \dots, \mathcal{D}et_{K_{p_l}}$. Since the group H preserves the colattice $k(C)$ in $\mathbb{A}_C$, the homomorphism $\mathcal{D}et_{\mathbb{A}_C}$ is isomorphic to the trivial one (by Lemma 4.7). Now using Remark 2.7 and Corollaries 2.9 and 2.10 we obtain (5-1). □

Remark 5.2. To obtain the triviality of homomorphism $\mathcal{D}et_{\mathbb{A}_C} : k(C)^\times \to \mathcal{P}ic^{\mathbb{Z}}$ in an explicit way, one has to use the following canonical isomorphism for any $g \in k(C)^\times$:

$$\mathcal{D}et_{\mathbb{A}_C}(g) \simeq \det(H^*(\mathcal{A}_C(g\mathbb{O}_C))) \otimes \det(H^*(\mathcal{A}_C(\mathbb{O}_C)))^{-1}, \tag{5-2}$$

where for any coherent sheaf $\mathcal{F}$ on C

$$\det(H^*(\mathcal{A}_C(\mathcal{F}))) := \det(H^0(\mathcal{A}_C(\mathcal{F}))) \otimes \det(H^1(\mathcal{A}_C(\mathcal{F})))^{-1} \simeq \det(H^0(C, \mathcal{F})) \otimes \det(H^1(C, \mathcal{F}))^{-1}.$$

(Formula (5-2) easily follows from adèle complexes and (4-5) if we change in (4-5) the lattices $\mathcal{L}$ and $g\mathcal{L}$ in $\mathbb{A}_C$ to any two lattices coming from nonzero coherent subsheaves $\mathcal{G} \subset \mathcal{H}$ of $k(C)$, and change correspondingly in formula (5-2) the sheaves $\mathbb{O}$ and $g\mathbb{O}$ to the sheaves $\mathcal{G} \subset \mathcal{H}$.) Now the homomorphism $\mathcal{D}et_{\mathbb{A}_C}$ is isomorphic to the trivial one by formula (5-2) and the fact that multiplication on an element $g \in k(C)^*$ gives a canonical isomorphism between adèle complexes $\mathcal{A}_C(\mathbb{O}_C)$ and $\mathcal{A}_C(g\mathbb{O}_C)$, which induce the canonical isomorphism between $\det(H^*(\mathcal{A}_C(\mathbb{O}_C)))$ and $\det(H^*(\mathcal{A}_C(g\mathbb{O}_C)))$.

5B. ***Parshin reciprocity laws.*** Let X be an algebraic surface over the field k. We assume, for simplicity, that X is a smooth connected surface.

We consider pairs $x \in C$, where C are irreducible curves on X and x are closed points on C. For every such pair one can define the ring $K_{x,C}$, which will be a finite product of two-dimensional local fields, as follows. Assume that the curve C on X has the formal branches $\mathbf{C}_1, \dots, \mathbf{C}_n$ at the point $x \in C$; that is,

$$C \mid_{\operatorname{Spec} \hat{\mathbb{O}}_x} = \bigcup_{1 \le i \le n} \mathbf{C}_i \ ,$$

where $\hat{\mathcal{O}}_x$ is the completion of the local ring $\mathcal{O}_x$ of a point $x \in X$, and $\mathbf{C}_i$ is irreducible in $\operatorname{Spec} \hat{\mathcal{O}}_x$ for any $1 \le i \le n$. (Since we assumed X is smooth, $\hat{\mathcal{O}}_x \simeq k(x)[\![t_1, t_2]\!]$.) Now every $\mathbf{C}_i$ defines a discrete valuation on the fraction field $\operatorname{Frac} \hat{\mathcal{O}}_x$. We define a two-dimensional local field $K_{x,\mathbf{C}_i}$ as the completion of the field $\operatorname{Frac} \hat{\mathcal{O}}_x$ with respect to this discrete valuation, and let $\hat{\mathcal{O}}_{x,\mathbf{C}_i}$ be the valuation ring. Then we define

$$K_{x,C} := \bigoplus_{1 \le i \le n} K_{x,\mathbf{C}_i}\,, \qquad \hat{\mathcal{O}}_{x,C} := \bigoplus_{1 \le i \le n} \hat{\mathcal{O}}_{x,\mathbf{C}_i}\,,$$

Observe that if $x \in C$ is a smooth point, then $\hat{\mathcal{O}}_{x,C} \simeq k(x)((t))[\![s]\!]$ and $K_{x,C} \simeq k(x)((t))((s))$. It is clear that the ring $\hat{\mathcal{O}}_x$ diagonally embeds into the ring $K_{x,C}$.

Let us also define $B_x \subset K_{x,C}$ as $\varinjlim_{n>0} s_C^{-n} \hat{\mathcal{O}}_x$, where a local equation $s_C = 0$ determines C on some open $X \supset V \ni x$. It is clear that the subring B_x of $K_{x,C}$ does not depend on the choice of such s_C when $V \ni x$. If $x \in C$ is a smooth point, and $K_{x,C} = k(x)((t))((s_C))$, where $s_C = 0$ is a local equation of the curve C on X near the point x and $t = 0$ defines a transversal curve locally on X near x, then $B_x = k(x)[\![t]\!]((s_C))$.

Any ring $K_{x,C}$ is a 2-Tate vector space over $k(x)$ (and therefore over k), and the ring $\hat{\mathcal{O}}_{x,C}$ is a lattice in $K_{x,C}$.

Let

$$f = \bigoplus_{1 \le i \le n} f_i, \quad g = \bigoplus_{1 \le i \le n} g_i, \quad h = \bigoplus_{1 \le i \le n} h_i.$$

be elements of $K_{x,C}^{\times}$. We define an element of $k(x)^{\times}$ by

$$\{f, g, h\}_{x,C} := \prod_{1 \le i \le n} \{f_i, g_i, h_i\}_{x,\mathbf{C}_i}, \tag{5-3}$$

where $\{f_i, g_i, h_i\}_{x,\mathbf{C}_i}$ is the two-dimensional tame symbol associated to the two-dimensional local field $K_{x,\mathbf{C}_i}$ (cf. Section 4A).

Fix a point $x \in X$. For any free finitely generated $\hat{\mathcal{O}}_x$-module subsheaf $\mathcal{F}$ of the constant sheaf $\operatorname{Frac} \hat{\mathcal{O}}_x$ on the scheme $\operatorname{Spec} \hat{\mathcal{O}}_x$ we consider the following adèle complex $\mathcal{A}_{X,x}(\mathcal{F})$:

$$\mathbb{A}_{X,x,0}(\mathcal{F}) \oplus \mathbb{A}_{X,x,1}(\mathcal{F}) \longrightarrow \mathbb{A}_{X,x,01}(\mathcal{F}).$$

This is the adèle complex on the one-dimensional scheme $U_x := \operatorname{Spec} \hat{\mathcal{O}}_x \setminus x$ for the sheaf $\mathcal{F}|_{U_x}$, and, hence, the cohomology groups of this complex coincide with the cohomology groups $H^*(U_x, \mathcal{F}|_{U_x})$. By definition, we have

$$\mathbb{A}_{X,x,0}(\mathcal{F}) = \operatorname{Frac} \hat{\mathcal{O}}_x, \quad \mathbb{A}_{X,x,1}(\mathcal{F}) = \prod_{\mathbf{C} \ni x} \hat{\mathcal{O}}_{x,\mathbf{C}} \otimes_{\hat{\mathcal{O}}_x} \mathcal{F}, \quad \mathbb{A}_{X,x,01}(\mathcal{F}) = \mathbb{A}_{X,x} = \prod_{\mathbf{C} \ni x}{}' K_{p,\mathbf{C}},$$

where the product is taken over all prime ideals $\mathbf{C}$ of height 1 of the ring $\hat{\mathcal{O}}_x$, and $\prod'$ denotes the restricted (adèle) product with respect to $\prod_{\mathbf{C}\ni x} \hat{\mathcal{O}}_{x,\mathbf{C}}$.

Observe that the adèle ring $\mathbb{A}_{X,x}$ is a 2-Tate vector space over the field $k(x)$. This is because

$$\mathbb{A}_{X,x} = \varprojlim_{\mathcal{G}\subset \mathrm{Frac}\,\hat{\mathcal{O}}_x} \varinjlim_{\mathcal{H}\subset \mathrm{Frac}\,\hat{\mathcal{O}}_x} \mathbb{A}_{X,x,1}(\mathcal{H})/\mathbb{A}_{X,x,1}(\mathcal{G}),$$

and $\mathbb{A}_{X,x,1}(\mathcal{H})/\mathbb{A}_{X,x,1}(\mathcal{G})$ is a 1-Tate vector space for free $\hat{\mathcal{O}}_x$-module subsheaves $0 \neq \mathcal{G} \subset \mathcal{H}$ of $\mathrm{Frac}\,\hat{\mathcal{O}}_x$. (We used that $\mathbb{A}_{X,x,1}(\mathcal{H})/\mathbb{A}_{X,x,1}(\mathcal{G}) = \bigoplus_{\mathbf{C}\ni x} \hat{\mathcal{O}}_{x,\mathbf{C}} \otimes_{\hat{\mathcal{O}}_x} \mathcal{H}/\mathcal{G}$.) For any free finitely generated $\hat{\mathcal{O}}_x$-module subsheaf $\mathcal{F}$ of $\mathrm{Frac}\,\hat{\mathcal{O}}_x$ the space $\mathbb{A}_{X,x,1}(\mathcal{F})$ is a lattice in the space $\mathbb{A}_{X,x}$.

From Proposition 8 in [Osipov 2005] it follows that the $k(x)$-vector spaces $H^0(\mathcal{A}_{X,x}(\mathcal{F}))$ and $H^1(\mathcal{A}_{X,x}(\mathcal{F}))$ are 1-Tate vector spaces. Indeed, since x is a smooth point of X,

$$H^0(\mathcal{A}_{X,x}(\mathcal{F})) = H^0(U_x, \mathcal{F}\,|_{U_x}) = \mathcal{F}$$

is a projective limit of finite-dimensional $k(x)$-vector spaces $\mathcal{F}/m_x^n\mathcal{F}$ (m_x is the maximal ideal of the ring $\hat{\mathcal{O}}_x$), and

$$H^1(\mathcal{A}_{X,x}(\mathcal{F})) = H^1(U_x, \mathcal{F}\,|_{U_x}) = \varinjlim_{n>0} \mathrm{Ext}^2_{\hat{\mathcal{O}}_x}(\hat{\mathcal{O}}_x/m_x^n, \mathcal{F}),$$

where for any $n > 0$ the space $\mathrm{Ext}^2_{\hat{\mathcal{O}}_x}(\hat{\mathcal{O}}_x/m_x^n, \mathcal{F})$ is a finite-dimensional over the field $k(x)$ vector space; see, for example, [Osipov 2005, Lemma 6].

Fix an irreducible projective curve C on X. For any invertible $\mathcal{O}_X$-subsheaf $\mathcal{F}$ of the constant sheaf $k(X)$ on X we consider the following adèle complex $\mathcal{A}_{X,C}(\mathcal{F})$

$$\mathbb{A}_{X,C,0}(\mathcal{F}) \oplus \mathbb{A}_{X,C,1}(\mathcal{F}) \longrightarrow \mathbb{A}_{X,C,01}(\mathcal{F}).$$

where $\mathbb{A}_{X,C,0}(\mathcal{F}) := K_C$, $\mathbb{A}_{X,C,01}(\mathcal{F}) := \mathbb{A}_{X,C} = \mathbb{A}_C((s_C))$, and

$$\mathbb{A}_{X,C,1}(\mathcal{F}) := \left(\prod_{x\in C} B_x \otimes_{\mathcal{O}_X} \mathcal{F}\right) \cap \mathbb{A}_{X,C}. \tag{5-4}$$

Here K_C is the completion of the field $k(X)$ with respect to the discrete valuation given by the curve C on X. (If $s_C = 0$ is a local equation of the curve C on some open subset V of X such that $V \cap C \neq \varnothing$, then $K_C = k(C)((s_C))$.) The ring $\mathbb{A}_{X,C}$ is a subring of $\prod_{x\in C} K_{x,C}$, and does not depend on the choice of s_C. The intersection (5-4) is taken in the ring $\prod_{x\in C} K_{x,C}$.

We note that from [Osipov 2005, § 5.1] it follows that the complex $\mathcal{A}_{X,C}(\mathcal{F})$ coincides with the following complex

$$\varinjlim_{n} \varprojlim_{m>n} \mathcal{A}_{(C,\mathcal{O}_X/J_C^{m-n})}(\mathcal{F} \otimes_{\mathcal{O}_X} J_C^n/J_C^m).$$

Here J_C is the ideal sheaf of the curve C on X, $(C, \mathcal{O}_X/J_C^{m-n})$ is a one-dimensional scheme which has the topological space C and the structure sheaf $\mathcal{O}_X/J_C^{m-n}$, and $\mathcal{A}_{(C,\mathcal{O}_X/J_C^{m-n})}(\mathcal{F} \otimes_{\mathcal{O}_X} J_C^n/J_C^m)$ is the adèle complex of the coherent sheaf $\mathcal{F} \otimes_{\mathcal{O}_X} J_C^n/J_C^m$ on the scheme $(C, \mathcal{O}_X/J_C^{m-n})$. Hence and from the proof of [Osipov 2005, Proposition 12] we obtain that

$$H^*(\mathcal{A}_{X,C}(\mathcal{F})) = \varinjlim_{n} \varprojlim_{m>n} H^*(C, \mathcal{F} \otimes_{\mathcal{O}_X} J_C^n/J_C^m),$$

where for $i = 0$ and $i = 1$ we have $\dim_k H^i(C, \mathcal{F} \otimes_{\mathcal{O}_X} J_C^n/J_C^m) < \infty$. For $i = 0$ and $i = 1$ the k-vector space $H^i(\mathcal{A}_{X,C}(\mathcal{F}))$ has the natural topology of inductive and projective limits. It is not difficult to see that the space $H^0(\mathcal{A}_{X,C}(\mathcal{F}))$ is a locally linearly compact k-vector space; i.e., it is a 1-Tate vector space. But the space $H^1(\mathcal{A}_{X,C}(\mathcal{F}))$ is not a Hausdorff space in this topology. Let $\tilde{H}^1(\mathcal{A}_{X,C}(\mathcal{F}))$ be the quotient space of $H^1(\mathcal{A}_{X,C}(\mathcal{F}))$ by the closure of zero. Then the space $\tilde{H}^1(\mathcal{A}_{X,C}(\mathcal{F}))$ is a locally linearly compact k-vector space, i.e., a 1-Tate vector space.

We note that for any invertible subsheaves $0 \neq \mathcal{G} \subset \mathcal{H}$ of $k(X)$ we have that the space $B_x \otimes_{\hat{\mathcal{O}}_x} (\mathcal{H}/\mathcal{G})$ is a 1-Tate vector space, which is equal to zero for almost all points $x \in C$. Hence, we obtain that the space

$$\mathbb{A}_{X,C,1}(\mathcal{H})/\mathbb{A}_{X,C,1}(\mathcal{G}) = \bigoplus_{x \in C} B_x \otimes_{\hat{\mathcal{O}}_x} (\mathcal{H}/\mathcal{G})$$

is a 1-Tate vector space.

For any point $x \in X$, we define a ring K_x as the localization of the ring $\hat{\mathcal{O}}_x$ with respect to the multiplicative system $\mathcal{O}_x \setminus 0$. (We note that inside of the field $\operatorname{Frac} \hat{\mathcal{O}}_x$ the ring K_x is defined as the product of two subrings: $\hat{\mathcal{O}}_x$ and $k(X)$.)

For any pair $x \in C$ (where C is an irreducible curve on X and $x \in C$ is a closed point), we have the natural embeddings $k(X) \hookrightarrow K_x$, $k(X) \hookrightarrow K_C$ (recall that K_C is the completion of the field $k(X)$ with respect to the discrete valuation given by the curve C). In addition, there are the natural embeddings $K_x, K_C \hookrightarrow K_{x,C}$. Therefore, we obtain

$$k(X) \hookrightarrow K_x \hookrightarrow \mathbb{A}_{X,x}, \qquad k(X) \hookrightarrow K_C \hookrightarrow \mathbb{A}_{X,C}.$$

Theorem 5.3 (Parshin reciprocity laws). (1) *Fix a point $x \in X$. Consider elements f, g, h of the group $K_x^\times$ of invertible elements of the ring K_x. Then the* *following product in* $k(x)^\times$ *contains only finitely many terms distinct from* 1 *and*

$$\prod_{C \ni x} \{f, g, h\}_{x,C} = 1. \tag{5-5}$$

(2) *Fix a projective irreducible curve* C *on* X. *Let elements* f, g, h *be from the group* $K_C^\times$. *Then the following product in* $k^\times$ *contains only finitely many terms distinct from* 1 *and*

$$\prod_{x \in C} \mathrm{Nm}_{k(x)/k}\{f, g, h\}_{x,C} = 1. \tag{5-6}$$

Proof. We first prove formula (5-5). By Theorem 4.11, for any $f, g, h \in K_{x,\mathbf{C}}^\times$ we have

$$\{f, g, h\}_{x,\mathbf{C}} = C_3^{\mathcal{D}et_{x,\mathbf{C}}}(f, g, h) \tag{5-7}$$

for all prime ideals $\mathbf{C}$ of height 1 of the ring $\hat{\mathcal{O}}_x$, where the central extension $\mathcal{D}et_{x,\mathbf{C}}$ of the group $K_{x,\mathbf{C}}^\times$ by the Picard groupoid $\mathcal{P}ic^{\mathbb{Z}}$ is constructed by formula (4-6) from the 2-Tate vector space $K_{x,\mathbf{C}}$ over the field $k(x)$ and the lattice $\hat{\mathcal{O}}_{x,\mathbf{C}}$ as in Section 4C. We note that for almost all prime ideals $\mathbf{C}$ of height 1 of ring $\hat{\mathcal{O}}_x$, and for any elements f, g, h from the group $\operatorname{Frac}\hat{\mathcal{O}}_x^\times$, we have $f\mathcal{O}_{x,\mathbf{C}} = \mathcal{O}_{x,\mathbf{C}}$, $g\mathcal{O}_{x,\mathbf{C}} = \mathcal{O}_{x,\mathbf{C}}$, and $h\mathcal{O}_{x,\mathbf{C}} = \mathcal{O}_{x,\mathbf{C}}$. Then by Lemma 4.10 and Corollary 2.19, for almost all prime ideals $\mathbf{C}$ of height 1 of ring $\hat{\mathcal{O}}_x$ we have $C_3^{\mathcal{D}et_{x,\mathbf{C}}}(f, g, h) = 1$.

We will prove that the central extension $\mathcal{D}et_x$ of $\operatorname{Frac}\hat{\mathcal{O}}_x^\times (\subset \mathrm{GL}(\mathbb{A}_{X,x}))$ by $\mathcal{P}ic^{\mathbb{Z}}$ constructed by the 2-Tate vector space $\mathbb{A}_{X,x}$ and the lattice $\mathbb{A}_{X,x,1}(\hat{\mathcal{O}}_x)$ using formula (4-6) can be trivialized in an explicit way. Observe that for any $d \in \operatorname{Frac}\hat{\mathcal{O}}_x^\times$, there is a canonical isomorphism of $\mathcal{P}ic^{\mathbb{Z}}$-torsors:

$$\mathcal{D}et(\mathbb{A}_{X,x,1}(\hat{\mathcal{O}}_x) \mid \mathbb{A}_{X,x,1}(d\hat{\mathcal{O}}_x)) \simeq \mathcal{D}et(H^*(\mathcal{A}_{X,x}(d\hat{\mathcal{O}}_x))) - \mathcal{D}et(H^*(\mathcal{A}_{X,x}(\hat{\mathcal{O}}_x))), \tag{5-8}$$

where for any free subsheaf $\mathcal{F}$ of $\operatorname{Frac}\hat{\mathcal{O}}_x$ on the scheme $\operatorname{Spec}\hat{\mathcal{O}}_x$

$$\mathcal{D}et(H^*(\mathcal{A}_{X,x}(\mathcal{F}))) := \mathcal{D}et(H^0(\mathcal{A}_{X,x}(\mathcal{F}))) - \mathcal{D}et(H^1(\mathcal{A}_{X,x}(\mathcal{F}))).$$

Indeed, isomorphism (5-8) follows from Proposition 3.8 applied to the long exact sequence (decomposed into the short exact sequences) associated with the following exact sequence of complexes of length 2 for any nonzero free subsheaves $\mathcal{G} \subset \mathcal{H}$ of $\operatorname{Frac}\hat{\mathcal{O}}_x$ on the scheme $\operatorname{Spec}\hat{\mathcal{O}}_x$:

$$0 \longrightarrow \mathcal{A}_{X,x}(\mathcal{G}) \longrightarrow \mathcal{A}_{X,x}(\mathcal{H}) \longrightarrow \mathcal{A}_{X,x,1}(\mathcal{H})/\mathbb{A}_{X,x,1}(\mathcal{G}) \longrightarrow 0,$$

where the last complex consists only of the group placed in degree zero. Now we have

$$\mathcal{D}et(H^*(\mathcal{A}_{X,x}(d\hat{\mathcal{O}}_x))) - \mathcal{D}et(H^*(\mathcal{A}_{X,x}(\hat{\mathcal{O}}_x))) \\ \simeq \mathrm{Hom}_{\mathcal{P}ic^{\mathbb{Z}}}(\mathcal{D}et(H^*(\mathcal{A}_{X,x}(\hat{\mathcal{O}}_x))), \mathcal{D}et(H^*(\mathcal{A}_{X,x}(d\hat{\mathcal{O}}_x))). \quad (5\text{-}9)$$

Multiplication by the element $d \in \operatorname{Frac} \hat{\mathcal{O}}_x^\times$ between adèle complexes $\mathcal{A}_{X,x}(\hat{\mathcal{O}}_x)$ and $\mathcal{A}_{X,x}(d\hat{\mathcal{O}}_x)$ gives a natural isomorphism of $\mathcal{P}ic^{\mathbb{Z}}$-torsor from formula (5-9) to the trivial torsor $\mathcal{P}ic^{\mathbb{Z}}$.

Let H be the subgroup of $\operatorname{Frac} \hat{\mathcal{O}}_x^\times$ generated by the elements $f, g, h \in \operatorname{Frac} \hat{\mathcal{O}}_x^\times$. Now we proceed as the proof of Weil reciprocity law (see Equation (5-1)), with the help of Lemma 4.9, Lemma 4.10, and Corollary 2.20. Then we obtain the following equality:

$$\prod_{\mathbf{C} \ni x} \{f, g, h\}_{x,\mathbf{C}} = 1.$$

Formula (5-5) follows from the last formula, since if a prime ideal $\mathbf{C}$ of height 1 in $\hat{\mathcal{O}}_x$ is not a formal branch at x of some irreducible curve C on X, then for any element $d \in K_x^\times$ we have $d\mathcal{O}_{x,\mathbf{C}} = \mathcal{O}_{x,\mathbf{C}}$. Hence, by formula (5-7), $\{f, g, h\}_{x,\mathbf{C}} = 1$ for such $\mathbf{C}$ and any $f, g, h \in K_x^\times$.

Next we will prove formula (5-6). We construct the central extension $\mathcal{D}et'_{x,C}$ of the group $k(X)^\times$ by the Picard groupoid $\mathcal{P}ic^{\mathbb{Z}}$ in the following way. We fix a point $x \in C$, and associate with the rings $B_x \subset K_{x,C}$ and with an element $d \in k(X)^\times$ the following $\mathcal{P}ic^{\mathbb{Z}}$-torsor:

$$\mathcal{D}et(B_x \mid dB_x) := \mathcal{D}et\left(\frac{dB_x}{B_x \cap dB_x}\right) - \mathcal{D}et\left(\frac{B_x}{B_x \cap dB_x}\right). \quad (5\text{-}10)$$

(We used that $B_x / B_x \cap dB_x$ is a 1-Tate vector space over the field k.) By the formula which is analogous to formula (4-7) we obtain that the central extension $\mathcal{D}et'_{x,C}$ is well defined. In a similar way we define the central extensions $\mathcal{D}et'_C$ and $\mathcal{D}et'_{C \setminus \{x_1,\dots,x_l\}}$ starting from the rings $\mathbb{A}_{X,C,1}(\mathcal{O}_X) \subset \mathbb{A}_{X,C}$ and $\mathbb{A}_{X,C\setminus\{x_1,\dots,x_l\},1}(\mathcal{O}_X) \subset \mathbb{A}_{X,C\setminus\{x_1,\dots,x_l\}}$, where $x_1, \dots, x_l$ are some points on the curve C.

Let the group H be generated in the group $k(X)^\times$ by the elements $f, g, h \in k(X)^\times$. For almost all points x of the curve C we have that the group H preserves the subring B_x. Therefore form formula (5-10) we obtain that the central extension $\mathcal{D}et'_{x,C}$ is isomorphic to the trivial one for almost all points x of the curve C. Therefore for almost all points x of the curve C we have $C_3^{\mathcal{D}et'_{x,C}}(f, g, h) = 1$.

We will prove that the central extension $\mathcal{D}et'_{x,C}$ is inverse (or dual) to the central extension $\mathcal{D}et_{x,C}$, where the last central extension is constructed by formula (4-6) from the lattice $\mathcal{O}_{x,C}$ in the 2-Tate vector space $K_{x,C}$. For any free subsheaf $\mathcal{F}$ of the constant sheaf $\operatorname{Frac} \hat{\mathcal{O}}_x$ on the scheme $\operatorname{Spec} \hat{\mathcal{O}}_x$ there is the following complex $\mathcal{A}_{X,C,x}(\mathcal{F})$:

$$(B_x \otimes_{\hat{\mathcal{O}}_x} \mathcal{F}) \oplus (\hat{\mathcal{O}}_{x,C} \otimes_{\hat{\mathcal{O}}_x} \mathcal{F}) \longrightarrow K_{x,C}.$$

We have canonically that $H^*(\mathcal{A}_{X,C,x}(\mathcal{F})) = H^*(U_x, \mathcal{F}|_{U_x})$, where we recall $U_x = \operatorname{Spec} \hat{\mathcal{O}}_x \setminus x$ (see the proof of [Osipov 2005, Proposition 13]). Therefore the cohomology groups of complex $\mathcal{A}_{X,C,x}(\mathcal{F})$ are 1-Tate vector spaces. Hence, there is a canonical isomorphism between the following $\mathcal{P}ic^{\mathbb{Z}}$-torsors for any $d \in k(X)^\times$:

$$\mathcal{D}et(B_x \mid dB_x) + \mathcal{D}et(\hat{\mathcal{O}}_{x,C} \mid d\hat{\mathcal{O}}_{x,C}),$$
$$\operatorname{Hom}_{\mathcal{P}ic^{\mathbb{Z}}}(\mathcal{D}et(H^*(\mathcal{A}_{X,C,x}(\hat{\mathcal{O}}_x))), \mathcal{D}et(H^*(\mathcal{A}_{X,C,x}(d\hat{\mathcal{O}}_x)))).$$

Now multiplication by the element d of adèle complexes gives a natural isomorphism from the last $\mathcal{P}ic^{\mathbb{Z}}$-torsor to the trivial one. Hence from Corollary 2.20 we have that

$$C_3^{\mathcal{D}et'_{x,C}}(f,g,h) = C_3^{\mathcal{D}et'_{x,C}}(f,g,h)^{-1} = \operatorname{Nm}_{k(x)/k}\{f,g,h\}_{x,C}^{-1}$$

for $f, g, h \in k(X)^\times$.

Now the proof of formula (5-6) for elements $f, g, h \in k(X)^\times$ follows by the same method as in the proof of formula (5-5), but we have to use the adèle ring $\mathbb{A}_{X,C}$ instead of the ring $\mathbb{A}_{X,x}$, and to use the central extension $\mathcal{D}et'_C$ instead of the central extension $\mathcal{D}et_x$. We need only to prove that the central extension $\mathcal{D}et'_C$ constructed by the analog of formula (5-10) from the rings $\mathbb{A}_{X,C,1}(\mathcal{O}_X) \subset \mathbb{A}_{X,C}$ is isomorphic the trivial central extension. This follows if we consider the following $\mathcal{P}ic^{\mathbb{Z}}$-torsors for $d \in k(X)^\times$

$$\operatorname{Hom}_{\mathcal{P}ic^{\mathbb{Z}}}(\mathcal{D}et(H^*(\mathcal{A}_{X,C}(\mathcal{O}_X))), \mathcal{D}et(H^*(\mathcal{A}_{X,C}(d\mathcal{O}_X)))), \tag{5-11}$$

where

$$\mathcal{D}et(H^*(\mathcal{A}_{X,C}(d\mathcal{O}_X)) := \mathcal{D}et(H^0(\mathcal{A}_{X,C}(d\mathcal{O}_X)) - \mathcal{D}et(\tilde{H}^1(\mathcal{A}_{X,C}(d\mathcal{O}_X)).$$

Multiplication by $d \in k(X)^\times$ of adèle complexes gives the triviality of the $\mathcal{P}ic^{\mathbb{Z}}$-torsor (5-11). (See analogous reasonings earlier in the proof of this theorem.)

To obtain formula (5-6) for elements $f, g, h \in K_C^\times$ we have to use that the field $k(X)$ is dense in the field K_C. Therefore for any element $f \in K_C^\times$ there is an element $\tilde{f} \in k(X)^\times$ such that $f = \tilde{f}m$, where the element m is from the subgroup $1 + m_C^n$ of the group $K_C^\times$ for some $n \geq 1$, and m_C is the maximal ideal of the valuation ring of discrete valuation field K_C. Then from formula (4-3) we have that $\{m, g, h\}_{x,\mathbf{C}} = 1$ for any point $x \in C$, and any formal branch $\mathbf{C}$ of the curve C at point x. Hence, from the trimultiplicativity of the two-dimensional tame symbol we obtain that

$$\{f, g, h\}_{x,\mathbf{C}} = \{\tilde{f}, g, h\}_{x,\mathbf{C}}.$$

Applying successively the same procedure to elements $g, h \in k_C^\times$ we obtain

$$\{f, g, h\}_{x,\mathbf{C}} = \{\tilde{f}, \tilde{g}, \tilde{h}\}_{x,\mathbf{C}},$$

where $\tilde{f}, \tilde{g}, \tilde{h} \in k(X)^{\times}$, and any point $x \in C$, and $\mathbf{C}$ is any formal branch of the curve C at point x. □

Remark 5.4. For the proof of Parshin reciprocity laws we used "semilocal" adèle complexes of length 2 connected with either points or irreducible curves on an algebraic surface. But for the formulation of these reciprocity laws we used the rings K_x and K_C which appear from the "global" adèle complex of length 3 on an algebraic surface. It would be interesting to find direct connections between the "global" adèle complex and "semilocal" adèle complexes of an algebrac surface.

Remark 5.5. We have a symmetric monoidal functor from the Picard torsor $\mathcal{P}ic^{\mathbb{Z}}$ to the Picard groupoid $\mathbb{Z}$ which sends every graded line to its grading element from $\mathbb{Z}$, where $\mathbb{Z}$ is considered as the groupoid with objects equal to $\mathbb{Z}$ and morphisms equal to identities morphisms. Under this functor a central extension of a group G by a $\mathcal{P}ic^{\mathbb{Z}}$-torsor goes to the usual central of the group G by the group $\mathbb{Z}$. In this way the map $\nu_{\mathbb{K}}$ for a two-dimensional local field $\mathbb{K}$ was obtained as the commutator of elements in this central extension in [Osipov 2005]. Also in this same reference the reciprocity laws for the map $\nu_{\mathbb{K}}$ were proved by the adèle complexes on an algebraic surface.

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Quantum differentiation and chain maps of bimodule complexes

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We consider a finite group acting on a vector space and the corresponding skew group algebra generated by the group and the symmetric algebra of the space. This skew group algebra illuminates the resulting orbifold and serves as a replacement for the ring of invariant polynomials, especially in the eyes of cohomology. One analyzes the Hochschild cohomology of the skew group algebra using isomorphisms which convert between resolutions. We present an explicit chain map from the bar resolution to the Koszul resolution of the symmetric algebra which induces various isomorphisms on Hochschild homology and cohomology, some of which have appeared in the literature before. This approach unifies previous results on homology and cohomology of both the symmetric algebra and skew group algebra. We determine induced combinatorial cochain maps which invoke quantum differentiation (expressed by Demazure–BGG operators).

1. Introduction

Let G be a finite group acting linearly on a finite-dimensional complex vector space V. The skew group algebra $S(V)\#G$ is a natural semi-direct product of G with the symmetric algebra $S(V)$ (a polynomial ring). It serves as a valuable, albeit noncommutative, replacement for the invariant ring $S(V)^G$ in geometric settings, as it encodes the abstract group structure of G as well as its action on V. The cohomology of $S(V)\#G$ informs various areas of mathematics (for example, geometry, combinatorics, representation theory, and mathematical physics). In particular, the Hochschild cohomology of $S(V)\#G$ governs its deformations, which include graded Hecke algebras, symplectic reflection algebras, and Cherednik algebras.

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The orbifold V/G may be realized as an algebraic variety whose coordinate ring is the ring of invariant functions $S(V^*)^G$ on the dual space V^*, which is the center of $S(V^*)\#G$ when G acts faithfully. (For details, see [Harris 1995].) The variety V/G is nonsingular exactly when the action of G on V is generated by reflections. Geometers and physicists are interested in resolving the singularities of V/G with a smooth variety X and examining the coordinate ring of X instead of $S(V)^G$. In situations they study, the skew group algebra $S(V)\#G$ serves as a replacement for the coordinate ring of X; indeed, Hochschild cohomology sees no difference between these rings [Căldăraru et al. 2004]. Connections with representation theory are still unfolding; for example, see [Gordon and Smith 2004].

The Hochschild cohomology $\mathrm{HH}^{\bullet}(A)$ of any algebra A over a field k is the space $\mathrm{Ext}^{\bullet}_{A\otimes A^{op}}(A, A)$. The cup product and Gerstenhaber bracket on Hochschild cohomology are both defined initially on the bar resolution, a natural $A\otimes A^{op}$-free resolution of A. The cup product has another description as Yoneda composition of extensions of modules, which can be transported to any other projective resolution. However, the Gerstenhaber bracket has resisted such a general description. Instead, one commonly computes $\mathrm{HH}^{\bullet}(A)$ using a more convenient resolution, then one finds and uses relevant chain maps to lift the Gerstenhaber bracket from the bar resolution. The case $A = S(V)\#G$ is complicated further because one does not work with resolutions of A directly, but instead one derives information from resolutions of the symmetric algebra $S(V)$.

In this paper, we begin this task by constructing explicit chain maps which encode traffic between resolutions used to describe the Hochschild cohomology of $A = S(V)\#G$. Our maps convert between the bar and Koszul resolutions of the polynomial ring $S(V)$, and serve as a tool for investigating the homology and cohomology of $S(V)$ with coefficients in any bimodule. Specifically, the Koszul resolution of the polynomial ring $S(V)$ embeds naturally into the bar resolution. We define an explicit chain map, depending on a choice of basis, giving a quasi-inverse to this embedding. We study in particular the induced maps on the Hochschild cohomology $\mathrm{HH}^{\bullet}(S(V), S(V)\#G)$. We give an elegant, combinatorial description of the induced map on cochains in terms of scaled Demazure (BGG) operators (or quantum partial differential operators, see Definition 3-4). We describe the induced maps on Hochschild homology as well. (These combinatorial descriptions are useful for computations, which we pursue in other articles.) The cohomology $\mathrm{HH}^{\bullet}(S(V)\#G)$ manifests as the G-invariant subspace of $\mathrm{HH}^{\bullet}(S(V), S(V)\#G)$ in characteristic 0. We thus obtain isomorphisms of homology and cohomology that allow one to transfer structures defined on the bar resolution to the complexes standardly used to describe $\mathrm{HH}^{\bullet}(S(V)\#G)$.

In Section 2, we establish notation and deploy the Hochschild cohomology $\mathrm{HH}^{\bullet}(S(V)\#G)$ in terms of both the Koszul and bar resolutions of $S(V)$. We introduce a combinatorial map Υ on cochains in Section 3. This combinatorial converter Υ takes vector forms (tagged by group elements) to twisted quantum differential operators. In Section 4, we give a technical formula for explicit chain maps from the bar resolution to the Koszul resolution (Definition 4-1), which is valid over an arbitrary ground field. These specific chain maps each induce an inverse to the embedding of the Koszul resolution into the bar resolution after taking homology or cohomology. (Indeed, after applying functors $\otimes$ or Hom, we recover some chain maps given in the literature for converting between complexes expressing Hochschild homology and cohomology—see Section 6.) In Section 5, we deduce that our combinatorial converter Υ defines automorphisms of cohomology by showing that it is induced by the chain maps of Section 4. We present similar automorphisms of homology (using quantum differentiation) in Section 6.

Our approach presents an immediate and obvious advantage: We define one primitive map between resolutions and then apply various functors that automatically give (co)chain maps in a variety of settings. We do not need to give separate proofs (depending on context) showing that these induced maps are chain maps, as such results follow immediately from the general theory. This uniform treatment provides a clear channel for navigating between chain and cochain complexes. Indeed, we use this channel in [Shepler and Witherspoon 2009; 2011] to explore the algebraic structure of $\mathrm{HH}^\bullet(S(V)\#G)$ under the cup product and the Gerstenhaber bracket.

Some results in this paper are valid over a field of arbitrary characteristic, while others assume the ground field is the complex numbers, $\mathbb{C}$. We have tried to state carefully requirements on the field throughout. The reader should note that whenever we work over $\mathbb{C}$, we could instead work over any field containing the eigenvalues of the action of G on V in which $|G|$ is invertible. All tensor and exterior products will be taken over the ground field unless otherwise indicated.

2. Preliminary material

In this section, we work over the complex numbers $\mathbb{C}$, although the definitions below of Hochschild cohomology, bar resolution, and Koszul resolution are valid over any ground field.

Let G be a finite group and V a (not necessarily faithful) $\mathbb{C}G$-module. Let ${}^g v$ denote the image of $v \in V$ under the action of $g \in G$. We work with the induced group action on all maps throughout this article: For any map θ and element $h \in \mathrm{GL}(V)$, we define the map ${}^h\theta$ by $({}^h\theta)(v) := {}^h(\theta({}^{h^{-1}}v))$ for all v. Let V^* denote the vector space dual to V with the contragredient (i.e., dual) representation. For any basis $v_1, \dots, v_n$ of V, let $v_1^*, \dots, v_n^*$ be the dual basis of V^*. Let $V^G = \{v \in V : {}^g v = v \text{ for all } g \in G\}$, the set of G-invariants in V. For any $g \in G$, let

$Z(g) = \{h \in G : gh = hg\}$, the centralizer of g in G, and let $V^g = \{v \in V : {}^g v = v\}$, the g-invariant subspace of V.

The *skew group algebra* $S(V)\#G$ is the vector space $S(V) \otimes \mathbb{C}G$ with multiplication given by

$$(a \otimes g)(b \otimes h) = a({}^g b) \otimes gh$$

for all $a, b \in S(V)$ and $g, h \in G$. We abbreviate $a \otimes g$ by $a\bar{g}$ ($a \in S(V)$, $g \in G$) and $a \otimes 1$, $1 \otimes g$ simply by a, $\bar{g}$, respectively. An element $g \in G$ acts on $S(V)$ by an inner automorphism in $S(V)\#G$: $\bar{g}a(\bar{g})^{-1} = ({}^g a)\bar{g}(\bar{g})^{-1} = {}^g a$ for all $a \in A$.

The *Hochschild cohomology* of a $\mathbb{C}$-algebra A (such as $A = S(V)\#G$), with coefficients in an A-bimodule M, is the graded vector space

$$\mathrm{HH}^{\bullet}(A, M) = \mathrm{Ext}^{\bullet}_{A^e}(A, M),$$

where $A^e = A \otimes A^{op}$ acts on A by left and right multiplication. This cohomology may be expressed in terms of the *bar resolution*, the following free A^e-resolution of A:

$$\cdots \xrightarrow{\delta_3} A^{\otimes 4} \xrightarrow{\delta_2} A^{\otimes 3} \xrightarrow{\delta_1} A^e \xrightarrow{m} A \to 0, \tag{2-1}$$

where

$$\delta_p(a_0 \otimes \cdots \otimes a_{p+1}) = \sum_{j=0}^{p} (-1)^j a_0 \otimes \cdots \otimes a_j a_{j+1} \otimes \cdots \otimes a_{p+1},$$

and $\delta_0 = m$ is multiplication. We apply $\mathrm{Hom}_{A^e}(-, M)$ to obtain a cochain complex whose homology is $\mathrm{HH}^{\bullet}(A, M)$. If $M = A$, we abbreviate $\mathrm{HH}^{\bullet}(A) = \mathrm{HH}^{\bullet}(A, A)$. For each p,

$$\mathrm{Hom}_{A^e}(A^{\otimes(p+2)}, A) \cong \mathrm{Hom}_{\mathbb{C}}(A^{\otimes p}, A),$$

and we identify these two spaces of p-cochains throughout this article. The graded vector space $\mathrm{HH}^{\bullet}(A)$ admits both a cup product and a graded Lie bracket under which it becomes a Gerstenhaber algebra. In this article, we develop automorphisms of cohomology converting between resolutions. These automorphisms will be used in later publications to explore the algebraic structure of $\mathrm{HH}^{\bullet}(S(V)\#G)$ under these two operations.

Hochschild cohomology of $S(V)\#G$. Farinati [2005] and Ginzburg and Kaledin [2004] determined the graded vector space structure of $\mathrm{HH}^{\bullet}(S(V)\#G)$ when G acts faithfully on V. The same techniques apply to nonfaithful actions. The following statements are valid only when the characteristic does not divide the order of G. (Otherwise, the cohomology is more complicated as the group algebra of G may itself not be semisimple.) Let $\mathscr{C}$ be a set of representatives of the conjugacy classes of G. A consequence of [Ştefan 1995, Corollary 3.4] posits a natural G-action giving the first of a series of isomorphisms of graded vector spaces:

$$\begin{aligned} \mathrm{HH}^{\bullet}(S(V)\#G) &\cong \mathrm{HH}^{\bullet}(S(V), S(V)\#G)^G \\ &\cong \Big(\bigoplus_{g\in G} \mathrm{HH}^{\bullet}(S(V), S(V)\bar{g})\Big)^G \\ &\cong \bigoplus_{g\in \mathscr{C}} \mathrm{HH}^{\bullet}\big(S(V), S(V)\bar{g}\big)^{Z(g)}. \end{aligned} \tag{2-2}$$

Specifically, the action of G on V extends naturally to the bar complex of $S(V)$ and commutes with the differentials, and so induces a natural action on Hochschild cohomology $\mathrm{HH}^{\bullet}(S(V), S(V)\#G)$ (for which we also use the action of G on $S(V)\#G$ by inner automorphisms). The subspace of G-invariants of this action is denoted by $\mathrm{HH}^{\bullet}(S(V), S(V)\#G)^G$. (Equivalently, this may also be defined via any other choice of G-compatible resolution used to compute cohomology; see [Ştefan 1995, Section 2], for example.)

