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Mutation classes of $\tilde{A}_n$-quivers and derived equivalence classification of cluster tilted algebras of type $\tilde{A}_n$

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We give an explicit description of the mutation classes of quivers of type $\tilde{A}_n$. Furthermore, we provide a complete classification of cluster tilted algebras of type $\tilde{A}_n$ up to derived equivalence. We show that the bounded derived category of such an algebra depends on four combinatorial parameters of the corresponding quiver.

1. Introduction

A few years ago, Fomin and Zelevinsky [2002] introduced the concept of cluster algebras, which rapidly became a successful research area. Cluster algebras nowadays link various areas of mathematics, like combinatorics, Lie theory, algebraic geometry, representation theory, integrable systems, Teichmüller theory, Poisson geometry and also string theory in physics (via recent work on quivers with superpotentials; see [Derksen et al. 2008; Labardini-Fragoso 2009]).

In an attempt to categorify cluster algebras, which a priori are combinatorially defined, cluster categories have been introduced by Buan, Marsh, Reineke, Reiten and Todorov [Buan et al. 2006]. For a quiver Q without loops and oriented 2-cycles and the corresponding path algebra KQ (over an algebraically closed field K), the cluster category $\mathscr{C}_Q$ is the orbit category of the bounded derived category $D^b(KQ)$ by the functor $\tau^{-1}[1]$, where τ denotes the Auslander–Reiten translation and $[1]$ is the shift functor on the triangulated category $D^b(KQ)$.

Important objects in cluster categories are the cluster-tilting objects. The endomorphism algebras of such objects in the cluster category $\mathscr{C}_Q$ are called cluster tilted algebras of type Q [Buan et al. 2007]. Cluster tilted algebras have several interesting properties; for example, their representation theory can be completely understood in terms of the representation theory of the corresponding path algebra of a quiver (ibid.). These algebras have been studied by various authors; see for instance [Assem et al. 2008a, 2008b; Buan et al. 2008; Caldero et al. 2006].

MSC2000: primary 16G20; secondary 18E35, 18E30.

Keywords: cluster tilted algebra, quiver mutation, derived equivalence.

In recent years, a focal point in the representation theory of algebras has been the investigation of derived equivalences of algebras. Since a lot of properties and invariants of rings and algebras are preserved by derived equivalences, it is important for many purposes to classify classes of algebras up to derived equivalence, instead of Morita equivalence. For self-injective algebras, the representation type is preserved under derived equivalences [Krause 1997; Rickard 1989a]. It has also been proved in [Rickard 1991] that the class of symmetric algebras is closed under derived equivalences. Additionally, derived equivalent algebras have the same number of pairwise nonisomorphic simple modules and isomorphic centers.

In this work, we are concerned with the problem of derived equivalence classification of cluster tilted algebras of type $\tilde{A}_n$. Such a classification was done for cluster tilted algebras of type A_n by Buan and Vatne [2008]; see also [Murphy 2010] on the more general case of m-cluster tilted algebras of type A_n.

Since the quivers of cluster tilted algebras of type $\tilde{A}_n$ are exactly the quivers in the mutation classes of $\tilde{A}_n$, our first aim in this paper is to give a description of the mutation classes of $\tilde{A}_n$-quivers; these mutation classes are known to be finite (for example, see [Fomin et al. 2008]). The second purpose of this note is to describe, when two cluster tilted algebras of type Q have equivalent derived categories, where Q is a quiver whose underlying graph is $\tilde{A}_n$.

In Definition 3.3 we present a class $\mathcal{Q}_n$ of quivers with $n+1$ vertices that includes all nonoriented cycles of length $n+1$. To show that this class contains all quivers mutation-equivalent to some quiver of type $\tilde{A}_n$ we first prove that this class is closed under quiver mutation. Furthermore, we define parameters $r_1,\ r_2,\ s_1$ and s_2 for any quiver $Q \in \mathcal{Q}_n$ in Definition 3.7 and prove that every quiver in $\mathcal{Q}_n$ with parameters $r_1,\ r_2,\ s_1$ and s_2 can be mutated to a normal form, see Figure 1, without changing the parameters.

With the help of the result above we can show that every quiver $Q \in \mathcal{Q}_n$ with parameters $r_1,\ r_2,\ s_1$ and s_2 is mutation-equivalent to some nonoriented cycle with $r := r_1 + 2r_2$ arrows in one direction and $s := s_1 + 2s_2$ arrows in the other. Hence, if two quivers Q_1 and Q_2 of $\mathcal{Q}_n$ have the parameters $r_1,\ r_2,\ s_1,\ s_2$, respectively $\tilde{r}_1,\ \tilde{r}_2,\ \tilde{s}_1,\ \tilde{s}_2$ and $r_1 + 2r_2 = \tilde{r}_1 + 2\tilde{r}_2,\ s_1 + 2s_2 = \tilde{s}_1 + 2\tilde{s}_2$ (or vice versa), then Q_1 is mutation-equivalent to Q_2.

The converse of this result — an explicit description of the mutation classes of quivers of type $\tilde{A}_n$ — can be shown with the help of [Fomin et al. 2008, Lemma 6.8].

The main result of the derived equivalence classification of cluster tilted algebras of type $\tilde{A}_n$ is the following theorem:

Theorem 1.1. *Two cluster tilted algebras of type $\tilde{A}_n$ are derived equivalent if and only if their quivers have the same parameters $r_1,\ r_2,\ s_1$ and s_2 (up to changing the roles of r_i and s_i, $i \in \{1, 2\}$).*

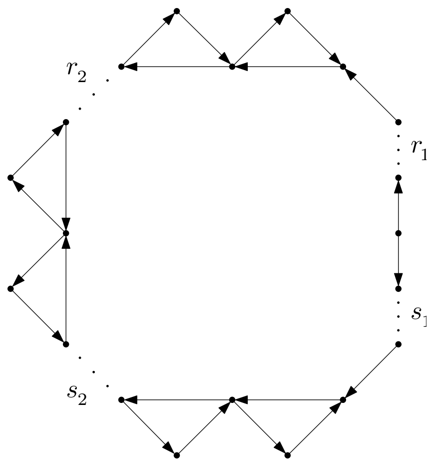


Figure 1. Normal form for quivers in $\mathcal{Q}_n$.

We prove that every cluster tilted algebra of type $\tilde{A}_n$ with parameters r_1, r_2, s_1 and s_2 is derived equivalent to a cluster tilted algebra corresponding to a quiver in normal form. Furthermore, we compute the parameters r_1, r_2, s_1 and s_2 as combinatorial derived invariants for a quiver $Q \in \mathcal{Q}_n$ with the help of an algorithm defined by Avella-Alaminos and Geiß [2008].

The paper is organized as follows. In Section 2 we collect some basic notions about quiver mutations. In Section 3 we present the set $\mathcal{Q}_n$ of quivers that can be obtained by iterated mutation from quivers whose underlying graph is of type $\tilde{A}_n$. Moreover, we describe, when two quivers of $\mathcal{Q}_n$ are in the same mutation class. In the fourth section we describe the cluster tilted algebras of type $\tilde{A}_n$ and their relations (as shown in [Assem et al. 2010]). In Section 5 we first briefly review the fundamental results on derived equivalences. Afterwards, we prove our main result, that is, we show, when two cluster tilted algebras of type $\tilde{A}_n$ are derived equivalent.

2. Quiver mutations

A *quiver* is a finite directed graph Q, consisting of a finite set of vertices Q_0 and a finite set of arrows Q_1 between them.

Let Q be a quiver and K be an algebraically closed field. We can form the *path algebra* KQ, where the basis of KQ is given by all paths in Q, including trivial paths e_i of length zero at each vertex i of Q. Multiplication in KQ is defined by concatenation of paths. Our convention is to read paths from right to left. For any path α in Q let $s(\alpha)$ denote its start vertex and $t(\alpha)$ its end vertex. Then the product of two paths α and β is defined to be the concatenated path $\alpha\beta$ if $s(\alpha) = t(\beta)$. The unit element of KQ is the sum of all trivial paths: $1_{KQ} = \sum_{i \in Q_0} e_i$.

We now recall the definition of quiver mutations.

Definition 2.1 [Fomin and Zelevinsky 2002]. Let Q be a quiver without loops and oriented 2-cycles. The *mutation* of Q at a vertex k to a new quiver Q^* can be described as follows:

(1) Add a new vertex k^*.

(2) If there are $r > 0$ arrows $i \to k$, $s > 0$ arrows $k \to j$ and $t \in \mathbb{Z}$ arrows $j \to i$ in Q, there are $t - rs$ arrows $j \to i$ in Q^*. (Here, a negative number of arrows means arrows in the opposite direction.)

(3) For any vertex i replace all arrows from i to k with arrows from k^* to i, and replace all arrows from k to i with arrows from i to k^*.

(4) Remove the vertex k.

Note that mutation at sinks or sources only means changing the direction of all incoming or outgoing arrows. Two quivers are called *mutation-equivalent* (or *sink/source equivalent*) if one can be obtained from the other by a finite sequence of mutations (at sinks or sources). The *mutation class* of a quiver Q is the class of all quivers mutation-equivalent to Q.

3. Mutation classes of $\tilde{A}_n$-quivers

Remark 3.1. Quivers of type $\tilde{A}_n$ are just cycles with $n+1$ vertices. If the cycle is oriented, we get the mutation class of D_{n+1} (see [Derksen and Owen 2008; Fomin et al. 2008, 2003] or Type IV in type D in [Vatne 2010]). If the cycle is nonoriented, we get what we call the mutation classes of $\tilde{A}_n$.

First, we have to fix one drawing (plane embedding) of this nonoriented cycle. Thus, we can speak of clockwise and anticlockwise oriented arrows. But we have to consider that this notation is only unique up to reflection of the cycle, i.e., up to changing the roles of clockwise and anticlockwise oriented arrows.

Lemma 3.2 [Fomin et al. 2008, Lemma 6.8]. *Let C_1 and C_2 be two nonoriented cycles, so that in C_1 there are s arrows oriented clockwise and r arrows oriented anticlockwise. Similarly, in C_2 there are $\tilde{s}$ arrows oriented clockwise and $\tilde{r}$ arrows oriented anticlockwise. Then C_1 and C_2 are mutation-equivalent if and only if the unordered pairs $\{r, s\}$ and $\{\tilde{r}, \tilde{s}\}$ coincide.*

Thus, two nonoriented cycles of length $n+1$ are mutation-equivalent if and only if they have the same parameters r and s (up to changing the roles of r and s).

