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Arithmetic theta lifting and L-derivatives for unitary groups, I

Yifeng Liu

We study cuspidal automorphic representations of unitary groups of $2n$ variables with ϵ-factor -1 and their central L-derivatives by constructing their arithmetic theta liftings, which are Chow cycles of codimension n on Shimura varieties of dimension $2n-1$ of certain unitary groups. We give a precise conjecture for the arithmetic inner product formula, originated by Kudla, which relates the height pairing of these arithmetic theta liftings and the central L-derivatives of certain automorphic representations. We also prove an identity relating the archimedean local height pairing and derivatives of archimedean Whittaker functions of certain Eisenstein series, which we call an arithmetic local Siegel–Weil formula for archimedean places. This provides some evidence toward the conjectural arithmetic inner product formula.

1. Introduction

Rallis [1982] developed a formula, called the Rallis inner product formula, to determine whether a certain theta lifting is vanishing. It is used to calculate the Petersson inner product of two automorphic forms on an orthogonal group lifted from those on a symplectic group through the Weil representation. It turns out, using the Siegel–Weil formula, that the inner product is related to a diagonal integral on the doubling symplectic group of the original automorphic forms with certain Eisenstein series. This doubling method was generalized to other cases by Gelbart et al. [1987]. If we assume that the automorphic forms we lift are

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cuspidal, this diagonal integral is in fact Eulerian, and decomposes into so-called local zeta integrals, which are closely related to the L-factors of the corresponding representations. In fact, Gelbart et al. prove in many cases that when everything is unramified, the local zeta integral is just the local Langlands L-factor divided by a product of certain Tate L-factors. Li [1992] extended this result for unitary groups.

Since, at that time, the key ingredient, the Siegel–Weil formula, was only known "above the convergence line", which means that the group we lift to should have a certain larger size than the group we lift from, the inner product formula could only regard the values of global L-functions at points far to the right of the central point. For example, if we lift forms from $\mathsf{Sp}(n)$ (rank $2n$ matrices) to $\mathsf{O}(2m)$ then $m > 2n+1$. In fact, using the nonvanishing result of these L-values, Li [1992] was able to prove some nonvanishing results for the cohomology of certain arithmetic quotients, which is an important and well-known application of the inner product formula. Kudla and Rallis [1994] extended the Siegel–Weil formula with great generality for symplectic-orthogonal pairs and Ichino [2004; 2007] did that for unitary pairs using the similar idea of Kudla and Rallis. Now we can extend Rallis' original inner product formula "below the convergence line" (after regularization if necessary) which enables us to say some words about the global L-values at other points, especially the central point $\frac{1}{2}$.

Now let us stick to a special case where the dual pair are unitary groups with the same even rank, hence the related L-value is the central value. Suppose that E/F is a quadratic imaginary extension of a totally real field with τ the nontrivial Galois involution. Let us denote by $H' \cong \mathsf{U}(n,n)_F$ the unique quasisplit unitary group of rank $2n$ (with respect to τ) and by H another unitary group of the same rank. Let π be an irreducible cuspidal automorphic representation of H' and let f be a nonzero form inside it. Choosing an auxiliary Schwartz function and using the Weil representation, we get an automorphic form on H called the (regularized, if necessary) theta lifting of f. If the global epsilon factor $\epsilon(\frac{1}{2},\pi) = 1$, then among all pure inner forms of H, the theta lifting of forms inside π should always vanish except for one possible $H = H(\pi)$. For this unitary group, the only obstruction to some theta lifting being nonvanishing is that $L(\frac{1}{2},\pi) = 0$.

The theory is not complete since we miss another half, ones whose $\epsilon(\frac{1}{2},\pi) = -1$. If this is the case, then $L(\frac{1}{2},\pi)$ is automatically 0 and all theta lifting to all possible unitary groups of the same rank should vanish. A great observation of Kudla [1997; 2002; 2003; Kudla et al. 2006] was that (in the symplectic-orthogonal case) there should exist some "arithmetic theta lifting" which is a cycle on certain (integral models of a) Shimura variety and an "arithmetic Siegel–Weil formula". This arithmetic Siegel–Weil formula should be related to the central derivative $L'(\frac{1}{2},\pi)$ of the global L-function instead of the central value $L(\frac{1}{2},\pi)$ via an arithmetic analogue of Rallis' inner product formula (see [Kudla 2003, Section 11]). For this direction, a particular form of the arithmetic inner product formula has already been obtained, for holomorphic cuspidal newforms of $\mathsf{PGL}(2)_{\mathbb{Q}}$ of weight 2 and level $\Gamma_0(N)$ for N square-free with epsilon factor -1, in [Kudla et al. 2006, Theorem 9.2.4], based on a lot of previous work.

In this paper, we will set up a general formulation and a more explicit formulation extending the above line. We will establish the general conjecture of an explicit form of the arithmetic inner product formula assuming some well-accepted properties of Arthur packets, the existence of Beilinson–Bloch height pairing when $F \neq \mathbb{Q}$, and some other auxiliary conjectures when $F = \mathbb{Q}$, all of which can be proved when $n = 1$. We also prove some partial results toward the general arithmetic inner product formula, namely the modularity theorem on the (noncompactified) generating series and the arithmetic analogue of the local Siegel–Weil formula at archimedean places. In the second part of this paper [Liu 2011], we will give a full proof of the arithmetic inner product formula for $n = 1$.

Before we state the main results, we would like to remark that the L-function appearing here is the so-called doubling L-series defined by Piatetski-Shapiro and Rallis (see [Harris et al. 1996] for a detailed definition for the unitary group case). This L-function is conjectured to coincide with the Langlands L-function of the standard base change $\mathsf{BC}(\pi)$ which is an irreducible automorphic representation of $\mathsf{GL}_{2n,E}$. Hence the set of central L-derivatives which can be computed by the arithmetic inner product formula at least contains those $L'(\frac{1}{2}, \Pi)$, where Π is an irreducible cuspidal automorphic representation of $\mathsf{GL}_{2n,E}$ such that $\Pi^\vee \cong \Pi^\tau$, $\Pi \otimes \epsilon_{E/F}$ is $\mathsf{GL}_{2n,F}$-distinguished, and Π_ι is the base change of the trivial representation of $\mathsf{U}(2n, 0)_{\mathbb{R}}$ for any archimedean place ι, where $\epsilon_{E/F}$ is the associated quadratic character by class field theory. In particular, when $n = 1$, this set is exactly the same as the one of central L-derivatives appearing in the (complete version of the) Gross–Zagier formula recently proved by X. Yuan, S.-W. Zhang and W. Zhang [Yuan et al. 2011].

More precisely, let E/F, τ, and $\epsilon_{E/F}$ be as above and ψ be an additive character of $F\backslash\mathbb{A}_F$, standard at archimedean places, which is used to define Weil representations and Fourier coefficients. For $n \geq 1$, let H_n be the unitary group over F such that for any F-algebra R, $H_n(R) = \{h \in \mathsf{GL}_{2n}(E \otimes_F R) \mid {}^th^\tau w_n h = w_n\}$ where

$$w_n = \begin{pmatrix} & \mathbf{1}_n \\ -\mathbf{1}_n & \end{pmatrix}.$$

The center of H_n is the F-torus $E^{\times,1} = \ker[\mathsf{Nm} : E^\times \to F^\times]$. Let π be an irreducible cuspidal automorphic representation of H_n and $\pi^\vee$ its contragredient. Let χ be a character of $\mathbb{A}_E^\times$ which is trivial on $E^\times\mathbb{A}_F^\times$. We can associate with χ a sequence of integers $\mathfrak{k}^\chi = (\mathfrak{k}_\iota^\chi)_\iota$ for each archimedean place ι of F whose definition is in Section 3A. In particular, they are all even integers for this χ.

By the theta dichotomy proved in [Paul 1998; Gong and Grenié 2011], we get a factor $\epsilon(\pi,\chi)$ (see Section 2D for a precise definition) which is the product of the local factors $\epsilon(\pi_v,\chi_v)$ for each place v of F, such that $\epsilon(\pi_v,\chi_v)\in\{\pm 1\}$ and $\epsilon(\pi_v,\chi_v)=1$ for almost all v. Although it is conjectured that this $\epsilon(\pi_v,\chi_v)$ is related to the local ϵ-factor in representation theory (see [Harris et al. 1996]), it is not by our definition. From these local factors, we can construct a hermitian space $\mathbb{V}(\pi,\chi)$ over $\mathbb{A}_E$ of rank $2n$ which is coherent (resp. incoherent) if $\epsilon(\pi,\chi)=1$ (resp. -1) (for this terminology, see Section 2A). When $\epsilon(\pi,\chi)=1$, we get the usual (extended) Rallis inner product formula. Moreover, we prove in Section 2 that when $\epsilon(\pi,\chi)=-1$, $L(\frac{1}{2},\pi,\chi)=0$.

Now let us assume $\epsilon(\pi,\chi)=-1$ and further assume that π_∞ is a *discrete series of weight* $(n-\mathfrak{k}^\chi/2, n+\mathfrak{k}^\chi/2)$. Careful readers may find that this is not standard terminology. We will now explain this in a more general situation. We say that a discrete series representation π of $\mathrm{U}(r,r)_\mathbb{R}$ ($r\geq 1$) is of weight (a,b) for some integers a,b such that $a+b$ is positive if the minimal type of the maximal compact subgroup $\mathrm{U}(r)_\mathbb{R}\times\mathrm{U}(r)_\mathbb{R}\subset\mathrm{U}(r,r)_\mathbb{R}$, for which we choose the standard embedding elaborated at the beginning of Section 4A (although we only write it for r even there, but it is the same for all r), is $\det_1^a\boxtimes\det_2^{-b}$, where $\det_i$ is the determinant on the i-th $\mathrm{U}(n)_\mathbb{R}$. One can prove that it is the theta correspondence (under certain Weil representation) of the trivial representation from $\mathrm{U}(a+b,0)_\mathbb{R}$ to $\mathrm{U}(r,r)_\mathbb{R}$. Finally, the first sentence in the paragraph means that for each ι, π_ι is a discrete series of weight $(n-\mathfrak{k}^\chi_\iota/2, n+\mathfrak{k}^\chi_\iota/2)$. For π as above, the corresponding $\mathbb{V}(\pi,\chi)$ is incoherent and totally positive definite.

Moreover, for any hermitian space $\mathbb{V}$ over $\mathbb{A}_E$ which is incoherent and totally positive-definite of rank $m\geq 2$, let $\mathbb{H}=\mathrm{Res}_{\mathbb{A}_F/\mathbb{A}}\mathrm{U}(\mathbb{V})$ be the corresponding unitary group. Then we can construct a projective system of unitary Shimura varieties $(\mathrm{Sh}_K(\mathbb{H}))_K$ of dimension $m-1$, smooth and quasiprojective over E where K is a sufficiently small open compact subgroup of $\mathbb{H}(\mathbb{A}_f)$ (for the construction, see Section 3A). Let χ be a character of $E^\times\backslash\mathbb{A}_E^\times$ such that $\chi\big|_{\mathbb{A}_F^\times}=\epsilon_{E/F}^m$ and $1\leq r<m$ another integer. For any Schwartz function $\phi\in\mathscr{S}(\mathbb{V}^r)^{\mathrm{U}_\infty K}$ (see Section 3A for notation), we can define Kudla's generating series $Z_\phi(g)$ for any $g\in H_r(\mathbb{A}_F)$ which takes values as formal sums in $\mathrm{CH}^r(\mathrm{Sh}(\mathbb{H}))_\mathbb{C}$: the inductive limit of Chow groups of codimension r cycles with complex coefficients on the Shimura varieties. For any linear functional ℓ of $\mathrm{CH}^r(\mathrm{Sh}(\mathbb{H}))_\mathbb{C}$, we can evaluate it on the generating series and hence obtain a smooth function $\ell(Z_\phi)(g)$ on $H_r(\mathbb{A}_F)$ provided that it is absolutely convergent. We prove in Section 3B the following theorem on the *modularity of the generating series*:

Theorem 3.5. (1) *If $\ell(Z_\phi)(g)$ is absolutely convergent, then it is an automorphic form of $H_r(\mathbb{A}_F)$. Moreover, $\ell(Z_\phi)_\infty$ is in a discrete series representation of weight $((m+\mathfrak{k}^\chi)/2, (m-\mathfrak{k}^\chi)/2)$.*

(2) *If* $r = 1$, *then* $\ell(Z_\phi)(g)$ *is absolutely convergent for any* ℓ.

There is also a version in the case of symplectic-orthogonal pairs which is proved in [Yuan et al. 2009]. The proof in the unitary case is similar to that of [Yuan et al. 2009] using the induction process on the codimension. Actually, the proof of the case $r = 1$ uses the result in the symplectic-orthogonal case. We will also state another version for the compactified generating series in Section 3C if the Shimura varieties are not proper, which happens in particular when $F = \mathbb{Q}$ and $m > 2$, but so far we are not able to prove it.

For simplicity, let us assume $F \neq \mathbb{Q}$ in the following discussion; then $\mathsf{Sh}_K(\mathbb{H})$ is projective. Let $m = 2n$. Similarly to [Kudla 2003; Kudla et al. 2006], for any $f \in \pi$ and Schwartz function $\phi \in \mathcal{S}(\mathbb{V}^n)^{\mathsf{U}_\infty K}$, we construct a cycle Θ^f_ϕ, called the arithmetic theta lifting, which is a cycle on $\mathsf{Sh}_K(\mathbb{H})$ of codimension n. On the contragredient side, we also have $\Theta^{f^\vee}_{\phi^\vee}$ for $f^\vee \in \pi^\vee$. The definition of Θ^f_ϕ is basically the integration of f with the generating series, that is,

$$\Theta^f_\phi = \int_{H_n(F)\backslash H_n(\mathbb{A}_F)} f(g) Z_\phi(g)\, dg,$$

which is a formal sum in $\mathsf{CH}^n(\mathsf{Sh}(\mathbb{H}))_{\mathbb{C}}$ but whose (Betti) cohomology class is well-defined. We show in Section 3D that it is cohomologically trivial assuming certain properties of Arthur packets. Hence we can consider the (conjectural if $n > 1$) Beilinson–Bloch height pairing (see [Bloch 1984; Beĭlinson 1987]) $\langle \Theta^f_\phi, \Theta^{f^\vee}_{\phi^\vee} \rangle_{\mathsf{BB}}$. Analogous to the coherent case, when $\mathbb{V} \neq \mathbb{V}(\pi, \chi)$, one easily shows that $\Theta^f_\phi = 0$. If $\mathbb{V} \cong \mathbb{V}(\pi, \chi)$, we conjecture the following:

Conjecture 3.11 (arithmetic inner product formula). *Let* π, χ *be as above* (*in particular*, $\epsilon(\pi, \chi) = -1$) *and* $\mathbb{V} \cong \mathbb{V}(\pi, \chi)$. *Then, for any* $f \in \pi$, $f^\vee \in \pi^\vee$ *and any* $\phi, \phi^\vee \in \mathcal{S}(\mathbb{V}^n)^{\mathsf{U}_\infty}$ *decomposable, we have*

$$\langle \Theta^f_\phi, \Theta^{f^\vee}_{\phi^\vee} \rangle_{\mathsf{BB}} = \frac{L'(\frac{1}{2}, \pi, \chi)}{\prod_{i=1}^{2n} L(i, \epsilon^i_{E/F})} \prod_v Z^*(0, \chi_v, f_v, f^\vee_v, \phi_v \otimes \phi^\vee_v),$$

where Z^* *are normalized local zeta integrals* (see Section 2C) *of which almost all are* 1.

We remark that this conjectural arithmetic inner product formula is different from that of Kudla (see, for example, [Kudla 2003, Section 11]) in the sense that our arithmetic theta lifting Θ^f_ϕ is canonically defined on the Shimura variety, not on any integral model. More important, it is cohomologically trivial, at least when the Shimura variety is proper, hence we can talk about its canonical height through the conjectural Beilinson–Bloch height pairing.

As we do in [Liu 2011], to prove the arithmetic inner product formula, we introduce analytic kernel functions and geometric kernel functions which carry over all cusp forms simultaneously. The former is the derivative of certain Eisenstein series on the doubling group which deals with derivatives of L-functions, while the latter is the height pairing of the generating series which deals with that of the arithmetic theta lifting. Both kernel functions can be essentially decomposed as a sum of local terms for each place v of F. Hence we should compare them place by place. At each archimedean place of F, it turns out that we need to compare the derivatives of certain Whittaker functions with the local height pairing of special subdomains of the hermitian symmetric domain D_{m-1} of $\mathrm{U}(m-1,1)_{\mathbb{R}}$. Let V be the complex hermitian space of signature $(m,0)$ and V' that of signature $(m-1,1)$. For any nonzero $x \in V'$, we can associate a hermitian symmetric subdomain $D_x \subset D_{m-1}$ and a Green's function $\xi(x)$ to it (see Section 4B). It is of codimension 1 if the inner product of x is positive and empty if not. The Green's function, originally constructed in [Kudla 1997], is related to the Kudla–Millson form [1986] and very closely related to derivatives of Whittaker functions. But this is not the admissible Green's function used in the Beilinson–Bloch height pairing (at an archimedean place); instead, they have a certain relation which will be elaborated in [Liu 2011] when $n=1$ and we expect that they relate for general n. For any $\vec{x}=(x_1,\dots,x_m)\in V'^m$ such that $x_1,\dots,x_m$ are linearly independent, we get an intersection number $H(\vec{x})_\infty$ (with respect to the Green's functions $\xi(x_i)$). It is clear that the moment matrix $T(\vec{x})$ is in $\mathrm{Her}_m(\mathbb{C})$: the set of complex hermitian matrices of rank m, and whose signature is $(m-1,1)$. The intersection number $H(\vec{x})_\infty$ only depends on $T=T(\vec{x})$; hence it makes sense to write it as $H(T)_\infty$. In Section 4, we prove the following *local arithmetic Siegel–Weil formula* at an archimedean place:

Proposition 4.5, Theorem 4.17. *Let $T\in \mathrm{Her}_m(\mathbb{C})$ be nonsingular with $\mathrm{sign}(T)=(p,q)$.*

(1) $\mathrm{ord}_{s=0} W_T(s,e,\Phi^0)\ge q$.

(2) *If T is positive definite, that is, $q=0$, we have*
$$W_T(0,e,\Phi^0)=\gamma_V\frac{(2\pi)^{m^2}}{\Gamma_m(m)}e^{-2\pi\,\mathrm{tr}\,T}.$$

(3) *If T is of signature $(m-1,1)$, we have*
$$W'_T(0,e,\Phi^0)=\gamma_V\frac{(2\pi)^{m^2}}{\Gamma_m(m)}e^{-2\pi\,\mathrm{tr}\,T}H(T)_\infty.$$

Here, $\Phi^0\in\mathcal{S}(V^m)$ is the standard Gaussian; $W_T(0,e,\Phi^0)$ is the T-th Whittaker integral at $s=0$ and $e\in\mathrm{U}(m,m)_{\mathbb{R}}$; $\gamma_V\in\{\pm1\}$ is the Weil constant and $\Gamma_m(m)$ is a product of certain usual gamma functions (see Lemma 4.3).

The study of derivatives of L-functions starts from the celebrated paper [Gross and Zagier 1986], which studied the relation between the central derivatives of Rankin L-series and the heights of Heegner points on modular curves thus obtaining the famous Gross–Zagier formula predicted by the Birch and Swinnerton-Dyer conjecture. Later, this was generalized to the case of Shimura curves over totally real fields in [Zhang 2001a; 2001b]. The complete version of the Gross–Zagier formula has been achieved in [Yuan et al. 2011]. Moreover, Bruinier and Yang [2009] used regularized theta lifting and related the inner product to L-derivatives to give another proof of the original Gross–Zagier formula. A certain p-adic (or rigid analytic) version of the Gross–Zagier formula has been studied in [Bertolini and Darmon 1997; 1998].

There is another approach to studying L-derivatives via doubling integrals and in general derivatives of Eisenstein series, discovered by Kudla [1997; 2002; 2003; Kudla et al. 2006]. He proposed the project of the arithmetic Siegel–Weil formula and proved a special form of the arithmetic inner product formula with Rapoport and Yang. Our work follows the second approach, establishing an explicit form of the arithmetic inner product formula and, together with [Liu 2011], proving the complete version of the arithmetic inner product formula in the case of unitary groups of two variables over totally real fields.

For applications of the arithmetic inner product formula, we are able to construct nontorsion Chow cycles instead of cohomology classes in the classical case if the central derivative is nonzero. In the case of the Gross–Zagier formula [Gross and Zagier 1986; Yuan et al. 2011] and the arithmetic triple product formula [Yuan et al. 2010], we have already seen many interesting and important applications of nontorsion cycles on certain Shimura varieties.

For the positivity of the global L-function at the central point, which is a consequence of the generalized Riemann hypothesis, it is obvious that the positivity of normalized local zeta integrals (at the point 0) will imply the positivity of the central value $L(\frac{1}{2}, \pi, \chi)$. Moreover, through the arithmetic inner product formula, the positivity of normalized local zeta integrals plus the (conjectural) positivity of the Beilinson–Bloch height pairing will imply the positivity of the central derivative $L'(\frac{1}{2}, \pi, \chi)$, which is again a consequence of the generalized Riemann hypothesis!

Now we state the outline of the paper. In Section 2, we review the classical Siegel–Weil formula; in Section 2A is its generalization of the work of Ichino, and in Section 2B the doubling integral introduced by Piatetski-Shapiro and Rallis. We introduce the definition of the L-function and its relation with the local zeta integral in Section 2C. In Section 2D, we introduce the Rallis inner product formula for the central L-value in the coherent case. In Section 2E, we derive a formula for central L-derivatives using derivatives of Eisenstein series in the incoherent case.

In Section 3, we treat the geometric part of the theory. We introduce the notion of Shimura varieties of unitary groups, Kudla's special cycle, and generating series in Section 3A. Section 3B is devoted to proving Theorem 3.5. We introduce the canonical smooth compactification in the case of higher dimensions in Section 3C. In Section 3D, we define the arithmetic theta lifting and formulate the conjecture above and two auxiliary conjectures.

Section 4 is devoted to proving Proposition 4.5, Theorem 4.17 and hence finishing the archimedean comparison on the Shimura variety in the global setting.

In the Appendix we calculate the theta correspondence for spherical representations at a nonarchimedean place. This result is a key step in the proof of Proposition 3.9. Our calculation for the unitary case follows exactly that of the symplectic-orthogonal case which is proved in [Rallis 1984].

The following conventions hold throughout this paper.

- $\mathbb{A}_f=\widehat{\mathbb{Z}}\otimes_{\mathbb{Z}}\mathbb{Q}=(\varprojlim_N \mathbb{Z}/N\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{Q}$ is the ring of finite adèles and $\mathbb{A}=\mathbb{A}_f\times\mathbb{R}$ is the ring of full adèles.
- For any number field K, $\mathbb{A}_K=\mathbb{A}\otimes_{\mathbb{Q}}K$, $\mathbb{A}_{f,K}=\mathbb{A}_f\otimes_{\mathbb{Q}}K$, $K_\infty=K\otimes_{\mathbb{Q}}\mathbb{R}$, and $\Gamma_K=\mathrm{Gal}(K^{\mathrm{ac}}/K)$ is the Galois group of K.
- As usual, for a subset S of places, $-_S$ (resp. $-^S$) means the S-component (resp. component away from S) for the corresponding (decomposable) adèlic object; $-_\infty$ (resp. $-_f$) is the infinite (resp. finite) part.
- The symbols Tr and Nm mean the trace (resp. reduced trace) and norm (resp. reduced norm) if they apply to fields or rings of adèles (resp. simple algebras), and tr means the trace for matrix and linear transforms.
- $\mathbf{1}_n$ and $\mathbf{0}_n$ are the $n\times n$ identity and zero matrices; ${}^tg$ is the transpose of a matrix g.
- All (skew-)hermitian spaces and quadratic spaces are assumed to be nondegenerate.
- For a scheme X over a field K, we let $\mathrm{Pic}(X)$ be the Picard group of X over K, not the Picard scheme.

2. Doubling method

2A. ***Siegel–Weil formulae.*** In this section, we will review the classical Siegel–Weil formula and some generalizations to be used later.

Let F be a totally real field and E a totally imaginary quadratic extension of F. We denote by τ the nontrivial element in $\mathrm{Gal}(E/F)$ and by $\epsilon_{E/F}:\mathbb{A}_F^\times/F^\times\to\{\pm1\}$ the associated character by class field theory. Let Σ (resp. Σ_f; resp. Σ_∞) be the set of all places (resp. finite places; resp. infinite places) of F, and Σ°, Σ°_f, and Σ°_∞ those of E. We fix a nontrivial additive character ψ of $\mathbb{A}_F/F$.

For positive integer r, we denote by W_r the standard skew-hermitian space over E with respect to the involution τ, which has a skew-hermitian form $\langle\,\cdot\,,\,\cdot\,\rangle$ such that there is an E-basis $\{e_1, \dots, e_{2r}\}$ satisfying $\langle e_i, e_j\rangle = 0$, $\langle e_{r+i}, e_{r+j}\rangle = 0$, and $\langle e_i, e_{r+j}\rangle = \delta_{ij}$ for $1 \le i, j \le r$. Let $H_r = \mathrm{U}(W_r)$ be the unitary group of W_r which is a reductive group over F. The group $H_r(F)$, in which F can be itself or its completion at some place, is generated by the standard parabolic subgroup $P_r(F) = N_r(F)M_r(F)$ and the element w_r. More precisely,

$$N_r(F) = \left\{ n(b) = \begin{pmatrix} \mathbf{1}_r & b \\ & \mathbf{1}_r \end{pmatrix} \;\middle|\; b \in \mathrm{Her}_r(E) \right\},$$

$$M_r(F) = \left\{ m(a) = \begin{pmatrix} a & \\ & {}^t a^{\tau,-1} \end{pmatrix} \;\middle|\; a \in \mathrm{GL}_r(E) \right\},$$

and

$$w_r = \begin{pmatrix} & \mathbf{1}_r \\ -\mathbf{1}_r & \end{pmatrix}.$$

Here $\mathrm{Her}_r(E) = \{b \in \mathrm{Mat}_r(E) \mid b^\tau = {}^t b\}$.

Degenerate principal series and Eisenstein series. We fix a place $v \in \Sigma$ and suppress it from the notation. Thus $F = F_v$ is a local field of characteristic zero, $E = E_v$ is a quadratic extension of F which may be split, and $H_r = H_{r,v} = H_r(F_v)$ is a local reductive group. Also, we denote by $\mathcal{K}_r$ its maximal compact subgroup which is the intersection of H_r with $\mathrm{GL}_{2n}(\mathcal{O}_E)$ (resp. is isomorphic to $\mathrm{U}(r) \times \mathrm{U}(r)$) if v is finite (resp. if v is infinite). For $s \in \mathbb{C}$ and a character χ of $E^\times$, we denote by $I_r(s, \chi) = \text{s-Ind}_{P_r}^{H_r}(\chi|\cdot|_E^{s+r/2})$ the degenerate principal series representation (see [Kudla and Sweet 1997]) of H_r, where s-Ind means the nonnormalized smooth induction. Precisely, it realizes on the space of $\mathcal{K}_r$-finite functions φ_s on H_r satisfying

$$\varphi_s(n(b)m(a)g) = \chi(\det a)|\det a|_E^{s+r/2}\varphi_s(g)$$

for all $g \in H_r$, $m(a) \in M_r$, and $n(b) \in N_r$. A (holomorphic) section φ_s of $I_r(s, \chi)$ is called *standard* if its restriction to $\mathcal{K}_r$ is independent of s. It is called *unramified* if it takes value 1 on $\mathcal{K}_r$.

Now we view F and E as number fields. For a character χ of $\mathbb{A}_E^\times$ which is trivial on $E^\times$ and $s \in \mathbb{C}$, we have an admissible representation $I_r(s, \chi) = \bigotimes' I_r(s, \chi_v)$ of $H_r(\mathbb{A}_F)$, where the restricted tensor product is taken with respect to the unramified sections. For any standard section $\varphi_s = \otimes\varphi_{s,v} \in I_r(s, \chi)$, we can define an Eisenstein series as

$$E(g, \varphi_s) = \sum_{\gamma \in P_r(F)\backslash H_r(F)} \varphi_s(\gamma g).$$

The series is absolutely convergent if $\Re(s) > r/2$ and has a meromorphic continuation to the entire complex plane which is holomorphic at $s = 0$ (see [Tan 1999, Proposition 4.1]).

Hermitian spaces, Weil representations, and theta functions. Let us have a quick review of the classification of (nondegenerate) hermitian spaces. Suppose $v \in \Sigma_f$ and E is nonsplit at v. Then, up to isometry, there are two different hermitian spaces over E_v of dimension $m \geq 1$: $V^{\pm}$, defined by

$$\epsilon(V^{\pm}) = \epsilon_{E/F}\big((-1)^{m(m-1)/2} \det V^{\pm}\big) = \pm 1.$$

Suppose $v \in \Sigma_f$ and E is split at v. Then, up to isometry, there is only one hermitian space V^+ over E_v of dimension m. Suppose $v \in \Sigma_\infty$. Then, up to isometry, there are $m+1$ different hermitian spaces over E_v of dimension m: V_s with signature $(s, m-s)$ where $0 \leq s \leq m$. In the latter two cases, we can still define $\epsilon(V)$ in the same way. In the global case, up to isometry, all hermitian spaces V over E of dimension m are classified by signatures at infinite places and $\det V \in F^\times/\mathsf{Nm}E^\times$; particularly, V is determined by all $V_v = V \otimes_F F_v$. In general, we will also consider a hermitian space $\mathbb{V}$ over $\mathbb{A}_E$ of rank m. In this case, $\mathbb{V}$ is nondegenerate if there is a basis under which the representing matrix is invertible in $\mathsf{GL}_m(\mathbb{A}_E)$. For any place $v \in \Sigma$, we let $\mathbb{V}_v = \mathbb{V} \otimes_{\mathbb{A}_F} F_v$, $\mathbb{V}_f = \mathbb{V} \otimes_{\mathbb{A}_F} \mathbb{A}_{f,F}$, and define $\Sigma(\mathbb{V}) = \{v \in \Sigma \mid \epsilon(\mathbb{V}_v) = -1\}$, which is a finite set, and $\epsilon(\mathbb{V}) = \prod \epsilon(\mathbb{V}_v)$. We say $\mathbb{V}$ is *coherent* (resp. *incoherent*) if the cardinality of $\Sigma(\mathbb{V})$ is even (resp. odd), that is, $\epsilon(\mathbb{V}) = 1$ (resp. -1). By the Hasse principle, there is a hermitian space V over E such that $\mathbb{V} \cong V \otimes_F \mathbb{A}_F$ if and only if $\mathbb{V}$ is coherent. These two terminologies are introduced in the orthogonal case in [Kudla and Rallis 1994]; see also [Kudla 1997].

We fix a place $v \in \Sigma$ and suppress it from the notation. For a hermitian space V of dimension m with hermitian form $(\,\cdot\,,\,\cdot\,)$ and a positive integer r, we can construct a symplectic space $\boldsymbol{W} = \mathsf{Res}_{E/F} W_r \otimes_E V$ of dimension $4rm$ over F with the skew-symmetric form $\frac{1}{2}\mathsf{Tr}_{E/F}\langle\,\cdot\,,\,\cdot\,\rangle^\tau \otimes (\,\cdot\,,\,\cdot\,)$. We let $H = \mathsf{U}(V)$ be the unitary group of V and $\mathscr{S}(V^r)$ the space of Schwartz functions on V^r. Given a character χ of $E^\times$ satisfying $\chi\big|_{F^\times} = \epsilon_{E/F}^m$, we have a splitting homomorphism

$$\tilde{\iota}_{(\chi,\mathbf{1})} : H_r \times H \to \mathsf{Mp}(\boldsymbol{W})$$

lifting the natural map $\iota : H_r \times H \to \mathsf{Sp}(\boldsymbol{W})$ (see [Harris et al. 1996, Section 1]). We thus have a Weil representation (with respect to ψ) $\omega_\chi = \omega_{\chi,\psi}$ of $H_r \times H$ on the space $\mathscr{S}(V^r)$. Explicitly, for $\phi \in \mathscr{S}(V^r)$ and $h \in H$,

- $\omega_\chi(n(b))\phi(x) = \psi(\operatorname{tr} bT(x))\phi(x)$,
- $\omega_\chi(m(a))\phi(x) = |\det a|_E^{m/2}\chi(\det a)\phi(xa)$,
- $\omega_\chi(w_r)\phi(x) = \gamma_V\widehat{\phi}(x)$, and
- $\omega_\chi(h)\phi(x) = \phi(h^{-1}x)$,

where $T(x) = \frac{1}{2}\big((x_i, x_j)\big)_{1\leq i,j\leq r}$ is the moment matrix of x, γ_V is the Weil constant associated to the underlying quadratic space of V (and also ψ), and $\widehat{\phi}$ is the Fourier transform

$$\widehat{\phi}(x) = \int_{V^r} \phi(y)\psi\big(\tfrac{1}{2}\mathrm{Tr}_{E/F}(x, {}^t y)\big)\, dy$$

using the self-dual measure dy on V^r with respect to ψ. Taking the restricted tensor product over all local Weil representations, we get a global $\mathscr{S}(V^r) := \bigotimes' \mathscr{S}(V_v^r)$ as a representation of $H_r(\mathbb{A}_F) \times H(\mathbb{A}_F)$.

Now for V over E, χ a character of $\mathbb{A}_E^\times/E^\times$ such that $\chi\big|_{\mathbb{A}_F^\times} = \epsilon_{E/F}^m$, and $\phi \in \mathscr{S}(V^r)$, we define the theta function

$$\theta(g, h; \phi) = \sum_{x \in V^r(E)} \omega_\chi(g, h)\phi(x),$$

which is a smooth, slowly increasing function of $H_r(F)\backslash H_r(\mathbb{A}_F) \times H(F)\backslash H(\mathbb{A}_F)$, and consider the integral

$$I_V(g, \phi) = \int_{H(F)\backslash H(\mathbb{A}_F)} \theta(g, h; \phi)\, dh$$

if it is absolutely convergent. Here we normalize dh so that $\mathrm{vol}(H(F)\backslash H(\mathbb{A}_F)) = 1$. It is absolutely convergent for all ϕ if $m > 2r$ or V is anisotropic.

Siegel–Weil formulae. It is easy to see that $\varphi_{\phi,s}(g) = \omega_\chi(g)\phi(0)\lambda_{P_r}(g)^{s-(m-r)/2}$ is a standard section in $I_r(s, \chi)$ for any $\phi \in \mathscr{S}(V^r)$, where $\lambda_{P_r}(g) = \lambda_{P_r}(n(b)m(a)k) = |\det a|_E$ under the Iwasawa decomposition with respect to P_r. Hence we can define an Eisenstein series $E(s, g, \phi) = E(g, \varphi_{\phi,s})$ and we have:

Theorem 2.1 (Siegel–Weil formula). *Let $s_0 = (m - r)/2$,*

(1) *If $m > 2r$, $E(s_0, g, \phi)$ is absolutely convergent and*

$$E(s_0, g, \phi) = I_V(g, \phi).$$

(2) *If $r < m \le 2r$ and V is anisotropic, $E(s, g, \phi)$ is holomorphic at s_0 and*

$$E(s, g, \phi)\big|_{s=s_0} = I_V(g, \phi).$$

(3) *If $m = r$ and V is anisotropic, $E(s, g, \phi)$ is holomorphic at $s_0 = 0$ and*

$$E(s, g, \phi)\big|_{s=0} = 2I_V(g, \phi).$$

In the above theorem, (1) is the classical Siegel–Weil formula. (2) and (3) are certain generalizations which appear in [Ichino 2007, Theorem 1.1] and [Ichino 2004, Theorem 4.2], respectively. In the following, we simply write $E(s_0, g, \phi)$ for $E(s, g, \phi)\big|_{s=s_0}$ if it is holomorphic at s_0.

Remark 2.2. In case (3), if V is isotropic, we still have a (regularized) Siegel–Weil formula. But then since the theta integral $I_V(g, \phi)$ is not necessarily convergent, a regularization process must be applied. The inner product introduced in the next section also requires a regularization process. Since the classical inner-product formula is not the purpose of this paper, we will always assume that V is anisotropic for simplicity, or pretend that the regularization process has been applied for general V in the following discussion.

2B. ***Doubling integrals.*** In this section, we will review the method of doubling integrals which is first introduced in [Gelbart et al. 1987].

We now let $m = 2n$ and $r = n$ with $n \geq 1$ and suppress n from the notation, except that we will use H' instead of H_n, P' instead of P_n, N' instead of N_n, and $\mathcal{K}'$ instead of $\mathcal{K}_n$. Hence $\chi\big|_{\mathbb{A}_F^\times} = \mathbf{1}$. Let $\pi = \bigotimes' \pi_v$ be an irreducible cuspidal automorphic representation of $H'(\mathbb{A}_F)$ contained in $L^2(H'(F)\backslash H'(\mathbb{A}_F))$ and $\pi^\vee$ realizes on the space of complex conjugation of functions in π.

We denote by $(-W)$ the skew-hermitian space over E with the form $-\langle\,\cdot\,,\,\cdot\,\rangle$. Hence we can find a basis $\{e_1^-, \dots, e_{2n}^-\}$ satisfying $\langle e_i^-, e_j^-\rangle = 0$, $\langle e_{r+i}^-, e_{r+j}^-\rangle = 0$, and $\langle e_i^-, e_{n+j}^-\rangle = -\delta_{ij}$ for $1 \leq i \leq n$. Let $W'' = W \oplus (-W)$ be the direct sum of two skew-hermitian spaces. There is a natural embedding $\iota : H' \times H' \hookrightarrow \mathrm{U}(W'')$ which is, under the basis $\{e_1, \dots, e_{2n}\}$ of W and $\{e_1, \dots, e_n; e_1^-, \dots, e_n^-; e_{n+1}, \dots, e_{2n}; -e_{n+1}^-, \dots, -e_{2n}^-\}$ of W'', given by $\iota(g_1, g_2) = \iota_0(g_1, g_2^\vee)$, where

$$g_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \quad g_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}, \quad g^\vee = \begin{pmatrix} \mathbf{1}_n & \\ & -\mathbf{1}_n \end{pmatrix} g \begin{pmatrix} \mathbf{1}_n & \\ & -\mathbf{1}_n \end{pmatrix}^{-1},$$

and

$$\iota_0(g_1, g_2) = \begin{pmatrix} a_1 & & b_1 & \\ & a_2 & & b_2 \\ c_1 & & d_1 & \\ & c_2 & & d_2 \end{pmatrix}.$$

For a complete polarization $W'' = W' \oplus \overline{W}'$, where $W' = \mathrm{span}_E\{e_1, \dots, e_n; e_1^-, \dots, e_n^-\}$ and $\overline{W}' = \mathrm{span}_E\{e_{n+1}, \dots, e_{2n}; -e_{n+1}^-, \dots, -e_{2n}^-\}$, there is a Weil representation of $\mathrm{U}(W'')$, denoted by ω''_χ (with respect to ψ), on the space $\mathcal{S}(V^{2n})$, such that $\iota^*\omega''_\chi \cong \omega_{\chi,\psi} \boxtimes \chi\omega^\vee_{\chi,\psi}$, which is realized on the space $\mathcal{S}(V^n) \otimes \mathcal{S}(V^n)$. Here we realize the contragredient representation $\omega^\vee_{\chi,\psi}$ on the space $\mathcal{S}(V^n)$ through the bilinear pairing

$$\langle \phi, \phi^\vee \rangle = \int_{V^n(\mathbb{A}_E)} \phi(x)\phi^\vee(x)\,dx$$

for $\phi, \phi^\vee \in \mathcal{S}(V^n)$. Then $\omega^\vee_{\chi,\psi}$ is identified with $\omega_{\chi^{-1},\psi^{-1}}$.