The second isomorphism of (2-2) surfaces simply because the $S(V)^e$-module $S(V)\#G$ decomposes into the direct sum of $S(V)^e$-modules $S(V)\bar{g}$, and cohomology preserves direct sums. (The isomorphism arises of course at the cochain level, as the Hom-functor preserves direct sums.) We identify $\mathrm{HH}^{\bullet}(S(V), S(V)\#G)$ with $\bigoplus_{g\in G} \mathrm{HH}^{\bullet}(S(V), S(V)\bar{g})$ when convenient throughout this article. Note that G permutes the components in the direct sum in accordance with the conjugation action of G on itself. Thus for each $g \in G$, the subgroup $Z(g)$ fixes the g-component $\mathrm{HH}^{\bullet}(S(V), S(V)\bar{g})$ setwise. The third isomorphism of (2-2) canonically projects onto a set of representative summands.

One may use the Koszul resolution for $S(V)$ to determine each g-component $\mathrm{HH}^{\bullet}(S(V), S(V)\bar{g})$ in the last line of (2-2) above. The *Koszul resolution*, denoted by $K_{\bullet}(S(V))$, is given by $K_0(S(V)) = S(V)^e$, $K_1(S(V)) = S(V)^e \otimes V$, and for each $p \geq 2$,

$$K_p(S(V)) = \bigcap_{j=0}^{p-2} S(V)^e \otimes (V^{\otimes j} \otimes R \otimes V^{\otimes(p-j-2)}), \tag{2-3}$$

where R is the subspace of $V \otimes V$ spanned by all $v \otimes w - w \otimes v$ ($v, w \in V$); see [Braverman and Gaitsgory 1996], for example. This is a subcomplex of the bar resolution (2-1) for $S(V)$. For any choice of basis $v_1, \ldots, v_n$ of V, it is equivalent to the Koszul resolution corresponding to the regular sequence $\{v_i \otimes 1 - 1 \otimes v_i\}_{i=1}^n$ in $S(V)^e$:

$$K_p(\{v_i \otimes 1 - 1 \otimes v_i\}_{i=1}^n) \cong S(V)^e \otimes \textstyle\bigwedge^p(V), \tag{2-4}$$

a free $S(V)^e$-resolution of $S(V)$; see [Weibel 1994, §4.5], for instance. The differentials are given by

$$d_p(1 \otimes 1 \otimes v_{j_1} \wedge \cdots \wedge v_{j_p}) \\ = \sum_{i=1}^{p} (-1)^{i+1} (v_{j_i} \otimes 1 - 1 \otimes v_{j_i}) \otimes (v_{j_1} \wedge \cdots \wedge \hat{v}_{j_i} \wedge \cdots \wedge v_{j_p}). \quad (2\text{-}5)$$

The canonical inclusion of the Koszul resolution (2-3) into the bar resolution (2-1) for $S(V)$ is then given on resolution (2-4) by the chain map

$$\Phi : S(V)^e \otimes \textstyle\bigwedge^{\bullet}(V) \to S(V)^{\otimes(\bullet+2)} ,$$

defined by

$$\Phi_p(1 \otimes 1 \otimes v_{j_1} \wedge \cdots \wedge v_{j_p}) = \sum_{\pi \in \mathrm{Sym}_p} \mathrm{sgn}(\pi) \otimes v_{j_{\pi(1)}} \otimes \cdots \otimes v_{j_{\pi(p)}} \otimes 1 \quad (2\text{-}6)$$

for all $v_{j_1}, \ldots, v_{j_p} \in V$, $p \geq 1$, where Sym_p denotes the symmetric group on p symbols. Note that by its definition, Φ is invariant under the action of $\mathrm{GL}(V)$, i.e., ${}^h\Phi = \Phi$ for all h in $\mathrm{GL}(V)$.

Any chain map $\Psi_p : S(V)^{\otimes(p+2)} \to S(V)^e \otimes \bigwedge^p V$ from the bar resolution to the Koszul resolution yields a commutative diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & S(V)^{\otimes 4} \arrow[r, "\delta_2"] \arrow[d, shift right, "\Psi_2"'] & S(V)^{\otimes 3} \arrow[r, "\delta_1"] \arrow[d, shift right, "\Psi_1"'] & S(V)^e \arrow[r, "m"] \arrow[d, leftrightarrow, "="'] & S(V) \arrow[r] \arrow[d, leftrightarrow, "="'] & 0 \\
\cdots \arrow[r] & S(V)^e \otimes \bigwedge^2 V \arrow[r, "d_2"] \arrow[u, shift right, "\Phi_2"'] & S(V)^e \otimes \bigwedge^1 V \arrow[r, "d_1"] \arrow[u, shift right, "\Phi_1"'] & S(V)^e \arrow[r, "m"] & S(V) \arrow[r] & 0.
\end{tikzcd}$$

(In Definition 4-1, we explicitly define a map Ψ depending on a choice of basis of V.) Such maps Φ and Ψ necessarily induce inverse isomorphisms on cohomology $\mathrm{HH}^{\bullet}(S(V), M)$ for any $S(V)$-bimodule M upon applying $\mathrm{Hom}_{S(V)^e}(-, M)$. (Similarly for homology; see Section 6.) Identifying $\mathrm{Hom}_{S(V)^e}(S(V)^{\otimes(p+2)}, M)$ with $\mathrm{Hom}_{\mathbb{C}}(S(V)^{\otimes p}, M)$ and $\mathrm{Hom}_{S(V)^e}(S(V)^e \otimes \bigwedge^p V, M)$ with $\mathrm{Hom}_{\mathbb{C}}(\bigwedge^p V, M)$ for all p, we obtain the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbb{C}}(S(V)^{\otimes p}, M) \arrow[r, "\delta^*_{p-1}"] \arrow[d, shift left, "\Phi^*_p"] & \mathrm{Hom}_{\mathbb{C}}(S(V)^{\otimes(p+1)}, M) \arrow[d, shift left, "\Phi^*_{p+1}"] \\
\mathrm{Hom}_{\mathbb{C}}(\bigwedge^p V, M) \arrow[r, "d^*_{p-1}"] \arrow[u, shift left, "\Psi^*_p"] & \mathrm{Hom}_{\mathbb{C}}(\bigwedge^{p+1} V, M) \arrow[u, shift left, "\Psi^*_{p+1}"]
\end{tikzcd} \quad (2\text{-}7)$$

The maps $\Phi\Psi$ and $\Psi\Phi$ are each homotopic to an identity map by the Comparison Theorem, and thus Φ^*_p and Ψ^*_p induce inverse automorphisms on the cohomology $\mathrm{HH}^p(S(V), M)$; see the proof of [Weibel 1994, Lemma 2.4.1]. In this paper, we primarily consider the $S(V)^e$-modules $M = S(V)\#G$ and $M = S(V)\overline{g}$ for g in G.

We transfer the map Φ and any chain map Ψ to the Hochschild cohomology of the full skew group algebra, $\mathrm{HH}^\bullet(S(V)\#G)$, using the isomorphisms of (2-2). Set $M = S(V)\#G$ and let Φ^* and Ψ^* denote the induced maps on the cohomology

$$\mathrm{HH}^\bullet(S(V), S(V)\#G) \cong \bigoplus_{g\in G} \mathrm{HH}^\bullet\big(S(V), S(V)\bar{g}\big).$$

For each g in G, denote the restrictions to $\mathrm{HH}^\bullet\big(S(V), S(V)\bar{g}\big)$ by Φ_g^* and Ψ_g^*, respectively, so that

$$\Phi^* = \bigoplus_{g\in G} \Phi_g^* \qquad \text{and} \qquad \Psi^* = \bigoplus_{g\in G} \Psi_g^*.$$

The maps Φ^* and Ψ^* behave nicely with respect to the action of G:

Proposition 2-8. *Let Ψ be any choice of chain map from the bar resolution (2-1) to the Koszul resolution (2-4). The cochain maps Φ^* and Ψ^* are inverse automorphisms on the cohomology $\mathrm{HH}^\bullet(S(V), S(V)\#G)$ converting between expressions arising from the Koszul resolution and from the bar resolution. In addition,*

(1) *For any $g \in G$, the maps Φ_g^* and Ψ_g^* on the cohomology $\mathrm{HH}^\bullet\big(S(V), S(V)\bar{g}\big)$ are invariant under the centralizer $Z(g)$ of g in G, and the maps Φ^* and Ψ^* on $\bigoplus_{g\in G} \mathrm{HH}^\bullet\big(S(V), S(V)\bar{g}\big)$ are invariant under G;*

(2) *The maps Φ^* and Ψ^* induce inverse automorphisms on the graded vector space*

$$\bigoplus_{g\in\mathscr{C}} \Big(\mathrm{HH}^\bullet\big(S(V), S(V)\bar{g}\big)\Big)^{Z(g)} \cong \bigg(\bigoplus_{g\in G} \mathrm{HH}^\bullet(S(V), S(V)\bar{g})\bigg)^{G} \cong \mathrm{HH}^\bullet(S(V)\#G).$$

Proof. As explained after Diagram (2-7), the maps Φ_g^* and Ψ_g^* are inverse isomorphisms on the cohomology $\mathrm{HH}^\bullet\big(S(V), S(V)\bar{g}\big)$. By its definition, Φ is invariant under the action of $\mathrm{GL}(V)$, and so the map Φ^* on $\mathrm{HH}^\bullet(S(V), S(V)\#G)$ is invariant under G, and the map Φ_g^* on $\mathrm{HH}^\bullet\big(S(V), S(V)\bar{g}\big)$ is invariant under $Z(g)$. Fix some h in $Z(g)$ and consider the map ${}^h(\Psi_g^*)$. As maps on the cohomology $\mathrm{HH}^\bullet\big(S(V), S(V)\bar{g}\big)$,

$$1 \;=\; {}^h(\Phi_g^*\,\Psi_g^*) \;=\; {}^h(\Phi_g^*)\,{}^h(\Psi_g^*) \;=\; \Phi_g^*\,{}^h(\Psi_g^*),$$

thus ${}^h(\Psi_g^*)$ is also inverse to Φ_g^*. Hence ${}^h(\Psi_g^*) = \Psi_g^*$ (since the inverse is unique) as maps on cohomology, for all h in $Z(g)$, and Ψ_g^* is also $Z(g)$-invariant. Thus statement (1) holds. As a consequence, we may restrict both Φ_g^* and Ψ_g^* to the graded vector space $\big(\mathrm{HH}^\bullet(S(V), S(V)\bar{g})\big)^{Z(g)}$. Applying the isomorphisms (2-2), we obtain (2). □

The cohomology $\mathrm{HH}^\bullet(S(V), S(V)\#G)$ arising from the Koszul resolution (2-4) of $S(V)$ may be viewed as a set of *vector forms* on V *tagged* by group elements of G. Indeed, we identify $\mathrm{Hom}_{\mathbb{C}}(\bigwedge^p V, S(V)\overline{g})$ with $S(V)\overline{g} \otimes \bigwedge^p V^*$ for each g in G, and recognize the set of *cochains* derived from the Koszul resolution as (see Diagram (3-3) below)

$$C^{\bullet} := \bigoplus_{g \in G} C^{\bullet}_g, \quad \text{where} \quad C^p_g := S(V)\overline{g} \otimes {\textstyle\bigwedge^p} V^* . \tag{2-9}$$

3. Quantum differentiation and a combinatorial converter map

One generally uses the Koszul resolution of $S(V)$ to compute Hochschild cohomology, but some of the algebraic structure of its cohomology is defined using the bar resolution instead. We thus define automorphisms of cohomology which convert between resolutions. In Equation (2-6), we defined the familiar inclusion map Φ from the Koszul resolution to the bar resolution. But in order to transfer algebraic structure, we need chain maps in both directions. In Section 4, we shall construct explicit chain maps Ψ from the bar resolution to the Koszul resolution, which will then induce cochain maps Ψ^*. These maps Ψ are somewhat unwieldy, however. Thus, in this section, we first define a more elegant and handy map Υ on cochains using quantum differential operators (alternatively, Demazure operators). In Theorem 5-1, we prove that $\Upsilon = \Psi^*$ as maps on cocycles, for our specific construction of a chain map Ψ from the bar resolution to the Koszul resolution of $S(V)$. This implies that the map Υ is itself a cochain map, and that Υ is in fact equal to Ψ^* on cohomology, for *any* choice of chain map Ψ from the bar resolution to the Koszul resolution of $S(V)$. This development allows us to deduce important properties of the expedient map Υ (useful for computations) from the elephantine map Ψ^*. In this section, we work over the complex numbers $\mathbb{C}$.

Given any basis $v_1, \dots, v_n$ of V, and any complex number $\epsilon \neq 1$, we define the ϵ-quantum partial differential operator with respect to $v := v_i$ as the scaled Demazure (BGG) operator $\partial_{v,\epsilon} : S(V) \to S(V)$ given by

$$\partial_{v,\epsilon}(f) \;=\; (1-\epsilon)^{-1}\, \frac{f - {}^s f}{v} \;=\; \frac{f - {}^s f}{v - {}^s v}\,, \tag{3-1}$$

where $s \in \mathrm{GL}(V)$ is the reflection whose matrix with respect to the basis $v_1, \dots, v_n$ is $\mathrm{diag}(1, \dots, 1, \epsilon, 1, \dots, 1)$ with ϵ in the ith slot. Set $\partial_{v,\epsilon} = \partial/\partial v$, the usual partial differential operator with respect to v, when $\epsilon = 1$.

Remark 3-2. The quantum partial differential operator $\partial_{v,\epsilon}$ above coincides with the usual definition of quantum partial differentiation: One takes the ordinary partial derivative with respect to v but instead of multiplying each monomial by its degree k in v, one multiplies by the quantum integer $[k]_\epsilon$, where

$$[k]_\epsilon := 1 + \epsilon + \epsilon^2 + \cdots + \epsilon^{k-1}.$$

Let us check explicitly that these two definitions coincide. For $v = v_1$, $\epsilon \neq 1$,

$$\begin{aligned}
\partial_{v,\epsilon}(v_1^{k_1} v_2^{k_2} \cdots v_n^{k_n}) &= \frac{(v_1^{k_1} v_2^{k_2} \cdots v_n^{k_n}) - {}^s(v_1^{k_1} v_2^{k_2} \cdots v_n^{k_n})}{v_1 - {}^s v_1} \\
&= \frac{v_1^{k_1} v_2^{k_2} \cdots v_n^{k_n} - \epsilon^{k_1} v_1^{k_1} v_2^{k_2} \cdots v_n^{k_n}}{v_1 - \epsilon v_1} \\
&= \frac{(1-\epsilon^{k_1}) v_1^{k_1} v_2^{k_2} \cdots v_n^{k_n}}{(1-\epsilon) v_1} \\
&= [k_1]_\epsilon \;\; v_1^{k_1 - 1} v_2^{k_2} \cdots v_n^{k_n}.
\end{aligned}$$

We are now ready to construct the map Υ taking vector forms (tagged by group elements) to twisted quantum differential operators. We define Υ on cochains $C^\bullet$ (see (2-9)) so that the following diagram commutes for $M = S(V)\#G$:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbb{C}}(S(V)^{\otimes p}, M) \arrow[r, "\delta^*_{p-1}"] \arrow[d, shift left, "\Phi^*_p"] & \mathrm{Hom}_{\mathbb{C}}(S(V)^{\otimes (p+1)}, M) \arrow[d, shift left, "\Phi^*_{p+1}"] \\
\bigoplus_{g\in G} S(V)\overline{g} \otimes \bigwedge^p V^* \arrow[r, "d^*_{p-1}"] \arrow[u, shift left, "\Upsilon_p"] & \bigoplus_{g\in G} S(V)\overline{g} \otimes \bigwedge^{p+1} V^* \arrow[u, shift left, "\Upsilon_{p+1}"]
\end{tikzcd} \tag{3-3}$$

First, some notation. For g in G, fix a basis $B_g = \{v_1, \ldots, v_n\}$ of V consisting of eigenvectors of g with corresponding eigenvalues $\epsilon_1, \ldots, \epsilon_n$. Decompose g into a product of reflections diagonal in this basis: Let $g = s_1 \cdots s_n$ where each s_i is either the identity or a reflection defined by ${}^{s_i} v_j = v_j$ for $j \neq i$ and ${}^{s_i} v_i = \epsilon_i v_i$. Let $\partial_i := \partial_{v_i, \epsilon_i}$, the quantum partial derivative (see Definition (3-1)) with respect to B_g.

Definition 3-4. We define a resolution converter map Υ from the dual Koszul complex to the dual bar complex with coefficients in $S(V)\#G$:

$$\Upsilon_p : \quad C^p \to \mathrm{Hom}_{\mathbb{C}}(S(V)^{\otimes p}, S(V)\#G)\,.$$

Let g lie in G with basis $B_g = \{v_1, \ldots, v_n\}$ of V as above. Let

$$\alpha = f_g \overline{g} \otimes v^*_{j_1} \wedge \cdots \wedge v^*_{j_p}$$

with $f_g \in S(V)$ and $1 \leq j_1 < \ldots < j_p \leq n$. Define $\Upsilon(\alpha) : S(V)^{\otimes p} \to S(V)\#G$ by

$$\Upsilon(\alpha)(f_1 \otimes \cdots \otimes f_p) = \left(\prod_{k=1}^{p} {}^{s_1 s_2 \cdots s_{j_k - 1}}(\partial_{j_k} f_k) \right) f_g \overline{g}\,.$$

By Theorem 5-1 below, Υ is a cochain map. Thus Υ induces a map on the cohomology $\mathrm{HH}^\bullet(S(V), S(V)\#G) \cong \bigoplus_{g\in G} \mathrm{HH}^\bullet(S(V), S(V)\overline{g})$, which we denote by Υ

as well. For each g in G, let Υ_g denote the restriction to $C^\bullet_g$ and the restriction to $\mathrm{HH}^\bullet(S(V), S(V)\bar{g})$, so that

$$\Upsilon = \bigoplus_{g\in G} \Upsilon_g .$$

Remark 3-5. For each g in G, the *cochain* map $\Upsilon_g = \Upsilon_{g,B}$ depends on the chosen basis $B = B_g$ of eigenvectors of g. But we shall see (in Corollary 5-3 below) that the induced automorphism on cohomology $\mathrm{HH}^\bullet\big(S(V), S(V)\bar{g}\big)$ does *not* depend on the choice of basis. This will imply that as an automorphism of $\mathrm{HH}^\bullet(S(V)\#G)$, the map Υ does not depend on choices of bases of V used in its definition.

Example 3-6. Let $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ be the Klein four-group consisting of elements $1, g, h, gh$. Let $V = \mathbb{C}^3$ with basis v_1, v_2, v_3 on which G acts by ${}^g v_1 = -v_1$, ${}^g v_2 = v_2$, ${}^g v_3 = -v_3$, ${}^h v_1 = -v_1$, ${}^h v_2 = -v_2$, ${}^h v_3 = v_3$. Let $\alpha = f_h\bar{h} \otimes v_1^* \wedge v_2^*$ for some $f_h \in S(V)$. Write $h = s_1 s_2$, a product of reflections with ${}^{s_1} v_1 = -v_1$, ${}^{s_2} v_2 = -v_2$. Then $\Upsilon(\alpha)$ is the function on $S(V)^{\otimes 2}$ given by

$$\Upsilon(\alpha)(f_1 \otimes f_2) = (\partial_1 f_1)({}^{s_1}\partial_2 f_2) f_h \bar{h}$$

for all $f_1, f_2 \in S(V)$. For example, $\Upsilon(\alpha)(v_1 \otimes v_2) = f_h\bar{h}$ while $\Upsilon(\alpha)(v_2 \otimes v_1) = 0$.

Remark 3-7. The map Υ transforms any decomposable vector form into a (twisted) quantum operator characterizing the same subspace: For the fixed basis $B_g = \{v_1, \dots, v_n\}$ and $\alpha = f_g\bar{g} \otimes v_{j_1}^* \wedge \cdots \wedge v_{j_p}^*$ in C_g^p (with $j_1 < \dots < j_p$), we have

$$\Upsilon(\alpha)(v_{i_1} \otimes \cdots \otimes v_{i_p}) = 0 \qquad \text{unless } i_1 = j_1, \dots, i_p = j_p .$$

Generally, $\Upsilon(\alpha)(f_1 \otimes \cdots \otimes f_p) = 0$ whenever $\dfrac{\partial}{\partial v_{j_k}}(f_k) = 0$ for some k.

The next proposition explains how Υ depends on our choices of bases as a map on cochains.

Proposition 3-8. *The maps $\Upsilon_{g,B}$ on cochains, for g in G, satisfy the following change of basis rule: For any a in G,*

$${}^a\Upsilon_{g,B} = \Upsilon_{aga^{-1},\, {}^aB} .$$

In particular, for a in the centralizer $Z(g)$, ${}^a\Upsilon_{g,B} = \Upsilon_{g,\, {}^aB}$.

Proof. One may check directly from (3-1) that quantum partial differentiation obeys the following transformation law: For all v in V and ϵ in $\mathbb{C}$,

$${}^a\partial_{v,\epsilon} = \partial_{{}^av,\epsilon} ,$$

where ${}^a\partial_{v,\epsilon}$ differentiates with respect to a basis B and $\partial_{{}^av,\epsilon}$ with respect to ${}^aB$.

Let $B = \{v_1, \dots, v_n\}$ be a basis of V of eigenvectors of g with corresponding eigenvalues $\epsilon_1, \dots, \epsilon_n$. Decompose g as a product of diagonal reflections s_i in $\mathrm{GL}(V)$ (for $i = 1, \dots, n$) in this basis; we retain the notation before Definition 3-4.

Let $g' = aga^{-1}$, $B' = {}^aB$, $v_i' = {}^a v_i$, $s_i' = as_ia^{-1}$, and $\partial_i' = {}^a\partial_i$. Then the s_i' similarly decompose g' in the basis B' with $\partial_i' = \partial_{v_i', \epsilon_i}$.

Consider $\alpha = f_g \otimes v_{j_1}^* \wedge \cdots \wedge v_{j_p}^*$ in C^p. For all f_i in $S(V)$,

$$\begin{aligned}
{}^a\big(\Upsilon_{g,B}(\alpha)\big)(f_1 \otimes \ldots \otimes f_p) &= {}^a\big(\Upsilon_{g,B}(\alpha)({}^{a^{-1}}f_1 \otimes \cdots \otimes {}^{a^{-1}}f_p)\big) \\
&= {}^a\big({}^{s_1 \cdots s_{j_1 - 1}}\partial_{j_1}({}^{a^{-1}}f_1) \,\cdots\, {}^{s_1 \cdots s_{j_p - 1}}(\partial_{j_p}({}^{a^{-1}}f_p)) f_g\, \overline{g}\big) \\
&= {}^{s_1' \cdots s_{j_1-1}'}({}^a\partial_{j_1} f_1) \,\cdots\, {}^{s_1' \cdots s_{j_p-1}'}({}^a\partial_{j_p} f_p)\; {}^af_g\, \overline{g}' \\
&= {}^{s_1' \cdots s_{j_1-1}'}(\partial_{j_1}' f_1) \,\cdots\, {}^{s_1' \cdots s_{j_p-1}'}(\partial_{j_p}' f_p)\; {}^af_g\, \overline{g}' \\
&= \Upsilon_{g',B'}({}^a\alpha)(f_1 \otimes \ldots \otimes f_p)\ ,
\end{aligned}$$

and the result follows. □

The above proposition can also be seen using Definition 4-1 below of the chain map Ψ_B, Theorem 5-1 below equating $\Upsilon_{g,B}$ and Ψ_B, and the straightforward fact that Ψ_B has a similar change of basis property.

4. Chain maps from the bar to the Koszul resolution

In this section, we define specific chain maps Ψ_B from the bar resolution of $S(V)$ to its Koszul resolution (see (2-1) and (2-4)) depending on bases B of V. By the Comparison Theorem, the resulting maps $(\Psi_B)^*$ on cohomology do not depend on the choice of B. In particular, we consider cohomology with coefficients in $S(V)\overline{g}$ and write $\Psi_{g,B}^*$ for the induced map $(\Psi_B)_g^*$ on $\mathrm{HH}^{\bullet}\big(S(V), S(V)\overline{g}\big)$. We shall show in Theorem 5-1 below that $\Psi_{g,B}^* = \Upsilon_{g,B}$ (recall Definition 3-4) for any choice B of basis of V consisting of eigenvectors of g used to define both maps. This will imply (see Corollary 5-3) that as maps on cohomology, Υ_g and Υ are automorphisms independent of choices of bases B used to define them at the cochain level. In this section, we work over any base field.

First, we introduce some notation. Let $\underline{\ell}$ denote an n-tuple $\underline{\ell} := (\ell_1, \ldots, \ell_n)$. Let $\underline{v}^{\underline{\ell}}$ be the monomial $\underline{v}^{\underline{\ell}} := v_1^{\ell_1} \cdots v_n^{\ell_n}$ where $v_1, \ldots, v_n$ is a chosen basis of V. Sometimes we further abbreviate a p-tuple $\underline{\ell}^1, \ldots, \underline{\ell}^p$ of n-tuples by $\underline{\ell}$ when no confusion will arise.

Definition 4-1. Let V be a vector space over an arbitrary field, and let $B = \{v_1, \ldots, v_n\}$ be a basis of V. Define an $S(V)^e$-map Ψ_B from the bar resolution to the Koszul resolution, $\Psi_B : S(V)^{\otimes(\bullet+2)} \to S(V)^e \otimes \textstyle\bigwedge^{\bullet}(V)$, as follows. Let $(\Psi_B)_0$ be the identity map. For each $p \geq 1$, define $(\Psi_B)_p$ by

$$(\Psi_B)_p(1 \otimes \underline{v}^{\underline{\ell}^1} \otimes \cdots \otimes \underline{v}^{\underline{\ell}^p} \otimes 1) = \sum_{1 \leq i_1 < \cdots < i_p \leq n}\ \sum_{0 \leq a_{i_j} < \ell_{i_j}^j} \underline{v}^{\underline{Q}(\underline{\ell}; i_1, \ldots, i_p)} \otimes \underline{v}^{\underline{\hat{Q}}(\underline{\ell}; i_1, \ldots, i_p)} \otimes v_{i_1} \wedge \cdots \wedge v_{i_p}, \tag{4-2}$$

where the second sum ranges over all $a_{i_1}, \dots, a_{i_p}$ such that $0 \le a_{i_j} < \ell^j_{i_j}$ for each $j \in \{1, \dots, p\}$ and the functions $\underline{Q}$ and $\underline{\hat{Q}}$ (indicating monomial degree) depend also on the choices a_{i_j} (this dependence is suppressed in the notation for brevity):

$$\underline{Q}(\underline{\ell}^1, \dots, \underline{\ell}^p; i_1, \dots, i_p)_i = \begin{cases} a_i + \ell^1_i + \cdots + \ell^{j-1}_i & \text{if } i = i_j, \\ \ell^1_i + \cdots + \ell^j_i & \text{if } i_j < i < i_{j+1}, \end{cases}$$

where we set $i_0 = 0$ and $i_{p+1} = n + 1$ for convenience. We define the n-tuple $\underline{\hat{Q}}(\underline{\ell}; i_1, \dots, i_p)$ to be complementary to $\underline{Q}(\underline{\ell}; i_1, \dots, i_p)$ in the sense that

$$\underline{v}^{\underline{Q}(\underline{\ell}; i_1, \dots, i_p)} \, \underline{v}^{\underline{\hat{Q}}(\underline{\ell}; i_1, \dots, i_p)} \, v_{i_1} \cdots v_{i_p} = \underline{v}^{\underline{\ell}^1} \cdots \underline{v}^{\underline{\ell}^p}.$$

We simply write $\underline{\hat{Q}}$ when it is clear with respect to which $\underline{Q}(\underline{\ell}; i_1, \dots, i_p)$ it is complementary.

For small values of p, the formula for $(\Psi_B)_p$ is less cumbersome. In particular, for $p = 1, 2$, such formulas were given in [Witherspoon 2006, (4.9), (4.10)]. We repeat them here:

$$\Psi_1(1 \otimes v_1^{\ell_1} \cdots v_n^{\ell_n} \otimes 1) = \sum_{i=1}^{n} \sum_{t=1}^{\ell_i} v_i^{\ell_i - t} v_{i+1}^{\ell_{i+1}} \cdots v_n^{\ell_n} \otimes v_1^{\ell_1} \cdots v_{i-1}^{\ell_{i-1}} v_i^{t-1} \otimes v_i,$$

$$\begin{aligned} &\Psi_2(1 \otimes v_1^{\ell_1} \cdots v_n^{\ell_n} \otimes v_1^{m_1} \cdots v_n^{m_n} \otimes 1) \\ &= \sum_{1 \le i < j \le n} \sum_{r=1}^{m_j} \sum_{t=1}^{\ell_i} \Big(v_i^{\ell_i - t} v_{i+1}^{\ell_{i+1}} \cdots v_{j-1}^{\ell_{j-1}} v_j^{\ell_j + m_j - r} v_{j+1}^{\ell_{j+1} + m_{j+1}} \cdots v_n^{\ell_n + m_n} \\ &\qquad \otimes v_1^{\ell_1 + m_1} \cdots v_{i-1}^{\ell_{i-1} + m_{i-1}} v_i^{m_i + t - 1} v_{i+1}^{m_{i+1}} \cdots v_{j-1}^{m_{j-1}} v_j^{r-1} \otimes v_i \wedge v_j \Big). \end{aligned}$$

Example 4-3. To illustrate, we compute Ψ_2 on a few monomials of small degree:

$$\begin{aligned} \Psi_2(1 \otimes v_1 \otimes v_2 \otimes 1) &= 1 \otimes 1 \otimes v_1 \wedge v_2, \\ \Psi_2(1 \otimes v_1 v_2 \otimes v_2^3 \otimes 1) &= (v_2^3 \otimes 1 + v_2^2 \otimes v_2 + v_2 \otimes v_2^2) \otimes v_1 \wedge v_2, \\ \Psi_2(1 \otimes v_1 v_2 \otimes v_2^2 v_3 \otimes 1) &= (v_2^2 v_3 \otimes 1 + v_2 v_3 \otimes v_2) \otimes v_1 \wedge v_2 \\ &\quad + 1 \otimes v_1 v_2^2 \otimes v_2 \wedge v_3 + v_2 \otimes v_2^2 \otimes v_1 \wedge v_3. \end{aligned}$$

Theorem 4-4. *For each choice of basis B of V, the map Ψ_B of Definition 4-1 is a chain map.*

We defer the proof of Theorem 4-4 to the Appendix as it is rather technical.

5. Merits of the combinatorial converter map

In the previous two sections, we examined two maps $\Upsilon_{g,B}$ and $\Psi^*_{g,B}$ which convert between cochain complexes: They each transform cochains procured from the Koszul resolution (2-4) of $S(V)$ into cochains procured from the bar resolution (2-1) of $S(V)$ (see Definitions 3-4 and 4-1). In this section, we show that the two maps $\Upsilon_{g,B}$ and $\Psi^*_{g,B}$ are identical on cochains, and hence also on cohomology, for any g in G and any basis B consisting of eigenvectors of g. This will imply that Υ is itself a cochain map. We deduce other salient properties of the map Υ using this connection between Υ and Ψ. We take our ground field to be $\mathbb{C}$ in this section.

Theorem 5-1. *Let g be in G and let B be a basis of V consisting of eigenvectors of g. Then*

$$\Upsilon_{g,B} = \Psi^*_{g,B}$$

as maps on cochains. Thus $\Upsilon_{g,B}$ is a cochain map.

Proof. We check that $\Upsilon_{g,B}$ and $\Psi^*_{g,B}$ agree on cochains: Let $\alpha = f_g\overline{g}\otimes v^*_{j_1}\wedge\cdots\wedge v^*_{j_p}$ be a cochain in C^p_g with $f_g \in S(V)$ and $j_1 < \ldots < j_p$, where $B = \{v_1, \ldots, v_n\}$. Let

$$f_1 = \underline{v}^{\underline{\ell}^1}, \quad \ldots, \quad f_p = \underline{v}^{\underline{\ell}^p}$$

be monomials in $S(V)$. Without loss of generality, it suffices to show that $\Psi^*_{g,B}(\alpha)$ and $\Upsilon_{g,B}(\alpha)$ agree on $f_1\otimes\cdots\otimes f_p$, since such elements form a basis for $S(V)^{\otimes p}$. By Definition 4-1,

$$\begin{aligned}
&\Psi^*_{g,B}(\alpha)(f_1\otimes\cdots\otimes f_p)\\
&\qquad = \alpha(\Psi_{g,B}(f_1\otimes\cdots\otimes f_p))\\
&\qquad = \alpha(\Psi_{g,B}(v_1^{\ell_1^1}\cdots v_n^{\ell_n^1}\otimes\cdots\otimes v_1^{\ell_1^p}\cdots v_n^{\ell_n^p}))\\
&\qquad = \alpha\Bigg(\sum_{1\le i_1<\cdots<i_p\le n}\ \sum_{0\le a_{i_k}<\ell^k_{i_k}} \underline{v}^{\underline{Q}(\underline{\ell};i_1,\ldots,i_p)}\otimes \underline{v}^{\hat{\underline{Q}}(\underline{\ell};i_1,\ldots,i_p)}\otimes v_{i_1}\wedge\cdots\wedge v_{i_p}\Bigg).
\end{aligned}$$

Since α has exterior part $v^*_{j_1}\wedge\cdots\wedge v^*_{j_p}$, each summand is zero save one (the summand with $i_k = j_k$ for $k = 1,\ldots p$). Then

$$\begin{aligned}
&\Psi^*_{g,B}(\alpha)(f_1\otimes\cdots\otimes f_p)\\
&= \sum_{0\le a_{j_k}<\ell^k_{j_k}} \underline{v}^{\underline{Q}(\underline{\ell};j_1,\ldots,j_p)} f_g\overline{g}\, \underline{v}^{\hat{\underline{Q}}(\underline{\ell};j_1,\ldots,j_p)}\\
&= \sum_{0\le a_{j_k}<\ell^k_{j_k}} \Bigg(\prod_{t=1}^{p} \epsilon_{j_t}^{(\ell^t_{j_t}-a_{j_t}-1)+\ell^{t+1}_{j_t}\cdots+\ell^p_{j_t}} \prod_{i_{t-1}<i<i_t} \epsilon_i^{\ell^t_i+\cdots+\ell^p_i}\Bigg) \underline{v}^{\underline{Q}(\underline{\ell};j_1,\ldots,j_p)}\underline{v}^{\hat{\underline{Q}}(\underline{\ell};j_1,\ldots,j_p)} f_g\overline{g}.
\end{aligned}$$

Recall that, by definition, $\underline{v}^{\underline{Q}(\underline{\ell};j_1,\ldots,j_p)}\underline{v}^{\hat{\underline{Q}}(\underline{\ell};j_1,\ldots,j_p)} v_{j_1}\cdots v_{j_p} = \underline{v}^{\underline{\ell}^1}\cdots\underline{v}^{\underline{\ell}^p}$. Thus the factor $\underline{v}^{\underline{Q}(\underline{\ell};j_1,\ldots,j_p)}\underline{v}^{\hat{\underline{Q}}(\underline{\ell};j_1,\ldots,j_p)} f_g\overline{g}$ in each term of the above sum does not depend on the values of a_{j_k}, and we may move the summation symbol inside the parentheses. Simplifying, we obtain

$$\begin{aligned}\Bigg(\prod_{t=1}^{p}[\ell^t_{j_t}]_{\epsilon_{j_t}}\epsilon_{j_t}^{\ell^{t+1}_{j_t}+\cdots+\ell^p_{j_t}}\prod_{i_{t-1}<i<i_t}\epsilon_i^{\ell^t_i+\cdots+\ell^p_i}\Bigg)\underline{v}^{Q(\underline{\ell};j_1,\dots,j_p)}\underline{v}^{\hat{Q}(\underline{\ell};j_1,\dots,j_p)}f_g\overline{g}& \\ =\Bigg(\prod_{k=1}^{p}{}^{s_1\cdots s_{j_k-1}}(\partial_{j_k}f_k)\Bigg)f_g\overline{g}& \\ =\Upsilon_{g,B}(\alpha)(f_1\otimes\cdots\otimes f_p),&\end{aligned}$$

by Definition 3-4. Hence $\Upsilon_{g,B}=\Psi^*_{g,B}$ as maps on cochains. $\square$

As a consequence, we obtain a statement about any chain map Ψ from the bar resolution to the Koszul resolution, at the level of cohomology, which has further implications for the map Υ:

Corollary 5-2. *Let Ψ be any chain map from the bar resolution (2-1) to the Koszul resolution (2-4) for $S(V)$. Then*:

(1) $\Upsilon_g=\Psi^*_g$ *as maps on the cohomology* $\mathrm{HH}^\bullet(S(V),S(V)\overline{g})$, *for all g in G.*

(2) $\Upsilon=\Psi^*$ *as maps on*

$$\mathrm{HH}^\bullet(S(V),S(V)\#G)\cong\bigoplus_{g\in G}\mathrm{HH}^\bullet\big(S(V),S(V)\overline{g}\big)$$

and on its G-invariant subalgebra,

$$\mathrm{HH}^\bullet(S(V),S(V)\#G)^G\cong\mathrm{HH}^\bullet(S(V)\#G).$$

Proof. We constructed a specific choice of chain map Ψ_B in Definition 4-1 above from the bar to the Koszul resolution of $S(V)$. Since Ψ and Ψ_B are homotopic by the Comparison Theorem, $\Psi^*_g=\Psi^*_{g,B}$ as maps on cohomology $\mathrm{HH}^\bullet\big(S(V),S(V)\overline{g}\big)$. But $\Psi^*_{g,B}=\Upsilon_{g,B}$ for any choice of g and B by Theorem 5-1, and hence $\Psi^*=\Upsilon$. By Proposition 2-8, these maps preserve G-invariant subspaces, and so $\Psi^*=\Upsilon$ on $\mathrm{HH}^\bullet(S(V)\#G)$ as well. $\square$

Corollary 5-3. *Let $g\in G$. On the cohomology* $\mathrm{HH}^\bullet(S(V),S(V)\overline{g})$, *the map $\Upsilon_g=\Upsilon_{g,B}$ is independent of choice of basis B of eigenvectors of g used in its definition. Hence, as a map on the cohomologies* $\mathrm{HH}^\bullet(S(V),S(V)\#G)$ *and* $\mathrm{HH}^\bullet(S(V)\#G)$, *$\Upsilon$ is independent of the choices of bases used in its definition.*

Proof. By Corollary 5-2, $\Upsilon_g=\Psi^*_g$ on cohomology for any choice of chain map Ψ from the bar complex to the Koszul complex of $S(V)$, independent of the choice of basis of eigenvectors of g used to define Υ_g. Hence, Υ is independent of choices of bases. $\square$

Corollary 5-4. *The maps Υ and Φ^* are inverse isomorphisms on the cohomology* $\mathrm{HH}^\bullet(S(V),S(V)\#G)$ *and on its G-invariant subalgebra* $\mathrm{HH}^\bullet(S(V),S(V)\#G)^G\cong\mathrm{HH}^\bullet(S(V)\#G)$.

Proof. Again, in Corollary 5-2, we found that $\Upsilon = \Psi^*$ on cohomology for any chain map Ψ from the bar to the Koszul complex. But any such Ψ induces an automorphism on cohomology inverse to Φ^* by the Comparison Theorem. □

This corollary actually follows from a stronger fact: Υ is a right-sided inverse to Φ^* on cochains, not merely on cohomology, for any choice of bases $\{B_g\}_{g\in G}$ defining Υ. Indeed, a calculation shows directly that $\Phi^*\Upsilon = 1$ on cochains $C^\bullet$. We can see this fact yet another way. One can check that $\Psi_B\Phi = 1$ on chains, and therefore $\Phi_g^*(\Psi_B)_g^* = 1$ on cochains, for every B and g. In Theorem 5-1, we saw that $\Upsilon_{g,B} = (\Psi_B)_g^* = \Psi_{g,B}^*$ as maps on cochains, for all g in G and for any basis B of eigenvectors of g, and hence $\Phi^*\Upsilon = 1$ as a map on cochains.

6. Hochschild homology

Our chain maps Ψ_B of Definition 4-1 are useful in settings other than the cohomology of $S(V)\#G$. In this section, we obtain induced maps on Hochschild homology, and compare our induced maps on homology and cohomology with those in the literature. The Hochschild-Kostant-Rosenberg Theorem states that for smooth commutative algebras, Hochschild homology is isomorphic to the module of differential forms (i.e., the exterior algebra generated by the Kähler differentials); e.g., see [Weibel 1994, §9.4.2]. For noncommutative algebras, Hochschild homology provides a generalization of the notion of "differential forms". It is interesting to note that for some types of algebras (in particular for $S(V)\#G$), Hochschild homology and cohomology are dual (see [van den Bergh 1998] for the general theory and [Farinati 2005] for the case $S(V)\#G$). In this section, we work over an arbitrary field initially, then over $\mathbb{C}$ in Theorem 6-4.

Let M be any $S(V)^e$-module. Then Ψ_B induces an isomorphism on Hochschild homology

$$\mathrm{HH}_\bullet(S(V), M) := \mathrm{Tor}_\bullet^{S(V)^e}(S(V), M)$$

and on Hochschild cohomology

$$\mathrm{HH}^\bullet(S(V), M) := \mathrm{Ext}^\bullet_{S(V)^e}(S(V), M)$$

by applying the functors $M \otimes_{S(V)^e} -$ and $\mathrm{Hom}_{S(V)^e}(-, M)$, respectively, to the bar resolution (2-1) and to the Koszul resolution (2-4). This approach to obtaining maps on homology and cohomology has advantages over previous approaches in the literature which we explain now.