Next we provide an explicit description of the mutation classes of $\tilde{A}_n$-quivers. For this we need a description of the mutation class of quivers of type A_k. We use one given in [Buan and Vatne 2008]:

- Each quiver has k vertices.
- All nontrivial cycles are oriented and of length 3.
- A vertex has at most four incident arrows.
- If a vertex has four incident arrows, then two of them belong to one oriented 3-cycle, and the other two belong to another oriented 3-cycle.
- If a vertex has three incident arrows, then two of them belong to an oriented 3-cycle, and the third arrow does not belong to any oriented 3-cycle.

(By a *cycle* in the second condition we mean a cycle in the underlying graph not passing through the same edge twice. In particular, this condition excludes multiple arrows.)

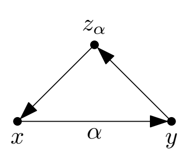
For another description of mutation classes of type A quivers, see [Seven 2007].

Now we can formulate the description of the mutation classes of $\tilde{A}_n$-quivers, similar to the description for Type IV in type D in [Vatne 2010].

Definition 3.3. Let $\mathcal{Q}_n$ be the class of connected quivers with $n+1$ vertices that satisfy the following conditions (see Figure 2 for an illustration):

(i) There exists precisely one full subquiver that is a nonoriented cycle of length ≥ 2. Thus, if the length is two, it is a double arrow.

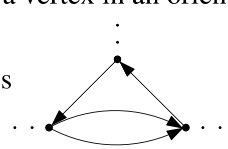
(ii) For each arrow $x \xrightarrow{\alpha} y$ in this nonoriented cycle, there may (or may not) be a vertex z_α not on the nonoriented cycle and such that there is an oriented 3-cycle of the form



Apart from the arrows of these oriented 3-cycles there are no other arrows incident to vertices on the nonoriented cycle.

(iii) If we remove all vertices in the nonoriented cycle and their incident arrows, the result is a disjoint union of quivers $Q_1, Q_2, \ldots$, one for each z_α (which we call Q_α). These are quivers of type A_{k_α} for $k_\alpha \geq 1$, and the vertices z_α have at most two incident arrows in these quivers. Furthermore, if a vertex z_α has two incident arrows in such a quiver, then z_α is a vertex in an oriented 3-cycle in Q_α.

Our convention is to choose only one of the double arrows to be part of the oriented 3-cycle in the case shown here:



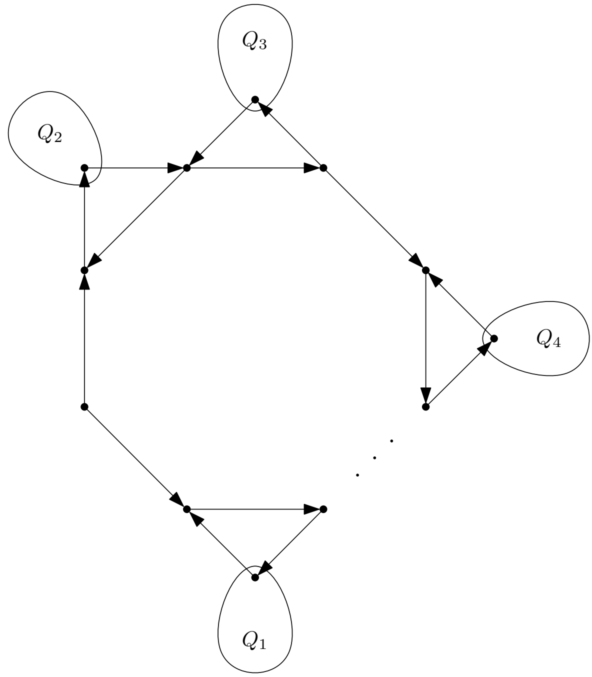
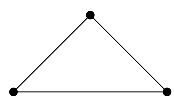


Figure 2. Quiver in $\mathcal{Q}_n$.

Notation 3.4. Whenever we draw an edge $\overset{j}{\bullet}\!\!-\!\!-\!\!-\!\!-\!\!\overset{k}{\bullet}$ the direction of the arrow between j and k is not important for this situation; and whenever we draw a cycle



it is an oriented 3-cycle.

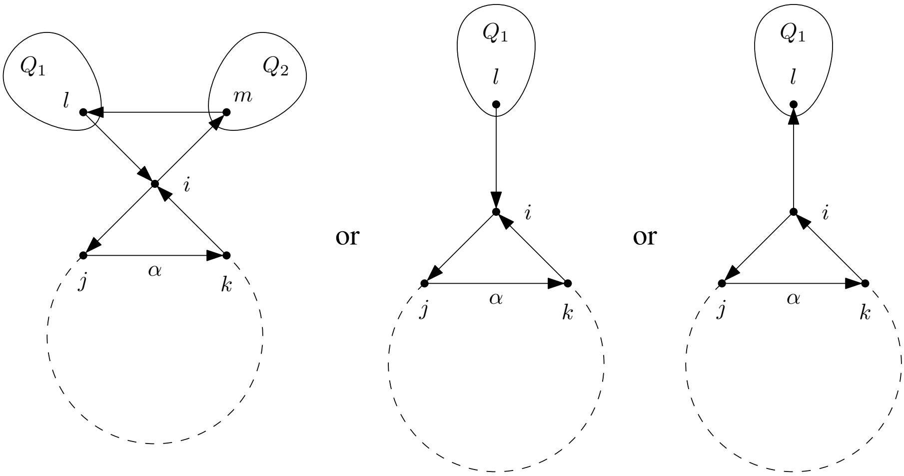
Lemma 3.5. *$\mathcal{Q}_n$ is closed under quiver mutation.*

Proof. Let Q be a quiver in $\mathcal{Q}_n$ and let i be some vertex of Q. The subquivers Q_1 and Q_2 highlighted in the pictures are quivers of type A.

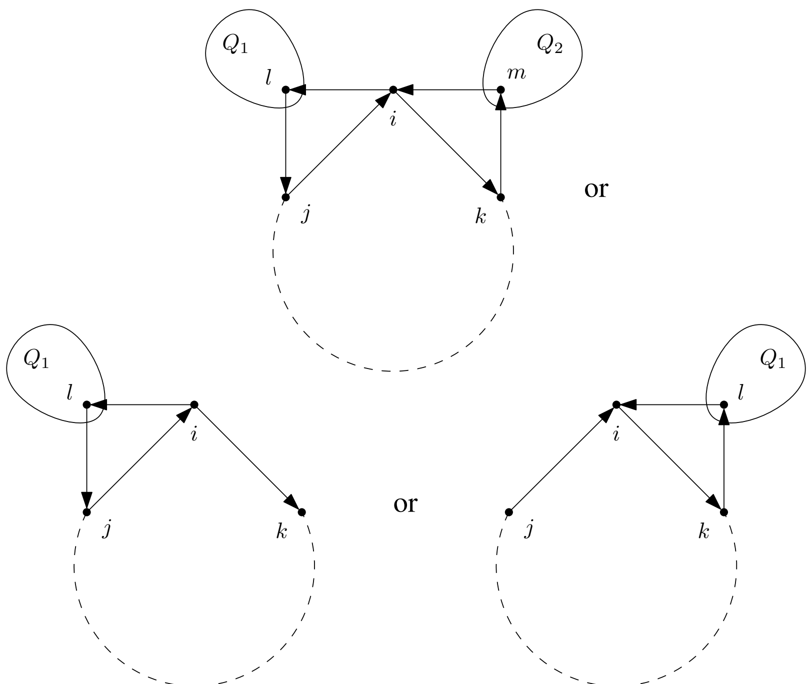
If i is a vertex in one of the quivers Q_α of type A, but not one of the vertices z_α connecting this quiver of type A to the rest of the quiver Q, then the mutation at i leads to a quiver $Q^* \in \mathcal{Q}_n$ since type A is closed under quiver mutation.

It therefore suffices to check what happens when we mutate at the other vertices, and we will consider four cases:

(1) Let i be one of the vertices z_α, hence not on the nonoriented cycle. For the situation where the quiver Q_α of type A attached to z_α consists only of one vertex, we can look at the first mutated quiver in case (2) below since quiver mutation is an involution. Thus, we have three cases:

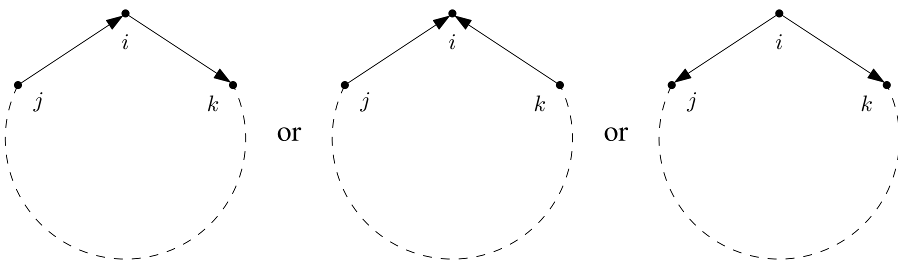


Then the mutation at i leads to the following three quivers, which have a nonoriented cycle one arrow longer than for Q, and this is indeed a nonoriented cycle since the arrows $j \to i \to k$ have the same orientation as α had before.

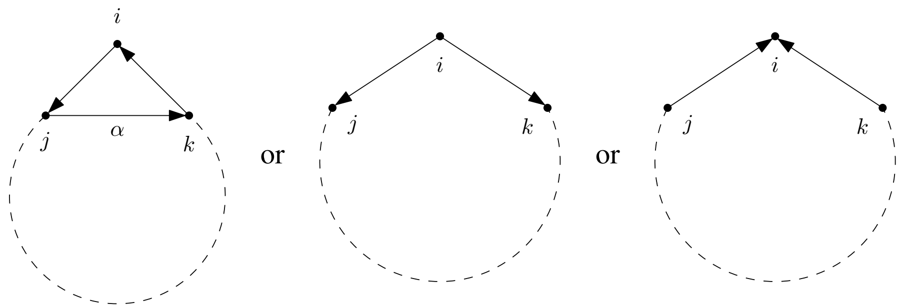


The vertices l and m have at most two incident arrows in the quivers Q_1 and Q_2 since they had at most four resp. three incident arrows in Q (see the description of quivers of type A). Furthermore, if l or m has two incident arrows in the quiver Q_1 or Q_2, then these two arrows form an oriented 3-cycle as in Q. Thus, the mutated quiver Q^* is also in $\mathfrak{Q}_n$.