For $\phi \in \mathcal{S}(V^n)$ and $f \in \pi$,

$$\theta_\phi^f(h) = \int_{H'(F)\backslash H'(\mathbb{A}_F)} \theta(g, h; \phi) f(g)\,dg$$

is a well-defined, slowly increasing function on $H(F)\backslash H(\mathbb{A}_F)$, where $dg=\otimes' dg_v$ such that $\mathcal{K}'_v$ gets volume 1 for any $v\in\Sigma$. Similarly, for $\phi^\vee\in\mathcal{S}(V^n)$ and $f^\vee\in\pi^\vee$, we have $\theta^{f^\vee}_{\phi^\vee}$. One should be careful that in the contragredient case, the Weil representation used to form the theta function should also be $\omega^\vee_\chi$. We have

$$
\begin{aligned}
\langle\theta^f_\phi,\theta^{f^\vee}_{\phi^\vee}\rangle_H &:= \int_{H(F)\backslash H(\mathbb{A}_F)} \theta^f_\phi(h)\theta^{f^\vee}_{\phi^\vee}(h)\,dh\\
&= \int_{H(F)\backslash H(\mathbb{A}_F)}\int_{[H'(F)\backslash H'(\mathbb{A}_F)]^2} \theta(g_1,h;\phi)f(g_1)\theta(g_2,h;\phi^\vee)\\
&\qquad\qquad\qquad\qquad\qquad\times f^\vee(g_2)\,dg_1\,dg_2\,dh\\
&= \int_{H(F)\backslash H(\mathbb{A}_F)}\int_{[H'(F)\backslash H'(\mathbb{A}_F)]^2} \theta(\iota(g_1,g_2),h;\phi\otimes\phi^\vee)\\
&\qquad\qquad\qquad\qquad\times f(g_1)f^\vee(g_2)\chi^{-1}(\det g_2)\,dg_1\,dg_2\,dh\\
&= \int_{[H'(F)\backslash H'(\mathbb{A}_F)]^2} f(g_1)f^\vee(g_2)\chi^{-1}(\det g_2)\\
&\qquad\qquad\times\int_{H(F)\backslash H(\mathbb{A}_F)} \theta(\iota(g_1,g_2),h;\phi\otimes\phi^\vee)\,dh\,dg_1\,dg_2.
\end{aligned}
\tag{2-1}
$$

We assume that V is anisotropic; then the inside integral in the last step is absolutely convergent and by Theorem 2.1(3), we have

$$
(2\text{-}1)=\frac{1}{2}\int_{[H'(F)\backslash H'(\mathbb{A}_F)]^2} f(g_1)f^\vee(g_2)\chi^{-1}(\det g_2)E(0,\iota(g_1,g_2),\phi\otimes\phi^\vee)\,dg_1dg_2.
$$

It should be mentioned that the Eisenstein series on $\mathrm{U}(W'')$ appearing above is formed with respect to the parabolic subgroup P fixing the subspace $\overline{W}'$ (with maximal unipotent subgroup N), that is,

$$
E(s,g,\Phi)=E(g,\varphi_{\Phi,s})=\sum_{\gamma\in P(F)\backslash \mathrm{U}(W'')(F)} \omega''_\chi(\gamma g)\Phi(0)\lambda_P(\gamma g)^s
$$

for $g\in\mathrm{U}(W'')(\mathbb{A}_F)$, $\Phi\in\mathcal{S}(V^{2n})$, and $\Re(s)>n$. The coset $P(F)\backslash\mathrm{U}(W'')(F)$ can be canonically identified with the space of isotropic n-planes in W''. Under the right action of $H'(F)\times H'(F)$ through ι, the orbit of an n-plane Z is determined by the invariant $d=\dim(Z\cap W)=\dim(Z\cap(-W))$. Let γ_d be a representative of the corresponding double coset where $0\le d\le n$. In particular, we take

$$
\gamma_0=\begin{pmatrix} & & \mathbf{1}_n & \\ & \mathbf{1}_n & & \\ -\mathbf{1}_n & \mathbf{1}_n & & \\ & & \mathbf{1}_n & \mathbf{1}_n \end{pmatrix} \text{ and } \gamma_n=\mathbf{1}_{4n}
$$

(see [Kudla and Rallis 2005]). Let St_d be the stabilizer of $P\gamma_d\iota(H'\times H')$ in $H'\times H'$. In particular $\mathrm{St}_0=\Delta(H')$ is the diagonal. Hence for a standard section

$\varphi_s \in I_{2n}(s,\chi)$ and $\Re(s) > n$,

$$\begin{aligned}\int_{[H'(F)\backslash H'(\mathbb{A}_F)]^2} & f(g_1) f^\vee(g_2)\chi^{-1}(\det g_2) E(\imath(g_1,g_2),\varphi_s)\, dg_1 dg_2 \\ &= \int_{[H'(F)\backslash H'(\mathbb{A}_F)]^2} (f\otimes f^\vee\chi^{-1})(g) \sum_{\gamma\in P(F)\backslash \mathrm{U}(W'')(F)} \varphi_s(\gamma\imath(g))\, dg \\ &= \sum_{d=0}^{n} \int_{\mathrm{St}_d(F)\backslash H'(\mathbb{A}_F)^2} (f\otimes f^\vee\chi^{-1})(g)\varphi_s(\gamma_d\imath(g))\, dg. \end{aligned} \tag{2-2}$$

When $d > 0$, St_d has a nontrivial unipotent radical. Since f and $f^\vee$ are cuspidal, we have

$$\begin{aligned}(2\text{-}2) &= \int_{\Delta(H'(F))\backslash H'(\mathbb{A}_F)^2} (f\otimes f^\vee\chi^{-1})(g)\varphi_s(\gamma_0\imath(g))\, dg \\ &= \int_{H'(F)\backslash H'(\mathbb{A}_F)} \int_{H'(\mathbb{A}_F)} f(g_1g_2) f^\vee(g_1)\chi^{-1}(\det g_1) \\ &\qquad\qquad\qquad\qquad \times\varphi_s(\gamma_0\imath(g_1g_2,g_1))\, dg_1dg_2 \\ &= \int_{H'(F)\backslash H'(\mathbb{A}_F)} \int_{H'(\mathbb{A}_F)} \pi(g_2) f(g_1) f^\vee(g_1)\chi^{-1}(\det g_1) \\ &\qquad\qquad\qquad\qquad \times\varphi_s(p(g_1)\gamma_0\imath(g_2,1))\, dg_1dg_2, \end{aligned} \tag{2-3}$$

where $p(g_1)\gamma_0 = \gamma_0\imath(g_1,g_1)$ having the property that under the Levi decomposition $p(g_1) = n(b)m(a) \in P(\mathbb{A}_F)$, we have $\det a = \det g_1$. Hence

$$\begin{aligned}(2\text{-}3) &= \int_{H'(\mathbb{A}_F)} \int_{H'(F)\backslash H'(\mathbb{A}_F)} \pi(g_2) f(g_1) f^\vee(g_1)\, dg_1 \varphi_s(\gamma_0\imath(g_2,1))\, dg_2 \\ &= \int_{H'(\mathbb{A}_F)} \langle \pi(g) f, f^\vee\rangle \varphi_s(\gamma_0\imath(g,1))\, dg \\ &= \prod_{v\in\Sigma} \int_{H'_v} \langle \pi_v(g_v) f_v, f_v^\vee\rangle \varphi_{s,v}(\gamma_0\imath(g_v,1))\, dg_v,\end{aligned}$$

where we assume f, $f^\vee$ and φ_s are all decomposable. In summary, we have:

Proposition 2.3. *Let f, $f^\vee$ and φ_s be as above. For $\Re(s) > n$, the integral*

$$\begin{aligned}\int_{[H'(F)\backslash H'(\mathbb{A}_F)]^2} & f(g_1) f^\vee(g_2)\chi^{-1}(\det g_2) E(\imath(g_1,g_2),\varphi_s)\, dg_1dg_2 \\ &= \prod_{v\in\Sigma} \int_{H'_v} \langle \pi_v(g_v) f_v, f_v^\vee\rangle \varphi_{s,v}(\gamma_0\imath(g_v,1))\, dg_v,\end{aligned}$$

which defines an element in

$$\mathrm{Hom}_{H'(\mathbb{A}_F)\times H'(\mathbb{A}_F)}(I_{2n}(s,\chi), \pi^\vee\boxtimes\chi\pi) = \bigotimes_v \mathrm{Hom}_{H'_v\times H'_v}(I_{2n}(s,\chi_v), \pi_v^\vee\boxtimes\chi_v\pi_v).$$

2C. *Local zeta integrals.* In this section, we will study the local functional we finally found in the last section.

Fix a finite place v of F and suppress it from the notation. For $f \in \pi$, $f^\vee \in \pi^\vee$, and holomorphic section $\varphi_s \in I_{2n}(s, \chi)$, define the local zeta integral

$$Z(\chi, f, f^\vee, \varphi_s) = \int_{H'} \langle \pi(g) f, f^\vee \rangle \varphi_s(\gamma_0 \iota(g, 1))\, dg,$$

which is absolutely convergent when $\Re(s) > 2n$. In [Harris et al. 1996, Section 6], the family of *good sections* is introduced. For any good section, the zeta integral $Z(\chi, f, f^\vee, \varphi_s)$ is a rational function in q^{-s}, where q is the cardinality of the residue field of F. In particular, it has a meromorphic continuation to the entire complex plane. Consider the family of zeta integrals

$$\{Z(\chi, f, f^\vee, \varphi_s) \mid f \in \pi,\ f^\vee \in \pi^\vee,\ \varphi_s \text{ is good}\}$$

and the fractional ideal $\mathcal{I}$ of the ring $\mathbb{C}[q^s, q^{-s}]$ in its fraction field generated by the above family. In fact, $\mathcal{I}$ is generated by $1/P(q^{-s})$, for a unique polynomial $P(X) \in \mathbb{C}[X]$ such that $P(0) = 1$. We let

$$L\big(s + \tfrac{1}{2}, \pi, \chi\big) = \frac{1}{P(q^{-s})}$$

be the local doubling L-series of Piatetski-Shapiro and Rallis. The same construction can also be applied to the archimedean case.

Now suppose E/F is unramified (including split) at v and ψ, χ, and π are also unramified. Let $f_0 \in \pi^{\mathcal{K}'}$, $f_0^\vee \in \pi^{\vee, \mathcal{K}'}$, and $\langle f_0, f_0^\vee \rangle = 1$, φ_s^0 be the unramified standard section. Then the calculation in [Gelbart et al. 1987] and [Li 1992] (see Theorem 3.1 of the latter reference) shows that

$$Z(\chi, f_0, f_0^\vee, \varphi_s^0) = \frac{L(s + \frac{1}{2}, \pi, \chi)}{b_{2n}(s)},$$

where

$$b_m(s) = \prod_{i=0}^{m-1} L(2s + m - i, \epsilon_{E/F}^i) \tag{2-4}$$

is a product of local Tate factors. For the general case,

$$\frac{b_{2n}(s) Z(\chi, f, f^\vee, \varphi_s)}{L(s + \frac{1}{2}, \pi, \chi)}$$

admits a meromorphic extension to the entire complex plane which is holomorphic at $s = 0$. Moreover, the *normalized zeta integral*

$$Z^*(\chi, f, f^\vee, \varphi_s) := \left.\frac{b_{2n}(s) Z(\chi, f, f^\vee, \varphi_s)}{L(s + \frac{1}{2}, \pi, \chi)}\right|_{s=0} \tag{2-5}$$

defines a nonzero element in $\mathrm{Hom}_{H'\times H'}(I_{2n}(0,\chi), \pi^\vee \boxtimes \chi\pi)$ (see [Harris et al. 1996, Proof of (1) of Theorem 4.3]).

Remark 2.4. It is conjectured (see, for example, [Harris et al. 1996]) that for all irreducible admissible representations π of H' and characters χ of $E^\times$, we have

$$L(s,\pi,\chi) = L(s, \mathrm{BC}(\pi)\otimes\chi).$$

This is known when E/F, χ, and π are all unramified due to [Li 1992] and (the similar argument for unitary groups in) [Kudla and Rallis 2005, Section 5]. It is also known when $n = 1$ due to [Harris 1993].

For further discussion, we need to recall a result on the degenerate principal series. In the following, we will use the notation H'' instead of $\mathrm{U}(W'')$ for short and recall our embedding $\iota : H' \times H' \hookrightarrow H''$. Let V be a hermitian space of dimension $2n$ over E. Then $\varphi_\phi(g) = \omega_\chi(g)\phi(0)$ defines an H''-intertwining map $\mathscr{S}(V^{2n}) \to I_{2n}(0,\chi)$ whose image $R(V,\chi)$ is isomorphic to $\mathscr{S}(V^{2n})_H$. Recall that we denote by $V^\pm$ the two nonisometric hermitian spaces of dimension $2n$ when v is finite nonsplit, by V^+ the only hermitian space of dimension $2n$ when v is finite split (up to isometry), and by V_s $(0 \le s \le 2n)$ the $2n+1$ nonisometric hermitian spaces of dimension $2n$ when v is infinite.

Proposition 2.5. (1) *If v is finite nonsplit, $R(V^+,\chi)$ and $R(V^-,\chi)$ are irreducible and inequivalent and $I_{2n}(0,\chi) = R(V^+,\chi)\oplus R(V^-,\chi)$.*

(2) *If v is finite split, $R(V,\chi)$ is irreducible and $I_{2n}(0,\chi) = R(V^+,\chi)$.*

(3) *If v is infinite, $R(V_s,\chi)$ are irreducible and inequivalent and $I_{2n}(0,\chi) = \bigoplus_{s=0}^{2n} R(V_s,\chi)$.*

Proof. (1) is [Kudla and Sweet 1997, Theorem 1.2], (2) is [Kudla and Sweet 1997, Theorem 1.3], and (3) is [Lee 1994, Section 6, Proposition 6.11]. □

2D. ***Central special values of L-functions.*** In this section, we will make a connection between the theta lifting θ_ϕ^f defined in Section 2B and the central special value of the L-function of the representation π.

Recall that we have an irreducible unitary cuspidal automorphic representation π of $H' = H_n$ and a hermitian space V over E of dimension $2n$. One key question in the theory of theta lifting is whether θ_ϕ^f is nonvanishing. A sufficient condition is to look at the local invariant functional as follows.

First, we have the following *theta dichotomy*.

Proposition 2.6. *For any nonsplit place $v\in\Sigma$, $\mathrm{Hom}_{H'_v\times H'_v}(R(V_v,\chi_v), \pi_v^\vee\boxtimes\chi_v\pi_v)$ is nonzero for exactly one hermitian space V_v (up to isometry) over E_v of dimension $2n$, which we denote by $V(\pi_v,\chi_v)$.*

Proof. If v is (real) archimedean, this is [Paul 1998, Theorem 2.9]. If v is nonarchimedean, it is due to [Gong and Grenié 2011, Theorem 2.10] and the nonvanishing of Z^*. □

In Proposition 2.3, if we let $\varphi_s = \varphi_{\phi\otimes\phi^\vee,s}$ and denote $Z^*(s, \chi_v, f_v, f_v^\vee, \phi_v \otimes \phi_v^\vee) = Z^*(\chi_v, f_v, f_v^\vee, \varphi_{\phi_v\otimes\phi_v^\vee,s})$, then both sides have meromorphic continuations to the entire complex plane that are actually holomorphic at the point $s = 0$; that is, we have

$$\langle \theta_\phi^f, \theta_{\phi^\vee}^{f^\vee} \rangle_H = \frac{L(\frac{1}{2}, \pi, \chi)}{2\prod_{i=1}^{2n} L(i, \epsilon_{E/F}^i)} \prod_{v\in S} Z^*(0, \chi_v, f_v, f_v^\vee, \phi_v \otimes \phi_v^\vee), \tag{2-6}$$

in which the product of normalized zeta integrals can actually be taken over a finite set S by the unramified calculation. In particular, for $v \notin S$, $V_v \cong V(\pi_v, \chi_v)$, that is, $\theta_{\chi_v}(\pi_v^\vee, V_v) \neq 0$. Then one necessary condition for θ_ϕ^f to be nonvanishing for some f and ϕ is that each local (normalized) zeta integral is not identically zero, which exactly means $V_v \cong V(\pi_v, \chi_v)$ for all $v \in \Sigma$. Let $\mathbb{V}(\pi, \chi)$ be the hermitian space over $\mathbb{A}_E$ such that $\mathbb{V}(\pi, \chi)_v \cong V(\pi_v, \chi_v)$ and let $\epsilon(\pi_v, \chi_v) = \epsilon(V(\pi_v, \chi_v))$ and $\epsilon(\pi, \chi) = \prod \epsilon(\pi_v, \chi_v)$. If $\epsilon(\pi, \chi) = -1$. Then $\mathbb{V}(\pi, \chi)$ is incoherent, hence for any V, the (possibly regularized) theta lifting θ_ϕ^f is always vanishing. If $\epsilon(\pi, \chi) = 1$, then $\mathbb{V}(\pi, \chi) \cong V(\pi, \chi) \otimes_F \mathbb{A}_F$ for some $V(\pi, \chi)$ over E. Assume $V(\pi, \chi)$ is anisotropic. Then there exist some $f \in \pi$ and $\phi \in \mathscr{S}(V(\pi, \chi)^n)$ such that $\theta_\phi^f \neq 0$ if and only if $L(\frac{1}{2}, \pi, \chi) \neq 0$.

We want to give another interpretation for the formula (2-6) when $\epsilon(\pi, \chi) = 1$, which is crucial for our proof in [Liu 2011]. For this purpose, let us assume the following conjecture raised by Kudla and Rallis (see [Harris et al. 1996]):

$$\dim \mathrm{Hom}_{H_v'\times H_v'}(I_{2n}(0, \chi_v), \pi_v^\vee \boxtimes \chi_v\pi_v) = 1 \tag{2-7}$$

for all components π_v of π. This is proved in [Liu 2011, Section 6B] when $n = 1$. Let $V = V(\pi, \chi)$ and $R(V, \chi) = \bigotimes' R(V_v, \chi_v)$; the functional

$$\beta(f, f^\vee, \phi, \phi^\vee) := \langle \theta_\phi^f, \theta_{\phi^\vee}^{f^\vee} \rangle_H$$

defines an element in

$$\mathrm{Hom}_{H'(\mathbb{A}_F)\times H'(\mathbb{A}_F)}(R(V, \chi), \pi^\vee \boxtimes \chi\pi) = \bigotimes_v \mathrm{Hom}_{H_v'\times H_v'}(R(V_v, \chi_v), \pi_v^\vee \boxtimes \chi_v\pi_v).$$

On the other hand, the functional

$$\alpha(f, f^\vee, \phi, \phi^\vee) := \prod_v Z^*(0, \chi_v, f_v, f_v^\vee, \phi_v \otimes \phi_v^\vee)$$

(when everything is decomposable, otherwise we take the linear combination) also defines an element in $\bigotimes \mathrm{Hom}_{H_v'\times H_v'}(R(V_v, \chi_v), \pi_v^\vee \boxtimes \chi_v\pi_v)$ which is nonzero. But by our assumption (2-7), this space is of dimension one. Hence β is a constant multiple of α. This constant, by (2-6), is

$$\frac{\beta}{\alpha}=\frac{L(\frac{1}{2},\pi,\chi)}{2\prod_{i=1}^{2n}L(i,\epsilon_{E/F}^{i})}.$$

In some sense, the vanishing of $L(\frac{1}{2},\pi,\chi)$ is the obstruction for β to be a nontrivial global invariant functional. This kind of formulation is first observed in [Yuan et al. 2011; 2010].

2E. ***Vanishing of central L-values.*** In this section, we will prove that the central L-value $L(\frac{1}{2},\pi,\chi)$ vanishes when $\epsilon(\pi,\chi)=-1$.

By Proposition 2.5, we have a decomposition of $H''(\mathbb{A}_F)$-admissible representation

$$I_{2n}(0,\chi)=\bigoplus_{\mathbb{V}}R(\mathbb{V},\chi)=\bigoplus_{\mathbb{V}}{\bigotimes_v}' R(\mathbb{V}_v,\chi_v),$$

where the direct sum is taken over all (isometry classes of) hermitian spaces over $\mathbb{A}_E$ of rank $2n$ and each $R(\mathbb{V},\chi)$ is irreducible. Recall the group $H''=\mathrm{U}(W'')$ and its standard parabolic subgroup P fixing $\overline{W}'$ whose unipotent radical is N as in Section 2B. First, we need some lemmas for local representations.

Fix any place v and suppress it from the notation. For $T\in\mathrm{Her}_{2n}(E)$, let $\Omega_T=\{x\in V^{2n}\mid T(x)=T\}$ and define a character ψ_T of $N\cong\mathrm{Her}_{2n}(E)$ by $\psi_T(n(b))=\psi(\operatorname{tr}Tb)$.

Lemma 2.7. (1) *Suppose v is finite, let $\mathcal{S}(V^{2n})_{N,\psi_T}$ (resp. $R(V,\chi)_{N,\psi_T}$) be the twisted Jacquet module of $\mathcal{S}(V^{2n})$ (resp. $R(V,\chi)$) associated to N and the character ψ_T.*

(a) *The quotient map $\mathcal{S}(V^{2n})\to\mathcal{S}(V^{2n})_{N,\psi_T}$ can be realized by the restriction $\mathcal{S}(V^{2n})\to\mathcal{S}(\Omega_T)$;*

(b) *If T is nonsingular, then*

$$\dim R(V,\chi)_{N,\psi_T}=\begin{cases}1 & \text{if } \Omega_T\neq\varnothing,\\ 0 & \text{otherwise.}\end{cases}$$

(2) *Suppose v is infinite, that is, $E/F=\mathbb{C}/\mathbb{R}$ and T is nonsingular, the space of H-invariant tempered distribution T on $\mathcal{S}(V^{2n})$ such that*

$$T(\omega_\chi(X)\Phi)=d\psi_T(X)T(\Phi)$$

for $X\in\mathfrak{n}=\mathrm{Lie}\,N$ is of dimension 1 (*resp.* 0) *if $\Omega_T\neq\varnothing$* (*resp.* $\Omega_T=\varnothing$).

Proof. (1) is [Rallis 1987, Lemma 4.2], (2) is [Rallis 1987, Lemma 4.2], and [Kudla and Rallis 1994, Proposition 2.9]. □

We now construct the twisted Jacquet module $R(V,\chi)_{N,\psi_T}$ or the invariant distribution explicitly if it is not trivial. For a standard section $\varphi_s \in I_{2n}(s,\chi)$, define the Whittaker integral

$$W_T(g,\varphi_s) = \int_N \varphi_s(wng)\psi_T(n)^{-1}\,dn,$$

where $w = w_{2n}$ and dn is self-dual with respect to ψ. The integral $W_T(g,\varphi_s)$ is absolutely convergent when $\Re(s) > n$. It is easy to see that $W_T(e,\cdot\,) : I_{2n}(s,\chi) \to \mathbb{C}_{N,\psi_T}$ is an N-intertwining map. Let $W_T(s,g,\Phi) = W_T(g,\varphi_{\Phi,s})$ for $\Phi \in \mathcal{S}(V^{2n})$. We have

Lemma 2.8. *Assume T is nonsingular.*

(1) *$W_T(g,\varphi_s)$ is entire.*

(2) *The integral $\Phi \mapsto W_T(0,e,\Phi)$ realizes the surjective N-intertwining map*

$$\mathcal{S}(V^{2n}) \to R(V,\chi) \to R(V,\chi)_{N,\psi_T}$$

or the invariant distribution in Lemma 2.7(2).

Proof. (1) is [Karel 1979, Corollary 3.6.1] for v finite and [Wallach 1988, Theorem 8.1] for v infinite; (2) is [Kudla and Rallis 1994, Proposition 2.7]. □

Lemma 2.9. *Suppose v is finite, E/F, ψ, and χ are all unramified, and $V = V^+$. Then for Φ^0 the characteristic function of $(\Lambda^+)^{2n}$ for a self-dual $\mathcal{O}_E$-lattice Λ^+ and $T \in \mathrm{Her}_{2n}(\mathcal{O}_F)$ with $\det T \in \mathcal{O}_F^\times$, we have*

$$W_T(s,e,\Phi^0) = b_{2n}(s)^{-1}.$$

Proof. This is [Tan 1999, Proposition 3.2]. □

Now suppose we are in the global situation. We denote by $\mathcal{A}(H'')$ the space of automorphic forms of H''. For $T \in \mathrm{Her}_{2n}(F)$, define the T-th Fourier coefficient of $f(g) \in \mathcal{A}(H'')$ as

$$W_T(g,f) = \int_{N(F)\backslash N(\mathbb{A}_F)} f(ng)\psi_T(n)^{-1}\,dn.$$

For any hermitian space $\mathbb{V}$ over $\mathbb{A}_E$ of rank $2n$, we have a series of linear maps

$$\begin{aligned}\mathcal{E}_s : R(\mathbb{V},\chi) &\to \mathcal{A}(H'')\\ \Phi &\mapsto E(s,g,\Phi) = E(g,\varphi_{\Phi,s})\end{aligned}$$

for s near 0. It is an $H''(\mathbb{A}_F)$-intertwining map exactly when $s = 0$. Then for T nonsingular (and s near 0), we have

$$E_T(s,g,\Phi) := W_T(g,\mathcal{E}_s(\Phi)) = \prod_{v\in\Sigma} W_T(s,g_v,\Phi_v). \tag{2-8}$$

Lemma 2.10. *For any $H''(\mathbb{A}_F)$-intertwining operator $\mathscr{E} : R(\mathbb{V}, \chi) \to \mathscr{A}(H'')$, if $W_T(g, \cdot) \circ \mathscr{E}$ vanishes for all nonsingular T, then $\mathscr{E} = 0$.*

Proof. Fix a finite place v; by Lemma 2.7(1), we can find a section $\Phi_0 = \Phi_{v,0}\Phi^v \in \mathscr{S}(\mathbb{V}^{2n})$ with nonzero projection in $R(\mathbb{V}, \chi)$ such that $\Phi_{v,0} \in \mathscr{S}(\mathbb{V}_v^{2n})_{\mathrm{reg}}$, the set consisting of functions supporting in the set $\{x \in \mathbb{V}_v^{2n} \mid \det T(x) \neq 0\}$. For any $g^v \in e_v H''(\mathbb{A}_F^v)$, the functional $\Phi_v \mapsto W_T(0, g^v, \Phi_v\Phi^v)$ factors through the twisted Jacquet module $\mathscr{S}(\mathbb{V}_v^{2n})_{N_v,\psi_T}$. If T is singular, then by our choice of $\Phi_{v,0}$ and Lemma 2.7(1-a), $W_T(0, g^v, \Phi_{v,0}\Phi^v) = 0$. Similarly, $W_T(0, g, \Phi_{v,0}\Phi^v) = 0$ for all $g \in P_v H''(\mathbb{A}_F^v)$ since P_v keeps the set $\mathscr{S}(\mathbb{V}_v^{2n})_{\mathrm{reg}}$. For T nonsingular, $W_T \equiv 0$ by the assumption. Hence $\mathscr{E}(\Phi_0)(g) = 0$ for $g \in P_v H''(\mathbb{A}_F^v)$. It follows that $\mathscr{E}(\Phi_0) = 0$ and $\mathscr{E} = 0$ by our choice of Φ_0 and the irreducibility of $R(\mathbb{V}, \chi)$. □

Proposition 2.11. (1) *If $\mathbb{V}$ is incoherent, then $\mathrm{Hom}_{H''(\mathbb{A}_F)}(R(\mathbb{V}, \chi), \mathscr{A}(H''))$ has dimension 0.*

(2) *If $\mathbb{V}$ is coherent, then $\mathrm{Hom}_{H''(\mathbb{A}_F)}(R(\mathbb{V}, \chi), \mathscr{A}(H''))$ has dimension 1 and $\mathscr{E}_0$ given above is a nontrivial element.*

Proof. For (1), assume that $\mathscr{E}$ is a nontrivial intertwining map. By Lemma 2.10, there is a nonsingular $T \in \mathrm{Her}_{2n}(F)$ such that $W_T(g, \cdot) \circ \mathscr{E}$ does not vanish. By parts (1-b) and (2) of Lemma 2.7, T is representable by $\mathbb{V}_v$ for any $v \in \Sigma$; that is, $\Omega_T \neq \varnothing$. But then $\mathbb{V}$ will be coherent which is a contradiction.

For (2), assume $\mathscr{E}$ and $\mathscr{E}'$ are both nontrivial intertwining maps. By Lemma 2.10, there is a nonsingular T such that $W_T(g, \cdot) \circ \mathscr{E}$ does not vanish. By parts (1-b) and (2) of Lemma 2.7, or the general fact that the Whittaker model with respect to a generic character is unique, there exists $c \in \mathbb{C}$ such that $W_T(g, \cdot) \circ \mathscr{E}' = c W_T(g, \cdot) \circ \mathscr{E}$. Furthermore, c is independent of nonsingular T since all of those which can be represented by $\mathbb{V}$ are in a single $M(F)$-orbit under the conjugation action on $N(F)$. Then by Lemma 2.10, $\mathscr{E}' - c\mathscr{E} = 0$, that is, $\dim \mathrm{Hom}_{H''(\mathbb{A}_F)}(R(\mathbb{V}, \chi), \mathscr{A}(H'')) \leq 1$.

For the rest, we need to prove that $\mathscr{E}_0$ is actually nontrivial. Choose a nonsingular $T \in \mathrm{Her}_{2n}(F)$ such that it is representable by $\mathbb{V}$ which exists since $\mathbb{V}$ is coherent. By (2-8) and Lemmas 2.8(2) and 2.9, we can find a suitable Φ such that $W_T(0, e, \Phi) \neq 0$; hence $\mathscr{E}_0 \neq 0$. □

Now we can state our main result in this section.

Theorem 2.12. *If $\epsilon(\pi, \chi) = -1$, then $L\left(\frac{1}{2}, \pi, \chi\right) = 0$.*

Proof. Let $\mathbb{V} = \mathbb{V}(\pi, \chi)$; then it is incoherent. We can choose suitable f_v, $f_v^\vee$, ϕ_v, and $\phi_v^\vee$ when one of E, ψ, χ, and π is ramified at v, such that

$$Z^*(0, \chi_v, f_v, f_v^\vee, \phi_v \otimes \phi_v^\vee) \neq 0.$$

Let f, $f^\vee$, ϕ, and $\phi^\vee$ be global vectors with these subscribed local components and unramified ones at the places where E, ψ, χ, and π are unramified. Then from

Proposition 2.3 (after analytic continuation), we have

$$\int_{[H'(F)\backslash H'(\mathbb{A}_F)]^2} f(g_1)f^\vee(g_2)\chi^{-1}(\det g_2)E(0,\iota(g_1,g_2),\phi\otimes\phi^\vee)\,dg_1dg_2 = \frac{L(\frac{1}{2},\pi,\chi)}{\prod_{i=1}^{2n}L(i,\epsilon_{E/F}^i)}\prod_{v\in S}Z^*(0,\chi_v,f_v,f_v^\vee,\phi_v\otimes\phi_v^\vee).$$

But $\mathscr{E}_0$ is zero on $R(\mathbb{V},\chi)$ by Proposition 2.11(1). We have $E(0,\iota(g_1,g_2),\phi\otimes\phi^\vee)\equiv 0$. Hence $L(\frac{1}{2},\pi,\chi)=0$ by our choices and the fact that the Tate L-values appearing here are finite. □

Since $L(\frac{1}{2},\pi,\chi)=0$, it leads us to consider its derivative at this point. In fact, we have

$$\begin{aligned}
&\int_{[H'(F)\backslash H'(\mathbb{A}_F)]^2} f(g_1)f^\vee(g_2)\chi^{-1}(\det g_2)\frac{d}{ds}\Big|_{s=0}E(s,\iota(g_1,g_2),\phi\otimes\phi^\vee)\,dg_1dg_2\\
&\quad=\frac{d}{ds}\Big|_{s=0}\int_{[H'(F)\backslash H'(\mathbb{A}_F)]^2} f(g_1)f^\vee(g_2)\chi^{-1}(\det g_2)\\
&\qquad\qquad\times E(s,\iota(g_1,g_2),\phi\otimes\phi^\vee)\,dg_1dg_2\\
&\quad=\frac{d}{ds}\Big|_{s=0}\frac{L(s+\frac{1}{2},\pi,\chi)}{\prod_{i=1}^{2n}L(2s+i,\epsilon_{E/F}^i)}\prod_{v\in S}Z^*(s,\chi_v,f_v,f_v^\vee,\phi_v\otimes\phi_v^\vee)\\
&\quad=\frac{L'(\frac{1}{2},\pi,\chi)}{\prod_{i=1}^{2n}L(i,\epsilon_{E/F}^i)}\prod_{v\in S}Z^*(0,\chi_v,f_v,f_v^\vee,\phi_v\otimes\phi_v^\vee)\\
&\qquad\qquad+L\Big(\frac{1}{2},\pi,\chi\Big)\frac{d}{ds}\Big|_{s=0}\frac{\prod_{v\in S}Z^*(s,\chi_v,f_v,f_v^\vee,\phi_v\otimes\phi_v^\vee)}{\prod_{i=1}^{2n}L(2s+i,\epsilon_{E/F}^i)}\\
&\quad=\frac{L'(\frac{1}{2},\pi,\chi)}{\prod_{i=1}^{2n}L(i,\epsilon_{E/F}^i)}\prod_{v\in S}Z^*(0,\chi_v,f_v,f_v^\vee,\phi_v\otimes\phi_v^\vee).
\end{aligned}\tag{2-9}$$

We call $E'(0,g,\Phi)=(d/ds)\big|_{s=0}E(s,g,\Phi)$ the *analytic kernel function* associated to the *test function* $\Phi\in\mathscr{S}(\mathbb{V}^{2n})$.

Recall that for $T\in\mathrm{Her}_{2n}(F)$, we let

$$E_T(s,g,\Phi)=W_T(g,\mathscr{E}_s(\Phi))$$

for s near 0. If T is nonsingular, then

$$W_T(g,\mathscr{E}_s(\Phi))=\prod_{v\in\Sigma}W_T(s,g_v,\Phi_v)$$

if $\Phi=\bigotimes\Phi_v$ is decomposable. Hence,

$$E(s,g,\Phi)=\sum_{T\text{ sing.}}E_T(s,g,\Phi)+\sum_{T\text{ nonsing.}}\prod_{v\in\Sigma}W_T(s,g_v,\Phi_v).$$

Taking the derivative at $s=0$, we have

$$\begin{aligned} E'(0,g,\Phi) &= \sum_{T \text{ sing.}} E'_T(0,g,\Phi) + \sum_{T \text{ nonsing.}} \sum_{v\in\Sigma} W'_T(0,g_v,\Phi_v) \prod_{v'\neq v} W_T(0,g_{v'},\Phi_{v'}) \\ &= \sum_{T \text{ sing.}} E'_T(0,g,\Phi) + \sum_{v\in\Sigma} \sum_{T \text{ nonsing.}} W'_T(0,g_v,\Phi_v) \prod_{v'\neq v} W_T(0,g_{v'},\Phi_{v'}). \end{aligned}$$

But we have $\prod_{v'\neq v} W_T(0,g_v,\Phi_v) \neq 0$ only if $\mathbb{V}_{v'}$ represents T for all $v' \neq v$ by Lemma 2.7(1-b). Since $\mathbb{V}$ is incoherent, $\mathbb{V}_v$ cannot represent T. For T nonsingular, there are only finitely many $v \in \Sigma$ such that T is not represented by $\mathbb{V}_v$, that is, there does not exist $x_1,\dots,x_{2n} \in \mathbb{V}_v$ whose moment matrix is T. We denote the set of such v by $\mathrm{Diff}(T,\mathbb{V})$. Then

$$E'(0,g,\Phi) = \sum_{T \text{ sing.}} E'_T(0,g,\Phi) + \sum_{v\in\Sigma} E_v(0,g,\Phi),$$

where

$$E_v(0,g,\Phi) = \sum_{\mathrm{Diff}(T,\mathbb{V})=\{v\}} W'_T(0,g_v,\Phi_v) \prod_{v'\neq v} W_T(0,g_{v'},\Phi_{v'}). \tag{2-10}$$

In fact, the second sum is only taken over those v which are nonsplit in E.

3. Arithmetic theta lifting

3A. *Shimura varieties of unitary groups and special cycles.* In this section, we will recall the notion of Shimura varieties of unitary groups and Kudla's special cycles on them. We fix an additive character $\psi : F\backslash\mathbb{A}_F \to \mathbb{C}$ such that ψ_ι is the standard $t \mapsto e^{2\pi i t}$ ($t \in F_\iota = \mathbb{R}$) for any $\iota \in \Sigma_\infty$ until the end of this paper. The basic references for this section are [Kudla and Millson 1990; Kottwitz 1992; Kudla 1997].

Shimura varieties of unitary groups. Let $m \geq 2$ and $1 \leq r < m$ be integers. Let $\mathbb{V}$ be a totally positive-definite incoherent hermitian space over $\mathbb{A}_E$ of rank m. Let $\mathbb{H} = \mathrm{Res}_{\mathbb{A}_F/\mathbb{A}} \mathrm{U}(\mathbb{V})$ be the unitary group which is a reductive group over $\mathbb{A}$ and $\mathbb{H}^{\mathrm{der}} = \mathrm{Res}_{\mathbb{A}_F/\mathbb{A}} \mathrm{SU}(\mathbb{V})$ its derived subgroup. Let $\mathbb{T} \cong \mathrm{Res}_{\mathbb{A}_F/\mathbb{A}} \mathbb{A}_E^{\times,1}$ be the maximal abelian quotient of $\mathbb{H}$ which is also isomorphic to its center. Let $T \cong \mathrm{Res}_{F/\mathbb{Q}} E^{\times,1}$ be the unique (up to isomorphism) $\mathbb{Q}$-torus such that $T \times_{\mathbb{Q}} \mathbb{A} \cong \mathbb{T}$. Then T has the property that $T(\mathbb{Q})$ is discrete in $T(\mathbb{A}_f)$. For any open compact subgroup K of $\mathbb{H}(\mathbb{A}_f)$, there is a Shimura variety $\mathrm{Sh}_K(\mathbb{H})$ of dimension $m-1$ defined over the reflex field E. For any embedding $\iota^\circ : E \hookrightarrow \mathbb{C}$ over $\iota \in \Sigma_\infty$, we have the ι°-adic uniformization

$$\mathrm{Sh}_K(\mathbb{H})^{\mathrm{an}}_{\iota^\circ} \cong H^{(\iota)}(\mathbb{Q})\backslash\bigl(\mathfrak{D}^{(\iota^\circ)} \times \mathbb{H}(\mathbb{A}_f)/K\bigr).$$

We briefly explain the notation above. Let $V^{(\iota)}$ be the *nearby E-hermitian space* of $\mathbb{V}$ at ι, that is, $V^{(\iota)}$ is the unique E-hermitian space (up to isometry) such that $V^{(\iota)}_v \cong \mathbb{V}_v$ for $v \neq \iota$ but $V^{(\iota)}_\iota$ is of signature $(m-1,1)$ and $H^{(\iota)} = \mathrm{Res}_{F/\mathbb{Q}}\mathrm{U}(V^{(\iota)})$. We identify $H^{(\iota)}(\mathbb{A}_f)$ and $\mathbb{H}(\mathbb{A}_f)$ through the corresponding hermitian spaces. Let $\mathfrak{D}^{(\iota^\circ)}$ be the symmetric hermitian domain consisting of all negative $\mathbb{C}$-lines in $V^{(\iota)}_\iota$ whose complex structure is given by the action of $F_\iota \otimes_F E$, which is isomorphic to $\mathbb{C}$ via ι°. The group $H^{(\iota)}(\mathbb{Q})$ diagonally acts on $\mathfrak{D}^{(\iota^\circ)}$ and $\mathbb{H}(\mathbb{A}_f)/K$ via the obvious way. In fact, $\mathfrak{D}^{(\iota^\circ)}$ is canonically identified with the $H^\iota(\mathbb{R})$-conjugacy class of the Hodge map $h^{(\iota)} : \mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\boldsymbol{G}_{\mathrm{m},\mathbb{C}} \to H^{(\iota)}_{\mathbb{R}} \cong \mathrm{U}(m-1,1)_{\mathbb{R}} \times \mathrm{U}(m,0)^{d-1}_{\mathbb{R}}$ given by

$$h^{(\iota)}(z) = \left(\begin{pmatrix} \mathbf{1}_{m-1} & \\ & \bar{z}/z \end{pmatrix}, \mathbf{1}_m, \dots, \mathbf{1}_m \right).$$

From now on, we assume that K *is contained in the principal congruence subgroup for* $N \geq 3$. Then $\mathrm{Sh}_K(\mathbb{H})$ is a quasiprojective nonsingular E-scheme. It is proper if and only if $F \neq \mathbb{Q}$ or $F = \mathbb{Q}$, $m = 2$, and $\Sigma(\mathbb{V}) \supsetneq \Sigma_\infty$. The set of geometric connected components of $\mathrm{Sh}_K(\mathbb{H})$ can be identified with $T(\mathbb{Q})\backslash T(\mathbb{A}_f)/\det(K)$.

For any other open compact subgroup $K' \subset K$, we have an étale covering map $\pi^{K'}_K : \mathrm{Sh}_{K'}(\mathbb{H}) \to \mathrm{Sh}_K(\mathbb{H})$. Let $\mathrm{Sh}(\mathbb{H})$ be the projective system of these $\mathrm{Sh}_K(\mathbb{H})$. On each $\mathrm{Sh}_K(\mathbb{H})$, we have a Hodge bundle $\mathscr{L}_K \in \mathrm{Pic}(\mathrm{Sh}_K(\mathbb{H}))_{\mathbb{Q}}$ which is ample. They are compatible under pull-backs of $\pi^{K'}_K$, and hence define an element $\mathscr{L} \in \mathrm{Pic}(\mathrm{Sh}(\mathbb{H}))_{\mathbb{Q}} := \varinjlim_K \mathrm{Pic}(\mathrm{Sh}_K(\mathbb{H}))_{\mathbb{Q}}$.