We obtain a map on Hochschild homology $\mathrm{HH}_\bullet(S(V)) := \mathrm{HH}_\bullet(S(V), S(V))$, denoted by $(\Psi_B)_*$, by setting $M = S(V)$. At the chain level,

$$(\Psi_B)_* : S(V) \otimes S(V)^{\otimes \bullet} \longrightarrow S(V) \otimes \textstyle\bigwedge^\bullet(V)\,.$$

A computation similar to that in the proof of Theorem 5-1 yields the following explicit formula for $(\Psi_B)_*$, valid over any ground field:

Theorem 6-1. *Let $B = \{v_1, \dots, v_n\}$ be a basis of V. Then as an automorphism on* $\mathrm{HH}_\bullet(S(V))$ *at the chain level,*

$$(\Psi_B)_*(f_0 \otimes f_1 \otimes \cdots \otimes f_p) = \sum_{1 \le i_1 < \cdots < i_p \le n} f_0 \frac{\partial f_1}{\partial v_{i_1}} \cdots \frac{\partial f_p}{\partial v_{i_p}} \otimes v_{i_1} \wedge \cdots \wedge v_{i_p} \tag{6-2}$$

for all $f_0, f_1, \dots, f_p \in S(V)$.

Proof. Without loss of generality, assume that $f_1, \dots, f_p$ are monomials, say

$$f_k = \underline{v}^{\underline{\ell}^k} = v_1^{\ell_1^k} \cdots v_n^{\ell_n^k}$$

for some n-tuple $\underline{\ell}^k = (\ell_1^k, \dots, \ell_n^k)$. Then

$$\begin{aligned}
&(\Psi_B)_*(f_0 \otimes \underline{v}^{\underline{\ell}^1} \otimes \cdots \otimes \underline{v}^{\underline{\ell}^p}) \\
&\qquad = f_0 \Psi_B(1 \otimes \underline{v}^{\underline{\ell}^1} \otimes \cdots \otimes \underline{v}^{\underline{\ell}^p}) \\
&\qquad = f_0 \Bigg(\sum_{1 \le i_1 < \cdots < i_p \le n} \; \sum_{0 \le a_{i_j} < \ell_{i_j}^j} \underline{v}^{\underline{Q}(\underline{\ell}; i_1, \dots, i_p)} \otimes \underline{v}^{\underline{\hat{Q}}(\underline{\ell}; i_1, \dots, i_p)} \otimes v_{i_1} \wedge \cdots \wedge v_{i_p} \Bigg) \\
&\qquad = \sum_{1 \le i_1 < \cdots < i_p \le n} \; \sum_{0 \le a_{i_j} < \ell_{i_j}^j} \underline{v}^{\underline{Q}(\underline{\ell}; i_1, \dots, i_p)} f_0 \, \underline{v}^{\underline{\hat{Q}}(\underline{\ell}; i_1, \dots, i_p)} \otimes v_{i_1} \wedge \cdots \wedge v_{i_p} \\
&\qquad = \sum_{1 \le i_1 < \cdots < i_p \le n} \; \sum_{0 \le a_{i_j} < \ell_{i_j}^j} f_0 \, \underline{v}^{\underline{Q}(\underline{\ell}; i_1, \dots, i_p)} \underline{v}^{\underline{\hat{Q}}(\underline{\ell}; i_1, \dots, i_p)} \otimes v_{i_1} \wedge \cdots \wedge v_{i_p} \\
&\qquad = \sum_{1 \le i_1 < \cdots < i_p \le n} \; \sum_{0 \le a_{i_j} < \ell_{i_j}^j} f_0 \, \underline{v}^{\underline{\ell}^1} \cdots \underline{v}^{\underline{\ell}^p} v_{i_1}^{-1} \cdots v_{i_p}^{-1} \otimes v_{i_1} \wedge \cdots \wedge v_{i_p} \\
&\qquad = \sum_{1 \le i_1 < \cdots < i_p \le n} \ell_{i_1}^1 \cdots \ell_{i_p}^p f_0 \, \underline{v}^{\underline{\ell}^1} \cdots \underline{v}^{\underline{\ell}^p} v_{i_1}^{-1} \cdots v_{i_p}^{-1} \otimes v_{i_1} \wedge \cdots \wedge v_{i_p},
\end{aligned}$$

where the product $\underline{v}^{\underline{\ell}^1} \cdots \underline{v}^{\underline{\ell}^p} v_{i_1}^{-1} \cdots v_{i_p}^{-1}$ is computed in the ring of Laurent polynomials in $v_1, \dots, v_n$. (Since $0 \le a_{i_j} < \ell_{i_j}^j$, the result lies in $S(V)$ when the corresponding sum is nonempty.) The expression above is precisely that claimed in the theorem. □

In case the ground field is $\mathbb{C}$ or $\mathbb{R}$, by the above theorem, our map $(\Psi_B)_*$ is precisely the map J of [Halbout 2001]. Halbout gave an explicit homotopy s showing that J is a homotopy inverse to the canonical embedding of the de Rham complex into the Hochschild complex. In contrast, we see immediately that $(\Psi_B)_*$ induces an isomorphism on homology since Ψ_B is itself a chain map.

For comparison, we give the map on Hochschild cohomology $\mathrm{HH}^\bullet(S(V))$; this is simply the case $g = 1$ of Definition 3-4, by Theorem 5-1:

Theorem 6-3. *Let $B = \{v_1, \dots, v_n\}$ be a basis of V. Then as an automorphism on* $\mathrm{HH}^\bullet(S(V))$ *at the chain level,*

$$(\Psi_B)^*(\alpha)(f_1 \otimes \cdots \otimes f_p) = f_0 \, \frac{\partial f_1}{\partial v_{j_1}} \cdots \frac{\partial f_p}{\partial v_{j_p}},$$

when $\alpha = f_0 \otimes v^*_{j_1} \wedge \cdots \wedge v^*_{j_p} \in \mathrm{Hom}_{\mathbb{C}}(S(V) \otimes \bigwedge^p V, S(V))$, $f_0, \dots, f_p \in S(V)$.

Now we restrict our choice of field again to $\mathbb{C}$. Let $M = S(V)\#G$, and note that Hochschild homology decomposes just as does Hochschild cohomology:

$$\mathrm{HH}_\bullet(S(V)\#G) \cong \mathrm{HH}_\bullet(S(V), S(V)\#G)^G \cong \Bigl(\bigoplus_{g \in G} \mathrm{HH}_\bullet(S(V), S(V)\bar{g})\Bigr)^G$$

(see [Farinati 2005; Ştefan 1995]). Thus one is interested in the components $\mathrm{HH}_\bullet(S(V), S(V)\bar{g}) = \mathrm{Tor}_\bullet^{S(V)^e}(S(V), S(V)\bar{g})$, for each g in G. A calculation similar to that in the proof of Theorem 5-1 yields the explicit formula in the next theorem for the induced map

$$(\Psi_B)_* : S(V)\bar{g} \otimes S(V)^{\otimes \bullet} \to S(V)\bar{g} \otimes {\textstyle\bigwedge}^\bullet(V)\,.$$

Note that quantum differential operators surface (compare with Definition 3-4 of Υ, which is equal to Ψ_B^* by Theorem 5-1). We have not found these maps in the literature on Hochschild homology.

For g in G, let $B = \{v_1, \dots, v_n\}$ be a basis of V consisting of eigenvectors of g with corresponding eigenvalues $\epsilon_1, \dots, \epsilon_n$. Write $g = s_1 \cdots s_n$ where ${}^{s_i}v_j = v_j$ for $j \neq i$ and ${}^{s_i}v_i = \epsilon_i v_i$. Recall the quantum partial differential operators $\partial_i := \partial_{v_i, \epsilon_i}$ of Definition (3-1).

Theorem 6-4. *Let $g \in G$ and let $B = \{v_1, \dots, v_n\}$ be a basis of V consisting of eigenvectors for g. Then as an automorphism on* $\mathrm{HH}_\bullet(S(V), S(V)\bar{g})$ *at the chain level,*

$$(\Psi_B)_*(f_0\bar{g} \otimes f_1 \otimes \cdots \otimes f_p) = \sum_{1 \le i_1 < \cdots < i_p \le n} f_0 \Bigl(\prod_{k=1}^{p} {}^{s_1 s_2 \cdots s_{i_{k-1}}}(\partial_{i_k} f_k)\Bigr) \bar{g} \otimes v_{i_1} \wedge \cdots \wedge v_{i_p}$$

for all $f_0, f_1, \dots, f_p \in S(V)$.

We make a few final comments about the appearance of our chain maps Ψ_B in Hochschild cohomology. Again let $M = S(V)\#G$, and consider the map $(\Psi_B)^*$ on the Hochschild cohomology $\mathrm{HH}^\bullet(S(V), S(V)\#G)$ for any basis B of V. We observed (as a consequence of Theorem 5-1 and Definition 3-4) that $(\Psi_B)^*$ is given by quantum partial differential operators. The reader should compare with maps given in [Halbout and Tang 2010]: these authors define functions directly on cochain complexes (without first defining chain maps on resolutions) and then must prove that these functions are cochain maps. Again, our approach presents an advantage: We instead define one primitive chain map Ψ_B from which induced cochain maps effortlessly spring. For example, $(\Psi_B)^* = \Upsilon$ is automatically a cochain map since Ψ_B is a chain map by Theorem 4-4. The reader is cautioned that Halbout and Tang [2010] work only over $\mathbb{R}$, in which case Hochschild cohomology has a specialized description (V and V^* are G-isomorphic in that setting, simplifying some aspects of homology and cohomology).

Appendix: Proof of Theorem 4-4

Let V be a finite-dimensional vector space over any field. Fix a basis

$$B = \{v_1, \dots, v_n\}$$

of V. Recall Definition 4-1 of the linear map $\Psi = \Psi_B$ from the bar resolution (2-1) to the Koszul resolution (2-4) of $S(V)$. We prove that Ψ is a chain map, that is, $\Psi_{p-1}\delta_p = d_p\Psi_p$ for all $p \geq 1$.

A straightforward but tedious calculation shows that $\Psi_0\delta_1 = d_1\Psi_1$, and we assume from now on that $p \geq 2$.

We first compute

$$d_p\Psi_p(1 \otimes \underline{v}^{\underline{\ell}^1} \otimes \cdots \otimes \underline{v}^{\underline{\ell}^p} \otimes 1).$$

For each $j \in \mathbb{N}$, let $\delta[j] : \mathbb{N} \to \{0, 1\}$ be the Kronecker delta function defined by

$$(\delta[j])_i = \delta[j](i) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

Then

$$\begin{aligned} &d_p\Psi_p(1 \otimes \underline{v}^{\underline{\ell}^1} \otimes \cdots \otimes \underline{v}^{\underline{\ell}^p} \otimes 1) \\ &= d_p\Bigg(\sum_{1 \leq i_1 < \cdots < i_p \leq n} \ \sum_{0 \leq a_{i_j} < \ell^j_{i_j}} \underline{v}^{\underline{Q}(\underline{\ell}; i_1, \dots, i_p)} \otimes \underline{v}^{\underline{\hat{Q}}(\underline{\ell}; i_1, \dots, i_p)} \otimes v_{i_1} \wedge \cdots \wedge v_{i_p} \Bigg) \\ &= \sum_{1 \leq i_1 < \cdots < i_p \leq n} \ \sum_{0 \leq a_{i_j} < \ell^j_{i_j}} \ \sum_{m=1}^{p} (-1)^{m+1} \Big(\underline{v}^{\underline{Q}+\delta[i_m]} \otimes \underline{v}^{\underline{\hat{Q}}} \otimes v_{i_1} \wedge \cdots \wedge \hat{v}_{i_m} \wedge \cdots \wedge v_{i_p} \\ &\qquad\qquad - \underline{v}^{\underline{Q}} \otimes \underline{v}^{\underline{\hat{Q}}+\delta[i_m]} \otimes v_{i_1} \wedge \cdots \wedge \hat{v}_{i_m} \wedge \cdots \wedge v_{i_p} \Big), \end{aligned}$$

where $\underline{Q} = \underline{Q}(\underline{\ell}; i_1, \dots, i_p)$ and $\underline{\hat{Q}} = \underline{\hat{Q}}(\underline{\ell}; i_1, \dots, i_p)$ are determined by the a_{i_j} as in Definition 4-1. Now fix m in the above expression. The factors $\underline{v}^{\underline{Q}+\delta[i_m]}$ and $\underline{v}^{\underline{Q}}$ differ only in the power of v_{i_m}. In the sum, the power a_{i_m} ranges over the set $\{0, \dots, \ell^m_{i_m} - 1\}$, and thus the corresponding terms cancel except for the first term when $a_{i_m} = \ell^m_{i_m} - 1$ and the second term when $a_{i_m} = 0$. After all such cancellations, for each $m = 1, \ldots, p$, what remains is

$$\sum_{1\le i_1<\cdots<i_p\le n} \sum_{m=1}^{p} \sum_{0\le a_{i_j}<\ell^j_{i_j}} (-1)^{m+1}\Big(\underline{v}^{\underline{R}^m(\underline{\ell};i_1,\ldots,i_p)} \otimes \underline{v}^{\underline{\hat{R}}^m} \otimes v_{i_1}\wedge\cdots\wedge\hat{v}_{i_m}\wedge\cdots\wedge v_{i_p} \\ - \underline{v}^{\underline{S}^m(\underline{\ell};i_1,\ldots,i_p)} \otimes \underline{v}^{\underline{\hat{S}}^m} \otimes v_{i_1}\wedge\cdots\wedge\hat{v}_{i_m}\wedge\cdots\wedge v_{i_p}\Big),$$

where the rightmost sum is over $a_i, \ldots, \hat{a}_{i_m}, \ldots, a_{i_p}$ and (abusing notation, as $\underline{R}$, $\underline{S}$ depend on fewer a_{i_j}'s than $\underline{Q}$)

$$\underline{R}^m(\underline{\ell}; i_1, \ldots, i_p)_i = \begin{cases} \ell^1_{i_m} + \cdots + \ell^m_{i_m} & \text{if } i = i_m, \\ \underline{Q}(\underline{\ell}; i_1, \ldots, i_p)_i & \text{if } i \neq i_m, \end{cases}$$

$$\underline{S}^m(\underline{\ell}; i_1, \ldots, i_p)_i = \begin{cases} \ell^1_{i_m} + \cdots + \ell^{m-1}_{i_m} & \text{if } i = i_m, \\ \underline{Q}(\underline{\ell}; i_1, \ldots, i_p)_i & \text{if } i \neq i_m, \end{cases}$$

and $\underline{\hat{R}}^m$, $\underline{\hat{S}}^m$ are defined by the equations

$$\underline{v}^{\underline{R}^m(\underline{\ell};i_1,\ldots,i_p)}\underline{v}^{\underline{\hat{R}}^m(\underline{\ell};i_1,\ldots,i_p)} v_{i_1}\cdots\hat{v}_{i_m}\cdots v_{i_p} = \underline{v}^{\underline{\ell}^1}\cdots\underline{v}^{\underline{\ell}^p},$$
$$\underline{v}^{\underline{S}^m(\underline{\ell};i_1,\ldots,i_p)}\underline{v}^{\underline{\hat{S}}^m(\underline{\ell};i_1,\ldots,i_p)} v_{i_1}\cdots\hat{v}_{i_m}\cdots v_{i_p} = \underline{v}^{\underline{\ell}^1}\cdots\underline{v}^{\underline{\ell}^p}.$$

Consider the leftmost sum over $1 \le i_1 < \cdots < i_p \le n$. If we replace a given i_m in $\underline{S}^m$ by $i_m + 1$ in $\underline{R}^m$ (provided $i_m + 1 < i_{m+1}$), keeping the others fixed, then

$$\underline{S}^m(\underline{\ell}; i_1, \ldots, i_p) = \underline{R}^m(\underline{\ell}; i_1, \ldots, i_{m-1}, i_m + 1, i_{m+1}, \ldots, i_p).$$

We thus have further cancellation, with the remaining terms coming from the first summand when $i_m = i_{m-1} + 1$ and the second summand when $i_m = i_{m+1} - 1$:

$$\sum_{m=1}^{p} \sum_{1\le i_1<\cdots<\hat{i}_m<\cdots<i_p\le n} \sum_{0\le a_{i_j}<\ell^j_{i_j}} (-1)^{m+1} \\ \Big(\underline{v}^{\underline{R}^m(\underline{\ell};i_1,\ldots,i_{m-1},i_{m-1}+1,i_{m+1},\ldots,i_p)} \otimes \underline{v}^{\underline{\hat{R}}^m} \otimes v_{i_1}\wedge\cdots\wedge\hat{v}_{i_m}\wedge\cdots\wedge v_{i_p} \\ - \underline{v}^{\underline{S}^m(\underline{\ell};i_1,\ldots,i_{m-1},i_{m+1}-1,i_{m+1},\ldots,i_p)} \otimes \underline{v}^{\underline{\hat{S}}^m} \otimes v_{i_1}\wedge\cdots\wedge\hat{v}_{i_m}\wedge\cdots\wedge v_{i_p}\Big),$$

where the rightmost sum is over all such $a_{i_1}, \ldots, \hat{a}_{i_m}, \ldots, a_{i_p}$.

Now consider the middle sum above ranging over all $1 \le i_1 < \cdots < \hat{i}_m < \cdots < i_p \le n$. If $m = 1$, this sum does not include $i_2 = 1$, due to the left out entry $\hat{i}_m$. Similarly, if $m = p$, this sum does not include $i_{p-1} = n$. For the sake of later comparison, we add and subtract terms in the $m = 1$ summand, corresponding to $i_2 = 1$, and in the $m = p$ summand, corresponding to $i_{p-1} = n$. These added and subtracted terms may be written with either notation, $\underline{R}$ or $\underline{S}$, so that the result looks the same as above except that now we include summands corresponding to $m=1$, $i_2=1$ and to $m=p$, $i_{p-1}=n$.

We next combine some of the terms. Consider the terms arising from a pair of subsequent indices m and $m+1$ in the leftmost sum. We pair each summand of type $\underline{S}^m$ with a summand of type $\underline{R}^{m+1}$. Fix an integer i and collect those summands (in the m-th sum) with $\underline{S}^m$-exponent for which $i_{m+1}=i$ and those summands (in the $(m+1)$-st sum) with $\underline{R}^{m+1}$-exponent for which $i_m=i$. Note that all these summands share the same sign. We compare the exponents

$$\underline{S}^m(\underline{\ell}; i_1, \dots, i_{m-1}, i_{m+1}-1, i_{m+1}, \dots, i_p)$$

and

$$\underline{R}^{m+1}(\underline{\ell}; i_1, \dots, i_m, i_m+1, i_{m+2}, \dots, i_p)$$

when $i_m=i=i_{m+1}$ and see that the power of v_i ranges from $\ell_i^1+\cdots+\ell_i^{m-1}$ to $\ell_i^1+\cdots+\ell_i^{m-1}+\ell_i^m-1$ and then again from $\ell_i^1+\cdots+\ell_i^{m-1}+\ell_i^m$ to $\ell_i^1+\cdots+\ell_i^m+\ell_i^{m+1}-1$ in this collection. Hence, we can simply rewrite the partial sum over this collection using the exponent

$$\underline{Q}(\underline{\ell}^1, \dots, \underline{\ell}^{m-1}, \underline{\ell}^m+\underline{\ell}^{m+1}, \underline{\ell}^{m+2}, \dots, \underline{\ell}^p; i_1, \dots, \hat{\imath}_{m+1}, \dots, i_p)$$

instead. We obtain the following, in which the $m=1$ (unmatched $\underline{R}^1$) and $m=p$ (unmatched $\underline{S}^p$) sums have been singled out:

$$\begin{aligned}
&\sum_{1\le i_2<\cdots<i_p\le n}\ \sum_{\substack{0\le a_{i_j}<\ell^j_{i_j}\\ (\text{for } j\in\{2,\dots,p\})}} \underline{v}^{\underline{Q}(\underline{\ell}^2,\cdots,\underline{\ell}^p;i_2,\cdots,i_p)+\underline{\ell}^1}\otimes \underline{v}^{\underline{\hat{Q}}}\otimes v_{i_2}\wedge\cdots\wedge v_{i_p}\\
&+\sum_{m=1}^{p-1}(-1)^m \sum_{1\le i_1<\cdots<\hat{\imath}_m<\cdots<i_p\le n}\ \sum_{\substack{0\le a_{i_j}<\ell^j_{i_j}\ (\text{for } j\in\{1,\dots,m-1\})\\ 0\le a_{i_{m+1}}\le \ell^m_{i_{m+1}}+\ell^{m+1}_{i_{m+1}}-1\\ 0\le a_{i_j}<\ell^{j+1}_{i_j}\ (\text{for } j\in\{m+2,\dots,p\})}}\\
&\qquad \underline{v}^{\underline{Q}(\underline{\ell}^1,\dots,\underline{\ell}^{m-1},\underline{\ell}^m+\underline{\ell}^{m+1},\underline{\ell}^{m+2},\dots,\underline{\ell}^p;i_1,\dots,\hat{\imath}_m,\dots,i_p)}\otimes\underline{v}^{\underline{\hat{Q}}}\otimes v_{i_1}\wedge\cdots\wedge\hat{v}_m\wedge\cdots\wedge v_{i_p}\\
&+(-1)^p\sum_{\substack{1\le i_1<\cdots\\ <i_{p-1}\le n}}\ \sum_{\substack{0\le a_{i_j}<\ell^j_{i_j}\\ (\text{for } j\in\{1,\dots,p-1\})}} \underline{v}^{\underline{Q}(\underline{\ell}^1,\cdots,\underline{\ell}^{p-1};i_2,\cdots,i_{p-1})}\otimes\underline{v}^{\underline{\hat{Q}}+\underline{\ell}^p}\otimes v_{i_1}\wedge\cdots\wedge v_{i_{p-1}}.
\end{aligned}$$

Now relabel indices so that each sum is taken over $1\le i_1<\cdots<i_{p-1}\le n$. We

obtain

$$\sum_{1\le i_1<\cdots<i_{p-1}\le n}\ \sum_{0\le a_{i_j}<\ell^j_{i_j}} \underline{v}^{\underline{Q}(\underline{\ell}^2,\cdots,\underline{\ell}^p;i_1,\cdots,i_{p-1})+\underline{\ell}^1}\otimes \underline{v}^{\underline{\hat{Q}}}\otimes v_{i_1}\wedge\cdots\wedge v_{i_{p-1}}$$

$$+\sum_{m=1}^{p-1}(-1)^m \sum_{1\le i_1<\cdots<i_{p-1}\le n}\ \sum_{\substack{0\le a_{i_j}<\ell^j_{i_j}\ (\text{for } j\in\{1,\dots,m-1\})\\ 0\le a_{i_m}\le \ell^m_{i_m}+\ell^{m+1}_{i_m}-1\\ 0\le a_{i_j}<\ell^{j+1}_{i_j}(\text{for } j\in\{m+1,\dots,p\})}}$$

$$\underline{v}^{\underline{Q}(\underline{\ell}^1,\dots,\underline{\ell}^{m-1},\underline{\ell}^m+\underline{\ell}^{m+1},\underline{\ell}^{m+2},\dots,\underline{\ell}^p;i_1,\dots,i_{p-1})}\otimes \underline{v}^{\underline{\hat{Q}}}\otimes v_{i_1}\wedge\cdots\wedge v_{i_{p-1}}$$

$$+(-1)^p\sum_{1\le i_1<\cdots<i_{p-1}\le n}\ \sum_{0\le a_{i_j}<\ell^j_{i_j}} \underline{v}^{\underline{Q}(\underline{\ell}^1,\cdots,\underline{\ell}^{p-1};i_2,\cdots,i_{p-1})}\otimes \underline{v}^{\underline{\hat{Q}}+\underline{\ell}^p}\otimes v_{i_1}\wedge\cdots\wedge v_{i_{p-1}}$$

$$=\ \Psi_{p-1}(\delta_p(1\otimes \underline{v}^{\underline{\ell}^1}\otimes\cdots\otimes \underline{v}^{\underline{\ell}^p}\otimes 1)).$$

This finishes the proof of Theorem 4-4.

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Toric-friendly groups

Mikhail Borovoi and Zinovy Reichstein

Let G be a connected linear algebraic group over a field k. We say that G is toric-friendly if for any field extension K/k and any maximal K-torus T in G the group $G(K)$ acts transitively on $(G/T)(K)$. Our main result is a classification of semisimple (and under certain assumptions on k, of connected) toric-friendly groups.

Introduction

Let k be a field and X be a homogeneous space of a connected linear algebraic group G defined over k. The first question one usually asks about X is whether or not it has a k-point. If the answer is "yes", then one often wants to know whether or not the set $X(k)$ of k-points of X forms a single orbit under the group $G(k)$.

In this paper we shall focus on the case where the geometric stabilizers for the G-action on X are maximal tori of $G_{\overline{k}} := G \times_k \overline{k}$ (here $\overline{k}$ stands for a fixed algebraic closure of k). Such homogeneous spaces arise, in particular, in the study of the adjoint action of a connected reductive group G on its Lie algebra or of the conjugation action of G on itself; see [Colliot-Thélène et al. 2011]. It is shown in Corollary 4.6 of the same reference (see also [Kottwitz 1982, Lemma 2.1]) that every homogeneous space X of this type has a k-point, assuming that G is split and $\operatorname{char}(k) = 0$. Therefore it is natural to ask if this point is unique up to translations by $G(k)$.

Definition 0.1. Let k be a field. We say that a connected linear k-group G is *toric-friendly* if for every field extension K/k the following condition is satisfied:

($*$) For every maximal K-torus T of $G_K := G \times_k K$, the group $G(K)$ has only one orbit in $(G_K/T)(K)$; equivalently, the natural map $\pi : G(K) \to (G_K/T)(K)$ is surjective.

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Examining the cohomology exact sequence associated to the K-subgroup T of G_K [Serre 1994, I.5.4, Proposition 36], we see that G is toric-friendly if and only if $\ker[H^1(K,T) \to H^1(K,G)] = 1$ for every field extension K/k and every maximal K-torus T of G_K.

Observe that G is toric-friendly if and only if condition $(*)$ of Definition 0.1 is satisfied for all *finitely generated* extensions K/k.

We are interested in classifying toric-friendly groups. In Section 1 we partially reduce this problem to the case where the group is semisimple. The rest of this paper will be devoted to proving the following classification theorem for semisimple toric-friendly groups.

Main Theorem 0.2. *Let k be a field. A connected semisimple k-group G is toric-friendly if and only if G is isomorphic to a direct product $\prod_i R_{F_i/k} G'_i$, where each F_i is a finite separable extension of k and each G'_i is an inner form of PGL_{n_i,F_i} for some integer n_i.*

Notation. Unless otherwise specified, k will denote an arbitrary field. For any field K we denote by K_s a separable closure of K.

By a k-group we mean an affine algebraic group scheme over k, not necessarily smooth or connected. However, when talking of a *reductive* or *semisimple* k-group, we implicitly assume smoothness and connectedness.

Let S be a k-group. We denote by $H^i(k,S)$ the i-th flat cohomology set for $i = 0, 1$ [Waterhouse 1979, 17.6]. If S is abelian, we denote by $H^i(k,S)$ the i-th flat cohomology group for $i \geq 0$ [Berhuy et al. 2007, Appendix B]. There are exact sequences for flat cohomology similar to those for Galois cohomology, [Waterhouse 1979, 18.1; Berhuy et al. 2007, Appendix B]. When S is smooth, the flat cohomology $H^i(k,S)$ can be identified with Galois cohomology.

1. First reductions

Lemma 1.1. *Let $1 \to U \to G \xrightarrow{\varphi} G' \to 1$ be an exact sequence of smooth connected k-groups, where U is unipotent. We assume that U is k-split, that is, has a composition series over k whose successive quotients are isomorphic to $\mathbb{G}_{\mathrm{a},k}$. Then G is toric-friendly if and only if G' is toric-friendly.*

Proof. Choose a field extension K/k and a maximal K-torus $T \subset G_K$. Set $T' = \varphi(T) \subset G'_K$, then T' is a maximal torus of G'_K. The map $\varphi^T : T \to T'$ is an isomorphism, because $T \cap U_K = 1$ (as U_K is unipotent). Conversely, let us start from a maximal torus T' of G'_K. The preimage

$$H = \varphi^{-1}(T') \subset G_K$$

of T' is smooth and connected, so any maximal torus T of H maps isomorphically onto T' and therefore it is maximal in G_K.

Now we have a commutative diagram

$$\begin{array}{ccc} H^1(K,T) & \longrightarrow & H^1(K,G) \\ {\scriptstyle \varphi_*^T}\downarrow & & \downarrow{\scriptstyle \varphi_*} \\ H^1(K,T') & \longrightarrow & H^1(K,G') \end{array}$$

Since $\varphi^T : T \to T'$ is an isomorphism of tori, the left vertical arrow φ_*^T is an isomorphism of abelian groups. On the other hand, by [Sansuc 1981, Lemma 1.13], the right vertical arrow φ_* is a bijective map. We see that the top horizontal arrow in the diagram is injective if and only if the bottom horizontal arrow is injective, which proves the lemma. □

Let k be a perfect field and G be a connected k-group. Recall that over a perfect field the unipotent radical of G makes sense; that is, the "geometric" unipotent radical over an algebraic closure is defined over k, by Galois descent. We denote the unipotent radical of G by $R_{\mathrm{u}}(G)$.

Corollary 1.2. *Let k be a perfect field, G be a connected k-group, and $R_{\mathrm{u}}(G)$ be its unipotent radical. Then G is toric-friendly if and only if the associated reductive k-group $G/R_{\mathrm{u}}(G)$ is toric-friendly.*

Proof. Since k is perfect, the smooth connected unipotent k-group $R_{\mathrm{u}}(G)$ is k-split [Borel 1991, Theorem 15.4], and the corollary follows from Lemma 1.1. □

Let k be a field. We recall that a k-group G is called *special* if $H^1(K,G) = 1$ for every field extension K/k. This notion was introduced by J.-P. Serre [1958]. Semisimple special groups over an algebraically closed field were classified by A. Grothendieck [1958]; we shall use his classification later on.

Recall that a k-torus T is called quasitrivial, if its character group $\mathbb{X}(T)$ is a permutation Galois module. Split tori and, more general, quasitrivial tori are special.

Proposition 1.3. *Let $1 \to C \to G \xrightarrow{\varphi} G' \to 1$ be an exact sequence of k-groups, where G and G' are reductive, and $C \subset G$ is central, hence of multiplicative type (not necessarily connected or smooth).*

(a) *If G is toric-friendly, so is G'.*

(b) *If C is a special k-torus, then G is toric-friendly if and only if G' is toric-friendly.*

Proof. Let K/k be a field extension. The map $T \mapsto T' := \varphi(T)$ is a bijection between the set of maximal K-tori $T \subset G_K$ and the set of maximal K-tori $T' \subset G'_K$

(the inverse map is $T' \mapsto T := \varphi^{-1}(T')$). For such T and $T' = \varphi(T)$ we have commutative diagrams

$$\begin{tikzcd} G_K \arrow[r, "\varphi"] \arrow[d, "\pi"'] & G'_K \arrow[d, "\pi'"] \\ G_K/T \arrow[r, "\varphi_*", "\cong"'] & G'_K/T' \end{tikzcd} \qquad \begin{tikzcd} G(K) \arrow[r, "\varphi"] \arrow[d, "\pi"'] & G'(K) \arrow[d, "\pi'"] \\ (G_K/T)(K) \arrow[r, "\varphi_*", "\cong"'] & (G'_K/T')(K) \end{tikzcd}$$

where $\varphi_* : G_K/T \xrightarrow{\sim} G'_K/T'$ is an isomorphism of K-varieties, and the induced map on K-points $\varphi_* : (G_K/T)(K) \to (G'_K/T')(K)$ is a bijection. Now, if G is toric-friendly, then the map $\pi : G(K) \to (G_K/T)(K)$ is surjective, and we see from the right-hand diagram that then the map $\pi' : G'(K) \to (G'_K/T')(K)$ is surjective as well. This shows that G' is toric-friendly, thus proving (a).

To prove (b), assume that G' is toric-friendly and C is a special k-torus. Then the map $\pi' : G'(K) \to (G'_K/T')(K)$ is surjective (because G' is toric-friendly) and the map $\varphi : G(K) \to G'(K)$ is surjective (because C is special). We see from the right-hand diagram that the map $\pi : G(K) \to (G_K/T)(K)$ is surjective as well. Hence G is toric-friendly. □

We record the following immediate corollary of Proposition 1.3(b).

Corollary 1.4. *Let G be a reductive k-group. Suppose that the radical* $\mathrm{R}(G)$ *is a special k-torus* (*in particular, this condition is satisfied if* $\mathrm{R}(G)$ *is a quasitrivial k-torus*). *Then G is toric-friendly if and only if the semisimple group $G/\mathrm{R}(G)$ is toric-friendly.* □

The next result follows from Corollaries 1.2 and 1.4. It partially reduces the problem of classifying toric-friendly groups G to the case where G is semisimple.

Corollary 1.5. *Let k be a perfect field. Let G be a connected k-group containing a split maximal torus. Then G is toric-friendly if and only if the semisimple group $G/\mathrm{R}(G)$ is toric-friendly.* □

The following two lemmas will be used to reduce the problem of classifying *adjoint* semisimple toric-friendly groups G to the case where G is an absolutely simple adjoint k-group.

Lemma 1.6. *A direct product $G = G' \times_k G''$ of connected k-groups is toric-friendly if and only if both G' and G'' are toric-friendly.*

Proof. Let K/k be a field extension. Let $T' \subset G'_K$ and $T'' \subset G''_K$ be maximal K-tori, then $T := T' \times_K T'' \subset G_K$ is a maximal K-torus, and every maximal K-torus in G_K is of this form. The commutative diagram

$$\begin{tikzcd}
G(K) \arrow[r, equal] \arrow[d] & G'(K) \times G''(K) \arrow[d] \\
(G_K/T)(K) \arrow[r, equal] & (G'_K/T')(K) \times (G''_K/T'')(K)
\end{tikzcd}$$

shows that every K-point of G_K/T lifts to G if and only if every K-point of G'_K/T' lifts to G' and every K-point of G''_K/T'' lifts to G''. □

Lemma 1.7. *Let l/k be a finite separable field extension, G' a connected l-group, and $G = R_{l/k}G'$. Then G is toric-friendly if and only if G' is toric-friendly.*

Proof. Let K/k be a field extension. Then $l \otimes_k K = L_1 \times \cdots \times L_r$, where L_i are finite separable extensions of K. It follows that $G_K = \prod_i R_{L_i/K} G'_{L_i}$. Let $T \subset G_K$ be a maximal K-torus, then $T = \prod_i R_{L_i/K} T'_i$, where T'_i is a maximal L_i-torus of G'_{L_i} for each i. We have

$$G(K) = G_K(K) = \Big(\prod_i R_{L_i/K} G'_{L_i}\Big)(K) = \prod_i G'_{L_i}(L_i) = \prod_i G'(L_i)$$

and similarly $(G_K/T)(K) = \prod_i (G'_{L_i}/T'_i)(L_i)$, yielding a commutative diagram

$$\begin{tikzcd}
G(K) \arrow[r, equal] \arrow[d] & \prod_i G'(L_i) \arrow[d] \\
(G_K/T)(K) \arrow[r, equal] & \prod_i (G'_{L_i}/T'_i)(L_i)
\end{tikzcd}$$

If G' is toric-friendly, then the right vertical arrow in the diagram is surjective, hence the left vertical arrow is surjective and G is toric-friendly.

Conversely, assume that G is toric-friendly. Let L/l be a field extension and $T' \subset G'_L$ a maximal L-torus. Set $K := L$ and $T := T'$ in the diagram above. Then we can identify L with one of L_i in the decomposition $l \otimes_k K = L_1 \times \cdots \times L_r$, say with L_1. In this way we identify G'_L with G'_{L_1} and G'_L/T' with G'_{L_1}/T'_1. Since G is toric-friendly, the left vertical arrow in the diagram is surjective, hence the right vertical arrow is also surjective. This means that the map $G'(L_i) \to (G'_{L_i}/T'_i)(L_i)$ is surjective for each i and in particular, for $i = 1$. Consequently, the map $G'(L) \to (G'_L/T')(L)$ is surjective, and G' is toric-friendly, as desired. □

2. The elementary obstruction

2.1. Let K be a field and X be a smooth geometrically integral K-variety. Write $\mathfrak{g} = \mathrm{Gal}(K_s/K)$, where K_s is a fixed separable closure of K. Recall from [Colliot-Thélène and Sansuc 1987, Definition 2.2.1] that the *elementary obstruction* $\mathrm{ob}(X)$

is the class in $\mathrm{Ext}^1_{\mathfrak{g}}(K_s(X)^*/K_s^*, K_s^*)$ of the extension

$$1 \to K_s^* \to K_s(X)^* \to K_s(X)^*/K_s^* \to 1.$$

In particular, $\mathrm{ob}(X) = 0$ if and only if this extension of $\mathfrak{g}$-modules splits. If X has a K-point, then $\mathrm{ob}(X) = 0$ [Colliot-Thélène and Sansuc 1987, Proposition 2.2.2(a)]. Conversely, if Y is a T-torsor over K for some K-torus T, and $\mathrm{ob}(Y) = 0$, then Y has a K-point, by Lemma 2.1(iv) of [Borovoi et al. 2008]. However, if X is an H-torsor over K for some simply connected semisimple K-group H, then $\mathrm{ob}(X) = 0$ even when X has no K-points; see Lemma 2.2(viii) of that same reference. (The standing assumption in [Borovoi et al. 2008] is that $\mathrm{char}(K) = 0$; however, the proofs of Lemmas 2.1(iv) and 2.2(viii) go through in arbitrary characteristic.)

The following key lemma was suggested to us by J.-L. Colliot-Thélène.

Lemma 2.2. *Let K be a field, T be a K-torus, H be a simply connected semisimple K-group, X be a H-torsor over K and Y be a T-torsor over K. If Y has an F-point over the function field $F = K(X)$ of X, then Y has a K-point.*

Proof. Since H is simply connected, $\mathrm{ob}(X) = 0$; see Section 2.1 above. Suppose Y has an F-point. This means that there exist a K-rational map $X \dashrightarrow Y$. By [Wittenberg 2008, Lemma 3.1.2], if we have a K-rational map $X \dashrightarrow Y$ between smooth geometrically integral K-varieties, then $\mathrm{ob}(X) = 0$ implies $\mathrm{ob}(Y) = 0$. Since T is a K-torus, if $\mathrm{ob}(Y) = 0$, then $Y(K) \neq \varnothing$; see Section 2.1 above. Thus in our situation Y has a K-point, as claimed. □

Lemma 2.3. *Let k be a field. Assume we have a commutative diagram of k-groups*

$$\begin{tikzcd} S \arrow[r] \arrow[d] & T \arrow[d] \\ H \arrow[r] & G \end{tikzcd}$$

where G is a smooth connected k-group, the vertical map $T \to G$ is the inclusion of a maximal k-torus T into G, and H is semisimple and simply connected. If there exists a field extension K/k such that the map

$$H^1(K, S) \to H^1(K, T)$$

is nontrivial, then G is not toric-friendly.

Proof. Choose K and $s \in H^1(K, S)$ such that the image $t \in H^1(K, T)$ of s in $H^1(K, T)$ is nontrivial. Let $h \in H^1(K, H)$ be the image of $s \in H^1(K, S)$ in $H^1(K, H)$, and let $g \in H^1(K, G)$ be the image of t (and of h) in $H^1(K, G)$, as shown in the commutative diagram below:

$$\begin{array}{ccc} H^1(K,S) & \longrightarrow & H^1(K,T) \\ \downarrow & & \downarrow \\ H^1(K,H) & \longrightarrow & H^1(K,G) \end{array} \qquad \begin{array}{ccc} s & \longrightarrow & t \\ \downarrow & & \downarrow \\ h & \longrightarrow & g \end{array}$$

Let X be an H-torsor over K representing h and let $F = K(X)$ be the function field of X. We denote by h_F the image of h in $H^1(F, H)$, and similarly we define s_F, t_F, and g_F. Clearly X has an F-point, hence $h_F = 1$ in $H^1(F, H)$ and therefore $g_F = 1$ in $H^1(F, G)$. On the other hand, by Lemma 2.2, $t_F \neq 1$. We conclude that the kernel of the natural map $H^1(F, T) \to H^1(F, G)$ contains $t_F \neq 1$ and hence, is nontrivial. This implies that G is not toric-friendly. □

2.4. Let G be a reductive k-group. Let G^{ss} be the derived group of G (it is semisimple), and let G^{sc} be the universal cover of G^{ss} (it is semisimple and simply connected). Consider the composed homomorphism $f : G^{\mathrm{sc}} \twoheadrightarrow G^{\mathrm{ss}} \hookrightarrow G$.

Let K/k be a field extension. There is a canonical bijective correspondence $T \leftrightarrow T^{\mathrm{sc}}$ between the set of maximal K-tori $T \subset G_K$ and the set of maximal K-tori $T^{\mathrm{sc}} \subset G^{\mathrm{sc}}$. Starting from a maximal K-torus $T \subset G_K$, we define a maximal K-torus $T^{\mathrm{sc}} := f^{-1}(T) \subset G^{\mathrm{sc}}_K$. Conversely, starting from a maximal K-torus $T^{\mathrm{sc}} \subset G^{\mathrm{sc}}_K$, we define a maximal K-torus $T := f(T^{\mathrm{sc}}) \cdot R(G)_K \subset G_K$, where $R(G)$ is the radical of G.

Proposition 2.5. *Let G be a reductive k-group. Let G^{sc} and $f : G^{\mathrm{sc}} \to G$ be as in Section 2.4 above. Let K/k be a field extension, $T \subset G_K$ be a maximal K-torus of G_K, and set $T^{\mathrm{sc}} = f^{-1}(T) \subset G^{\mathrm{sc}}_K$ as above. If the natural map $H^1(K, T^{\mathrm{sc}}) \to H^1(K, T)$ is nontrivial, then G is not toric-friendly.*

Proof. Immediate from Lemma 2.3. □

Proposition 2.6. *Let G be a semisimple k-group, $f : G^{\mathrm{sc}} \to G$ be the universal covering and $C := \ker(f)$. Then the following conditions are equivalent:*

(a) *G is toric-friendly.*

(b) *The map $H^1(K, T^{\mathrm{sc}}) \to H^1(K, T)$ is trivial (identically zero) for every field extension K/k and every maximal K-torus T^{sc} of G^{sc}. Here $T := f(T^{\mathrm{sc}})$.*

(c) *The map $H^1(K, C) \to H^1(K, T^{\mathrm{sc}})$ is surjective for every field extension K/k and every maximal K-torus T^{sc} of G^{sc}.*

(d) *The connecting homomorphism $\partial_T : H^1(K, T) \to H^2(K, C)$ is injective for every field extension K/k and every maximal K-torus T of G.*

(e) *The natural map $H^1(K, T) \to H^1(K, G)$ is injective for every field extension K/k and every maximal K-torus T of G.*

Proof. (a) ⇒ (b) by Proposition 2.5. Examining the cohomology sequence

$$H^1(K,C) \to H^1(K,T^{\mathrm{sc}}) \to H^1(K,T) \to H^2(K,C)$$

associated to the exact sequence $1 \to C \to T^{\mathrm{sc}} \to T \to 1$ of k-groups, we see that (b), (c) and (d) are equivalent.