(2) Let i be a vertex on the nonoriented cycle, and not part of any oriented 3-cycle. Three cases can occur:



and mutation at i leads to



If i is a sink or a source in Q, the nonoriented cycle in Q^* is of the same length as before and Q^* is in $\mathcal{Q}_n$. If there is a path $j \to i \to k$ in Q, mutation at i leads to a quiver Q^*, which has a nonoriented cycle one arrow shorter than in Q.

Note that in this case the nonoriented cycle in Q consists of at least three arrows and thus, the nonoriented cycle in Q^* has at least two arrows. Thus, the mutated quiver Q^* is also in $\mathcal{Q}_n$.

(3) Let i be a vertex on the nonoriented cycle that is part of exactly one oriented 3-cycle. Then four cases can occur, but two of them have been dealt with by the second and third mutated quiver in case (1) since quiver mutation is an involution. Thus, we only have to consider the two situations shown in Figure 3 and their special cases where the nonoriented cycle is a double arrow. (The two-headed arrows indicate mutation at i.)

After mutating at vertex i, the nonoriented cycle has the same length as before. Moreover, l has the same number of incident arrows as before. Thus, Q^* is in $\mathcal{Q}_n$.

(4) Let i be a vertex on the nonoriented cycle that is part of two oriented 3-cycles. Then three cases can occur, but one of them has been dealt with by the first mutated quiver in case (1). Thus, we have to consider only the situations in Figure 4 and their special cases where the nonoriented cycle is a double arrow.

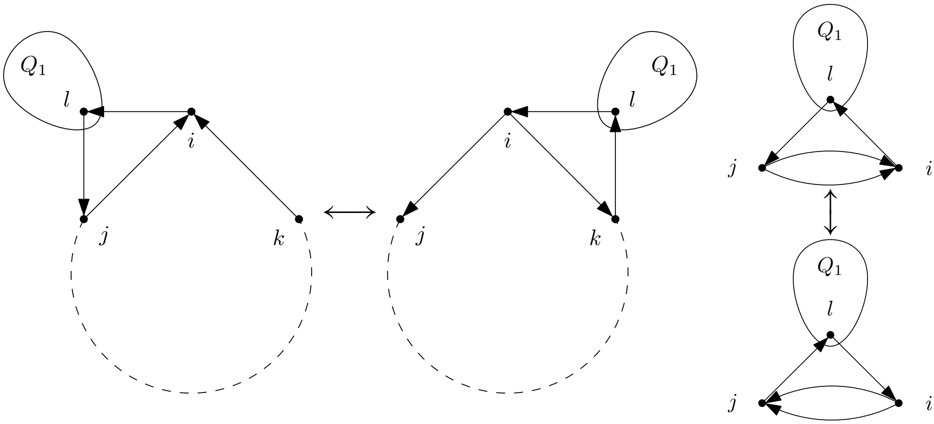


Figure 3. Possibilities in case (3).

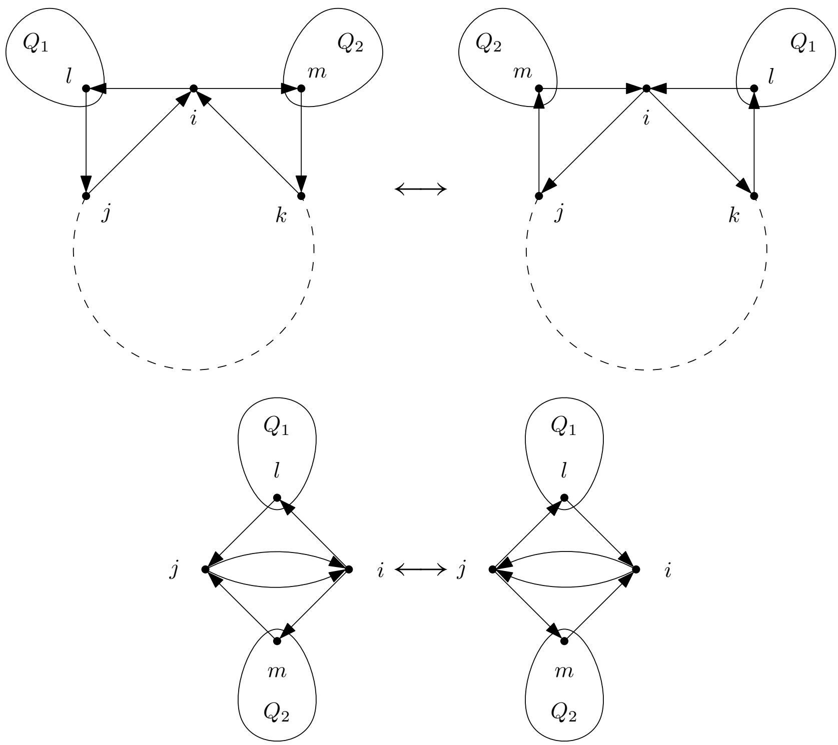


Figure 4. Possibilities in case (4).

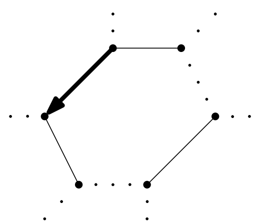
The nonoriented cycle has the same length as before. Moreover, l and m have the same number of incident arrows as before. Thus, again, the mutated quiver Q^* belongs to $\mathcal{Q}_n$. $\square$

Remark 3.6. It is easy to see that all orientations of a circular quiver of type $\tilde{A}_n$ are in $\mathcal{Q}_n$ (except the oriented case; but this leads to the mutation class of D_{n+1}). Since $\mathcal{Q}_n$ is closed under quiver mutation every quiver mutation-equivalent to some quiver of type $\tilde{A}_n$ is in $\mathcal{Q}_n$, too.

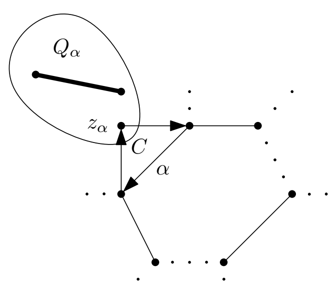
Now we fix one drawing of a quiver $Q \in \mathcal{Q}_n$, without arrow crossing. Thus, we can again speak of clockwise and anticlockwise oriented arrows of the nonoriented cycle. But we have to consider that this notation is only unique up to reflection of the nonoriented cycle, that is, up to changing the roles of clockwise and anticlockwise oriented arrows. We define four parameters r_1, r_2, s_1 and s_2 for a quiver $Q \in \mathcal{Q}_n$ as follows:

Definition 3.7. • Let r_1 be the number of arrows that are not part of any oriented 3-cycle and that fulfill one of two conditions:

(1) The arrow is part of the nonoriented cycle and is oriented anticlockwise:



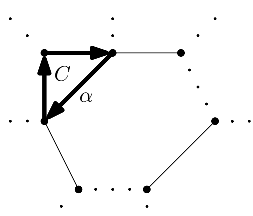
(2) The arrow is not part of the nonoriented cycle, but is attached to an oriented 3-cycle C sharing with the nonoriented cycle one arrow α that is oriented anticlockwise (see figure on the right).



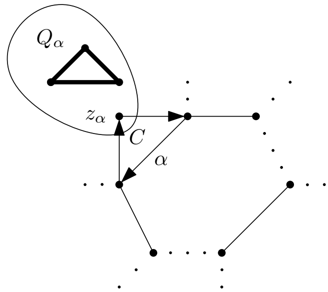
In this sense, "attached" means that the arrow is part of the quiver Q_α of type A that shares the vertex z_α with the cycle C (see Definition 3.3).

• Let r_2 be the number of oriented 3-cycles that fulfill one of two conditions:

(1) The cycle shares with the nonoriented cycle one arrow α that is oriented anticlockwise:



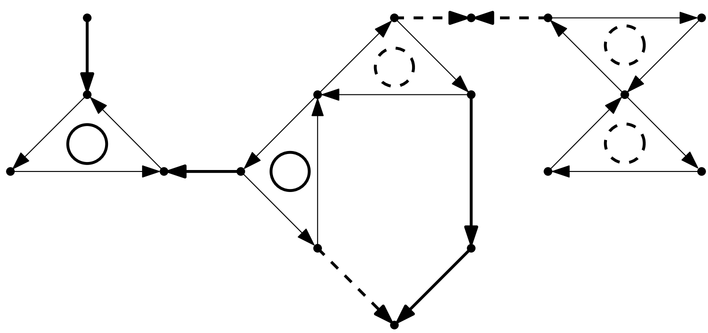
(2) The cycle is attached to an oriented 3-cycle C sharing one arrow α with the nonoriented cycle and α is oriented anticlockwise:



Here, “attached” is in the same sense as above.

- Similarly we define the parameters s_1 and s_2 with “clockwise” instead of “anticlockwise”.

Example 3.8. We denote the arrows that count for the parameter r_1 by $\dashrightarrow$ and the arrows that count for s_1 by $\longrightarrow$. Furthermore, the oriented 3-cycles of r_2 are denoted by and the oriented 3-cycles of s_2 are denoted by . Let $Q \in \mathcal{Q}_{16}$ be a quiver of the form



Then $r_1 = 3,\ r_2 = 3,\ s_1 = 4$ and $s_2 = 2$.