Special cycles. Let V_1 be an E-subspace of $\mathbb{V}_f = \mathbb{V} \otimes_{\mathbb{A}_F} \mathbb{A}_{f,F}$. We say that V_1 is *admissible* if $(\,\cdot\,,\,\cdot\,)\big|_{V_1}$ takes values in E and for any nonzero $x \in V_1$, (x,x) is totally positive. We have

Lemma 3.1. *V_1 is admissible if and only if for any $\iota \in \Sigma_\infty$, there is an $h \in \mathbb{H}^{\mathrm{der}}(\mathbb{A}_f)$ such that $hV_1 \subset V^{(\iota)} \subset \mathbb{V}_f$ and is totally positive definite.*

Proof. One direction is obvious. For the other direction, let us assume that V_1 is admissible and fix any ι. Take $v_1 \in V_1$ with nonzero norm. Then $q(v_1) = \frac{1}{2}(v_1, v_1)$ is locally a norm for the hermitian form on $V^{(\iota)}$ by the definition of admissibility and the fact about signatures of $\mathbb{V}$ and $V^{(\iota)}$. Thus it is a norm for some $v \in V^{(\iota)}$ by the Hasse–Minkowski theorem. Now we apply Witt's theorem to find an element $h_1 \in \mathrm{U}(\mathbb{V}_f) = \mathbb{H}(\mathbb{A}_f)$ such that $h_1 v_1 = v$ as elements in $\mathbb{V}_f$. Choose any vector $v' \in \langle v \rangle^\perp \subset V^{(\iota)}$ with nonzero norm. Let $h' \in \mathbb{H}(\mathbb{A}_f)$ fixing $\langle v' \rangle^\perp$ and multiplying $(\det h_1)^{-1}$ in the $\mathbb{A}_{f,E}$-line spanned by v'. Then $h' h_1 v_1 = h' v = v$ for $h = h' h_1 \in \mathrm{SU}(\mathbb{V}_f) = \mathbb{H}^{\mathrm{der}}(\mathbb{A}_f)$.

Replacing V_1 by hV_1 we can assume that $v_1 = v \in \mathbb{V}_f$. Since $\dim V_1 < m$, we can use induction on r by considering the orthogonal complement of v in V_1 and $V^{(\iota)}$ to find an $h \in \mathbb{H}^{\mathrm{der}}(\mathbb{A}_f)$ such that $hV_1 \subset V^{(\iota)} \subset \mathbb{V}_f$. $\square$

For admissible V_1, let $\mathbb{V}_1$ be a totally positive-definite (incoherent) hermitian space over $\mathbb{A}_E$ such that $\mathbb{V}_{1,f} \cong V_1^{\perp} \subset \mathbb{V}_f$. Let $\mathbb{H}_1$ be the corresponding unitary group. We have a finite morphism between Shimura varieties

$$\varsigma^{V_1} : \mathsf{Sh}_{K_1}(\mathbb{H}_1) \longrightarrow \mathsf{Sh}_K(\mathbb{H}), \tag{3-1}$$

where $K_1 = K \cap \mathbb{H}_1(\mathbb{A}_f)$, such that the image of the map is represented, under the uniformization at some ι, by the points $(z, h_1 h) \in \mathfrak{D}^{(\iota^\circ)} \times \mathbb{H}(\mathbb{A}_f)$ where h is as in Lemma 3.1 (with respect to ι), $z \perp hV_1$, and h_1 fixes all elements in hV_1. The image defines a *Kudla's special cycle* $Z(V_1)_K \in \mathsf{CH}^r(\mathsf{Sh}_K(\mathbb{H}))_{\mathbb{Q}}$. It only depends on the class KV_1.

For $x \in \mathbb{V}_f^r$, let V_x be the E-subspace of $\mathbb{V}_f$ generated by the components of x. We define

$$Z(x)_K = \begin{cases} Z(V_x)_K c_1(\mathscr{L}_K^\vee)^{r-\dim_E V_x} & \text{if } V_x \text{ is admissible,} \\ 0 & \text{otherwise.} \end{cases}$$

Generating series. First, we need a restriction of the space $\mathscr{S}(\mathbb{V}_\iota^r)$ of Weil representation when ι is infinite. We define a subspace $\mathscr{S}(\mathbb{V}_\iota^r)^{\mathsf{U}_\iota} \subset \mathscr{S}(\mathbb{V}_\iota^r)$ which consists of functions of the form

$$P(T(x))e^{-2\pi \operatorname{tr} T(x)},$$

where P is a polynomial function on $\mathsf{Her}_r(\mathbb{C})$. It is a $(\mathsf{Lie}\, H_{r,\iota}, \mathscr{K}_{r,\iota})$-module generated by the Gaussian

$$\phi_\infty^0(x) = e^{-2\pi \operatorname{tr} T(x)}.$$

Let $\mathscr{S}(\mathbb{V}^r)^{\mathsf{U}_\infty} = (\bigotimes_{\iota\in\Sigma_\infty} \mathscr{S}(\mathbb{V}_\iota^r)^{\mathsf{U}_\iota}) \otimes \mathscr{S}(\mathbb{V}_f^r)$ and $\mathscr{S}(\mathbb{V}^r)^{\mathsf{U}_\infty K} = (\bigotimes_{\iota\in\Sigma_\infty} \mathscr{S}(\mathbb{V}_\iota^r)^{\mathsf{U}_\iota}) \otimes \mathscr{S}(\mathbb{V}_f^r)^K$ for an open compact subgroup K of $\mathbb{H}(\mathbb{A}_f)$. Recall that we have a Weil representation ω_χ of $H_r(\mathbb{A}_F)$, where $\chi : E^\times\backslash\mathbb{A}_E^\times \to \mathbb{C}^\times$ such that $\chi\big|_{\mathbb{A}_F^\times} = \epsilon_{E/F}^m$. Associated to this χ, we get a sequence $\mathfrak{k}^\chi = (\mathfrak{k}_\iota^\chi)_\iota \in \mathbb{Z}^{\Sigma_\infty}$ determined by $\chi_\iota(z) = z^{\mathfrak{k}_\iota^\chi}$ for $z \in E_\iota^{\times,1} = \mathbb{C}^{\times,1}$, hence m and $\mathfrak{k}_\iota^\chi$ have the same parity.

For $\phi \in \mathscr{S}(\mathbb{V}^r)^{\mathsf{U}_\infty K}$, we define *Kudla's generating series* to be

$$Z_\phi(g) = \sum_{x\in K\backslash\mathbb{V}_f^r} \omega_\chi(g)\phi(T(x), x) Z(x)_K$$

as a series with values in $\mathsf{CH}^r(\mathsf{Sh}_K(\mathbb{H}))_{\mathbb{C}}$ for $g \in H_r(\mathbb{A}_F)$. Here for $\phi = \phi_\infty\phi_f$, we denote $\phi(T(x), x) = \phi_\infty(y)\phi_f(x)$ for any $y \in \mathbb{V}_\infty^r$ with $T(y) = T(x)$ which does not depend on the choice of y. This makes sense since $Z(x)_K \neq 0$ only for V_x admissible and hence $T(x)$ is totally semipositive definite. It is easy to see that $Z_\phi(g)$ is compatible under pull-backs of $\pi_K^{K'}$, hence defines a series with values in $\mathsf{CH}^r(\mathsf{Sh}(\mathbb{H}))_{\mathbb{C}} := \varinjlim_K \mathsf{CH}^r(\mathsf{Sh}_K(\mathbb{H}))_{\mathbb{C}}$.

3B. ***Modularity of the generating series.*** In this section, we are going to prove the modularity of the generating series. This is the only section where we use Shimura varieties of orthogonal groups.

Shimura varieties of orthogonal groups. The $\mathbb{A}_F$-module $\mathbb{V}$ is also a totally positive definite quadratic space over $\mathbb{A}_F$ of rank $2m$ with quadratic form $\frac{1}{2}\mathrm{Tr}_{\mathbb{A}_E/\mathbb{A}_F}(\,\cdot\,,\,\cdot\,)$. Then its discriminant is rational and it is incoherent. Let $\mathbb{G}=\mathrm{Res}_{\mathbb{A}_F/\mathbb{A}}\mathrm{GSpin}(\mathbb{V})$ be the special Clifford group of $\mathbb{V}$ with adjoint (quotient) group $\mathbb{G}^{\mathrm{ad}}=\mathrm{Res}_{\mathbb{A}_F/\mathbb{A}}\mathrm{SO}(\mathbb{V})$ and the derived subgroup $\mathbb{G}^{\mathrm{der}}=\mathrm{Res}_{\mathbb{A}_F/\mathbb{A}}\mathrm{Spin}(\mathbb{V})$. For any open compact subgroup K' of $\mathbb{G}(\mathbb{A}_f)$, there is a Shimura variety $\mathrm{Sh}_{K'}(\mathbb{G})$ defined over the reflex field F such that, for any embedding $\iota: F\hookrightarrow\mathbb{C}$, we have the following ι-adic uniformization:

$$\mathrm{Sh}_{K'}(\mathbb{G})^{\mathrm{an}}_{\iota}\cong G^{(\iota)}(\mathbb{Q})\backslash\big(\mathfrak{D}'^{(\iota)}\times\mathbb{G}(\mathbb{A}_f)/K'\big),$$

where the notation is similarly defined as in the unitary case. In particular, now the symmetric hermitian domain $\mathfrak{D}'^{(\iota)}$ consists of all *oriented* negative definite 2-planes in $V_{\iota}^{(\iota)}$. We denote the corresponding Hodge bundles by $\mathscr{L}'_{K'}$, special cycles $Z'(x)_{K'}$ for $x\in\mathbb{V}_f^r$ and the generating series $Z'_{\phi}(g')$ for $\phi\in\mathscr{S}(\mathbb{V}^r)^{\mathrm{U}_{\infty}K'}$ and $g'\in G_r(\mathbb{A}_f)$ (see [Yuan et al. 2009]). Here we introduce the standard skew-symmetric F-space W'_r (comparing to the space W_r in Section 2A) which has a basis $\{e_1,\dots,e_{2r}\}$ with symmetric form $\langle e_i,e_j\rangle=0$, $\langle e_{r+i},e_{r+j}\rangle=0$, and $\langle e_i,e_{r+j}\rangle=\delta_{ij}$ for $1\le i,j\le r$, and $G_r=\mathrm{Sp}(W'_r)$ which is an F-reductive group. Similarly, when defining the generating series, we have used the Weil representation ω (with respect to ψ) of $G_r(\mathbb{A}_F)\times\mathbb{G}(\mathbb{A})$ on $\mathscr{S}(\mathbb{V}^r)$.

Pull-back formulae. In this subsection, we will fix an embedding $\iota^{\circ}: E\hookrightarrow\mathbb{C}$ over ι and suppress the latter from the notation of nearby objects: $V=V^{(\iota)}$, $H=H^{(\iota)}$, $\mathfrak{D}=\mathfrak{D}^{(\iota^{\circ})}$, ... Hence we have our usual notions of Shimura variety $\mathrm{Sh}_K(H,X)$ (resp. $\mathrm{Sh}_{K'}(G,X')$ with a connected component X^+ of X') which is defined (to be precise) over $\iota^{\circ}(E)$ (resp. $\iota(F)$). The neutral component is the connected Shimura variety $\mathrm{Sh}^{\circ}_K(H^{\mathrm{der}},\overline{X})$ (resp. $\mathrm{Sh}^{\circ}_{K'}(G^{\mathrm{der}},\overline{X}^+)$ attached to the connected Shimura datum $(H^{\mathrm{der}},\overline{X})$ (resp. $(G^{\mathrm{der}},\overline{X}^+)$) which is defined over E_K (resp. $E_{K'}$), a finite abelian extension of $\iota(F)$ in $\mathbb{C}$. The canonical embedding $H^{\mathrm{der}}\hookrightarrow G^{\mathrm{der}}$ (see Remark 3.3(a)) between reductive groups and the embedding $\mathfrak{D}\hookrightarrow\mathfrak{D}'$ by forgetting the E-action define an injective map of connected Shimura data $(H^{\mathrm{der}},\overline{X})\to(G^{\mathrm{der}},\overline{X}^+)$ which hence gives an embedding $i_{K'}:\mathrm{Sh}^{\circ}_K(H^{\mathrm{der}},\overline{X})\hookrightarrow\mathrm{Sh}^{\circ}_{K'}(G^{\mathrm{der}},\overline{X}^+)$ which is defined over E_K providing $K\cap H^{\mathrm{der}}(\mathbb{A}_f)=K'\cap H^{\mathrm{der}}(\mathbb{A}_f)$ and K' is sufficiently small. Let $Z(x)^{\circ}_K$ (resp. $Z'(x)^{\circ}_{K'}$, $Z_{\phi}(g)^{\circ}$, $Z'_{\phi}(g')^{\circ}$) be the restriction of $Z(x)_K$ (resp. $Z'(x)_{K'}$, $Z_{\phi}(g)$, $Z'_{\phi}(g')$) to the neutral component.

Proposition 3.2. *Assume K' is small enough and $K\cap H^{\mathrm{der}}(\mathbb{A}_f)=K'\cap H^{\mathrm{der}}(\mathbb{A}_f)$. For $x\in\mathbb{V}_f$, the pull-back of the special divisor $i^*_{K'}Z'(x)^{\circ}_{K'}$ is the sum of $Z(x_1)^{\circ}_K$ indexed by the classes x_1 in $K\backslash K'x$, both considered as elements in Chow groups.*

Proof. If $x=0$, the only class in $K\backslash K'x$ is $x_1=0$; the proposition follows from the compatibility of Hodge bundles under pull-backs induced by maps between (connected) Shimura data. Now we assume that $\langle x,x\rangle\in E$ which is totally positive. Suppose that $(z,h)\in\mathfrak{D}\times H^{\mathrm{der}}(\mathbb{A}_f)$ represents a $\mathbb{C}$-point in the scheme-theoretic intersection $\mathrm{Sh}^\circ_K(H^{\mathrm{der}},\overline{X})\cap Z'(x_1)^\circ_{K'}$ for some $x_1\in K'x$. Let $g\in G(\mathbb{A}_f)$ such that $gx_1=x_1'\in V\subset\mathbb{V}_f$. Then $z\perp\gamma x_1'$ for some $\gamma\in G(\mathbb{Q})$ and $h\in\gamma G(\mathbb{A}_f)_{x_1'}gk'$ for some $k'\in K'$, where $G(\mathbb{A}_f)_{x_1'}$ is the subgroup of $G(\mathbb{A}_f)$ fixing x_1'. We now show that $\gamma G(\mathbb{A}_f)_{x_1'}gk'\cap H^{\mathrm{der}}(\mathbb{A}_f)=G(\mathbb{A}_f)_{\gamma x_1'}\gamma gk'\cap H^{\mathrm{der}}(\mathbb{A}_f)\neq\varnothing$, that is, $G(\mathbb{A}_f)_{\gamma x_1'}\cap H^{\mathrm{der}}(\mathbb{A}_f)k'^{-1}g^{-1}\gamma^{-1}\neq\varnothing$ which is true by Lemma 3.1. Hence (z,h) represents a $\mathbb{C}$-point in the special cycle $Z(h^{-1}E\langle\gamma gx_1\rangle)^\circ_K$ of $\mathrm{Sh}^\circ_K(H^{\mathrm{der}},\overline{X})$. If we write $h=g_1\gamma gk'$ with some $g_1\in G(\mathbb{A}_f)_{\gamma x_1'}$, then

$$h^{-1}E\langle\gamma gx_1\rangle=E\langle h^{-1}\gamma gx_1\rangle=E\langle k'^{-1}g^{-1}\gamma^{-1}g_1^{-1}\gamma gx_1\rangle=E\langle k'^{-1}x_1\rangle.$$

Hence the scheme-theoretic intersection is indexed by the classes x_1 in $K\backslash K'x$. This is also true in the Chow group since the intersection is proper. □

Remark 3.3. (a) The canonical embedding $H^{\mathrm{der}}\hookrightarrow G^{\mathrm{der}}$ is given in the following way: first, we have an embedding $H^{\mathrm{der}}\hookrightarrow H\hookrightarrow G^{\mathrm{ad}}$ by forgetting the E-action on $V=V^{(\iota)}$. Since H^{der} is simply connected, we have a canonical lifting $H^{\mathrm{der}}\hookrightarrow G$. Since H^{der} has no nontrivial abelian quotient, the image is in fact contained in G^{der}.

(b) In the proof of Proposition 3.2, we can still use the adèlic description of the $\mathbb{C}$-points of $\mathrm{Sh}^\circ_K(H^{\mathrm{der}},\overline{X})$ (resp. $\mathrm{Sh}^\circ_{K'}(G^{\mathrm{der}},\overline{X}^+)$) which is compatible with that of $\mathrm{Sh}_K(\mathbb{H})$ (resp. $\mathrm{Sh}_{K'}(\mathbb{G})$) since H^{der} (resp. G^{der}) is semisimple, of noncompact type, and simply connected.

The group G_r is canonically embedded in H_r by identifying the basis $\langle e_1,\dots,e_{2r}\rangle$ of W_r' and W_r and hence $\omega_\chi\big|_{G_r}=\omega$. From Proposition 3.2, we have

Corollary 3.4. *Let $r=1$ and K, K' as in Proposition 3.2. Then $i^*_{K'}Z'_\phi(g')^\circ=Z_\phi(g')^\circ$ for $g'\in G_1(\mathbb{A}_F)$ and $\phi\in\mathcal{S}(\mathbb{V})^{\mathrm{U}_\infty K'}$.*

Modularity. For a linear functional $\ell\in\mathrm{CH}^r(\mathrm{Sh}(\mathbb{H}))^*_{\mathbb{C}}$, we have a complex-valued series

$$\ell(Z_\phi)(g)=\sum_{x\in K\backslash\mathbb{V}_f^r}\omega_\chi(g)\phi(T(x),x)\ell(Z(x)_K)$$

for any K such that ϕ is invariant under K (which is of course independent of such choice). Our main theorem in this section is this:

Theorem 3.5 (modularity of the generating series). (1) *If $\ell(Z_\phi)(g)$ is absolutely convergent, then it is an automorphic form of $H_r(\mathbb{A}_F)$. Moreover, $\ell(Z_\phi)_\infty$ is in a discrete series representation of weight $((m+\mathfrak{k}^\chi)/2,(m-\mathfrak{k}^\chi)/2)$.*

(2) *If $r=1$, then $\ell(Z_\phi)(g)$ is absolutely convergent for any ℓ.*

Proof. (1) We proceed as in [Yuan et al. 2009, Section 4]. First, we can assume that $\phi = \phi_\infty^0 \otimes \phi_f$ since other cases will follow from the $(\mathrm{Lie}\, H_{r,\infty}, \mathcal{K}_{r,\infty})$-action. Assuming the absolute convergence of $\ell(Z_\phi)(g)$, we only need to check the automorphy, that is, the invariance under left translation of $H_r(F)$. The weight part is clear.

It is easy to check the invariance under $n(b)$ and $m(a)$. For $b \in \mathrm{Her}_r(E)$, the matrix $bT(x)$ is F-rational if $Z(x)_K \neq 0$, hence $\ell(Z_\phi)(n(b)g) = \ell(Z_\phi)(g)$ for all $g \in H_r(\mathbb{A}_F)$. For $a \in \mathrm{GL}_r(E)$, we have $Z(xa)_K = Z(x)_K$, hence

$$\ell(Z_\phi)(m(a)g) = \ell(Z_\phi)(g).$$

Since $H_r(F)$ is generated by $n(b)$, $m(a)$ and $w_{r,r-1}$, where

$$w_{r,d} = \begin{pmatrix} \mathbf{1}_d & & & \\ & & & \mathbf{1}_{r-d} \\ & & \mathbf{1}_d & \\ & -\mathbf{1}_{r-d} & & \end{pmatrix}, \quad 0 \le d \le r, \tag{3-2}$$

we only need to check that $\ell(Z_\phi)(w_{r,r-1}g) = \ell(Z_\phi)(g)$ for all $g \in H_r(\mathbb{A}_F)$. Assuming this for $r = 1$ (see Lemma 3.6), we now prove it for general $r > 1$, following [Yuan et al. 2009; Zhang 2009].

Recall that we have assumed that $\phi = \phi_\infty^0$ and we suppress ℓ from the notation for simplicity. Then for K sufficiently small, we have

$$\begin{aligned} Z_\phi(w_{r,r-1}g) &= \sum_{x \in K\backslash \mathbb{V}_f^r} \omega_\chi(w_{r,r-1}g)\phi(T(x), x)Z(x)_K \\ &= \sum_{x \in K\backslash \mathbb{V}_f^{r-1}} \sum_{y \in K_x\backslash \mathbb{V}_f} \omega_\chi(w_{r,r-1}g)\phi(T(x, y), (x, y))Z((x, y))_K, \end{aligned} \tag{3-3}$$

where K_x is the stabilizer of x in K, and

$$\begin{aligned} (3\text{-}3) = \sum_{x \in K\backslash \mathbb{V}_f^{r-1}} \sum_{y_1 \in K_x\backslash \mathbb{V}_f^x} \sum_{y_2 \in V_x} &\omega_\chi(w_{r,r-1}g) \\ &\times \phi(T(x, y_1 + y_2), (x, y_1 + y_2))Z((x, y_1 + y_2))_K, \end{aligned} \tag{3-4}$$

where $\mathbb{V}_f^x$ is the orthogonal complement of $V_x = E\langle x\rangle$ in $\mathbb{V}_f$. Recall the morphism ς^{V_x} in (3-1); we have

$$\begin{aligned} (3\text{-}4) = \sum_{x \in K\backslash \mathbb{V}_f^{r-1}} \sum_{y_1 \in K_x\backslash \mathbb{V}_f^x} \sum_{y_2 \in V_x} &\omega_\chi(w_{r,r-1})(\omega_\chi(g)\phi)(T(x, y_1 + y_2), (x, y_1 + y_2)) \\ &\times \varsigma_*^{V_x} Z(y_1)_{K_x}. \end{aligned} \tag{3-5}$$

Applying the case $r=1$ to the special cycle $Z(V_x)_{K_x}$, we have

$$\text{(3-5)} = \sum_{x\in K\backslash \mathbb{V}_f^{r-1}} \sum_{y_1\in K_x\backslash \mathbb{V}_f^x} \sum_{y_2\in V_x} \omega_\chi(w_{r,r-1})\widehat{\omega_\chi(g)\phi}^{y_1}(T(x,y_1+y_2),(x,y_1+y_2)) \times \varsigma_*^{V_x} Z(y_1)_{K_x}, \tag{3-6}$$

where the superscript y_1 means taking the Fourier transformation with respect to the y_1 coordinate. Applying the Poisson formula (recall that ϕ_∞ is the Gaussian), we have

$$\begin{aligned}
\text{(3-6)} &= \sum_{x\in K\backslash \mathbb{V}_f^{r-1}} \sum_{y_1\in K_x\backslash \mathbb{V}_f^x} \sum_{y_2\in V_x} \omega_\chi(w_{r,r-1})\widehat{\omega_\chi(g)\phi}^{y_1,y_2}(T(x,y_1+y_2),(x,y_1+y_2)) \\
&\qquad\qquad \times\varsigma_*^{V_x} Z(y_1)_{K_x} \\
&= \sum_{x\in K\backslash \mathbb{V}_f^{r-1}} \sum_{y_1\in K_x\backslash \mathbb{V}_f^x} \sum_{y_2\in V_x} \omega_\chi(w_{r,r-1})\widehat{\omega_\chi(g)\phi}^{y}(T(x,y_1+y_2),(x,y_1+y_2)) \\
&\qquad\qquad \times\varsigma_*^{V_x} Z(y_1)_{K_x} \\
&= \sum_{x\in K\backslash \mathbb{V}_f^{r-1}} \sum_{y_1\in K_x\backslash \mathbb{V}_f^x} \sum_{y_2\in V_x} \omega_\chi(g)\phi(T(x,y_1+y_2),(x,y_1+y_2))\varsigma_*^{V_x} Z(y_1)_{K_x} \\
&= \sum_{x\in K\backslash \mathbb{V}_f^{r}} \omega_\chi(g)\phi(T(x),x)Z(x)_K = Z_\phi(g).
\end{aligned}$$

This finishes the proof of (1).

(2) follows from the argument in Lemma 3.6, following Corollary 3.4 and [Yuan et al. 2009, Theorem 1.3], which uses the result in [Kudla and Millson 1990]. $\square$

Lemma 3.6. *If $r=1$, then $\ell(Z_\phi)(w_1 g)=\ell(Z_\phi)(g)$ for all $g\in H_1(\mathbb{A}_F)$.*

Proof. We suppress ℓ from the notation. Further, we fix any $\iota^\circ\in\Sigma_\infty^\circ$ over $\iota\in\Sigma_\infty$ and suppress them as in the previous subsection. It is clear that we only need to prove that $Z_\phi(w_1g)=Z_\phi(g)$ for $g\in G_1(\mathbb{A}_f)$ since $G_1(\mathbb{A}_{\infty,F})\mathscr{K}_{1,\infty}=H_1(\mathbb{A}_{\infty,F})$. As before, we assume that ϕ_∞ is the Gaussian and K is sufficiently small. Recall that $\pi_0(\mathrm{Sh}_K(H,X)_{\iota,\mathbb{C}})\cong T(\mathbb{Q})\backslash T(\mathbb{A}_f)/\det(K)$. We have the following inclusion:

$$\mathrm{CH}^1(\mathrm{Sh}_K(H,X))_{\mathbb{C}} \hookrightarrow \bigoplus_{\{t\}\in T(\mathbb{Q})\backslash T(\mathbb{A}_f)/\det(K)} \mathrm{CH}^1(\mathrm{Sh}_K(H,X)_{\{t\}})_{\mathbb{C}}, \tag{3-7}$$

where $\mathrm{Sh}_K(H,X)_{\{t\}}$ is the (canonical model of the) corresponding (geometric) connected component. Let $h\in H(\mathbb{A}_f)$ such that $\det(h)=t$ and let T_h be the Hecke operator. Then $T_h:\mathrm{Sh}_{K^h}(H,X)\to\mathrm{Sh}_K(H,X)$ induces

$$T_h^\circ:\mathrm{Sh}^\circ_{K^h}(H^{\mathrm{der}},\overline{X})=\mathrm{Sh}_{K^h}(H,X)_{\{1\}}\xrightarrow{\sim}\mathrm{Sh}_K(H,X)_{\{h\}}\hookrightarrow\mathrm{Sh}_K(H,X),$$

where $K^h = hKh^{-1}$. We have $T_h^{\circ,*} Z_\phi(g) = Z_\phi(g)^\circ$ which is the image of $Z_\phi(g)$ under the projection to $\mathrm{CH}^1(\mathrm{Sh}_K(H, X)_{\{t\}})_{\mathbb{C}}$ under (3-7). Here $Z_\phi(g)^\circ$ is the generating series on $\mathrm{Sh}^\circ_{K^h}(H^{\mathrm{der}}, \overline{X})$. Now shrinking K^h if necessary such that we can apply Corollary 3.4, we have $Z_\phi(g)^\circ = i^*_{K'} Z'_\phi(g)^\circ$ for $g \in G_1(\mathbb{A}_f)$. Applying [Yuan et al. 2009, Theorem 1.2 or Theorem 1.3], we conclude that $Z_\phi(w_1 g)^\circ = Z_\phi(g)^\circ$. The lemma follows by (3-7). □

3C. ***Smooth compactification of unitary Shimura varieties.*** In this section, we introduce the canonical smooth compactification of the unitary Shimura varieties if they are not proper and the compactified generating series on them.

Let $m \geq 2$ be an integer, $E = \mathbb{Q}(\sqrt{-D}) \subset \mathbb{C}$ for some square-free integer $D > 0$, let $\mathbb{O}_E$ be its ring of integers, and let τ be the nontrivial Galois involution on E. Let $(V, (\cdot\,,\,\cdot))$ be a hermitian space of dimension m over E whose signature is $(m-1, 1)$. If $m = 2$, we further assume that $\det V \in \mathrm{Nm}\, E^\times$. Let $H = \mathrm{U}(V)$ be the unitary group; we have the Hodge map $h : \mathbb{S} \to H_{\mathbb{R}} \cong \mathrm{U}(m-1, 1)_{\mathbb{R}}$ given by

$$h(z) = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & \bar{z}/z \end{pmatrix}.$$

Then we have the notion of the Shimura variety $\mathrm{Sh}_K(H, h)$ for any open compact subgroup K of $H(\mathbb{A}_f)$. For K sufficiently small, it is smooth and quasiprojective but nonproper over E of dimension $m-1$. Hence we need to construct a smooth compactification of $\mathrm{Sh}_K(H, h)$ such that we can do height pairing. When $m = 2$, it is trivial since we only need to add cusps. When $m = 3$ and H is quasisplit, a canonical smooth compactification (even of the integral model) has been constructed in [Larsen 1992]. In fact, the same construction works in the more general case (just for compactification of the canonical model), namely any H appearing here. We should mention that, if the signature of V is (a, b) such that $a \geq b > 1$ or V is over a totally real field but not $\mathbb{Q}$ and indefinite at any archimedean place, then we should not hope that there exists a canonical smooth compactification.

Now let us assume $m > 2$. Since we are going to use modular interpretations, we should work with the group of unitary similitude. For any $v, w \in V$,

$$(v, w)' = \mathrm{Tr}_{E/\mathbb{Q}}(\sqrt{-D}(v, w))$$

defines an alternating form of V satisfying $(ev, w) = (v, e^\tau w)$ for any $e \in E$. Let $GH = \mathrm{GU}(V)$ such that for any $\mathbb{Q}$-algebra R,

$$GH(R) = \{h \in \mathrm{GL}_m(E \otimes_{\mathbb{Q}} R) \mid (hv, hw)' = \lambda(h)(v, w)' \text{ for some } \lambda(h) \in R^\times\}$$

and the Hodge map $Gh : \mathbb{S} \to GH_{\mathbb{R}} \cong \mathrm{GU}(m-1, 1)_{\mathbb{R}}$ is given by

$$Gh(z) = \begin{pmatrix} z & & & \\ & \ddots & & \\ & & z & \\ & & & \bar{z} \end{pmatrix}.$$

For any sufficiently small open compact subgroup K of $GH(\mathbb{A}_f)$, we have the Shimura variety $\mathrm{Sh}_K(GH, Gh)$ which is smooth and quasiprojective but nonproper over E of dimension $m-1$. Although we don't have a map between Shimura data, $\mathrm{Sh}_{K\cap H(\mathbb{A}_f)}(H, h)$ and $\mathrm{Sh}_K(GH, Gh)$ have the same neutral component for sufficiently small K. Hence it is the same to give a canonical smooth compactification of $\mathrm{Sh}_K(GH, Gh)$ instead of the original one. In fact, $\mathrm{Sh}_K(GH, Gh)$ is a moduli space of abelian varieties of certain PEL type. We fix a lattice $V_{\mathbb{Z}}$ of V such that $V_{\mathbb{Z}} \subset V_{\mathbb{Z}}^{\perp}$ and let $V_{\widehat{\mathbb{Z}}} = V_{\mathbb{Z}} \otimes \widehat{\mathbb{Z}}$. Then Sh_K represents the following functor: for any (S, s) where S is a connected, locally noetherian E-scheme with a geometric point s, $\mathrm{Sh}_K(GH, Gh)(S, s)$ is the isomorphism classes of quadruples $(A, \theta, i, \bar{\eta})$, where

- A is an abelian scheme over S of dimension m;
- $\theta : A \to A^{\vee}$ is a polarization;
- $i : \mathbb{O}_E \hookrightarrow \mathrm{End}(A)$ such that $\mathrm{tr}(i(e); \mathrm{Lie}_S(A)) = (m-1)e + e^{\tau}$ and $\theta \circ i(e) = i(e^{\tau})^{\vee} \circ \theta$ for all $e \in \mathbb{O}_E$;
- $\bar{\eta}$ is a $\pi_1(S, s)$-invariant K-class of $\mathbb{O}_E \otimes \widehat{\mathbb{Z}}$-linear symplectic similitude $\eta : V_{\widehat{\mathbb{Z}}} \to \mathrm{H}_1^{\text{ét}}(A_s, \widehat{\mathbb{Z}})$, where the pairing on the latter space is the θ-Weil pairing; hence the degree of θ is $[V_{\mathbb{Z}}^{\perp} : V_{\mathbb{Z}}]$.

Here in the third condition, $1 \in \mathbb{O}_E$ goes to the identity endomorphism and we view $(m-1)e + e^{\tau}$ as a constant section of $\mathbb{O}_S$ via the structure map $E \to \mathbb{O}_S$.

In the theory of toroidal compactification (see [Ash et al. 1975]), we need to choose a rational polyhedral cone decomposition. But in our case, we only have one unique choice, namely a torus in an affine line. We claim that there is a scheme $\mathrm{Sh}_K^{\sim}(GH, Gh)$ with these properties:

- $\mathrm{Sh}_K^{\sim}(GH, Gh)$ is smooth and proper over E.
- $\tilde{i}_K : \mathrm{Sh}_K(GH, Gh) \hookrightarrow \mathrm{Sh}_K^{\sim}(GH, Gh)$ is an open immersion and for $K' \subset K$ there is a morphism $\tilde{\pi}_K^{K'}$ such that the diagram

$$\begin{array}{ccc} \mathrm{Sh}_{K'}(GH, Gh) & \xrightarrow{\tilde{i}_{K'}} & \mathrm{Sh}_{K'}^{\sim}(GH, Gh) \\ \downarrow{\scriptstyle \pi_K^{K'}} & & \downarrow{\scriptstyle \tilde{\pi}_K^{K'}} \\ \mathrm{Sh}_K(GH, Gh) & \xrightarrow[\tilde{i}_K]{} & \mathrm{Sh}_K^{\sim}(GH, Gh) \end{array}$$

  commutes.

- The boundary $GY_K = \mathrm{Sh}_K^{\sim}(GH, Gh) - \mathrm{Sh}_K(GH, Gh)$ is a smooth divisor defined over E and each geometric component is isomorphic to an extension of an abelian variety of dimension $m-2$ by a finite group.

The boundary part GY_K parametrizes the degeneration of abelian varieties with the above PEL data. We consider a semiabelian variety G with $i : \mathbb{O}_E \hookrightarrow \mathrm{End}(G)$ such that $\mathrm{tr}(i(e); \mathrm{Lie}(G)) = (m-1)e + e^\tau$; then for any $e \in \mathbb{O}_E$, we have the following commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & T \arrow[r, "t"] & G \arrow[r, "\alpha"] \arrow[d, "i(e)"] & A \arrow[r] & 0 \\ 0 \arrow[r] & T \arrow[r, "t"] & G \arrow[r, "\alpha"] & A \arrow[r] & 0. \end{tikzcd}$$

Then the composition $\alpha \circ i(e) \circ t$ is trivial. Thus i induces actions of $\mathbb{O}_E$ on both torus part T and abelian variety part A. Suppose $X(T) = \mathbb{Z}^r$ with $r > 0$. Then r is even since E is quadratic imaginary. Further assuming $\mathrm{tr}(i(e); \mathrm{Lie}(T)) = ae + be^\tau$, we have $a+b=r$ and $a=b$. Hence there is only one possibility, namely $a=b=1$ and $r=2$. Then T is of rank 2 and A is an abelian variety of dimension $m-2$ such that $\mathbb{O}_E$ acts on A and $\mathrm{tr}(i(e); \mathrm{Lie}(A)) = (m-2)e$. Let A_1 be an elliptic curve of CM type $(\mathbb{O}_E, e \mapsto e)$. Then A is isogenous to A_1^{m-2}. Each geometric point s of GY_K corresponds to a semiabelian variety $G_s = (T_s \hookrightarrow G_s \to A_s)$ as above with certain level structure which will be defined later. For two geometric points s, s' in the same geometric connected component, the abelian variety part $A_s \cong A_{s'}$ and the rank 1 $\mathbb{O}_E$-modules $X(T_s)$ and $X(T_{s'})$ are isomorphic. It is easy to see that if A and T are fixed, then the set of such G, up to isomorphism, is parametrized by $X(T) \otimes_{\mathbb{O}_E} A$ which is an abelian variety of dimension $m-2$.

To include the level structure, we only stick in one geometric component since it is same to others. This means that we fix T and A with $\mathbb{O}_E$-actions but, of course, not G. Let us fix a maximal isotropic subspace W of $V_{\mathbb{Z}}$. Then W is of rank 1. We have a filtration $0 \subset W \subset W^\perp \subset V_{\mathbb{Z}}$. Let B_W be the subgroup of $H(\mathbb{A}_f)$ that preserves this filtration, $N_W \subset B_W$ that acts trivially on the associated graded modules, and $U_W \subset N_W$ that acts trivially on $W^\perp$ and $V_W = N_W/U_W$. We also fix a generator w of W. On the other hand, fix an $\mathbb{O}_E \otimes \widehat{\mathbb{Z}}$ generator w_T of $\mathrm{H}_1^{\text{ét}}(T, \widehat{\mathbb{Z}})$ and a polarization $\theta_A : A \to A^\vee$ such that there exists a symplectic similitude between $\mathrm{H}_1^{\text{ét}}(A, \widehat{\mathbb{Z}})$ and $V_W \otimes \widehat{\mathbb{Z}}$. For a sufficiently small open compact subgroup $K \subset H(\mathbb{A}_f)$, let $N_{W,K} = N_W \cap K$, $U_{W,K} = U_W \cap K$, and $V_{W,K} = N_{W,K}/U_{W,K}$. Then the level structure of $(G, T \hookrightarrow G \to A)$ with respect to K is a $V_{W,K}$-class of isomorphisms $W^\perp \otimes \widehat{\mathbb{Z}} \to \mathrm{H}_1^{\text{ét}}(G, \widehat{\mathbb{Z}})$ which sends w to w_T and induces a symplectic similitude between $V_W \otimes \widehat{\mathbb{Z}}$ and $\mathrm{H}_1^{\text{ét}}(A, \widehat{\mathbb{Z}}) = \mathrm{H}_1^{\text{ét}}(G, \widehat{\mathbb{Z}})/\mathbb{O}_E \cdot w_T$. We conclude that any geometric component of GY_K is isomorphic to (a connected component of) an extension of $X(T) \otimes_{\mathbb{O}_E} A$ by $V_W/V_{W,H}$ for some T and A as above. There is a universal object $\pi:\mathcal{G}\to \mathrm{Sh}^{\sim}_K(GH,Gh)$ which is a semiabelian scheme of relative dimension m.

Now we come back to the Shimura variety $\mathrm{Sh}_K:=\mathrm{Sh}_K(H,h)$. As we have said, the canonical smooth compactification above gives a canonical smooth compactification for Sh_K, which we denote by $\mathrm{Sh}^{\sim}_K$, $Y_K=\mathrm{Sh}^{\sim}_K-\mathrm{Sh}_K$; then they will have the same property as above. We also apply the notation above to the trivial case $m=2$. We let $\mathscr{L}^{\sim}_K$ be the line bundle on $\mathrm{Sh}^{\sim}_K$ induced from $\bigwedge^m\pi_*\Omega_{\mathcal{G}/\mathrm{Sh}^{\sim}_K(GH,Gh)}$ on $\mathrm{Sh}^{\sim}_K(GH,Gh)$ which is an extension of the Hodge bundle $\mathscr{L}_K$ on Sh_K (see Section 3A). By the canonicality of the compactification, $(\mathscr{L}^{\sim}_K)_K$ defines an element in $\mathrm{Pic}(\mathrm{Sh}^{\sim})_{\mathbb{C}}=\varinjlim_K\mathrm{Pic}(\mathrm{Sh}^{\sim}_K)_{\mathbb{C}}$. We also need to extend special cycles and hence the generating series. For $1\le r<m$ and $x\in V^r_f:=V^r\otimes_{\mathbb{Q}}\mathbb{A}_f$, we define the compactified special cycle as

$$Z(x)^{\sim}_K=\begin{cases} Z(V_x)^{\sim}_K c_1(\mathscr{L}^{\sim,\vee}_K)^{r-\dim_E V_x} & \text{if } V_x \text{ is admissible,}\\ 0 & \text{otherwise,}\end{cases}$$

where $Z(V_x)^{\sim}_K$ is just the Zariski closure of $Z(V_x)_K$ in $\mathrm{Sh}^{\sim}_K$. We define the compactified generating series by formal series in $\mathrm{CH}^r(\mathrm{Sh}^{\sim})_{\mathbb{C}}=\varinjlim_K\mathrm{CH}^r(\mathrm{Sh}^{\sim}_K)_{\mathbb{C}}$:

$$Z^{\sim}_{\phi}(g)=\begin{cases} \displaystyle\sum_{x\in K\backslash V^r_f}\omega_\chi(g)\phi(T(x),x)Z(x)^{\sim}_K & \text{if } m>2,\\ \displaystyle\sum_{x\in K\backslash V_f}\omega_\chi(g)\phi(T(x),x)Z(x)^{\sim}_K+W_0\big(\tfrac12,g,\phi\big)c_1(\mathscr{L}^{\sim,\vee}_K) & \text{if } r=1, m=2,\end{cases}$$

for $g\in H_r(\mathbb{A})$ and $\phi\in\mathscr{S}(V^r)^{\mathrm{U}_\infty}$. Here, $W_0(s,g,\phi)=\prod_v W_0(s,g_v,\phi_v)$, which is holomorphic at $s=\frac12$. Moreover, we define the following *positive partial compactified generating series* as

$$Z^{\sim}_{\phi,+}(g)=\sum_{\substack{x\in K\backslash V^r_f\\ T(x)\gg\mathbf{0}_r}}\omega_\chi(g)\phi(T(x),x)Z(x)^{\sim}_K,$$

where the sum is taken over all x such that $T(x)$ is totally positive definite. We would like to propose the following conjecture on the *modularity of the compactified generating series*:

Conjecture 3.7. *Let ℓ be a linear functional on $\mathrm{CH}^r(\mathrm{Sh}^{\sim})_{\mathbb{C}}$ such that $\ell(Z^{\sim}_\phi)(g)$ is absolutely convergent. Then if $1\le r\le m-2$, $\ell(Z^{\sim}_\phi)(g)$ is a holomorphic automorphic form of $H_r(\mathbb{A}_F)$; if $r=1, m=2$, $\ell(Z^{\sim}_\phi)(g)$ is an automorphic form of $H_1(\mathbb{A}_F)$, not necessarily holomorphic; in general, if $r=m-1$, $\ell(Z^{\sim}_{\phi,+})(g)$ is the sum of the positive-definite Fourier coefficients of an automorphic form of $H_{m-1}(\mathbb{A}_F)$.*

The case $m=2$ $(r=1)$ will be proved in [Liu 2011, Section 3B] and is actually not far from Theorem 3.5 as we point out there.