(d) ⇒ (e): The diagram

$$\begin{tikzcd} 1 \arrow[r] & C \arrow[r] \arrow[d, equal] & T^{\mathrm{sc}} \arrow[r] \arrow[d, hook] & T \arrow[r] \arrow[d, hook] & 1 \\ 1 \arrow[r] & C \arrow[r] & G^{\mathrm{sc}} \arrow[r] & G \arrow[r] & 1 \end{tikzcd}$$

of K-groups induces compatible connecting morphisms

$$\begin{tikzcd} H^1(K,T) \arrow[dd] \arrow[dr, "\partial_T"] & \\ & H^2(K,C) \\ H^1(K,G) \arrow[ur, "\partial_G"] & \end{tikzcd}$$

Suppose $\alpha, \beta \in H^1(K,T)$ map to the same element in $H^1(K,G)$. Then the diagram above shows that $\partial_T(\alpha) = \partial_T(\beta)$ in $H^2(K,C)$. Part (d) now tells us that $\alpha = \beta$.

(e) ⇒ (a) is obvious, since (a) is equivalent to the assertion that $H^1(K,T) \to H^1(K,G)$ has trivial kernel for every K and T; see Definition 0.1. □

Corollary 2.7. *With the assumptions and notation of Proposition 2.6, if G is toric-friendly and quasisplit, then*

(a) *the map $H^1(K,G^{\mathrm{sc}}) \to H^1(K,G)$ is trivial for every K/k,*

(b) *the map $H^1(K,C) \to H^1(K,G^{\mathrm{sc}})$ is surjective for every K/k,*

(c) *the connecting map $\partial_G : H^1(K,G) \to H^2(K,C)$ has trivial kernel for every K/k.*

Proof. Examining the cohomology sequence

$$H^1(K,C) \to H^1(K,G^{\mathrm{sc}}) \to H^1(K,G) \to H^2(K,C)$$

associated to the exact sequence $1 \to C \to G^{\mathrm{sc}} \to G \to 1$, we see that (a), (b) and (c) are equivalent.

To prove (a), recall that since G_K is quasisplit, by a theorem of Steinberg [1965, Theorem 1.8] every $x^{\mathrm{sc}} \in H^1(K,G^{\mathrm{sc}})$ lies in the image of the map $H^1(K,T^{\mathrm{sc}}) \to$

$H^1(K, G^{\mathrm{sc}})$ for some maximal K-torus T^{sc} of G^{sc}_K. Since G is toric-friendly, by Proposition 2.6 the map $H^1(K, T^{\mathrm{sc}}) \to H^1(K, T)$ is trivial. The commutative diagram

$$\begin{tikzcd} H^1(K, T^{\mathrm{sc}}) \arrow[r] \arrow[d] & H^1(K, T) \arrow[d] \\ H^1(K, G^{\mathrm{sc}}) \arrow[r] & H^1(K, G) \end{tikzcd}$$

now shows that the image of x^{sc} in $H^1(K, G)$ is 1. Thus the map $H^1(K, G^{\mathrm{sc}}) \to H^1(K, G)$ is trivial. $\square$

Theorem 2.8. *Let G be a split semisimple k-group and $f : G^{\mathrm{sc}} \to G$ be its universal covering map. If G is toric-friendly, then G^{sc} is special.*

Proof. Let T^{sc} be a split maximal torus of G^{sc}. Recall that T^{sc} is special (as is any split torus). Set $C = \ker f$, then $C \subset T^{\mathrm{sc}}$. For any field extension K/k, the map $H^1(K, C) \to H^1(K, G^{\mathrm{sc}})$ factors through $H^1(K, T^{\mathrm{sc}}) = 1$ and hence is trivial. By Corollary 2.7(b) this map is also surjective. This shows that $H^1(K, G^{\mathrm{sc}}) = 1$ for every K/k, that is, G^{sc} is special. $\square$

Remark 2.9. Our proof of Theorem 2.8 goes through for any (not necessarily split) semisimple k-group G, as long as G^{sc} contains a special maximal k-torus T^{sc}. In particular, Theorem 2.8 remains valid for any quasisplit semisimple k-group G, in view of Lemma 2.10 below. This lemma is a special case of [Colliot-Thélène et al. 2004, Lemma 5.6]; however, for the sake of completeness we supply a short self-contained proof.

Lemma 2.10. *Let G be a semisimple, simply connected, quasisplit k-group over a field k. Let $B \subset G$ be a Borel subgroup defined over k, and let $T \subset B \subset G$ be a maximal k-torus of G contained in B. Then T is a quasitrivial k-torus.*

Proof. We write $\bar{k}$ for a fixed algebraic closure of k. Let $\mathbb{X}^\vee(T)$ denote the group of cocharacters of T. Let $R^\vee = R^\vee(G_{\bar{k}}, T_{\bar{k}}) \subset \mathbb{X}^\vee(T)$ denote the coroot system of $G_{\bar{k}}$ with respect to $T_{\bar{k}}$, and let $\Pi^\vee \subset R^\vee$ denote the basis of $R^\vee$ corresponding to B. The Galois group $\mathrm{Gal}(k_s/k)$ acts on $\mathbb{X}^\vee(T)$. Since T, G, and B are defined over k, the subsets $R^\vee$ and $\Pi^\vee$ of $\mathbb{X}^\vee(T)$ are invariant under this action. Since G is simply connected, $\Pi^\vee$ is a $\mathbb{Z}$-basis of $\mathbb{X}^\vee(T)$. Thus $\mathrm{Gal}(k_s/k)$ permutes the $\mathbb{Z}$-basis $\Pi^\vee$ of $\mathbb{X}^\vee(T)$; in other words, T is a quasitrivial torus. $\square$

Remark 2.11. A similar assertion for *adjoint* quasisplit groups was proved by G. Prasad [1989, Proof of Lemma 2.0].

3. Examples in type A

Let k be a field and A a central simple k-algebra of dimension n^2. We write $\mathrm{GL}_{1,A}$ for the k-group with $\mathrm{GL}_{1,A}(R) = (A \otimes_k R)^*$ for any unital commutative k-algebra

R (here $(\ )^*$ denotes the group of invertible elements). The k-group $\mathrm{GL}_{1,A}$ is an inner form of $\mathrm{GL}_{n,k}$.

Let K be a field. Recall that an n-dimensional commutative étale K-algebra is a finite product $E = \prod_i L_i$, where each L_i is a finite separable field extension of K and $\sum_i [L_i : K] = n$. For such $E = \prod_i L_i$ we define a K-torus $R_{E/K}\mathbb{G}_{m,E} := \prod_i R_{L_i/K}\mathbb{G}_{m,L_i}$, then $(R_{E/K}\mathbb{G}_{m,E})(K) = E^*$. Clearly the K-torus $R_{E/K}\mathbb{G}_{m,E}$ is quasitrivial.

Proposition 3.1. *Let k be a field, and let A/k be a central simple k-algebra of dimension n^2.*

(a) *The k-group $G = \mathrm{GL}_{1,A}$ is toric-friendly.*

(b) *The k-group $\mathrm{PGL}_{1,A} := \mathrm{GL}_{1,A}/\mathbb{G}_{m,k}$ is toric-friendly.*

(c) *In particular, $\mathrm{GL}_{n,k}$ and $\mathrm{PGL}_{n,k}$ are toric-friendly.*

Proof. (a) Let K/k be a field extension and let

$$T \subset G_K = \mathrm{GL}_{1,A\otimes_k K}$$

be a maximal K-torus. Let E be the centralizer of T in $A\otimes_k K$. An easy calculation over a separable closure K_s of K shows that E is an n-dimensional commutative étale K-subalgebra of $A \otimes_k K$ and that $T = R_{E/K}\mathbb{G}_{m,E}$. It follows that T is quasitrivial, hence special. Since all maximal K-tori $T \subset G_K$ are special, G is toric-friendly.

(b) follows from (a) and Corollary 1.4. To deduce (c) from (a) and (b), set $A = M_n(k)$ (the matrix algebra). □

We now come to the main result of this section, which asserts that a toric-friendly semisimple groups of type A is necessarily an adjoint group.

Proposition 3.2. *Let k be a field. Consider a k-group $G = (\mathrm{SL}_{n_1} \times \cdots \times \mathrm{SL}_{n_r})/C$, where $C \subset \mu := \mu_{n_1} \times \cdots \times \mu_{n_r}$ is a central subgroup of $G^{\mathrm{sc}} = \mathrm{SL}_{n_1} \times \cdots \times \mathrm{SL}_{n_r}$, not necessarily smooth. If $C \neq \mu$, then G is not toric-friendly.*

Before proceeding with the proof, we fix some notation. Let L/K be a finite separable field extension of degree n. Set

$$R^1_{L/K}(\mathbb{G}_m) := \ker[N_{L/K} : R_{L/K}\mathbb{G}_{m,L} \to \mathbb{G}_{m,K}],$$

where $N_{L/K}$ is the norm map. Clearly $R^1_{L/K}(\mathbb{G}_m)$ can be embedded into $\mathrm{SL}_{n,K}$ as a maximal K-torus. The embedding $K \hookrightarrow L$ induces an embedding $\mu_{n,K} \hookrightarrow R^1_{L/K}\mathbb{G}_m$, where $n = [L : K]$.

The following two lemmas are undoubtedly known. We include short proofs below because we have not been able to find appropriate references.

Lemma 3.3. *There is a commutative diagram*

$$\begin{tikzcd} K^*/K^{*n} \arrow[r, "\cong"] \arrow[d] & H^1(K, \mu_n) \arrow[d] \\ K^*/N_{L/K}(L^*) \arrow[r, "\cong"] & H^1(K, R^1_{L/K}\mathbb{G}_m) \end{tikzcd} \tag{1}$$

where the horizontal arrows are canonical isomorphisms, the right vertical arrow is induced by the embedding $\mu_n \hookrightarrow R^1_{L/K}\mathbb{G}_m$, and the left vertical arrow is the natural projection.

Proof. Apply the flat cohomology functor to the commutative diagram of commutative K-groups

$$\begin{tikzcd} 1 \arrow[r] & \mu_{n,K} \arrow[r] \arrow[d, hook] & \mathbb{G}_{m,K} \arrow[r, "n"] \arrow[d, hook] & \mathbb{G}_{m,K} \arrow[r] \arrow[d, "\mathrm{id}"] & 1 \\ 1 \arrow[r] & R^1_{L/K}\mathbb{G}_m \arrow[r] & R_{L/K}\mathbb{G}_m \arrow[r, "N_{L/K}"] & \mathbb{G}_{m,K} \arrow[r] & 1 \end{tikzcd}$$

and use Hilbert's Theorem 90. □

Lemma 3.4. *Suppose $r \mid n$. Then there is a commutative diagram*

$$\begin{tikzcd} K^*/K^{*n} \arrow[r, "\cong"] \arrow[d] & H^1(K, \mu_n) \arrow[d, "(n/r)_*"] \\ K^*/K^{*r} \arrow[r, "\cong"] & H^1(K, \mu_r)\,, \end{tikzcd}$$

where the horizontal arrows are canonical isomorphisms, the right vertical arrow is induced by the homomorphism $\mu_n \xrightarrow{n/r} \mu_r$ given by $x \mapsto x^{n/r}$, and the left vertical arrow is the natural projection.

Proof. Similar to that of Lemma 3.3, using the commutative diagram

$$\begin{tikzcd} 1 \arrow[r] & \mu_n \arrow[r] \arrow[d, "n/r"] & \mathbb{G}_m \arrow[r, "n"] \arrow[d, "n/r"] & \mathbb{G}_m \arrow[r] \arrow[d, "\mathrm{id}"] & 1 \\ 1 \arrow[r] & \mu_r \arrow[r] & \mathbb{G}_m \arrow[r, "r"] & \mathbb{G}_m \arrow[r] & 1 \end{tikzcd}$$

□

Example 3.5. The group $G = \mathrm{SL}_{n,k}$ $(n \geq 2)$ is not toric-friendly.

Proof. Since SL_n is special, it suffices to construct an extension K/k and a maximal K-torus $T := R^1_{L/K}(\mathbb{G}_m)$ such that $H^1(K, T) \neq 1$. In view of Lemma 3.3 it suffices to show that $N_{L/K}(L^*) \neq K^*$ for some field extension K/k and some finite separable field extension L/K of degree n. This is well known; see for example the proof of [Rowen 1980, Proposition 3.1.46]. We include a short proof below as a way of motivating a related but more complicated argument at the end of the proof of Proposition 3.2.

Let $L := k(x_1, \dots, x_n)$, where $x_1, \dots, x_n$ are independent variables, and $K := L^\Gamma$, where Γ is the cyclic group of order n that acts on L by cyclically permuting $x_1, \dots, x_n$. For $0 \neq a \in k[x_1, \dots, x_n]$, let $\deg(a) \in \mathbb{N}$ denote the degree of a as a polynomial in $x_1, \dots, x_n$. If $a \in k(x_1, \dots, x_n)$ is of the form $a = b/c$ with nonzero $b, c \in k[x_1, \dots, x_n]$, then we define $\deg(a) = \deg(b) - \deg(c)$. This yields the usual degree homomorphism $\deg : L^* \to \mathbb{Z}$. Since $\mathrm{N}_{L/K}(a) = \prod_{\gamma \in \Gamma} \gamma(a)$, we see that $\deg(\mathrm{N}_{L/K}(a)) = n \deg(a)$ is divisible by n, for every $a \in L^*$. On the other hand, $s_1 = x_1 + \dots + x_n \in K$ has degree 1. This shows that $N_{L/K}(L^*) \neq K^*$, as claimed. $\square$

3.6. ***Proof of Proposition 3.2.*** Let K/k be a field extension. For each $i = 1, \dots, r$, let L_i be a separable field extension of degree n_i over K, and let $T = T_1 \times \dots \times T_r$ be a maximal K-torus of G^{sc}, where $T_i := R^1_{L_i/K}(\mathbb{G}_m)$. By Proposition 2.6 it suffices to show that the composition

$$H^1(K, C) \to H^1(K, \mu) \to H^1(K, T) \tag{2}$$

is not surjective for some choice of extensions K/k and L_i/K_i. Since $C \subsetneq \mu$, there exist a prime p and a nontrivial character $\chi : \mu \to \mu_p$ such that $\chi(C) = 1$. By Proposition 1.3(a) we may assume that $C = \ker(\chi)$. For notational simplicity, let us suppose that $n_1, \dots, n_s$ are divisible by p and $n_{s+1}, \dots, n_r$ are not, for some $0 \le s \le r$. Then it is easy to see that χ is of the form

$$\chi(c_1, \dots, c_r) = c_1^{d_1 n_1/p} \cdots c_s^{d_s n_s/p}$$

for some integers $d_1, \dots, d_s$. Since χ is nontrivial on μ, we have $s \ge 1$ and d_i is not divisible by p for some $i = 1, \dots, s$, say for $i = 1$. That is, we may assume that d_1 is not divisible by p.

Lemma 3.3 gives a concrete description of the second map in (2). To determine the image of the map $H^1(K, C) \to H^1(K, \mu)$, we examine the cohomology exact sequence

$$\begin{tikzcd} H^1(K, C) \arrow[r] & H^1(K, \mu) \arrow[r, "\chi_*"] \arrow[d, equal] & H^1(K, \mu_p) \arrow[d, equal] \\ & \prod_{i=1}^r K^*/K^{*n_i} \arrow[r, "\chi_*"] & K/K^{*p} \end{tikzcd}$$

induced by the exact sequence $1 \to C \to \mu \xrightarrow{\chi} \mu_p \to 1$. The image of $H^1(K, C)$ in $H^1(K, \mu)$ is the kernel of χ_*. By Lemma 3.4, χ_* maps the class of $(a_1, \dots, a_r)$

in $H^1(K,\mu)=\prod_{i=1}^r K^*/K^{*n_i}$ to the class of $a_1^{d_1}\cdots a_s^{d_s}$ in $H^1(K,\mu_p)=K/K^{*p}$. In other words, the image of $H^1(K,C)$ in $H^1(K,\mu)$ is the subgroup of classes of r-tuples $(a_1,\dots,a_r)$ in $H^1(K,\mu)=\prod_{i=1}^r K^*/K^{*n_i}$ such that $a_1^{d_1}\dots a_s^{d_s}\in K^{*p}$. Hence, the image of $H^1(K,C)$ in $H^1(K,T)=\prod_{i=1}^r K^*/N_{L_i/K}(L_i^*)$ consists of classes of r-tuples $(a_1,\dots,a_r)$ such that $a_1^{d_1}\dots a_s^{d_s}\in K^{*p}$.

It remains to construct a field extension K/k, separable field extensions L_i/K of degree n_i for $i=1,\dots,r$, and an element $\alpha\in H^1(K,T)=\prod_{i=1}^r K^*/N_{L_i/K}(L_i^*)$, which cannot be represented by $(a_1,\dots,a_r)\in(K^*)^r$ such that $a_1^{d_1}\cdots a_s^{d_s}\in K^{*p}$. This will show that the map $H^1(K,C)\to H^1(K,T)$ is not surjective, as claimed.

Set $L:=k(x_1,\dots,x_n)$, where $n=n_1+\dots+n_r$ and $x_1,\dots,x_n$ are independent variables. The symmetric group S_n acts on L by permuting these variables; we embed $S_{n_1}\times\dots\times S_{n_r}$ into S_n in the natural way, by letting S_{n_1} permute the first n_1 variables, S_{n_2} permute the next n_2 variables, etc. Set $K:=L^{S_{n_1}\times\dots\times S_{n_r}}$, $s_1:=x_1+\dots+x_n\in K$ and

$$L_1:=K(x_1),\ \ L_2:=K(x_{n_1+1}),\ \ \dots\ \ L_r:=K(x_{n_1+\dots+n_{r-1}+1}).$$

Clearly $[L_i:K]=n_i$. We claim the class of $(s_1,1,\dots,1)$ in $\prod_{i=1}^r K^*/N_{L_i/K}(L_i^*)$ cannot be represented by any $(a_1,\dots,a_r)\in(K^*)^r$ with $a_1^{d_1}\cdots a_s^{d_s}\in K^{*p}$.

Let $\deg: L^*\to\mathbb{Z}$ be the degree map, as in Example 3.5. Arguing as we did there, we see that $\deg(N_{L_i/K}(a))$ is divisible by n_i for every $i=1,\dots,r$ and every $a\in L_i^*$. In particular, $(a_1,\dots,a_r)\mapsto\deg(a_i)+n_i\mathbb{Z}$ is a well-defined function $\prod_{i=1}^r K^*/N_{L_i/K}(L_i^*)\to\mathbb{Z}/n_i\mathbb{Z}$, and consequently,

$$f(a_1,\dots,a_n):=d_1\deg(a_1)+\dots+d_s\deg(a_s)+p\mathbb{Z}$$

is a well-defined function $H^1(K,T)\to\mathbb{Z}/p\mathbb{Z}$. We have

$$f(a_1,\dots,a_n)=\deg(a_1^{d_1}\cdots a_s^{d_s}).$$

If $a_1^{d_1}\cdots a_s^{d_s}\in K^{*p}$, then $f(a_1,\dots,a_r)=0$ in $\mathbb{Z}/p\mathbb{Z}$. On the other hand, since $\deg(1)=0$, $\deg(s_1)=1$ and d_1 is not divisible by p, we conclude that $f(s_1,1,\dots,1)$ is nonzero in $\mathbb{Z}/p\mathbb{Z}$. This proves the claim and the proposition. □

4. Groups of type C_n and outer forms of A_n

Proposition 4.1. *No absolutely simple k-group of type C_n $(n\geq 2)$ is toric-friendly.*

Proof. Clearly we may assume that k is algebraically closed. We may also assume that G is adjoint, see Proposition 1.3(a). We see that $G=\mathrm{PSp}_{2n}$ and $G^{\mathrm{sc}}=\mathrm{Sp}_{2n}$. By Example 3.5, SL_2 is not toric-friendly. This means that there exist a field extension K/k, a maximal K-torus $S\subset\mathrm{SL}_{2,K}$, and a cohomology class $a_S\in H^1(K,S)$ such that $a_S \neq 1$. We consider the standard embedding

$$(\mathrm{SL}_2)^n = (\mathrm{Sp}_2)^n \hookrightarrow \mathrm{Sp}_{2n}, \quad n \geq 2.$$

Set $T^{\mathrm{sc}} = S^n \subset (\mathrm{Sp}_2)^n \subset \mathrm{Sp}_{2n} = G^{\mathrm{sc}}$. Let $\iota : S \hookrightarrow T^{\mathrm{sc}} = S^n$ be the embedding as the first factor. Set $a^{\mathrm{sc}} = \iota_*(a_S) \in H^1(K, T^{\mathrm{sc}})$. Let T be the image of T^{sc} in $G = \mathrm{PSp}_{2n}$, and let a be the image of a^{sc} in $H^1(K, T)$.

Now observe that the homomorphism

$$\chi : T^{\mathrm{sc}} = S^n \to S, \quad (x_1, \ldots, x_n) \mapsto x_1 x_2^{-1},$$

factors through T (recall that $n \geq 2$). Since $\chi \circ \iota = \mathrm{id}_S$, we see that $a \neq 1$. On the other hand, the image of a^{sc} in $H^1(K, G^{\mathrm{sc}})$ is 1 (because $G^{\mathrm{sc}} = \mathrm{Sp}_{2n}$ is special), hence $a \in \ker[H^1(K, T) \to H^1(K, G)]$, and we see that $G = \mathrm{PSp}_{2n}$ is not toric-friendly. □

Proposition 4.2. *No absolutely simple k-group of outer type A_n ($n \geq 2$) is toric-friendly.*

Lemma 4.3. *Let k be a field, K/k a separable quadratic extension, and D/K a central division algebra of dimension r^2 over K with an involution σ of the second kind (i.e., σ acts nontrivially on K and trivially on k). Then there exists a finite separable field extension F/k such that $K_F := K \otimes_k F$ is a field and $D \otimes_K K_F$ is split, that is, K_F-isomorphic to the matrix algebra $M_r(K_F)$.*

Proof of the lemma. Since there are no nontrivial central division algebras over finite fields, we may assume that k and K are infinite. Let

$$H = \{x \in D \mid x^\sigma = x\}$$

denote the k-space of Hermitian elements of D. Consider the embedding $D \hookrightarrow M_r(K_s)$ induced by an isomorphism $D \otimes_K K_s \cong M_r(K_s)$, where K_s is a separable closure of K. An element x of D is called semisimple regular if its image in $D \otimes_K K_s \cong M_r(K_s)$ is a semisimple matrix with r distinct eigenvalues. A standard argument using an isomorphism $D \otimes_k K_s \cong M_r(K_s) \times M_r(K_s)$ shows that there is a dense open subvariety H_{reg} in the space H, consisting of semisimple regular elements. Clearly H_{reg} is defined over k and contains k-points.

Let $x \in H_{\mathrm{reg}}(k) \subset D$ be a semisimple regular Hermitian element. Let L be the centralizer of x in D. Since x is Hermitian (σ-invariant), the k-algebra L is σ-invariant. Since x is semisimple and regular, the algebra L is a commutative étale K-subalgebra of D of dimension r over K, as is easily seen by passing to K_s. Clearly L is a field, $[L : K] = r$, and L is separable over k. Since $L \subset D$ and $[L : K] = r$, the field L is a splitting field for D; see, for example, [Pierce 1982, Corollary 13.3].

Since $L \supset K$, we see that σ acts nontrivially on L. Let $F = L^{\langle\sigma\rangle}$ denote the subfield of L consisting of elements fixed by σ. Then $[L:F]=2$ and $[F:k]=r$. Clearly F is separable over k. Since $F \cap K = k$ and $FK = L$, we conclude that $L = K \otimes_k F := K_F$. This completes the proof of the lemma. □

4.4. ***Proof of Proposition 4.2.*** By Proposition 1.3(a) we may assume that G is adjoint. By Lemma 4.3 there is a finite separable field extension F/k such that $G_F \cong \mathrm{PSU}(L^{n+1}, h)$, where L/F is a separable quadratic extension and h is a Hermitian form on L^{n+1}. It suffices to prove that $G_F = \mathrm{PSU}(L^{n+1}, h)$ is not toric-friendly.

Set $S = R^1_{L/F}\mathbb{G}_m$. We set $G_F^{\mathrm{sc}} = \mathrm{SU}(L^{n+1}, h)$. We may assume that h is a diagonal form [Knus 1991, Proposition 6.2.4(1); Scharlau 1985, Theorem 7.6.3]. Consider the diagonal torus $S^{n+1} \subset \mathrm{U}(L^{n+1}, h)$ and set $T^{\mathrm{sc}} = S^{n+1} \cap \mathrm{SU}(L^{n+1}, h)$.

We claim that there exists a field extension K/F such that $H^1(K, S) \neq 1$. Indeed, take $K = F((t))$, the field of formal Laurent series over F. Then by [Serre 1968, Proposition V.2.3(c)], $H^1(K, S) \cong H^1(F, S) \times \mathbb{Z}/2\mathbb{Z} \neq 1$.

Now let $a_S \in H^1(K, S)$, $a_S \neq 1$, and consider the embedding

$$\iota : S \hookrightarrow T^{\mathrm{sc}} \subset S^{n+1}, \quad x \mapsto (x, x^{-1}, 1, \ldots, 1).$$

Set $a^{\mathrm{sc}} = \iota_*(a_S) \in H^1(K, T^{\mathrm{sc}})$. Let T be the image of T^{sc} in $G_F = \mathrm{PSU}(L^{n+1}, h)$ and a be the image of a^{sc} in $H^1(K, T)$.

Note that the homomorphism

$$\chi : T^{\mathrm{sc}} \to S, \quad (x_1, \ldots, x_n, x_{n+1}) \mapsto x_1 x_3^{-1},$$

factors through T (recall that $n \geq 2$). Since $\chi \circ \iota = \mathrm{id}_S$, we see that $a \neq 1$. Now by Proposition 2.5, G_F and hence G are not toric-friendly. □

5. Classification of semisimple toric-friendly groups

Lemma 5.1. *Let k be an algebraically closed field. If a semisimple k-group G is toric-friendly, then it is adjoint of type A, that is, $G \cong \prod_i \mathrm{PGL}_{n_i}$ for some integers $n_i \geq 2$.*

Proof. First assume that G is simple. By Theorem 2.8 the simply connected cover G^{sc} of G is special. By a theorem of Grothendieck [1958, Theorem 3], G^{sc} is special if and only if G is of type A_n, $n \geq 1$ or C_n, $n \geq 2$. Proposition 4.1 rules out the second possibility. Thus G is of type A.

Now let G be semisimple. By Proposition 1.3(a), G^{ad} is toric-friendly. Write $G^{\mathrm{ad}} = \prod_i G_i$, where each G_i is an adjoint simple group, then by Lemma 1.6 each G_i is toric-friendly. As we have seen, this implies that each G_i is of type A, that is, isomorphic to PGL_{n_i} for some n_i. By Proposition 3.2, G is adjoint, that is, $G = G^{\mathrm{ad}} = \prod_i \mathrm{PGL}_{n_i}$. □

5.2. ***Proof of the Main Theorem 0.2.*** If G is toric-friendly, then clearly $G_{\bar{k}}$ is toric-friendly, where $\bar{k}$ is an algebraic closure of k. By Lemma 5.1, G is adjoint of type A. Write $G = \prod_i R_{F_i/k} G'_i$, where each F_i/k is a finite separable extension and G'_i is a form of PGL_{n_i,F_i}. By Lemmas 1.6 and 1.7, each G'_i is toric-friendly, and by Proposition 4.2, G'_i is an *inner* form of PGL_{n_i,F_i}.

Conversely, by Proposition 3.1 an inner form G'_i of PGL_{n_i,F_i} is toric-friendly. By Lemmas 1.6 and 1.7, the product $G = \prod_i R_{F_i/k} G'_i$ is toric-friendly. □

Corollary 5.3. *Let G be a nontrivial semisimple k-group. Then there exist a field extension K/k and a maximal K-torus $T \subset G$ that is not special. Equivalently, there exist a field extension K/k and a maximal K-torus T of G such that $H^1(K,T) \neq 1$.*

Proof. Assume the contrary, that is, that for any field extension K/k, any maximal K-torus $T \subset G_K$ is special. We may and shall assume that G is split. Recall that for a (quasi)split group, by [Steinberg 1965, Theorem 11.1], every element of $H^1(K,G)$ lies in the image of the map $H^1(K,T) \to H^1(K,G)$ for some maximal K-torus T of G. Thus, under our assumption we have $H^1(K,G) = 1$ for every field extension K/k, that is, G is special. By [Grothendieck 1958, Theorem 3], this is only possible if G is simply connected and has components only of types A and C. On the other hand, G is clearly toric-friendly (see Definition 0.1), and by the Main Theorem 0.2 no nontrivial simply connected semisimple group can be toric-friendly, a contradiction. □

The next result follows immediately from the Main Theorem 0.2 and Corollary 1.4.

Corollary 5.4. *Let G be a split reductive k-group. The group G is toric-friendly if and only if it satisfies these two conditions*:

(a) *the center $Z(G)$ of G is a k-torus, and*

(b) *the adjoint group $G^{\mathrm{ad}} := G/Z(G)$ is a direct product of simple adjoint groups of type A.* □

Note that in condition (a) we allow the trivial k-torus $\{1\}$.

By Corollary 1.4 if G is a reductive k-group such that $G/R(G)$ is toric-friendly and $R(G)$ is special, then G is toric-friendly. The example below shows that when $G/R(G)$ is toric-friendly but $R(G)$ is not special, G need not be toric-friendly.

Example 5.5. Let $k=\mathbb{R}$, $G=\mathrm{U}_2$, the unitary group in two complex variables. Then $Z(G)$ is the group of scalar matrices in G, it is connected, hence $R(G) = Z(G)$ and $G/R(G) = G^{\mathrm{ad}} = \mathrm{PSU}_2$. Since PSU_2 is an inner form of $\mathrm{PGL}_{2,\mathbb{R}}$, by the Main Theorem 0.2 it is toric-friendly. However, the group $G = \mathrm{U}_2$ is not toric-friendly. This does not contradict Corollary 1.4, because $R(G) = Z(G)$ is not special: $H^1(\mathbb{R}, Z(G)) = \mathbb{R}^*/N_{\mathbb{C}/\mathbb{R}}(\mathbb{C}^*) \cong \mathbb{Z}/2\mathbb{Z}$.

To show that $G = \mathrm{U}_2$ is not toric-friendly, set $S = R^1_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m$. Let T be the diagonal maximal $\mathbb{R}$-torus of U_2. Set $G^{\mathrm{sc}} = \mathrm{SU}_2$, $T^{\mathrm{sc}} = T \cap \mathrm{SU}_2$, then $T^{\mathrm{sc}} \cong S$.

Let $a^{\mathrm{sc}} \in H^1(\mathbb{R}, T^{\mathrm{sc}})$ be the cohomology class of the cocycle given by the element $-1 \in T^{\mathrm{sc}}(\mathbb{R})$ of order 2. Let $a \in H^1(\mathbb{R}, T)$ be the image of a^{sc} in $H^1(\mathbb{R}, T)$. Clearly $a \neq 1$. By Proposition 2.5, G is not toric-friendly. □

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Reflexivity and rigidity for complexes, II Schemes

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We prove basic facts about reflexivity in derived categories over noetherian schemes, and about related notions such as semidualizing complexes, invertible complexes, and Gorenstein-perfect maps. Also, we study a notion of rigidity with respect to semidualizing complexes, in particular, relative dualizing complexes for Gorenstein-perfect maps. Our results include theorems of Yekutieli and Zhang concerning rigid dualizing complexes on schemes. This work is a continuation of part I (*Algebra and Number Theory* **4**:1 (2010), 47–86), which dealt with commutative rings.

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Introduction

This paper is concerned with properties of complexes over noetherian schemes, that play important roles in duality theory. Some such properties, like (derived) reflexivity, have been an integral part of the theory since its inception; others, like rigidity, appeared only recently. Our main results reveal new aspects of such concepts and establish novel links between them.

Similar questions over commutative rings were examined in [Avramov et al. 2010a]. Additional topics treated there are semidualizing complexes, complexes of finite Gorenstein dimension, perfect complexes, invertible complexes, and rigidity with respect to semidualizing complexes, as well as versions of these notions relative to essentially-finite-type ring-homomorphisms that have finite flat dimension or, more generally, finite Gorenstein dimension. In this sequel we globalize such considerations, that is, extend them to the context of schemes.

This work is a substantial application of Grothendieck duality theory, seen as the study of a twisted inverse image pseudofunctor $(-)^!$ defined on appropriate categories of schemes. Duality theory provides interpretations of the local facts, a technology to globalize them, and suggestions for further directions of development.

To place our work in context, we review two methods for proving existence of $(-)^!$ for noetherian schemes and separated scheme-maps of finite type. The original approach of Grothendieck involves the construction of a "coherent family" of dualizing complexes; details are presented in [Hartshorne 1966] and revised in [Conrad 2000]. An alternative method, based on Nagata compactifications and sketched in [Deligne 1966] and [Verdier 1969], is developed in [Lipman 2009]. Recent extensions of these approaches to maps essentially of finite type provide a principal object of this study — the concept of rigidity — and one of our main tools.

Indeed, rigid dualizing complexes over rings, introduced by van den Bergh [1997] in the context of noncommutative algebraic geometry, are used by Yekutieli and Zhang [2008; 2009] in an ongoing project aiming to simplify Grothendieck's construction of $(-)^!$, and extend it to schemes essentially of finite type over a regular ring of finite Krull dimension. On the other hand, Nayak [2009] proved an analog of Nagata's compactification theorem and extended the pseudofunctor $(-)^!$ to the category of all noetherian schemes and their separated maps essentially of finite type. We work in this category.

Next we describe in some detail the notions and results of the paper. Comparison with earlier work is postponed until the end of this Introduction.

All schemes are assumed to be noetherian; all scheme-maps are assumed to be essentially of finite type and separated. Let X be a scheme, $\mathsf{D}(X)$ the derived category of the category of $\mathcal{O}_X$-modules, and $\mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X) \subset \mathsf{D}(X)$ the full subcategory whose objects are the complexes with coherent homology that vanishes in all but finitely many degrees.

For F and A in $\mathsf{D}(X)$, we say that F is *derived A-reflexive* if both F and $\mathsf{R}\mathcal{H}om_X(F, A)$ are in $\mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X)$, and if the canonical $\mathsf{D}(X)$-map is an isomorphism

$$F \xrightarrow{\sim} \mathsf{R}\mathcal{H}om_X\big(\mathsf{R}\mathcal{H}om_X(F, A), A\big).$$

When $\mathcal{O}_X$ itself is derived A-reflexive the complex A is said to be *semidualizing*. (The classical notion of *dualizing complex* includes the additional requirement that A be isomorphic, in $\mathsf{D}(X)$, to a bounded complex of injective sheaves.)

In Chapter 1 we prove basic results about semidualizing complexes in $\mathsf{D}(X)$, and examine their interplay with perfect complexes, that is, complexes $F \in \mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X)$ such that for every $x \in X$ the stalk F_x is isomorphic in $\mathsf{D}(\mathcal{O}_{X,x})$ to a bounded complex of flat $\mathcal{O}_{X,x}$-modules (or equivalently, such that F is isomorphic in $\mathsf{D}(X)$ to a bounded complex of flat $\mathcal{O}_X$-modules).

In Chapter 2 we explore conditions on a scheme-map $f: X \to Y$ that allow for the transfer of properties, like reflexivity, along standard functors $\mathsf{D}(Y) \to \mathsf{D}(X)$. These basic global notions turn out to be local, not only in the Zariski topology but also in the flat topology; that is, we find that they behave rather well under faithfully flat maps. (This opens the way to examination of more general sites, not undertaken here.)

One such condition involves the notion of *perfection relative to* f, defined for F in $\mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X)$ by replacing $\mathcal{O}_{X,x}$ with $\mathcal{O}_{Y,f(x)}$ in the definition of perfection. If this condition holds with $F = \mathcal{O}_X$, then f is said to be perfect (aka finite flat dimension, or finite tor-dimension). Flat maps are classical examples.

As a sample of results concerning ascent and descent along perfect maps, we quote from Theorem 2.2.5 and Corollary 2.2.6:

Theorem 1. *Let $f : X \to Y$ be a perfect map and B a complex in $\mathsf{D}^{+}_{\mathsf{c}}(Y)$.*

If $M \in \mathsf{D}(Y)$ is derived B-reflexive, then the complex $\mathsf{L}f^*M$ in $\mathsf{D}(X)$ is both derived $\mathsf{L}f^*B$-reflexive and derived $f^!B$-reflexive. For $M = \mathcal{O}_Y$ this says that if B is semidualizing then so are $\mathsf{L}f^*B$ and $f^!B$.

For each of these four statements, the converse holds if M and B are in $\mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(Y)$, and f is faithfully flat, or f is perfect, proper and surjective.

The perfection of f can be recognized by its *relative dualizing complex*, $f^!\mathcal{O}_Y$. Indeed, f is perfect if and only if $f^!\mathcal{O}_Y$ is relatively perfect. Furthermore, if f is perfect, then every perfect complex in $\mathsf{D}(X)$ is derived $f^!\mathcal{O}_Y$-reflexive.

In particular, when f is perfect the complex $f^!\mathcal{O}_Y$ is semidualizing. We take this condition as the definition of *G-perfect* maps. (Here G stands for Gorenstein.) In the affine case, these are just maps of finite G-dimension (see Example 2.3.6). They form a class significantly larger than that of perfect maps. For instance, when the scheme Y is Gorenstein *every* scheme map $X \to Y$ is G-perfect. In Section 2.3, we prove some basic properties of such maps, and, more generally, of $\mathcal{O}_X$-complexes that are derived $f^!\mathcal{O}_Y$-reflexive. For such complexes, called *G-perfect relative to* f, there exist nice dualities with respect to the relative dualizing complex (see Corollary 2.3.12).

Quasi-Gorenstein maps are defined by the condition that $f^!\mathcal{O}_Y$ is perfect. A very special case has been extensively studied: a *flat* map is quasi-Gorenstein if and only if all its fibers are Gorenstein schemes. On the other hand, *every* map of Gorenstein schemes is quasi-Gorenstein. Every quasi-Gorenstein map is G-perfect.

The conditions of relative perfection and G-perfection interact in many pleasing ways with composition and base change of scheme-maps, as explicated mainly in Section 2.5. Included there are a number of additional results about ascent and descent along perfect maps. Application to the case of structure sheaves produces facts, such as the following — all taken from Section 2.5, about the behavior of perfect, of G-perfect and of quasi-Gorenstein maps.

Theorem 2. *Let $Z \xrightarrow{g} X \xrightarrow{f} Y$ be scheme-maps, with g perfect.*

(i) *If f is perfect (resp. G-perfect) then so is fg. The converse holds if g is faithfully flat.*

(ii) *Suppose that fg is quasi-Gorenstein. Then so is g; and if g is faithfully flat, then also f is quasi-Gorenstein.*

Theorem 3. *Let $Z \xrightarrow{g} X \xrightarrow{f} Y$ be scheme-maps, with f quasi-Gorenstein.*

The composition fg is G-perfect if and only if so is g.

Also, if g is quasi-Gorenstein then so is fg.

Theorem 4. *Let there be given a fiber square, with u flat:*

$$\begin{array}{ccc} \bullet & \xrightarrow{\ v\ } & \bullet \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle f} \\ \bullet & \xrightarrow[\ u\ ]{} & \bullet \end{array}$$

(i) *If f is G-perfect then so is h. The converse holds if u is faithfully flat.*

(ii) *If f is quasi-Gorenstein then so is h. The converse holds if u is faithfully flat.*

In Chapter 3 we define *rigidity* with respect to an arbitrary semidualizing complex $A \in \mathsf{D}(X)$. An A-rigid structure on F in $\mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X)$ is a $\mathsf{D}(X)$-isomorphism

$$\rho : F \xrightarrow{\ \sim\ } \mathsf{R}\mathcal{H}om_X(\mathsf{R}\mathcal{H}om_X(F, A), F).$$

We say that (F, ρ) is an *A-rigid pair*; $F \in \mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X)$ is an *A-rigid complex* if such an isomorphism ρ exists. Morphisms of rigid pairs are defined in the obvious way.

In Theorem 3.1.7 we establish the basic fact about rigid pairs:

Theorem 5. *Let A be a semidualizing complex in $\mathsf{D}(X)$.*

For each quasicoherent $\mathcal{O}_X$-ideal I such that $I^2 = I$, there exists a canonical A-rigid structure on IA; and every A-rigid pair is uniquely isomorphic in $\mathsf{D}(X)$ to such an IA along with its canonical structure.

The theorem validates the term "rigid", as it implies that the only automorphism of a rigid pair is the identity. It also shows that isomorphism classes of A-rigid complexes correspond bijectively to the open-and-closed subsets of X. A more precise description — in terms of those subsets — of the skeleton of the category of rigid pairs appears in Theorem 3.2.6.

In the derived category, gluing over open coverings is usually not possible; but it is for idempotent ideals (Proposition C.8). Consequently the uniqueness expressed by Theorem 5 leads to gluing for rigid pairs, in the following strong sense:

Theorem 6. *For any open cover (U_α) of X and family (F_α, ρ_α) of $A|_{U_\alpha}$-rigid pairs such that for all α, α' the restrictions of (F_α, ρ_α) and $(F_{\alpha'}, \rho_{\alpha'})$ to $U_\alpha \cap U_{\alpha'}$ are isomorphic, there is a unique (up to unique isomorphism) A-rigid pair (F, ρ), such that for each α, $(F, \rho)|_{U_\alpha} \simeq (F_\alpha, \rho_\alpha)$.*

This gluing property holds even under the flat topology, see Theorem 3.2.9.

In Section 3.3 we study complexes that are *relatively rigid*, that is, rigid with respect to the relative dualizing complex $f^!\mathcal{O}_Y$ of a G-perfect map $f\colon X \to Y$ (a complex that is, by the definition of such maps, semidualizing). As a consequence of gluing for rigid complexes under the flat topology, gluing for relatively rigid complexes holds under the étale topology, see Proposition 3.3.1.