Lemma 3.9. *If Q_1 and Q_2 are quivers in $\mathcal{Q}_n$, and Q_1 and Q_2 have the same parameters $r_1,\ r_2,\ s_1$ and s_2 (up to interchanging r_1 with s_1 and r_2 with s_2), then Q_2 can be obtained from Q_1 by iterated mutation, where all the intermediate quivers have the same parameters as well.*

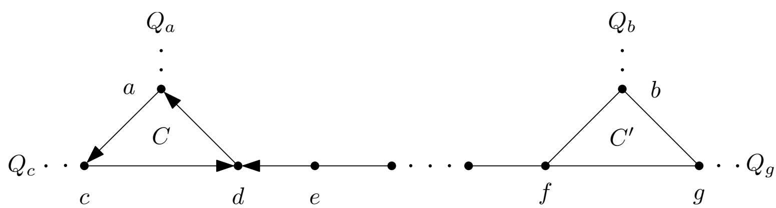
Proof. It is enough to show that all quivers in $\mathcal{Q}_n$ with parameters $r_1,\ r_2,\ s_1\, s_2$ can be mutated to a quiver in *normal form* (see Figure 1) without changing the parameters $r_1,\ r_2,\ s_1\, s_2$. In such a quiver, r_1 is the number of anticlockwise arrows in the nonoriented cycle that do not share any arrow with an oriented 3-cycle and

s_1 is the number of clockwise arrows in the nonoriented cycle that do not share any arrow with an oriented 3-cycle. Furthermore, r_2 is the number of oriented 3-cycles sharing one arrow α with the nonoriented cycle and α is oriented anticlockwise and s_2 is the number of oriented 3-cycles sharing one arrow β with the nonoriented cycle and β is oriented clockwise (see Definition 3.7).

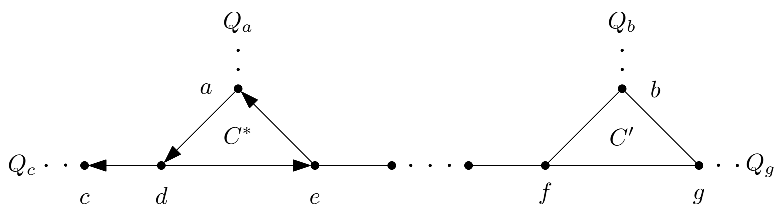
We divide this process into five steps.

Step 1: Let Q be a quiver in $\mathcal{Q}_n$. We move all oriented 3-cycles of Q sharing no arrow with the nonoriented cycle towards the oriented 3-cycle that is attached to them and that shares one arrow with the nonoriented cycle.

Method: Let C and C' be a pair of neighboring oriented 3-cycles in Q (i.e., no arrow in the path between them is part of an oriented 3-cycle) such that the length of the path between them is at least one. We want to move C and C' closer together by mutation.



In the picture the Q_i are subquivers of Q. Mutating at d will produce a quiver Q^* looking like this:

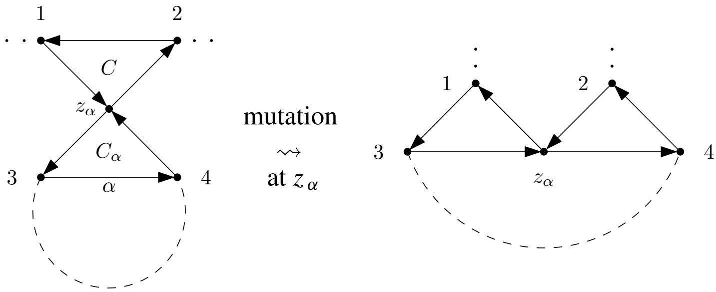


Thus, the length of the path between C^* and C' decreases by 1 and there is a path of length one between C^* and Q_c. The arguments for a quiver with arrow $d \to e$ are analogous and these mutations can also be used if the arrows between d and f are part of the nonoriented cycle (see Step 4).

In this procedure, the parameters r_1, r_2, s_1 and s_2 are left unchanged since we are not changing the number of arrows and the number of oriented 3-cycles which are attached to an oriented 3-cycle sharing one arrow with the nonoriented cycle.

Step 2: We move all oriented 3-cycles onto the nonoriented cycle.

Method: Let C be an oriented 3-cycle that shares one vertex z_α with an oriented 3-cycle C_α sharing an arrow α with the nonoriented cycle. Then we mutate at the vertex z_α:



Hence, both of the oriented 3-cycles share one arrow with the nonoriented cycle and these arrows are oriented as α was before. Thus, the parameters r_1, r_2, s_1 and s_2 are left unchanged. Furthermore, the length of the nonoriented cycle increases by 1. By iterated mutation of that kind, we produce a quiver Q^*, where all the oriented 3-cycles share an arrow with the nonoriented cycle.

Step 3: We move all arrows onto the nonoriented cycle.

Method: This is a similar process as in Step 2: Let C_α be an oriented 3-cycle that shares an arrow α with the nonoriented cycle. All arrows attached to C_α can be moved into the nonoriented cycle by iteratively mutating at vertex z_α. After mutating, all these arrows have the same orientation as α in the nonoriented cycle. Thus, the parameters r_1, r_2, s_1 and s_2 are left unchanged.

Step 4: Move oriented 3-cycles along the nonoriented cycle.

Method: First, we number all oriented 3-cycles by $C_1, \ldots, C_{r_2+s_2}$ in such a way that C_{i+1} follows C_i anticlockwise. As in Step 1, we can move an oriented 3-cycle C_i towards C_{i+1} without changing the orientation of the arrows, that is, without changing the parameters r_1, r_2, s_1 and s_2.

If the nonoriented cycle includes the vertex a in the pictures of Step 1, the arrows between the two cycles move to the top of C_{i+1}, that is, they are no longer part of the nonoriented cycle. However, we can reverse their directions by mutating at the new sinks or sources and insert these arrows into the nonoriented cycle between C_{i+1} and C_{i+2} by mutations like in Step 3 (if C_{i+2} exists).

Doing this iteratively, we produce a quiver Q^* as in Figure 5, with r_1+s_1 arrows that are not part of any oriented 3-cycle and r_2+s_2 oriented 3-cycles sharing one arrow with the nonoriented cycle.

Step 5: Change orientation on the nonoriented cycle to the orientation of Figure 1.

Method: The part of the nonoriented cycle without oriented 3-cycles can be moved to the desired orientation of Figure 1 via sink/source mutations, without mutating

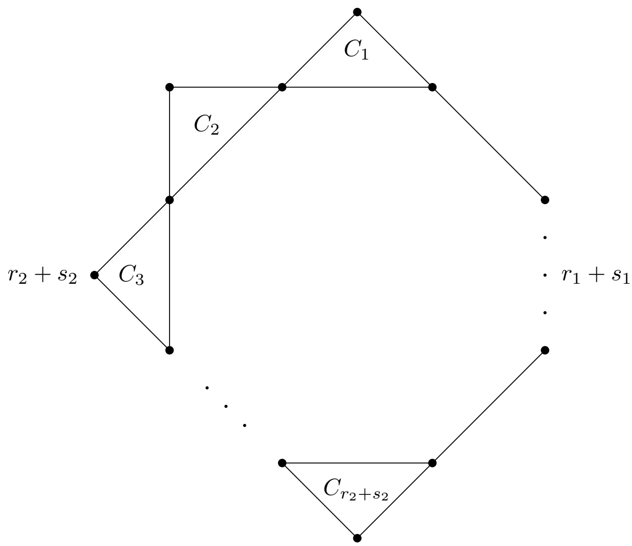
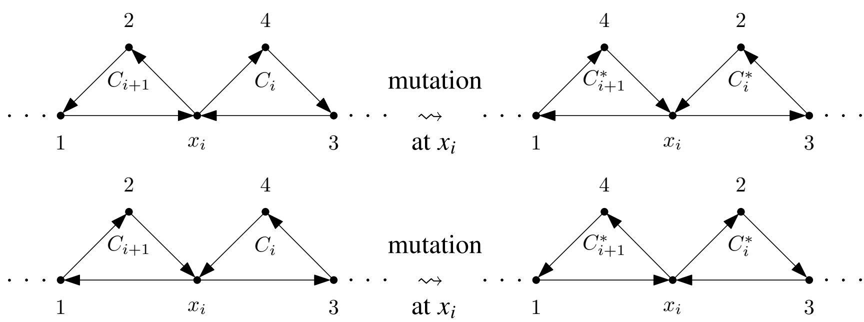


Figure 5. Normal form of Step 4.

at the end vertices that are attached to oriented 3-cycles. Thus, the parameters r_1 and s_1 are left unchanged.

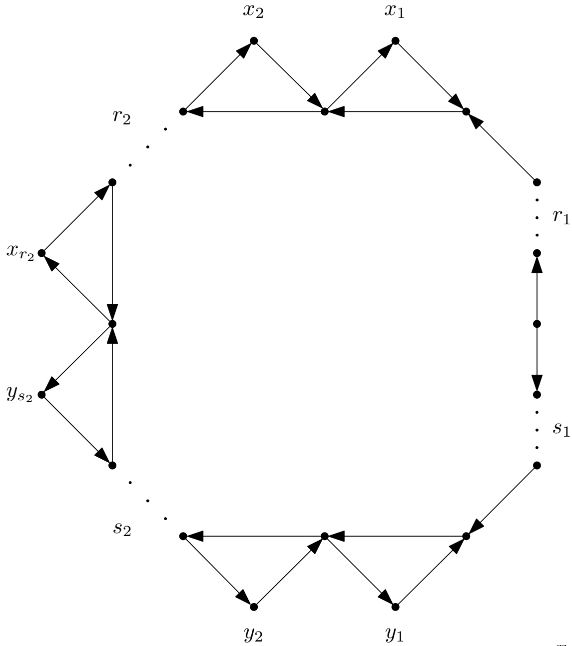
Each oriented 3-cycle shares one arrow with the nonoriented cycle. If all of these arrows are oriented in the same direction, the quiver is in the required form. Thus, we can assume that there are at least two arrows of two oriented 3-cycles C_i and C_{i+1} having opposite orientations. If we mutate at the connecting vertex of C_i and C_{i+1}, the directions of these arrows are changed:

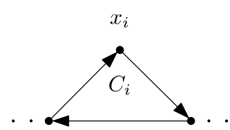


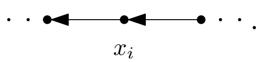
Hence, these mutations act like sink/source mutations at the nonoriented cycle and the parameters r_2 and s_2 are left unchanged. Thus, we can mutate at such connecting vertices as in the part without oriented 3-cycles to reach the desired orientation of Figure 1. $\square$

Theorem 3.10. *Let $Q \in \mathcal{Q}_n$ with parameters r_1, r_2, s_1 and s_2. Then Q is mutation-equivalent to a nonoriented cycle of length $n+1$ with parameters $r = r_1 + 2r_2$ and $s = s_1 + 2s_2$.*

Proof. We can assume that Q is in normal form (see Lemma 3.9) and we label the vertices z_α as follows:

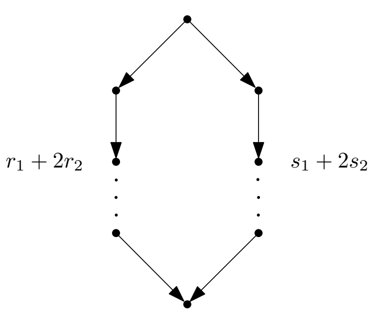


Mutation at the vertex x_i of an oriented 3-cycle 

leads to two arrows of the form 

.