Fix a rational prime ℓ. There are class maps $\mathrm{cl}:\mathrm{CH}^r(\mathrm{Sh}^{\sim}_K)_{\mathbb{C}}\to \mathrm{H}^{2r}_{\text{ét}}(\mathrm{Sh}^{\sim}_K\times_E E^{\mathrm{ac}},\mathbb{Z}_\ell(r))^{\Gamma_E}\otimes_{\mathbb{Z}_\ell}\mathbb{C}$ compatible under $\tilde{\pi}^{K'}_K$, which induces $\mathrm{cl}:\mathrm{CH}^r(\mathrm{Sh}^{\sim})_{\mathbb{C}}\to \mathrm{H}^{2r}_{\text{ét}}(\mathrm{Sh}^{\sim}\times_E E^{\mathrm{ac}},\mathbb{Z}_\ell(r))^{\Gamma_E}\otimes_{\mathbb{Z}_\ell}\mathbb{C}\subset \mathrm{H}^{2r}_{\mathrm{B}}(\mathrm{Sh}^{\sim},\mathbb{C})$ (the Betti cohomology) where the two cohomology groups are defined as inductive limits as K varies. Let $\mathrm{H}^\bullet_Y(\mathrm{Sh}^{\sim},\mathbb{C}) = \varinjlim_K \mathrm{H}^\bullet_{Y_K}(\mathrm{Sh}^{\sim}_K(\mathbb{C}),\mathbb{C})$ be the inductive limit of cohomology groups with support in Y_K as K varies. Then since Y is a smooth divisor, we have $\mathrm{H}^\bullet_Y(\mathrm{Sh}^{\sim},\mathbb{C})\cong \mathrm{H}^{\bullet-2}_{\mathrm{B}}(Y,\mathbb{C})=\varinjlim_K \mathrm{H}^{\bullet-2}_{\mathrm{B}}(Y_K(\mathbb{C}),\mathbb{C})$.

On the other hand, let us denote by $\mathrm{Sh}^{\#}_K$ the Baily–Borel compactification of Sh_K. Hence we have the commutative diagram

$$\begin{tikzcd} \mathrm{Sh}^{\sim}_K \arrow[rr, "j_K"] & & \mathrm{Sh}^{\#}_K, \\ & \mathrm{Sh}_K \arrow[ul, hook, "\tilde{i}_K"] \arrow[ur, hook, "i^{\#}_K"'] & \end{tikzcd}$$

which is compatible when K varies, and, more importantly, Hecke equivariant. We also denote by $\mathrm{IH}^\bullet(\mathrm{Sh}^{\#},\mathbb{C})=\varinjlim_K \mathrm{IH}^\bullet(\mathrm{Sh}^{\#}_K(\mathbb{C}),\mathbb{C})$ the inductive limit of the intersection cohomology groups. Then by [Beĭlinson et al. 1982, Théorème 6.2.5], we have the exact sequence

$$\mathrm{H}^\bullet_Y(\mathrm{Sh}^{\sim},\mathbb{C}) \longrightarrow \mathrm{H}^\bullet_{\mathrm{B}}(\mathrm{Sh}^{\sim},\mathbb{C}) \xrightarrow{j_*} \mathrm{IH}^\bullet(\mathrm{Sh}^{\#},\mathbb{C}) \longrightarrow 0. \tag{3-8}$$

Let $\mathrm{H}^\bullet_\partial(\mathrm{Sh}^{\sim})$ be the image of the first map which is isomorphic to a quotient of $\mathrm{H}^{\bullet-2}_{\mathrm{B}}(Y,\mathbb{C})$.

3D. ***Arithmetic theta lifting and inner product formula.*** We next define the arithmetic theta lifting and prove its cohomological triviality under certain assumptions. We then formulate the conjectural arithmetic inner product formula in general.

Arithmetic theta lifting. We assume Conjecture 3.7 and the following assumptions on A-packets which are a certain part of the Langlands–Arthur conjecture (see [Arthur 1984; 1989]); they should be proved by a similar method to that in [Arthur 2012] (which handles the case of symplectic and orthogonal groups):

- A-packets are defined for all unitary groups $\mathrm{U}(m)_{/F}$. We denote by $\mathrm{AP}(\mathrm{U}(m)_{\mathbb{A}_F})$ the set of A-packets of $\mathrm{U}(m)$ and by $\mathrm{AP}(\mathrm{U}(m)_{\mathbb{A}_F})_{\mathrm{disc}}\subset \mathrm{AP}(\mathrm{U}(m)_{\mathbb{A}_F})$ the subset of discrete A-packets.
- If Π_1 and Π_2 are in $\mathrm{AP}(\mathrm{U}(m)_{\mathbb{A}_F})_{\mathrm{disc}}$ such that for almost all $v\in\Sigma$, $\Pi_{1,v}$ and $\Pi_{2,v}$ contain the same unramified representation, then $\Pi_1=\Pi_2$.
- Let $\mathrm{U}(m)^*$ be the quasisplit unitary group. Then we have the correspondence between A-packets of inner forms: $\mathrm{JL}:\mathrm{AP}(\mathrm{U}(m)_{\mathbb{A}_F})_{\mathrm{disc}}\to \mathrm{AP}(\mathrm{U}(m)^*_{\mathbb{A}_F})_{\mathrm{disc}}$.

Definition 3.8. Let π be an irreducible cuspidal automorphic representation of $H_r(\mathbb{A}_F)$ realized in $L^2(H_r(F)\backslash H_r(\mathbb{A}_F))$. We assume that $1 \le r \le m-2$ or $r = 1$, $m = 2$. For any $\phi \in \mathscr{S}(\mathbb{V}^r)^{\mathsf{U}_\infty}$ and any cusp form $f \in \pi$, the integral

$$\Theta_\phi^f = \begin{cases} \int_{H_r(F)\backslash H_r(\mathbb{A}_F)} f(g) Z_\phi(g)\, dg \in \mathsf{CH}^r(\mathsf{Sh})_{\mathbb{C}} & \text{if } \mathsf{Sh} \text{ is proper,} \\ \int_{H_r(F)\backslash H_r(\mathbb{A}_F)} f(g) Z_\phi^{\sim}(g)\, dg \in \mathsf{CH}^r(\mathsf{Sh}^{\sim})_{\mathbb{C}} & \text{otherwise,} \end{cases}$$

is called the *arithmetic theta lifting* of f which is a (formal sum of) codimension r cycle(s) on a certain (compactified) Shimura variety of dimension $m-1$. Its cohomology class (restricted to Sh) is well-defined due to [Kudla and Millson 1990]. The original idea of this construction comes from Kudla; see [Kudla 2003, Section 8] or [Kudla et al. 2006, Section 9.1]. He constructed the arithmetic theta series as an Arakelov divisor on a certain integral model of a Shimura curve.

In the following discussion, let $m = 2n$ and $r = n$. Let π be an irreducible cuspidal automorphic representation of $H_n(\mathbb{A}_F)$, χ a character of $E^\times\mathbb{A}_F^\times\backslash\mathbb{A}_E^\times$ such that π_∞ is a discrete series representation of weight $(n - \mathfrak{k}^\chi/2, n + \mathfrak{k}^\chi/2)$, and $\epsilon(\pi,\chi) = -1$. Then the (equal-rank) theta correspondence of π_ι (under ω_χ) is the trivial representation of $\mathsf{U}(2n,0)_{\mathbb{R}}$ for any archimedean place ι. Hence $\mathbb{V}(\pi,\chi)$ is a totally positive-definite incoherent hermitian space over $\mathbb{A}_E$. Now we fix an incoherent hermitian space $\mathbb{V}$ which is totally positive-definite of rank $2n$ and let $(\mathsf{Sh}_K)_K$ be the associated Shimura varieties.

We fix an embedding $\iota^\circ : E \hookrightarrow \mathbb{C}$ inducing $\iota : F \hookrightarrow \mathbb{C}$ if $F \ne \mathbb{Q}$. Then similarly we have the class map $\mathsf{cl} : \mathsf{CH}^\bullet(\mathsf{Sh}_K)_{\mathbb{C}} \to \mathsf{H}_{\mathsf{B}}^{2\bullet}(\mathsf{Sh}_{K,\iota^\circ}(\mathbb{C}),\mathbb{C})$. By a theorem in [Zucker 1982, Section 6] concerning the L^2-cohomology and the intersection cohomology, we have a (compatible system of) Hecke equivariant isomorphisms:

$$\mathsf{H}_{(2)}^\bullet(\mathsf{Sh}_K) = \begin{cases} \mathsf{H}_{\mathsf{B}}^\bullet(\mathsf{Sh}_{K,\iota^\circ}(\mathbb{C}),\mathbb{C}) & \text{if } \mathsf{Sh}_K \text{ is proper,} \\ \mathsf{IH}^\bullet(\mathsf{Sh}_K^\#,\mathbb{C}) & \text{otherwise.} \end{cases}$$

We let

$$\mathsf{H}_{(2)}^\bullet(\mathsf{Sh}) = \varinjlim_K \mathsf{H}_{(2)}^\bullet(\mathsf{Sh}_K).$$

In the nonproper case, we compose the map j_* in (3-8) to get a class map still denoted by $\mathsf{cl} : \mathsf{CH}^\bullet(\mathsf{Sh}^{\sim}) \to \mathsf{H}_{(2)}^{2\bullet}(\mathsf{Sh})$.

Proposition 3.9. *The class* $\mathsf{cl}(\Theta_\phi^f) = 0$ *in* $\mathsf{H}_{(2)}^{2n}(\mathsf{Sh})$, *that is, if* Sh *is proper* (*resp. nonproper*), Θ_ϕ^f *is cohomologically trivial* (*resp. such that* $\mathsf{cl}(\Theta_\phi^f) \in \mathsf{H}_\partial^{2n}(\mathsf{Sh}^{\sim})$).

Proof. If Sh is nonproper, we can assume $n > 1$. By our definition of the arithmetic theta lifting, for $\phi = \phi_\infty\phi_f$ with fixed ϕ_∞, $\mathsf{cl}(\Theta_\phi^f)$ defines an element in

$$\mathrm{Hom}_{H_n(\mathbb{A}_{f,F})}\big(\mathscr{S}(\mathbb{V}_f^n) \otimes \pi_f, \mathsf{H}_{(2)}^{2n}(\mathsf{Sh})\big),$$

where $H_n(\mathbb{A}_{f,F})$ acts trivially on the L^2-cohomology.

Let $V^{(\iota)}$ be the nearby hermitian space of $\mathbb{V}$ at ι (see Section 3A) and $H^{(\iota)} = \mathrm{U}(V^{\iota})$. Then since $Z^{\sim}_{\omega(h)\phi}(g) = T_h^* Z^{\sim}_{\phi}(g)$ for all $h \in H^{(\iota)}(\mathbb{A}_{f,F})$ where T_h is the Hecke operator of h, we see that $\mathrm{cl}(\Theta^f_\phi)$ in fact defines an element

$$\begin{aligned} H_{\Theta,\phi_\infty} &\in \mathrm{Hom}_{H_n(\mathbb{A}_{f,F})\times H^{(\iota)}(\mathbb{A}_{f,F})}\big(\mathscr{S}(\mathbb{V}^n_f)\otimes\pi_f,\ \mathrm{H}^{2n}_{(2)}(\mathrm{Sh})\big) \\ &= \mathrm{Hom}_{H_n(\mathbb{A}_{f,F})\times H^{(\iota)}(\mathbb{A}_{f,F})}\big(\mathscr{S}(\mathbb{V}^n_f),\ \pi^\vee_f\otimes\mathrm{H}^{2n}_{(2)}(\mathrm{Sh})\big), \end{aligned}$$

where $H_n(\mathbb{A}_{f,F}) \times H^{(\iota)}(\mathbb{A}_{f,F})$ acts on $\mathscr{S}(\mathbb{V}^n_f)$ through the Weil representation ω_χ and $H^{(\iota)}(\mathbb{A}_{f,F})$ acts on $\mathrm{H}^{2\bullet}_{(2)}(\mathrm{Sh})$ through Hecke operators and on π_f trivially. As an $H^{(\iota)}(\mathbb{A}_{f,F})$-representation, we have the following well-known decomposition (see, for example, [Borel and Wallach 2000, Chapter XIV]):

$$\mathrm{H}^{2n}_{(2)}(\mathrm{Sh}) = \bigoplus_\sigma m_{\mathrm{disc}}(\sigma)\mathrm{H}^{2n}\big(\mathrm{Lie}\, H^{(\iota)}_\infty, K_{H^{(\iota)}_\infty}; \sigma_\infty\big)\otimes\sigma_f,$$

where the direct sum is taken over all irreducible discrete automorphic representations of $H^{(\iota)}(\mathbb{A}_F)$. If the invariant functional $H_{\Theta,\phi_\infty} \neq 0$, then some σ_f with

$$m_{\mathrm{disc}}(\sigma)\mathrm{H}^{2n}\big(\mathrm{Lie}\, H^{(\iota)}_\infty, K_{H^{(\iota)}_\infty}; \sigma_\infty\big) \neq 0$$

is the theta correspondence $\theta(\pi^\vee_f)$ of $\pi^\vee_f$.

We define a character $\tilde{\chi}$ of $E^{\times,1}\backslash\mathbb{A}^{\times,1}_E$ in the following way. For any $x \in \mathbb{A}^{\times,1}_E$, we can write $x = e/e^\tau$ for some $e \in \mathbb{A}^\times_E$ and define $\tilde{\chi}(x) = \chi(e)$ which is well-defined since $\chi\big|_{\mathbb{A}^\times_F} = 1$.

For all finite places v such that $v \nmid 2$ and ψ_v, χ_v, and π_v are unramified, we have $H^{(\iota)}_v \cong H_{n,v}$. Let $\Sigma \in \mathrm{AP}(H^{(\iota)}_{\mathbb{A}_F})_{\mathrm{disc}}$ be the A-packet containing σ and $\Pi \in \mathrm{AP}(H_{n,\mathbb{A}_F})_{\mathrm{disc}}$ containing π. Then by Corollary A.6, we have that for v as above, $\mathrm{JL}(\Sigma)_v = \mathrm{JL}(\Sigma_v) = \Sigma_v$ and $\Pi_v \otimes \tilde{\chi}_v$ contain the same unramified representation, hence $\mathrm{JL}(\Sigma)$ and $\Pi\otimes\tilde{\chi}$ coincide. In particular,

$$\mathrm{JL}(\Sigma_\infty) = \mathrm{JL}(\Sigma)_\infty = \Pi_\infty\otimes\tilde{\chi}_\infty,$$

which implies that Σ_∞ is a discrete series L-packet (see [Adams 2011]). This contradicts our assumption since for any discrete series representation σ_∞, we can have $\mathrm{H}^\bullet(\mathrm{Lie}\, H^{(\iota)}_\infty, K_{H^{(\iota)}_\infty}; \sigma_\infty) \neq 0$ only in the middle dimension, which is $2n-1$, not $2n$ (see [Borel and Wallach 2000, Chapter II, Theorem 5.4]). Thus $H_{\Theta,\phi_\infty} = 0$ and we prove the proposition. □

The proposition says that Θ^f_ϕ is automatically cohomologically trivial at least in the proper case.

Conjecture 3.10. *When* Sh *is nonproper*, $\mathrm{cl}(\Theta^f_\phi) \in \mathrm{H}^{2n}_\partial(\mathrm{Sh}^\sim)$ *is* 0 *for any cusp form* $f \in \pi$ *and* ϕ *as above.*

When $n=1$, this is proved in [Liu 2011] just by computing the degree of the generating series which is the linear combination of an Eisenstein series and (possibly) an automorphic character (that is, one-dimensional automorphic representation) of $H_1(\mathbb{A})$. Hence $\mathrm{cl}(\Theta_\phi^f)$ is zero since f is cuspidal. For the general case, we believe that the same phenomenon will happen.

Main conjecture. Let us assume Conjecture 3.10 and the existence of Beilinson–Bloch height pairing on smooth proper schemes over number fields. Then Θ_ϕ^f is cohomologically trivial and we let

$$\langle \Theta_\phi^f, \Theta_{\phi^\vee}^{f^\vee} \rangle_{\mathrm{BB}}^K$$

be the Beilinson–Bloch height on Sh_K (resp. $\widetilde{\mathrm{Sh}_K}$) if it is proper (resp. nonproper) for sufficiently small K. Let $\mathrm{vol}(K)$ be the volume with respect to the Haar measure defined in the proof of Theorem 4.20. Then

$$\langle \Theta_\phi^f, \Theta_{\phi^\vee}^{f^\vee} \rangle_{\mathrm{BB}} := \mathrm{vol}(K) \langle \Theta_\phi^f, \Theta_{\phi^\vee}^{f^\vee} \rangle_{\mathrm{BB}}^K$$

is a well-defined number which is independent of K.

If $\mathbb{V} \ncong \mathbb{V}(\pi, \chi)$, then $\langle \Theta_\phi^f, \Theta_{\phi^\vee}^{f^\vee} \rangle_{\mathrm{BB}} = 0$ for any f, $f^\vee$ and ϕ, $\phi^\vee$ since otherwise, it defines a nonzero functional

$$\gamma(f, f^\vee, \phi, \phi^\vee) \in \mathrm{Hom}_{H_n(\mathbb{A}_{f,F}) \times H_n(\mathbb{A}_{f,F})}(R(\mathbb{V}_f, \chi_f), \pi_f^\vee \boxtimes \chi_f \pi_f)$$

which contradicts the fact that the latter space is zero. This will imply that, assuming the conjecture that the Beilinson–Bloch height pairing is nondegenerate, any arithmetic theta lifting $\Theta_\phi^f = 0$.

If $\mathbb{V} \cong \mathbb{V}(\pi)$, we have the following main conjecture:

Conjecture 3.11 (arithmetic inner product formula). *Let π be an irreducible cuspidal automorphic representation of $H_n(\mathbb{A}_F)$, χ a character of $E^\times \mathbb{A}_F^\times \backslash \mathbb{A}_E^\times$ such that π_∞ is a discrete series representation of weight $(n - \mathfrak{k}^\chi/2, n + \mathfrak{k}^\chi/2)$, $\epsilon(\pi, \chi) = -1$, and $\mathbb{V} \cong \mathbb{V}(\pi, \chi)$. Then, for any $f \in \pi$, $f^\vee \in \pi^\vee$, and any $\phi, \phi^\vee \in \mathscr{S}(\mathbb{V}_f^n)^{\mathrm{U}_\infty}$ decomposable, we have*

$$\langle \Theta_\phi^f, \Theta_{\phi^\vee}^{f^\vee} \rangle_{\mathrm{BB}} = \frac{L'(\frac{1}{2}, \pi, \chi)}{\prod_{i=1}^{2n} L(i, \epsilon_{E/F}^i)} \prod_v Z^*(0, \chi_v, f_v, f_v^\vee, \phi_v \otimes \phi_v^\vee),$$

where in the last product almost all factors are 1.

We remark that when $n=1$, the height pairing $\langle \Theta_\phi^f, \Theta_{\phi^\vee}^{f^\vee} \rangle_{\mathrm{BB}}$ is just the Néron–Tate height pairing, hence is defined without any assumption.

4. Comparison at infinite places

4A. ***Archimedean Whittaker integrals.*** In this section, we will calculate certain Whittaker integrals $W_T(s, g, \Phi)$ and their first derivatives (at $s = 0$) at an archimedean place.

Elementary reductions. We fix an archimedean place $\iota : F \hookrightarrow \mathbb{C}$ and suppress it from the notation. Hence we have $H' \cong \mathrm{U}(n, n)$ and $H'' \cong \mathrm{U}(2n, 2n)$ with parabolic subgroup P, $V \cong \mathbb{V}_\iota$ the standard positive definite $2n$-dimensional complex hermitian space, and χ a character of $\mathbb{C}^\times$ which is trivial on $\mathbb{R}^\times$. In this section, we write a^* instead of ${}^t a^\tau$ for a complex matrix. Recall that we are always writing the elements in H'' in matrix form with respect to the basis $\{e_1, \dots, e_n; e_1^-, \dots, e_n^-; e_{n+1}, \dots, e_{2n}; -e_{n+1}^-, \dots, -e_{2n}^-\}$ under which P is the standard maximal parabolic subgroup. The map

$$\mathrm{U}(2n) \times \mathrm{U}(2n) \longrightarrow H'' \cong \mathrm{U}(2n, 2n)$$

$$(k_1, k_2) \longmapsto [k_1, k_2] := \frac{1}{2}\begin{pmatrix} k_1 + k_2 & -ik_1 + ik_2 \\ ik_1 - ik_2 & k_1 + k_2 \end{pmatrix}$$

is an isomorphism onto its image which is a maximal compact subgroup of H'', denoted by $\mathcal{K}$. Let Φ^0 be the standard Gaussian in $\mathcal{S}(V^{2n})$ and $\chi(z) = z^{2\ell}$ ($|z| = 1$) for some $\ell \in \mathbb{Z}$. Then $\omega_\chi([k_1, k_2])\Phi^0 = (\det k_1)^{n+\ell}(\det k_2)^{-n+\ell}\Phi^0$. Our first goal is to analyze the integral

$$W_T(s, g, \Phi^0) = \int_{\mathrm{Her}_{2n}(\mathbb{C})} \varphi_{\Phi^0,s}(wn(u)g)\psi_T(n(u))^{-1}\, du \tag{4-1}$$

for $T \in \mathrm{Her}_{2n}(\mathbb{C})$ and $\Re(s) > n$, where du is the self-dual measure with respect to (the standard) ψ. Write $g = n(b)m(a)[k_1, k_2]$ under the Iwasawa decomposition. Then

$$\begin{aligned}
(4\text{-}1) &= \int_{\mathrm{Her}_{2n}(\mathbb{C})} \omega_\chi(wn(u)n(b)m(a)[k_1, k_2])\Phi^0(0) \\
&\qquad\qquad \times \lambda_P(wn(u)n(b)m(a)[k_1, k_2])^s \psi(-\operatorname{tr} Tu)\, du \\
&= \psi(\operatorname{tr} Tb)(\det k_1)^{n+\ell}(\det k_2)^{-n+\ell} \\
&\qquad \times \int_{\mathrm{Her}_{2n}(\mathbb{C})} \omega_\chi(wn(u)m(a))\Phi^0(0)\lambda_P(wn(u)m(a))^s \psi(-\operatorname{tr} Tu)\, du. \qquad (4\text{-}2)
\end{aligned}$$

Since $wn(u)m(a) = wm(a)n(a^{-1}ua^{*,-1}) = m(a^{*,-1})wn(a^{-1}ua^{*,-1})$ and after changing variable $du = |\det a|_{\mathbb{C}}^{2n} d(a^{-1}ua^{*,-1})$, we have

$$\begin{aligned}
(4\text{-}2) &= \psi(\operatorname{tr} Tb)|\det a|_{\mathbb{C}}^{n-s}\chi(\det a)(\det k_1)^{n+\ell}(\det k_2)^{-n+\ell} \\
&\qquad \times \int_{\mathrm{Her}_{2n}(\mathbb{C})} \omega_\chi(wn(u))\Phi^0(0)\lambda_P(wn(u))^s \psi(-\operatorname{tr} a^*Tau)\, du
\end{aligned}$$

$$=\psi(\operatorname{tr} Tb)|\det a|_{\mathbb{C}}^{n-s}\chi(\det a)(\det k_1)^{n+\ell}(\det k_2)^{-n+\ell}W_{a^*Ta}(s,e,\Phi^0). \quad (4\text{-}3)$$

Hence we only need to consider $W_T(s,e,\Phi^0)$. From now on, we will not restrict ourselves to considering the case of dimension $2n$. Let $m\geq 1$ be an integer, V the totally positive-definite complex hermitian space of dimension m, and Φ^0 the standard Gaussian. For $T\in\mathrm{Her}_m(\mathbb{C})$, we can still consider $W_T(s,e,\Phi^0)$ which has the usual integral presentation for $\Re(s)>m/2$.

Lemma 4.1. *For $u\in\mathrm{Her}_m(\mathbb{C})$, $\omega_\chi(wn(u))\Phi^0(0)=\gamma_V\det(\mathbf{1}_m-iu)^m$, where γ_V is the Weil constant.*

Proof. By definition,

$$\gamma_V^{-1}\omega_\chi(wn(u))\Phi^0(0)=\int_{V^m}\omega_\chi(n(u))\Phi^0(x)\,dx=\int_{V^m}\psi(\operatorname{tr} uT(x))\Phi^0(x)\,dx, \quad (4\text{-}4)$$

where γ_V is the Weil constant. Write $u=k\,\mathrm{diag}[\dots,u_j,\dots]k^*$ with $u_j\in\mathbb{R}$ ($j=1,\dots,m$) and $k\in\mathrm{U}(m)$. Then

$$\begin{aligned}(4\text{-}4)&=\int_{V^m}\psi(\operatorname{tr} k\,\mathrm{diag}[\dots,u_j,\dots]k^*T(x)kk^{-1})\Phi^0(x)\,dx\\ &=\int_{V^m}\psi(\operatorname{tr}\mathrm{diag}[\dots,u_j,\dots]T(xk))e^{-2\pi\operatorname{tr}T(x)}\,dx.\end{aligned} \quad (4\text{-}5)$$

Changing the variable $x\mapsto xk$ and using $\operatorname{tr}T(x)=\operatorname{tr}T(xk)$, we have

$$\begin{aligned}(4\text{-}5)&=\int_{V^m}\exp(2\pi i\operatorname{tr}\mathrm{diag}[\dots,u_j,\dots]T(x)-2\pi\operatorname{tr}T(x))\,dx\\ &=\prod_{j=1}^m\int_V\exp(2\pi iu_jT(x_j)-2\pi T(x_j))\,dx_j.\end{aligned} \quad (4\text{-}6)$$

We identify V with $\mathbb{C}^m$ and $(\cdot,\cdot)$ with the usual hermitian form on $\mathbb{C}^m$, and then the self-dual measure dx_j on V is just the usual Lebesgue measure $d\boldsymbol{x}$ on $\mathbb{C}^m\cong\mathbb{R}^{2m}$. Hence

$$\begin{aligned}(4\text{-}6)&=\prod_{j=1}^m\int_{\mathbb{R}^{2m}}e^{-\pi(1-iu_j)\|\boldsymbol{x}\|^2}\,d\boldsymbol{x}=\prod_{j=1}^m\biggl(\int_{-\infty}^{\infty}e^{-\pi(1-iu_j)t^2}\,dt\biggr)^{2m}\\ &=\prod_{j=1}^m(1-iu_j)^{-m}=\det(\mathbf{1}_m-iu)^{-m}.\end{aligned}$$

□

Lemma 4.2. *For $u\in\mathrm{Her}_m(\mathbb{C})$, $\lambda_P(wn(u))=\det(\mathbf{1}_m+u^2)^{-1}$.*

Proof. We have

$$wn(u)\begin{pmatrix}i\mathbf{1}_m\\ \mathbf{1}_m\end{pmatrix}=\begin{pmatrix}&\mathbf{1}_m\\ -\mathbf{1}_m&\end{pmatrix}\begin{pmatrix}\mathbf{1}_m&u\\ &\mathbf{1}_m\end{pmatrix}\begin{pmatrix}i\mathbf{1}_m\\ \mathbf{1}_m\end{pmatrix}=\begin{pmatrix}\mathbf{1}_m\\ -i\mathbf{1}_m-u\end{pmatrix}.$$

Then,

$$\mathbf{1}_m(-i\mathbf{1}_m-u)^{-1}=-u(\mathbf{1}_m+u^2)^{-1}+i(\mathbf{1}_m+u^2)^{-1}.$$

Hence $\lambda_P(wn(u))=\det(\mathbf{1}_m+u^2)^{-1}$ which is a positive real number. □

Combining Lemmas 4.1 and 4.2, we have for $\Re(s)>m/2$,

$$\gamma_V^{-1}W_T(s,e,\Phi^0)=\int_{\mathrm{Her}_m(\mathbb{C})}\psi(-\operatorname{tr}Tu)\det(\mathbf{1}_m+iu)^{-s}\det(\mathbf{1}_m-iu)^{-s-m}\,du.$$

Now we proceed as in [Shimura 1982, Case II]. First, we need to introduce some new notation which may be different from that in [Shimura 1982]. Let

$$\begin{aligned}\mathrm{Her}_m^+(\mathbb{C})&=\{x\in\mathrm{Her}_m(\mathbb{C})\mid x>0\},\\ \mathfrak{h}_m&=\{x+iy\mid x\in\mathrm{Her}_m(\mathbb{C}),\ y\in\mathrm{Her}_m^+(\mathbb{C})\},\\ \mathfrak{h}'_m&=\{x+iy\mid x\in\mathrm{Her}_m^+(\mathbb{C}),\ y\in\mathrm{Her}_m(\mathbb{C})\}.\end{aligned}$$

Lemma 4.3 (Siegel; see [Shimura 1982, Section 1]). (1) *For $z\in\mathfrak{h}'_m$ and $\Re(s)>m-1$, we have*

$$\int_{\mathrm{Her}_m^+(\mathbb{C})}e^{-\operatorname{tr}(zx)}\det(x)^{s-m}\,dx=\Gamma_m(s)\det(z)^{-s},$$

where dx is induced from the self-dual measure on $\mathrm{Her}_m(\mathbb{C})$ and

$$\Gamma_m(s)=(2\pi)^{m(m-1)/2}\prod_{j=0}^{m-1}\Gamma(s-j).$$

(2) *For $x\in\mathrm{Her}_m(\mathbb{C})$, $b\in\mathrm{Her}_m^+(\mathbb{C})$ and $\Re(s)>2m-1$, we have*

$$\Gamma_m(s)\int_{\mathrm{Her}_m(\mathbb{C})}e^{2\pi i\operatorname{tr}(ux)}\det(b+2\pi iu)^{-s}\,du=\begin{cases}e^{-\operatorname{tr}(xb)}\det(x)^{s-m} & \text{if } x\in\mathrm{Her}_m^+(\mathbb{C}),\\ 0 & \text{if } x\notin\mathrm{Her}_m^+(\mathbb{C}).\end{cases}$$

Now for $\Re(s)>m-1$, by Lemma 4.3(1),

$$\begin{aligned}\gamma_V^{-1}W_T(s,e,\Phi^0)&=\int_{\mathrm{Her}_m(\mathbb{C})}\psi(-\operatorname{tr}Tu)\frac{1}{\Gamma_m(s)}\\ &\qquad\times\int_{\mathrm{Her}_m^+(\mathbb{C})}e^{-\operatorname{tr}(\mathbf{1}_m+iu)x}\det(x)^{s-m}\det(\mathbf{1}_m-iu)^{-s-m}\,dx\,du\\ &=\frac{1}{\Gamma_m(s)}\int_{\mathrm{Her}_m^+(\mathbb{C})}e^{-\operatorname{tr}x}\det(x)^{s-m}\\ &\qquad\times\int_{\mathrm{Her}_m(\mathbb{C})}e^{-i\operatorname{tr}(x+2\pi T)u}\det(\mathbf{1}_m-iu)^{-s-m}\,du\,dx.\end{aligned}\tag{4-7}$$

Apply Lemma 4.3(2) with $b=\mathbf{1}_m$, $x=x+2\pi T$, and $s=s+m$, and perform the change of variable $u \mapsto -u/(2\pi)$ to obtain

$$\text{(4-7)} = \frac{1}{\Gamma_m(s)} \times \int_{x>0,x+2\pi T>0} e^{-\operatorname{tr} x} \det(x)^{s-m} \times \frac{(2\pi)^{m^2}}{\Gamma_m(s+m)} e^{-\operatorname{tr}(x+2\pi T)} \det(x+2\pi T)^s \, dx. \tag{4-8}$$

We change the variable $x \mapsto x/\pi + T$,

$$\begin{aligned}\text{(4-8)} &= \frac{(2\pi)^{m^2}\pi^{2ms}}{\Gamma_m(s)\Gamma_m(s+m)} \int_{x>-T,x>T} e^{-\operatorname{tr} 2\pi x} \det(x+T)^s \det(x-T)^{s-m} \, dx \\ &= \frac{(2\pi)^{m^2}\pi^{2ms}}{\Gamma_m(s)\Gamma_m(s+m)} \eta(2\pi \mathbf{1}_m, T; s+m, s),\end{aligned} \tag{4-9}$$

where the function $\eta(g, h; \alpha, \beta)$ for $g \in \mathrm{Her}_m^+(\mathbb{C})$, $h \in \mathrm{Her}_m(\mathbb{C})$, $\Re(\alpha) \gg 0$, and $\Re(\beta) \gg 0$ is introduced in [Shimura 1982, (1.26)]. In what follows, we assume that T is nonsingular with $\mathrm{sign}(T) = (p, q)$ for $p + q = m$. We write $T = k\mathrm{diag}[t_1 \dots, t_p, -t_{p+1}, \dots, -t_m]k^*$ with $k \in \mathrm{U}(m)$, $t_j \in \mathbb{R}_{>0}$ and let $a = k\mathrm{diag}[\dots, t_j^{1/2}, \dots]$. Then $T = a\varepsilon_{p,q}a^*$ where $\varepsilon_{p,q} = \mathrm{diag}[\mathbf{1}_p, -\mathbf{1}_q]$. It is easy to see that we have

$$\eta(g, T; \alpha, \beta) = |\det T|^{\alpha+\beta-m} \eta(a^* g a, \varepsilon_{p,q}; \alpha, \beta), \tag{4-10}$$

$$\eta(g, \varepsilon_{p,q}; \alpha, \beta) = 2^{m(\alpha+\beta-m)} e^{-\operatorname{tr} g} \zeta_{p,q}(2g; \alpha, \beta). \tag{4-11}$$

We introduce $\zeta_{p,q}$ as in [Shimura 1982, (4.16)]. For $g \in \mathrm{Her}_m^+(\mathbb{C})$ and $p+q = m$, let $\varepsilon_p = \mathrm{diag}[\mathbf{1}_p, \mathbf{0}_q]$ and $\varepsilon'_q = \mathrm{diag}[\mathbf{0}_p, \mathbf{1}_q]$. Then

$$\zeta_{p,q}(g; \alpha, \beta) = \int_{X_{p,q}} e^{-\operatorname{tr}(gx)} \det(x+\varepsilon_p)^{\alpha-m} \det(x+\varepsilon'_q)^{\beta-m} \, dx,$$

where $X_{p,q} := \{x \in \mathrm{Her}_m(\mathbb{C}) \mid x + \varepsilon_p > 0, x + \varepsilon'_q > 0\}$ with the measure induced from the self-dual one on $\mathrm{Her}_m(\mathbb{C})$. Then $X_{m,0} = \mathrm{Her}_m^+(\mathbb{C})$. If $q = 0$, we simply write $\zeta_m = \zeta_{m,0}$.

Analytic continuation. Following [Shimura 1982, (4.17)], we let

$$\omega_{p,q}(g; \alpha, \beta) = \Gamma_q(\alpha-p)^{-1}\Gamma_p(\beta-q)^{-1} \det{}^+(\varepsilon_{p,q} g)^{\beta-q/2} \times \det{}^-(\varepsilon_{p,q} g)^{\alpha-p/2} \zeta_{p,q}(g; \alpha, \beta), \tag{4-12}$$

where $\det^\pm$ denotes the absolute value of the product of positive or negative eigenvalues (equal to 1 if there are no such eigenvalues) of a nonsingular element in $\mathrm{Her}_m(\mathbb{C})$. It is proved in [Shimura 1982, Section 4] that $\omega_{p,q}(g; \alpha, \beta)$ can be continued as a holomorphic function in (α, β) to the whole $\mathbb{C}^2$ and satisfies the functional equation $\omega_{p,q}(g; m-\beta, m-\alpha) = \omega_{p,q}(g; \alpha, \beta)$. Also, if $q = 0$, we simply write $\omega_m = \omega_{m,0}$.

Lemma 4.4. *If $q = 0$, then $\omega_m(g; m, \beta) = \omega_m(g; \alpha, 0) = 1$.*

Proof. The integral defining $\zeta_m(g; m, \beta)$,

$$\int_{\mathrm{Her}_m^+(\mathbb{C})} e^{-\operatorname{tr}(gx)} \det(x)^{\beta-m}\, dx,$$

is absolutely convergent for $\Re(\beta) > m-1$ and is equal to $\Gamma_m(\beta)\det(g)^{-\beta}$ by Lemma 4.3(1). Hence $\omega_m(g; m, \beta) \equiv 1$, which proves the lemma by the functional equation. □

Proposition 4.5. *Let $T \in \mathrm{Her}_m(\mathbb{C})$ be nonsingular with $\mathrm{sign}(T) = (p, q)$.*

(1) $\mathrm{ord}_{s=0} W_T(s, e, \Phi^0) \geq q$.

(2) *If T is positive definite, that is, $q = 0$, then*

$$W_T(0, e, \Phi^0) = \gamma_V \frac{(2\pi)^{m^2}}{\Gamma_m(m)} e^{-2\pi \operatorname{tr} T}.$$

Proof. (1) Combining (4-9), (4-10), (4-11), and (4-12), we have

$$\gamma_V^{-1} W_T(s, e, \Phi^0) = \frac{\Gamma_q(m+s-p)\Gamma_p(s-q)}{\Gamma_m(s)\Gamma_m(s+m)} (2\pi)^{m^2+2ms} |\det T|^{2s} e^{-2\pi \operatorname{tr}(a^*a)} \\ \times \det{}^+(4\pi T)^{q/2-s} \det{}^-(4\pi T)^{p/2-m-s} \omega_{p,q}(4\pi a^*a; m+s, s). \tag{4-13}$$

All terms except the gamma factors are holomorphic for any $s \in \mathbb{C}$. But

$$\frac{\Gamma_q(m+s-p)\Gamma_p(s-q)}{\Gamma_m(s)\Gamma_m(s+m)} = \frac{(2\pi)^{-pq-m(m-1)/2}}{\Gamma(s)\cdots\Gamma(s-q+1) \times \Gamma(s+m)\cdots\Gamma(s+m-p+1)}.$$

Hence $\mathrm{ord}_{s=0} W_T(s, e, \Phi^0) \geq -\mathrm{ord}_{s=0}\Gamma(s)\cdots\Gamma(s-q+1) = q$.