Relative rigidity behaves naturally with respect to (G-)perfect maps, in the sense that certain canonical isomorphisms from duality theory, involving relative dualizing complexes, respect the additional rigid structure. In Corollary 3.3.5 we show that, when g is perfect, the twisted inverse image functor $g^!$ preserves relative rigidity; and also, for a composition $Z \xrightarrow{g} X \xrightarrow{f} Y$ where f is G-perfect, we demonstrate the interaction of rigidity with the canonical isomorphism

$$g^!\mathcal{O}_X \otimes^{\mathsf{L}}_Z \mathsf{L}g^* f^!\mathcal{O}_Y \xrightarrow{\;\sim\;} (fg)^!\mathcal{O}_Y.$$

In Corollary 3.3.7 we do the same with respect to flat base change. Such results are obtained as applications of simple necessary and sufficient condition for additive functors of rigid complexes to be liftable to rigid pairs, detailed in Theorem 3.3.2.

The results above can be applied to complete some work started in [Avramov et al. 2010b]. In that paper, we associated a relative dualizing complex to each essentially-finite-type homomorphism of commutative rings, but did not touch upon the functoriality properties of that complex. This aspect of the construction can now be supplied by using the fact that the sheafification of the complex in [Avramov et al. 2010b] is a relative dualizing complex for the corresponding map of spectra; see Example 2.3.2. One can then use the results in Section 3.3, discussed above, to enrich the reduction isomorphism [Avramov et al. 2010b, 4.1] to a functorial one. For such applications, it is crucial to work with scheme-maps that are *essentially* of finite type; this is one of our reasons for choosing this category in the setup for this paper.

Notions and notation related to scheme-maps, as well as pertinent material from Grothendieck duality theory, as used in this paper, are surveyed in the Appendices.

We finish the introduction by reviewing connections to earlier work.

The results in Chapter 1 are, for the most part, extensions to the global situation of results proved over commutative rings in [Avramov et al. 2010a]; the transfer is fairly straightforward.

Homomorphisms of commutative noetherian rings that track Gorenstein-type properties were introduced and studied in [Avramov and Foxby 1992; Avramov and Foxby 1997; Iyengar and Sather-Wagstaff 2004], without finiteness hypotheses. Those papers are based on Auslander and Bridger's [1969] theory of Gorenstein dimension, which is defined in terms of resolutions by finite modules or projective modules, and so does not globalize. The scheme-maps defined and studied in Chapter 2 are based on a different description of finite Gorenstein dimension for ring-homomorphisms essentially of finite type, obtained in [Avramov et al. 2010a, 2.2].

The developments in Chapter 3 are largely motivated and inspired by work of Yekutieli and Zhang [2004; 2008; 2009] (see also [Yekutieli 2010]). One of their goals was to construct a new foundation for Grothendieck duality theory. Making extensive use of differential graded algebras (DGAs), Yekutieli and Zhang [2008; 2009] extended van den Bergh's construction [1997] of rigid *dualizing* complexes to schemes *essentially of finite type over a regular ring of finite Krull dimension*, and analyzed the behavior of such complexes under some types of perfect maps. Theirs is a novel approach, especially with regard to the introduction of DGAs into the subject. However, it remains to be seen whether, once all the details are fully exposed, it will prove to be simpler than the much more generally applicable theory presented, for example, in [Lipman 2009].

We come to rigidity from the opposite direction, presupposing duality theory and making no use of DGAs. The concept obtained in this way applies to *semi*dualizing complexes over arbitrary schemes, and behaves well under *all* perfect scheme-maps. In the setup of [Yekutieli and Zhang 2009], the regularity of the base ring implies that relative dualizing complexes are actually dualizing. To compare results, one also needs to know that, when both apply, our concept of rigidity coincides with Yekutieli and Zhang's. This follows from the Reduction Theorem [Avramov et al. 2010b, 4.1]; see [Avramov et al. 2010a, 8.5.5].

1. Derived reflexivity over schemes

Rings are assumed to be commutative, and both rings and schemes are assumed to be noetherian.

1.1. ***Standard homomorphisms.*** Let $(X, \mathscr{O}_X)$ be a scheme and $\mathsf{D}(X)$ the derived category of the category of sheaves of $\mathscr{O}_X$-modules.

Let $\mathsf{D}^+(X)$, resp. $\mathsf{D}^-(X)$, be the full subcategory of $\mathsf{D}(X)$ having as objects those complexes whose cohomology vanishes in all but finitely many negative, resp. positive, degrees; set $\mathsf{D}^{\mathsf{b}}(X) := \mathsf{D}^+(X) \cap \mathsf{D}^-(X)$. For $\bullet = +, -$ or b, let $\mathsf{D}^\bullet_{\mathsf{c}}(X)$, resp. $\mathsf{D}^\bullet_{\mathsf{qc}}(X)$, be the full subcategory of $\mathsf{D}(X)$ with objects those complexes all of whose cohomology sheaves are *coherent,* resp. *quasicoherent*.

To lie in $\mathsf{D}^\bullet_*(X)$ ($* = \mathsf{c}$ or qc, and $\bullet = +, -$ or b) is a *local condition*: if (U_α) is an open cover of X, then $F \in \mathsf{D}(X)$ lies in $\mathsf{D}^\bullet_*(X)$ if and only if for all α the restriction $F|_{U_\alpha}$ lies in$\mathsf{D}^\bullet_*(U_\alpha)$.

A number of canonical homomorphisms play a fundamental role in this paper.

Remark 1.1.1. There is a standard trifunctorial isomorphism, relating the derived tensor and sheaf-homomorphism functors (see e.g., [Lipman 2009, §2.6]):

$$\mathsf{R}\mathscr{H}om_X\big(E \otimes^{\mathsf{L}}_X F, G\big) \xrightarrow{\sim} \mathsf{R}\mathscr{H}om_X\big(E, \mathsf{R}\mathscr{H}om_X(F, G)\big) \quad (E, F, G \in \mathsf{D}(X)) \tag{1.1.1.1}$$

from which one gets, by application of the composite functor $\mathrm{H}^0\mathsf{R}\Gamma(X, -)$,

$$\mathrm{Hom}_{\mathsf{D}(X)}\big(E \otimes^{\mathsf{L}}_X F, G\big) \xrightarrow{\sim} \mathrm{Hom}_{\mathsf{D}(X)}\big(E, \mathsf{R}\mathscr{H}om_X(F, G)\big). \tag{1.1.1.2}$$

The map corresponding via (1.1.1.2) to the identity map of $\mathsf{R}\mathscr{H}om_X(F, G)$

$$\varepsilon = \varepsilon^F_G : \mathsf{R}\mathscr{H}om_X(F, G) \otimes^{\mathsf{L}}_X F \to G \qquad (F, G \in \mathsf{D}(X)) \tag{1.1.1.3}$$

is called *evaluation*. When F is a flat complex in $\mathsf{D}^-(X)$ (or more generally, any q-flat complex in $\mathsf{D}(X)$, see [Lipman 2009, §2.5]), and G is an injective complex in $\mathsf{D}^+(X)$ (or more generally, any q-injective complex in $\mathsf{D}(X)$, see [Lipman 2009, §2.3]), one verifies that ε is induced by the family of maps of complexes

$$\varepsilon(U) : \mathrm{Hom}_{\mathscr{O}_X(U)}(F(U), G(U)) \otimes_{\mathscr{O}_X(U)} F(U) \to G(U) \qquad (U \subseteq X \text{ open})$$

where, for homogeneous $\alpha \in \mathrm{Hom}_{\mathscr{O}_X(U)}(F(U), G(U))$ and $b \in F(U)$,

$$\varepsilon(U)(\alpha \otimes b) = \alpha(b).$$

Basic properties of supports of complexes are recalled for further reference.

Remark 1.1.2. For any $F \in \mathsf{D}(X)$, the *support of* F is the set

$$\operatorname{Supp}_X F := \{\, x \in X \mid H^n(F_x) \neq 0 \text{ for some } n \,\}. \tag{1.1.2.1}$$

If $F \in \mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X)$, then $\operatorname{Supp}_X F$ is a *closed subset* of X. Also, for all F and G in $\mathsf{D}^{-}_{\mathsf{c}}(X)$, it follows from, e.g., [Avramov et al. 2010a, A.6] that

$$\operatorname{Supp}_X(F \otimes^{\mathsf{L}}_X G) = \operatorname{Supp}_X F \cap \operatorname{Supp}_X G. \tag{1.1.2.2}$$

Note that $\operatorname{Supp}_X F = \varnothing$ if and only if $F = 0$ in $\mathsf{D}(X)$.

The following example opens the door to applications of the results in [Avramov et al. 2010a].

Example 1.1.3. Let R be a ring. Let $\mathsf{D}(R)$ be the derived category of the category of R-modules, and define, as above, its full subcategories $\mathsf{D}^{\bullet}(R)$ for $\bullet = +$, $-$ or b. Let $\mathsf{D}^{\bullet}_{\mathsf{f}}(R)$ be the full subcategory of $\mathsf{D}^{\bullet}(R)$ having as objects those complexes whose cohomology modules are all *finite*, i.e., finitely generated, over R.

For the affine scheme $X = \operatorname{Spec} R$, the functor that associates to each complex $M \in \mathsf{D}(R)$ its sheafification $M^{\sim}$ is an *equivalence of categories* $\mathsf{D}^{\bullet}_{\mathsf{f}}(R) \xrightarrow{\approx} \mathsf{D}^{\bullet}_{\mathsf{c}}(X)$, see [Bökstedt and Neeman 1993, 5.5]; when $\bullet = +$ or b, see also [Hartshorne 1966, p. 133, 7.19].

There is a natural bifunctorial isomorphism

$$(M \otimes^{\mathsf{L}}_R N)^{\sim} \xrightarrow{\ \sim\ } M^{\sim} \otimes^{\mathsf{L}}_X N^{\sim} \qquad \big(M,\ N \in \mathsf{D}(R)\big); \tag{1.1.3.1}$$

to define it one may assume that M and N are suitable flat complexes, so that $\otimes^{\mathsf{L}}$ becomes ordinary $\otimes$, see [Lipman 2009, §2.5 and (2.6.5)].

There is also a natural bifunctorial map

$$\mathrm{RHom}_R(M, N)^{\sim} \longrightarrow \mathsf{R}\mathscr{H}om_X\big(M^{\sim}, N^{\sim}\big), \tag{1.1.3.2}$$

defined to be the one that corresponds via (1.1.1.2) to the composite map

$$\mathrm{RHom}_R(M, N)^{\sim} \otimes^{\mathsf{L}}_X M^{\sim} \xrightarrow{\ \sim\ } (\mathrm{RHom}_R(M, N) \otimes^{\mathsf{L}}_R M)^{\sim} \xrightarrow{\ \varepsilon^{\sim}\ } N^{\sim},$$

where the isomorphism comes from (1.1.3.1), and the *evaluation map* ε corresponds to the identity map of $\mathrm{RHom}_R(M, N)$ via the analog of (1.1.1.2) over $\mathsf{D}(R)$.

The map (1.1.3.2) is an *isomorphism* if $M \in \mathsf{D}^{-}_{\mathsf{f}}(R)$ and $N \in \mathsf{D}^{+}(R)$. To show this for variable M and fixed N one can use the "way-out" Lemma [Hartshorne 1966, p. 68, 7.1], with A the opposite of the category of R-modules and P the family $(R^n)_{n>0}$, to reduce to the case $M = R$, where, one checks, the map is the obvious isomorphism.

1.2. ***Derived multiplication by global functions.*** Let $(X, \mathcal{O}_X)$ be a scheme. Here we discuss some technicalities about the natural action of $\mathrm{H}^0(X, \mathcal{O}_X)$ on $\mathsf{D}(X)$.

We identify $\mathrm{H}^0(X, \mathcal{O}_X)$ with $\mathrm{Hom}_{\mathsf{D}(X)}(\mathcal{O}_X, \mathcal{O}_X)$ via the ring isomorphism that takes $\alpha \in \mathrm{H}^0(X, \mathcal{O}_X)$ to multiplication by α. For $\alpha \in \mathrm{H}^0(X, \mathcal{O}_X)$ and $F \in \mathsf{D}(X)$, let $\mu_F(\alpha)$ ("multiplication by α in F") be the natural composite $\mathsf{D}(X)$-map

$$F \simeq \mathcal{O}_X \otimes^{\mathsf{L}}_X F \xrightarrow{\alpha \otimes^{\mathsf{L}}_X 1} \mathcal{O}_X \otimes^{\mathsf{L}}_X F \simeq F,$$

or equivalently,

$$F \simeq F \otimes^{\mathsf{L}}_X \mathcal{O}_X \xrightarrow{1 \otimes^{\mathsf{L}}_X \alpha} F \otimes^{\mathsf{L}}_X \mathcal{O}_X \simeq F.$$

Clearly, for any $\mathsf{D}(X)$-map $\phi\colon F \to C$,

$$\phi\alpha := \phi \circ \mu_F(\alpha) = \mu_C(\alpha) \circ \phi =: \alpha\phi.$$

Furthermore, using the obvious isomorphism $(\mathcal{O}_X \otimes^{\mathsf{L}}_X F)[1] \xrightarrow{\sim} \mathcal{O}_X \otimes^{\mathsf{L}}_X F[1]$ one sees that $\mu_F(\alpha)$ commutes with translation, that is, $\mu_F(\alpha)[1] = \mu_{F[1]}(\alpha)$.

Thus, the family $(\mu_F)_{F \in \mathsf{D}(X)}$ maps $\mathrm{H}^0(X, \mathcal{O}_X)$ into the ring C_X consisting of endomorphisms of the identity functor of $\mathsf{D}(X)$ that commute with translation—the *center* of $\mathsf{D}(X)$. It is straightforward to verify that this map is an injective ring homomorphism onto the subring of *tensor-compatible* members of C_X, that is, those $\eta \in \mathsf{C}_X$ such that for all $F, G \in \mathsf{D}(X)$,

$$\eta(F \otimes^{\mathsf{L}}_X G) = \eta(F) \otimes^{\mathsf{L}}_X \mathrm{id}^G = \mathrm{id}^G \otimes^{\mathsf{L}}_X \eta(G).$$

The category $\mathsf{D}(X)$ is C_X-linear: for all F, $G \in \mathsf{D}(X)$, $\mathrm{Hom}_{\mathsf{D}(X)}(F, G)$ has a natural structure of C_X-module, and composition of maps is C_X-bilinear. So $\mathsf{D}(X)$ is also $\mathrm{H}^0(X, \mathcal{O}_X)$-linear, via μ.

Lemma 1.2.1. *For any $F, G \in \mathsf{D}(X)$ and $\mathsf{D}(X)$-homomorphism $\alpha\colon \mathcal{O}_X \to \mathcal{O}_X$, and $\mu_\bullet(\alpha)$ as above, there are equalities*

$$\mathsf{R}\mathcal{H}om_X(\mu_F(\alpha), G) = \mu_{\mathsf{R}\mathcal{H}om_X(F,G)}(\alpha) = \mathsf{R}\mathcal{H}om_X(F, \mu_G(\alpha))).$$

Proof. Consider, for any $E \in \mathsf{D}(X)$, the natural trifunctorial isomorphism

$$\tau\colon \mathrm{Hom}_{\mathsf{D}(X)}(E \otimes^{\mathsf{L}}_X F, G) \xrightarrow{\sim} \mathrm{Hom}_{\mathsf{D}(X)}(E, \mathsf{R}\mathcal{H}om_X(F, G)).$$

From tensor-compatibility in the image of μ, and $\mathrm{H}^0(X, \mathcal{O}_X)$-linearity of $\mathsf{D}(X)$, it follows that for any $\alpha \in \mathrm{H}^0(X, \mathcal{O}_X)$, the map $\mu_E(\alpha)$ induces multiplication by α in both the source and target of τ. Functoriality shows then that τ is an isomorphism of $\mathrm{H}^0(X, \mathcal{O}_X)$-modules.

Again, tensor-compatibility implies that $\mu_F(\alpha)$ induces multiplication by α in the source of the $\mathrm{H}^0(X, \mathcal{O}_X)$-linear map τ, hence also in the target. Thus, by functoriality, $\mathsf{R}\mathcal{H}om_X(\mu_F(\alpha), G)$ induces multiplication by α in the target of τ. For $E = \mathsf{R}\mathcal{H}om_X(F, G)$, this gives $\mathsf{R}\mathcal{H}om_X(\mu_F(\alpha), G) = \mu_{\mathsf{R}\mathcal{H}om_X(F,G)}(\alpha)$. One shows similarly that $\mathsf{R}\mathcal{H}om_X(F, \mu_G(\alpha)) = \mu_{\mathsf{R}\mathcal{H}om_X(F,G)}(\alpha)$. $\square$

1.3. ***Derived reflexivity.*** Let $(X, \mathcal{O}_X)$ be a scheme.

One has, for all A and F in $\mathsf{D}(X)$, a *biduality morphism*

$$\delta_F^A : F \to \mathsf{R}\mathcal{H}om_X\big(\mathsf{R}\mathcal{H}om_X(F, A), A\big),$$

corresponding via (1.1.1.2) to the natural composition

$$F \otimes_X^{\mathsf{L}} \mathsf{R}\mathcal{H}om_X(F, A) \xrightarrow{\ \sim\ } \mathsf{R}\mathcal{H}om_X(F, A) \otimes_X^{\mathsf{L}} F \xrightarrow{\varepsilon_A^F} A.$$

The map δ_F^A "commutes" with restriction to open subsets (use [Lipman 2009, 2.4.5.2]).

When A is a q-injective complex in $\mathsf{D}(X)$, δ_F^A is induced by the family

$$\delta(U) : F(U) \to \mathrm{Hom}_{\mathcal{O}_X(U)}(\mathrm{Hom}_{\mathcal{O}_X(U)}(F(U), A(U)), A(U)) \qquad (U \subseteq X \text{ open})$$

of maps of complexes, where, for each $n \in F(U)$ of degree b, the map $\delta(U)(n)$ is

$$\alpha \mapsto (-1)^{ab}\alpha(n)\,,$$

for $\alpha \in \mathrm{Hom}_{\mathcal{O}_X(U)}(F(U), A(U))$ homogeneous of degree a.

Definition 1.3.1. Given A and F in $\mathsf{D}(X)$, we say that F is *derived A-reflexive* if both F and $\mathsf{R}\mathcal{H}om_R(F, A)$ are in $\mathsf{D}_{\mathsf{c}}^{\mathsf{b}}(X)$ and δ_F^A is an isomorphism.

This is a *local condition*: for any open cover (U_α) of X, F is derived A-reflexive if and only if the same is true over every U_α for the restrictions of F and A. Also, as indicated below, if U is affine, say $U := \operatorname{Spec} R$, and $C, M \in \mathsf{D}_{\mathsf{f}}^{\mathsf{b}}(R)$, then

$$M^\sim \text{ is derived } C^\sim\text{-reflexive in } \mathsf{D}(U) \iff M \text{ is derived } C\text{-reflexive in } \mathsf{D}(R).$$

Example 1.3.2. When $X = \operatorname{Spec} R$ and $M, C \in \mathsf{D}(R)$, it follows that with δ_M^C as in [Avramov et al. 2010a, (2.0.1)], the map $\delta_{M^\sim}^{C^\sim}$ factors naturally as

$$M^\sim \xrightarrow{(\delta_M^C)^\sim} \big(\mathsf{R}\mathrm{Hom}_R(\mathsf{R}\mathrm{Hom}_R(M, C), C)\big)^\sim \xrightarrow{\ s\ } \mathsf{R}\mathcal{H}om_X\big(\mathsf{R}\mathcal{H}om_X(M^\sim, C^\sim), C^\sim\big),$$

where, as in (1.1.3.2), the map s is an isomorphism if $M \in \mathsf{D}_{\mathsf{f}}^{-}(R)$, $C \in \mathsf{D}^{+}(R)$ and $\mathsf{R}\mathrm{Hom}_R(M, C) \in \mathsf{D}_{\mathsf{f}}^{\mathsf{b}}(R)$. Thus, derived reflexivity globalizes the notion in [Avramov et al. 2010a, §2].

From [Avramov et al. 2010a, 2.1 and 2.3] one now gets:

Proposition 1.3.3. *Let X be a noetherian scheme, and let $A, F \in \mathsf{D}_{\mathsf{c}}^{\mathsf{b}}(X)$. Then the following conditions are equivalent.*

(i) *F is derived A-reflexive.*

(ii) $\mathsf{R}\mathcal{H}om_X(F, A) \in \mathsf{D}^{-}(X)$ *and there exists an isomorphism in* $\mathsf{D}(X)$

$$F \xrightarrow{\ \sim\ } \mathsf{R}\mathcal{H}om_X\big(\mathsf{R}\mathcal{H}om_X(F, A), A\big).$$

(iii) $\mathsf{R}\mathcal{H}om_X(F, A)$ *is derived A-reflexive and* $\operatorname{Supp}_X F \subseteq \operatorname{Supp}_X A$. $\square$

Remark 1.3.4. For $A = \mathcal{O}_X$ the theorem above shows that $F \in \mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X)$ is derived $\mathcal{O}_X$-reflexive if and only if so is $\mathsf{R}\mathcal{H}om_X(F, \mathcal{O}_X)$.

In the affine case, $X = \operatorname{Spec} R$, for any $M \in \mathsf{D}^{\mathsf{b}}_{\mathsf{f}}(R)$, the derived $\mathcal{O}_X$-reflexivity of $M^{\sim}$ is equivalent to finiteness of the Gorenstein dimension of M, as defined by Auslander and Bridger [1969].

Definition 1.3.5. An $\mathcal{O}_X$-complex A is *semidualizing* if $\mathcal{O}_X$ is derived A-reflexive. In other words, $A \in \mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X)$ and the map $\chi^A : \mathcal{O}_X \to \mathsf{R}\mathcal{H}om_X(A, A)$ corresponding via (1.1.1.2) to the natural map $\mathcal{O}_X \otimes^{\mathsf{L}}_X A \to A$ is an isomorphism.

As above, this condition is local on X. When $X = \operatorname{Spec} R$, a complex $C \in \mathsf{D}^{\mathsf{b}}_{\mathsf{f}}(R)$ is semidualizing in the commutative-algebra sense (that is, R is derived C-reflexive; see, e.g., [Avramov et al. 2010a, §3]) if and only if $C^{\sim}$ is semidualizing in the present sense.

Lemma 1.3.6. *If $A \in \mathsf{D}(X)$ is semidualizing then each $\mathsf{D}(X)$-endomorphism of A is multiplication by a uniquely determined $\alpha \in \mathrm{H}^0(X, \mathcal{O}_X)$.*

Proof. With $\chi^A : \mathcal{O}_X \to \mathsf{R}\mathcal{H}om_X(A, A)$ as in Definition 1.3.5, the map μ_A is easily seen to factor as follows:

$$\begin{aligned}\operatorname{Hom}_{\mathsf{D}(X)}(\mathcal{O}_X, \mathcal{O}_X) \xrightarrow{\text{via } \chi^A} &\operatorname{Hom}_{\mathsf{D}(X)}\big(\mathcal{O}_X, \mathsf{R}\mathcal{H}om_X(A, A)\big)\\ &\cong \operatorname{Hom}_{\mathsf{D}(X)}(\mathcal{O}_X \otimes^{\mathsf{L}}_X A, A)\\ &\cong \operatorname{Hom}_{\mathsf{D}(X)}(A, A).\end{aligned}$$

The assertion results. $\square$

Lemma 1.3.7. *Let X be a noetherian scheme. If A is a semidualizing $\mathcal{O}_X$-complex, then* $\operatorname{Supp}_X A = X$. *Furthermore, if there is an isomorphism $A \simeq A_1 \oplus A_2$ then* $\operatorname{Supp}_X A_1 \cap \operatorname{Supp}_X A_2 = \varnothing$.

Proof. The $\mathcal{O}_X$-complex $\mathsf{R}\mathcal{H}om_X(A, A)$, which is isomorphic in $\mathsf{D}(X)$ to $\mathcal{O}_X$, is acyclic over the open set $X \setminus \operatorname{Supp}_X A$. This implies $\operatorname{Supp}_X A = X$.

As to the second assertion, taking stalks at arbitrary $x \in X$ reduces the problem to showing that if R is a local ring, and M_1 and M_2 in $\mathsf{D}(R)$ are such that the natural map

$$R \to \operatorname{RHom}_R(M_1 \oplus M_2, M_1 \oplus M_2) = \bigoplus\nolimits_{i,j=1}^{2} \operatorname{RHom}_R(M_i, M_j)$$

is an isomorphism, then either $M_1 = 0$ or $M_2 = 0$.

But clearly, R being local, at most one of the direct summands $\operatorname{RHom}_R(M_i, M_j)$ can be nonzero, so for $i = 1$ or $i = 2$ the identity map of M_i is 0, whence the conclusion. $\square$

1.4. ***Perfect complexes.*** Again, $(X, \mathcal{O}_X)$ is a scheme.

Definition 1.4.1. An $\mathcal{O}_X$-complex P is *perfect* if X is a union of open subsets U such that the restriction $P|_U$ is $\mathsf{D}(U)$-isomorphic to a bounded complex of finite-rank locally free $\mathcal{O}_U$-modules.

From [Illusie 1971, p. 115, 3.5 and p. 135, 5.8.1], one gets:

Remark 1.4.2. The complex P is perfect if and only if $P \in \mathsf{D_c}(X)$ and P is isomorphic in $\mathsf{D}(X)$ to a bounded complex of flat $\mathcal{O}_X$-modules.

Perfection is a local condition. If $X = \operatorname{Spec} R$ and $M \in \mathsf{D}(R)$ then $M^\sim$ is perfect if and only if N is isomorphic in $\mathsf{D}(R)$ to a bounded complex of finite projective R-modules; cf. [Avramov et al. 2010a, §4]. The next result is contained in [Christensen 2000, 2.1.10]; see also [Avramov et al. 2010a, 4.1].

Theorem 1.4.3. *$P \in \mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X)$ is perfect if and only if so is $\mathsf{R}\mathcal{H}om_X(P, \mathcal{O}_X)$.* □

Proposition 1.4.4. *Let A and P be in $\mathsf{D}(X)$, with P perfect.*

If $F \in \mathsf{D}(X)$ is derived A-reflexive then so is $P \otimes^{\mathsf{L}}_X F$; in particular, P is derived $\mathcal{O}_X$-reflexive. If A is semidualizing then P is derived A-reflexive.

Proof. The assertion being local, we may assume that P is a bounded complex of finite-rank free $\mathcal{O}_X$-modules. If two vertices of a triangle are derived A-reflexive then so is the third, whence an easy induction on the number of degrees in which P is nonzero shows that if F is A-reflexive then so is $P \otimes^{\mathsf{L}}_X F$. To show that P is derived $\mathcal{O}_X$-reflexive, take $A = \mathcal{O}_X = F$.

For the final assertion, take $F = \mathcal{O}_X$. □

A partial converse is given by the next result:

Theorem 1.4.5. *Let $F \in \mathsf{D_c}(X)$, let $A \in \mathsf{D}^{+}_{\mathsf{c}}(X)$, and let P be a perfect $\mathcal{O}_X$-complex with* $\operatorname{Supp}_X P \supseteq \operatorname{Supp}_X F$.*If $P \otimes^{\mathsf{L}}_X F$ is in $\mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X)$, or $P \otimes^{\mathsf{L}}_X F$ is perfect, or $P \otimes^{\mathsf{L}}_X F$ is derived A-reflexive, then the corresponding property holds for F.*

Proof. The assertions are all local, and the local statements are proved in [Avramov et al. 2010a, 4.3, 4.4, and 4.5], modulo sheafification; see Example 1.1.3. □

We'll need the following isomorphisms, for which cf. [Illusie 1971, pp. 152–153, 7.6 and 7.7].

Let E, F and G be complexes in $\mathsf{D}(X)$, and consider the map

$$\mathsf{R}\mathcal{H}om_X(E,\, F) \otimes^{\mathsf{L}}_X G \to \mathsf{R}\mathcal{H}om_X(E,\, F \otimes^{\mathsf{L}}_X G), \tag{1.4.5.1}$$

corresponding via (1.1.1.2) to the natural composition

$$(\mathsf{R}\mathcal{H}om_X(E,\, F) \otimes^{\mathsf{L}}_X G) \otimes^{\mathsf{L}}_X E \xrightarrow{\ \sim\ } (\mathsf{R}\mathcal{H}om_X(E,\, F) \otimes^{\mathsf{L}}_X E) \otimes^{\mathsf{L}}_X G \xrightarrow{\varepsilon \otimes^{\mathsf{L}}_X 1} F \otimes^{\mathsf{L}}_X G.$$

where ε is the evaluation map from (1.1.1.3).

Lemma 1.4.6. *Let E, F and G be complexes in $\mathsf{D}(X)$.*

(1) *When either E or G is perfect, the map (1.4.5.1) is an isomorphism*

$$\mathsf{R}\mathcal{H}om_X(E, F) \otimes^{\mathsf{L}}_X G \xrightarrow{\sim} \mathsf{R}\mathcal{H}om_X(E, F \otimes^{\mathsf{L}}_X G).$$

(2) *When G is perfect, there is a natural isomorphism*

$$\mathsf{R}\mathcal{H}om_X(E \otimes^{\mathsf{L}}_X G, F) \xrightarrow{\sim} \mathsf{R}\mathcal{H}om_X(E, F) \otimes^{\mathsf{L}}_X \mathsf{R}\mathcal{H}om_X(G, \mathcal{O}_X).$$

Proof. (1). Whether the map (1.4.5.1) is an isomorphism is a local question, so if E is perfect then one may assume that E is a bounded complex of finite-rank free $\mathcal{O}_X$-modules. The affirmative answer is then given by a simple induction on the number of degrees in which E is nonzero.

A similar argument applies when G is perfect.

(2). Setting $\check{G} := \mathsf{R}\mathcal{H}om_X(G, \mathcal{O}_X)$, we get from (1), with (E, F, G) changed to $(G, \mathcal{O}_X, F)$, an isomorphism

$$F \otimes^{\mathsf{L}}_X \check{G} \simeq \check{G} \otimes^{\mathsf{L}}_X F \xrightarrow{\sim} \mathsf{R}\mathcal{H}om_X(G, F).$$

This induces the second isomorphism below:

$$\begin{aligned}
\mathsf{R}\mathcal{H}om_X(E \otimes^{\mathsf{L}}_X G, F) &\xrightarrow{\sim} \mathsf{R}\mathcal{H}om_X(E, \mathsf{R}\mathcal{H}om_X(G, F))\\
&\xrightarrow{\sim} \mathsf{R}\mathcal{H}om_X(E, F \otimes^{\mathsf{L}}_X \check{G})\\
&\xrightarrow{\sim} \mathsf{R}\mathcal{H}om_X(E, F) \otimes^{\mathsf{L}}_X \check{G};
\end{aligned}$$

the first isomorphism comes from (1.1.1.1) and the third from (1), since $\check{G}$ is also perfect, by Theorem 1.4.3. The desired map is the composite isomorphism. □

1.5. *Invertible complexes.* Again, $(X, \mathcal{O}_X)$ is a scheme.

Definition 1.5.1. A complex in $\mathsf{D}(X)$ is *invertible* if it is semidualizing and perfect.

This condition is local. If $X = \operatorname{Spec} R$ and $M \in \mathsf{D}(R)$, then M is invertible in the sense of [Avramov et al. 2010a, §5] if and only if $M^{\sim}$ is invertible in the present sense.

Recall that Σ denotes the usual translation (suspension) operator on complexes.

Theorem 1.5.2. *For $L \in \mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X)$ the following conditions are equivalent.*

(i) *L is invertible.*

(ii) *$L^{-1} := \mathsf{R}\mathcal{H}om_X(L, \mathcal{O}_X)$ is invertible.*

(iii) *Each $x \in X$ has an open neighborhood U_x such that for some integer r_x, there is a $\mathsf{D}(U_x)$-isomorphism $L|_{U_x} \simeq \Sigma^{r_x}\mathcal{O}_{U_x}$.*

(iii′) *For each connected component U of X there is an integer r, a locally free rank-one $\mathcal{O}_U$-module $\mathcal{L}$, and a $\mathsf{D}(U)$-isomorphism $L|_U \simeq \Sigma^r\mathcal{L}$.*

(iv) *For some* $F \in \mathsf{D_c}(X)$ *there is an isomorphism* $F \otimes^{\mathsf{L}}_X L \simeq \mathcal{O}_X$.

(v) *For all* $G \in \mathsf{D}(X)$ *the evaluation map* ε *from* (1.1.1.3) *is an isomorphism*

$$\mathsf{R}\mathcal{H}om_X(L, G) \otimes^{\mathsf{L}}_X L \xrightarrow{\sim} G.$$

(v′) *For all* G *and* G' *in* $\mathsf{D}(X)$, *the natural composite map* (see (1.1.1.1))

$$\begin{aligned}\mathsf{R}\mathcal{H}om_X(G' \otimes^{\mathsf{L}}_X L, G) \otimes^{\mathsf{L}}_X L &\xrightarrow{\sim} \mathsf{R}\mathcal{H}om_X(L \otimes^{\mathsf{L}}_X G', G) \otimes^{\mathsf{L}}_X L\\ &\xrightarrow{\sim} \mathsf{R}\mathcal{H}om_X(L, \mathsf{R}\mathcal{H}om_X(G', G)) \otimes^{\mathsf{L}}_X L\\ &\xrightarrow[\varepsilon]{} \mathsf{R}\mathcal{H}om_X(G', G)\end{aligned}$$

is an isomorphism.

Proof. When (i) holds, Lemma 1.4.6(2), with $E = \mathcal{O}_X$ and $G = L = F$, yields:

$$\mathcal{O}_X \xrightarrow{\sim} \mathsf{R}\mathcal{H}om_X(L, L) \xrightarrow{\sim} L \otimes^{\mathsf{L}}_X L^{-1}. \tag{1.5.2.1}$$

(i) ⇔ (ii). By Theorem 1.4.3, the $\mathcal{O}_X$-complex L is perfect if and only if so is L^{-1}. If (i) holds, then (1.5.2.1), Proposition 1.4.4 (with $A = \mathcal{O}_X = F$, $P = L$), and Lemma 1.4.6(1) give isomorphisms

$$\mathcal{O}_X \xrightarrow{\sim} L \otimes^{\mathsf{L}}_X L^{-1} \xrightarrow{\sim} \mathsf{R}\mathcal{H}om_X(L^{-1}, \mathcal{O}_X) \otimes^{\mathsf{L}}_X L^{-1} \xrightarrow{\sim} \mathsf{R}\mathcal{H}om_X(L^{-1}, L^{-1}),$$

so that by Proposition 1.3.3(ii) (with $F = \mathcal{O}_X$ and $A = L^{-1}$), the $\mathcal{O}_X$-module L^{-1} is semidualizing; since it also perfect (ii) holds.

The same argument with L and L^{-1} interchanged establishes that (ii) ⇒ (i).

(i) ⇒ (iii). One may assume here that X is affine. Then, since L is invertible, §5.1 of [Avramov et al. 2010a] gives that the stalk at x of the cohomology of L vanishes in all but one degree, where it is isomorphic to $\mathcal{O}_{X,x}$. The cohomology of L is bounded and coherent, therefore there is an open neighborhood U_x of x over which the cohomology of L vanishes in all but one degree, where it is isomorphic to $\mathcal{O}_{U_x}$, i.e., (iii) holds.

(iii) ⇒ (iv). If (iii) holds then the evaluation map (1.1.1.3) (with $A = L$ and $G = \mathcal{O}_X$) is an isomorphism $L^{-1} \otimes^{\mathsf{L}}_X L \xrightarrow{\sim} \mathcal{O}_X$.

(iv) ⇒ (i). This is a local statement that is established (along with some other unstated equivalences) in [Avramov et al. 2010a, 5.1]; see also [Frankild et al. 2009, 4.7].

(iii) ⇒ (iii′). The function $x \mapsto r_x$ must be locally constant, so of constant value, say r, on U; and then in $\mathsf{D}(U)$, $L \simeq \Sigma^r H^{-r}(L)$.

(iii′) ⇒ (iii). This implication is clear.

(i) ⇒ (v). The first of the following isomorphisms comes from Lemma 1.4.6(2) (with $(E, F, G) = (L, G, \mathcal{O}_X)$), and the second from (1.5.2.1):

$$\mathsf{R}\mathcal{H}om_X(L, G) \otimes^{\mathsf{L}}_X L \xrightarrow{\sim} L^{-1} \otimes^{\mathsf{L}}_X G \otimes^{\mathsf{L}}_X L \xrightarrow{\sim} G.$$

That this composite isomorphism is ε is essentially the definition of the isomorphism

$$L^{-1} \otimes^{\mathsf{L}}_X G = \mathsf{R}\mathscr{H}om_X(L, \mathscr{O}_X) \otimes^{\mathsf{L}}_X G \xrightarrow{\sim} \mathsf{R}\mathscr{H}om_X(L, G)\,;$$

see the proof of Lemma 1.4.6.

(v) $\Rightarrow$ (iv). Set $F := L^{-1}$, and apply (v) for $G = \mathscr{O}_X$.

(v) $\Leftrightarrow$ (v$'$). Replace G in (v) by $\mathsf{R}\mathscr{H}om_X(G', G)$; or G' in (v$'$) by $\mathscr{O}_X$. □

Corollary 1.5.3. *Let L_1 and L_2 be complexes in $\mathsf{D_c}(X)$.*

(1) *If L_1 and L_2 are invertible, then so is $L_1 \otimes^{\mathsf{L}}_X L_2$.*

(2) *If L_1 is in $\mathsf{D^b_c}(X)$ and $L_1 \otimes^{\mathsf{L}}_X L_2$ is invertible, then L_1 is invertible.*

(3) *For any scheme-map $g : Z \to X$, if L_1 is invertible then so is $\mathsf{L}g^*L_1$.*

Proof. For (1), use Theorem 1.5.2(iii$'$); for (2), Theorem 1.5.2(iv) — noting that the F there may be taken to be the invertible complex L^{-1}, and that tensoring with an invertible complex takes $\mathsf{D_c}(X)$ into itself; and for (3), the fact that g maps any connected component of Z into a connected component of X. □

Corollary 1.5.4. *Let A, L and F be complexes in $\mathsf{D^b_c}(X)$, with L invertible.*

(1) *F is derived A-reflexive if and only if it is derived $L \otimes^{\mathsf{L}}_X A$-reflexive.*

(2) *F is derived A-reflexive if and only if $F \otimes^{\mathsf{L}}_X L$ is derived A-reflexive.*

(3) *A is semidualizing if and only if $L \otimes^{\mathsf{L}}_X A$ is semidualizing.*

Proof. From, say, Theorem 1.5.2(iii$'$) and Lemma 1.4.6(1), one gets

$$\mathsf{R}\mathscr{H}om_X(F, A) \in \mathsf{D^b_c}(X) \iff \mathsf{R}\mathscr{H}om_X(F, L \otimes^{\mathsf{L}}_X A) \in \mathsf{D^b_c}(X).$$

Since $L^{-1} \otimes^{\mathsf{L}}_X L \simeq \mathscr{O}_X$, (1) follows now from Lemma 1.4.6; (2) follows from Theorem 1.5.2(iii); and (3) follows from (1). □

Remark 1.5.5. A complex $A \in \mathsf{D^b_c}(X)$ is *pointwise dualizing* if every $F \in \mathsf{D^b_c}(X)$ is derived A-reflexive (see [Avramov et al. 2010a, 6.2.2]). Such an A is semidualizing: take $F = \mathscr{O}_X$.

It is proved in [Avramov et al. 2010a, 8.3.1] that $\mathscr{O}_X$ *is pointwise dualizing if and only if X is a Gorenstein scheme* (i.e., the local ring $\mathscr{O}_{X,x}$ is Gorenstein for all $x \in X$).

It follows from [Avramov et al. 2010a, 5.7] that invertible complexes can be characterized as those that are semidualizing and derived $\mathscr{O}_X$-reflexive. Hence *when X is Gorenstein,*

$$A \in \mathsf{D^b_c}(X) \textit{ is semidualizing} \iff A \textit{ is pointwise dualizing} \iff A \textit{ is invertible.}$$

2. Gorenstein-type properties of scheme-maps

All schemes are assumed to be noetherian; all scheme-maps are assumed to be essentially of finite type (see Appendix A) *and separated.*

2.1. *Perfect maps.* Let $f : X \to Y$ be a scheme-map.

Let $f_0 : X \to Y$ denote the underlying map of topological spaces, and f_0^{-1} the left adjoint of the direct image functor f_{0*} from sheaves of abelian groups on X to sheaves of abelian groups on Y. There is then a standard way of making $f_0^{-1}\mathcal{O}_Y$ into a sheaf of commutative rings on X, whose stalk at any point $x \in X$ is $\mathcal{O}_{Y,\,f(x)}$.

Definition 2.1.1. An $\mathcal{O}_X$-complex F is *perfect relative to* f — or, as we will write, *perfect over* f — if it is in $\mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X)$, and in the derived category of the category of $f_0^{-1}\mathcal{O}_Y$-modules F is isomorphic to a bounded complex of flat $f_0^{-1}\mathcal{O}_Y$-modules.

The map f is *perfect* if $\mathcal{O}_X$ is perfect over f [Illusie 1971, p. 250, définition 4.1].

Perfection over id^X is equivalent to perfection in $\mathsf{D}(X)$; see Remark 1.4.2.

Remark 2.1.2. Using [Illusie 1971, p. 242, 3.3], one sees that perfection over f is local on X, in the sense that F has this property if and only if every $x \in X$ has an open neighborhood U such that $F|_U$ is perfect over $f|_U$.

Since f is essentially of finite type, there is always such a U for which $f|U$ factors as (essentially smooth) $\circ$ (closed immersion). If $X \xrightarrow{i} W \to Y$ is such a factorization, then F is perfect over f if and only if i_*F is perfect over id^W: the proof of [Illusie 1971, pp. 252, 4.4] applies here (see Remark A.3).

Remark 2.1.3. Each complex that is perfect over f is derived $f^!\mathcal{O}_Y$-reflexive. In particular, if the map f is perfect, then $\mathcal{O}_X$ is derived $f^!\mathcal{O}_Y$-reflexive.

This is given by [Illusie 1971, p. 259, 4.9.2], in whose proof "smooth" can be replaced by "essentially smooth"; see [Avramov et al. 2010b, 5.1].

Let $\mathsf{P}(f)$ be the full subcategory of $\mathsf{D}(X)$ whose objects are all the complexes that are perfect over f; and let $\mathsf{P}(X) := \mathsf{P}(\mathrm{id}^X)$ be the full subcategory of $\mathsf{D}(X)$ whose objects are all the perfect $\mathcal{O}_X$-complexes.

Example 2.1.4. If the map $f : X = \operatorname{Spec} S \to \operatorname{Spec} K = Y$ corresponds to a homomorphism of noetherian rings $\sigma : K \to S$, then $\mathsf{P}(f)$ is equivalent to the full subcategory $\mathsf{P}(\sigma) \subseteq \mathsf{D}^{\mathsf{b}}_{\mathsf{f}}(S)$ with objects those complexes M that are isomorphic in $\mathsf{D}(K)$ to some bounded complex of flat K-modules; this follows from [Illusie 1971, p. 168, 2.2.2.1 and p. 242, 3.3], in view of the standard equivalence, given by sheafification, between finite S-modules and coherent $\mathcal{O}_X$-modules.