Thus, after mutating at all the x_i, the parameter r_2 is zero and we have a new parameter $r = r_1 + 2r_2$. Similarly, we get $s = s_1 + 2s_2$. Hence, mutating at all the x_i and y_i leads to a quiver with underlying graph $\tilde{A}_n$ as follows:



Since there is a nonoriented cycle in every $Q \in \mathcal{Q}_n$, both r and s are nonzero. Thus, the cycle above is also nonoriented. Hence, Q is mutation-equivalent to some quiver of type $\tilde{A}_n$ with parameters $r = r_1 + 2r_2$ and $s = s_1 + 2s_2$. $\square$

Corollary 3.11. *Let $Q_1, Q_2 \in \mathcal{Q}_n$ with parameters $r_1,\ r_2,\ s_1$ and s_2, respectively $\tilde{r}_1,\ \tilde{r}_2,\ \tilde{s}_1$ and $\tilde{s}_2$. If $r_1 + 2r_2 = \tilde{r}_1 + 2\tilde{r}_2$ and $s_1 + 2s_2 = \tilde{s}_1 + 2\tilde{s}_2$, or vice versa, then Q_1 is mutation-equivalent to Q_2.*

Theorem 3.12. *Let $Q_1, Q_2 \in \mathcal{Q}_n$ with parameters $r_1,\ r_2,\ s_1$ and s_2, respectively $\tilde{r}_1,\ \tilde{r}_2,\ \tilde{s}_1$ and $\tilde{s}_2$. Then Q_1 is mutation-equivalent to Q_2 if and only if*

$$r_1 + 2r_2 = \tilde{r}_1 + 2\tilde{r}_2 \quad \textit{and} \quad s_1 + 2s_2 = \tilde{s}_1 + 2\tilde{s}_2$$

or

$$r_1 + 2r_2 = \tilde{s}_1 + 2\tilde{s}_2 \quad \textit{and} \quad s_1 + 2s_2 = \tilde{r}_1 + 2\tilde{r}_2.$$

The "only if" part follows from Theorem 3.10 and Lemma 3.2.

4. Cluster tilted algebras of type $\tilde{A}_n$

In general, cluster tilted algebras arise as endomorphism algebras of cluster-tilting objects in a cluster category [Buan et al. 2007]. Since a cluster tilted algebra A of type $\tilde{A}_n$ is finite dimensional over an algebraically closed field K, there exists a quiver Q which is in the mutation classes of $\tilde{A}_n$ [Buan et al. 2008] and an admissible ideal I of the path algebra KQ of Q such that $A \cong KQ/I$. A nonzero linear combination $k_1\alpha_1 + \cdots + k_m\alpha_m,\ k_i \in K\backslash\{0\}$, of paths α_i of length at least two, with the same starting point and the same end point, is called a *relation* in Q. If $m = 1$, we call such a relation a *zero-relation*. Any admissible ideal of KQ is generated by a finite set of relations in Q.

From [Assem et al. 2010] and [Assem and Redondo 2009], we know that a cluster tilted algebra A of type $\tilde{A}_n$ is gentle, a notion whose definition we recall:

Definition 4.1. We call $A = KQ/I$ a *special biserial algebra* if these properties hold:

(1) Each vertex of Q is the starting point of at most two arrows and the end point of at most two arrows.

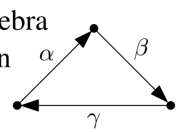
(2) For each arrow α in Q there is at most one arrow β such that $\alpha\beta \notin I$, and at most one arrow γ such that $\gamma\alpha \notin I$.

A is *gentle* if moreover:

(3) The ideal I is generated by paths of length 2.

(4) For each arrow α in Q there is at most one arrow β' with $t(\alpha) = s(\beta')$ such that $\beta'\alpha \in I$, and there is at most one arrow γ' with $t(\gamma') = s(\alpha)$ such that $\alpha\gamma' \in I$.

Also from the same references, all relations in a cluster tilted algebra A of type $\tilde{A}_n$ occur in the oriented 3-cycles (cycles of the form on the right with (zero-)relations $\alpha\gamma,\ \beta\alpha$ and $\gamma\beta$).



Remark 4.2. According to our convention in Definition 3.3 there are only three (zero-)relations in the quiver

$$\begin{tikzcd} & \bullet \arrow[dl, "\beta"'] & \\ \bullet \arrow[rr, bend left, "\delta"] \arrow[rr, bend right, "\gamma"'] & & \bullet \arrow[ul, "\alpha"'] \end{tikzcd}$$

and here, these are $\alpha\delta$, $\beta\alpha$ and $\delta\beta$.

For the next section, we need the notion of Cartan matrices of an algebra A (for example, see [Holm 2005]). Let K be a field and $A = KQ/I$. Since $\sum_{i \in Q_0} e_i + I$ is the unit element in A we get $A = A \cdot 1 = \bigoplus_{i \in Q_0} Ae_i$; hence the (left) A-modules $P_i := Ae_i$ are the indecomposable projective A-modules. The *Cartan matrix* $C = (c_{ij})$ of A is a $|Q_0| \times |Q_0|$-matrix defined by setting $c_{ij} = \dim_K \operatorname{Hom}_A(P_j, P_i)$. Any homomorphism $\varphi : Ae_j \to Ae_i$ of left A-modules is uniquely determined by $\varphi(e_j) \in e_j Ae_i$, the K-vector space generated by all paths in Q from vertex i to vertex j that are nonzero in A. In particular, $c_{ij} = \dim_K e_j Ae_i$.

That means that computing entries of the Cartan matrix for A reduces to counting paths in Q that are nonzero in A.

5. Derived equivalence classification of cluster tilted algebras of type $\tilde{A}_n$

We briefly review the fundamental results on derived equivalences. For a K-algebra A the bounded derived category of A-modules is denoted by $D^b(A)$. Recall that two algebras A, B are called derived equivalent if $D^b(A)$ and $D^b(B)$ are equivalent as triangulated categories. By a celebrated theorem of Rickard (Theorem 5.2), derived equivalences can be found using the concept of tilting complexes.

Definition 5.1. A tilting complex T over A is a bounded complex of finitely generated projective A-modules satisfying the following conditions:

(i) $\operatorname{Hom}_{D^b(A)}(T, T[i]) = 0$ for all $i \neq 0$, where $[\,\cdot\,]$ denotes the shift functor in $D^b(A)$.

(ii) The category $\operatorname{add}(T)$ (i.e., the full subcategory consisting of direct summands of direct sums of T) generates the homotopy category $K^b(P_A)$ of projective A-modules as a triangulated category.

Theorem 5.2 [Rickard 1989b]. *Two algebras A and B are derived equivalent if and only if there exists a tilting complex T for A such that the endomorphism algebra* $\operatorname{End}_{D^b(A)}(T) \cong B$.

For calculating the endomorphism algebra $\operatorname{End}_{D^b(A)}(T)$ we can use the following alternating sum formula, which gives a general method for computing the Cartan matrix of an endomorphism algebra of a tilting complex from the Cartan matrix of the algebra A.

Proposition 5.3 [Happel 1988]. *For an algebra A let $Q = (Q^r)_{r\in\mathbb{Z}}$ and $R = (R^s)_{s\in\mathbb{Z}}$ be bounded complexes of projective A-modules. Then*

$$\sum_i (-1)^i \dim \operatorname{Hom}_{D^b(A)}(Q, R[i]) = \sum_{r,s} (-1)^{r-s} \dim \operatorname{Hom}_A(Q^r, R^s).$$

In particular, if Q and R are direct summands of the same tilting complex, then

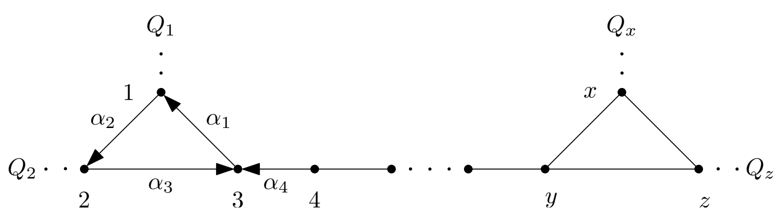
$$\dim \operatorname{Hom}_{D^b(A)}(Q, R) = \sum_{r,s} (-1)^{r-s} \dim \operatorname{Hom}_A(Q^r, R^s).$$

Lemma 5.4. *Let $A = KQ/I$ be a cluster tilted algebra of type $\tilde{A}_n$. Let r_1, r_2, s_1 and s_2 be the parameters of Q that are defined in Definition 3.7. Then A is derived equivalent to a cluster tilted algebra corresponding to a quiver in normal form as in Figure 1.*

Proof. First, the number of oriented 3-cycles with full relations is invariant under derived equivalence for gentle algebras [Holm 2005], so the number $r_2 + s_2$ is an invariant. From [Avella-Alaminos and Geiss 2008, Proposition B], we know that the number of arrows is also invariant under derived equivalence, so the number $r_1 + s_1$ is an invariant, too. Later, we show in the proof of Theorem 5.5 that the single parameters r_1, r_2, s_1 and s_2 are invariant under derived equivalence.