(2) If T is positive definite, then $\operatorname{tr}(a^*a) = \operatorname{tr} T$. By (4-13) and Lemma 4.4, we have

$$\gamma_V^{-1} W_T(0, e, \Phi^0) = \frac{(2\pi)^{m^2}}{\Gamma_m(m)} e^{-2\pi \operatorname{tr} T} \omega_m(4\pi a^*a; m, 0) = \frac{(2\pi)^{m^2}}{\Gamma_m(m)} e^{-2\pi \operatorname{tr} T}.$$ □

The case $q = 1$. By Proposition 4.5(1), the T-th coefficient will not contribute to the analytic kernel function $E'(0, g, \Phi)$ if $\mathrm{sign}(T) = (p, q)$ with $q \geq 2$. For this reason, we now focus on the case $q = 1$, that is, the functions $\zeta_{m-1,1}(g; \alpha, \beta)$ and $\omega_{m-1,1}(g; \alpha, \beta)$. We can assume that $g = \mathrm{diag}[a, b]$ with $a \in \mathrm{Her}_{m-1}^+(\mathbb{C})$ and $b \in \mathbb{R}_{>0}$. We write elements in X in the form

$$\begin{pmatrix} x & z \\ z^* & y \end{pmatrix}, \quad x \in \mathrm{Her}_{m-1}(\mathbb{C}), \quad y \in \mathbb{R}, \quad z \in \mathrm{Mat}_{m-1,1}(\mathbb{C}).$$

Then (see [Shimura 1982, p. 288])

$$X=\{(x,y,z)\mid x>0,\ y>0,\ x+\mathbf{1}_{m-1}>zy^{-1}z^*,\ y+1>z^*x^{-1}z\}$$
$$=\{(x,y,z)\mid x+\mathbf{1}_{m-1}>0,\ y+1>0,\ x>z(y+1)^{-1}z^*,\ y>z^*(x+\mathbf{1}_{m-1})^{-1}z\}.$$

We have

$$\zeta_{m-1,1}(g;\alpha,\beta)$$
$$=\int_X e^{-\operatorname{tr}(ax)-by}\det\begin{pmatrix} x+\mathbf{1}_{m-1} & z\\ z^* & z\end{pmatrix}^{\alpha-m}\det\begin{pmatrix} x & z\\ z^* & y+1\end{pmatrix}^{\beta-m}dx\,dy\,dz, \tag{4-14}$$

where we use the self-dual measure dx on $\mathrm{Her}_{m-1}(\mathbb{C})$, the Lebesgue measure dy on $\mathbb{R}$, and $dz=2^{m-1}\times$ the Lebesgue measure on $\mathrm{Mat}_{m-1,1}(\mathbb{C})$. Now we make a change of variables as in [Shimura 1982, p. 289] as follows. Put

$$f=(x+\mathbf{1}_{m-1})^{-1/2}z(y+1)^{-1/2}.$$

Then $\mathbf{1}_{m-1}-ff^*>0$. Put $r=(1-f^*f)^{1/2}$ and $s=(\mathbf{1}_{m-1}-ff^*)^{1/2}$; also $w=s^{-1}f=fr^{-1}$, $u=x-ww^*$, and $v=y-w^*w$. The map $(x,y,z)\mapsto(u,v,w)$ maps X bijectively onto $Y=\mathrm{Her}^+_{m-1}(\mathbb{C})\times\mathbb{R}_{>0}\times\mathrm{Mat}_{m-1,1}(\mathbb{C})$. Then the Jacobian

$$\frac{\partial(x,y,z)}{\partial(u,v,w)}=\det(\mathbf{1}_{m-1}+x)(1+y)^{m-1}(1+w^*w)^{-m}$$

for the measure $\partial(u,v,w)$ on Y induced from that on $\mathrm{Her}_{m-1}(\mathbb{C})\times\mathbb{R}\times\mathrm{Mat}_{m-1,1}(\mathbb{C})$ as an open subset. We have

$$\det\begin{pmatrix} x+\mathbf{1}_{m-1} & z\\ z^* & y\end{pmatrix}=\det(u+\mathbf{1}_{m-1}+ww^*)v\det(\mathbf{1}_{m-1}ww^*)^{-1},$$
$$\det\begin{pmatrix} x & z\\ z^* & y+1\end{pmatrix}=(v+1+w^*w)\det(u)\det(\mathbf{1}_{m-1}ww^*)^{-1}.$$

We obtain that

$$\text{(4-14)}=\int_Y e^{-\operatorname{tr}(au+aww^*)-(bv+bw^*w)}\det(\mathbf{1}_{m-1}+ww^*)^{m-\alpha-\beta}$$
$$\times\det(u+\mathbf{1}_{m-1}+ww^*)^{\alpha-m+1}\det(u)^{\beta-m}\times(v+1+w^*w)^{\beta-1}v^{\alpha-m}\,du\,dv\,dw$$
$$=\int_{\mathrm{Mat}_{m-1,1}(\mathbb{C})}e^{-\operatorname{tr}(aww^*)-bw^*w}\zeta_1(b(1+w^*w);\beta,\alpha-m+1)$$
$$\times\int_{\mathrm{Her}^+_{m-1}(\mathbb{C})}e^{-\operatorname{tr}(au)}\det(u+\mathbf{1}_{m-1}+ww^*)^{\alpha-m-1}\det(u)^{\beta-m}\,du\,dw. \tag{4-15}$$

On the other hand, again by (4-10), (4-11), and (4-12), we have

$$\gamma_V^{-1}W_T(s,e,\Phi^0)=\frac{(2\pi)^{m^2+2ms}|\det T|^{2s}}{\Gamma_m(s)\Gamma_m(s+m)}e^{-2\pi\operatorname{tr}(a^*a)}\zeta_{m-1,1}(4\pi a^*a;m+s,s). \tag{4-16}$$

Assuming $T \sim \mathrm{diag}[a_1, \dots, a_{m-1}, -b]$ with $a_j, b \in \mathbb{R}_{>0}$, then $a^*a = \mathrm{diag}[a_1, \dots, a_{m-1}, b]$. By (4-16) and Proposition 4.5(1),

$$\gamma_V^{-1} W_T'(0, e, \Phi^0) = \lim_{s\to 0} \frac{(2\pi)^{m^2}}{s\Gamma_m(s)\Gamma_m(m)} \times e^{-2\pi(a_1+\dots+a_{m-1}+b)} \zeta_{m-1,1}(4\pi\,\mathrm{diag}[a_1, \dots, a_{m-1}, b]; m, s). \quad (4\text{-}17)$$

Plugging in (4-15) for $(\alpha, \beta) = (m, s)$:

$$(4\text{-}17) = \lim_{s\to 0} \frac{2^{m-1}(2\pi)^{m^2}}{s\Gamma_m(s)\Gamma_m(m)} e^{-2\pi(a_1+\dots+a_{m-1}+b)} \times \int_{\mathbb{C}^{m-1}} e^{-4\pi[(a_1+a_m)w_1\bar{w}_1+\dots+(a_{m-1}+a_m)w_{m-1}\bar{w}_{m-1}]} \zeta_1(4\pi b(1+w^*w); 0, 1) \times \int_{\mathrm{Her}_{m-1}^+(\mathbb{C})} e^{-4\pi\,\mathrm{tr}\,\mathrm{diag}[a_1,\dots,a_{m-1}]u} \times \det(u + \mathbf{1}_{m-1} + ww^*)\det(u)^{s-m}\, du\, dw_1 \cdots dw_{m-1}. \quad (4\text{-}18)$$

It is easy to see that

$$\zeta_1(4\pi b(1+w^*w); 0, 1) = -e^{4\pi b(1+w^*w)} \mathrm{Ei}(-4\pi b(1+w^*w)), \quad (4\text{-}19)$$

where Ei is the classical exponential integral

$$\mathrm{Ei}(z) := -\int_1^\infty \frac{e^{zt}}{t}\, dt.$$

The main difficulty is calculating the inside integral, the one over $\mathrm{Her}_{m-1}^+(\mathbb{C})$. We temporarily let $g_0 = 4\pi\,\mathrm{diag}[a_1, \dots, a_{m-1}]$ and consider the integral

$$\int_{\mathrm{Her}_{m-1}^+(\mathbb{C})} e^{-\mathrm{tr}(ug_0)} \det(u + \mathbf{1}_{m-1} + ww^*)\det(u)^{s-m}\, du. \quad (4\text{-}20)$$

We define a differential operator $\Delta = \det\Bigl(\dfrac{\partial}{\partial g_{jk}}\Bigr)_{j,k=1}^{m-1}$. Then

$$\Delta e^{-\mathrm{tr}(ug)} = (-1)^{m-1} \det(u) e^{-\mathrm{tr}(ug)}.$$

Hence

$$\begin{aligned}(4\text{-}20) &= e^{\mathrm{tr}(\mathbf{1}_{m-1}+ww^*)g_0} \times \int_{\mathrm{Her}_{m-1}^+(\mathbb{C})} e^{-\mathrm{tr}(u+\mathbf{1}_{m-1}+ww^*)g_0} \det(u + \mathbf{1}_{m-1} + ww^*)\det(x)^{s-m}\, du \\ &= (-1)^{m-1} e^{\mathrm{tr}(\mathbf{1}_{m-1}+ww^*)g_0} \times \int_{\mathrm{Her}_{m-1}^+(\mathbb{C})} \Delta\big|_{g=g_0} e^{-\mathrm{tr}(u+\mathbf{1}_{m-1}+ww^*)g} \det(x)^{s-m}\, du. \quad (4\text{-}21)\end{aligned}$$

We can exchange Δ and the integration by analytic continuation; then

$$\begin{aligned}
\text{(4-21)} &= (-1)^{m-1} e^{\operatorname{tr}(\mathbf{1}_{m-1}+ww^*)g_0} \Delta\big|_{g=g_0} \int_{\mathrm{Her}^+_{m-1}(\mathbb{C})} e^{-\operatorname{tr}(u+\mathbf{1}_{m-1}+ww^*)g} \det(x)^{s-m}\, du \\
&= (-1)^{m-1} e^{\operatorname{tr}(\mathbf{1}_{m-1}+ww^*)g_0} \Delta\big|_{g=g_0} \big(e^{-\operatorname{tr}(\mathbf{1}_{m-1}+ww^*)g} \zeta_{m-1}(g; m-1, s-1)\big) \\
&= (-1)^{m-1} e^{\operatorname{tr}(\mathbf{1}_{m-1}+ww^*)g_0} \Delta\big|_{g=g_0} \big(e^{-\operatorname{tr}(\mathbf{1}_{m-1}+ww^*)g} \det(g)^{1-s} \Gamma_{m-1}(s-1)\big) \\
&= (-1)^{m-1} \Gamma_{m-1}(s-1) e^{\operatorname{tr}(\mathbf{1}_{m-1}+ww^*)g_0} \Delta\big|_{g=g_0} \\
&\qquad\qquad \times \big(e^{-\operatorname{tr}(\mathbf{1}_{m-1}+ww^*)g} \det(g)^{1-s}\big). \qquad (4\text{-}22)
\end{aligned}$$

We plug (4-19) and (4-22) in (4-18):

$$\begin{aligned}
\text{(4-18)} &= \lim_{s\to 0} \frac{\Gamma_{m-1}(s-1)(-2)^{m-1}(2\pi)^{m^2}}{s\Gamma_m(s)\Gamma_m(m)} e^{-2\pi \operatorname{tr} T} \\
&\quad \times \int_{\mathbb{C}^{m-1}} e^{-4\pi(a_1 w_1 \bar{w}_1 + \cdots + a_{m-1} w_{m-1} \bar{w}_{m-1})} e^{\operatorname{tr}(\mathbf{1}_{m-1}+ww^*)g_0} \Delta\big|_{g=g_0} \\
&\quad \times \big(e^{-\operatorname{tr}(\mathbf{1}_{m-1}+ww^*)g} \det(g)\big)(-\mathrm{Ei})(-4\pi b(1+w^*w))\, dw_1 \cdots dw_{m-1} \\
&= \frac{(2\pi)^{m^2}(-2)^{m-1}}{\Gamma_m(m)(2\pi)^{m-1}} e^{-2\pi \operatorname{tr} T} \int_{\mathbb{C}^{m-1}} e^{-4\pi(a_1 w_1 \bar{w}_1 + \cdots + a_{m-1} w_{m-1} \bar{w}_{m-1})} \\
&\qquad \times e^{\operatorname{tr}(\mathbf{1}_{m-1}+ww^*)g_0} \Delta\big|_{g=g_0} \big(e^{-\operatorname{tr}(\mathbf{1}_{m-1}+ww^*)g} \det(g)\big) \\
&\qquad\qquad \times (-\mathrm{Ei})(-4\pi b(1+w^*w))\, dw_1 \cdots dw_{m-1}. \qquad (4\text{-}23)
\end{aligned}$$

Now we make a change of variables. Let

$$D_{m-1} = \{z = (z_1, \dots, z_{m-1}) \in \mathbb{C}^{m-1} \mid z\bar{z} := z_1\bar{z}_1 + \cdots + z_{m-1}\bar{z}_{m-1} < 1\}$$

be the open unit disc in $\mathbb{C}^{m-1}$. Then the map

$$w_j = \frac{z_j}{(1-z\bar{z})^{1/2}}, \qquad j = 1, \dots, m-1 \qquad (4\text{-}24)$$

is a C^∞-homeomorphism from $\mathbb{C}^{m-1}$ to D_{m-1}. To calculate the Jacobian, let $w_j = u_j + v_j i$ and $z_j = x_j + y_j i$ be the corresponding real and imaginary parts. Then

$$\begin{aligned}
&\frac{\partial u_j}{\partial x_k} = \frac{x_j x_k}{(1-z\bar{z})^{3/2}}, \quad k \neq j; \qquad \frac{\partial u_j}{\partial x_j} = \frac{x_j^2}{(1-z\bar{z})^{3/2}} + \frac{1}{(1-z\bar{z})^{1/2}}; \qquad \frac{\partial u_j}{\partial y_k} = \frac{x_j y_k}{(1-z\bar{z})^{3/2}}; \\
&\frac{\partial v_j}{\partial y_k} = \frac{y_j y_k}{(1-z\bar{z})^{3/2}}, \quad k \neq j; \qquad \frac{\partial v_j}{\partial y_j} = \frac{y_j^2}{(1-z\bar{z})^{3/2}} + \frac{1}{(1-z\bar{z})^{1/2}}; \qquad \frac{\partial v_j}{\partial x_k} = \frac{y_j x_k}{(1-z\bar{z})^{3/2}}.
\end{aligned}$$

Let $c_j = x_j$ and $c_{m-1+j} = y_j$ for $j = 1, \dots, m-1$. Then by Lemma 4.6(2) below with $n = 2m+2$, $c = {}^t(c_1, \dots, c_{2m-2})$, and $\epsilon = 1 - z\bar{z}$,

$$
\begin{aligned}
&\frac{\partial(u_1, v_1, \dots, u_{m-1}, v_{m-1})}{\partial(x_1, y_1, \dots, x_{m-1}, y_{m-1})} \\
&\quad = \frac{\partial(u_1, \dots, u_{m-1}; v_1, \dots, v_{m-1})}{\partial(x_1, \dots, x_{m-1}; y_1, \dots, y_{m-1})} \\
&\quad = (1 - z\bar{z})^{-3(m-1)} \det((1 - z\bar{z})\mathbf{1}_{2m-2} + cc^*) \\
&\quad = (1 - z\bar{z})^{-3(m-1)} (1 - z\bar{z})^{2m-3} (1 - z\bar{z} + x_1^2 + \cdots + x_{m-1}^2 + y_1^2 + \cdots + y_{m-1}^2) \\
&\quad = (1 - z\bar{z})^{-m}.
\end{aligned} \tag{4-25}
$$

Lemma 4.6. *Let* $c \in \mathrm{Mat}_{n \times 1}(\mathbb{C})$. *Then*

(1) $\det(\mathbf{1}_n + cc^*) = 1 + c^*c$ *and*

(2) *for* $\epsilon > 0$, $\det(\epsilon\mathbf{1}_n + cc^*) = \epsilon^{n-1}(\epsilon + c^*c)$.

Proof. (1) is [Shimura 1982, Lemma 2.2]. Since it is not difficult, we will give a proof here, following Shimura. We claim that $\det(\mathbf{1}_n + scc^*) = 1 + sc^*c$ for all $c \in \mathbb{R}$. Since they are both polynomials in s, we only need to prove this for $s < 0$. We have

$$
\begin{pmatrix} \mathbf{1}_n & -\sqrt{-s}c \\ & 1 \end{pmatrix} \begin{pmatrix} \mathbf{1}_n & \sqrt{-s}c \\ \sqrt{-s}c^* & 1 \end{pmatrix} \begin{pmatrix} \mathbf{1}_n & \\ -\sqrt{-s}c^* & 1 \end{pmatrix} = \begin{pmatrix} \mathbf{1}_n + scc^* & \\ & 1 \end{pmatrix},
$$
$$
\begin{pmatrix} \mathbf{1}_n & \\ -\sqrt{-s}c^* & 1 \end{pmatrix} \begin{pmatrix} \mathbf{1}_n & \sqrt{-s}c \\ \sqrt{-s}c^* & 1 \end{pmatrix} \begin{pmatrix} \mathbf{1}_n & -\sqrt{-s}c \\ & 1 \end{pmatrix} = \begin{pmatrix} \mathbf{1}_n & \\ & 1 + sc^*c \end{pmatrix}.
$$

Hence $\det(\mathbf{1}_n + scc^*) = 1 + sc^*c$. (2) follows from (1) immediately. □

Now we write the Lebesgue measure $dz_1 \cdots dz_{m-1}$ in the differential form of degree $(m-1, m-1)$ on D_{m-1} which is

$$
dz_1 \cdots dz_{m-1} = \frac{1}{(-2i)^{m-1}} \Omega := \frac{1}{(-2i)^{m-1}} \bigwedge_{j=1}^{m-1} dz_j \wedge d\bar{z}_j,
$$

where in the latter we view dz_j as a $(1,0)$-form, not the Lebesgue measure anymore. By (4-25), we have

$$
\begin{aligned}
\text{(4-23)} = \frac{(2\pi)^{m^2}}{\Gamma_m(m)(2\pi i)^{m-1}} e^{-2\pi \operatorname{tr} T} \int_{D_{m-1}} e^{-4\pi(a_1 w_1 \bar{w}_1 + \cdots + a_{m-1} w_{m-1} \bar{w}_{m-1})} (1 - z\bar{z})^{-m} \\
\times e^{\operatorname{tr}(\mathbf{1}_{m-1} + ww^*)g_0} \Delta\big|_{g=g_0} \big(e^{-\operatorname{tr}(\mathbf{1}_{m-1} + ww^*)g} \det(g) j\big)(-\mathrm{Ei})(-4\pi b(1 + w^*w))\Omega,
\end{aligned} \tag{4-26}
$$

where w_j are as in (4-24). The final step is finished by the following lemma.

Lemma 4.7. *For* $g_0 = 4\pi\,\mathsf{diag}[a_1, \dots, a_{m-1}]$,

$$\Delta\big|_{g=g_0}\big(e^{-\operatorname{tr}(\mathbf{1}_{m-1}+ww^*)g}\det(g)\big) = e^{-\operatorname{tr}(\mathbf{1}_{m-1}+ww^*)g} \times \sum_{1\le s_1<\cdots<s_t\le m-1} (-4\pi)^t(m-1-t)!(a_{s_1}\cdots a_{s_t})(1+w_{s_1}\bar{w}_{s_1}+\cdots+w_{s_t}\bar{w}_{s_t}),$$

where the sum is taken over all subsets of $\{1, \dots, m-1\}$.

Proof. Let $u_{jk} = -(1+w_j\bar{w}_k)$ and $g = (g_{jk})_{j,k=1}^{m-1}$ be the matrix variables. For short, we also use $|g|$ to indicate the determinant of the square matrix g. For subsets $I, J \subset \{1, \dots, m-1\}$ of the same cardinality, we denote by $g_{J,K}$ (resp. $g^{J,K}$) the (square) matrix obtained by keeping (resp. omitting) the rows indexed in J and the columns indexed in K. Let $\mathfrak{S}_{m-1}$ be the group of $(m-1)$-permutations; for $\sigma \in \mathfrak{S}_{m-1}$ and a subset $J = \{j_1 < \cdots < j_t\} \subset \{1, \dots, m-1\}$, let $\epsilon_J(\sigma) \in \{\pm 1\}$ be a factor which only depends on J and σ. This factor comes from the combinatorics in taking successive partial derivatives. In our application, we only need to know its value in the case σ maps J to itself. In this case, let σ_J be the restriction of σ to J. Then $\epsilon_J(\sigma) = (-1)^{|\sigma_J|}$. Now we compute

$$\frac{\partial}{\partial g_{1,\sigma(1)}} e^{\operatorname{tr}(ug)}|g| = u_{\sigma(1),1}e^{\operatorname{tr}(ug)}|g| + \epsilon_{\{1\}}(\sigma)e^{\operatorname{tr}(ug)}|g^{\{1\},\{\sigma(1)\}}|,$$

$$\frac{\partial}{\partial g_{2,\sigma(2)}}\frac{\partial}{\partial g_{1,\sigma(1)}} e^{\operatorname{tr}(ug)}|g| = u_{\sigma(2),2}u_{\sigma(1),1}e^{\operatorname{tr}(ug)}|g| + \epsilon_{\{2\}}(\sigma)u_{\sigma(1),1}e^{\operatorname{tr}(ug)}|g^{\{2\},\{\sigma(2)\}}| + \epsilon_{\{1\}}(\sigma)u_{\sigma(2),2}e^{\operatorname{tr}(ug)}|g^{\{1\},\{\sigma(1)\}}| + \epsilon_{\{1,2\}}(\sigma)e^{\operatorname{tr}(ug)}|g^{\{1,2\},\{\sigma(1),\sigma(2)\}}|.$$

By induction, we have

$$\frac{\partial}{\partial g_{m-1,\sigma(m-1)}}\cdots\frac{\partial}{\partial g_{1,\sigma(1)}} e^{\operatorname{tr}(ug)}|g| = \sum_{1\le j_1<\cdots<j_t\le m-1} \epsilon_{\{j_1,\dots,j_t\}}(\sigma)u_{\sigma(s_{m-1-t}),s_{m-1-t}} \cdots u_{\sigma(s_1),s_1}e^{\operatorname{tr}(ug)}|g^{\{j_1,\dots,j_t\},\{\sigma(j_1),\dots,\sigma(j_t)\}}|,$$

where $\{s_1 < \cdots < s_{m-1-t}\}$ is the complement of $\{j_1, \dots, j_t\}$. Summing over σ, we have

$$\Delta\big|_{g=g_0} e^{\operatorname{tr}(ug)}|g| = e^{\operatorname{tr}(ug_0)} \sum_{\sigma\in\mathfrak{S}_{m-1}} (-1)^{|\sigma|} \sum_{1\le j_1<\cdots<j_t\le m-1} \epsilon_{\{j_1,\dots,j_t\}}(\sigma)u_{\sigma(s_{m-1-t}),s_{m-1-t}} \cdots u_{\sigma(s_1),s_1}|g_0^{\{j_1,\dots,j_t\},\{\sigma(j_1),\dots,\sigma(j_t)\}}|.$$

Changing the order of summation, since g_0 is diagonal, we have

$$\Delta\big|_{g=g_0} e^{\operatorname{tr}(ug)}|g| = e^{\operatorname{tr}(ug_0)} \sum_{J=\{j_1<\cdots<j_t\}} \sum_{\sigma(J)=J} (-1)^{|\sigma|}(-1)^{|\sigma_J|}u_{\sigma(s_{m-1-t}),s_{m-1-t}} \cdots u_{\sigma(s_1),s_1}|g_0^{J,J}|$$

$$= e^{\mathrm{tr}(ug_0)} \sum_{J=\{j_1<\cdots<j_t\}} t!|g_0^{J,J}| \sum_{\sigma':\bar{J}\to\bar{J}} (-1)^{|\sigma'|} u_{\sigma(s_{m-1-t}),s_{m-1-t}} \cdots u_{\sigma(s_1),s_1}$$

$$= e^{\mathrm{tr}(ug_0)} \sum_{J=\{j_1<\cdots<j_t\}} t!|g_0^{J,J}||u^{J,J}| = e^{\mathrm{tr}(ug_0)} \sum_{J'=\{s_1<\cdots<s_t\}} (m-1-t)!|(g_0)_{J',J'}||u_{J',J'}|.$$

The lemma follows by Lemma 4.6(1). □

In conclusion, using (4-26), we obtain our main result.

Proposition 4.8. *For $T \sim \mathrm{diag}[a_1, \dots, a_{m-1}, -b]$ of signature $(m-1, 1)$, we have*

$$W'_T(0, e, \Phi^0) = \gamma_V \frac{(2\pi)^{m^2}}{\Gamma_m(m)(2\pi i)^{m-1}} e^{-2\pi \,\mathrm{tr}\, T} \int_{D_{m-1}} e^{-4\pi(a_1 w_1 \bar{w}_1 + \cdots + a_{m-1} w_{m-1} \bar{w}_{m-1})}$$
$$\times \sum_{1\le s_1<\cdots<s_t\le m-1} (-4\pi)^t (m-1-t)!(a_{s_1}\cdots a_{s_t})(1 + w_{s_1}\bar{w}_{s_1} + \cdots + w_{s_t}\bar{w}_{s_t})$$
$$\times (-\mathrm{Ei})(-4\pi b(1 + w^* w))(1 - z\bar{z})^{-m}\Omega,$$

where w_j are C^∞-functions in z as in (4-24) and Ω is the volume form in z.

4B. ***An archimedean local height function.*** In this section, we will introduce a notion of height on the symmetric domain which will finally contribute to the local height pairing at an archimedean place. We will also prove some important properties of this height. A basic reference for archimedean Green's currents and height pairing is [Soulé 1992, Chapter II].

Green's functions. We still let $m \ge 2$ be an integer and V' the complex hermitian space of signature $(m-1, 1)$. We identify V' with $\mathbb{C}^m$ equipped with the hermitian form $(z, z') = z_1\bar{z}'_1 + \cdots + z_{m-1}\bar{z}'_{m-1} - z_m\bar{z}'_m$ for $z = (z_1, \dots, z_m)$ and $z' = (z'_1, \dots, z'_m)$ in $\mathbb{C}^m$. The hermitian domain $\mathfrak{D}$ of $\mathrm{U}(V') \cong \mathrm{U}(n-1, 1)$ can be identified with the $(m-1)$-dimensional complex open unit disc D_{m-1} through

$$z = [z_1 : \cdots : z_m] \in \mathfrak{D} \mapsto \left(\frac{z_1}{z_m}, \dots, \frac{z_{m-1}}{z_m}\right) \in D_{m-1}$$

and we will not distinguish them anymore. Given any $x \ne 0 \in V'^r$ $(1 \le r \le m-1)$ with nonsingular moment matrix $T(x)$, let D_x be the subspace of D_{m-1} consisting of lines perpendicular to all components in x which is nonempty if and only if $T(x)$ is positive definite. Now suppose $r = 1$, for $z \in D_{m-1}$, and let $x = x_z + x^z$ be the orthogonal decomposition with respect to the line z, that is, $x_z \in z$ and $x^z \perp z$. Let $R(x, z) = -(x_z, x_z)$ which is nonnegative since z is negative definite and $R(x, z) = 0$ if and only if $x = 0$ or $z \in D_x$. Explicitly, let $x = (x_1, \dots, x_m) \in V'$ and $z = (z_1, \dots, z_{m-1}) \in D_{m-1}$; then

$$R(x, z) = \frac{(x_1\bar{z}_1 + \cdots + x_{m-1}\bar{z}_{m-1} - x_m)(\bar{x}_1 z_1 + \cdots + \bar{x}_{m-1} z_{m-1} - \bar{x}_m)}{1 - z\bar{z}},$$

where we recall that $z\bar{z} = z_1\bar{z}_1 + \cdots + z_{m-1}\bar{z}_{m-1}$. We define

$$\xi(x, z) = -\mathrm{Ei}(-2\pi R(x, z)).$$

For each $x \neq 0 \in V'$, $\xi(x, \cdot)$ is a smooth function on $D_{m-1} - D_x$ and has logarithmic growth along D_x if not empty. Hence we can view it as a current $[\xi(x, \cdot)]$ on D. On the other hand, we recall the *Kudla–Millson form* $\varphi \in [\mathcal{S}(V'^r) \otimes A^{r,r}(D_{m-1})]^{\mathrm{U}(V')}$ $(1 \leq r \leq m-1)$ constructed in [Kudla and Millson 1986] and let $\omega(x, \cdot) = e^{2\pi \operatorname{tr} T(x)}\varphi(x, \cdot)$. Then we have

Proposition 4.9. *Let $x \neq 0 \in V'$, as currents; we have*

$$dd^c[\xi(x)] + \delta_{D_x} = [\omega(x)].$$

Proof. We will only give a proof for $m = 2$ since the proof for general m is similar but involves tedious computations.

First, we prove that $dd^c\xi(x) = \omega(x)$ holds away from D_x. Let $x = (x_1, x_2)$ and $z \in D_1 - D_x$. Sometimes we simply write R for $R(x)$. We have the formula

$$dd^c\xi(x) = \frac{1}{2\pi i}\left\{\frac{e^{-2\pi R}}{R^2}(R\partial\bar{\partial}R - \partial R \wedge \bar{\partial}R) - 2\pi\frac{e^{-2\pi R}}{R}\partial R \wedge \bar{\partial}R\right\}. \tag{4-27}$$

Computing each term, we have

$$\begin{aligned}
R(x, z) &= \frac{(x_1\bar{z} - x_2)(\bar{x}_1 z - \bar{x}_2)}{1 - z\bar{z}},\\
\bar{\partial}R &= \frac{x_1(\bar{x}_1 z - \bar{x}_2)(1 - z\bar{z}) + (x_1\bar{z} - x_2)(\bar{x}_1 z - \bar{x}_2)z}{(1 - z\bar{z})^2}\, d\bar{z},\\
\partial R &= \frac{\bar{x}_1(x_1\bar{z} - x_2)(1 - z\bar{z}) + (x_1\bar{z} - x_2)(\bar{x}_1 z - \bar{x}_2)\bar{z}}{(1 - z\bar{z})^2}\, dz,\\
\partial\bar{\partial}R &= \left(\frac{x_1\bar{x}_1}{1 - z\bar{z}} + \frac{x_1\bar{z}(\bar{x}_1 z - \bar{x}_2) + (x_1\bar{z} - x_2)(2\bar{x}_1 z - \bar{x}_2) + 2Rz\bar{z}}{(1 - z\bar{z})^2}\right) dz \wedge d\bar{z},\\
\partial R \wedge \bar{\partial}R &= \left(\frac{x_1\bar{x}_1 R}{1 - z\bar{z}} + \frac{\bar{x}_1 z(x_1\bar{z} - x_2)R + x_1\bar{z}(\bar{x}_1 z - \bar{x}_2)R + R^2 z\bar{z}}{(1 - z\bar{z})^2}\right) dz \wedge d\bar{z}.
\end{aligned}$$

Hence,

$$R\partial\bar{\partial}R - \partial R \wedge \bar{\partial}R = \left(\frac{(x_1\bar{z} - x_2)(\bar{x}_1 z - \bar{x}_2)R}{(1 - z\bar{z})^2} - \frac{R^2 z\bar{z}}{(1 - z\bar{z})^2}\right) dz \wedge d\bar{z} = R^2\frac{dz \wedge d\bar{z}}{(1 - z\bar{z})^2} \tag{4-28}$$

and

$$\begin{aligned}\partial R\wedge\bar\partial R&=(x_1\bar x_1(1-z\bar z)+\bar x_1 z(x_1\bar z-x_2)+x_1\bar z(\bar x_1 z-\bar x_2)+Rz\bar z)\frac{R\,dz\wedge d\bar z}{(1-z\bar z)^2}\\&=(x_1\bar x_1+\bar x_1 z x_2+x_1\bar z\bar x_2+x_1\bar x_1 z\bar z)\frac{R\,dz\wedge d\bar z}{(1-z\bar z)^2}\\&=((x,x)+(\bar x_1 z-\bar x_2)(x_1\bar z-x_2)+Rz\bar z)\frac{R\,dz\wedge d\bar z}{(1-z\bar z)^2}\\&=(R(x,z)+(x,x))\frac{R\,dz\wedge d\bar z}{(1-z\bar z)^2}.\end{aligned}\tag{4-29}$$

Plugging in (4-28) and (4-29), we have

$$(4\text{-}27)=\bigl(1-2\pi(R(x,z)+(x,x))\bigr)e^{-2\pi R(x,z)}\frac{dz\wedge d\bar z}{2\pi i(1-z\bar z)^2}=\omega(x,z).$$

The rest is the same as the proof of Proposition 11.1 of [Kudla 1997], from Lemma 11.2 on page 606. We omit it. □

The proposition says that $\xi(x)$ is a Green's function of logarithmic type for D_x. Now we consider $x=(x_1,\dots,x_r)\in V'^r$ with nonsingular moment matrix $T(x)$. Then using the star product of the Green's current, we have a Green's current $\Xi_x:=[\xi(x_1)]*\cdots*[\xi(x_r)]$ for D_x and as currents of degree (r,r) and

$$dd^c([\xi(x_1)]*\cdots*[\xi(x_r)])+\delta_{D_x}=[\omega(x_1)\wedge\cdots\wedge\omega(x_r)]=[\omega(x)].$$

A height function. For $x=(x_1,\dots,x_m)\in V'^m$ with nonsingular moment matrix $T(x)$, we define the *height function*

$$H(x)_\infty:=\langle 1,\,\Xi_x\rangle=\langle 1,[\xi(x_1)]*\cdots*[\xi(x_m)]\rangle.$$

Since $\xi(hx,hz)=\xi(x,z)$ for $h\in\mathrm{U}(V')$, the height function satisfies $H(hx)_\infty=H(x)_\infty$ and thus only depends on the (nonsingular) moment matrix $T(x)$. We sometimes simply write $H(T)_\infty$ for this function. Our main result is this:

Proposition 4.10 (invariance under $\mathrm{U}(m)$). *The height function $H(T)_\infty$ only depends on the eigenvalues of T, that is, for any $k\in\mathrm{U}(m)$, $H(kTk^*)_\infty=H(T)_\infty$.*

Proof. We prove this by induction on m. We will treat the case $m=2$ in the next subsection. Now suppose $m\geq 3$ and the proposition holds for $m-1$. Since $\mathrm{U}(m)$ is generated by the elements

$$\begin{pmatrix}k'&\\&1\end{pmatrix},\qquad k'\in\mathrm{U}(m-1)$$

and elements of the form $k=(k_{i,j})_{i,j=1}^m$ with entries $k_{i,\sigma(i)}\in\mathrm{U}(1)$ and zero for others for some $\sigma\in\mathfrak{S}_m$. We only need to prove that $H((x'k',x_m))_\infty=H((x',x_m))_\infty$

where $x=(x',x_m)\in V'^{m-1}\oplus V'=V'^m$ with $T(x)=T$. By definition,

$$\begin{aligned}H((x',x_m))_\infty&=\langle 1,[\xi(x_1)]*\dots*[\xi(x_{m-1})]\big|_{D_{x_m}}\rangle+\int_{D_{m-1}}\omega(x_1)\wedge\dots\wedge\omega(x_{m-1})\wedge\xi(x_m)\\&=H(x')_\infty+\int_{D_{m-1}}\omega(x')\wedge\xi(x_m).\end{aligned}$$

By induction, $H(x'k')_\infty=H(x')_\infty$. Moreover, $\omega(x')=\omega(x'k')$, by [Kudla and Millson 1986, Theorem 3.2(ii)]. Hence $H((x'k',x_m))_\infty=H((x',x_m))_\infty$. $\square$

Invariance under $\mathsf{U}(2)$*: A calculus exercise.* Now we consider the case $m=2$. Suppose

$$T=\begin{pmatrix}d_1&m\\\bar{m}&d_2\end{pmatrix},\qquad d_1,d_2\in\mathbb{R}.$$

Choose a complex number ϵ with norm 1 such that $\epsilon^2m\in\mathbb{R}$. Then

$$\begin{pmatrix}\epsilon&\\&\epsilon^{-1}\end{pmatrix}\begin{pmatrix}d_1&m\\\bar{m}&d_2\end{pmatrix}\begin{pmatrix}\epsilon&\\&\epsilon^{-1}\end{pmatrix}^*=\begin{pmatrix}d_1&\epsilon^2m\\\epsilon^2m&d_2\end{pmatrix}\in\mathsf{Sym}_2(\mathbb{R}).$$

Now we write the elements of $\mathsf{SO}(2)$ in the form

$$k_\theta:=\begin{pmatrix}\cos\theta&\sin\theta\\-\sin\theta&\cos\theta\end{pmatrix},\qquad\theta\in\mathbb{R}$$

and write $T[\theta]=k_\theta Tk_\theta^*$. Since $\xi(\epsilon x)=\xi(x)$ for any $x\in V'$, we have reduced the problem to proving this:

Proposition 4.11. *For any $T\in\mathsf{Sym}_2(\mathbb{R})$, we have $H(T[\theta])_\infty=H(T)_\infty$.*

Proof. The proof is similar to that in [Kudla 1997, Section 13]. Here is the idea. We construct a differentiable map

$$\alpha:\mathbb{R}\times D_1\to\mathsf{Her}_2(\mathbb{C})^{\det=0}$$

and a 2-form Ξ on the latter space such that the integration of $\alpha_{\theta_1}^*(\Xi)-\alpha_{\theta_0}^*(\Xi)$ on D_1 calculates the difference $H(T[\theta_1])_\infty-H(T[\theta_0])_\infty$, where $\alpha_\theta=\alpha(\theta,\cdot)$. Then we try to apply Stokes' theorem. The difficulty is that $\alpha^*(\Xi)$ has singularities along the real axis, hence a limit process should be taken to get the correct result. The difference between our proof and Kudla's is that we have different symmetric domains. Although they are conformal to each other, we need to take different 2-forms and limits of the integration domains.

Suppose that $T=\mathsf{diag}[a,-b]$ with $a,b>0$. Let

$$T[\theta]=\begin{pmatrix}d_{1,\theta}&m_\theta\\m_\theta&d_{2,\theta}\end{pmatrix}\in\mathsf{Sym}_2(\mathbb{R})$$

and let $x_0 = (\sqrt{2a}, 0) \in V'$, $y_0 = (0, \sqrt{2b}) \in V'$. For $\theta \in \mathbb{R}$, let

$$x_\theta = x_0 k_\theta = (x_{1,\theta}, x_{2,\theta}) = \cos\theta \cdot x_0 - \sin\theta \cdot y_0,$$
$$y_\theta = y_0 k_\theta = (y_{1,\theta}, y_{2,\theta}) = \sin\theta \cdot x_0 + \cos\theta \cdot y_0.$$

We have $dx_\theta/d\theta = -y_\theta$, $dy_\theta/d\theta = x_\theta$, and $H(T[\theta])_\infty = H((x_\theta, y_\theta))_\infty$. Let $z_{x,\theta} = x_{2,\theta}/x_{1,\theta}$ and $z_{y,\theta} = y_{2,\theta}/y_{1,\theta}$). Then $D_{x_\theta} = [z_{x,\theta}, 1]$ and $D_{y_\theta} = [z_{y,\theta}, 1]$, if not empty. We make the convention that if $|z| \geq 1$, then $f(z)$ is 0 for any function f.