Recall that an exact functor $\mathsf{F} : \mathsf{D}(Y) \to \mathsf{D}(X)$ is said to be *bounded below* if there is an integer d such that for all $M \in \mathsf{D}(Y)$ and $n \in \mathbb{Z}$ the following holds:

$$H^i(M) = 0 \text{ for all } i < n \implies H^j(\mathsf{F}(M)) = 0 \text{ for all } j < n - d,$$

By substituting $>$ for $<$ in the preceding definition one obtains the notion of *bounded above*. If F is bounded below, then clearly $\mathsf{F}\,\mathsf{D}^{+}(Y) \subseteq \mathsf{D}^{+}(Y)$; likewise, if F is bounded above, then $\mathsf{F}\,\mathsf{D}^{-}(Y) \subseteq \mathsf{D}^{-}(Y)$.

Remark 2.1.5. For *every* scheme-map f the functor $\mathsf{L}f^*$ is bounded above. It is bounded below if and only if f is perfect. When f is perfect, one has

$$\mathsf{L}f^*\mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(Y) \subseteq \mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X)\,.$$

For, $\mathsf{L}f^*$ is bounded above and below, hence, as above, $\mathsf{L}f^*\mathsf{D}^{\mathsf{b}}(Y) \subseteq \mathsf{D}^{\mathsf{b}}(X)$. Also, $\mathsf{L}f^*\mathsf{D}_{\mathsf{c}}(Y) \subseteq \mathsf{D}_{\mathsf{c}}(X)$; see [Hartshorne 1966, p. 99, 4.4], whose proof uses 7.3 on page 73 of the same reference as well as compatibility of $\mathsf{L}f^*$ with open base change to reduce to the assertion that $\mathsf{L}f^*\mathcal{O}_Y = \mathcal{O}_X$.

The following characterization of perfection of f, in terms of the twisted inverse image functor $f^!$, was proved for finite-type f in [Lipman 2009, 4.9.4] and then extended to the essentially finite-type case in [Nayak 2009, 5.9].

Remark 2.1.6. For any scheme-map $f : X \to Y$, and for all M, N in $\mathsf{D}^{+}_{\mathsf{qc}}(Y)$, there is defined in [Lipman 2009, §4.9] and [Nayak 2009, 5.7–5.8] a functorial $\mathsf{D}(X)$-map

$$f^!M \otimes^{\mathsf{L}}_X \mathsf{L}f^*N \to f^!(M \otimes^{\mathsf{L}}_Y N). \tag{2.1.6.1}$$

The following conditions on f are equivalent:

(i) The map f is perfect.

(ii) The functor $f^! : \mathsf{D}^{+}_{\mathsf{qc}}(Y) \to \mathsf{D}^{+}_{\mathsf{qc}}(X)$ is bounded above and below.

(iii) The complex $f^!\mathcal{O}_Y$ is perfect over f.

(iv) When M is perfect, $f^!M$ is perfect over f; and when $M \otimes^{\mathsf{L}}_Y N$ is in $\mathsf{D}^{+}_{\mathsf{qc}}(Y)$, the natural map (2.1.6.1) is an isomorphism

$$f^!M \otimes^{\mathsf{L}}_X \mathsf{L}f^*N \xrightarrow{\ \sim\ } f^!(M \otimes^{\mathsf{L}}_Y N). \tag{2.1.6.2}$$

From (ii) one gets, as above, $f^!\mathsf{D}^{\mathsf{b}}_{\mathsf{qc}}(Y) \subseteq \mathsf{D}^{\mathsf{b}}(X)$; and the last paragraph in §5.4 of [Nayak 2005] gives

$$f^!\mathsf{D}^{+}_{\mathsf{c}}(Y) \subseteq \mathsf{D}^{+}_{\mathsf{c}}(X). \tag{2.1.6.3}$$

Thus, for perfect f, one has

$$f^!\mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(Y) \subseteq \mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X). \tag{2.1.6.4}$$

Next we establish some further properties of perfect maps for later use.

Lemma 2.1.7. *Let $f : X \to Y$ be a scheme-map, and M, B complexes in $\mathsf{D}(Y)$.*

If f is an open immersion, or if f is perfect, M is in $\mathsf{D}^{-}_{\mathsf{c}}(Y)$ and B is in $\mathsf{D}^{+}_{\mathsf{qc}}(Y)$, then there are natural isomorphisms

$$\mathsf{L}f^*\mathsf{R}\mathcal{H}om_Y(M, B) \xrightarrow{\ \sim\ } \mathsf{R}\mathcal{H}om_X(\mathsf{L}f^*M, \mathsf{L}f^*B)\,, \tag{2.1.7.1}$$

$$f^!\mathsf{R}\mathcal{H}om_Y(M, B) \xrightarrow{\ \sim\ } \mathsf{R}\mathcal{H}om_X(\mathsf{L}f^*M, f^!B)\,. \tag{2.1.7.2}$$

Proof. As a map in $\mathsf{D}(X)$, (2.1.7.1) comes from (B.1.5). To show that it is an isomorphism we may assume that Y is affine, say $Y = \operatorname{Spec} R$. Then by [Bökstedt and Neeman 1993, 5.5] and [Hartshorne 1966, p. 42, 4.6.1 (dualized)], any $M \in \mathsf{D}^-_{\mathsf{c}}(Y)$ is isomorphic to the sheafification of a complex of finite-rank free R-modules, vanishing in all large degrees; so [Lipman 2009, p. 181, (4.6.7)] gives the desired assertion.

For (2.1.7.2), use [Lipman 2009, 4.2.3(e)] when f is proper; and then in the general case, compactify, see Appendix A. □

Remark 2.1.8. Let $f : X \to Y$ be a perfect and proper scheme map.

One has $\mathsf{R}f_*(\mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X)) \subseteq \mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(Y)$, by [Illusie 1971, p. 237, 2.2.1]. Moreover, if $F \in \mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X)$ is perfect, then so is $\mathsf{R}f_*F$; see Remark 1.4.2 and [Illusie 1971, p. 250, Proposition 3.7.2].

Remark 2.1.9. In $\mathsf{D}(X)$ there is a natural map

$$\alpha(E, F, G) : \mathsf{R}\mathcal{H}om_X(E, F) \to \mathsf{R}\mathcal{H}om_X(E \otimes^{\mathsf{L}}_X G, F \otimes^{\mathsf{L}}_X G) \qquad \big(E, F, G \in \mathsf{D}(X)\big),$$

corresponding via (1.1.1.2) to

$$(\mathsf{R}\mathcal{H}om_X(E, F) \otimes^{\mathsf{L}}_X E) \otimes^{\mathsf{L}}_X G \xrightarrow{\varepsilon \otimes^{\mathsf{L}}_X \mathbf{1}} F \otimes^{\mathsf{L}}_X G$$

where ε is evaluation (1.1.1.3).

Assume now that f is perfect. By Remark 2.1.6 there is a natural isomorphism

$$\mathsf{L}f^*N \otimes^{\mathsf{L}}_X f^!\mathcal{O}_Y \simeq f^!N \qquad (N \in \mathsf{D}^+_{\mathsf{qc}}(Y)). \tag{2.1.9.1}$$

Hence $\alpha(\mathsf{L}f^*M, \mathsf{L}f^*N, f^!\mathcal{O}_Y)$ gives rise to a natural map, for all $M, N \in \mathsf{D}^+_{\mathsf{qc}}(Y)$,

$$\beta(M, N, f) : \mathsf{R}\mathcal{H}om_X(\mathsf{L}f^*M, \mathsf{L}f^*N) \to \mathsf{R}\mathcal{H}om_X(f^!M, f^!N). \tag{2.1.9.2}$$

Lemma 2.1.10. *When $f : X \to Y$ is perfect, M is in $\mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(Y)$, and N is in $\mathsf{D}^+_{\mathsf{qc}}(Y)$, the map $\beta(M, N, f)$ is an isomorphism.*

Proof. One checks, using B.3(i) and Lemma 2.1.7, that the question is local on both X and Y. Hence, via [Hartshorne 1966, p. 133, 7.19], one may assume that Y is affine, that M is a bounded-above complex of finite-rank free $\mathcal{O}_Y$-modules, and that N is a quasicoherent complex in $\mathsf{D}^+(X)$.

By Remarks 2.1.5 and 2.1.6, respectively, the functors $\mathsf{L}f^*$ and $f^!$ are bounded (both above and below). Therefore, for every fixed N, the source and target of $\beta(M, N, f)$ are bounded-below functors of M. So one can argue as in the proof of [Hartshorne 1966, p.69, (iv)] to reduce the problem to the case $M = \mathcal{O}_Y$. This case can be dealt with as follows (cf. [Lipman 2009, p. 239, (c)].

The question being local on X, one may assume there is a factorization $f = pi$ as in Remark 2.1.2 ($i : X \to W$ a closed immersion, $p : W \to Y$ essentially smooth), with $i_*\mathcal{O}_X$ a perfect complex. Since the functor i_* preserves stalks of sheaves, it suffices then to show that the composite map

$$\begin{aligned} i_*\mathsf{L}i^*p^*N &\cong i_*\mathsf{R}\mathcal{H}om_X(\mathsf{L}f^*\mathcal{O}_Y,\,\mathsf{L}f^*N)\\ &\xrightarrow{i_*\beta} i_*\mathsf{R}\mathcal{H}om_X(f^!\mathcal{O}_Y,\,f^!N) && \big(\beta=\beta(\mathcal{O}_Y,N,f)\big)\\ &\cong i_*\mathsf{R}\mathcal{H}om_X(i^!p^!\mathcal{O}_Y,\,i^!p^!N)\\ &\cong \mathsf{R}\mathcal{H}om_W(i_*i^!p^!\mathcal{O}_Y,\,p^!N) && (\text{see (B.6.1)})\\ &\cong \mathsf{R}\mathcal{H}om_W\big(\mathsf{R}\mathcal{H}om_W(i_*\mathcal{O}_X,\,p^!\mathcal{O}_Y),\,p^!N\big) && (\text{see (B.6.1)}) \end{aligned}$$

is an isomorphism in $\mathsf{D}(W)$.

Since both $i_*\mathcal{O}_X$ and $p^!\mathcal{O}_Y$ are perfect complexes (see (B.5.1), therefore the target of $i_*\beta$ is a bounded functor of N, that preserves direct sums. (This well-known fact about perfect complexes P can be shown by an easy induction on the number of degrees in which P is nonzero.) Also, using (B.1.3), one gets

$$i_*i^*p^*N \cong i_*(\mathcal{O}_X \otimes^{\mathsf{L}}_X i^*p^*N) \cong i_*\mathcal{O}_X \otimes^{\mathsf{L}}_W p^*N;$$

and hence the source of $i_*\beta$ is a bounded functor of N, that preserves direct sums.

Every quasicoherent $\mathcal{O}_Y$-module being a homomorphic image of a free one, arguing as in [Hartshorne 1966, p.69, (iii) and (iv)(dualized)] reduces the isomorphism question to the case $N=\mathcal{O}_Y$. It remains to observe that $\beta(\mathcal{O}_Y,\mathcal{O}_Y,f)$ is isomorphic to the natural map $\mathcal{O}_X\to \mathsf{R}\mathcal{H}om_X(f^!\mathcal{O}_Y,\,f^!\mathcal{O}_Y)$, a map that, by Remark 2.1.3, is indeed an isomorphism. □

Lemma 2.1.11. *Let $f : X \to Y$ be a perfect map.*

When M is in $\mathsf{D}^-_{\mathsf{c}}(Y)$ and B is in $\mathsf{D}^+_{\mathsf{c}}(Y)$, the complex $\mathsf{L}f^*M$ is derived $\mathsf{L}f^*B$-reflexive if and only if it is derived $f^!B$-reflexive.

Proof. We deal first with the boundedness conditions in Definition 1.3.1. The condition $\mathsf{L}f^*M \in \mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X)$ holds throughout, by assumption.

Assume that $\mathsf{R}\mathcal{H}om_X(\mathsf{L}f^*M,\mathsf{L}f^*B)$ is in $\mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X)$. As $\mathsf{R}\mathcal{H}om_Y(M,B)\in \mathsf{D}^+_{\mathsf{c}}(Y)$ (see [Hartshorne 1966, p. 92, 3.3]), one gets from Remark 2.1.6 and (2.1.7.1) an isomorphism

$$f^!\mathcal{O}_Y \otimes^{\mathsf{L}}_X \mathsf{R}\mathcal{H}om_X(\mathsf{L}f^*M,\mathsf{L}f^*B) \simeq f^!\mathsf{R}\mathcal{H}om_Y(M,B)\,. \tag{2.1.11.1}$$

By Remark 2.1.6(iii), $f^!\mathcal{O}_Y\in\mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X)$, so it follows that $f^!\mathsf{R}\mathcal{H}om_Y(M,B)\in\mathsf{D}^-_{\mathsf{c}}(X)$. On the other hand, by (2.1.6.3), $f^!\mathsf{R}\mathcal{H}om_Y(M,B)\in\mathsf{D}^+_{\mathsf{c}}(X)$. We conclude that $f^!\mathsf{R}\mathcal{H}om_Y(M,B)\in\mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X)$, and so by (2.1.7.2), that $\mathsf{R}\mathcal{H}om_X(\mathsf{L}f^*M,f^!B)\in\mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X)$.

Suppose, conversely, that $\mathsf{R}\mathcal{H}om_X(\mathsf{L}f^*M,\,f^!B)\in\mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X)$, so that by (2.1.7.2), there is an integer n such that

$$H^i\big(f^!\mathsf{R}\mathcal{H}om_Y(M,B)\big)=0 \ \text{ for all }\ i>n.$$

Using (2.1.7.1) and Remark 2.1.5 one gets

$$\mathsf{R}\mathcal{H}om_X(\mathsf{L}f^*M, \mathsf{L}f^*B) \simeq \mathsf{L}f^*\mathsf{R}\mathcal{H}om_Y(M, B) \in \mathsf{D_c}(X).$$

Also, $f^!\mathcal{O}_Y \in \mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X)$, by Remark 2.1.6, and it follows from an application of (i)–(iii) in B.3 to a local factorization of f as (essentially smooth)∘(closed immersion) — or from Proposition 2.3.9 — that $\mathrm{Supp}_X\, f^!\mathcal{O}_Y = X$. So except for the trivial case where X is empty, there is an integer m such that

$$H^m f^!\mathcal{O}_Y \neq 0 \quad \text{and} \quad H^j f^!\mathcal{O}_Y = 0 \;\text{ for all }\; j > m.$$

Hence, by (2.1.11.1), for each x in X and for all $k > n-m$, [Avramov et al. 2010a, A.4.3] gives $(H^k\mathsf{R}\mathcal{H}om_X(\mathsf{L}f^*M, \mathsf{L}f^*B))_x = 0$. It follows that

$$\mathsf{R}\mathcal{H}om_X(\mathsf{L}f^*M, \mathsf{L}f^*B) \in \mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X).$$

The desired assertions now result from the isomorphisms

$$\begin{aligned}\mathsf{R}\mathcal{H}om_X(\mathsf{R}\mathcal{H}om_X(\mathsf{L}f^*M, \mathsf{L}f^*B), \mathsf{L}f^*B) &\xrightarrow{\sim} \mathsf{R}\mathcal{H}om_X(\mathsf{L}f^*\mathsf{R}\mathcal{H}om_X(M, B), \mathsf{L}f^*B)\\ &\xrightarrow{\sim} \mathsf{R}\mathcal{H}om_X(f^!\mathsf{R}\mathcal{H}om_X(M, B), f^!B)\\ &\xrightarrow{\sim} \mathsf{R}\mathcal{H}om_X(\mathsf{R}\mathcal{H}om_X(\mathsf{L}f^*M, f^!B), f^!B),\end{aligned}$$

given by formula (2.1.7.1), Lemma 2.1.10, and formula (2.1.7.2), respectively. □

2.2. ***Ascent and descent.*** Let $f : X \to Y$ be a scheme-map.

Remark 2.2.1. Recall that f is said to be *faithfully flat* if it is flat and surjective; and that for any flat f, the canonical map to f^* from its left-derived functor $\mathsf{L}f^*$ is an isomorphism — in brief, $\mathsf{L}f^* = f^*$.

Lemma 2.2.2. *Let $f : X \to Y$ be a perfect scheme-map and M a complex in $\mathsf{D}(Y)$. If M is in $\mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(Y)$ then $\mathsf{L}f^*M$ is in $\mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X)$. The converse holds when M is in $\mathsf{D_c}(Y)$ and f is faithfully flat, or proper and surjective.*

Proof. The forward implication is contained in Remark 2.1.5. For the converse, when f is faithfully flat there are isomorphisms $H^n(f^*M) \cong f^*H^n(M)$ $(n \in \mathbb{Z})$; so it suffices that $f^*H^n(M) = 0$ imply $H^n(M) = 0$. This can be seen stalkwise, where we need only recall, for a flat local homomorphism $R \to S$ of local rings and any R-module P, that $P \otimes_R S = 0$ implies $P = 0$.

When f is proper then by Remark 2.1.8, $\mathsf{R}f_*(\mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X)) \subseteq \mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(Y)$ and $\mathsf{R}f_*\mathcal{O}_X$ is perfect. Furthermore, surjectivity of f implies that

$$\mathrm{Supp}_Y\, \mathsf{R}f_*\mathcal{O}_X \supseteq \mathrm{Supp}_Y\, H^0\mathsf{R}f_*\mathcal{O}_X = \mathrm{Supp}_Y\, f_*\mathcal{O}_X = Y.$$

In view of the projection isomorphism

$$\mathsf{R}f_*\mathsf{L}f^*M \simeq \mathsf{R}f_*\mathcal{O}_X \otimes^{\mathsf{L}}_Y M,$$

see (B.1.4), the desired converse follows from Theorem 1.4.5. □

Proposition 2.2.3. *Let $f : X \to Y$ be a scheme-map and $M \in \mathsf{D}^-_{\mathsf{c}}(Y)$.*

If M is perfect, then $\mathsf{L}f^*M$ is perfect. The converse holds if f is faithfully flat, or if f is perfect, proper and surjective.

Proof. Suppose M is perfect in $\mathsf{D}(Y)$. One may assume, after passing to a suitable open cover, that M is a bounded complex of finite-rank free $\mathcal{O}_Y$-modules. Then $\mathsf{L}f^*M = f^*M$ is a bounded complex of finite-rank free $\mathcal{O}_X$-modules. Thus if M is perfect then so is $\mathsf{L}f^*M$.

For the *converse,* when f is faithfully flat we use the following characterization of perfection ([Illusie 1971, p. 135, 5.8.1]): *$M \in \mathsf{D}(Y)$ is perfect if and only if $M \in \mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(Y)$ and there are integers $m \le n$ such that for all $\mathcal{O}_Y$-modules E and all i outside the interval $[m, n]$, $H^i(E \otimes^{\mathsf{L}}_Y M) = 0$.*

Writing f^* in place of $\mathsf{L}f^*$ (see Remark 2.2.1) we have, as in the proof of Lemma 2.2.2, that for any i, the vanishing of

$$H^i(f^*E \otimes^{\mathsf{L}}_X f^*M) = H^i(f^*(E \otimes^{\mathsf{L}}_Y M)) \cong f^*H^i(E \otimes^{\mathsf{L}}_Y M)$$

implies that of $H^i(E \otimes^{\mathsf{L}}_Y M)$. Hence the converse holds.

When f is perfect, proper and surjective, one can argue as in the last part of the proof of Lemma 2.2.2 to show that if $\mathsf{L}f^*M$ is perfect then M is perfect. $\square$

Proposition 2.2.4. *Let $f : X \to Y$ be a proper scheme-map and $B \in \mathsf{D}^+_{\mathsf{qc}}(Y)$.*

If $F \in \mathsf{D}(X)$ is derived $f^!B$-reflexive then $\mathsf{R}f_*F$ is derived B-reflexive.

Proof. Since F and $\mathsf{R}\mathcal{H}om_X(F, f^!B)$ are in $\mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X)$, it follows from Remark 2.1.8 that $\mathsf{R}f_*F$ is in $\mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(Y)$, and (via (B.6.1)) that

$$\mathsf{R}\mathcal{H}om_Y(\mathsf{R}f_*F, B) \simeq \mathsf{R}f_*\mathsf{R}\mathcal{H}om_X(F, f^!B) \in \mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(Y).$$

Now apply the functor $\mathsf{R}f_*$ to the assumed isomorphism

$$\delta^{f^!B}_F : F \xrightarrow{\sim} \mathsf{R}\mathcal{H}om_X(\mathsf{R}\mathcal{H}om_X(F, f^!B), f^!B),$$

and use the duality isomorphism (B.6.1) twice, to get the isomorphisms

$$\begin{aligned}\mathsf{R}f_*F &\xrightarrow{\sim} \mathsf{R}f_*\mathsf{R}\mathcal{H}om_X(\mathsf{R}\mathcal{H}om_X(F, f^!B), f^!B)\\ &\xrightarrow{\sim} \mathsf{R}\mathcal{H}om_Y(\mathsf{R}f_*\mathsf{R}\mathcal{H}om_X(F, f^!B), B)\\ &\xrightarrow{\sim} \mathsf{R}\mathcal{H}om_Y(\mathsf{R}\mathcal{H}om_Y(\mathsf{R}f_*F, B), B).\end{aligned}$$

Their composition is actually $\delta^B_{\mathsf{R}f_*F}$, though that doesn't seem so easy to show. Fortunately, owing to Proposition 1.3.3(ii) we needn't do so to conclude that $\mathsf{R}f_*F$ is derived B-reflexive. $\square$

Theorem 2.2.5. *Let $f : X \to Y$ be a perfect scheme-map, M a complex in $\mathsf{D}^-_{\mathsf{c}}(Y)$, and B a complex in $\mathsf{D}^+_{\mathsf{c}}(Y)$.*

If M is derived B-reflexive, then $\mathsf{L}f^*M$ is derived $\mathsf{L}f^*B$-reflexive and derived $f^!B$-reflexive. If f is faithfully flat, or proper and surjective, and $\mathsf{L}f^*M$ is derived $\mathsf{L}f^*B$-reflexive or derived $f^!B$-reflexive, then M is derived B-reflexive.

Proof. Suppose first that M is derived B-reflexive, so that, by definition, both M and $\mathsf{R}\mathcal{H}om_Y(M, B)$ are in $\mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(Y)$. Then (2.1.7.1) and Remark 2.1.5 show that $\mathsf{L}f^*M$ and $\mathsf{R}\mathcal{H}om_X(\mathsf{L}f^*M, \mathsf{L}f^*B)$ are in $\mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X)$. Moreover, application of the functor $\mathsf{L}f^*$ to the $\mathsf{D}(Y)$-isomorphism $M \simeq \mathsf{R}\mathcal{H}om_Y(\mathsf{R}\mathcal{H}om_Y(M, B), B)$ yields provides a $\mathsf{D}(X)$-isomorphism

$$\mathsf{L}f^*M \simeq \mathsf{R}\mathcal{H}om_X(\mathsf{R}\mathcal{H}om_X(\mathsf{L}f^*M, \mathsf{L}f^*B), \mathsf{L}f^*B).$$

Proposition 1.3.3(ii) then gives that $\mathsf{L}f^*M$ is derived $\mathsf{L}f^*B$-reflexive. When B is in $\mathsf{D}^{+}_{\mathsf{c}}(Y)$, Lemma 2.1.11 yields that $\mathsf{L}f^*M$ is derived $f^!B$-reflexive.

Suppose, conversely, that $\mathsf{L}f^*M$ is derived $\mathsf{L}f^*B$-reflexive, or equivalently, that $\mathsf{L}f^*M$ is derived $f^!B$-reflexive (see Lemma 2.1.11). Then, first, $\mathsf{L}f^*M \in \mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X)$ and, by (2.1.7.1), $\mathsf{L}f^*\mathsf{R}\mathcal{H}om_Y(M, B) \in \mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X)$. Lemma 2.2.2 then gives $M \in \mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(Y)$; similarly, since $\mathsf{R}\mathcal{H}om_Y(M, B) \in \mathsf{D}_{\mathsf{c}}(Y)$ (see [Hartshorne 1966, p. 92, 3.3]), we obtain $\mathsf{R}\mathcal{H}om_Y(M, B) \in \mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(Y)$.

Next, when f is faithfully flat (so that $\mathsf{L}f^* = f^*$, see Remark 2.2.1), one checks, with moderate effort, that if

$$\delta := \delta^B_M : M \to \mathsf{R}\mathcal{H}om_Y(\mathsf{R}\mathcal{H}om_Y(M, B), B)$$

is the canonical $\mathsf{D}(Y)$-map, then $f^*\delta$ is identified, via (2.1.7.1), with the canonical $\mathsf{D}(X)$-map $\delta^{f^*B}_{f^*M}$. The latter being an isomorphism, therefore so are all the maps $H^n(f^*\delta) = f^*H^n(\delta)$. Verifying that a sheaf-map is an isomorphism can be done stalkwise, and so, f being faithfully flat, local considerations show that the maps $H^n(\delta)$ are isomorphisms. Therefore, δ is an isomorphism.

Finally, when f is proper and surjective and $\mathsf{L}f^*M$ is derived $f^!B$-reflexive, whence, by Proposition 2.2.4, $\mathsf{R}f_*M$ is derived B-reflexive, one argues as in the last part of the proof of Lemma 2.2.2 to deduce that M is derived B-reflexive. □

Taking $M = \mathcal{O}_Y$ one gets:

Corollary 2.2.6. *Let $f : X \to Y$ be a perfect scheme-map and $B \in \mathsf{D}^{+}_{\mathsf{c}}(Y)$.*

If B is semidualizing, then so are $\mathsf{L}f^*B$ and $f^!B$. Conversely, if f is faithfully flat, or proper and surjective, and $\mathsf{L}f^*B$ or $f^!B$ is semidualizing, then so is B. □

Corollary 2.2.7. *Let $f : X \to Y$ be a perfect scheme-map and M a complex in $\mathsf{D}^{-}_{\mathsf{c}}(Y)$. Consider the following properties:*

(a) *M is semidualizing.*

(b) *M is derived $\mathcal{O}_Y$-reflexive.*

(c) *M is invertible.*

Each of these properties implies the corresponding property for $\mathsf{L}f^*M$ *in* $\mathsf{D}(X)$. *The converse holds when* f *is faithfully flat, or proper and surjective.*

Proof. Note that, given Lemma 2.2.2, we may assume that M is in $\mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(Y)$. The assertions about properties (a) and (b) are the special cases $(M, B) = (\mathcal{O}_Y, M)$ and $(M, B) = (M, \mathcal{O}_Y)$, respectively, of Theorem 2.2.5. The assertion about (c) follows from the assertion about (a) together with Proposition 2.2.3. □

2.3. ***Gorenstein-perfect maps.*** Let $f : X \to Y$ be a scheme-map.

Definition 2.3.1. A *relative dualizing complex* for f is any $\mathcal{O}_X$-complex isomorphic in $\mathsf{D}(X)$ to $f^!\mathcal{O}_Y$.

Any relative dualizing complex is in $\mathsf{D}^+_{\mathsf{c}}(X)$. Indeed, § B.3(i) and B.4 reduce the assertion to the case of maps between affine schemes, where the desired assertion follows from the following example.

Example 2.3.2. For a homomorphism $\tau : K \to P$ of commutative rings denote the P-module of relative differentials by Ω_τ, and set

$$\Omega^n_\tau = \textstyle\bigwedge^n_P \Omega_\tau \quad \text{for each } 0 \le n \in \mathbb{Z}.$$

Let $\sigma : K \to S$ be a homomorphism of rings that is essentially of finite type; thus, there exists a factorization

$$K \xrightarrow{\dot\sigma} P \xrightarrow{\sigma'} S \tag{2.3.2.1}$$

where $\dot\sigma$ is *essentially smooth of relative dimension* d and σ' is *finite*, see A.1. As in [Avramov et al. 2010a, (8.0.2)], we set

$$D^\sigma := \Sigma^d \operatorname{RHom}_P(S, \Omega^d_{\dot\sigma}) \in \mathsf{D}(S). \tag{2.3.2.2}$$

With $f : X = \operatorname{Spec} S \to \operatorname{Spec} K = Y$ the scheme-map corresponding to σ, the complex of $\mathcal{O}_X$-modules $(D^\sigma)^\sim$ is a relative dualizing complex for f; in particular, up to isomorphism, D^σ *depends only on* σ, and not on the factorization (2.3.2.1).

Indeed, there is a $\mathsf{D}_{\mathsf{qc}}(X)$-isomorphism

$$f^!\mathcal{O}_Y \simeq (D^\sigma)^\sim; \tag{2.3.2.3}$$

for, if $f = \dot f f'$ is the factorization corresponding to (2.3.2.1) then

$$f^!\mathcal{O}_Y \simeq f'^!\dot f^!\mathcal{O}_Y \simeq f'^!(\Sigma^d\Omega^d_{\dot\sigma})^\sim \simeq \Sigma^d \operatorname{RHom}_P(S, \Omega^d_{\dot\sigma})^\sim = (D^\sigma)^\sim,$$

the second isomorphism coming from B.5, and the third from (B.6.2).

Definition 2.3.3. A complex F in $\mathsf{D}(X)$ is said to be *G-perfect* (for *Gorenstein-perfect*) *relative to* f if F is derived $f^!\mathcal{O}_Y$-reflexive. The full subcategory of $\mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X)$, whose objects are the complexes that are G-perfect relative to f is denoted $\mathsf{G}(f)$.

In particular, F is in $\mathsf{G}(\mathrm{id}^X)$ if and only if F is derived $\mathcal{O}_X$-reflexive. We set

$$\mathsf{G}(X) := \mathsf{G}(\mathrm{id}^X)\,.$$

In view of (2.3.2.3), in the affine case G-perfection can be expressed in terms of *finite G-dimension* in the sense of [Auslander and Bridger 1969]; see [Avramov et al. 2010a, §6.3 and 8.2.1].

As is the case for perfection (Remark 2.1.2), G-perfection can be tested locally.

Remark 2.3.4. A complex F in $\mathsf{D}(X)$ is in $\mathsf{G}(f)$ if and only if every $x \in X$ has an open neighborhood U such that $F|_U$ is in $\mathsf{G}(f|_U)$.

If f factors as

$$X \xrightarrow{i} W \xrightarrow{h} Y$$

with i a closed immersion and h essentially smooth, then F is in $\mathsf{G}(f)$ if and only if i_*F is in $\mathsf{G}(W)$. It suffices to show this locally; and then this is [Avramov et al. 2010a, 8.2.1], in view of the equivalence of categories in Example 1.1.3.

Definition 2.3.5. The map $f : X \to Y$ is said to be *G-perfect* (for *Gorenstein perfect*) if $f^!\mathcal{O}_Y$ is semidualizing, that is, if $\mathcal{O}_X$ is in $\mathsf{G}(f)$.

A local theory of such maps already exists:

Example 2.3.6. If $X = \operatorname{Spec} S$ and $Y = \operatorname{Spec} K$, where K and S are noetherian rings, and $\sigma : K \to S$ is the ring-homomorphism corresponding to f, then f is *G-perfect* if and only if σ is of *finite G-dimension* in the sense of [Avramov and Foxby 1997]; see [Avramov et al. 2010a, 8.4.1].

Recall from Remark 2.1.6 that f is perfect if and only if $f^!\mathcal{O}_Y$ is in $\mathsf{P}(f)$, the full subcategory of $\mathsf{D}(X)$ whose objects are all the complexes that are perfect with respect to f. There is a similar description of G-perfection:

Remark 2.3.7. The map f is G-perfect if and only if $f^!\mathcal{O}_Y \in \mathsf{G}(f)$. This follows from Proposition 1.3.3, since for all $x \in X$, the stalk at x satisfies $(f^!\mathcal{O}_Y)_x \not\simeq 0$; see (2.3.2.3).

Remark 2.3.8. When Y is Gorenstein, *every* map $f : X \to Y$ is G-perfect, since [Avramov et al. 2010a, 8.3.1] and (2.3.2.3) together imply that $\mathsf{G}(f) = \mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X)$.

Via (2.3.2.3), a slight generalization of [Illusie 1971, p. 258, 4.9*ff*] globalizes [Avramov et al. 2010b, 1.2]:

Proposition 2.3.9. *Let $f : X \to Y$ be a scheme-map.*

The following inclusion holds: $\mathsf{P}(f) \subseteq \mathsf{G}(f)$.

If $M \in \mathsf{P}(Y)$ then the functor $\mathsf{R}\mathcal{H}om_X(-, f^!M)$ *takes* $\mathsf{P}(f)$ (*resp.* $\mathsf{G}(f)$) *into itself*; *and if $M \in \mathsf{G}(Y)$ then* $\mathsf{R}\mathcal{H}om_X(-, f^!M)$ *takes* $\mathsf{P}(f)$ *into* $\mathsf{G}(f)$.

Proof. The first assertion is a restatement of Remark 2.1.3.

The second assertion is local on X, so one may suppose f factors as $X \xrightarrow{i} W \xrightarrow{h} Y$ with i a closed immersion and h essentially smooth. For any $F \in \mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X)$ and $M \in \mathsf{D}^{+}_{\mathsf{qc}}(Y)$ one has, using formula (B.6.1), B.5 and Lemma 1.4.6,

$$i_*\mathsf{R}\mathscr{H}om_X(F, i^!h^!M) \simeq \mathsf{R}\mathscr{H}om_W(i_*F, h^!M) \simeq \mathsf{R}\mathscr{H}om_W(i_*F, h^*M) \otimes^{\mathsf{L}}_W h^!\mathscr{O}_Y,$$

where $h^!\mathscr{O}_Y$ is invertible. Consequently, by Remark 2.1.2,

$$\begin{aligned}\mathsf{R}\mathscr{H}om_X(F, f^!M) \in \mathsf{P}(f) &\iff i_*\mathsf{R}\mathscr{H}om_X(F, f^!M) \in \mathsf{P}(W)\\ &\iff \mathsf{R}\mathscr{H}om_W(i_*F, h^*M) \in \mathsf{P}(W).\end{aligned}$$

Similarly, by Remark 2.3.4 and Corollary 1.5.4(2),

$$\begin{aligned}\mathsf{R}\mathscr{H}om_X(F, f^!M) \in \mathsf{G}(f) &\iff i_*\mathsf{R}\mathscr{H}om_X(F, f^!M) \in \mathsf{G}(h)\\ &\iff \mathsf{R}\mathscr{H}om_W(i_*F, h^*M) \text{ is derived } \mathscr{O}_W\text{-reflexive.}\end{aligned}$$

If $F \in \mathsf{P}(f)$ then i_*F is a perfect $\mathscr{O}_W$-complex, and by Lemma 1.4.6(2),

$$\mathsf{R}\mathscr{H}om_W(i_*F, h^*M) \simeq h^*M \otimes^{\mathsf{L}}_W \mathsf{R}\mathscr{H}om_W(i_*F, \mathscr{O}_W), \tag{2.3.9.1}$$

where $\mathsf{R}\mathscr{H}om_W(i_*F, \mathscr{O}_W)$ is perfect (see Theorem 1.4.3).

If $M \in \mathsf{P}(Y)$ then by Proposition 2.2.3, $h^*M \in \mathsf{P}(W)$, and then (2.3.9.1) shows that $\mathsf{R}\mathscr{H}om_W(i_*F, h^*M) \in \mathsf{P}(W)$. Thus $\mathsf{R}\mathscr{H}om_X(F, f^!M) \in \mathsf{P}(f)$.

If $M \in \mathsf{G}(Y)$, then h^*M is derived $\mathscr{O}_W$-reflexive, hence so is $\mathsf{R}\mathscr{H}om_W(i_*F, h^*M)$; see Theorem 2.2.5, (2.3.9.1) and Proposition 1.4.4. So $\mathsf{R}\mathscr{H}om_X(F, f^!M) \in \mathsf{G}(f)$.

If $F \in \mathsf{G}(f)$ and $M \in \mathsf{P}(Y)$ then $i_*F \in \mathsf{G}(h)$ is $\mathscr{O}_W$-reflexive and h^*M is perfect; so by Lemma 1.4.6(1), (2.3.9.1) still holds, so $\mathsf{R}\mathscr{H}om_W(i_*F, h^*M)$ is $\mathscr{O}_W$-reflexive, by Remark 1.3.4 and Proposition 1.4.4. So again, $\mathsf{R}\mathscr{H}om_X(F, f^!M) \in \mathsf{G}(f)$. □

From Proposition 2.3.9 one gets the following result. It can also be seen as the special case $g = \mathrm{id}^Y$ of Proposition 2.5.2 below.

Corollary 2.3.10. *Any perfect map is G-perfect.* □

Applying Proposition 2.3.9 to $\mathsf{R}\mathscr{H}om_X(\mathscr{O}_X, f^!F)$, one gets:

Corollary 2.3.11. *If $f : X \to Y$ is perfect then $f^!\mathsf{P}(Y) \subseteq \mathsf{P}(f)$ and furthermore $f^!\mathsf{G}(Y) \subseteq \mathsf{G}(f)$. If f is G-perfect then $f^!\mathsf{P}(Y) \subseteq \mathsf{G}(f)$.* □

Also, in view of Proposition 1.3.3(iii):

Corollary 2.3.12. *For any scheme-map $f : X \to Y$, the* relative dualizing functor $\mathsf{R}\mathscr{H}om_X(-, f^!\mathscr{O}_Y)$ *induces a commutative diagram of categories, where horizontal arrows represent equivalences*:

$$\begin{array}{ccc} \mathsf{G}(f)^{\mathsf{op}} & \xleftrightarrow{\ \equiv\ } & \mathsf{G}(f) \\ \cup\!\mid & & \cup\!\mid \\ \mathsf{P}(f)^{\mathsf{op}} & \xleftrightarrow{\ \equiv\ } & \mathsf{P}(f) \end{array}$$

These equivalences are dualities, in the sense of [Avramov et al. 2010a, §6]. □

2.4. *Quasi-Gorenstein maps.* For the following notion of quasi-Gorenstein map, cf. [Avramov and Iyengar 2008, 2.2] and [Avramov et al. 2010a, §8.6.1]. For the case when f is flat, see also [Hartshorne 1966, p. 298, Exercise 9.7], which can be done, e.g., along the lines of the proof of [Lipman 1979, Lemma 1].)

Definition 2.4.1. A map $f : X \to Y$ is *quasi-Gorenstein* if $f^!\mathcal{O}_Y$ is invertible. If, in addition, f is perfect, then f is said to be a *Gorenstein map*.

If $f : X \to Y$ is quasi-Gorenstein then, clearly, $\mathcal{O}_X \in \mathsf{G}(f)$, i.e., f *is G-perfect.* More generally, Corollary 1.5.4 shows that $\mathsf{G}(f) = \mathsf{G}(X)$.

Example 2.4.2. Let $f : X \to Y$ be a scheme map. If X is Gorenstein and f is G-perfect, then f is quasi-Gorenstein; see Remark 1.5.5. Remark 2.3.8 shows then that when X and Y are both Gorenstein f is quasi-Gorenstein.

One has the following globalization of the *flat* case of [Avramov et al. 2010a, 8.6.2], see also [Avramov and Iyengar 2008, 2.4]:

Proposition 2.4.3. *If $f : X \to Y$ is a flat Gorenstein map, with diagonal map $\delta : X \to X \times_Y X$, then there are natural isomorphisms*

$$W_f := \mathcal{H}om_X(\delta^!(\mathcal{O}_{X\times_Y X}), \mathcal{O}_X) \xrightarrow[\nu]{\ \sim\ } \mathsf{R}\mathcal{H}om_X(\delta^!(\mathcal{O}_{X\times_Y X}), \mathcal{O}_X) \xrightarrow{\ \sim\ } f^!\mathcal{O}_Y .$$

If furthermore $g : Z \to X$ is finite, then (B.6.1) *gives a natural isomorphism*

$$g_*(fg)^!\mathcal{O}_Y \cong \mathsf{R}g_*g^!f^!\mathcal{O}_Y \xrightarrow{\ \sim\ } \mathsf{R}\mathcal{H}om_X(g_*\mathcal{O}_Z, W_f)\,.$$

Proof. For *any* flat scheme-map $f : X \to Y$ there is a natural isomorphism

$$\delta^!(\mathcal{O}_{X\times_Y X}) \xrightarrow{\ \sim\ } \mathsf{R}\mathcal{H}om_X(f^!\mathcal{O}_Y, \mathcal{O}_X)$$

(see Corollary 6.5 in [Avramov et al. 2010b], with $M = \mathcal{O}_X = N$).

It follows, when $f^!\mathcal{O}_Y$ is invertible, that the complex $\delta^!(\mathcal{O}_{X\times_Y X})$ is invertible, and that there is a natural $\mathsf{D}(X)$-isomorphism

$$f^!\mathcal{O}_Y \xrightarrow{\ \sim\ } \mathsf{R}\mathcal{H}om_X(\delta^!(\mathcal{O}_{X\times_Y X}), \mathcal{O}_X).$$

That the natural map ν is an isomorphism holds true with any perfect complex in place of $\delta^!(\mathcal{O}_{X\times_Y X})$: the assertion is local, hence reduces to the corresponding (obvious) assertion for rings.

For the final assertion, note that the natural map is an isomorphism

$$g_*(fg)^!\mathscr{O}_Y \xrightarrow{\sim} \mathsf{R}g_*(fg)^!\mathscr{O}_Y$$

because the equivalence of categories given in [Hartshorne 1966, p. 133, 7.19] allows us to work exclusively with quasicoherent sheaves, on which the functor g_* is exact. □

2.5. ***Composition, decomposition, and base change.*** We turn now to the behavior of relative perfection and G-perfection, especially vis-à-vis the derived direct- and inverse-image functors and the twisted inverse image functor, when several maps are involved.

Generalizing Proposition 2.2.3 (which is the special case $f = \mathrm{id}^X$), one has:

Proposition 2.5.1 (cf. [Illusie 1971, pp. 253–254, 4.5.1]). *Let $Z \xrightarrow{g} X$ and $X \xrightarrow{f} Y$ be scheme-maps, with g perfect.*

Then $\mathsf{L}g^*\mathsf{P}(f) \subseteq \mathsf{P}(fg)$. In particular, if f is perfect then so is fg.

Conversely, if g is faithfully flat, or if g is proper and surjective and $F \in \mathsf{D}_{\mathsf{c}}(X)$, then $\mathsf{L}g^*F \in \mathsf{P}(fg) \implies F \in \mathsf{P}(f)$. In particular, if fg is perfect then so is f.