Our strategy in this proof is to go through the proof of Lemma 3.9 and define a tilting complex for each mutation in Steps 1 and 2. We can omit the other three steps since these are just the same situations as in the first two steps. We show that if we mutate at some vertex of the quiver Q and obtain a quiver Q^*, then the two corresponding cluster tilted algebras are derived equivalent.

Step 1: Let A be a cluster tilted algebra with corresponding quiver



We can compute the Cartan matrix to be $\begin{pmatrix} 1 & 1 & 0 & 0 & \dots \\ 0 & 1 & 1 & 0 & \dots \\ 1 & 0 & 1 & 0 & \dots \\ 1 & 0 & 1 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$.

Since we are dealing with left modules and read paths from right to left, a nonzero path from vertex i to j gives a homomorphism $P_j \to P_i$ by right multiplication. Thus, two arrows $\alpha : i \to j$ and $\beta : j \to k$ give a path $\beta\alpha$ from i to k and a homomorphism $\alpha\beta : P_k \to P_i$.

In the situation above, we have homomorphisms $P_3 \xrightarrow{\alpha_3} P_2$ and $P_3 \xrightarrow{\alpha_4} P_4$.

Let $T = \bigoplus_{i=1}^{n+1} T_i$ be the following bounded complex of projective A-modules, where $T_i : 0 \to P_i \to 0$, $i \in \{1, 2, 4, \dots, n+1\}$, are complexes concentrated in degree zero and

$$T_3 : 0 \to P_3 \xrightarrow{(\alpha_3,\alpha_4)} P_2 \oplus P_4 \to 0$$

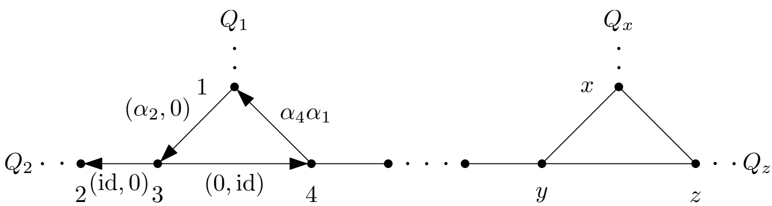
is a complex concentrated in degrees -1 and 0.

We leave it to the reader to verify that this is indeed a tilting complex.

By Rickard's Theorem 5.2, $E := \mathrm{End}_{D^b(A)}(T)$ is derived equivalent to A. Using the alternating sum formula of the Proposition 5.3 of Happel we can compute the Cartan matrix of E to be

$$\begin{pmatrix} 1 & 1 & 1 & 0 & \dots \\ 0 & 1 & 0 & 0 & \dots \\ 0 & 1 & 1 & 1 & \dots \\ 1 & 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

We define homomorphisms in E as follows:



Now we have to check the relations, up to homotopy.

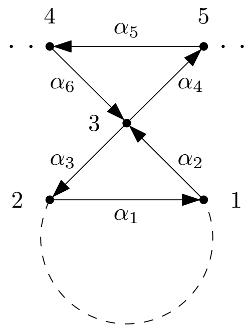
Clearly, the homomorphism $(\alpha_4\alpha_1\alpha_2, 0)$ in the oriented 3-cycle containing the vertices 1, 3 and 4 is zero since $\alpha_1\alpha_2$ was zero in A. Furthermore, the composition of $(\alpha_2, 0)$ and $(0, \mathrm{id})$ yields a zero-relation. The last zero-relation in this oriented 3-cycle is the concatenation of $(0, \mathrm{id})$ and $\alpha_4\alpha_1$ since this homomorphism is homotopic to zero:

$$\begin{tikzcd} & 0 \arrow[r] & P_1 \arrow[r] \arrow[dl, "\alpha_1"'] \arrow[d, "(0,\alpha_4\alpha_1)"] & 0 \\ 0 \arrow[r] & P_3 \arrow[r, "(\alpha_3,\alpha_4)"'] & P_2 \oplus P_4 \arrow[r] & 0 \end{tikzcd}$$

The relations in all other oriented 3-cycles of this quiver are the same as in the quiver of A.

Thus, we have defined homomorphisms between the summands of T corresponding to the arrows of the quiver that we obtain after mutating at vertex 3 in the quiver of A. We have shown that they satisfy the defining relations of this algebra and the Cartan matrices agree. Thus, A is derived equivalent to E and A^{op} is derived equivalent to E^{op}, where the quiver of E is the same as the quiver we obtain after mutating at vertex 3 in the quiver of A. Furthermore, the quivers of A^{op} and E^{op} are the quivers in the other case in Step 1.

Step 2: Let A be a cluster tilted algebra with corresponding quiver



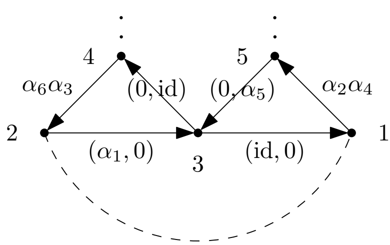
We define a tilting complex T as the bounded complex of projective A-modules $T = \bigoplus_{i=1}^{n+1} T_i$, where $T_i : 0 \to P_i \to 0$, for $i \in \{1, 2, 4, \dots, n+1\}$, are complexes concentrated in degree zero and $T_3 : 0 \to P_3 \xrightarrow{(\alpha_2,\alpha_6)} P_1 \oplus P_4 \to 0$ is a complex concentrated in degrees -1 and 0.

By Rickard's theorem, $E := \mathrm{End}_{D^b(A)}(T)$ is derived equivalent to A. Using Happel's alternating sum formula (Proposition 5.3), we can compute the Cartan matrix of E to be

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 & \dots \\ 1 & 1 & 1 & 0 & 0 & \dots \\ 1 & 0 & 1 & 1 & 0 & \dots \\ 0 & 1 & 0 & 1 & 0 & \dots \\ 0 & 0 & 1 & 1 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

(This deals with the case where not all the arrows between 2 and 1 along the nonoriented cycle are oriented in the same direction. The case where they are can be handled similarly.)

We define homomorphisms in E as follows:



Thus, A is derived equivalent to E and A^{op} is derived equivalent to E^{op}, where the quiver of E is the same as the quiver we obtain after mutating at 3.

In Steps 3 and 4 of the proof of Lemma 3.9 we mutate at a vertex with three incident arrows as in Step 1. In Step 5 we mutate at sinks, sources and at vertices with four incident arrows as in Step 2.

Thus, we obtain a quiver of a derived equivalent cluster tilted algebra by all mutations in the proof of Lemma 3.9. Hence, every cluster tilted algebra $A = KQ/I$ of type $\tilde{A}_n$ is derived equivalent to a cluster tilted algebra with a quiver in normal form having the same parameters as Q. $\square$

Our next aim is to prove the main result:

Theorem 5.5. *Two cluster tilted algebras of type $\tilde{A}_n$ are derived equivalent if and only if their quivers have the same parameters r_1, r_2, s_1 and s_2, up to changing the roles of r_i and s_i for $i \in \{1, 2\}$.*

But first, we recall some background from [Avella-Alaminos and Geiss 2008]. Let $A = KQ/I$ be a gentle algebra, where $Q = (Q_0, Q_1)$ is a connected quiver. A *permitted path* of A is a path $C = \alpha_l \dots \alpha_2\alpha_1$ that contains no zero-relations. A permitted path C is called a *nontrivial permitted thread* if for all $\beta \in Q_1$ neither $C\beta$ nor βC is a permitted path. Similarly a *forbidden path* of A is a sequence $\Pi = \alpha_l \dots \alpha_2\alpha_1$ formed by pairwise different arrows in Q with $\alpha_{i+1}\alpha_i \in I$ for all $i \in \{1, 2, \dots, l-1\}$. A forbidden path Π is called a *nontrivial forbidden thread* if for all $\beta \in Q_1$ neither $\Pi\beta$ nor $\beta\Pi$ is a forbidden path. Let $v \in Q_0$ such that $\#\{\alpha \in Q_1 : s(\alpha) = v\} \leq 1$, $\#\{\alpha \in Q_1 : t(\alpha) = v\} \leq 1$ and if $\beta, \gamma \in Q_1$ are such that $s(\gamma) = v = t(\beta)$, then $\gamma\beta \notin I$. Then we consider e_v a *trivial permitted thread* in v and denote it by h_v. Let $\mathcal{H}_A$ be the set of all permitted threads of A, trivial and nontrivial. Similarly, for $v \in Q_0$ such that $\#\{\alpha \in Q_1 : s(\alpha) = v\} \leq 1$, $\#\{\alpha \in Q_1 : t(\alpha) = v\} \leq 1$ and if $\beta, \gamma \in Q_1$ are such that $s(\gamma) = v = t(\beta)$, then $\gamma\beta \in I$, we consider e_v a *trivial forbidden thread* in v and denote it by p_v. Note that certain paths can be permitted and forbidden threads simultaneously.

Now, one can define functions $\sigma, \varepsilon : Q_1 \to \{1, -1\}$ that satisfy these conditions:

(1) If $\beta_1 \neq \beta_2$ are arrows with $s(\beta_1) = s(\beta_2)$, then $\sigma(\beta_1) = -\sigma(\beta_2)$.

(2) If $\gamma_1 \neq \gamma_2$ are arrows with $t(\gamma_1) = t(\gamma_2)$, then $\varepsilon(\gamma_1) = -\varepsilon(\gamma_2)$.

(3) If β and γ are arrows with $s(\gamma) = t(\beta)$ and $\gamma\beta \notin I$, then $\sigma(\gamma) = -\varepsilon(\beta)$.

We can extend these functions to threads of A as follows: for a nontrivial thread $H = \alpha_l \dots \alpha_2\alpha_1$ of A define $\sigma(H) := \sigma(\alpha_1)$ and $\varepsilon(H) := \varepsilon(\alpha_l)$. If there is a trivial permitted thread h_v for some $v \in Q_0$, the connectivity of Q assures the existence of some $\gamma \in Q_1$ with $s(\gamma) = v$ or some $\beta \in Q_1$ with $t(\beta) = v$. In the first case, we define $\sigma(h_v) = -\varepsilon(h_v) := -\sigma(\gamma)$, for the second case $\sigma(h_v) = -\varepsilon(h_v) := \varepsilon(\beta)$. If there is a trivial forbidden thread p_v for some $v \in Q_0$, we know that there exists $\gamma \in Q_1$ with $s(\gamma) = v$ or $\beta \in Q_1$ with $t(\beta) = v$. In the first case, we define $\sigma(p_v) = \varepsilon(h_v) := -\sigma(\gamma)$, for the second case $\sigma(p_v) = \varepsilon(h_v) := -\varepsilon(\beta)$.