Lemma 4.12 [Kudla 1997, Lemma 11.4]. *We have*

$$\begin{aligned} H((x_\theta, y_\theta))_\infty &= \xi(x_\theta, z_{y,\theta}) + \int_{D_1} \xi(y_\theta)\omega(x_\theta) \\ &= \xi(y_\theta, z_{x,\theta}) + \int_{D_1} \xi(x_\theta)\omega(y_\theta) \\ &= \xi(x_\theta, z_{y,\theta}) + \xi(y_\theta, z_{x,\theta}) - \int_{D_1} d\xi(x_\theta) \wedge d^c\xi(y_\theta). \end{aligned}$$

We now write $x = (x_1, x_2) \in V'$, $y = (y_1, y_2) \in V'$ and $R_1 = R(x)$, $R_2 = R(y)$ and consider the following integral in general:

$$\begin{aligned} I(T) = I((x, y)) &:= -\int_{D_1} d\xi(x) \wedge d^c\xi(y) \\ &= -\frac{1}{4\pi i}\int_{D_1} (\partial + \bar\partial)\xi(x) \wedge (\partial - \bar\partial)\xi(y) \\ &= -\frac{i}{4\pi}\int_{D_1} \partial\xi(x) \wedge \bar\partial\xi(y) + \partial\xi(y) \wedge \bar\partial\xi(x) \\ &= -\frac{i}{4\pi}\int_{D_1} \frac{e^{-2\pi(R_1+R_2)}}{R_1 R_2}(\partial R_1 \wedge \bar\partial R_2 + \partial R_2 \wedge \bar\partial R_1). \end{aligned}$$

For $z \in D_1$, letting $x(z) = (1 - z\bar z)^{-1/2}(z, 1) \in V'$ and $M = (x, x(z))\overline{(y, x(z))}$, we have

Lemma 4.13. *Let* $2m = (x, y)$. *Then*

$$\partial R_1 \wedge \bar\partial R_2 + \partial R_2 \wedge \bar\partial R_1 = 2(R_1 R_2 + \bar m M + m\overline{M})\frac{dz \wedge d\bar z}{(1 - z\bar z)^2},$$
$$\partial R_1 \wedge \bar\partial R_2 - \partial R_2 \wedge \bar\partial R_1 = 2(\bar m M - m\overline{M})\frac{dz \wedge d\bar z}{(1 - z\bar z)^2}.$$

Proof. By definition,

$$R_1 = \frac{(x_1\bar z - x_2)(\bar x_1 z - \bar x_2)}{1 - z\bar z}.$$

Hence,

$$\partial R_1 = \frac{(x_1\bar z - x_2)\bar x_1 + \bar z R_1}{1 - z\bar z}\,dz$$

and similarly for ∂R_2, $\bar\partial R_1$, and $\bar\partial R_2$. We compute

$$\begin{aligned}
&\partial R_1\wedge\bar\partial R_2\\
&=\{(x_1\bar z-x_2)(\bar y_1 z-\bar y_2)\bar x_1 y_1+(\bar y_1 z-\bar y_2)y_1\bar z R_1+(x_1\bar z-x_2)\bar x_1 z R_2+z\bar z R_1R_2\}\frac{dz\wedge d\bar z}{(1-z\bar z)^2}\\
&=\{(x_1\bar z-x_2)(\bar y_1 z-\bar y_2)\bar x_1 y_1+(\bar y_1 z-\bar y_2)y_2 R_1+(x_1\bar z-x_2)\bar x_1 z R_2+R_1R_2\}\frac{dz\wedge d\bar z}{(1-z\bar z)^2}\\
&=\left\{(x_1\bar z-x_2)(\bar y_1 z-\bar y_2)\left(\bar x_1 y_1+\frac{y_2(\bar x_1 z-\bar x_2)}{1-z\bar z}\right)+(x_1\bar z-x_2)\bar x_1 z R_2+R_1R_2\right\}\frac{dz\wedge d\bar z}{(1-z\bar z)^2}\\
&=\left\{(x_1\bar z-x_2)(\bar y_1 z-\bar y_2)\frac{\bar x_1 y_1-\bar x_2 y_2-\bar x_1 z(y_1\bar z-y_2)}{1-z\bar z}H(x_1\bar z-x_2)\bar x_1 z R_2+R_1R_2\right\}\\
&\qquad\qquad\times\frac{dz\wedge d\bar z}{(1-z\bar z)^2}\\
&=(2\bar m M+R_1R_2)\frac{dz\wedge d\bar z}{(1-z\bar z)^2}.
\end{aligned}$$

The lemma follows from a similar calculation for $\partial R_2\wedge\bar\partial R_1$. □

We define a morphism $\alpha:\mathbb{R}\times D_1\longrightarrow \mathsf{Her}_2(\mathbb{C})^{\det=0}$ between two 3-dimensional real analytic spaces where

$$\alpha(\theta,z)=\begin{pmatrix}R_1 & M\\ \overline{M} & R_2\end{pmatrix}=\begin{pmatrix}(x_\theta,x(z))\overline{(x_\theta,x(z))} & (x_\theta,x(z))\overline{(y_\theta,x(z))}\\ \overline{(x_\theta,x(z))}(y_\theta,x(z)) & (y_\theta,x(z))\overline{(y_\theta,x(z))}\end{pmatrix}$$

and $\alpha_\theta:=\alpha(\theta,\cdot)$. By an easy computation, we see that

$$\frac{dR_1}{d\theta}=-(M+\overline{M}),\qquad \frac{dR_2}{d\theta}=M+\overline{M},\qquad \frac{dM}{d\theta}=R_1+R_2. \tag{4-30}$$

Hence R_1+R_2 and $M-\overline{M}$, which are the values at $\theta=0$, are independent of θ:

$$R_1+R_2=\frac{2az\bar z+2b}{1-z\bar z}=-2a+\frac{2(a+b)}{1-z\bar z},\qquad M-\overline{M}=\frac{2\sqrt{ab}(z-\bar z)}{1-z\bar z}. \tag{4-31}$$

By Lemma 4.13 and the fact that $2m_\theta=(x_\theta,y_\theta)\in\mathbb{R}$, we have

$$\begin{aligned}
I(T[\theta])&=-\frac{i}{2\pi}\int_{D_1}\frac{e^{-2\pi(R_1+R_2)}}{R_1R_2}(R_1R_2+m(M+\overline{M}))\frac{dz\wedge d\bar z}{(1-z\bar z)^2}\\
&=\left(-\frac{i}{2\pi}\int_{D_1}e^{-2\pi(R_1+R_2)}\frac{dz\wedge d\bar z}{(1-z\bar z)^2}\right)\\
&\qquad\qquad+\left(-\frac{i}{2\pi}\int_{D_1}\frac{e^{-2\pi(R_1+R_2)}}{R_1R_2}m(M+\overline{M})\frac{dz\wedge d\bar z}{(1-z\bar z)^2}\right)\\
&=:I'(T[\theta])+I''(T[\theta]).
\end{aligned}\tag{4-32}$$

By (4-31), the integral $I'(T[\theta])$ is independent of θ; hence we now only consider the second one, $I''(T[\theta])$. We define a differential form of degree two on (the smooth locus of) $\mathrm{Her}_2(\mathbb{C})^{\det=0}$:

$$\Xi=-\frac{i}{4\pi}\frac{e^{-2\pi(R_1+R_2)}}{R_1R_2}\frac{M+\overline{M}}{M-\overline{M}}\,dR_1\wedge dR_2$$

which has singularities along $R_1R_2(M-\overline{M})=0$.

Lemma 4.14. (1) *For a fixed* $\theta\in\mathbb{R}$,

$$\alpha_\theta^*(\Xi)=-\frac{i}{2\pi}\frac{e^{-2\pi(R_1+R_2)}}{R_1R_2}m(M+\overline{M})\frac{dz\wedge d\bar{z}}{(1-z\bar{z})^2};$$

(2) *On* $\mathrm{Her}_2(\mathbb{C})^{\det=0}$, *we have*

$$d\Xi=\frac{i}{\pi}\frac{e^{-2\pi(R_1+R_2)}}{(M-\overline{M})^2(M+\overline{M})}d(M-\overline{M})\wedge dR_1\wedge dR_2.$$

Proof. (1) follows from Lemma 4.13. For (2), by the relation

$$(M+\overline{M})^2-(M-\overline{M})^2=4R_1R_2,$$

we have

$$\frac{d}{d(M-\overline{M})}\frac{M+\overline{M}}{M-\overline{M}}=-\frac{4R_1R_2}{(M-\overline{M})^2(M+\overline{M})}.$$ □

Let $D_1^+=\{z\in D_1\mid \Im(z)\geq 0\}$. Since $\alpha_\theta^*(\Xi)/dz\wedge d\bar{z}$ is invariant under $z\mapsto\bar{z}$, by (4-32) and Lemmas 4.12 and 4.14(1), we have

$$\begin{aligned}
&H(T[\theta_1])_\infty-H(T[\theta_0])_\infty\\
&=\xi(x_{\theta_1},z_{y,\theta_1})+\xi(y_{\theta_1},z_{x,\theta_1})-\xi(x_{\theta_0},z_{y,\theta_0})-\xi(y_{\theta_0},z_{x,\theta_0})+I(T[\theta])-I(T[\theta_0])\\
&=\xi(x_{\theta_1},z_{y,\theta_1})+\xi(y_{\theta_1},z_{x,\theta_1})-\xi(x_{\theta_0},z_{y,\theta_0})-\xi(y_{\theta_0},z_{x,\theta_0})+I''(T[\theta])-I''(T[\theta_0])\\
&=\xi(x_{\theta_1},z_{y,\theta_1})+\xi(y_{\theta_1},z_{x,\theta_1})-\xi(x_{\theta_0},z_{y,\theta_0})-\xi(y_{\theta_0},z_{x,\theta_0})+\int_{D_1}\alpha_{\theta_1}^*(\Xi)-\int_{D_1}\alpha_{\theta_0}^*(\Xi)\\
&=\xi(x_{\theta_1},z_{y,\theta_1})+\xi(y_{\theta_1},z_{x,\theta_1})-\xi(x_{\theta_0},z_{y,\theta_0})-\xi(y_{\theta_0},z_{x,\theta_0})\\
&\qquad\qquad+2\int_{D_1^+}\alpha_{\theta_1}^*(\Xi)-2\int_{D_1^+}\alpha_{\theta_0}^*(\Xi).
\end{aligned}\tag{4-33}$$

We see that the form $\alpha_\theta^*(\Xi)$ has (possible) singularities when $R_1R_2=0$, that is, the (possible) points $z_{x,\theta}$, $z_{y,\theta}$. An easy calculation shows that

$$z_{x,\theta}=\frac{x_{2,\theta}}{x_{1,\theta}}=-\tan\theta\cdot\sqrt{\frac{b}{a}}\in\mathbb{R},\qquad z_{y,\theta}=\frac{y_{2,\theta}}{y_{1,\theta}}=\cot\theta\cdot\sqrt{\frac{b}{a}}\in\mathbb{R}.$$

Now we assume that $[\theta_0,\theta_1]\subset(0,\pi/2)$. Then 0 will not be a singular point for $\theta\in[\theta_0,\theta_1]$. Our goal is to calculate the value

$$\int_{D_1^+}\alpha^*_{\theta_0}(\Xi)-\int_{D_1^+}\alpha^*_{\theta_1}(\Xi).$$

For any $\epsilon>0$ small enough, let $B_{1,\epsilon}$ be the (oriented) path $\{z=re^{i\epsilon}\mid r\in[0,1)\}$ from $r=0$ to $r=1$, $B_{2,\epsilon}$ the path $\{z=re^{i(\pi-\epsilon)}\mid r\in[0,1)\}$ from $r=1$ to $r=0$, and $D_\epsilon\subset D_1^+$ the area containing points on or above the lines $B_{1,\epsilon}$ and $B_{2,\epsilon}$. By our assumption, $\alpha^*_\theta(\Xi)$ is nonsingular on D_ϵ for any $\theta\in[\theta_0,\theta_1]$. By Stokes' theorem and the fact that $e^{-2\pi(R_1+R_2)}$ decays rapidly as $|z|$ goes to 1, we have

$$\int_{[\theta_0,\theta_1]\times D_\epsilon}\alpha^*(d\Xi)=\int_{D_\epsilon}\alpha^*_{\theta_1}(\Xi)-\int_{D_\epsilon}\alpha^*_{\theta_0}(\Xi)+\int_{[\theta_0,\theta_1]\times(B_{2,\epsilon}+B_{1,\epsilon})}\alpha^*(\Xi).\tag{4-34}$$

Lemma 4.15.
$$\int_{[\theta_0,\theta_1]\times D_\epsilon}\alpha^*(d\Xi)=0.$$

Proof. By (4-30) and (4-31), we have

$$\begin{aligned}dR_1&=\partial R_1+\bar\partial R_1-(M+\overline{M})\,d\theta,\\ dR_2&=\partial R_2+\bar\partial R_2+(M+\overline{M})\,d\theta,\\ d(M-\overline{M})&=2\sqrt{ab}\left(\partial\frac{z-\bar z}{1-z\bar z}+\bar\partial\frac{z-\bar z}{1-z\bar z}\right).\end{aligned}$$

Hence

$$\begin{aligned}&\alpha^*(d(M-\overline{M})\wedge dR_1\wedge dR_2)\\ &\qquad=2\sqrt{ab}(M+\overline{M})\left(\partial\frac{z-\bar z}{1-z\bar z}\wedge\bar\partial(R_1+R_2)-\partial(R_1+R_2)\wedge\bar\partial\frac{z-\bar z}{1-z\bar z}\right)\\ &\qquad=4\sqrt{ab}(a+b)(M+\overline{M})\left(\partial\frac{z-\bar z}{1-z\bar z}\wedge\bar\partial\frac{1}{1-z\bar z}-\partial\frac{1}{1-z\bar z}\wedge\bar\partial\frac{z-\bar z}{1-z\bar z}\right)\end{aligned}$$

and by Lemma 4.14(2),

$$\begin{aligned}\alpha^*(d\Xi)&=\frac{4i\sqrt{ab}(a+b)}{\pi}\frac{e^{-2\pi(R_1+R_2)}}{(M-\overline{M})^2}\left(\partial\frac{z-\bar z}{1-z\bar z}\wedge\bar\partial\frac{1}{1-z\bar z}-\partial\frac{1}{1-z\bar z}\wedge\bar\partial\frac{z-\bar z}{1-z\bar z}\right)\\ &=\frac{4i\sqrt{ab}(a+b)}{\pi}\frac{e^{-2\pi(R_1+R_2)}}{(M-\overline{M})^2}\frac{z+\bar z}{(1-z\bar z)^3}\,dz\wedge d\bar z.\end{aligned}$$

Since $z\mapsto-\bar z$ keeps the domain $[\theta_0,\theta_1]\times D_\epsilon$ and maps $\alpha^*(d\Xi)/dz\wedge d\bar z$ to its negative, the integral is zero. □

Hence by (4-34),

$$\int_{D_1^+}\alpha^*_{\theta_0}(\Xi)-\int_{D_1^+}\alpha^*_{\theta_1}(\Xi)=\lim_{\epsilon\to 0}\int_{D_\epsilon}\alpha^*_{\theta_0}(\Xi)-\lim_{\epsilon\to 0}\int_{D_\epsilon}\alpha^*_{\theta_1}(\Xi)$$
$$=\lim_{\epsilon\to 0}\int_{[\theta_0,\theta_1]\times(B_{1,\epsilon}+B_{2,\epsilon})}\alpha^*(\Xi). \quad (4\text{-}35)$$

A simple computation shows that on $[\theta_0,\theta_1]\times(B_{2,\epsilon}+B_{1,\epsilon})$,

$$\begin{aligned}\alpha^*(\Xi)&=\frac{-i(a+b)}{\pi}\,\frac{e^{-2\pi(R_1+R_2)}}{R_1R_2}\,\frac{(M+\overline{M})^2}{M-\overline{M}}\,\frac{r}{(1-r^2)^2}\,dr\wedge d\theta\\ &=\frac{-i(a+b)}{\pi}\,\frac{e^{-2\pi(R_1+R_2)}(M-\overline{M})}{R_1R_2}\,\frac{r}{(1-r^2)^2}\,dr\wedge d\theta\\ &\qquad+\frac{-4i(a+b)}{\pi}\,\frac{e^{-2\pi(R_1+R_2)}}{M-\overline{M}}\,\frac{r}{(1-r^2)^2}\,dr\wedge d\theta.\end{aligned}$$

Since the integrations of the second form on the two paths cancel each other, we have

$$\begin{aligned}(4\text{-}35)&=\int_{\theta_0}^{\theta_1}d\theta\cdot\frac{-i(a+b)}{\pi}\lim_{\epsilon\to 0}\int_{B_{1,\epsilon}+B_{2,\epsilon}}\frac{e^{-2\pi(R_1+R_2)}(M-\overline{M})}{R_1R_2}\,\frac{r}{(1-r^2)^2}\,dr\\ &=\int_{\theta_0}^{\theta_1}d\theta\cdot\frac{4\sqrt{ab}(a+b)}{\pi}\lim_{\epsilon\to 0}\sin\epsilon\int_{B_{1,\epsilon}+B_{2,\epsilon}}\frac{e^{-2\pi(R_1+R_2)}}{R_1R_2}\,\frac{r^2}{(1-r^2)^3}\,dr.\end{aligned} \quad (4\text{-}36)$$

To proceed, we need the following lemma.

Lemma 4.16. *Let $f(r)$ be a C^∞-function on $[0,1)$ which is rapidly decreasing as $r\to 1$. Then for any $c_1,c_2,d_1,d_2>0$,*

$$\lim_{\epsilon\to 0^+}\int_0^1\frac{\sin\epsilon}{(c_1^2r^2+c_2^2-2c_1c_2r\cos\epsilon)(d_1^2r^2+d_2^2+2d_1d_2r\cos\epsilon)}f(r)\,dr=\begin{cases}\dfrac{\pi c_1}{c_2(c_1d_2+c_2d_1)^2}f\Big(\dfrac{c_2}{c_1}\Big), & c_1>c_2,\\ 0, & c_1\le c_2.\end{cases}$$

Proof. The case $c_1\le c_2$ follows from the fact that f is rapidly decreasing. To prove the first case, we only need to prove that

$$\lim_{\epsilon\to 0^+}\int_0^1\frac{\sin\epsilon}{c_1^2r^2+c_2^2-2c_1c_2r\cos\epsilon}\,dr=\frac{\pi}{c_1c_2}. \quad (4\text{-}37)$$

The integral of the left-hand side of (4-37) (for small $\epsilon > 0$) equals

$$\begin{aligned}\sin\epsilon\int_0^1 &\frac{1}{(c_1r-c_2\cos\epsilon)^2+c_2^2(1-\cos\epsilon)}\\ &=\frac{\sin\epsilon}{c_1c_2\sqrt{1-\cos\epsilon}}\int_{-\frac{\cos\epsilon}{\sqrt{1-\cos\epsilon}}}^{\frac{c_1-c_2\cos\epsilon}{c_2\sqrt{1-\cos\epsilon}}}\frac{1}{\left(\frac{c_1r-c_2\cos\epsilon}{c_2\sqrt{1-\cos\epsilon}}\right)^2+1}\,d\frac{c_1r-c_2\cos\epsilon}{c_2\sqrt{1-\cos\epsilon}}\\ &=\frac{\sin\epsilon}{c_1c_2\sqrt{1-\cos\epsilon}}\left(\arctan\frac{c_1r-c_2\cos\epsilon}{c_2\sqrt{1-\cos\epsilon}}+\arctan\frac{\cos\epsilon}{\sqrt{1-\cos\epsilon}}\right).\end{aligned}$$

Let $\epsilon\to 0^+$, and the limit is π/c_1c_2. □

Applying the lemma, we have

$$(4\text{-}36)=\int_{\theta_0}^{\theta_1}\sqrt{ab}(a+b)\left(\frac{e^{-2\pi R_1(z_{y,\theta})}}{R_1(z_{y,\theta})}\frac{y_{1,\theta}y_{2,\theta}}{d_{2,\theta}^2}+\frac{e^{-2\pi R_2(z_{x,\theta})}}{R_2(z_{x,\theta})}\frac{x_{1,\theta}x_{2,\theta}}{d_{1,\theta}^2}\right)d\theta. \tag{4-38}$$

But

$$\begin{aligned}\frac{dR_1(z_{y,\theta})}{d\theta}&=\frac{d}{d\theta}(R_1(z_{y,\theta})+R_2(z_{y,\theta}))=\frac{4(a+b)r}{(1-r^2)^2}\bigg|_{r=y_{2,\theta}/y_{1,\theta}}\cdot\frac{d}{d\theta}\left(\frac{y_{2,\theta}}{y_{1,\theta}}\right)\\ &=2\sqrt{ab}(a+b)\frac{y_{1,\theta}y_{2,\theta}}{d_{2,\theta}^2},\\ \frac{dR_2(z_{x,\theta})}{d\theta}&=2\sqrt{ab}(a+b)\frac{x_{1,\theta}x_{2,\theta}}{d_{1,\theta}^2}.\end{aligned}$$

Hence

$$\begin{aligned}(4\text{-}38)&=\frac{1}{2}\left(\int_{R_1(z_{y,\theta_0})}^{R_1(z_{y,\theta_1})}\frac{e^{-2\pi R_1(z_{y,\theta})}}{R_1(z_{y,\theta})}\,dR_1(z_{y,\theta})+\int_{R_2(z_{x,\theta_0})}^{R_2(z_{x,\theta_1})}\frac{e^{-2\pi R_2(z_{x,\theta})}}{R_2(z_{x,\theta})}\,dR_2(z_{x,\theta})\right)\\ &=\frac{1}{2}(\xi(x_{\theta_1},z_{y,\theta_1})+\xi(y_{\theta_1},z_{x,\theta_1})-\xi(x_{\theta_0},z_{y,\theta_0})-\xi(y_{\theta_0},z_{x,\theta_0}))\end{aligned}$$

which, by (4-33), implies that

$$H(T[\theta_1])_\infty-H(T[\theta_0])_\infty=0$$

for $[\theta_0,\theta_1]\in(0,\pi/2)$. The same argument works for other intervals and the constancy of $H(T[\theta])_\infty$ for all $\theta\in\mathbb{R}$ follows from the continuity. This finishes the proof of Proposition 4.11. □

4C. ***An arithmetic local Siegel–Weil formula.*** In this section, we will find a relation between derivatives of Whittaker functions and the height functions defined above. Further, we will prove a local arithmetic analogue of the Siegel–Weil formula at an archimedean place for general dimensions.

Comparison on the hermitian domain. We are going to prove a relation between $W'_T(0, e, \Phi^0)$ and $H(T)_\infty$. Now suppose $T \sim \mathrm{diag}[a_1, \dots, a_{m-1}, -b]$ which is hermitian of signature $(m-1, 1)$. By Proposition 4.10, $H(T)_\infty$ only depends on $a_1, \dots, a_{m-1}, b$. Hence, if we let $x_j = (\dots, \sqrt{2a_j}, \dots) \in \mathbb{C}^m \cong V'$ with the j-th entry $\sqrt{2a_j}$ and all others zero for $j = 1, \dots, m-1$ and $x_m = (0, \dots, 0, \sqrt{2b})$, then $H(T)_\infty = H((x_1, \dots, x_m))_\infty$. Since $(x_m, x_m) < 0$, we have $D_{x_m} = \varnothing$ and

$$H(T)_\infty = \int_{D_{m-1}} \omega(x_1) \wedge \cdots \wedge \omega(x_{m-1}) \wedge \xi(x_m).$$

Our main result is the following *local arithmetic Siegel–Weil formula* at an archimedean place:

Theorem 4.17. *For $T \in \mathrm{Her}_m(\mathbb{C})$ of signature $(m-1, 1)$, we have*

$$W'_T(0, e, \Phi^0) = \gamma_V \frac{(2\pi)^{m^2}}{\Gamma_m(m)} e^{-2\pi \operatorname{tr} T} H(T)_\infty.$$

Proof. By the above discussion, we can assume $T = \mathrm{diag}[a_1, \dots, a_{m-1}, -b]$ and, by Proposition 4.8, we need to prove that

$$\begin{aligned}(2\pi i)^{m-1} \int_{D_{m-1}} \omega(x_1) \wedge \cdots \wedge \omega(x_{m-1}) \wedge \xi(x_m) = \int_{D_{m-1}} e^{-4\pi(a_1 w_1 \bar{w}_1 + \cdots + a_{m-1} w_{m-1} \bar{w}_{m-1})} \\ \times \sum_{1 \le s_1 < \cdots < s_t \le m-1} (-4\pi)^t (m-1-t)! (a_{s_1} \cdots a_{s_t})(1 + w_{s_1} \bar{w}_{s_1} + \cdots + w_{s_t} \bar{w}_{s_t}) \\ \times (-\mathrm{Ei})(-4\pi b(1 + w^* w))(1 - z\bar{z})^{-m} \Omega. \quad (4\text{-}39)\end{aligned}$$

By definition and (4-24), we have

$$\begin{aligned} R_j(z) &:= R(x_j, z) = \frac{2a_j z_j \bar{z}_j}{1 - z\bar{z}} = 2a_j w_j \bar{w}_j, \quad j = 1, \dots, m-1, \\ R_m(z) &:= R(x_m, z) = \frac{-2b}{1 - z\bar{z}} = -2b(1 + w^* w). \end{aligned}$$

Hence $\xi(x_m) = -\mathrm{Ei}(-4\pi b(1 + w^* w))$. Now we need an explicit formula for $\omega(x_j)$. By (4-27), we need to calculate $\bar{\partial} R_j$, ∂R_j, and $\partial \bar{\partial} R_j$ for $j = 1, \dots, m-1$. We have

$$\begin{aligned} & (1 - z\bar{z}) R_j = 2a_j z_j \bar{z}_j \\ \Longrightarrow \quad & \bar{\partial}(1 - z\bar{z}) R_j + (1 - z\bar{z}) \bar{\partial} R_j = 2a_j z_j \, d\bar{z}_j, \quad (4\text{-}40) \\ \Longrightarrow \quad & \bar{\partial} R_j = \frac{2a_j z_j \, d\bar{z}_j + R_j \bar{\partial}(z\bar{z})}{1 - z\bar{z}}. \quad (4\text{-}41) \end{aligned}$$

Similarly,

$$\partial R_j = \frac{2a_j \bar{z}_j \, dz_j + R_j \partial(z\bar{z})}{1 - z\bar{z}}. \quad (4\text{-}42)$$

Differentiating (4-40) again and plugging in (4-41) and (4-42), we have

$$\partial\bar\partial(1-z\bar z)R_j+\partial R_j\bar\partial(1-z\bar z)+\partial(1-z\bar z)\bar\partial R_j+(1-z\bar z)\partial\bar\partial R_j=2a_j\,dz_jd\bar z_j$$

which implies that

$$R_j=\frac{1}{(1-z\bar z)^2}\big(2a_j(1-z\bar z)\,dz_jd\bar z_j+2a_j\bar z_j\,dz_j\bar\partial(z\bar z)+2a_jz_j\partial(z\bar z)\,d\bar z_j \\ +2R_j\partial(z\bar z)\bar\partial(z\bar z)+R_j(1-z\bar z)\partial\bar\partial(z\bar z)\big). \quad (4\text{-}43)$$

Taking the wedge of (4-41) and (4-42), we have

$$\partial R_j\wedge\bar\partial R_j \\ =\frac{4a_j^2z_j\bar z_jdz_jd\bar z_j+2a_jR_j\bar z_jdz_j\bar\partial(z\bar z)+2a_jR_jz_j\partial(z\bar z)d\bar z_j+R_j^2\partial(z\bar z)\bar\partial(z\bar z)}{(1-z\bar z)^2}. \quad (4\text{-}44)$$

Combining (4-43) and (4-44), we have

$$\frac{1}{R_j^2}(R_j\partial\bar\partial R_j-\partial R_j\wedge\bar\partial R_j)=\frac{\partial(z\bar z)\bar\partial(z\bar z)}{(1-z\bar z)^2}+\frac{\partial\bar\partial(z\bar z)}{1-z\bar z}. \quad (4\text{-}45)$$

For simplicity, we make some substitutions. Let

$$\omega=\partial(z\bar z)\bar\partial(z\bar z)+(1-z\bar z)\partial\bar\partial(z\bar z),$$
$$\omega_j=(1-z\bar z)\bar z_jdz_jd\bar z_j+\bar z_jdz_j\bar\partial(z\bar z)+z_j\partial(z\bar z)d\bar z_j \\ +w_j\bar w_j\partial(z\bar z)\bar\partial(z\bar z),\qquad j=1,\dots,m-1.$$

Then $2\pi i\omega(x_j)=-\partial\bar\partial\xi(x_j)=e^{-4\pi a_jw_j\bar w_j}(\omega-4\pi a_j\omega_j)(1-z\bar z)^2$. Hence to prove (4-39), we only need to prove the following equality between $(m-1,m-1)$-forms on D_{m-1}:

$$\bigwedge_{j=1}^{m-1}(\omega-4\pi a_j\omega_j) \\ =\sum_{s_1<\dots<s_t}(-4\pi)^t(m-1-t)!(a_{s_1}\cdots a_{s_t})(1+w_{s_1}\bar w_{s_1}+\dots+w_{s_t}\bar w_{s_t})(1-z\bar z)^{m-2}\Omega$$

which follows from the claim that for any subset $\{s_1<\dots<s_t\}\subset\{1,\dots,m-1\}$, we have

$$\omega_{s_1}\wedge\dots\wedge\omega_{s_t}\wedge\omega^{m-1-t} \\ =(m-1-t)!(1+w_{s_1}\bar w_{s_1}+\dots+w_{s_t}\bar w_{s_t})(1-z\bar z)^{m-2}\Omega. \quad (4\text{-}46)$$

This will be proved in the next lemma where, without loss of generality, we assume that $s_j=j$. The theorem follows. □

Lemma 4.18. *Let w_j, Ω, ω, and ω_j be as above; for any integer $0 \le t \le m-1$, we have the following equality between $(m-1, m-1)$-forms:*

$$\Big(\bigwedge_{j=1}^{t} \omega_j\Big) \wedge \omega^{m-1-t} = (m-1-t)!\Big(1 + \sum_{j=1}^{t} w_j \bar{w}_j\Big)(1 - z\bar{z})^{m-2}\Omega.$$

Proof. For $j = 1, \dots, m-1$, we let

$$\sigma_j = \bar{z}_j\, dz_j \bar{\partial}(z\bar{z}), \quad \sigma'_j = z_j \partial(z\bar{z})\, d\bar{z}_j, \quad \delta_j = (1 - z\bar{z})\, dz_j d\bar{z}_j.$$

Then

$$\sum_{k=1}^{m-1} \sigma_k = \sum_{k=1}^{m-1} \sigma'_k, \quad \omega = \sum_{k=1}^{m-1} (\sigma_k + \delta_k), \quad \omega_j = \delta_j + \sigma_j + \sigma'_j + w_j \bar{w}_j \sum_{k=1}^{m-1} \sigma_k,$$

and

$$\sigma_j \wedge \sigma_j = 0, \quad \sigma'_j \wedge \sigma'_j = 0, \quad \delta_j \wedge \delta_j = 0.$$

We introduce the $(m-1) \times (m-1)$ matrix

$$Z = \begin{pmatrix} \bar{z}_1 z_1 & \bar{z}_2 z_1 & \cdots & \bar{z}_{m-1} z_1 \\ \bar{z}_1 z_2 & \bar{z}_2 z_2 & \cdots & \bar{z}_{m-1} z_2 \\ \vdots & \vdots & \ddots & \vdots \\ \bar{z}_1 z_{m-1} & \bar{z}_2 z_{m-1} & \cdots & \bar{z}_{m-1} z_{m-1} \end{pmatrix}$$

and recall the notation $Z_{J,K}$ (see the proof of Lemma 4.7) for subsets $J, K \subset \{1, \dots, m-1\}$ with $|J| = |K|$. It is easy to see that $|Z_{J,K}| \neq 0$ only if $|J| \le 1$ where $|Z_{\{j\},\{k\}}| = \bar{z}_j z_k$ and $|Z_{\varnothing,\varnothing}| = 1$.

Now we consider three subsets $I, J, K \subset \{1, \dots, m-1\}$ with $|I| + |J| + |K| = m-1$. Writing

$$\sigma_I = \bigwedge_{i \in I} \sigma_i$$

and similarly for σ'_J and δ_K, we have the following equalities

$$\sigma_I \sigma'_J \delta_K := \sigma_I \wedge \sigma'_J \wedge \delta_K = \begin{cases} \epsilon_{I,J,K} |Z_{I,\overline{J \cup K}}||Z_{\overline{I \cup K},J}|(1 - z\bar{z})^{|K|}\Omega, & (I \cup J) \cap K = \varnothing, \\ 0, & (I \cup J) \cap K \neq \varnothing, \end{cases}$$

where $\epsilon_{I,J,K} \in \{\pm 1\}$ is a factor only depending on I, J, K. This is not zero only if $|I| \le 1$ and $|J| \le 1$. Explicitly,

$$\sigma_I \sigma'_J \delta_K = \begin{cases} \bar{z}_i z_i \bar{z}_j z_j (1 - z\bar{z})^{m-3}\Omega, & i \neq j, I = \{i\}, J = \{j\}, K = \overline{I \cup J}, \\ -\bar{z}_i z_j \bar{z}_j z_i (1 - z\bar{z})^{m-3}\Omega, & i \neq j, I = J = \{i\}, K = \overline{I \cup \{j\}}, \\ \bar{z}_i z_i (1 - z\bar{z})^{m-2}\Omega, & I \cup J = \{i\}, K = \overline{\{i\}}, \\ (1 - z\bar{z})^{m-1}\Omega, & I = J = \varnothing, K = \{1, \dots, m-1\}. \end{cases}$$

Now we compute

$$
\begin{aligned}
&\Big(\bigwedge_{j=1}^{t}\omega_j\Big)\wedge\omega^{m-1-t}\\
&=\bigwedge_{j=1}^{t}\Big(\delta_j+\sigma_j+\sigma'_j+w_j\bar{w}_j\sum_{k=1}^{m-1}\sigma_k\Big)\wedge\Big(\sum_{k=1}^{m-1}\sigma_k+\sum_{k=1}^{m-1}\delta_k\Big)^{m-1-t}\\
&=\Big(\sum_{L\sqcup M\sqcup N\sqcup P=\{1,\dots,t\}}\delta_L\sigma_M\sigma'_N w_P\bar{w}_P\Big(\sum_{k=1}^{m-1}\sigma_k\Big)^{|P|}\Big)\\
&\qquad\times\Big(\sum_{\substack{Q\subset 1,\dots,m-1\\ |Q|\le m-1-t}}\frac{(m-1-t)!}{(m-1-t-|Q|)!}\Big(\sum_{k=1}^{m-1}\sigma_k\Big)^{m-1-t-|Q|}\delta_Q\Big)\\
&=\sum_{L,M,N,P,Q}\frac{(m-1-t)!}{(m-1-t-|Q|)!}\delta_{L\cup Q}\sigma_M\sigma'_N w_P\bar{w}_P\Big(\sum_{k=1}^{m-1}\sigma_k\Big)^{|P|+m-1-t-|Q|}\\
&=:\sum_{L,M,N,P,Q}T_{L,M,N,P,Q},
\end{aligned}\tag{4-47}
$$

where $w_P=\prod_{p\in P}w_p$ and similarly for $\bar{w}_P$. We now classify and calculate all the terms $T_{L,M,N,P,Q}$ which are not zero. It is easy to see that $|Q|\ge m-2-t$ if $T_{L,M,N,P,Q}\ne 0$. We now list all cases where $T_{L,M,N,P,Q}$ may not be zero.

Case I: $|Q|=m-1-t$. Then $|P|\le 1$:

Case I-1: $|P|=0$. Then $Q=\{t+1,\dots,m-1\}$ and $|M|\le 1$, $|N|\le 1$:

Case I-1a: $M=\{m\}$ and $N=\{n\}$ for $m\ne n\in\{1,\dots,t\}$. Then the sum of corresponding terms is

$$\sum T_{L,M,N,P,Q}=(m-1-t)!\sum_{\substack{m,n=1\\ m\ne n}}^{t}z_m\bar{z}_m z_n\bar{z}_n(1-z\bar{z})^{m-3}\Omega.\tag{4-48}$$

Case I-1b: $M\cup N=\{m\}$ for $1\le m\le t$. Then the sum of corresponding terms is

$$\sum T_{L,M,N,P,Q}=2(m-1-t)!\sum_{m=1}^{t}z_m\bar{z}_m(1-z\bar{z})^{m-2}\Omega.\tag{4-49}$$

Case I-1c: $M=N=\varnothing$. The corresponding term is

$$T_{L,M,N,P,Q}=T_{\{1,\dots,t\},\varnothing,\varnothing,\varnothing,\{t+1,\dots,m-1\}}=(m-1-t)!(1-z\bar{z})^{m-1}\Omega.\tag{4-50}$$

Case I-2: $|P|=1$. Then $M=N=\varnothing$. Suppose $P=\{p\}$ for $1\le p\le t$. Then $Q=\{p,t+1,\dots,m-1\}-\{q\}$ for some q inside. The sum of the corresponding terms is

$$\sum T_{L,M,N,P,Q}=(m-1-t)!\sum_{p=1}^{t} w_p\bar{w}_p\Big(z_p\bar{z}_p+\sum_{q=t+1}^{m-1} z_q\bar{z}_q\Big)(1-z\bar{z})^{m-2}\Omega. \tag{4-51}$$

Case II: $|Q|=m-2-t$. Then $M=N=P=\varnothing$ and $|Q|=\{t+1,\dots,m-1\}-\{q\}$ for some q inside. The sum of the corresponding terms is

$$\sum T_{L,M,N,P,Q}=(m-1-t)!\sum_{q=t+1}^{m-1} z_q\bar{z}_q(1-z\bar{z})^{m-2}\Omega. \tag{4-52}$$

Taking the sum from (4-48) to (4-52), we have

$$\begin{aligned}
\text{(4-47)} &= (m-1-t)!(1-z\bar{z})^{m-2}\Omega\Bigg\{\sum_{p=1}^{t} w_p\bar{w}_p\Big(z_p\bar{z}_p+\sum_{q=t+1}^{m-1} z_q\bar{z}_q\Big)\\
&\qquad\qquad +\frac{\sum_{\substack{m,n=1\\ m\neq n}}^{t} z_m\bar{z}_m z_n\bar{z}_n}{1-z\bar{z}}+2\sum_{m=1}^{t} z_m\bar{z}_m+\sum_{q=t+1}^{m-1} z_q\bar{z}_q+(1-z\bar{z})\Bigg\}\\
&= (m-1-t)!(1-z\bar{z})^{m-2}\Omega\Bigg\{1+\sum_{m=1}^{t} z_m\bar{z}_m\\
&\qquad\qquad +\frac{\sum_{\substack{m,n=1\\ m\neq n}}^{t} z_m\bar{z}_m z_n\bar{z}_n+\sum_{p=1}^{t} z_p\bar{z}_p\big(z_p\bar{z}_p+\sum_{q=t+1}^{m-1} z_q\bar{z}_q\big)}{1-z\bar{z}}\Bigg\}\\
&= (m-1-t)!(1-z\bar{z})^{m-2}\Omega\frac{\sum_{m=1}^{t} z_m\bar{z}_m}{1-z\bar{z}}\\
&= (m-1-t)!\Big(1+\sum_{j=1}^{t} w_j\bar{w}_j\Big)(1-z\bar{z})^{m-2}\Omega.
\end{aligned}$$

This finishes the proof of the lemma. □

Remark 4.19. W. Zhang has proved Theorem 4.17 independently (unpublished) using a similar method, assuming invariance under $\mathrm{U}(2)$ (Proposition 4.11).

Comparison on the Shimura variety. Now we use the previous results to compute the archimedean local height pairing on the unitary Shimura varieties with respect to suitable Green's currents. Let $n\ge 1$ be a positive integer. We recall the notation for groups in Section 2B and the notation for Shimura varieties in Section 3A with $m=2n$ and $r=n$. We let M_K be the variety $\mathrm{Sh}_K(\mathbb{H})$ for simplicity. For decomposable $\phi_i=\phi_\infty^0\phi_{i,f}$ with the Gaussian ϕ_∞^0 at infinite places and $\phi_{i,f}\in$

$\mathcal{S}(\mathbb{V}_f^n)^K$ $(i=1,2)$ such that $\phi_{1,v}\otimes\phi_{2,v}\in\mathcal{S}(\mathbb{V}_v^{2n})_{\mathrm{reg}}$ for some $v\in\Sigma_f$, the generating series $Z_{\phi_1}(g_1)$ and $Z_{\phi_2}(g_2)$, defined by the Weil representation $\omega_{\chi,\psi}$ with standard ψ_∞, will not intersect on M_K providing $g_i\in P'_vH'(\mathbb{A}_F^v)$. For any infinite place $\iota^\circ\in\Sigma_\infty^\circ$ over $\iota\in\Sigma_\infty$, we will attach a Green's current $\Xi_{\phi_i}(g_i)_{\iota^\circ}$ to $Z_{\phi_i}(g_i)\in\mathrm{CH}^n(M_K)$ and consider the local height pairing

$$\mathrm{vol}(K)\langle(Z_{\phi_1}(g_1),\,\Xi_{\phi_1}(g_1)_{\iota^\circ}),\,(Z_{\phi_2}(g_2),\,\Xi_{\phi_2}(g_2)_{\iota^\circ})\rangle_{M_K}.$$

Our main theorem is the following:

Theorem 4.20. *Let ϕ_i, g_i $(i=1,2)$ and ι°, ι be as above. Then there is a unique Haar measure on $\mathbb{H}(\mathbb{A}_f)$ which only depends on ψ_f such that*

$$E_\iota(0,\iota(g_1,g_2^\vee),\phi_1\otimes\phi_2)=-2\mathrm{vol}(K)\langle(Z_{\phi_1}(g_1),\,\Xi_{\phi_1}(g_1)_{\iota^\circ}),\,(Z_{\phi_2}(g_2),\,\Xi_{\phi_2}(g_2)_{\iota^\circ})\rangle_{M_K}$$

where E_ι is given in (2-10) and $\mathrm{vol}(K)$ is the volume of K determined by this measure.

Proof. We can assume that $K=\prod K_v$ is decomposable and sufficiently small. To do this, we consider the uniformization of M_K at ι° and suppress the superscript (ι) for the nearby objects such as H, $\mathfrak{D}$, and V. We have

$$(M_K)^{\mathrm{an}}_{\iota^\circ}\cong H(\mathbb{Q})\backslash(\mathfrak{D}\times H(\mathbb{A}_f)/K)=\coprod_{\{h\}}M_{K,\{h\}},$$

and $M_{K,\{h\}}=\Gamma_{\{h\}}\backslash\mathfrak{D}$ is a geometric connected component with $\Gamma_{\{h\}}:=H(\mathbb{Q})\cap hKh^{-1}$ viewed as a subgroup of $H(\mathbb{Q})$, where $\{h\}$ goes through the double cosets $H(\mathbb{Q})\backslash H(\mathbb{A}_f)/K$.