Proof. Let $F \in \mathsf{P}(f)$. By Lemma 2.2.2, $\mathsf{L}g^*F \in \mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(Z)$. Hence by [Illusie 1971, p. 242, 3.3, p. 251, 4.3 and p. 115, 3.5(b)] (whose proofs are easily made to apply to essentially finite-type maps of noetherian schemes), for $\mathsf{L}g^*F$ to be in $\mathsf{P}(fg)$ it suffices that there be integers $m \le n$ such that for any $\mathscr{O}_Y$-module M and any integer $j \notin [m, n]$,

$$0 = H^j(\mathsf{L}g^*F \otimes^{\mathsf{L}}_Z \mathsf{L}(fg)^*M) \cong H^j(\mathsf{L}g^*(F \otimes^{\mathsf{L}}_X \mathsf{L}f^*M)).$$

But by *loc. cit.* this holds because F is in $\mathsf{P}(g)$ and $\mathsf{L}g^*$ is bounded.

Taking $M = \mathscr{O}_Y$ one gets that if f is perfect then fg is perfect.

For the converse, if g is faithfully flat (so that $\mathsf{L}g^* = g^*$) then for any $\mathscr{O}_X$-module F and any $j \in \mathbb{Z}$, one sees stalkwise that

$$H^j(g^*F) \cong g^*H^j(F) = 0 \Longleftrightarrow H^j(F) = 0.$$

Hence if $F \in \mathsf{D}_{\mathsf{c}}(X)$ and $g^*F \in \mathsf{P}(fg) \subseteq \mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(Z)$ — whence $F \in \mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X)$ — then by an argument like that above, $F \in \mathsf{P}(f)$.

In the remaining case one argues as in the proof of Proposition 2.2.3. (It should be noted that the relevant part of Theorem 1.4.5 is proved via the above criterion for relative perfection, so it applies not only to perfection but more generally to relative perfection.) □

Analogously, for $A := f^!\mathscr{O}_Y$ one has $(fg)^!\mathscr{O}_Y \simeq g^!A$, so Theorem 2.2.5 gives

Proposition 2.5.2 (cf. [Avramov and Foxby 1997, 4.7]). *Let $Z \xrightarrow{g} X \xrightarrow{f} Y$ be scheme-maps, with g perfect.*

Then $\mathsf{L}g^*\mathsf{G}(f)\subseteq\mathsf{G}(fg)$. *In particular, if f is G-perfect then so is fg.*

Conversely, if g is faithfully flat and $F\in\mathsf{D}^-_{\mathsf{c}}(X)$, or if g is proper and surjective and $F\in\mathsf{D}_{\mathsf{c}}(X)$, then $\mathsf{L}g^*F$ in $\mathsf{G}(fg)$ implies $F\in\mathsf{G}(f)$. □

The next proposition generalizes parts of Proposition 2.3.9. The proof is quite similar, and so is omitted.

Proposition 2.5.3. *Let $Z\xrightarrow{g}X\xrightarrow{f}Y$ be scheme-maps, $P\in\mathsf{P}(g)$, F, $A\in\mathsf{D}(X)$.*

If $F\in\mathsf{P}(f)$ then $\mathsf{R}\mathscr{H}om_Z(P,g^!F)\in\mathsf{P}(fg)$. (Cf. [Illusie 1971, p. 258, 4.9].) *In other words, the functor $\mathsf{R}\mathscr{H}om_Z(-,g^!F)$ takes $\mathsf{P}(g)$ to $\mathsf{P}(fg)$.*

If F is A-reflexive then $\mathsf{R}\mathscr{H}om_Z(P,g^!F)$ is $g^!A$-reflexive. For $A=f^!\mathcal{O}_Y$ this gives that $\mathsf{R}\mathscr{H}om_X(-,g^!F)$ takes $\mathsf{P}(g)$ to $\mathsf{G}(fg)$. □

Proposition 2.5.4. *Let $Z\xrightarrow{g}X\xrightarrow{f}Y$ be scheme-maps, with g perfect.*

Then $g^!\mathsf{P}(f)\subseteq\mathsf{P}(fg)$ and $g^!\mathsf{G}(f)\subseteq\mathsf{G}(fg)$.

Conversely, if g is proper and surjective, F is in $\mathsf{D}^+_{\mathsf{c}}(X)$, and $g^!F$ is in $\mathsf{P}(fg)$ (resp. $\mathsf{G}(fg)$) then F is in $\mathsf{P}(f)$ (resp. $\mathsf{G}(f)$).

Proof. The direct assertions are obtained from Proposition 2.5.3 by taking $P=\mathcal{O}_Z$.

If g is perfect then $g^!\mathcal{O}_X\in\mathsf{P}(g)$ and

$$\mathsf{R}g_*g^!F\simeq\mathsf{R}g_*(g^!\mathcal{O}_X\otimes^{\mathsf{L}}_Z\mathsf{L}g^*F)\simeq\mathsf{R}g_*g^!\mathcal{O}_X\otimes^{\mathsf{L}}_X F\,;$$

see Remark 2.1.6. If g is also proper then $\mathsf{R}g_*g^!\mathcal{O}_X$ is perfect [Illusie 1971, p. 257, 4.8(a)]. One can then argue as at the end of the proof of Proposition 2.5.1. □

Proposition 2.5.5. *Let $Z\xrightarrow{g}X\xrightarrow{f}Y$ be scheme-maps, with g proper.*

Then $\mathsf{R}g_*\mathsf{P}(fg)\subseteq\mathsf{P}(f)$ and $\mathsf{R}g_*\mathsf{G}(fg)\subseteq\mathsf{G}(f)$.

Proof. For P one can proceed as in [Illusie 1971, p. 257, 4.8]. (This ultimately uses the projection isomorphism (B.1.3).)

For G apply Proposition 2.2.4 with $B=f^!\mathcal{O}_Y$. □

Proposition 2.5.6 (cf. [Iyengar and Sather-Wagstaff 2004, 5.2]). *Let $Z\xrightarrow{g}X\xrightarrow{f}Y$ be scheme-maps, with f quasi-Gorenstein.*

Then $\mathsf{G}(fg)=\mathsf{G}(g)$. In particular, fg is G-perfect if and only if so is g.

Also, if g is quasi-Gorenstein then so is fg.

Proof. For any invertible $F\in\mathsf{D}(X)$ the natural map (see (2.1.6.1))

$$g^!\mathcal{O}_X\otimes^{\mathsf{L}}_Z\mathsf{L}g^*F\to g^!F$$

is an *isomorphism*: the question being local (see B.4), one reduces via 1.5.2(iii′) to the simple case $F=\mathcal{O}_X$.

When F is the invertible complex $f^!\mathcal{O}_Y$, there results an isomorphism

$$g^!\mathcal{O}_X\otimes^{\mathsf{L}}_Z\mathsf{L}g^*f^!\mathcal{O}_Y\to g^!f^!\mathcal{O}_Y\simeq(fg)^!\mathcal{O}_Y.$$

The first assertion follows from Corollary 1.5.4(1) (with $A = g^!\mathcal{O}_X$, $L = \mathsf{L}g^*f^!\mathcal{O}_Y$); and the last holds because if $g^!\mathcal{O}_X$ is invertible then by Corollary 1.5.3, $(fg)^!\mathcal{O}_Y$ is invertible as well. □

The last assertion of Proposition 2.5.6 expresses a composition property of quasi-Gorenstein homomorphisms. Here is a decomposition property:

Proposition 2.5.7 (cf. [Avramov and Foxby 1992, 4.6], [Iyengar and Sather-Wagstaff 2004, 5.5]). *Let $Z \xrightarrow{g} X \xrightarrow{f} Y$ be scheme-maps, with g perfect.*

If fg is quasi-Gorenstein then g is Gorenstein.

Suppose g is faithfully flat, or proper and surjective. If fg is quasi-Gorenstein (resp. Gorenstein) then so is f.

Proof. By Remark 2.1.6, one has $g^!\mathcal{O}_X \in \mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(Z)$ and an isomorphism

$$g^!\mathcal{O}_X \otimes^{\mathsf{L}}_Z \mathsf{L}g^*f^!\mathcal{O}_Y \to g^!f^!\mathcal{O}_Y \simeq (fg)^!\mathcal{O}_Y.$$

Also, the paragraph immediately before §5.5 in [Nayak 2009] yields $f^!\mathcal{O}_Y \in \mathsf{D}_{\mathsf{c}}(X)$, whence $\mathsf{L}g^*f^!\mathcal{O}_Y \in \mathsf{D}_{\mathsf{c}}(Z)$. Now Corollary 1.5.3(2) gives the first assertion. It also shows that $\mathsf{L}g^*f^!\mathcal{O}_Y$ is invertible, whence so is $f^!\mathcal{O}_Y$ if g is faithfully flat, or proper and surjective (see Corollary 2.2.7), giving the quasi-Gorenstein part of the second assertion. The last assertion in Proposition 2.5.2 now gives the Gorenstein part. □

From Propositions 2.5.2, 2.5.4 and 2.5.6 one gets:

Corollary 2.5.8. *Let there be given a commutative diagram*

$$\begin{array}{ccc} X' & \xrightarrow{v} & X \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[u]{} & Y \end{array}$$

with u quasi-Gorenstein and v perfect.

Then $\mathsf{L}v^*\mathsf{G}(f) \subseteq \mathsf{G}(h)$ and $v^!\mathsf{G}(f) \subseteq \mathsf{G}(h)$. Thus, when f is G-perfect so is h. □

It is shown in [Illusie 1971, p. 245, 3.5.2] that relative perfection is preserved under tor-independent base change. Here is an analog (and more) for relative G-perfection.

Proposition 2.5.9. *Let there be given a tor-independent fiber square* (*see* §B.2)

$$\begin{array}{ccc} X' & \xrightarrow{v} & X \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[u]{} & Y \end{array}$$

If the map u is Gorenstein, or flat, or if u is perfect and f is proper, then $\mathsf{L}v^*\mathsf{G}(f) \subseteq \mathsf{G}(h)$. *In particular, if f is G-perfect then so is h.*

Conversely, suppose that u is faithfully flat, or that u is perfect, proper, and surjective and f is proper. If $F \in \mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X)$ and $\mathsf{L}v^*F \in \mathsf{G}(h)$, then $F \in \mathsf{G}(f)$.

Proof. In all cases, u is perfect, whence so is v [Illusie 1971, p. 245, 3.5.2].

If u is Gorenstein, the assertion is contained in Corollary 2.5.8.

By Lemma 2.2.2, if F is $f^!\mathcal{O}_Y$-reflexive then $\mathsf{L}v^*F$ is $\mathsf{L}v^*f^!\mathcal{O}_Y$-reflexive.

If u (hence v) is flat then by B.4, one has

$$\mathsf{L}v^*f^!\mathcal{O}_Y \cong h^!\mathsf{L}u^*\mathcal{O}_Y = h^!\mathcal{O}_{Y'}. \tag{2.5.9.1}$$

Thus v^*F is $h^!\mathcal{O}_{Y'}$-reflexive, i.e., $v^*F \in \mathsf{G}(h)$.

The case when u is perfect and f is proper is treated similarly through the tor-independent base-change theorem [Lipman 2009, 4.4.3].

For the converse, the assumption is, in view of the isomorphism (2.5.9.1), that $\mathsf{L}v^*F$ is derived $\mathsf{L}v^*f^!\mathcal{O}_Y$-reflexive. Formula (2.1.6.3) gives that $f^!\mathcal{O}_Y \in \mathsf{D}^+_{\mathsf{c}}(X)$. So since v satisfies all the same hypotheses as u does, Theorem 2.2.5 yields that F is $f^!\mathcal{O}_Y$-reflexive, as asserted. □

Proposition 2.5.10. *Let there be given a tor-independent fiber square* (*see* B.2)

$$\begin{tikzcd} X' \arrow[r, "v"] \arrow[d, "h"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "u"'] & Y \end{tikzcd}$$

with either u flat, or u perfect and f proper.

If the map f is quasi-Gorenstein (*resp. Gorenstein*) *then so is h.*

The converse holds if u (*hence v*) *is faithfully flat, or if u* (*hence v*) *is perfect, proper and surjective and f is proper.*

Proof. As in the proof of Proposition 2.5.9, one has the isomorphism (2.5.9.1). Hence if $f^!\mathcal{O}_Y$ is invertible then so is $h^!\mathcal{O}_{Y'}$ (see Corollary 1.5.3(3)), whence the first quasi-Gorenstein assertion, whose converse follows from Corollary 2.2.7(c). Also, by [Illusie 1971, p. 245, 3.5.2], if f is perfect then so is h, whence the first Gorenstein assertion, whose converse follows from the preceding converse and Proposition 2.5.1 (since u perfect and h perfect implies $hu = fv$ perfect). □

3. Rigidity over schemes

As in previous sections, schemes are assumed to be noetherian, and scheme-maps to be essentially of finite type, and separated.

3.1. ***Rigid complexes.*** Fix a scheme X and a semidualizing $\mathcal{O}_X$-complex A, and for any $F \in \mathsf{D}(X)$ set

$$F^\dagger := \mathsf{R}\mathcal{H}om_X(F, A).$$

Definition 3.1.1. An *A-rigid pair* (F, ρ) is one where $F \in \mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X)$ and ρ is a $\mathsf{D}(X)$-isomorphism

$$\rho : F \xrightarrow{\sim} \mathsf{R}\mathcal{H}om_X(F^\dagger, F).$$

An $\mathcal{O}_X$-complex F is *A-rigid* if there exists a ρ such that (F, ρ) is an A-rigid pair. Such a ρ is called an *A-rigidifying isomorphism for F*.

A *morphism of A-rigid pairs* $(F, \rho) \to (G, \sigma)$ is a $\mathsf{D}(X)$-map $\phi : F \to G$ such that the following diagram, with $\tilde{\phi} : \mathsf{R}\mathcal{H}om_X(F^\dagger, F) \to \mathsf{R}\mathcal{H}om_X(G^\dagger, G)$ the map induced by ϕ, commutes:

$$\begin{tikzcd} F \arrow[r, "\rho"] \arrow[d, "\phi"'] & \mathsf{R}\mathcal{H}om_X(F^\dagger, F) \arrow[d, "\tilde{\phi}"] \\ G \arrow[r, "\sigma"'] & \mathsf{R}\mathcal{H}om_X(G^\dagger, G) \end{tikzcd}$$

The terminology "rigid" is motivated by the fact, contained in Theorem 3.2.1, that *the only automorphism of an A-rigid pair is the identity.*

Example 3.1.2. If R is a ring, $X = \operatorname{Spec} R$, and $M, C \in \mathsf{D}^{\mathsf{b}}_{\mathsf{f}}(R)$ are such that $\mathsf{RHom}_R(M, C) \in \mathsf{D}^{\mathsf{b}}_{\mathsf{f}}(R)$, then by Example 1.1.3, M is C-rigid in the sense of [Avramov et al. 2010a, §7] if and only if $M^\sim$ is $C^\sim$-rigid in the present sense.

Since $\mathsf{R}\mathcal{H}om$ commutes with restriction to open subsets, an A-rigid pair restricts over any open $U \subseteq X$ to an $A|_U$-rigid pair. However, rigidity is *not* a local condition: any invertible sheaf F is F-rigid, but $\mathcal{O}_X$ is not F-rigid unless $F \cong \mathcal{O}_X$.

On the other hand, *rigid pairs glue,* in the sense explained in Theorem 6 of the Introduction, and generalized in Theorem 3.2.9 below.

The central result of this section, Theorem 3.1.7, a globalization of [Avramov et al. 2010a, 7.2], is that any A-rigid F is isomorphic in $\mathsf{D}(X)$ to i_*i^*A, with i the inclusion into X of some open-and-closed subscheme — necessarily the support of F, see (1.1.2.1); or equivalently, $F \simeq IA$ for some idempotent $\mathcal{O}_X$-ideal I, uniquely determined by F (see Appendix C); or equivalently, F is, in $\mathsf{D}(X)$, a direct summand of A.

Example 3.1.3. The pair (A, ρ^A) with ρ^A the natural composite isomorphism

$$\rho^A : A \xrightarrow{\sim} \mathsf{R}\mathcal{H}om_X(\mathcal{O}_X, A) \xrightarrow{\sim} \mathsf{R}\mathcal{H}om_X(\mathsf{R}\mathcal{H}om_X(A, A), A),$$

is A-rigid.

Extending this example a little leads to:

Example 3.1.4. Let $U \subseteq X$ be an open-and-closed subset, and $i : U \hookrightarrow X$ the inclusion. Recall that the $\mathcal{O}_U$-module i^*A is semidualizing; see Corollary 2.2.6. If $F \in \mathsf{D}(U)$ is i^*A-rigid then i_*F is A-rigid.

Indeed, if ρ is an i^*A-rigidifying isomorphism for F, then one has isomorphisms

$$\begin{aligned} i_*F &\xrightarrow[i_*\rho]{\sim} i_*\mathsf{R}\mathcal{H}om_U\big(\mathsf{R}\mathcal{H}om_U(F, i^*A), F\big) \\ &\xrightarrow{\sim} i_*\mathsf{R}\mathcal{H}om_U\big(i^*\mathsf{R}\mathcal{H}om_X(i_*F, A), F\big) \\ &\xrightarrow{\sim} \mathsf{R}\mathcal{H}om_X\big(\mathsf{R}\mathcal{H}om_X(i_*F, A), i_*F\big), \end{aligned}$$

where the second comes from (B.1.5) (since $i^*i_*F = F$), and the third is a special case of [Lipman 2009, p. 98, (3.2.3.2)] (or see [Lipman 2009, §3.5.4], or just reason directly, using that i_*F vanishes outside U).

The composition of these isomorphisms is A-rigidifying for i_*F.

Definition 3.1.5. The *U-canonical A-rigid pair* $(i_*i^*A, \rho^{i_*i^*A})$ is the one constructed in Example 3.1.4 out of the i^*A-rigid pair (i^*A, ρ^{i^*A}) in Example 3.1.3.

It is well known that *any* monomorphism (resp. epimorphism) in $\mathsf{D}(X)$ is split, i.e., has a left (resp. right) inverse (see e.g., [Lipman 2009, 1.4.2.1]). Thus, when we speak of mono- or epimorphisms, the adjective "split" will usually be omitted.

Lemma 3.1.6. *Let $\theta : F \hookrightarrow A$ be a monomorphism in $\mathsf{D}(X)$. Let (A, ρ^A) be the canonical A-rigid pair in Example 3.1.3. There exists a unique A-rigidifying isomorphism ρ for F such that θ is a morphism of rigid pairs $(F, \rho) \to (A, \rho^A)$.*

Proof. It suffices to deal with the situation separately over each connected component of X; so we may assume that X is connected. Then, by Lemma 1.3.7, either $F = 0$ or θ is an isomorphism. In either case the assertion is obvious. □

Theorem 3.1.7. *For any $F \in \mathsf{D}(X)$, the following conditions are equivalent.*

(i) *F is A-rigid.*

(ii) *In $\mathsf{D}(X)$, $F \simeq I \otimes^{\mathsf{L}}_X A \simeq IA \simeq I^\dagger$ for some idempotent $\mathcal{O}_X$-ideal I.*

(iii) *There is an open-and-closed $U \subseteq X$ such that, $i : U \hookrightarrow X$ being the inclusion, $F \simeq i_*i^*A$ in $\mathsf{D}(X)$, whence $U = \operatorname{Supp}_X F$.*

(iv) *There is, in $\mathsf{D}(X)$, a monomorphism $F \to A$.*

When they hold, there is a unique ideal I satisfying condition (ii).

Proof. (iii) ⇒ (i). In view of Example 3.1.3, this is contained in Example 3.1.4.

(i) ⇒ (iii). For the last assertion in (iii), since i_*i^*A vanishes outside U, and since for all $x \in U$ one has, in $\mathsf{D}(\mathcal{O}_{U,x})$,

$$0 \neq \mathcal{O}_{U,x} \simeq (\mathsf{R}\mathcal{H}om_U(i^*A, i^*A))_x \simeq \mathsf{R}\mathcal{H}om_{\mathcal{O}_{U,x}}((i_*i^*A)_x, (i_*i^*A)_x)$$

therefore $U = \operatorname{Supp}_X(i_*i^*A)$.

Now let F be A-rigid. Then $U := \operatorname{Supp}_X F$ *is an open-and-closed subset of* X. For, X is covered by open subsets of the form $V = \operatorname{Spec} R$; and with $j : V \hookrightarrow X$ the inclusion, the j^*A-rigid complex j^*F (resp. its homology) is the sheafification of $F_V := \mathsf{R}\Gamma(V, F)$ (resp. its homology), so $(\operatorname{Supp}_X F) \cap V = \operatorname{Supp}_R F_V$. But F_V is $\mathsf{R}\Gamma(V, A)$-rigid (since $(F_V)^\sim \cong j^*F$ is j^*A-rigid), so by [Avramov et al. 2010a, 7.2], $\operatorname{Supp}_R F_V = U \cap V$ is an open-and-closed subset of V. That U is open-and-closed follows.

Hence, the natural map $F \to i_*i^*F$ is a $\mathsf{D}(X)$-isomorphism; so to prove the theorem we can replace (X, A, F) by (U, i^*A, i^*F), i.e., *we may assume* $\operatorname{Supp}_X F = X$.

In $\mathsf{D}(X)$, the complex $L := F^\dagger$ is isomorphic to H^0L, which is an invertible sheaf: this assertion need only be checked locally, i.e., for affine X, where it is given by [Avramov et al. 2010a, 4.9]. (The assumptions of that theorem are satisfied because F and A are both in $\mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X)$.) The invertible complex L is derived A-reflexive (take $F = \mathcal{O}_X$ in 1.5.4(2)); similarly, so is $L \otimes^{\mathsf{L}}_X L$. Since $\operatorname{Supp}_X A = X$, by Lemma 1.3.7, therefore Proposition 1.3.3(iii) yields that F is derived A-reflexive. So $L^\dagger \simeq F$, and

$$L^\dagger \simeq \mathsf{R}\mathcal{H}om_X(L, L^\dagger) \simeq (L \otimes^{\mathsf{L}}_X L)^\dagger \qquad \text{(see (1.1.1.1))}.$$

Applying the functor † to these isomorphisms we get $L \otimes^{\mathsf{L}}_X L \simeq L$. Tensoring with L^{-1} shows then that $L \simeq \mathcal{O}_X$. Thus $F \simeq L^\dagger \simeq A$.

(iii) $\Rightarrow$ (ii). Associated to any open-and closed $U \subseteq X$ is the unique idempotent $\mathcal{O}_X$-ideal I that is isomorphic to $i_*\mathcal{O}_U$ (Corollary C.3). For this I we have natural isomorphisms, the second from (B.1.3) and the last two from Corollary C.4:

$$i_*i^*A \simeq i_*(\mathcal{O}_U \otimes^{\mathsf{L}}_U i^*A) \simeq i_*\mathcal{O}_U \otimes^{\mathsf{L}}_X A \simeq I \otimes^{\mathsf{L}}_X A \simeq IA \simeq I^\dagger.$$

(ii) $\Rightarrow$ (iii). Given I as in (ii), let $U = \operatorname{Supp}_X I$, with inclusion $i : U \hookrightarrow X$, and use the preceding isomorphisms (see Corollary C.3).

(iii) $\Rightarrow$ (iv). If i is as in (iii), then i_*i^*A is a direct summand of A.

(iv) $\Rightarrow$ (i). See Lemma 3.1.6.

It remains to note that the uniqueness of I in (ii) results from

$$\operatorname{Supp}_X IA = \operatorname{Supp}_X(I \otimes^{\mathsf{L}}_X A) = \operatorname{Supp}_X I \cap \operatorname{Supp}_X A = \operatorname{Supp}_X I \cap X = \operatorname{Supp}_X I,$$

see (1.1.2.2). The proof of Theorem 3.1.7 is now completed. □

Define a *direct decomposition* of $F \in \mathsf{D}(X)$ to be a $\mathsf{D}(X)$-isomorphism

$$F \simeq F_1 \oplus F_2 \oplus \cdots \oplus F_n \tag{3.1.7.1}$$

such that no F_i vanishes; call F *indecomposable* if $F \neq 0$ and in any direct decomposition of F, one has $n = 1$. Say that (3.1.7.1) is an *orthogonal decomposition* of F if, in addition, $F_i \otimes^{\mathsf{L}}_X F_j = 0$ for all $i \neq j$.

Corollary 3.1.8. *Let $F \neq 0$ be an A-rigid complex. Let* $\operatorname{Supp}_X F = \bigsqcup_{s=1}^n U_s$ *be a decomposition into disjoint nonempty connected closed subsets, and* $i_s : U_s \hookrightarrow X$ $(1 \le s \le n)$ *the canonical inclusions.*

The U_s are then connected components of X, and there is an orthogonal decomposition into indecomposable A-rigid complexes:

$$F \simeq \bigoplus_{s=1}^{n} (i_s)_*(i_s)^*A.$$

If $F \simeq F_1 \oplus \cdots \oplus F_r$ is a direct decomposition with each F_t indecomposable, then $r = n$ and (after renumbering) there is for each s an isomorphism $F_s \simeq (i_s)_*(i_s)^*A$.

Proof. Since by Theorem 3.1.7(iii), $\operatorname{Supp}_X F$ is open and closed in X, therefore each U_s is a connected component of X. Moreover, if $i\colon \operatorname{Supp}_X F \hookrightarrow X$ is the inclusion, then i^*A is semidualizing (Corollary 2.2.6), and compatibility of $\mathsf{R}\mathcal{H}om$ with open immersions (to see which, use [Lipman 2009, 2.4.5.2]) implies that i^*F is i^*A-rigid. It follows then from Theorem 3.1.7(iii) that we may assume $F = A$.

The decomposition $X = \bigsqcup_{s=1}^n U_s$ now yields a decomposition of $F \in \mathsf{D}(X)$:

$$F \simeq \bigoplus_{s=1}^{n} (i_s)_*(i_s)^*F = \bigoplus_{s=1}^{n} (i_s)_*(i_s)^*A\,.$$

As before, $(i_s)^*A$ is a semidualizing complex of $\mathcal{O}_{U_s}$-modules, so its support is U_s, and it is indecomposable; see Lemma 1.3.7. Hence $(i_s)_*(i_s)^*A$ is indecomposable, and has support U_s. It then follows from (1.1.2.2) that the decomposition above is orthogonal. Moreover, the complexes $(i_s)_*(i_s)^*A$ are A-rigid; see Definition 3.1.5.

Let $F \simeq F_1 \oplus \cdots \oplus F_r$ be a direct decomposition. It results from Lemma 1.3.7 that this decomposition is orthogonal. Hence $X = \operatorname{Supp}_X F = \bigsqcup_{t=1}^r V_t$. Furthermore, $F \in \mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X) \implies F_t \in \mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X)$ for all t. Hence $V_t = \operatorname{Supp}_X F_t$ is open and closed; and since F_t is indecomposable, V_t is connected. Thus the V_t are the connected components of X. In particular, $r = n$, and, after renumbering, one may assume $V_t = U_t$ for each t. It remains to observe that $F_s \simeq (i_s)_*(i_s)^*F \simeq (i_s)_*(i_s)^*A$. □

3.2. ***Morphisms of rigid complexes.*** We present elaborations of Theorem 3.1.7, leading to a simple description of the skeleton of the category of rigid pairs; see Theorem 3.2.6 and Remark 3.2.7.

The result below involves the $\mathrm{H}^0(X, \mathcal{O}_X)$ action on $\mathsf{D}(X)$ described in 1.2.

Theorem 3.2.1. *If (F, ρ), (F', ρ') are A-rigid pairs with* $\operatorname{Supp}_X F = \operatorname{Supp}_X F'$ *then there exists a unique isomorphism $(F, \rho) \xrightarrow{\ \sim\ } (F', \rho')$. In particular, any A-rigid pair (F, ρ) admits a unique isomorphism into a U-canonical one, for some open-and-closed $U \subseteq X$, necessarily the support of F.*

Moreover, if $F' = F$ then with $U_F := \mathrm{Supp}_X F$, there is a unique unit u in the ring $\mathrm{H}^0(U_F, \mathcal{O}_{U_F})$ such that $\rho' = \rho\bar{u}$, where $\bar{u} \in \mathrm{H}^0(X, \mathcal{O}_X)$ is u extended by 0, and the unique isomorphism $(F, \rho) \xrightarrow{\sim} (F, \rho')$ is multiplication in F by $\bar{u}$.

For any endomorphism ϕ of the A-rigid pair (F, ρ) there is a uniquely determined idempotent $u \in \mathrm{H}^0(U_F, \mathcal{O}_{U_F})$ such that ϕ is multiplication by $\bar{u}$.

Proof. Modulo Theorem 3.1.7, the proof is basically that of [Avramov et al. 2010a, 7.3]. Indeed, Theorem 3.1.7(iii) implies that we may assume $F = F'$, and that furthermore, we may replace X by U, i.e., assume $F = A$ (so that $\bar{u} = u$).

Each endomorphism of F is multiplication by a unique element u in $\mathrm{H}^0(X, \mathcal{O}_X)$. From Lemma 1.2.1 it follows that multiplication by u induces multiplication by u on $F^\dagger$ and multiplication by u^2 on $\mathsf{R}\mathcal{H}om_X(F^\dagger, F)$. With u_F, resp. u_H, multiplication by u on F, resp. on $\mathsf{R}\mathcal{H}om_X(F^\dagger, F)$, we have then that $u_H\rho = \rho u_F$, see 1.2, so that $u_H^2\rho = u_H\rho u_F = \rho u_F^2$.

In view of this identity, one gets that u_F is an isomorphism from the rigid pair (F, ρ) to the rigid pair $(F, \rho') \iff \rho' u_F = u_H^2\rho \iff \rho' u_F = \rho u_F^2 \iff \rho' = \rho u_F$. Thus the sought-after u is the unique one such that u_F is the automorphism $\rho^{-1}\rho'$.

In the same vein, when u_F induces an endomorphism of the rigid pair (F, ρ) one gets a relation $\rho u = \rho u^2$, whence, ρ being an isomorphism, $u^2 = u$. □

Corollary 3.2.2. *For any A-rigid complex F, the group of automorphisms of F acts faithfully and transitively on the set of rigidifying isomorphisms ρ of F.*
□

Corollary 3.2.3. *If X is connected then every nonzero morphism of A-rigid pairs is an isomorphism.* □

Definition 3.2.4. For any $\mathsf{D}(X)$-map $\phi : F \to F'$ of A-rigid pairs, $\mathrm{Supp}_X \phi$ is the union of those connected components of X to which the restriction of ϕ is nonzero.

By Corollary 3.2.3, if X is connected then nonzero maps of A-rigid pairs are isomorphisms. So for a composable pair (ϕ, ψ) of maps of A-rigid pairs,

$$\mathrm{Supp}_X(\phi\psi) = \mathrm{Supp}_X \phi \cap \mathrm{Supp}_X \psi. \tag{3.2.4.1}$$

Corollary 3.2.5. *Let (F, ρ) and (F', ρ') be A-rigid pairs.*

(1) *Suppose that $\mathrm{Supp}_X F \subseteq \mathrm{Supp}_X F'$. Then there is a unique monomorphism $(F, \rho) \hookrightarrow (F', \rho')$ and a unique epimorphism $(F', \rho') \twoheadrightarrow (F, \rho)$.*

(2) *For any morphism $\phi : (F, \rho) \to (F', \rho')$, if (G, σ) is an A-rigid pair with $\mathrm{Supp}_X G = \mathrm{Supp}_X \phi$ then ϕ factors uniquely as*

$$(F, \rho) \overset{\phi'}{\twoheadrightarrow} (G, \sigma) \overset{\phi''}{\hookrightarrow} (F', \rho')$$

with ϕ' an epimorphism and ϕ'' a monomorphism.

Thus ϕ is uniquely determined by its source, target and support.

Proof. Looking at connected components separately, one reduces to where X is connected; the assertions then follow from Corollary 3.2.3 and Theorem 3.2.1. $\square$

Here is a *structure theorem* for the category $\mathsf{Rp}_A(X)$ of A-rigid pairs.

Theorem 3.2.6. *Let* $\mathsf{OC}(X)$ *be the category whose objects are the open-and-closed subsets of* X, *and whose maps* $U \to V$ *are the open-and-closed subsets of* $U \cap V$, *the composition of* $S \subseteq U \cap V$ *and* $T \subseteq V \cap W$ *being* $S \cap T \subseteq U \cap W$.

Let $\Psi : \mathsf{Rp}_A(X) \to \mathsf{OC}(X)$ *be the functor taking* $(F, \rho) \in \mathsf{Rp}_A(X)$ *to* $\operatorname{Supp}_X F$, *and taking a morphism* $\phi \in \mathsf{Rp}_A(X)$ *to* $\operatorname{Supp}_X \phi$ *(see* (3.2.4.1)*).*

This Ψ *is an equivalence of categories.*

Proof. Let (F, ρ) and (F', ρ') be A-rigid pairs, $U := \operatorname{Supp}_X F$, $V := \operatorname{Supp}_X F'$, and S an open-and-closed subset of $U \cap V$. It follows from Corollary 3.2.5, with (G, σ) the S-canonical pair, that there is a unique map of A-rigid pairs $\phi : (F, \rho) \to (F', \rho')$ such that $\operatorname{Supp}_X \phi = S$, whence the conclusion. $\square$

Remark 3.2.7. A quasi-inverse Φ of Ψ can be constructed as follows:

$\Phi : \mathsf{OC}(X) \to \mathsf{Rp}_A(X)$ takes an open-and-closed $U \subseteq X$ to an arbitrarily chosen rigid pair (F, ρ) with $\operatorname{Supp}_X F = U$; and then, for any $\mathsf{OC}(X)$-map $S \subseteq U \cap V$, $\Phi(S)$ is the unique epimorphism $\Phi U \twoheadrightarrow \Phi S$ followed by the unique monomorphism $\Phi S \hookrightarrow \Phi V$ (see Corollary 3.2.5).

That this describes a functor is, modulo (3.2.4.1), straightforward to see.

Taking into account that the map $S \subseteq U \cap V$ factors as a split epimorphism (namely $S \subseteq U \cap S$) followed by a split monomorphism (namely $S \subseteq S \cap V$), and that any functor respects left and right inverses, one sees that in fact *all* quasi-inverses of Ψ have the preceding form.

In particular, there is a canonical Φ, associating to each U the U-canonical pair. Thus $\mathsf{OC}(X)$ is *canonically isomorphic* to the category of canonical A-rigid pairs.

The next result is in preparation for establishing a gluing property for rigid pairs.

Lemma 3.2.8. *If* $g : Z \to X$ *is a perfect map and* F *is an* A*-rigid complex in* $\mathsf{D}(X)$, *then* $\mathsf{L}g^*A \in \mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(Z)$ *is semidualizing and* $\mathsf{L}g^*F$ *is* $(\mathsf{L}g^*A)$*-rigid.*

Proof. That $\mathsf{L}g^*A$ is semidualizing is given by Corollary 2.2.6.

If ρ is an A-rigidifying isomorphism for $F \in \mathsf{D}(X)$, then, abusing notation, we let $\mathsf{L}g^*\rho$ be the composed isomorphism

$$\begin{aligned} \mathsf{L}g^*F &\xrightarrow{\sim} \mathsf{L}g^*\mathsf{R}\mathcal{H}om_X(F^\dagger, F) \\ &\xrightarrow{\sim} \mathsf{R}\mathcal{H}om_Z(\mathsf{L}g^*F^\dagger, \mathsf{L}g^*F) \\ &\xrightarrow{\sim} \mathsf{R}\mathcal{H}om_Z(\mathsf{R}\mathcal{H}om_Z(\mathsf{L}g^*F, \mathsf{L}g^*A), \mathsf{L}g^*F), \end{aligned}$$

where the first isomorphism is the result of applying the functor $\mathsf{L}g^*$ to ρ, and the other two come from (2.1.7.1). Thus $\mathsf{L}g^*\rho$ is $(\mathsf{L}g^*A)$-rigidifying for $\mathsf{L}g^*F$. $\square$

Theorem 3.2.9. *Let $g : Z \to X$ be a faithfully flat scheme-map, $W := Z \times_X Z$, $\pi_1 : W \to Z$ and $\pi_2 : W \to Z$ the canonical projections.*

Let $A \in \mathsf{D}(X)$ be semidualizing. If (G, σ) is a (g^*A)-rigid pair such that there exists an isomorphism $\pi_1^*G \simeq \pi_2^*G$, then there is, up to unique isomorphism, a unique A-rigid pair (F, ρ) such that $(G, \sigma) \simeq (g^*F,\ g^*\rho)$.

Proof. (Uniqueness.) If $g^*F \simeq g^*F'$ then, since

$$g^{-1}\operatorname{Supp}_X F = \operatorname{Supp}_Z g^*F = \operatorname{Supp}_Z g^*F' = g^{-1}\operatorname{Supp}_X F',$$

and g is surjective, therefore $\operatorname{Supp}_X F = \operatorname{Supp}_X F'$; and so by Theorem 3.2.1, there is a unique isomorphism $(F,\ \rho) \xrightarrow{\sim} (F',\ \rho')$.

(Existence.) In view of Theorem 3.1.7, we may assume that $G = Jg^*A$ for some idempotent $\mathcal{O}_Z$-ideal J. Then, for $i = 1, 2$, Corollaries C.4 and C.7 yield

$$\begin{aligned}\operatorname{Supp}_W \pi_i^*G &= \operatorname{Supp}_W(\pi_i^*J \otimes^{\mathsf{L}}_W \pi_i^*g^*A)\\ &= \operatorname{Supp}_W \pi_i^*J \cap \operatorname{Supp}_W \pi_i^*g^*A\\ &= \operatorname{Supp}_W \pi_i^*J.\end{aligned}$$

So π_1^*J and π_2^*J, being isomorphic to idempotent ideals with the same support, must be isomorphic. Hence by Proposition C.8, there is a unique idempotent $\mathcal{O}_X$-ideal I such that $J = I\mathcal{O}_Z$. If $F = IA$ then $G \simeq g^*F$.

Let ρ be a rigidifying isomorphism for F, so that $(g^*F,\ g^*\rho)$ is a (g^*A)-rigid pair. By Theorem 3.2.1, there is a unique isomorphism $(g^*F,\ g^*\rho) \xrightarrow{\sim} (G, \sigma)$. □

Remark 3.2.10. In view of 3.2.1 and 3.2.8, the hypothesis $\pi_1^*G \simeq \pi_2^*G$ in 3.2.9 means simply that $\operatorname{Supp}_W \pi_1^*G = \operatorname{Supp}_W \pi_2^*G$.

3.3. *Relative rigidity.* With reference to a G-perfect map $f : X \to Y$, we take particular interest in those complexes that are $f^!\mathcal{O}_Y$-rigid — complexes we will simply call *f-rigid.*

For g any essentially étale map (so that, by Proposition 2.5.2, fg is G-perfect), there is a natural isomorphism of functors $(fg)^! \simeq g^*f^!$ (see B.3). By Lemma 3.2.8, if P is f-rigid then g^*P is (fg)-rigid.

The following *étale gluing* result (where for simplicity we omit mention of rigidifying isomorphisms) is an immediate consequence of Theorem 3.2.9.

Proposition 3.3.1. *Let $Z \xrightarrow{g} X \xrightarrow{f} Y$ be scheme-maps, where f is G-perfect and g is essentially étale and surjective. Let $W := Z \times_X Z$, with canonical projections $\pi_1 : W \to Z$ and $\pi_2 : W \to Z$. If P is an (fg)-rigid complex such that there exists an isomorphism $\pi_1^*P \simeq \pi_2^*P$, then there exists, up to isomorphism, a unique f-rigid complex F with $g^*F \simeq P$.* □

Fix a semidualizing complex A on a scheme X. The main result in this section, Theorem 3.3.2, is that for any additive functor from A-rigid complexes to the derived category of some scheme, that takes A to a semidualizing complex C — and hence, by Theorem 3.1.7(iv), takes A-rigid complexes to C-rigid complexes — there is a unique lifting to the category of A-rigid pairs that takes the canonical pair (A, ρ^A) to (C, ρ^C), provided that the functor "respects intersection of supports".

From Theorem 3.3.2 we will derive the behavior of relatively rigid complexes with respect to perfect maps (Corollaries 3.3.4 and 3.3.5). These results generalize, and were inspired by, results in [Yekutieli and Zhang 2004, Sections 3 and 6].

Let $\mathsf{Rc}_A(X) \subseteq \mathsf{D}(X)$ be the full subcategory of A-rigid complexes, and let $\mathsf{Rp}_A(X)$ be the category of A-rigid pairs. Let $\boldsymbol{\varphi}_X : \mathsf{Rp}_A(X) \to \mathsf{D}(X)$ be the functor taking (F, ρ) to $F \in \mathsf{Rc}_A(X)$. The rigid pair (A, ρ^A) is defined in Example 3.1.3.

Theorem 3.3.2. *Let X and Z be schemes, let $A \in \mathsf{D}(X)$ be semidualizing, and let $\mathsf{F} : \mathsf{Rc}_A(X) \to \mathsf{D}(Z)$ be an additive functor such that $\mathsf{F}A$ is semidualizing.*

There exists at most one functor $\bar{\mathsf{F}} : \mathsf{Rp}_A(X) \to \mathsf{Rp}_{\mathsf{F}A}(Z)$, such that

$$\boldsymbol{\varphi}_Z \bar{\mathsf{F}} = \mathsf{F}\boldsymbol{\varphi}_X \quad \textit{and} \quad \bar{\mathsf{F}}(A, \rho^A) = (\mathsf{F}A, \rho^{\mathsf{F}A}).$$

For such an $\bar{\mathsf{F}}$ to exist it is necessary that for any idempotent $\mathcal{O}_X$-ideals I, J,

$$\operatorname{Supp}_Z \mathsf{F}(IJA) = \operatorname{Supp}_Z \mathsf{F}(IA) \cap \operatorname{Supp}_Z \mathsf{F}(JA), \tag{3.3.2.1}$$

and it is sufficient that (3.3.2.1) hold whenever $IJ = 0$.

Remark 3.3.3. Let $a, b \in \mathrm{H}^0(X, \mathcal{O}_X)$ be the idempotents such that $I = a\mathcal{O}_X$ and $J = b\mathcal{O}_X$. Since IA admits a monomorphism into A, therefore $\mathsf{F}(IA)$ admits a monomorphism into $\mathsf{F}A$, and it follows from Theorem 3.1.7 that there is a unique idempotent $f(a) \in \mathrm{H}^0(Z, \mathcal{O}_Z)$ with $\mathsf{F}(IA) \simeq f(a)\mathsf{F}A$. By (1.1.2.2), Corollary C.4, and the fact that a semidualizing complex on a scheme is supported at every point of the underlying space, see Lemma 1.3.7, condition (3.3.2.1) amounts then to $f(ab) = f(a)f(b)$.