We next use a combinatorial algorithm to produce certain pairs of natural numbers, using only the quiver with relations which defines a gentle algebra. In the algorithm we go forward through permitted threads and backwards through forbidden threads in such a way that each arrow and its inverse are used exactly once.

Algorithm 5.6 [Avella-Alaminos and Geiss 2008].

(1) Begin with a permitted thread H_0 of A.

- If H_i is defined, consider Π_i the forbidden thread that ends in $t(H_i)$ and such that $\varepsilon(H_i) = -\varepsilon(\Pi_i)$.
- Let H_{i+1} be the permitted thread that starts in $s(\Pi_i)$ and such that $\sigma(H_{i+1}) = -\sigma(\Pi_i)$.

The process stops when $H_k = H_0$ for some natural number k. Set

$$m = \sum_{1 \leq i \leq k} l(\Pi_{i-1}),$$

where $l(\cdot)$ is the length (number of arrows) of a path. We obtain the pair (k, m).

(2) Repeat the first step of the algorithm until all permitted threads of A have been considered.

(3) If there are oriented cycles in which each pair of consecutive arrows form a relation, we add a pair $(0, m)$ for each of those cycles, where m is the length of the cycle.

(4) Define $\phi_A : \mathbb{N}^2 \to \mathbb{N}$, where $\phi_A(k, m)$ is the number of times the pair (k, m) arises in the algorithm.

This function ϕ is invariant under derived equivalence:

Lemma 5.7 [Avella-Alaminos and Geiss 2008]. *Let A and B be gentle algebras. If A and B are derived equivalent, then $\phi_A = \phi_B$.*

Example 5.8. Figure 6 shows the quiver of a cluster tilted algebra A of type $\tilde{A}_{18}$, where $r_1 = 2,\ r_2 = 3,\ s_1 = 3$ and $s_2 = 4$ and thus, $r := r_1 + r_2 = 5$ and $s := s_1 + s_2 = 7$.

Define the functions σ and ε for all arrows in Q:

$$\begin{aligned}
\sigma(\alpha_i) &= 1, & \varepsilon(\alpha_i) &= -1 & &\text{for all } i = 1, \dots, 5,\\
\sigma(\alpha_i) &= -1, & \varepsilon(\alpha_i) &= 1 & &\text{for all } i = 6, \dots, 12,\\
\sigma(\beta_{j,1}) &= 1, & \varepsilon(\beta_{j,1}) &= 1 & &\text{for all } j = 1, \dots, 3,\\
\sigma(\beta_{j,2}) &= -1, & \varepsilon(\beta_{j,2}) &= 1 & &\text{for all } j = 1, \dots, 3,\\
\sigma(\gamma_{l,1}) &= -1, & \varepsilon(\gamma_{l,1}) &= -1 & &\text{for all } l = 1, \dots, 4,\\
\sigma(\gamma_{l,2}) &= 1, & \varepsilon(\gamma_{l,2}) &= -1 & &\text{for all } l = 1, \dots, 4.
\end{aligned}$$

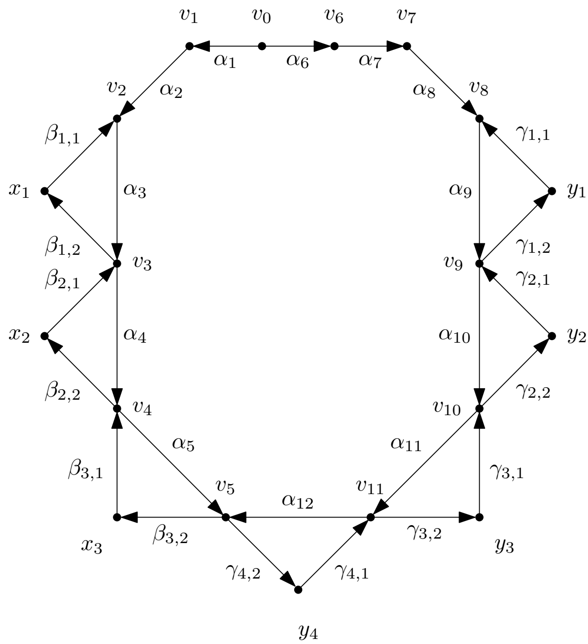


Figure 6. Quiver for Example 5.8.

Then $\mathscr{H}_A$ is formed by h_{v_1}, h_{v_6}, h_{v_7}, $\gamma_{4,2}\alpha_5\alpha_4\alpha_3\alpha_2\alpha_1$, $\beta_{3,2}\alpha_{12}\alpha_{11}\alpha_{10}\alpha_9\alpha_8\alpha_7\alpha_6$, $\beta_{1,1}$, $\beta_{1,2}\beta_{2,1}$, $\beta_{2,2}\beta_{3,1}$, $\gamma_{1,1}$, $\gamma_{1,2}\gamma_{2,1}$, $\gamma_{2,2}\gamma_{3,1}$ and $\gamma_{3,2}\gamma_{4,1}$. The forbidden threads of A are p_{x_1}, p_{x_2}, p_{x_3}, p_{y_1}, p_{y_2}, p_{y_3}, p_{y_4}, α_1, α_2, α_6, α_7, α_8 and all the oriented 3-cycles.

Moreover, we can write

$$\begin{aligned}
\sigma(h_{v_1}) &= -\varepsilon(h_{v_1}) = -\sigma(\alpha_2) = \varepsilon(\alpha_1) = -1,\\
\sigma(h_{v_6}) &= -\varepsilon(h_{v_6}) = -\sigma(\alpha_7) = \varepsilon(\alpha_6) = 1,\\
\sigma(h_{v_7}) &= -\varepsilon(h_{v_7}) = -\sigma(\alpha_8) = \varepsilon(\alpha_7) = 1
\end{aligned}$$

for the trivial permitted threads and

$$\begin{aligned}
\sigma(p_{x_i}) &= \varepsilon(p_{x_i}) = -\sigma(\beta_{i,1}) = -\varepsilon(\beta_{i,2}) = -1 \quad \text{for all } i = 1, 2, 3,\\
\sigma(p_{y_i}) &= \varepsilon(p_{y_i}) = -\sigma(\gamma_{i,1}) = -\varepsilon(\gamma_{i,2}) = 1 \quad \text{for all } i = 1, 2, 3, 4
\end{aligned}$$

for the trivial forbidden threads.

Let $H_0 = h_{v_1}$ and $\Pi_0 = \alpha_1$ with $\varepsilon(h_{v_1}) = -\varepsilon(\alpha_1) = 1$. Then H_1 is the permitted thread that starts in $s(\Pi_0) = v_0$ and $\sigma(H_1) = \sigma(\alpha_6) = -\sigma(\Pi_0) = -1$, that is, $\beta_{3,2}\alpha_{12}\alpha_{11}\alpha_{10}\alpha_9\alpha_8\alpha_7\alpha_6$. Now $\Pi_1 = p_{x_3}$ since it is the forbidden thread that ends in x_3 and $\varepsilon(\Pi_1) = -\varepsilon(H_1) = -\varepsilon(\beta_{3,2}) = -1$. Then $H_2 = \beta_{2,2}\beta_{3,1}$ is the permitted thread starting in x_3 and $\sigma(\Pi_1) = -\sigma(H_2) = -\sigma(\beta_{3,1}) = -1$. Thus, $\Pi_2 = p_{x_2}$ with $\varepsilon(H_2) = \varepsilon(\beta_{2,2}) = -\varepsilon(\Pi_2) = 1$.

In the same way we can define the missing threads and we get

$$\begin{array}{ll} H_0 = h_{v_1} & \Pi_0^{-1} = \alpha_1^{-1} \\ H_1 = \beta_{3,2}\alpha_{12}\alpha_{11}\alpha_{10}\alpha_9\alpha_8\alpha_7\alpha_6 & \Pi_1^{-1} = p_{x_3} \\ H_2 = \beta_{2,2}\beta_{3,1} & \Pi_2^{-1} = p_{x_2} \\ H_3 = \beta_{1,2}\beta_{2,1} & \Pi_3^{-1} = p_{x_1} \\ H_4 = \beta_{1,1} & \Pi_4^{-1} = \alpha_2^{-1} \\ H_5 = H_0 & \end{array}$$

$$\to (5, 2)$$

where α_1^{-1} is defined by $s(\alpha_1^{-1}) := t(\alpha_1),\ t(\alpha_1^{-1}) := s(\alpha_1)$ and $(\alpha_1^{-1})^{-1} = \alpha_1$.

If we continue with the algorithm we obtain the second pair $(7, 3) = (s, s_1)$ in the following way:

$$\begin{array}{ll} H_0 = h_{v_6} & \Pi_0^{-1} = \alpha_6^{-1} \\ H_1 = \gamma_{4,2}\alpha_5\alpha_4\alpha_3\alpha_2\alpha_1 & \Pi_1^{-1} = p_{y_4} \\ H_2 = \gamma_{3,2}\gamma_{4,1} & \Pi_2^{-1} = p_{y_3} \\ H_3 = \gamma_{2,2}\gamma_{3,1} & \Pi_3^{-1} = p_{y_2} \\ H_4 = \gamma_{1,2}\gamma_{2,1} & \Pi_4^{-1} = p_{y_1} \\ H_5 = \gamma_{1,1} & \Pi_5^{-1} = \alpha_8^{-1} \\ H_6 = h_{v_7} & \Pi_6^{-1} = \alpha_7^{-1} \\ H_7 = H_0 & \end{array}$$

$$\to (7, 3)$$

Finally, we have to add seven pairs $(0, 3)$ for the seven oriented 3-cycles. Thus, we get $\phi_A(5, 2) = 1,\ \phi_A(7, 3) = 1$, and $\phi_A(0, 3) = 7$.

Now we can extend this example to general quivers of cluster tilted algebras of type $\tilde{A}_n$ in normal form.