By our assumption on ϕ_i and g_i, we may write the generating series in the following way:

$$Z_{\phi_i}(g_i)=\sum_{T_i\in\mathrm{Her}_n^+(E)}\;\sum_{h_i\in H_{x_{T_i}}(\mathbb{A}_f)\backslash H(\mathbb{A}_f)/K}\omega_\chi(g_i)\phi_i(T_i,h_i^{-1}x_{T_i})Z(h_i^{-1}x_{T_i})_K$$

for $i=1,2$, where $\mathrm{Her}_n^+(E)$ is the set of totally positive-definite hermitian matrices in $\mathrm{Her}_n(E)$ and $x_{T_i}\in V^n$ is any element (if it exists) such that $T(x_{T_i})=T_i$ since $H(\mathbb{Q})$ acts transitively on $\Omega_T:=\{x\in V^n\mid T(x)=T\}$ for $T\in\mathrm{Her}_n^+(E)$. By definition,

$$Z(h_i^{-1}x_{T_i})_K=\sum_h Z_{x_{T_i},h},$$

where $Z_{x_{T_i},h}$ is a cycle in $\mathrm{CH}^n(M_{K,\{h\}})$ represented by the points (z,h) with $z\in D_{x_{T_i}}$ and the sum takes over a set of representatives h in the double coset

$$H_{x_{T_i}}(\mathbb{Q})\backslash H_{x_{T_i}}(\mathbb{A}_f)h_iK/K.$$

Then we have

$$Z_{\phi_i}(g_i)=\sum_{T_i\in\mathrm{Her}_n^+(E)}\ \sum_{h_i\in H_{x_{T_i}}(\mathbb{Q})\backslash H(\mathbb{A}_f)/K}\omega_\chi(g_i)\phi_i(T_i,h_i^{-1}x_{T_i})Z_{x_{T_i},h_i}.$$

Writing $g_{i,\iota}=n(b_i)m(a_i)[k_{i,1},k_{i,2}]$ for the Iwasawa decomposition as in Section 4A, let $\Xi_{x_ia_i,h_i}$ be the $(n-1,n-1)$ Green's current of $D_{x_ia_i}$ on the hermitian symmetric domain $(\mathfrak{D},h)\subset\mathfrak{D}\times H(\mathbb{A}_f)/K$. We define a current

$$\Xi_{\phi_i}(g_i)_{\iota^\circ}=\sum_{x_i}\ \sum_{h_i\in H(\mathbb{Q})\backslash H(\mathbb{A}_f)/K}\omega_\chi(g_i)\phi_i(T(x_i),h_i^{-1}x_i)\,\Xi_{x_ia_i,h_i}$$

on $\mathfrak{D}\times H(\mathbb{A}_f)/K$, where x_i is taken over all elements in V^n whose components are linearly independent. It It projects to a current on $H(\mathbb{Q})\backslash\mathfrak{D}\times H(\mathbb{A}_f)/K$, which is a Green's current for $Z_{\phi_i}(g_i)$. Then we have

$$\begin{aligned}
&\langle(Z_{\phi_1}(g_1),\Xi_{\phi_1}(g_1)_{\iota^\circ}),(Z_{\phi_2}(g_2),\Xi_{\phi_2}(g_2)_{\iota^\circ})\rangle_{M_K}\\
&\quad=\sum_{x_1,x_2}\ \sum_{h\in H(\mathbb{Q})\backslash H(\mathbb{A}_f)/K}\omega''_\chi(\iota(g_1,g_2^\vee))\phi_1\otimes\phi_2\bigl(T(x_1)\oplus T(x_2),(h^{-1}x_1,h^{-1}x_2)\bigr)\\
&\qquad\qquad\times\int_{H(\mathbb{Q})\backslash(\mathfrak{D}\times H(\mathbb{A}_f)/K)}\Xi_{x_1a_1,h}*\Xi_{x_2a_2,h}\\
&\quad=\sum_T\ \sum_{h\in H(\mathbb{A}_f)/K}\omega''_\chi(\iota(g_1,g_2^\vee))\phi_1\otimes\phi_2(T,h^{-1}x_T)H(a^*Ta)_\infty\\
&\quad=\sum_T H(a^*Ta)_\infty\prod_{v\in\Sigma_\infty}\omega''_{\chi_v}(\iota(g_{1,v},g_{2,v}^\vee))\Phi_v^0(T)\\
&\qquad\qquad\times\prod_{v\in\Sigma_f}\ \sum_{h_v\in H_v/K_v}\omega''_{\chi_v}(\iota(g_{1,v},g_{2,v}^\vee))\phi_{1,v}\otimes\phi_{2,v}(h_v^{-1}x_T),
\end{aligned}\tag{4-53}$$

where the sum is taken over all nonsingular $T\in\mathrm{Her}_{2n}(E)$ that are moment matrices of some $x_T\in V^{2n}$ and $a=a_1\oplus a_2\in\mathrm{GL}_{2n}(\mathbb{C})$. We compute for each v.

Case I: $v=\iota$, by (4-3) and Theorem 4.17 for the coefficient a^*Ta, we have

$$H(a^*Ta)_\infty\omega''_{\chi_\iota}(\iota(g_{1,\iota},g_{2,\iota}^\vee))\Phi_\iota^0(T)=\gamma_{\mathbb{V}_\iota}^{-1}\frac{\Gamma_{2n}(2n)}{(2\pi)^{4n^2}}W'_T(0,\iota(g_{1,\iota},g_{2,\iota}^\vee),\Phi_\iota^0).$$

Recalling the local Tate factors (2-4), we have

$$H(a^*Ta)_\infty\omega''_{\chi_\iota}(\iota(g_{1,\iota},g_{2,\iota}^\vee))\Phi_\iota^0(T)=\gamma_{\mathbb{V}_\iota}^{-1}b_{2n,\iota}(0)W'_T(0,\iota(g_{1,\iota},g_{2,\iota}^\vee),\Phi_\iota^0).\tag{4-54}$$

Case II: $v\in\Sigma_\infty$, $v\neq\iota$. By (4-3) and Proposition 4.5(2), we have

$$\omega''_{\chi_v}(\iota(g_{1,v},g_{2,v}^\vee))\Phi_v^0(T)=\gamma_{\mathbb{V}_v}^{-1}b_{2n,v}(0)W_T(0,\iota(g_{1,v},g_{2,v}^\vee),\Phi_v^0).\tag{4-55}$$

Case III: $v\in\Sigma_f$. Recalling the set Ω_T defined in Section 2E, it is easy to see that $\Omega_T\neq\varnothing$ is a single orbit of the left translation by H_v whose stabilizer at any point is trivial. Hence any Haar measure $d'h_v$ on H_v induces a measure $d'x$ on Ω_T. We have

$$\sum_{h_v\in H_v/K_v} \omega''_{\chi_v}(\iota(g_{1,v}, g^\vee_{2,v}))\phi_{1,v}\otimes\phi_{2,v}(h_v^{-1}x_T) \\ = \mathrm{vol}'(K_v)^{-1}\int_{H_v}\omega''_{\chi_v}(\iota(g_{1,v}, g^\vee_{2,v}))\phi_{1,v}\otimes\phi_{2,v}(h_v^{-1}x_T)\,d'h_v,$$

where $\mathrm{vol}'(K_v)$ is the volume of K_v under the measure $d'h_v$. By [Rallis 1987, Lemma 4.2], we can choose a unique measure $d'h_v$ such that

$$W_T(0, \iota(g_{1,v}, g^\vee_{2,v}), \phi_{1,v}\otimes\phi_{2,v}) = W_T\big(0, e, \omega''_{\chi_v}(\iota(g_{1,v}, g^\vee_{2,v}))\phi_{1,v}\otimes\phi_{2,v}\big) \\ = \gamma_{\mathbb{V}_v} b_{2n,v}(0)^{-1}\int_{H_v}\omega''_{\chi_v}(\iota(g_{1,v}, g^\vee_{2,v}))\phi_{1,v}\otimes\phi_{2,v}(h_v^{-1}x_T)\,d'h_v. \quad (4\text{-}56)$$

By Lemma 2.9, for almost all v, we have $\mathrm{vol}'(K_v) = 1$.

Now taking the product of (4-54), (4-55), and (4-56), we have

$$(4\text{-}53) = -b_{2n}(0)\mathrm{vol}'(K)^{-1}E_\iota(0, \iota(g_1, g_2^\vee), \phi_1\otimes\phi_2).$$

Now we take the modified measure $(2b_{2n}(0))^{-1}\prod_{v\in\Sigma_f} d'h_v$ under which we have the desired identity in Theorem 4.20. □

Appendix: Theta correspondence of spherical representations

In this appendix, we consider the theta correspondence of spherical representations for unitary groups since we cannot find literature in this case. We follow [Rallis 1984] where the symplectic-orthogonal case was discussed.

Let $F/\mathbb{Q}_p$ be a finite field extension with $p\neq 2$ and E/F an unramified quadratic field extension with $\mathrm{Gal}(E/F) = \{1, \tau\}$. Let $\mathbb{O}_F$ (resp. $\mathbb{O}_E$) be the ring of integers of F (resp. E), ϖ a uniformizer of $\mathbb{O}_F$, and q the cardinality of $\mathbb{O}_F/\varpi\mathbb{O}_F$. Let ψ be the unramified additive character of F which determines an additive character of E by composing with $\frac{1}{2}\mathrm{Tr}_{E/F}$. Let dx be the Haar measure of E which is self-dual with respect to $\psi\circ\frac{1}{2}\mathrm{Tr}_{E/F}$ and $d^\times x = dx/|x|_E$ the Haar measure of $E^\times$, normalized such that $|\varpi|_E = q^{-2}$.

Let $n, m \geq 1$ be two integers and $r = \min\{m, n\}$. Let $(W_n, \langle\,\cdot\,,\,\cdot\,\rangle)$ be a skew hermitian space over E whose skew hermitian form is given by

$$\begin{pmatrix} & \mathbf{1}_n \\ -\mathbf{1}_n & \end{pmatrix}$$

under the basis $\{e_1, \dots, e_n; e_1^*, \dots, e_n^*\}$ and $(V_m, (\cdot\,,\,\cdot))$ a hermitian space over E whose hermitian form is given by

$$\begin{pmatrix} & \mathbf{1}_m \\ \mathbf{1}_m & \end{pmatrix}$$

under the basis $\{f_1, \dots, f_m; f_1^*, \dots, f_m^*\}$. Let $H_n' = \mathrm{U}(W_n)$, $H_m = \mathrm{U}(V_m)$ and $K_n' = \mathrm{U}(W_n) \cap \mathrm{GL}_{2n}(\mathbb{O}_E)$, $K_m = \mathrm{U}(V_m) \cap \mathrm{GL}_{2m}(\mathbb{O}_E)$ be hyperspecial maximal compact subgroups. We have a Weil representation $\omega = \omega_{\mathbf{1},\psi}$ of $H_n' \times H_m$ on the space of Schwartz functions $\mathcal{S}(V_m^n)$ defined as

- $\omega\left(\begin{pmatrix} A & \\ & {}^tA^{\tau,-1} \end{pmatrix}\right)\phi(x) = |\det A|_E^m \phi(xA)$,
- $\omega\left(\begin{pmatrix} \mathbf{1}_n & B \\ & \mathbf{1}_n \end{pmatrix}\right)\phi(x) = \psi(\mathrm{tr}\, BT(x))\phi(x)$,
- $\omega\left(\begin{pmatrix} & \mathbf{1}_n \\ -\mathbf{1}_n & \end{pmatrix}\right)\phi(x) = \widehat{\phi}(x)$,
- $\omega(h)\phi(x) = \phi(h^{-1}x)$,

where $\phi \in \mathcal{S}(V_m^n)$, $A \in \mathrm{GL}_n(E)$, $B \in \mathrm{Her}_n(E)$, $h \in H_m$, and $\widehat{\phi}$ is the Fourier transform with respect to $\psi \circ \frac{1}{2}\mathrm{Tr}_{E/F}$ and dx.

Let $W_{n,i}^* = \mathrm{span}_E\{e_{i+1}^*, \dots, e_n^*\}$ for $0 \le i \le n$ and $V_{m,j}^* = \mathrm{span}_E\{f_{j+1}^*, \dots, f_m^*\}$ for $0 \le j \le m$. Then we have filtration of the maximal isotropic subspaces $W_{n,0}^*$ and $V_{m,0}^*$:

$$W_{n,0}^* \supset W_{n,1}^* \supset \cdots \supset W_{n,n}^* = \{0\}, \qquad V_{m,0}^* \supset V_{m,1}^* \supset \cdots \supset V_{m,m}^* = \{0\}.$$

Then, up to conjugacy, the maximal parabolic subgroups of $H_n' \times H_m$ are precisely those subgroups $P_{n,i}' \times P_{m,j}$ consisting of elements (h', h) stabilizing the subspace $W_{n,i}^* \otimes V_{m,j}^*$. Let $N_{n,i}' \times N_{m,j}$ be its unipotent radical. Also the Levi factor of $P_{n,i}' \times P_{m,j}$ is isomorphic to $(\mathrm{GL}_{n-i}(E) \times H_i') \times (\mathrm{GL}_{m-j}(E) \times H_j)$. We also define the algebraic closed subsets Σ_t of V_m^n for $0 \le t \le n$ to be

$$\Sigma_t = \{x = (x_1, \dots, x_n) \in V_m^n \mid (x_i, x_j) = 0 \text{ for } t+1 \le j \le n\}.$$

We say that a function $\phi \in \mathcal{S}(V_m^n)$ is *spherical* if it is invariant under the action of $K_n' \times K_m$. Then we have

Lemma A.1. *Let ϕ be a spherical function in $\mathcal{S}(V_m^n)$ such that for any $h' \in H_n'$, $\omega(h')\phi$ vanishes on the subset Σ_0, then $\omega(h')\phi$ vanishes identically.*

Proof. The proof follows exactly that of [Rallis 1984, Proposition 2.2]. □

Now we identify V_m^n with $\mathrm{Mat}_{2m\times n}(E)$ via the basis $\{f_1, \dots, f_m; f_1^*, \dots, f_m^*\}$. Then as a $\mathrm{GL}_n(E) \times H_m$-module, the action is given by $(A, h).X = hXA^{-1}$. We have the following version of [Rallis 1984, Lemma 3.1]:

Lemma A.2. *Let $\Sigma_0^{(i)} = \{X \in \Sigma_0 \mid \mathsf{rank}(X) = i\}$. Then $\Sigma_0^{(i)}$ (if nonempty) is an orbit under $\mathsf{GL}_n(E) \times H_m$ and Σ_0 is a disjoint union of orbits of the form $\Sigma_0^{(i)}$ for $i = 0, 1, \dots, r$ where $\Sigma_0^{(r)}$ is the unique open one.*

Let us review some facts about spherical representations of $H'_n \times H_m$. Consider the minimal parabolic subgroup $B'_n \times B_{m,r}$ of $H'_n \times H_m$ defined as follows. Let

$$B'_n = \left\{ \begin{pmatrix} A & \\ & {}^tA^{\tau,-1} \end{pmatrix} \begin{pmatrix} \mathbf{1}_n & B \\ & \mathbf{1}_n \end{pmatrix} \;\middle|\; A \text{ is lower triangular and } B \text{ is hermitian} \right\},$$

which has a decomposition $B'_n = T'_n \cdot U'_n$, with

$$T'_n = \{\mathsf{diag}[t_1, \dots, t_n, t_1^{\tau,-1}, \dots, t_n^{\tau,-1}] \mid t_i \in E^\times\}$$

and U'_n the unipotent radical of B'_n. Let

$$B_{m,r} = \left\{ \begin{pmatrix} A & \\ & {}^tA^{\tau,-1} \end{pmatrix} \begin{pmatrix} \mathbf{1}_m & B \\ & \mathbf{1}_m \end{pmatrix} \;\middle|\; A = \begin{pmatrix} A_1 & A_2 \\ & A_3 \end{pmatrix} \right\},$$

where $A_1 \in \mathsf{Mat}_{r\times r}(E)$ is lower triangular, $A_3 \in \mathsf{Mat}_{(m-r)\times(m-r)}(E)$ is upper triangular, $A_2 \in \mathsf{Mat}_{r\times(m-r)}(E)$, and B is skew-hermitian. We have a decomposition $B_{m,r} = T_m \cdot U_{m,r}$ where $T_m = T'_m$ and $U_{m,r}$ is the unipotent radical of $B_{m,r}$.

For $\nu = (\nu_1, \dots, \nu_n) \in \mathbb{C}^n$, we define the space $I'(\nu)$ consisting of all locally constant functions $\varphi : H'_n \to \mathbb{C}$ satisfying

$$\varphi(h' \cdot t' \cdot u') = \delta_n'^{-1/2}(t') \prod_{i=1}^{n} |t'_i|_E^{\nu_i} \varphi(h')$$

for all $h' \in H'_n$, $t \in T'_n$ and $u' \in U'_n$ where δ'_n is the modulus function of B'_n. We have $\delta'_n(t') = \prod_{i=1}^n |t'_i|_E^{2i-1}$. These $I'(\nu)$ give all spherical principal series of H'_n. Let $\mathcal{S}(H'_n /\!/ K'_n)$ be the spherical Hecke algebra of H'_n. Then we have the Fourier–Satake isomorphism $\mathsf{FS} : \mathcal{S}(H'_n /\!/ K'_n) \to \mathbb{C}[X_1, X_1^{-1}, \dots, X_n, X_n^{-1}]^{W(H'_n)}$ such that for any $f' \in \mathcal{S}(H'_n /\!/ K'_n)$,

$$\mathsf{FS}(f')(q^{2\nu_1}, q^{-2\nu_1}, \dots, q^{2\nu_n}, q^{-2\nu_n}) = \mathsf{trace}_{I'(\nu)}(f').$$

For $\mu = (\mu_1, \dots, \mu_m) \in \mathbb{C}^m$, we define the space $I(\mu)$ consisting of all locally constant functions $\varphi : H_m \to \mathbb{C}$ satisfying

$$\varphi(h \cdot t \cdot u) = \delta_{m,r}^{-1/2}(t) \prod_{j=1}^{m} |t_j|_E^{\mu_j} \varphi(h)$$

for all $h \in H_m$, $t \in T_m$, and $u \in U_{m,r}$, where $\delta_{m,r}$ is the modulus function of $B_{m,r}$. We have $\delta_{m,r}(t) = \prod_{j=1}^r |t_j|_E^{2m-2r+2j-1} \prod_{j=r+1}^m |t_j|_E^{2m-2j+1}$. These $I(\mu)$ give all spherical principal series of H_m. Let $\mathcal{S}(H_m /\!/ K_m)$ be the spherical Hecke algebra of H_m. Then we have the Fourier–Satake isomorphism $\mathsf{FS}:\mathcal{S}(H_m//K_m)\to \mathbb{C}[X_1,X_1^{-1},\dots,X_m,X_m^{-1}]^{W(H_m)}$ such that for any $f\in\mathcal{S}(H_m//K_m)$,

$$\mathsf{FS}(f)(q^{2\mu_1},q^{-2\mu_1},\dots,q^{2\mu_m},q^{-2\mu_m})=\mathsf{trace}_{I(\mu)}(f).$$

Now we are going to construct a certain explicit intertwining operator from $\mathcal{S}(V_m^n)$ to $I'(\nu)\otimes I(\mu)$. To do this, we introduce the subgroup $Y_r=A_r\cdot L_r$ of $\mathsf{GL}_r(E)$, where

$$A_r=\{\mathsf{diag}[a_1,\dots,a_r]\mid a_i\in E^\times\},\qquad L_r=\left\{\begin{pmatrix}1 & & \\ & \ddots & \\ l_{ij} & & 1\end{pmatrix}\,\middle|\, l_{ij}\in E\right\}.$$

It has a right invariant measure given by

$$dy_r=\prod_{i=1}^r|a_i|_E^{2i-(r+1)}d^\times a_i\prod_{1\le j<i\le r}\tilde{d}l_{i,j},$$

where $\tilde{d}l_{i,j}$ is a certain measure on L_r normalized as in [Rallis 1984, p. 490]. For $\sigma=(\sigma_1,\dots,\sigma_r)\in\mathbb{C}^r$ such that $\Re(\sigma_i)\gg 0$, the integral

$$Z_\sigma(\phi)=\int_{Y_r}\phi\left(\begin{pmatrix}y_r & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{pmatrix}\right)\prod_{i=1}^r|a_i|_E^{\sigma}\,dy_r$$

is absolutely convergent. We define a functional $\mathfrak{Z}_\sigma$ sending ϕ to the function $(h',h)\mapsto Z_\sigma(\omega(h'^{-1},h^{-1})\phi)$. It is a nonzero $H'_n\times H_m$-intertwining map from $\mathcal{S}(V_m^n)$ to $\mathcal{S}(H'_n\times H_m)$; moreover:

Lemma A.3. *For $\Re(\sigma_i)\gg 0$, the image of the above intertwining map $\mathfrak{Z}_\sigma$ lies in $I'(\nu)\otimes I(\mu)$ where*

$$\nu=\left(2+\sigma_1-m-\tfrac{3}{2},\dots,2r+\sigma_r-m-\tfrac{3}{2},(r+1)-m-\tfrac{1}{2},\dots,n-m-\tfrac{1}{2}\right),$$
$$\mu=\left(-2-\sigma_1+m+\tfrac{3}{2},\dots,-2r+\sigma_r+m+\tfrac{3}{2},-(r+1)+m+\tfrac{1}{2},\dots,\tfrac{1}{2}\right).$$

Proof. We have

$$\begin{aligned}\mathfrak{Z}_\sigma(\phi)(h't'u',htu)&=\int_{Y_r}\omega(u'^{-1}t'^{-1}h'^{-1},u^{-1}t^{-1}h^{-1})\phi\left(\begin{pmatrix}y_r & \\ & \end{pmatrix}\right)\prod_{i=1}^r|a_i|_E^{\sigma}\,dy_r\\ &=\int_{Y_r}\omega(t'^{-1}h'^{-1},t^{-1}h^{-1})\phi\left(\begin{pmatrix}y_r & \\ & \end{pmatrix}\right)\prod_{i=1}^r|a_i|_E^{\sigma}\,dy_r\\ &=\int_{Y_r}|\det t'|_E^{-m}\omega(h'^{-1},h^{-1})\phi\left(t\begin{pmatrix}y_r & \\ & \end{pmatrix}t'^{-1}\right)\prod_{i=1}^r|a_i|_E^{\sigma}dy_r.\end{aligned}\tag{A.1}$$

Changing the variable $y_r \mapsto y''_r = \mathrm{diag}[t_1, \ldots, t_r] y_r$, we have

$$\prod_{i=1}^{r} |a_i|_E^{\sigma} \, dy_r = \prod_{i=1}^{r} |t_i|_E^{r-3i-\sigma_i+2} \prod_{i=1}^{r} |t_i a_i|_E^{\sigma} \, dy''_r.$$

Changing the variable $y_r \mapsto y'_r = \mathrm{diag}[t_1'^{-1}, \ldots, t_r'^{-1}]$, then

$$\prod_{i=1}^{r} |a_i|_E^{\sigma} \, dy_r = \prod_{i=1}^{r} |t_i|_E^{i+\sigma_i-1} \prod_{i=1}^{r} |t_i'^{-1} a_i|_E^{\sigma} \, dy'_r.$$

Hence

$$\begin{aligned}(\mathrm{A}.1) = \prod_{i=1}^{r} |t'_i|_E^{i+\sigma_i-m-1} \prod_{i=r+1}^{n} |t'_i|_E^{-m} \prod_{j=1}^{r} |t_j|_E^{r-3j-\sigma_j+2} & \\ \times \int_{Y_r} \omega(h'^{-1}, h^{-1}) \phi\left(\begin{pmatrix} y_r & \\ & \end{pmatrix}\right) \prod_{i=1}^{r} |a_i|_E^{\sigma} \, dy_r & \\ = \prod_{i=1}^{r} |t'_i|_E^{i+\sigma_i-m-1} \prod_{i=r+1}^{n} |t'_i|_E^{-m} \prod_{j=1}^{r} |t_j|_E^{r-3j-\sigma_j+2} \mathfrak{L}_{\sigma}(\phi)(h', h). & \qquad \square\end{aligned}$$

From this lemma, it is easy to see the following. If $m \geq n = r$, there is a surjective homomorphism $\Phi_{m,n} : \mathcal{S}(H_m//K_m) \to \mathcal{S}(H'_n//K'_n)$ which has the property

$$\mathfrak{L}_{\sigma} \circ (\Phi_{m,n}(f) - f) = 0$$

for all $f \in \mathcal{S}(H_m//K_m)$ and $\Re(\sigma_i) \gg 0$. Using the Fourier–Satake isomorphism, the map $\Phi_{m,n}$ is given by

$$\mathbb{C}[X_1, X_1^{-1}, \ldots, X_m, X_m^{-1}]^{W(H_m)} \longrightarrow \mathbb{C}[X_1, X_1^{-1}, \ldots, X_n, X_n^{-1}]^{W(H'_n)},$$

where

$$\begin{aligned} \log_q X_j &\mapsto \log_q X_j, & j &= 1, \ldots, n, \\ \log_q X_j &\mapsto 2m - 2j + 1, & j &= n+1, \ldots, m. \end{aligned}$$

In particular, when $m = n$, $\Phi_{m,n}$ is the identity map.

If $n > m = r$, similarly there is a surjective homomorphism $\Phi'_{n,m} : \mathcal{S}(H'_n//K'_n) \to \mathcal{S}(H_m//K_m)$ which has the property

$$\mathfrak{L}_{\sigma} \circ (f' - \Phi'_{n,m}(f')) = 0$$

for all $f' \in \mathcal{S}(H'_n//K'_n)$ and $\Re(\sigma_i) \gg 0$. Using the Fourier–Satake isomorphism, the map $\Phi'_{n,m}$ is given by

$$\mathbb{C}[X_1, X_1^{-1}, \ldots, X_n, X_n^{-1}]^{W(H'_n)} \longrightarrow \mathbb{C}[X_1, X_1^{-1}, \ldots, X_m, X_m^{-1}]^{W(H_m)},$$

where

$$\log_q X_i \mapsto \log_q X_i, \qquad i = 1, \dots, m,$$
$$\log_q X_i \mapsto 2m - 2i + 1, \qquad i = m+1, \dots, n.$$

In fact, we have

Lemma A.4. *Let $\phi \in \mathscr{S}(V_m^n)$ be spherical and $Z_\sigma(\phi) \equiv 0$ for all $\sigma \in \mathbb{C}^r$ with $\Re(\sigma_i) \gg 0$. Then $\omega(h')\phi$ vanishes on Σ_0 for all $h' \in H_n'$.*

Proof. It suffices to show that $\omega(h')\phi$ vanishes on $\Sigma_0^{(r)}$ since it is dense open in Σ_0. Since $\Sigma_0^{(r)}$ is a $\mathsf{GL}_n(E) \times H_m$-orbit, we only need to show that

$$\omega(h', h)\phi\left(\begin{pmatrix} \mathbf{1}_r & \end{pmatrix}\right) \equiv 0$$

for all (h', h). But we can write $h' = b'k'$ with $b' \in B_n'$, $k' \in K_n'$, and $h = bk$ with $b \in B'_{m,r}$ and $k \in K_m$. Then since ϕ is spherical, we have

$$\omega(h', h)\phi\left(\begin{pmatrix} \mathbf{1}_r & \end{pmatrix}\right) = \omega(b', b)\phi\left(\begin{pmatrix} \mathbf{1}_r & \end{pmatrix}\right) = \phi\left(\begin{pmatrix} X & \end{pmatrix}\right)$$

with $X \in \mathsf{Mat}_{r\times r}(E)$. Hence the lemma follows from [Rallis 1984, Lemma 5.2] for $k = E$. □

Combining Lemmas A.1, A.3, and A.4, we have

Proposition A.5. *The ideal*

$$\mathscr{I}_{n,m} = \{f \in \mathscr{S}(H_n'//K_n') \otimes \mathscr{S}(H_m//K_m) \mid \omega(f) \equiv 0\}$$

is generated by

$$\{\Phi_{m,n}(f) - f \mid f \in \mathscr{S}(H_m//K_m)\} \quad (\textit{resp. } \{f' - \Phi'_{n,m}(f') \mid f' \in \mathscr{S}(H_n'//K_n')\})$$

if $m \geq n$ (resp. $m < n$).

We have a similar result for the Weil representation of $\mathsf{GL}_n(F) \times \mathsf{GL}_m(F)$ on $\mathscr{S}(\mathsf{Mat}_{m\times n}(F))$ given by $\omega(g', g)\phi(x) = \phi(g^{-1}xg')$ (see [Rallis 1984, Section 6]). Without lost of generality, we assume that $n \geq m$; then the ideal

$$\mathscr{I}_{n,m} = \{f \in \mathscr{S}(\mathsf{GL}_n(F)//\mathsf{GL}_n(\mathbb{O}_F)) \otimes \mathscr{S}(\mathsf{GL}_m(F)//\mathsf{GL}_m(\mathbb{O}_F)) \mid \omega(f) \equiv 0\}$$

is generated by

$$\{f - \Psi_{n,m}(f) \mid f \in \mathscr{S}(\mathsf{GL}_n(F)//\mathsf{GL}_n(\mathbb{O}_F))\}.$$

In terms of the Fourier–Satake isomorphism, the surjective homomorphism $\Psi_{n,m}$ is given by

$$\mathbb{C}[X_1, X_1^{-1}, \dots, X_n, X_n^{-1}]^{W(\mathsf{GL}_n(F))} \longrightarrow \mathbb{C}[X_1, X_1^{-1}, \dots, X_m, X_m^{-1}]^{W(\mathsf{GL}_m(F))},$$

where

$$\begin{aligned} \log_q X_i &\mapsto -\log_q X_i + \frac{n-m}{2}, & i &= 1, \ldots, m, \\ \log_q X_i &\mapsto -i + \frac{n+1}{2}, & i &= m+1, \ldots, n. \end{aligned}$$

Corollary A.6. (1) *If π is an unramified irreducible admissible representation of H'_n, then the theta correspondence of π to $H_n \cong H'_n$ is nontrivial and isomorphic to π, that is, $\theta_{\mathbf{1}}(\pi, V_n) = \pi$.*

(2) *If π is an unramified irreducible admissible representation of $\mathsf{GL}_n(F)$ and χ an unramified character of $F^\times$, then the theta correspondence of π to $\mathsf{GL}_n(F)$ through the Weil representation ω_χ, where $\omega_\chi(g', g)\phi(x) = \chi(\det g')\phi(g^{-1}xg')$ for $\phi \in \mathcal{S}(\mathsf{Mat}_{n\times n}(F))$, is nontrivial and isomorphic to $\pi^\vee \otimes \chi$.*

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Arithmetic theta lifting and L-derivatives for unitary groups, II

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We prove the arithmetic inner product formula conjectured in the first paper of this series for $n = 1$, that is, for the group $\mathsf{U}(1, 1)_F$ unconditionally. The formula relates central L-derivatives of weight-2 holomorphic cuspidal automorphic representations of $\mathsf{U}(1, 1)_F$ with ϵ-factor -1 with the Néron–Tate height pairing of special cycles on Shimura curves of unitary groups. In particular, we treat all kinds of ramification in a uniform way. This generalizes the arithmetic inner product formula obtained by Kudla, Rapoport, and Yang, which holds for certain cusp eigenforms of $\mathsf{PGL}(2)_{\mathbb{Q}}$ of square-free level.

1. Introduction

The Birch–Swinnerton-Dyer conjecture predicts a deep relation between rational points on rational elliptic curves and the associated analytic object called the Hasse–Weil zeta function or L-function. This conjecture has also been generalized to higher dimensions and to more general varieties and motives by Beilinson, Bloch and others. Gross and Zagier [1986] studied the relation between the central derivative of the L-function of a rational elliptic curve and the height pairing of Heegner points on it, through the arithmetic theory of modular curves and Rankin L-series. After elaborate computations, they obtained the famous Gross–Zagier formula, which is exactly predicted by the Birch–Swinnerton-Dyer conjecture. This was

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generalized by S.-W. Zhang [2001a; 2001b] to the case of Shimura curves over totally real fields. The complete version of the Gross–Zagier formula was recently achieved by X. Yuan, S.-W. Zhang and W. Zhang [Yuan et al. 2011]. Bruinier and Yang [2009] used regularized theta lifting and relate the inner product to L-derivatives to give another proof of the original Gross–Zagier formula. Certain p-adic (or rigid analytic) versions of the Gross–Zagier formula have been studied in [Bertolini and Darmon 1997; 1998].

Kudla [1997; 2002; 2003; Kudla et al. 2006] found another way to study L-derivatives, or more generally, derivatives of Siegel Eisenstein series. It was his great discovery that the theory of doubling integrals established in [Gelbart et al. 1987] can be used instead of the classical Rankin–Selberg convolution and that derivatives of (Siegel) Eisenstein series are also related to the height pairing of certain arithmetic objects. His project on the arithmetic Siegel–Weil formula sheds new light on this area. More importantly, the idea should work for higher dimensions and in both symplectic-orthogonal and unitary cases. Kudla, Rapoport, and Yang [Kudla et al. 2006] have also proved a special form of the arithmetic inner product formula for quaternion Shimura curves over $\mathbb{Q}$ of minimal level.

Extending that work, we set up in [Liu 2011] a general, explicit formulation of arithmetic theta lifting. Conjecture 3.11 of that paper gave an arithmetic inner product formula for unitary groups; we also proved the modularity theorem for the generating series (Theorem 3.5) and an archimedean arithmetic Siegel–Weil formula for any dimension (Theorems 4.17 and 4.20) predicted by this formulation. In this second paper, we prove the complete version of the arithmetic inner product formula for unitary groups of two variables over totally real fields.

The following is a detailed introduction. Let F be a totally real field, E/F a quadratic imaginary extension, τ the nontrivial Galois involution, $\epsilon_{E/F}$ the associated quadratic character by class field theory, and ψ an additive character of $F\backslash\mathbb{A}_F$, standard at archimedean places. For $n \geq 1$, let H_n be the unitary group over F such that for any F-algebra R, $H_n(R) = \{h \in \mathrm{GL}_{2n}(E \otimes_F R) \mid {}^t h^\tau w_n h = w_n\}$ where

$$w_n = \begin{pmatrix} & \mathbf{1}_n \\ -\mathbf{1}_n & \end{pmatrix}.$$

The center of H_n is the F-torus $E^{\times,1} = \ker[\mathrm{Nm} : E^\times \to F^\times]$. Let π be an irreducible cuspidal automorphic representation of H_n and $\pi^\vee$ its contragredient. Let χ be a character of $\mathbb{A}_E^\times$ which is trivial on $E^\times \mathbb{A}_F^\times$.

By the theta dichotomy proved in [Paul 1998; Gong and Grenié 2011], we get a factor $\epsilon(\pi, \chi)$ (see Section 2A for a precise definition) which is the product of local ones $\epsilon(\pi_v, \chi_v)$ for each place v of F, such that $\epsilon(\pi_v, \chi_v) \in \{\pm 1\}$ and $\epsilon(\pi_v, \chi_v) = 1$ for almost all v. Although it is conjectured that this $\epsilon(\pi_v, \chi_v)$ is related to the local ϵ-factor in representation theory (see [Harris et al. 1996]), it is not the same, according to our definition. From these local factors, we can construct a hermitian space $\mathbb{V}(\pi,\chi)$ over $\mathbb{A}_E$ of rank $2n$ which is coherent (resp. incoherent) if $\epsilon(\pi,\chi)=1$ (resp. -1). When $\epsilon(\pi,\chi)=1$, we get the usual (generalized) Rallis inner product formula (see [Kudla and Rallis 1994; Ichino 2004; 2007], also [Liu 2011, Section 2] in our setting).

Now let us assume $n=1$ and $\epsilon(\pi,\chi)=-1$. Then the central L-value $L(\frac{1}{2},\pi,\chi)$ vanishes where the global L-function $L(s,\pi,\chi)=\prod_v(s,\pi_v,\chi_v)$ is the product of local ones, which are essentially defined as the common denominators of local zeta integrals by Piatetski-Shapiro and Rallis (see [Gelbart et al. 1987; Harris et al. 1996]; this will be recalled in Section 2A). It is natural to ask the value of $L'(\frac{1}{2},\pi,\chi)$. For this purpose, we further assume that for any archimedean place ι of F, π_ι is a discrete series representation of weight 2 such that its central character $\omega_{\pi_\iota}=\chi_\iota^{-1}$. Then the corresponding $\mathbb{V}(\pi,\chi)$ is incoherent and totally positive definite of rank 2. Now for any hermitian space $\mathbb{V}$ over $\mathbb{A}_E$ which is incoherent and totally positive definite of rank 2, let $\mathbb{H}=\mathrm{Res}_{\mathbb{A}_F/\mathbb{A}}\mathrm{U}(\mathbb{V})$ be the corresponding unitary group. Then we can construct a projective system of unitary Shimura curves $(\mathrm{Sh}_K(\mathbb{H}))_K$, smooth and quasiprojective over E, where K is a sufficiently small open compact subgroup of $\mathbb{H}(\mathbb{A}_f)$. These curves are nonproper if and only if $F=\mathbb{Q}$ and $\epsilon(\pi_v,\chi_v)=1$ for all finite places v of F. In any case, we denote by $(M_K)_K$ the (compactified, if necessary) system of unitary Shimura curves for simplicity. For any $f\in\pi$ and Schwartz function $\phi\in\mathcal{S}(\mathbb{V})^{\mathrm{U}_\infty}$ (see Section 3B), we construct a cycle Θ_ϕ^f, called the arithmetic theta lifting, which is a divisor on M_K of degree 0 for any K fixing ϕ, through the Weil representation ω_χ. On the contragredient side, we also have $\Theta_{\phi^\vee}^{f^\vee}$ for $f^\vee\in\pi^\vee$ (but through $\omega_\chi^\vee$). We prove the following *arithmetic inner product formula for* $\mathrm{U}(1,1)_F$:

Theorem 1.1. *Let π, χ be as above and let $\mathbb{V}$ be any totally positive-definite incoherent hermitian space over $\mathbb{A}_E$ of rank* 2. *Then*

(1) *If $\mathbb{V}\not\cong\mathbb{V}(\pi,\chi)$, then the arithmetic theta lifting Θ_ϕ^f is a torsion class for any $f\in\pi$ and $\phi\in\mathcal{S}(\mathbb{V})^{\mathrm{U}_\infty}$.*

(2) *If $\mathbb{V}\cong\mathbb{V}(\pi,\chi)$, then for any $f\in\pi$, $f^\vee\in\pi^\vee$ and any $\phi,\phi^\vee\in\mathcal{S}(\mathbb{V})^{\mathrm{U}_\infty}$ decomposable, we have*

$$\langle\Theta_\phi^f,\Theta_{\phi^\vee}^{f^\vee}\rangle_{\mathrm{NT}}=\frac{L'(\frac{1}{2},\pi,\chi)}{L_F(2)L(1,\epsilon_{E/F})}\prod_v Z^*(0,\chi_v,f_v,f_v^\vee,\phi_v\otimes\phi_v^\vee),$$

where we take the Néron–Tate height pairing on some M_K (same as Beilinson–Bloch pairing on curves) such that ϕ and $\phi^\vee$ are invariant under K and we normalize it by a volume factor such that the resulting pairing is independent of the K we choose. The terms Z^* in the product are normalized local zeta integrals defined in Section 2A, of which almost all are 1.

We remark that the L-function $L(s, \pi, \chi)$ defined by Piatetski-Shapiro and Rallis (see [Harris et al. 1996] for a detailed definition for the unitary group case) coincides with $L(s, \mathrm{BC}(\pi) \otimes \chi)$ when $n = 1$; this is conjectured to be true for any n. In particular, the set of L-derivatives appearing in the above main theorem is exactly the same as in the Gross–Zagier formula in full generality, recently proved in [Yuan et al. 2011].

Our basic idea is similar to that of [Kudla 1997; Kudla et al. 2006]. The difference is that those works consider a certain integral model of the Shimura curve associated to a $\mathbb{Q}$-quaternion algebra and view the generating series and hence the arithmetic theta lifting as Arakelov divisors on that integral model. It has a canonical integral model in their minimal level case. But for general-level structures and even higher dimensional Shimura varieties, it is not all known. Instead, we work over canonical models of (unitary) Shimura varieties over reflex fields and define the generating series and the arithmetic theta lifting as usual Chow (co)cycles. In this way, we can formulate a precise version of the arithmetic inner product formula assuming that the Beilinson–Bloch height pairing, which is just the Néron–Tate pairing in the case of curves, is well-defined. At least in the case of $\mathrm{U}(1, 1)_F$, everything is well-defined.

For the proof, we use theories of Siegel Eisenstein series, Arakelov geometry, local heights, and p-divisible groups. The geometry part of this method actually goes back to [Gross and Zagier 1986]. Instead of explicit place-by-place computation (which is possible in the minimal level case) as in [Kudla et al. 2006], we greatly use the theory of theta lifting, certain multiplicity one results, modularity of the generating series, and various techniques for choosing test functions to avoid explicit computations at bad places which are almost impossible in the case of general levels. This allows us to prove the result for all kinds of ramification, from both representations and geometry, in a uniform way. This new idea was first proposed by Yuan, Zhang, and Zhang, and was used in their recent work on the general Gross-Zagier formula and the arithmetic triple product formula [Yuan et al. 2010; 2011].

The paper is organized as follows. In Section 2, we start by reviewing the method of doubling integrals, especially the integral presentation of L-functions and L-derivatives for unitary groups. In particular, we recall the analytic kernel function $E'(0, g, \Phi)$. Usually, it is extremely difficult to calculate its Fourier coefficients explicitly. But we prove later in the section that for a certain "nice" choice of test functions, we can kill all irregular Fourier coefficients and even arbitrary finitely many derivatives of regular ones. This nice choice is quite delicate and hence not easy to describe at this point. Finally, we have the following decomposition for nice Φ and g in a subgroup of $H_{2n}(\mathbb{A}_F)$ which is dense in $H_{2n}(F)\backslash H_{2n}(\mathbb{A}_F)$:

$$E'(0, g, \Phi) = \sum_{v \notin S} E_v(0, g, \Phi),$$

where S is a certain finite set of finite places of F which are "bad". The term $E_v(0, g, \Phi)$ is a sum of products of local Whittaker functions away from v and their derivatives at v; it is 0 if v is split in E.