Before proving Theorem 3.3.2, we gather together some examples. Part (1) of the next corollary elaborates Lemma 3.2.8.

Recall that if $g : Z \to X$ is perfect then both $\mathsf{L}g^*B$ and $g^!A$ are semidualizing; see Corollary 2.2.6. If $L \in \mathsf{D}(X)$ is invertible then $L \otimes^{\mathsf{L}}_X A$ is semidualizing, by Corollary 1.5.4(3); and if $F \in \mathsf{D}^+_{\mathsf{qc}}(X)$, then there is as in (2.1.6.2) a natural isomorphism $g^!L \otimes^{\mathsf{L}}_Z \mathsf{L}g^*F \xrightarrow{\sim} g^!(L \otimes^{\mathsf{L}}_X F)$.

Corollary 3.3.4. *Let $g : Z \to X$ be a perfect map, and $A \in \mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X)$ semidualizing.*

(1) *There is a unique functor $g^{**} : \mathsf{Rp}_A(X) \to \mathsf{Rp}_{\mathsf{L}g^*A}(Z)$ such that*

$$\boldsymbol{\varphi}_Z\, g^{**} = \mathsf{L}g^* \quad \text{and} \quad g^{**}(A, \rho^A) = (\mathsf{L}g^*A, \rho^{\mathsf{L}g^*A}).$$

(2) *There is a unique functor* $g^{!!}: \mathsf{Rp}_A(X) \to \mathsf{Rp}_{g^!A}(Z)$ *such that*

$$\boldsymbol{\varphi}_Z\, g^{!!} = g^! \quad \text{and} \quad g^{!!}(A, \rho^A) = (g^!A, \rho^{g^!A}).$$

(3) *For each invertible* $L \in \mathsf{D}(X)$ *there is a unique bifunctor*

$$g^{\otimes}: \mathsf{Rp}_{g^!L}(Z) \times \mathsf{Rp}_A(X) \to \mathsf{Rp}_{g^!(L \otimes^{\mathsf{L}}_X A)}(Z)$$

such that

$$\boldsymbol{\varphi}_Z\, g^{\otimes}(P, F) = P \otimes^{\mathsf{L}}_Z \mathsf{L}g^*F$$

and

$$g^{\otimes}\big((g^!L, \rho^{g^!L}), (A, \rho^A)\big) = (g^!(L \otimes^{\mathsf{L}}_X A), \rho^{g^!(L \otimes^{\mathsf{L}}_X A)})\,.$$

Proof. Corollary C.7 implies that for either functor, one has in Remark 3.3.3 that $f(a)$ is the image of a under the natural map $\mathrm{H}^0(X, \mathcal{O}_X) \to \mathrm{H}^0(Z, \mathcal{O}_Z)$. Thus $f(ab) = f(a)f(b)$ holds, and so (1) and (2) result from Theorem 3.3.2.

For (3) replace X in Theorem 3.3.2 by the disjoint union $Z \sqcup X$. For $P \in \mathsf{D}(Z)$ and $F \in \mathsf{D}(X)$, let $(P, F) \in \mathsf{D}(Z \sqcup X)$ be the complex whose restriction to Z is P and to X is F. There is an obvious functor $\mathsf{F}: \mathsf{D}(Z \sqcup X) \to \mathsf{D}(Z)$ taking (P, F) to $P \otimes^{\mathsf{L}}_Z \mathsf{L}g^*F$. This takes the semidualizing complex $(g^!L, A)$ to the semidualizing complex $g^!L \otimes^{\mathsf{L}}_Z \mathsf{L}g^*A \simeq g^!(L \otimes^{\mathsf{L}}_X A)$. Using (1.1.2.2) and Remark 3.3.3, one verifies that (3.3.2.1) holds; and so (3) results. □

Recall that if $Z \xrightarrow{g} X \xrightarrow{f} Y$ are maps such that g is perfect and f is G-perfect then fg is G-perfect (Proposition 2.5.2). Taking $A = f^!\mathcal{O}_Y$ and $L = \mathcal{O}_X$ in (2) and (3) of Corollary 3.3.4 one gets:

Corollary 3.3.5. *Let* $g: Z \to X$ *be perfect, and* $f: X \to Y$ *G-perfect.*

(1) *If* F *is* f*-rigid then* $g^!F$ *is* fg*-rigid.*

(2) *If* P *is* g*-rigid and* F *is* f*-rigid then* $P \otimes^{\mathsf{L}}_Z \mathsf{L}g^*F$ *is* fg*-rigid.* □

Corollary 3.3.6. *Let* $g: Z \to X$ *be a proper map such that the natural map is an isomorphism* $\mathcal{O}_X \xrightarrow{\sim} \mathsf{R}g_*\mathcal{O}_Z$. *Let* $A \in \mathsf{D}^+_{\mathsf{qc}}(X)$ *be such that* $g^!A$ *is semidualizing.*

Then A *is semidualizing, the canonical map is an isomorphism* $\mathsf{R}g_*g^!A \xrightarrow{\sim} A$, *and there is a unique functor* $g_{**}: \mathsf{Rp}_{g^!A}(Z) \to \mathsf{Rp}_A(X)$ *such that*

$$\boldsymbol{\varphi}_X\, g_{**} = \mathsf{R}g_*\boldsymbol{\varphi}_Z \quad \text{and} \quad g_{**}(g^!A, \rho^{g^!A}) = (\mathsf{R}g_*g^!A, \rho^{\mathsf{R}g_*g^!A}).$$

Hence, if $f: X \to Y$ *is such that* fg *is G-perfect then* f *is G-perfect, and if* P *is* fg*-rigid then* $\mathsf{R}g_*P$ *is* f*-rigid.*

Proof. That A is semidualizing is given by Proposition 2.2.4.

There are, for $E \in \mathsf{D}_{\mathsf{qc}}(X)$, natural isomorphisms, the second from B.3(ii), and the third from (B.1.3),

$$\begin{aligned}\mathrm{Hom}_{\mathsf{D}(X)}(E, \mathsf{R}g_*g^!A) &\cong \mathrm{Hom}_{\mathsf{D}(Z)}(\mathsf{L}g^*E, g^!A)\\ &\cong \mathrm{Hom}_{\mathsf{D}(X)}(\mathsf{R}g_*(\mathcal{O}_Z \otimes^{\mathsf{L}}_Z \mathsf{L}g^*E), A)\\ &\cong \mathrm{Hom}_{\mathsf{D}(X)}(\mathsf{R}g_*\mathcal{O}_Z \otimes^{\mathsf{L}}_X E, A) \cong \mathrm{Hom}_{\mathsf{D}(X)}(E, A).\end{aligned}$$

It follows, via [Lipman 2009, 3.4.7(ii)], that the canonical map is an isomorphism

$$\mathsf{R}g_*g^!A \xrightarrow{\ \sim\ } A.$$

By assumption, one has the natural isomorphism $\mathrm{H}^0(X, \mathcal{O}_X) \xrightarrow{\ \sim\ } \mathrm{H}^0(Z, \mathcal{O}_Z)$. So there is a bijection between the idempotents in these two rings; and also, g is surjective. Hence g^{-1} gives a bijection from the open-and-closed subsets of X to the open-and-closed subsets of Z. Furthermore, for any $P \in \mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(Z)$, the support $\mathrm{Supp}_Z P$ is closed, whence, g being proper, $U := X \setminus g(\mathrm{Supp}_Z P)$ is open; and the restriction of P to $g^{-1}U$ is acyclic. Thus $\mathrm{Supp}_X \mathsf{R}g_*P \subseteq g(\mathrm{Supp}_Z P)$. One now easily checks (3.3.2.1), with $\mathsf{F} = \mathsf{R}g_*$ and A replaced by $g^!A$, when $IJ = 0$ — so that $\mathrm{Supp}_Z(Ig^!A)$ and $\mathrm{Supp}_Z(Jg^!A)$ are disjoint open-and-closed subsets of Z. The existence and uniqueness of g_{**} follows then from Theorem 3.3.2.

For the last assertion, take $A = f^!\mathcal{O}_Y$. $\square$

Corollary 3.3.7. *Let there be given a tor-independent fiber square* (*see* B.2)

$$\begin{array}{ccc} X' & \xrightarrow{\ v\ } & X \\ {\scriptstyle h}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ Y' & \xrightarrow[\ u\ ]{} & Y \end{array}$$

in which f is G-perfect.

If u is flat, or if u is perfect and f is proper, then h is G-perfect and for any f-rigid $\mathcal{O}_X$-complex F, $\mathsf{L}v^*F$ is h-rigid.

Proof. Proposition 2.5.9 and [Illusie 1971, p. 245, 3.5.2] imply h is G-perfect and v is perfect. By Corollary 3.3.4(i), $\mathsf{L}v^*F$ is $\mathsf{L}v^*f^!\mathcal{O}_Y$-rigid, i.e., $h^!\mathcal{O}_{Y'}$-rigid; see (2.5.9.1). $\square$

Proof of Theorem 3.3.2. (Uniqueness.) Let (G, σ) be an A-rigid pair.

Set $(\mathsf{F}G, \tau) := \bar{\mathsf{F}}(G, \sigma)$. Let ϕ_G be the unique (split) monomorphism from (G, σ) to the canonical pair (A, ρ^A), so that $\bar{\mathsf{F}}(\phi_G)$ is a (split) monomorphism, necessarily the unique one from $(\mathsf{F}G, \tau)$ to $(\mathsf{F}A, \sigma^{\mathsf{F}A})$, see Corollary 3.2.5. It follows then from Lemma 3.1.6 that τ depends only on F and (G, σ).

Also, for any morphism ϕ of A-rigid pairs, $\boldsymbol{\varphi}_Z\bar{\mathsf{F}} = \mathsf{F}$ implies $\bar{\mathsf{F}}\phi = \mathsf{F}\phi$.

(Necessity of (3.3.2.1)). Let $\Psi_Z : \mathsf{Rp}_{\mathsf{F}A}(Z) \to \mathsf{OC}(Z)$ be as in Theorem 3.2.6. Let $\Phi : \mathsf{OC}(X) \to \mathsf{Rp}_A(X)$ be as in Remark 3.2.7, sending an open-and-closed $U \subseteq X$ to $I_U A$, where I_U is the idempotent $\mathcal{O}_X$-ideal that is $\mathcal{O}_U$ over U and (0) elsewhere. Then $\Psi_Z\bar{\mathsf{F}}\Phi : \mathsf{OC}(X) \to \mathsf{OC}(Z)$ respects composition of maps, i.e., (3.3.2.1) holds.

(Existence.) Since any functor preserves a map's property of being split—mono or epi—Theorem 3.1.7(iv) shows that F takes A-rigid complexes to $\mathsf{F}A$-rigid complexes; and the preceding uniqueness argument shows how $\bar{\mathsf{F}}(G, \sigma)$ must be defined. It remains to prove that for any morphism $\phi : (G, \sigma) \to (G', \sigma')$ of A-rigid pairs, $\mathsf{F}\phi$ is a morphism of $\mathsf{F}A$-rigid pairs.

Let $U_1, \dots, U_n$ be the connected components of X. For each j, let V_j be the support of the $\mathsf{F}A$-rigid complex $\mathsf{F}(I_{U_j}A)$ (see above). The condition (3.3.2.1), for $IJ = 0$, guarantees that if $j \neq k$ then the open-and-closed subsets V_j and V_k are disjoint. So we need only show that

($*$) *the restriction of* $\mathsf{F}\phi$ *over each* V_j *is a morphism of* $\mathsf{F}A|_{V_j}$*-rigid pairs.*

Corollary 3.1.8 shows that $\phi = \sum_{j=1}^n \phi_j$ where for each j, the source and target of ϕ_j each have support that, if not empty, is U_j. Then, since F is additive, $\mathsf{F}\phi = \sum_{j=1}^n \mathsf{F}\phi_j$; and the source and target of $\mathsf{F}\phi_j$ each have support contained in V_j (see the first assertion in Theorem 3.2.1). Hence the restriction of $\mathsf{F}\phi$ over V_j is $\mathsf{F}\phi_j$. Proving ($*$) is thus reduced to the case where X is connected, so that by Corollary 3.2.3, ϕ is either 0 or an isomorphism.

If $\phi = 0$, ($*$) is obvious. If ϕ (hence $\mathsf{F}\phi$) is an isomorphism consider the diagram, where $(\mathsf{F}G, \tau) := \bar{\mathsf{F}}(G, \sigma)$, $(\mathsf{F}G', \tau') := \bar{\mathsf{F}}(G', \sigma')$, where $\phi_{G'}$ is as above, and where the maps on the right are induced by those on the left:

$$\begin{tikzcd}
\mathsf{F}G \arrow[r, "\tau"] \arrow[d, "\mathsf{F}\phi"'] & \mathsf{R}\mathcal{H}om_Z(\mathsf{R}\mathcal{H}om_Z(\mathsf{F}G, \mathsf{F}A), \mathsf{F}G) \arrow[d, "\xi"] \\
\mathsf{F}G' \arrow[r, "\tau'"] \arrow[d, "\mathsf{F}\phi_{G'}"'] & \mathsf{R}\mathcal{H}om_Z(\mathsf{R}\mathcal{H}om_Z(\mathsf{F}G', \mathsf{F}A), \mathsf{F}G') \arrow[d, "\xi'"] \\
\mathsf{F}A \arrow[r, "\sigma^{\mathsf{F}A}"'] & \mathsf{R}\mathcal{H}om_Z(\mathsf{R}\mathcal{H}om_Z(\mathsf{F}A, \mathsf{F}A), \mathsf{F}A)
\end{tikzcd}$$

By the above-indicated definition of τ and τ', the bottom square commutes, as does the square obtained by erasing τ'. Since ξ' is a monomorphism, therefore the top square commutes too. Thus $\mathsf{F}\phi$ is a map of $\mathsf{F}A$-rigid pairs. $\square$

Remark 3.3.8. One would naturally like more concrete definitions of the functors in Corollary 3.3.4.

One does find in [Yekutieli and Zhang 2004, §3] some explicitly formulated — in DGA terms — versions of special cases of these functors. (Indeed, that's what suggested Corollary 3.3.4.) But getting from here to there does not appear to be a simple matter. One might well have to go via the Reduction Theorem [Avramov et al. 2010b, 4.1], the main result of that paper, cf. [Avramov et al. 2010a, 8.5.5]); and, say for smooth maps, make use of nontrivial formal properties of Verdier's isomorphism (B.5).

In Duality Land the well-cultivated concrete and abstract plains are not presently known to be connected other than by forbidding mountain passes, that can only be traversed by hard slogging.

Appendices: Background

We review background concepts and basic facts having to do with scheme-maps, insofar as needed in the main text. Of special import is the *twisted inverse-image pseudofunctor*, a fundamental object in Grothendieck duality theory.

Rings and schemes are assumed throughout to be noetherian.

A. Essentially finite-type maps

A.1. A homomorphism $\sigma : K \to S$ of commutative rings is *essentially of finite type* if σ can be factored as a composition of ring-homomorphisms

$$K \hookrightarrow K[x_1, \dots, x_d] \to V^{-1}K[x_1, \dots, x_d] \twoheadrightarrow S\,,$$

where $x_1, \dots, x_d$ are indeterminates, $V \subseteq K[x_1, \dots, x_d]$ is a multiplicatively closed set, the first two maps are canonical and the third is surjective. The map σ is *of finite type* if one can choose $V = \{1\}$; the map σ is *finite* if it turns S into a finite (that is, finitely generated) R-module.

A homomorphism $\dot\sigma : K \to P$ is (*essentially*) *smooth* if it is flat and (essentially) of finite type, and if for each homomorphism of rings $K \to k$, where k is a field, the ring $k \otimes_K P$ is regular. By [Grothendieck and Dieudonné 1967, 17.5.1], this notion of smoothness is equivalent to the one defined in terms of lifting of homomorphisms.

When $\dot\sigma$ is essentially smooth the P-module $\Omega_{\dot\sigma}$ of relative Kähler differentials is finite projective; we say $\dot\sigma$ has *relative dimension* d if for every $p \in \operatorname{Spec} S$, the free S_p-module $(\Omega_{\dot\sigma})_p$ has rank d.

A.2. A scheme-map $f : X \to Y$ is *essentially of finite type* if every $y \in Y$ has an affine open neighborhood $V = \operatorname{Spec}(A)$ such that $f^{-1}V$ can be covered by finitely many affine open sets $U_i = \operatorname{Spec}(C_i)$ so that the corresponding ring homomorphisms $A \to C_i$ are essentially of finite type.

If, moreover, there exists for each i a multiplicatively closed subset $V_i \subseteq A$ such that $A \to C_i$ factors as $A \to V_i^{-1}A \xrightarrow{\sim} C_i$ where the first map is canonical and the second is an isomorphism (in other words, $A \to C_i$ is a *localization* of A), then we say that f is *localizing*. If the scheme-map f is localizing and also set-theoretically injective, then we say that f is a *localizing immersion*.

The map f is *essentially smooth* (of relative dimension d) if it is essentially of finite type and the above data $A \to C_i$ can be chosen to be essentially smooth ring homomorphisms (of relative dimension d). The map f is *essentially étale* if it is essentially smooth of relative dimension 0. Equivalently, f is essentially smooth (resp. étale) if it is essentially of finite type and formally smooth (resp. étale); see [Grothendieck and Dieudonné 1967, §17.1]. For example, any localizing map is essentially étale.

Remark A.3. We will refer a few times to proofs in [Illusie 1971] that make use of the fact that the diagonal of a smooth map is a quasiregular immersion. To ensure that those proofs apply here, we note that the same property for *essentially smooth* maps is given by [Grothendieck and Dieudonné 1967, 16.10.2 and 16.9.4].

Nayak [2009, 4.1], extending a compactification theorem of Nagata, shows that *every essentially-finite-type separated map f of noetherian schemes factors as $f = \bar{f}u$ with $\bar{f}$ proper and u a localizing immersion.*

Example A.4. (Local compactification.) A map $f : X = \operatorname{Spec} S \to \operatorname{Spec} K = Y$ coming from an essentially finite-type homomorphism of rings $K \to S$ factors as

$$X \xrightarrow{j} W \overset{i}{\hookrightarrow} \bar{W} \xrightarrow{\pi} Y,$$

where W is the Spec of a finitely generated K-algebra T of which S is a localization, j being the corresponding map, where i is an open immersion, and where π is a projective map, so that π is proper and ij is a localizing immersion.

B. Review of global duality theory

All scheme-maps are assumed to be essentially of finite type and separated.

We recall some global duality theory, referring to [Lipman 2009] and [Nayak 2009] for details.

B.1. To any scheme-map $f : X \to Y$ one associates the right-derived direct-image functor $\mathsf{R}f_* : \mathsf{D}_{\mathsf{qc}}(X) \to \mathsf{D}_{\mathsf{qc}}(Y)$ and its left adjoint, the left-derived inverse-image functor $\mathsf{L}f^* : \mathsf{D}_{\mathsf{qc}}(Y) \to \mathsf{D}_{\mathsf{qc}}(X)$ [Lipman 2009, 3.2.2, 3.9.1, 3.9.2]. These functors interact with the left-derived tensor product $\otimes^{\mathsf{L}}$ via a natural isomorphism

$$\mathsf{L}f^*(M \otimes^{\mathsf{L}}_Y N) \xrightarrow{\sim} \mathsf{L}f^*M \otimes^{\mathsf{L}}_X \mathsf{L}f^*N \qquad \big(M, N \in \mathsf{D}(Y)\big), \tag{B.1.1}$$

see [Lipman 2009, 3.2.4]; via the functorial map

$$\mathsf{R}f_*F \otimes^{\mathsf{L}}_Y \mathsf{R}f_*G \to \mathsf{R}f_*(F \otimes^{\mathsf{L}}_X G) \qquad \big(F, G \in \mathsf{D}(X)\big) \tag{B.1.2}$$

adjoint to the natural composite map

$$\mathsf{L}f^*(\mathsf{R}f_*F \otimes^{\mathsf{L}}_Y \mathsf{R}f_*G) \xrightarrow{\sim} \mathsf{L}f^*\mathsf{R}f_*F \otimes^{\mathsf{L}}_X \mathsf{L}f^*\mathsf{R}f_*G \longrightarrow F \otimes^{\mathsf{L}}_X G;$$

and via the *projection isomorphism*

$$\mathsf{R}f_*F \otimes^{\mathsf{L}}_Y M \xrightarrow{\sim} \mathsf{R}f_*(F \otimes^{\mathsf{L}}_X \mathsf{L}f^*M) \qquad \big(F \in \mathsf{D}_{\mathsf{qc}}(X),\ M \in \mathsf{D}_{\mathsf{qc}}(Y)\big), \tag{B.1.3}$$

defined qua map to be the natural composition

$$\mathsf{R}f_*F \otimes^{\mathsf{L}}_Y M \to \mathsf{R}f_*F \otimes^{\mathsf{L}}_Y \mathsf{R}f_*\mathsf{L}f^*M \to \mathsf{R}f_*(F \otimes^{\mathsf{L}}_X \mathsf{L}f^*M).$$

see [Lipman 2009, 3.9.4)]. The projection isomorphism yields a natural isomorphism

$$\mathsf{R}f_*\mathsf{L}f^*M \simeq \mathsf{R}f_*(\mathcal{O}_X \otimes^{\mathsf{L}}_X \mathsf{L}f^*M) \simeq \mathsf{R}f_*\mathcal{O}_X \otimes^{\mathsf{L}}_Y M. \tag{B.1.4}$$

Interactions with the derived (sheaf-)homomorphism functor $\mathsf{R}\mathcal{H}om$ occur via natural bifunctorial maps:

$$\mathsf{L}f^*\mathsf{R}\mathcal{H}om_Y(M, N) \to \mathsf{R}\mathcal{H}om_X(\mathsf{L}f^*M, \mathsf{L}f^*N) \qquad \big(M, N \in \mathsf{D}(Y)\big), \tag{B.1.5}$$

(see [Lipman 2009, 3.5.6(a)]) which is an *isomorphism* if f is an open immersion [Lipman 2009, p. 190, end of §4.6]; and

$$\mathsf{R}f_*\mathsf{R}\mathcal{H}om_X(F, G) \to \mathsf{R}\mathcal{H}om_Y(\mathsf{R}f_*F, \mathsf{R}f_*G) \qquad \big(F, G \in \mathsf{D}(X)\big), \tag{B.1.6}$$

the latter corresponding via (1.1.1.2) to the natural composition

$$\mathsf{R}f_*\mathsf{R}\mathcal{H}om_X(F, G) \otimes^{\mathsf{L}}_Y \mathsf{R}f_*F \to \mathsf{R}f_*\big(\mathsf{R}\mathcal{H}om_X(F, G) \otimes^{\mathsf{L}}_X F\big) \xrightarrow{\mathsf{R}f_*\varepsilon} \mathsf{R}f_*G,$$

where the first map comes from (B.1.2), and ε is the evaluation map (1.1.1.3).

B.2. For any commutative square of scheme-maps

$$\begin{tikzcd} X' \arrow[r, "v"] \arrow[d, "h"'] \arrow[dr, phantom, "\Xi"] & X \arrow[d, "f"] \\ Y' \arrow[r, "u"'] & Y \end{tikzcd} \tag{B.2.1}$$

one has the map $\theta_\Xi : \mathsf{L}u^*\mathsf{R}f_* \to \mathsf{R}h_*\mathsf{L}v^*$ adjoint to the natural composite map

$$\mathsf{R}f_* \longrightarrow \mathsf{R}f_*\mathsf{R}v_*\mathsf{L}v^* \xrightarrow{\sim} \mathsf{R}u_*\mathsf{R}h_*\mathsf{L}v^*.$$

When Ξ is a *fiber square* (which means that the map associated to Ξ is an isomorphism $X' \xrightarrow{\sim} X \times_Y Y'$), and u is *flat*, then θ_Ξ is an *isomorphism*. In fact, for any fiber square Ξ, θ_Ξ *is an isomorphism* $\iff$ Ξ *is tor-independent* [Lipman 2009, 3.10.3].

B.3. Duality theory focuses on the *twisted inverse-image pseudofunctor*

$$f^! : \mathsf{D}^+_{\mathsf{qc}}(Y) \to \mathsf{D}^+_{\mathsf{qc}}(X),$$

where "pseudofunctoriality" (also known as "2-functoriality") entails, in addition to functoriality, a family of functorial isomorphisms $c_{g,f} : (fg)^! \xrightarrow{\sim} g^! f^!$, one for each composable pair $Z \xrightarrow{g} X \xrightarrow{f} Y$, satisfying a natural "associativity" property vis-à-vis any composable triple; see, e.g., [Lipman 2009, 3.6.5].

This pseudofunctor is uniquely determined up to isomorphism by the following three properties:

(i) If f is essentially étale then $f^!$ is the usual restriction functor f^*.

(ii) If f is proper then $f^!$ is right-adjoint to $\mathsf{R}f_*$.

(iii) If in a fiber square Ξ as in (B.2.1) the map f (and hence h) is proper and u is essentially étale, then the functorial *base-change map*

$$\beta_\Xi(M) : v^* f^! M \to h^! u^* M \qquad \bigl(M \in \mathsf{D}^+_{\mathsf{qc}}(Y)\bigr), \tag{B.3.1}$$

defined to be adjoint to the natural composition

$$\mathsf{R}h_* v^* f^! M \xrightarrow[\theta_\Xi^{-1}]{\sim} u^* \mathsf{R}f_* f^! M \longrightarrow u^* M,$$

is identical with the natural composite *isomorphism*

$$v^* f^! M = v^! f^! M \xrightarrow{\sim} (fv)^! M = (uh)^! M \xrightarrow{\sim} h^! u^! M = h^! u^* M.$$

For the existence of such a pseudofunctor, see [Nayak 2009, section 5.2].

B.4. Nayak's theorem [2009, 5.3] (as elaborated in [Nayak 2005, 7.1.6]) shows that one can associate, in a unique way, to *every* fiber square Ξ as in (B.2.1) with u (and hence v) flat, a functorial isomorphism

$$\beta_\Xi(M) : v^* f^! M \xrightarrow{\sim} h^! u^* M \qquad (M \in \mathsf{D}^+_{\mathsf{qc}}(Y)),$$

equal to (B.3.1) when f is proper, and to the natural isomorphism $v^* f^* \xrightarrow{\sim} h^* u^*$ when f is essentially étale.

B.5. Generalizing (i) in B.3, let $f : X \to Y$ be essentially smooth, so that by [Grothendieck and Dieudonné 1967, 16.10.2] the relative differential sheaf Ω_f is locally free over $\mathcal{O}_X$. On any connected component W of X, the rank of Ω_f is a constant, denoted $d(W)$.

There is a *functorial isomorphism*

$$f^! M \xrightarrow{\sim} \Sigma^d \Omega^d_f \otimes_{\mathcal{O}_X} f^* M \qquad \bigl(M \in \mathsf{D}_{\mathsf{qc}}(Y)\bigr), \tag{B.5.1}$$

with $\Sigma^d\Omega^d_f$ the complex whose restriction to any W is $\Sigma^{d(W)}\bigwedge^{d(W)}_{\mathcal{O}_W}(\Omega_f|_W)$. ($\Sigma$ is the usual translation automorphism of $\mathsf{D}(X)$; and $\bigwedge$ denotes exterior power.)

To prove this, one may assume that X itself is connected, and set $d := d(X)$. Noting that the diagonal $\Delta : X \to X \times_Y X$ is defined locally by a regular sequence of length d (see Remark A.3), so that $\Delta^!\mathcal{O}_{X\times_Y X} \otimes^{\mathsf{L}} \mathsf{L}\Delta^* G \cong \Delta^! G$ for all $G \in \mathsf{D}_{\mathsf{qc}}(X \times_Y X)$ [Hartshorne 1966, p. 180, 7.3], one can imitate the proof of [Verdier 1969, p. 397, Thm. 3], where, in view of (a) above, one can drop the properness condition and take $U = X$, and where finiteness of Krull dimension is superfluous.

B.6. The fact that $\beta_\Xi(M)$ in (B.3.1) is an isomorphism for all M whenever u is an open immersion and f is proper, is shown in [Lipman 2009, §4.6, part V] to be equivalent to *sheafified duality,* which is that *for any proper $f : X \to Y$, and any $F \in \mathsf{D}_{\mathsf{qc}}(X)$, $M \in \mathsf{D}^+_{\mathsf{qc}}(Y)$, the natural composition, in which the first map comes from (B.1.6),*

$$\mathsf{R}f_*\mathcal{H}om_X(F,\ f^!M) \to \mathsf{R}\mathcal{H}om_Y(\mathsf{R}f_*F,\ \mathsf{R}f_*f^!M) \to \mathsf{R}\mathcal{H}om_Y(\mathsf{R}f_*F, M), \tag{B.6.1}$$

is an isomorphism.

Moreover, if the proper map f has finite flat dimension, then sheafified duality holds for *all* $M \in \mathsf{D}_{\mathsf{qc}}(Y)$, see [Lipman 2009, 4.7.4].

If f is a *finite* map, then (B.6.1) with $F = \mathcal{O}_X$ determines the functor $f^!$. (See also [Conrad 2007, §2.2].) In particular, if $f : \operatorname{Spec} B \to \operatorname{Spec} A$ corresponds to a finite ring homomorphism $A \to B$, and $^\sim$ is the standard sheafification functor, then for an A-complex N, $f^!(N^\sim)$ is the B-complex

$$f^!(N^\sim) = \mathsf{R}\mathrm{Hom}_A(B, N)^\sim, \tag{B.6.2}$$

where $\mathsf{R}\mathrm{Hom}_A(B, -)$ denotes the right-derived functor of the functor $\mathrm{Hom}_A(B, -)$ from A-modules to B-modules.

C. Idempotent ideal sheaves

Definition C.1. Let $(X, \mathcal{O}_X)$ be a local-ringed space, i.e., X is a topological space and $\mathcal{O}_X$ is a sheaf of commutative rings whose stalk at each point is a local ring (not necessarily noetherian). An $\mathcal{O}_X$-ideal is *idempotent* if it is of finite type (i.e., locally finitely generated) and satisfies the equivalent conditions in the next proposition.

Proposition C.2. *Let $(X, \mathcal{O}_X)$ be a local-ringed space. Consider the following conditions on an $\mathcal{O}_X$-ideal I.*

(i) *There is an $a \in H^0(X, \mathcal{O}_X)$ such that $a^2 = a$ and $I = a\mathcal{O}_X$.*

(i′) *The identity map of I extends to an $\mathcal{O}_X$-homomorphism $\pi : \mathcal{O}_X \to I$.*

(ii) *There is an open and closed $U \subseteq X$, with inclusion, say, $i : U \hookrightarrow X$, and an $\mathcal{O}_X$-isomorphism $i_*\mathcal{O}_U \simeq I$.*

(iii) *The $\mathcal{O}_X$-module $\mathcal{O}_X/I$ is flat.*

(iv) *For all $\mathcal{O}_X$-modules F, the natural map is an isomorphism $I \otimes_X F \xrightarrow{\sim} IF$.*

(v) *For all $\mathcal{O}_X$-ideals J, $IJ = I \cap J$.*

(vi) $I^2 = I$.

One has the implications

$$\text{(i)} \iff \text{(i}'\text{)} \iff \text{(ii)} \implies \text{(iii)} \iff \text{(iv)} \iff \text{(v)} \implies \text{(vi)};$$

and if I is of finite type then (vi) $\implies$ (i).

Proof. (i) $\Leftrightarrow$ (i$'$). If (i) holds, let π be the map taking $1 \in \mathrm{H}^0(X, \mathcal{O}_X)$ to a. Conversely, given (i$'$), let $a = \pi(1)$.

(ii) $\Rightarrow$ (i). Let a be the global section that is 1 over U and 0 over $X \setminus U$.

(i) $\Rightarrow$ (vi). Trivial.

(vi) $\Rightarrow$ (ii) when I is of finite type (whence (i) $\Rightarrow$ (ii) always). The support of I, $U := \{\, x \in X \mid I_x \neq 0\,\}$, is closed when I is of finite type. For any $x \in U$, since I_x is a finitely generated $\mathcal{O}_{X,x}$-ideal such that $I_x = I_x^2$, therefore Nakayama's lemma shows that $I_x = \mathcal{O}_{X,x}$. So $X \setminus U = \{\, x \in X \mid \mathcal{O}_{X,x}/I_x \neq 0\,\}$ is closed, and thus U is open as well as closed. Clearly, $I|_U = \mathcal{O}_U$ and $I|_{X\setminus U} = 0$, whence $I \simeq i_*\mathcal{O}_U$.

(i) $\Rightarrow$ (iii). If (i) holds then the germ of a at any $x \in X$ is 1 or 0, so $(\mathcal{O}/I)_x$ is either (0) or $\mathcal{O}_{X,x}$, both of which are flat over $\mathcal{O}_{X,x}$.

The remaining implications can be tested stalkwise, and so reduce to the corresponding well-known implications for ideals I, J in a local ring R, and R-modules F:

(iii) $\Rightarrow$ (iv). The surjection $I \otimes_R F \twoheadrightarrow IF \subseteq R \otimes_R F$ has kernel $\mathrm{Tor}_1^R(R/I, F) = 0$.

(iv) $\Rightarrow$ (v). $(I \cap J)/IJ$ is the kernel of the natural injective (by (iv)) map

$$R/IJ \cong I \otimes_R R/J \to R \otimes_R (R/J) = R/J.$$

(v) $\Rightarrow$ (iii). Flatness of R/I is implied by injectivity, for all R-ideals J, of the natural map $J/IJ \cong J \otimes_R (R/I) \to R \otimes_R (R/I) = R/I$, with kernel $(I \cap J)/IJ$.

(v) $\Rightarrow$ (vi). Take $J = I$. □

Corollary C.3. (1) *Taking a to $a\mathcal{O}_X$ gives a bijection from the set of idempotent elements of $H^0(X, \mathcal{O}_X)$ to the set of idempotent $\mathcal{O}_X$-ideals.*

(2) *There is a bijection that associates to each idempotent $\mathcal{O}_X$-ideal its support — an open-and-closed subset of X — and to each open-and-closed $U \subseteq X$, with inclusion map i, the unique idempotent $\mathcal{O}_X$-ideal isomorphic to $i_*\mathcal{O}_U$, that is, the ideal whose restriction to U is $\mathcal{O}_U$ and to $X \setminus U$ is* (0). □

Corollary C.4. *A finite-type $\mathcal{O}_X$-ideal I is idempotent if and only if for each $G \in \mathsf{D}(X)$ there exist $\mathsf{D}(X)$-isomorphisms, functorial in G,*

$$\mathsf{R}\mathcal{H}om_X(I, G) \simeq I \otimes^{\mathsf{L}}_X G \simeq IG.$$

Proof. If I is idempotent then over the open set $U := \mathrm{Supp}_X I$ one has $I = \mathcal{O}_U$, and over the disjoint open set $X \setminus U$, $I \simeq 0$, so the asserted isomorphisms obviously exist over $X = U \sqcup (X \setminus U)$.

Conversely, if these isomorphisms hold for all members of the natural triangle

$$I \to \mathcal{O}_X \to \mathcal{O}_X/I \xrightarrow{+}$$

then, since $I(\mathcal{O}_X/I) = 0$, application of the functor $\mathsf{R}\mathcal{H}om(I, -)$ yields that the natural map is an isomorphism $I \simeq I^2$ in $\mathsf{D}(X)$, hence in $\mathcal{O}_X$, i.e., $I = I^2$. □

Corollary C.5. *Let X be a locally noetherian scheme. For a complex $L \in \mathsf{D}(X)$ the following conditions are equivalent.*

(i) *L is isomorphic in $\mathsf{D}(X)$ to an idempotent $\mathcal{O}_X$-ideal.*

(ii) *$L \in \mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X)$ and there exists a $\mathsf{D}(X)$-isomorphism $L \otimes^{\mathsf{L}}_X L \xrightarrow{\sim} L$.*

Proof. If (i) holds then $L \in \mathsf{D}^{\mathsf{b}}_{\mathsf{c}}(X)$ is clear; and taking $G = I$ in C.4, one gets (ii).

When (ii) holds, (i) follows easily from [Avramov et al. 2010a, 4.9]. □

Proposition C.6. *Let $g : Z \to X$ be a morphism of local ringed spaces (so that for each $z \in Z$ the associated stalk homomorphism $\mathcal{O}_{X,gz} \to \mathcal{O}_{Z,z}$ is a local homomorphism of local rings). Let I be an $\mathcal{O}_X$-ideal. If I is idempotent then so is $I\mathcal{O}_Z \cong g^*I \simeq \mathsf{L}g^*I$. The converse holds if g is flat and surjective.*

Proof. If $I = I^2$ then $I\mathcal{O}_Z = (I\mathcal{O}_Z)^2$. Flatness of $\mathcal{O}_X/I$ implies that I is flat and that the natural map $g^*I \to g^*\mathcal{O}_X = \mathcal{O}_Z$ is injective, and thus $\mathsf{L}g^*I \simeq g^*I \cong I\mathcal{O}_Z$.

If g is flat and surjective then for each $x \in X$ there is a $z \in Z$ such that $g(z) = x$, and then there is a flat local homomorphism $\mathcal{O}_{X,x} \to \mathcal{O}_{Z,z}$. Hence if $I\mathcal{O}_Z = (I\mathcal{O}_Z)^2$ then $I_x O_{Z,z} = I_x^2 O_{Z,z}$, i.e., $I_x = I_x^2$. As this holds for all x, therefore $I = I^2$. □

Corollary C.7. *Let $g : Z \to X$ be a morphism of local ringed spaces, and I an idempotent $\mathcal{O}_X$-ideal.*

(1) *For any $E \in \mathsf{D}(X)$, there is a unique isomorphism*

$$\mathsf{L}g^*(IE) \simeq I\,\mathsf{L}g^*E$$

whose composition with the natural map $I\,\mathsf{L}g^*E \to \mathsf{L}g^*E$ is the map obtained by applying $\mathsf{L}g^*$ to the natural map $IE \to E$.

(2) *If g is a perfect scheme-map then for any $E \in \mathsf{D}^{+}_{\mathsf{qc}}(X)$, there exists a unique isomorphism $g^!(IE) \simeq Ig^!E$ whose composition with the natural map $Ig^!E \to g^!E$ is the map obtained by applying $g^!$ to the natural map $IE \to E$.*

Proof. Uniqueness holds because, $I\mathcal{O}_Z$ being idempotent, $I\,\mathsf{L}g^*E \simeq I\mathcal{O}_Z \otimes_Z \mathsf{L}g^*E$ is a direct summand of $\mathcal{O}_Z \otimes_Z \mathsf{L}g^*E \simeq \mathsf{L}g^*E$ (Proposition C.2, (iv) and (i′)).

Since both I and $\mathcal{O}_X/I$ are flat over $\mathcal{O}_X$, there are for all $F \in \mathsf{D}(X)$ natural isomorphisms $\mathsf{L}g^*I \otimes^{\mathsf{L}}_Z F \simeq g^*I \otimes_Z F \cong IF$. So for all $E \in \mathsf{D}(X)$,

$$\mathsf{L}g^*(IE) \simeq \mathsf{L}g^*(I \otimes^{\mathsf{L}}_X E) \simeq \mathsf{L}g^*I \otimes^{\mathsf{L}}_Z \mathsf{L}g^*E \simeq I\,\mathsf{L}g^*E.$$

The composition of these isomorphisms has the property asserted in (1).

Similarly, if g is a perfect scheme-map then, using Remark 2.1.6, one gets natural isomorphisms for all $E \in \mathsf{D}^+_{\mathsf{qc}}(X)$,

$$g^!(IE) \simeq g^!(I \otimes^{\mathsf{L}}_X E) \simeq \mathsf{L}g^*I \otimes^{\mathsf{L}}_Z \mathsf{L}g^*E \otimes^{\mathsf{L}}_Z g^!\mathcal{O}_X \simeq \mathsf{L}g^*I \otimes^{\mathsf{L}}_Z g^!E \simeq I\,g^!E,$$

that compose to the isomorphism needed for (2). □

The next result is to the effect that *idempotence satisfies faithfully flat descent* (without any "cocycle condition").

Proposition C.8. *Let $g : Z \to X$ be a faithfully flat map, and let $\pi_1 : Z \times_X Z \to Z$ and $\pi_2 : Z \times_X Z \to Z$ be the canonical projections. If J is an idempotent $\mathcal{O}_Z$-ideal such that there exists an isomorphism $\pi_1^*J \cong \pi_2^*J$ then there is a unique idempotent $\mathcal{O}_X$-ideal such that $J = I\mathcal{O}_Z$.*

Proof. (Uniqueness.) If $J = I\mathcal{O}_Z = I'\mathcal{O}_Z$ where I and I' are idempotent $\mathcal{O}_X$-ideals with respective supports U and U', then $g^{-1}U = g^{-1}U'$ (both being the support of J), and since g is surjective, therefore $U = U'$, so $I = I'$.

(Existence.) Let V be the support of J. The support of π_1^*J is $\pi_1^{-1}V = V \times_X Z$, and similarly that of π_1^*J is $Z \times_X V$. Hence, since $\pi_1^*J \cong \pi_2^*J$, the following subsets of $Z \times_X Z$ are all the same:

$$V \times_X Z = Z \times_X V = (V \times_X Z) \cap (Z \times_X V) = V \times_X V.$$

If $v \in V$ and $w \in Z$ are such that $g(v) = g(w)$, then there is a field K and a map $\gamma : \operatorname{Spec} K \to V \times_X Z = V \times_X V$ such that the set-theoretic images of $\pi_1\gamma$ and $\pi_2\gamma$ are v and w respectively, so $w \in V$. Thus $V = g^{-1}g(V)$.

We claim that $g(V)$ *is open and closed in* X. For this it suffices to show that for each connected component $X' \subseteq X$, $g(V \cap g^{-1}X') = X'$. Without loss of generality, then, we may assume that X is connected, so $X' = X$.

Since g is flat, if $y \in g(V)$ then the generic point x_1 of any irreducible component X_1 of X containing y is also in $g(V)$. In fact $X_1 \subseteq g(V)$, else the preceding argument applied to $\bar{V} := Z \setminus V$ would show that $x_1 \in g(\bar{V}) = X \setminus g(V)$. It results that some open neighborhood of y is in $g(V)$; and thus $g(V)$ is open. Similarly, $g(\bar{V}) = X \setminus g(V)$ is open, so $g(V)$ is closed.

The conclusion follows, with I the idempotent $\mathcal{O}_X$-ideal corresponding to the open-and-closed set $g(V) \subseteq X$. □

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