Proof of Theorem 5.5. We know from Lemma 5.4 that every cluster tilted algebra $A = KQ/I$ of type $\tilde{A}_n$ with parameters $r_1,\ r_2,\ s_1$ and s_2 is derived equivalent to a cluster tilted algebra with a quiver in normal form, as shown in Figure 1, where r_1 is the number of arrows anticlockwise that do not share any arrow with an oriented 3-cycle and s_1 is the number of arrows clockwise that do not share any arrow with an oriented 3-cycle. Moreover, r_2 is the number of oriented 3-cycles that share one arrow α with the nonoriented cycle and α is oriented anticlockwise and s_2 is the number of oriented 3-cycles that share one arrow β with the nonoriented cycle and β is oriented clockwise (see Definition 3.7). Thus, $r := r_1 + r_2$ is the number of anticlockwise arrows of the nonoriented cycle and $s := s_1 + s_2$ is the number of clockwise arrows of the nonoriented cycle.

We consider the quiver Q in normal form with notation as given in Figure 7 and define the functions σ and ε for all arrows in Q:

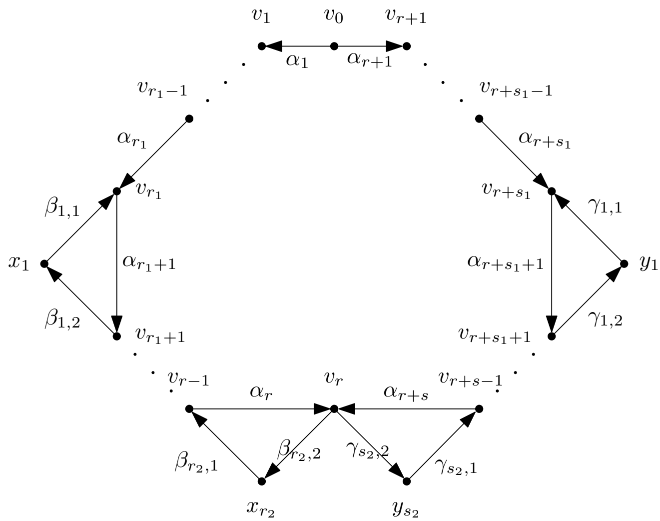


Figure 7. A quiver in normal form.

$$\begin{array}{llll}
\sigma(\alpha_i) = 1, & \varepsilon(\alpha_i) = -1 & \text{for all} & i = 1, \dots, r, \\
\sigma(\alpha_i) = -1, & \varepsilon(\alpha_i) = 1 & \text{for all} & i = r+1, \dots, r+s, \\
\sigma(\beta_{j,1}) = 1, & \varepsilon(\beta_{j,1}) = 1 & \text{for all} & j = 1, \dots, r_2, \\
\sigma(\beta_{j,2}) = -1, & \varepsilon(\beta_{j,2}) = 1 & \text{for all} & j = 1, \dots, r_2, \\
\sigma(\gamma_{l,1}) = -1, & \varepsilon(\gamma_{l,1}) = -1 & \text{for all} & l = 1, \dots, s_2, \\
\sigma(\gamma_{l,2}) = 1, & \varepsilon(\gamma_{l,2}) = -1 & \text{for all} & l = 1, \dots, s_2.
\end{array}$$

Here $\mathcal{H}_A$ is formed by

$$h_{v_1}, \dots, h_{v_{r_1-1}}, h_{v_{r+1}}, \dots, h_{v_{r+s_1-1}}, \gamma_{s_2,2}\alpha_r\alpha_{r-1}\dots\alpha_2\alpha_1, \\ \beta_{r_2,2}\alpha_{r+s}\alpha_{r+s-1}\dots\alpha_{r+2}\alpha_{r+1}, \beta_{1,1}, \beta_{1,2}\beta_{2,1}, \dots, \beta_{r_2-1,2}\beta_{r_2,1}, \\ \gamma_{1,1}, \gamma_{1,2}\gamma_{2,1}, \dots, \gamma_{s_2-1,2}\gamma_{s_2,1}.$$

The forbidden threads of A are $p_{x_1}, \dots, p_{x_{r_2}}, p_{y_1}, \dots, p_{y_{s_2}}, \alpha_1, \dots, \alpha_{r_1}, \alpha_{r+1}, \dots, \alpha_{r+s_1}$ and all the oriented 3-cycles.

Moreover, we can write

$$\begin{array}{lllll}
\sigma(h_{v_1}) & = -\varepsilon(h_{v_1}) & = -\sigma(\alpha_2) & = \varepsilon(\alpha_1) & = -1, \\
 & \vdots & & & \\
\sigma(h_{v_{r_1-1}}) & = -\varepsilon(h_{v_{r_1-1}}) & = -\sigma(\alpha_{r_1}) & = \varepsilon(\alpha_{r_1-1}) & = -1, \\
\sigma(h_{v_{r+1}}) & = -\varepsilon(h_{v_{r+1}}) & = -\sigma(\alpha_{r+2}) & = \varepsilon(\alpha_{r+1}) & = 1, \\
 & \vdots & & & \\
\sigma(h_{v_{r+s_1-1}}) & = -\varepsilon(h_{v_{r+s_1-1}}) & = -\sigma(\alpha_{r+s_1}) & = \varepsilon(\alpha_{r+s_1-1}) & = 1
\end{array}$$

for the trivial permitted threads and

$$\begin{aligned}
\sigma(p_{x_i}) = \varepsilon(p_{x_i}) = -\sigma(\beta_{i,1}) = -\varepsilon(\beta_{i,2}) &= -1 \quad \text{for all } i = 1, \ldots, r_2, \\
\sigma(p_{y_i}) = \varepsilon(p_{y_i}) = -\sigma(\gamma_{i,1}) = -\varepsilon(\gamma_{i,2}) &= 1 \quad \text{for all } i = 1, \ldots, s_2
\end{aligned}$$

for the trivial forbidden threads.

Thus, we can apply Algorithm 5.6 as follows:

$$\begin{array}{llll}
H_0 & = h_{v_1} & \Pi_0^{-1} & = \alpha_1^{-1} \\
H_1 & = \beta_{r_2,2}\alpha_{r+s}\alpha_{r+s-1}\ldots\alpha_{r+2}\alpha_{r+1} & \Pi_1^{-1} & = p_{x_{r_2}} \\
H_2 & = \beta_{r_2-1,2}\beta_{r_2,1} & \Pi_2^{-1} & = p_{x_{r_2-1}} \\
\vdots & & \vdots & \\
H_{r_2} & = \beta_{1,2}\beta_{2,1} & \Pi_{r_2}^{-1} & = p_{x_1} \\
H_{r_2+1} & = \beta_{1,1} & \Pi_{r_2+1}^{-1} & = \alpha_{r_1}^{-1} \\
H_{r_2+2} & = h_{v_{r_1-1}} & \Pi_{r_2+2}^{-1} & = \alpha_{r_1-1}^{-1} \\
\vdots & & \vdots & \\
H_{r-1} & = h_{v_2} & \Pi_{r-1}^{-1} & = \alpha_2^{-1} \\
H_r & = H_0 & &
\end{array}$$

$$\begin{aligned}
m &= l(\Pi_0) + l(\Pi_{r_2+1}) + l(\Pi_{r_2+2}) + \cdots + l(\Pi_{r-1}) \\
&= 1 + \underbrace{1 + 1 + \cdots + 1}_{(r-1)-r_2 \text{ times}} \\
&= 1 + (r-1) - r_2 \\
&= r - r_2 \\
&= r_1
\end{aligned}$$

$$\rightarrow (r, r_1)$$

If we continue with the algorithm we obtain the second pair (s, s_1) as follows:

$$\begin{array}{llll}
H_0 & = h_{v_{r+1}} & \Pi_0^{-1} & = \alpha_{r+1}^{-1} \\
H_1 & = \gamma_{s_2,2}\alpha_r\alpha_{r-1}\ldots\alpha_2\alpha_1 & \Pi_1^{-1} & = p_{y_{s_2}} \\
H_2 & = \gamma_{s_2-1,2}\gamma_{s_2,1} & \Pi_2^{-1} & = p_{y_{s_2-1}} \\
\vdots & & \vdots & \\
H_{s_2} & = \gamma_{1,2}\gamma_{2,1} & \Pi_{s_2}^{-1} & = p_{y_1} \\
H_{s_2+1} & = \gamma_{1,1} & \Pi_{s_2+1}^{-1} & = \alpha_{r+s_1}^{-1} \\
H_{s_2+2} & = h_{v_{r+s_1-1}} & \Pi_{s_2+2}^{-1} & = \alpha_{r+s_1-1}^{-1} \\
\vdots & & \vdots & \\
H_{s-1} & = h_{v_{r+2}} & \Pi_{s-1}^{-1} & = \alpha_{r+2}^{-1} \\
H_s & = H_0 & &
\end{array}$$

$$\rightarrow (s, s_1)$$

Finally, we have to add $r_2 + s_2$ pairs $(0, 3)$ for the oriented 3-cycles. Thus, we have $\phi_A(r, r_1) = 1$, $\phi_A(s, s_1) = 1$ and $\phi_A(0, 3) = r_2 + s_2$, where $r = r_1 + r_2$ and $s = s_1 + s_2$.

Now, let A and B be two cluster tilted algebras of type $\tilde{A}_n$ with parameters r_1, r_2, s_1, s_2, respectively $\tilde{r}_1$, $\tilde{r}_2$, $\tilde{s}_1$, $\tilde{s}_2$. From above we can conclude that $\phi_A = \phi_B$ if and only if $r_1 = \tilde{r}_1$, $r_2 = \tilde{r}_2$, $s_1 = \tilde{s}_1$ and $s_2 = \tilde{s}_2$ or $r_1 = \tilde{s}_1$, $r_2 = \tilde{s}_2$, $s_1 = \tilde{r}_1$ and $s_2 = \tilde{r}_2$ (which ends up being the same quiver).

Hence, if A is derived equivalent to B, we know from Lemma 5.7 that $\phi_A = \phi_B$ and thus, that the parameters are the same. Otherwise, if A and B have the same parameters, they are both derived equivalent to the same cluster tilted algebra with a quiver in normal form. □

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