In Section 3, we review the definition of Néron–Tate and Beilinson–Bloch height pairing on curves over number fields. Using this, we have a parallel construction of the kernel function for the height-pairing side when $n = 1$, namely the geometric kernel function $\mathbb{E}(g_1, g_2; \phi_1 \otimes \phi_2)$ which is essentially the height pairing of two generating series $Z_{\phi_1}(g_1)$ and $Z_{\phi_2}(g_2)$. Thanks to the theorem of modularity of the generating series proved in [Liu 2011], it is not difficult to see that $\mathbb{E}(g_1, g_2; \phi_1 \otimes \phi_2)$ is an automorphic form of $H_1 \times H_1$. Analogous to the analytic side, for nice choice of $\phi_1 \otimes \phi_2$, we have a decomposition for g_i inside a subgroup of $H_1(\mathbb{A}_F)$ which is dense in $H_1(F)\backslash H_1(\mathbb{A}_F)$:

$$\begin{aligned}\mathbb{E}(g_1, g_2; \phi_1 \otimes \phi_2) = -\mathsf{vol}(K) \sum_{v^\circ \in \Sigma^\circ} \langle \widehat{Z}_{\phi_1}(g_1), \widehat{Z}_{\phi_2}(g_2) \rangle_{v^\circ} \\ + \text{Eisenstein series and (possibly) automorphic characters,}\end{aligned}$$

where v° takes over all places of E and the local height pairing is taken over a certain integral model of M_K. The terms of automorphic characters appear only in the case where the original Shimura curve is nonproper due to the nonvanishing of a certain intertwining operator.

Section 4 is dedicated to comparing the corresponding terms in two kernel functions for good finite places, namely the analytic side $E_v(0, \iota(g_1, g_2^\vee), \phi_1 \otimes \phi_2)$ and the geometric side $\langle \widehat{Z}_{\phi_1}(g_1), \widehat{Z}_{\phi_2}(g_2) \rangle_{v^\circ}$ with $v^\circ | v$.

Section 5 is dedicated to treating bad places appearing only on the geometric side. We prove that, for nicely chosen $\phi_1 \otimes \phi_2$, these (finitely many nonzero) height pairings are Eisenstein series and theta series.

We reach the final stage of the proof in Section 6. First, we introduce the notion of holomorphic projection and compute that for the analytic kernel function. By the comparison theorem at infinite places proved in [Liu 2011, Section 4], it turns out that after doing holomorphic projection, we will get the correct Green's function. Second, the difference between the (holomorphic projection of the) analytic kernel function and the geometric kernel function

$$\mathsf{Pr}\big(E'(0, \iota(g_1, g_2^\vee), \phi_1 \otimes \phi_2)\big) - \mathbb{E}(g_1, g_2; \phi_1 \otimes \phi_2)$$

is now a linear combination of Eisenstein series, automorphic characters (that is, one-dimensional automorphic representations), and theta series for (g_1, g_2) inside a subgroup of $H_1(\mathbb{A}_F) \times H_1(\mathbb{A}_F)$ which is dense in $H_1(F)\backslash H_1(\mathbb{A}_F) \times H_1(F)\backslash H_1(\mathbb{A}_F)$.

But the key thing is that they are both automorphic forms; hence they really differ by a linear combination of Eisenstein series, automorphic characters, and theta series. Now we integrate automorphic forms $f \in \pi$ and $f^\vee \in \pi^\vee$ with this difference and get zero since π is cuspidal and $\epsilon(\pi, \chi) = -1$! This has already implied the arithmetic inner product formula but only for nicely chosen $\phi \otimes \phi^\vee$. To obtain the full formula, we need to use the multiplicity-one result proved in Section 6B. We introduce functionals

$$\alpha(f, f^\vee, \phi, \phi^\vee) := \prod_v Z^*(0, \chi_v, f_v, f_v^\vee, \phi_v \otimes \phi_v^\vee),$$
$$\gamma(f, f^\vee, \phi, \phi^\vee) := \langle \Theta_\phi^f, \Theta_{\phi^\vee}^{f^\vee} \rangle_{\mathsf{NT}},$$

which are obviously inside $\mathrm{Hom}_{H_1(\mathbb{A}_F) \times H_1(\mathbb{A}_F)}(R(\mathbb{V}, \chi), \pi^\vee \boxtimes \chi\pi)$, whose dimension is 1 when $\mathbb{V} = \mathbb{V}(\pi, \chi)$. Moreover, by [Harris et al. 1996] we know that as a functional, $\alpha \neq 0$. Hence γ is a constant multiple of α. To calculate this constant, we only need to plug in certain $f, f^\vee, \phi, \phi^\vee$ such that $\alpha(f, f^\vee, \phi, \phi^\vee) \neq 0$. By the density result proved in Section 2D, we can choose nice $\phi \otimes \phi^\vee$ and $f, f^\vee$ such that $\alpha(f, f^\vee, \phi, \phi^\vee) \neq 0$ where the constant has already been computed. As a consequence, we obtain the arithmetic inner product formula for any $f, f^\vee, \phi, \phi^\vee$.

The following conventions hold throughout this paper.

- $\mathbb{A}_f = \widehat{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q} = (\varprojlim_N \mathbb{Z}/N\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}$ is the ring of finite adèles, $\mathbb{A} = \mathbb{A}_f \times \mathbb{R}$ is the ring of full adèles.
- For any number field K, $\mathbb{A}_K = \mathbb{A} \otimes_{\mathbb{Q}} K$, $\mathbb{A}_{f,K} = \mathbb{A}_f \otimes_{\mathbb{Q}} K$, $K_\infty = K \otimes_{\mathbb{Q}} \mathbb{R}$, and $\Gamma_K = \mathrm{Gal}(K^{\mathrm{ac}}/K)$ is the Galois group of K.
- As usual, for a subset S of places, $-_S$ (resp. $-^S$) means the S-component (resp. component away from S) for the corresponding (decomposable) adèlic object; $-_\infty$ (resp. $-_f$) is the infinite (resp. finite) part.
- The symbols Tr and Nm mean the trace (resp. reduced trace) and norm (resp. reduced norm) if they apply to fields or rings of adèles (resp. simple algebras), and tr means the trace for matrices and linear transformations.
- $\mathbf{1}_n$ and $\mathbf{0}_n$ are the $n \times n$ identity and zero matrices; ${}^t g$ is the transpose of a matrix g.
- All (skew-)hermitian spaces and quadratic spaces are assumed to be nondegenerate.
- For a ring R, sometimes R also stands for its spectrum $\mathrm{Spec}\, R$ or $\mathrm{Spf}\, R$ (if it causes no confusion) according to the context.
- For a scheme X over a field K, we let $\mathrm{Pic}(X)$ be the Picard group of X over K, not the Picard scheme.

2. Analytic kernel functions

2A. ***Doubling method.*** We briefly recall results mainly from [Gelbart et al. 1987; Li 1992; Harris et al. 1996] with the setups and notation of [Liu 2011, Section 2].

Let F be a totally real field and E a totally imaginary quadratic extension of F. We denote by τ the nontrivial element in $\mathrm{Gal}(E/F)$ and $\epsilon_{E/F} : \mathbb{A}_F^\times/F^\times \to \{\pm 1\}$ the associated character by class field theory. Let Σ (resp. Σ_f; resp. Σ_∞) be the set of all places (resp. finite places; resp. infinite places) of F, and Σ°, Σ_f°, and Σ_∞° those of E. We fix a nontrivial additive character ψ of $\mathbb{A}_F/F$.

For a positive integer r, we denote by W_r the standard skew-hermitian space over E with respect to the involution τ, which has a skew-hermitian form $\langle\,\cdot\,,\cdot\rangle$ such that there is an E-basis $\{e_1, \dots, e_{2r}\}$ satisfying $\langle e_i, e_j\rangle = 0$, $\langle e_{r+i}, e_{r+j}\rangle = 0$ and $\langle e_i, e_{r+j}\rangle = \delta_{ij}$ for $1 \le i, j \le r$. Let $H_r = \mathrm{U}(W_r)$ be the unitary group of W_r which is a reductive group over F. The group $H_r(F)$, in which F can be itself or its completion at some place, is generated by the standard parabolic subgroup $P_r(F) = N_r(F)M_r(F)$ and the element w_r. Precisely,

$$N_r(F) = \left\{ n(b) = \begin{pmatrix} \mathbf{1}_r & b \\ & \mathbf{1}_r \end{pmatrix} \;\middle|\; b \in \mathrm{Her}_r(E) \right\},$$

$$M_r(F) = \left\{ m(a) = \begin{pmatrix} a & \\ & {}^t a^{\tau,-1} \end{pmatrix} \;\middle|\; a \in \mathrm{GL}_r(E) \right\},$$

and

$$w_r = \begin{pmatrix} & \mathbf{1}_r \\ -\mathbf{1}_r & \end{pmatrix},$$

where $\mathrm{Her}_r(E) = \{ b \in \mathrm{Mat}_r(E) \mid b^\tau = {}^t b \}$.

We fix a place $v \in \Sigma$ and suppress it from the notation. Thus $F = F_v$ is a local field of characteristic zero, $E = E_v$ is a quadratic extension of F which may be split and $H_r = H_{r,v} = H_r(F_v)$ is a local reductive group. Also, we denote by $\mathcal{K}_r$ the maximal compact subgroup of H_r which is the intersection of H_r with $\mathrm{GL}_{2n}(\mathcal{O}_E)$ (resp. is isomorphic to $\mathrm{U}(r) \times \mathrm{U}(r)$) if v is finite (resp. if v is infinite). For $s \in \mathbb{C}$ and a character χ of $E^\times$, we denote by $I_r(s, \chi) = \text{s-Ind}_{P_r}^{H_r}(\chi|\cdot|_E^{s+r/2})$ the degenerate principal series representation (see [Kudla and Sweet 1997]) of H_r, where s-Ind means the unnormalized smooth induction. Precisely, it realizes on the space of $\mathcal{K}_r$-finite functions φ_s on H_r satisfying

$$\varphi_s(n(b)m(a)g) = \chi(\det a)|\det a|_E^{s+r/2}\varphi_s(g)$$

for all $g \in H_r$, $m(a) \in M_r$, and $n(b) \in N_r$. A (holomorphic) section φ_s of $I_r(s, \chi)$ is called *standard* if its restriction to $\mathcal{K}_r$ is independent of s. It is called *unramified* if it takes value 1 on $\mathcal{K}_r$. Now we view F and E as number fields. For a character χ of $\mathbb{A}_E^\times$ which is trivial on $E^\times$ and $s \in \mathbb{C}$, we have an admissible representation

$I_r(s,\chi)=\bigotimes' I_r(s,\chi_v)$ of $H_r(\mathbb{A}_F)$, where the restricted tensor product is taken with respect to the unramified sections.

Let us have a quick review of the classification of (nondegenerate) hermitian spaces. If $v\in\Sigma_f$ and E is nonsplit at v, there are, up to isometry, two different hermitian spaces over E_v of dimension $m\geq 1$: $V^{\pm}$ defined by

$$\epsilon(V^{\pm})=\epsilon_{E/F}\big((-1)^{m(m-1)/2}\det V^{\pm}\big)=\pm 1.$$

If $v\in\Sigma_f$ and E is split at v, there is, up to isometry, only one hermitian space V^+ over E_v of dimension m. If $v\in\Sigma_\infty$, there are, up to isometry, $m+1$ different hermitian spaces over E_v of dimension m: V_s with signature $(s,m-s)$ where $0\leq s\leq m$. In the later two cases, we can still define $\epsilon(V)$ in the same way. In the global case, up to isometry, all hermitian spaces V over E of dimension m are classified by signatures at infinite places and $\det V\in F^\times/\mathsf{Nm}E^\times$; particularly, V is determined by all $V_v=V\otimes_F F_v$. In general, we will also consider a hermitian space $\mathbb{V}$ over $\mathbb{A}_E$ of rank m. In this case, $\mathbb{V}$ is nondegenerate if there is a basis under which the representing matrix is invertible in $\mathsf{GL}_m(\mathbb{A}_E)$. For any place $v\in\Sigma$, we let $\mathbb{V}_v=\mathbb{V}\otimes_{\mathbb{A}_F}F_v$, $\mathbb{V}_f=\mathbb{V}\otimes_{\mathbb{A}_F}\mathbb{A}_{f,F}$, and define $\Sigma(\mathbb{V})=\{v\in\Sigma\mid\epsilon(\mathbb{V}_v)=-1\}$, which is a finite set, and $\epsilon(\mathbb{V})=\prod\epsilon(\mathbb{V}_v)$. We say $\mathbb{V}$ is *coherent* (resp. *incoherent*) if the cardinality of $\Sigma(\mathbb{V})$ is even (resp. odd), that is, $\epsilon(\mathbb{V})=1$ (resp. -1). By the Hasse principle, there is a hermitian space V over E such that $\mathbb{V}\cong V\otimes_F\mathbb{A}_F$ if and only if $\mathbb{V}$ is coherent. These two terminologies are introduced in the orthogonal case in [Kudla and Rallis 1994]; see also [Kudla 1997].

We fix a place $v\in\Sigma$ and suppress it from the notation. For a hermitian space V of dimension m with hermitian form $(\,\cdot\,,\cdot\,)$ and a positive integer r, we can construct a symplectic space $\boldsymbol{W}=\mathsf{Res}_{E/F}W_r\otimes_E V$ of dimension $4rm$ over F with the skew-symmetric form $\frac{1}{2}\mathsf{Tr}_{E/F}\langle\,\cdot\,,\cdot\,\rangle^\tau\otimes(\,\cdot\,,\cdot\,)$. We let $H=\mathsf{U}(V)$ be the unitary group of V and $\mathcal{S}(V^r)$ the space of Schwartz functions on V^r. Given a character χ of $E^\times$ satisfying $\chi\big|_{F^\times}=\epsilon_{E/F}^m$, we have a splitting homomorphism

$$\tilde{\iota}_{(\chi,\mathbf{1})}:H_r\times H\to\mathsf{Mp}(\boldsymbol{W})$$

lifting the natural map $\iota:H_r\times H\to\mathsf{Sp}(\boldsymbol{W})$ (see [Harris et al. 1996, Section 1]). We thus have a Weil representation (with respect to ψ) $\omega_\chi=\omega_{\chi,\psi}$ of $H_r\times H$ on the space $\mathcal{S}(V^r)$. Explicitly, for $\phi\in\mathcal{S}(V^r)$ and $h\in H$, we have:

- $\omega_\chi(n(b))\phi(x)=\psi(\mathrm{tr}\,bT(x))\phi(x)$.
- $\omega_\chi(m(a))\phi(x)=|\det a|_E^{m/2}\chi(\det a)\phi(xa)$.
- $\omega_\chi(w_r)\phi(x)=\gamma_V\widehat{\phi}(x)$.
- $\omega_\chi(h)\phi(x)=\phi(h^{-1}x)$,

where $T(x)=\frac{1}{2}\left((x_i,x_j)\right)_{1\le i,j\le r}$ is the moment matrix of x, γ_V is the Weil constant associated to the underlying quadratic space of V (and also ψ), and $\widehat{\phi}$ is the Fourier transform

$$\widehat{\phi}(x)=\int_{V^r}\phi(y)\psi\left(\tfrac{1}{2}\mathrm{Tr}_{E/F}(x,{}^t y)\right)dy,$$

using the self-dual measure dy on V^r with respect to ψ. Taking the restricted tensor product over all local Weil representations, we get a global $\mathcal{S}(V^r):=\bigotimes'\mathcal{S}(V_v^r)$ as a representation of $H_r(\mathbb{A}_F)\times H(\mathbb{A}_F)$.

We now let $m=2n$ and $r=n$ with $n\ge 1$ and suppress n from our notation, except that we will use H' instead of H_n, P' instead of P_n, N' instead of N_n, and $\mathcal{K}'$ instead of $\mathcal{K}_n$. Hence $\chi\big|_{\mathbb{A}_F^\times}=\mathbf{1}$. Let $\pi=\bigotimes'\pi_v$ be an irreducible cuspidal automorphic representation of $H'(\mathbb{A}_F)$ contained in $L^2(H'(F)\backslash H'(\mathbb{A}_F))$ and $\pi^\vee$ realizes on the space of complex conjugation of functions in π.

We denote by $(-W)$ (recall that $W=W_n$) the skew-hermitian space over E with the form $-\langle\cdot,\cdot\rangle$. Hence we can find a basis $\{e_1^-,\dots,e_{2n}^-\}$ satisfying $\langle e_i^-,e_j^-\rangle=0$, $\langle e_{r+i}^-,e_{r+j}^-\rangle=0$ and $\langle e_i^-,e_{n+j}^-\rangle=-\delta_{ij}$ for $1\le i\le n$. Let $W''=W\oplus(-W)$ be the direct sum of two skew-hermitian spaces. There is a natural embedding $\iota: H'\times H'\hookrightarrow H'':=\mathrm{U}(W'')$ given, under the basis $\{e_1,\dots,e_{2n}\}$ of W and $\{e_1,\dots,e_n;e_1^-,\dots,e_n^-;e_{n+1},\dots,e_{2n};-e_{n+1}^-,\dots,-e_{2n}^-\}$ of W'', by $\iota(g_1,g_2)=\iota_0(g_1,g_2^\vee)$, where

$$g_1=\begin{pmatrix}a_1&b_1\\c_1&d_1\end{pmatrix},\quad g_2=\begin{pmatrix}a_2&b_2\\c_2&d_2\end{pmatrix},\quad g^\vee=\begin{pmatrix}\mathbf{1}_n&\\&-\mathbf{1}_n\end{pmatrix}g\begin{pmatrix}\mathbf{1}_n&\\&-\mathbf{1}_n\end{pmatrix}^{-1},$$

and

$$\iota_0(g_1,g_2)=\begin{pmatrix}a_1&&b_1&\\&a_2&&b_2\\c_1&&d_1&\\&c_2&&d_2\end{pmatrix}.$$

For a complete polarization $W''=W'\oplus\overline{W}'$, where $W'=\mathrm{span}_E\{e_1,\dots,e_n;e_1^-,\dots,e_n^-\}$ and $\overline{W}'=\mathrm{span}_E\{e_{n+1},\dots,e_{2n};-e_{n+1}^-,\dots,-e_{2n}^-\}$, there is a Weil representation of H'', denoted by ω''_χ (with respect to ψ), on the space $\mathcal{S}(V^{2n})$, such that $\iota^*\omega''_\chi\cong\omega_{\chi,\psi}\boxtimes\chi\omega^\vee_{\chi,\psi}$, which is realized on the space $\mathcal{S}(V^n)\otimes\mathcal{S}(V^n)$. Let P be the parabolic subgroup of H'' fixing the subspace $\overline{W}'$ whose maximal unipotent subgroup is denoted by N.

Let $\mathbb{V}$ be a hermitian space over $\mathbb{A}_E$ of rank $2n$. We have a linear map

$$\mathcal{S}(\mathbb{V}^{2n})\to I_{2n}(s,\chi)$$

given by $\varphi_{\Phi,s}(g)=\omega''_\chi(g)\Phi(0)\lambda_P(g)^s$. When $s=0$, it is $H''(\mathbb{A}_F)$-equivariant and we denote by $R(\mathbb{V},\chi)=\bigotimes'_v R(\mathbb{V}_v,\chi)$ the image of this map. We define the

Eisenstein series

$$E(g,\varphi_s)=\sum_{\gamma\in P(F)\backslash H''(F)}\varphi_s(\gamma g)$$

for any standard section φ_s and $E(s,g,\Phi)=E(g,\varphi_{\Phi,s})$. It is absolutely convergent when $\Re(s)>n$ and has a meromorphic continuation to the entire complex plane, holomorphic at $s=0$ (see [Tan 1999, Proposition 4.1]).

By [Gelbart et al. 1987; Li 1992] (see also [Liu 2011, Proposition 2.3]), for any $f\in\pi$, $f^\vee\in\pi^\vee$, and standard $\varphi_s\in I_{2n}(s,\chi)$ which are decomposable, we have for $\Re(s)>n$

$$\int_{[H'(F)\backslash H'(\mathbb{A}_F)]^2} f(g_1)f^\vee(g_2)\chi^{-1}(\det g_2)E(\iota(g_1,g_2),\varphi_s)\,dg_1\,dg_2 = \prod_{v\in\Sigma}Z(\chi_v,f_v,f_v^\vee,\varphi_{s,v}),$$

where

$$Z(\chi_v,f_v,f_v^\vee,\varphi_{s,v})=\int_{H'_v}\langle\pi_v(g_v)f_v,f_v^\vee\rangle\varphi_{s,v}(\gamma_0\iota(g_v,1))\,dg_v$$

is the *local zeta integral*, which has a meromorphic continuation to the entire complex plane. Here,

$$\gamma_0=\begin{pmatrix} & & \mathbf{1}_n & \\ & \mathbf{1}_n & & \\ -\mathbf{1}_n & \mathbf{1}_n & & \\ & & \mathbf{1}_n & \mathbf{1}_n\end{pmatrix}.$$

Let us temporarily suppress v in the following. As in [Harris et al. 1996] (see also [Liu 2011, Section 2C]), we define the local L-factors $L(s,\pi,\chi)$ through these local zeta integrals and define the normalized one to be

$$Z^*(\chi,f,f^\vee,\varphi_s):=\left.\frac{b_{2n}(s)Z(\chi,f,f^\vee,\varphi_s)}{L(s+\frac{1}{2},\pi,\chi)}\right|_{s=0},\tag{2-1}$$

which is a nonzero element in $\mathrm{Hom}_{H'\times H'}(I_{2n}(0,\chi),\pi^\vee\boxtimes\chi\pi)$ (see [Harris et al. 1996, Proof of (1), Theorem 4.3]), where

$$b_m(s)=\prod_{i=0}^{m-1}L(2s+m-i,\epsilon_{E/F}^i).\tag{2-2}$$

We let $Z^*(s,\chi,f,f^\vee,\Phi)=Z^*(\chi,f,f^\vee,\varphi_{\Phi,s})$.

When everything is unramified, $Z^*(s,\chi,f,f^\vee,\Phi)=1$, by [Li 1992]. Hence, globally (and assuming everything is decomposable; otherwise we take a linear combination), the assignment

$$\alpha(f, f^\vee, \Phi) := \prod_v Z^*(0, \chi_v, f_v, f_v^\vee, \Phi_v)$$

defines an element in

$$\mathrm{Hom}_{H'(\mathbb{A}_F)\times H'(\mathbb{A}_F)}(R(\mathbb{V}, \chi), \pi^\vee \boxtimes \chi\pi) = \bigotimes_v \mathrm{Hom}_{H'_v\times H'_v}(R(\mathbb{V}_v, \chi_v), \pi_v^\vee \boxtimes \chi_v\pi_v).$$

By analytic continuation, we have

$$\int_{[H'(F)\backslash H'(\mathbb{A}_F)]^2} f(g_1)f^\vee(g_2)\chi^{-1}(\det g_2)E(s, \iota(g_1, g_2), \Phi)\, dg_1\, dg_2 \\ = \frac{L(s+\frac{1}{2}, \pi, \chi)}{\prod_{i=1}^{2n} L(2s+i, \epsilon_{E/F}^i)} \prod_v Z^*(s, \chi_v, f_v, f_v^\vee, \Phi_v)$$

for any $s \in \mathbb{C}$, where in the last product almost all factors are 1.

By the theta dichotomy proved in [Paul 1998] in the archimedean case and [Gong and Grenié 2011] in the nonarchimedean case (see [Liu 2011, Proposition 2.6] for our statement), we have $\mathrm{Hom}_{H'_v\times H'_v}(R(V_v, \chi_v), \pi_v^\vee \boxtimes \chi_v\pi_v) \neq 0$ for exactly one V_v (up to isometry) over E_v of dimension $2n$. We denote this hermitian space by $V(\pi_v, \chi_v)$ and $\epsilon(\pi_v, \chi_v) = \epsilon(V(\pi_v, \chi_v))$. Let $\mathbb{V}(\pi, \chi)$ be the hermitian space over $\mathbb{A}_E$, unique up to isometry, such that $\mathbb{V}(\pi, \chi)_v \cong V(\pi_v, \chi_v)$ for any $v \in \Sigma$ and $\epsilon(\pi, \chi) = \prod \epsilon(\pi_v, \chi_v)$. Hence we can choose $f \in \pi$, $f^\vee \in \pi^\vee$, and $\Phi \in \mathscr{S}(\mathbb{V}(\pi, \chi)^{2n})$ such that $\alpha(f, f^\vee, \Phi) \neq 0$. If $\epsilon(\pi, \chi) = -1$, then $\mathbb{V}(\pi, \chi)$ is incoherent and $E(0, g, \Phi) = 0$ by [Liu 2011, Proposition 2.11(1)]. Then $L(\frac{1}{2}, \pi, \chi) = 0$ and we have

$$\int_{[H'(F)\backslash H'(\mathbb{A}_F)]^2} f(g_1)f^\vee(g_2)\chi^{-1}(\det g_2)E'(0, \iota(g_1, g_2), \Phi)\, dg_1\, dg_2 \\ = \frac{L'(\frac{1}{2}, \pi, \chi)}{\prod_{i=1}^{2n} L(i, \epsilon_{E/F}^i)} \prod_v Z^*(0, \chi_v, f_v, f_v^\vee, \Phi_v). \quad (2\text{-}3)$$

We call $E'(0, g, \Phi) = d/ds|_{s=0}E(s, g, \Phi)$ the *analytic kernel function* associated to the *test function* $\Phi \in \mathscr{S}(\mathbb{V}^{2n})$.

For any $T \in \mathrm{Her}_{2n}(E)$, we have the T-th Fourier coefficient

$$E_T(s, g, \Phi) = \int_{N(F)\backslash N(\mathbb{A}_F)} E(s, ng, \Phi)\psi_T(n)^{-1}\, dn,$$

where $\psi_T(n(b)) = \psi(\operatorname{tr} Tb)$ and locally dn_v is the self-dual measure with respect to ψ_v. It turns out that for nonsingular T, the Fourier coefficient has a decomposition as

$$E_T(s, g, \Phi) = W_T(s, g, \Phi) := \prod_{v\in\Sigma} W_T(s, g_v, \Phi_v) \quad (2\text{-}4)$$

if $\Phi = \bigotimes \Phi_v$ is decomposable. Here, locally for any standard section $\varphi_{s,v}$ in $I_{2n}(s, \chi_v)$, we define the Whittaker integral as

$$W_T(g_v, \varphi_{s,v}) = \int_{N_v} \varphi_{s,v}(wn_v g_v)\psi_T(n_v)^{-1}\, dn_v,$$

where $w = w_{2n}$ and $W_T(s, g_v, \Phi_v) := W_T(g, \varphi_{\Phi_v,s})$ for any $T \in \mathrm{Her}_{2n}(E_v)$.

Hence we have

$$E(s, g, \Phi) = \sum_{T \text{ sing.}} E_T(s, g, \Phi) + \sum_{T \text{ nonsing.}} \prod_{v\in\Sigma} W_T(s, g_v, \Phi_v).$$

Taking the derivative at $s = 0$, we have

$$\begin{aligned} E'(0, g, \Phi) &= \sum_{T \text{ sing.}} E'_T(0, g, \Phi) + \sum_{T \text{ nonsing.}} \sum_{v\in\Sigma} W'_T(0, g_v, \Phi_v) \prod_{v'\neq v} W_T(0, g_{v'}, \Phi_{v'}) \\ &= \sum_{T \text{ sing.}} E'_T(0, g, \Phi) + \sum_{v\in\Sigma} \sum_{T \text{ nonsing.}} W'_T(0, g_v, \Phi_v) \prod_{v'\neq v} W_T(0, g_{v'}, \Phi_{v'}). \end{aligned}$$

But $\prod_{v'\neq v} W_T(0, g_v, \Phi_v) \neq 0$ only if $\mathbb{V}_{v'}$ represents T for all $v' \neq v$. Since $\mathbb{V}$ is incoherent, $\mathbb{V}_v$ cannot represent T. For T nonsingular, there are only finitely many $v \in \Sigma$ such that T is not represented by $\mathbb{V}_v$, that is, there does not exist $x_1, \dots, x_{2n} \in \mathbb{V}_v$ whose moment matrix is T. We denote the set of such v by $\mathrm{Diff}(T, \mathbb{V})$. Then

$$E'(0, g, \Phi) = \sum_{T \text{ sing.}} E'_T(0, g, \Phi) + \sum_{v\in\Sigma} E_v(0, g, \Phi),$$

where

$$E_v(0, g, \Phi) = \sum_{\mathrm{Diff}(T,\mathbb{V})=\{v\}} W'_T(0, g_v, \Phi_v) \prod_{v'\neq v} W_T(0, g_{v'}, \Phi_{v'}). \tag{2-5}$$

In fact, the second sum is only taken over those v which are nonsplit in E.

2B. ***Regular test functions.*** In this section, we will prove that the summation of $E'_T(0, g, \Phi)$ over singular T's vanishes for a certain choice of Φ and g in a suitable subset of $H''(\mathbb{A}_F)$. We follow the ideas in [Yuan et al. 2010].

For a finite place v, recall the definition

$$\mathcal{S}(\mathbb{V}_v^{2n})_{\mathrm{reg}} = \{\Phi_v \in \mathcal{S}(\mathbb{V}_v^{2n}) \mid \Phi_v(x) = 0 \text{ if } \det T(x) = 0\}.$$

We call the elements in this set *regular test functions*.

Fix a finite subset $S \subset \Sigma_f$ with $|S| = k > 0$ and let $\mathcal{S}(\mathbb{V}_S^{2n})_{\mathrm{reg}} = \bigotimes_{v\in S} \mathcal{S}(\mathbb{V}_v^{2n})_{\mathrm{reg}}$. Our main result in this section is:

Proposition 2.1. *For $\Phi = \Phi_S\Phi^S \in \mathcal{S}(\mathbb{V}_S^{2n})_{\mathrm{reg}} \otimes \mathcal{S}(\mathbb{V}^{S,2n})$, $\mathrm{ord}_{s=0} E_T(s, g, \Phi) \geq k$ for T singular and $g \in P(\mathbb{A}_{F,S})H''(\mathbb{A}_F^S)$.*

We can assume that $\Phi = \bigotimes \Phi_v$ is decomposable with $\Phi_v \in \mathcal{S}(\mathbb{V}_v^{2n})_{\mathrm{reg}}$ for $v \in S$ and $\operatorname{rank} T = 2n - r < 2n$. Choose $a \in \mathrm{GL}_{2n}(E)$ such that

$$aT\,{}^t a^\tau = \begin{pmatrix} \mathbf{0}_r & \\ & \widetilde{T} \end{pmatrix}, \tag{2-6}$$

with $\widetilde{T} \in \mathrm{Her}_{2n-r}(E)$. Then

$$E_T(s, g, \Phi) = E_{aT\,{}^t a^\tau}(s, m(a)g, \Phi).$$

Hence we can assume that T is of the form (2-6).

First, we need a more explicit formula for the singular coefficient E_T. By definition, for $\Re(s) > n$

$$\begin{aligned} E_T(s, g, \Phi) &= \int_{N(F)\backslash N(\mathbb{A}_F)} \sum_{\gamma \in P(F)\backslash H''(F)} \varphi_{\Phi,s}(\gamma n g)\psi_T(n)^{-1}\, dn \\ &= \int_{N(F)\backslash N(\mathbb{A}_F)} \sum_{\gamma \in P(F)\backslash H''(F)} r(g)\varphi_{\Phi,s}(\gamma n)\psi_T(n)^{-1}\, dn, \end{aligned} \tag{2-7}$$

where $r(g)$ means the H'' action on $I_{2n}(s, \chi)$ by right translation. We need to unfold this summation. Let

$$w_{2n,d} = \begin{pmatrix} \mathbf{1}_d & & & \\ & & & \mathbf{1}_{2n-d} \\ & & \mathbf{1}_d & \\ & -\mathbf{1}_{2n-d} & & \end{pmatrix} \tag{2-8}$$

for $0 \le d \le 2n$ be a set of representatives of the double coset $W_{P/N}\backslash W_{H''}/W_{P/N}$ of Weyl groups, thus $w_{2n,0} = w_{2n} = w$. We have a Bruhat decomposition

$$H''(F) = \coprod_{d=0}^{2n} P(F) w_{2n,d} P(F),$$

where F can be the global field or its local completions.

Lemma 2.2. *If $v \in S$ and $g_v \in P_v$, the support of $r(g_v)\varphi_{\Phi_v,s}$ is contained in $P(F_v)wN(F_v)$.*

Proof. It suffices to prove that $\varphi_{\Phi_v,s}$ vanishes on $P(F_v)w_{2n,d}P(F_v)$ for $d > 0$ since $g_v \in P(F_v)$. For $g = n(b_1)m(a_1)w_{2n,d}n(b_2)m(a_2) \in P(F_v)w_{2n,d}P(F_v)$, we have

$$\begin{aligned} \varphi_{\Phi_v,s}(g) &= \omega_{\chi_v}(g)\Phi(0)\lambda(g)^s \\ &= \chi_v(\det a_1 a_2)|\det a_1 a_2|_{E_v}^n \lambda(g)^s \int_{\mathbb{V}_v^{2n-d}} \psi_{b_2}(T(x))\Phi_v(xa_2)\, dx, \end{aligned}$$

where $\mathbb{V}_v^{2n-d}$ is viewed, via $(x_1, \dots, x_{2n-d}) \mapsto (0, \dots, 0, x_1, \dots, x_{2n-d})$, as a subset of $\mathbb{V}_v^{2n}$. Since Φ_v is regular and $d > 0$, $\Phi_v(xa_2) = 0$ for $x \in \mathbb{V}_v^{2n-d}$. $\square$

By the lemma, we have for $g \in P(\mathbb{A}_{F,S})H''(\mathbb{A}_F^S)$,

$$\begin{aligned}
\text{(2-7)} &= \int_{N(F)\backslash N(\mathbb{A}_F)} \sum_{\gamma \in P(F)\backslash P(F)wP(F)} r(g)\varphi_{\Phi,s}(\gamma n)\psi_T(n)^{-1}\,dn \\
&= \int_{N(F)\backslash N(\mathbb{A}_F)} \sum_{\gamma \in wN(F)} r(g)\varphi_{\Phi,s}(\gamma n)\psi_T(n)^{-1}\,dn \\
&= \int_{N(\mathbb{A}_F)} r(g)\varphi_{\Phi,s}(wn)\psi_T(n)^{-1}\,dn \\
&= \int_{N(\mathbb{A}_F)} \varphi_s(wn)\psi_T(n)^{-1}\,dn \\
&= \prod_{v\in\Sigma} \int_{N_v} \varphi_{v,s}(wn_v)\psi_T(n_v)^{-1}\,dn_v, \qquad (2\text{-}9)
\end{aligned}$$

where we write φ_s instead of $r(g)\varphi_{\Phi,s}$ for simplicity. Let $S' \subset \Sigma$ be the finite subset containing all infinite places and ramified places, away from which χ_v and ψ_v are unramified; $\varphi_{v,s}$ is the (unique) unramified section in $I_{2n}(s, \chi_v)$ (hence $S' \supset S$) and $\det \widetilde{T} \in \mathbb{O}_{F_v}^\times$. Then

$$\text{(2-9)} = \left(\prod_{v\in S'} W_T(e, \varphi_{v,s})\right) W_T(e, \varphi_s^{S'}). \qquad (2\text{-}10)$$

By [Kudla and Rallis 1994, p. 36] and [Tan 1999, Proposition 3.2], we have

$$W_T(e, \varphi_s^{S'}) = \frac{a_{2n}^{S'}(s)}{a_{2n-r}^{S'}(s - \frac{1}{2}r)b_{2n}^{S'}(s)},$$

where

$$a_{m,v}(s) = \prod_{i=0}^{m-1} L_v(2s+i-m+1, \epsilon_{E/F}^i) \quad \text{and} \quad b_{m,v}(s) = \prod_{i=0}^{m-1} L_v(2s+m-i, \epsilon_{E/F}^i),$$

as in (2-2). Hence $W_T(e, \varphi_s^{S'})$ has a meromorphic continuation to the entire complex plane. For $v \in S'$, we normalize the Whittaker functional to be

$$W_T^*(e, \varphi_{v,s}) = \frac{a_{2n-r,v}(s - \frac{1}{2}r)b_{2n,v}(s)}{a_{2n,v}(s)} W_T(e, \varphi_{v,s}).$$

Using the argument and notation of [Kudla and Rallis 1994, p. 35], we have

$$W_T(e, \varphi_{v,s}) = W_{\widetilde{T}}(e, i^* \circ U_{r,v}(s)\varphi_{v,s}).$$

By [Piatetski-Shapiro and Rallis 1987, Section 4], the (local) intertwining operator $U_{r,v}(s)$ has a meromorphic extension to the entire complex plane. By [Liu 2011, Lemma 2.8(1)], which combines results from [Karel 1979] and [Wallach 1988],

$W_T(e,\varphi_{v,s})$ and hence $W_T^*(e,\varphi_{v,s})$ have meromorphic continuations to the entire complex plane. Together with the meromorphic continuation of W_T away from S' and W_T^* in S', (2-10) has a meromorphic continuation which equals

$$\frac{a_{2n}(s)}{a_{2n-r}(s-\frac{1}{2}r)b_{2n}(s)}\prod_{v\in S'} W_T^*(e,\varphi_{v,s}).$$

Proof of Proposition 2.1. Consider the point $s=0$, $b_{2n}(0)=\prod_{i=1}^{2n} L(i,\epsilon_{E/F}^i)\in\mathbb{C}^\times$, and

$$\frac{a_{2n}(0)}{a_{2n-r}(-\frac{1}{2}r)}=\prod_{i=0}^{r-1} L(-i,\epsilon_{E/F}^{i+1})\in\mathbb{C}^\times.$$

Let $\kappa_v=\mathrm{ord}_{s=0}W_T^*(e,\cdot)$ be the order of the functional at $s=0$ for $v\in S'$ and $\kappa_v'=\mathrm{ord}_{s=0}W_T^*(s,e,\cdot)|_{\mathcal{S}(\mathbb{V}_v^{2n})_{\mathrm{reg}}}$ for $v\in S$. Since $E_T(e,\varphi_\Phi)=0$ if $\Phi=\bigotimes\Phi_v$ for at least one Φ_v regular, by (2-10) and the proof of [Liu 2011, Lemma 2.10], we have $\kappa_{v_0}'+\sum_{v_0\neq v\in S'}\kappa_v\geq 1$ for any v_0 in S. Also by the definition of W_T, we see that

$$\varphi_{v,0}\mapsto s^{-\kappa_v}W_T^*(e,\varphi_{v,s})\big|_{s=0}$$

is a nontrivial N-intertwining map from $I_{2n}(0,\chi)$ to $\mathbb{C}_{N,\psi_T}$. Now if $v\in S$, our $\varphi_{v,0}=\varphi_{\Phi_v,0}$ for a regular test function $\Phi_v\in\mathcal{S}(\mathbb{V}_v^{2n})_{\mathrm{reg}}$. By [Rallis 1987, Lemma 4.2] stated as [Liu 2011, Lemma 2.7(1-a)], $\varphi_{v,0}$ goes to 0 under the above map, that is, $\kappa_v'\geq\kappa_v+1$ for $v\in S$. Hence

$$\mathrm{ord}_{s=0}\prod_{v\in S'}W_T^*(e,\varphi_{v,s})\geq\sum_{v\in S}\kappa_v'+\sum_{v\in S'-S}\kappa_v\geq k-1+\kappa_{v_0}'+\sum_{v_0\neq v\in S'}\kappa_v\geq k.\quad\square$$

In conclusion, if we choose S such that $|S|\geq 2$ and a decomposable test function

$$\Phi=\Phi_S\Phi^S\in\mathcal{S}(\mathbb{V}_S^{2n})_{\mathrm{reg}}\otimes\mathcal{S}(\mathbb{V}^{S,2n}),$$

then for $g\in P(\mathbb{A}_{F,S})H''(\mathbb{A}_F^S)$,

$$E'(0,g,\Phi)=\sum_{v\in\Sigma}E_v(0,g,\Phi).\tag{2-11}$$

2C. ***Test functions of higher discriminant.*** In this section, we will show that if we have a better choice of Φ_v for $v\in S$, we can even make $W_T'(0,e,\Phi_v)=0$ for any nonsingular T which is not representable by $\mathbb{V}_v$. We follow the ideas in [Yuan et al. 2010].

Since the argument is local, we fix one $v\in S$ which is nonsplit and suppress it from the notation in this section. Let V be one of $V^\pm$ and V' the other one which is not isometric to V. For $d \in \mathbb{Z}$, let

$$\begin{aligned}\mathsf{Her}^0_{2n}(E) &= \{T \in \mathsf{Her}_{2n}(E) \mid \det T \neq 0\},\\ \mathcal{H} &= \{b \in \mathsf{Her}^0_{2n}(E) \mid b = T(x) \text{ for some } x \in V^{2n}\},\\ \mathcal{H}' &= \{b' \in \mathsf{Her}^0_{2n}(E) \mid b' = T(x') \text{ for some } x' \in V'^{2n}\},\\ \mathcal{H}'_d &= \{b' + b'' \mid b' \in \mathcal{H}' \text{ and } b'' \in \mathsf{Her}_{2n}(\mathfrak{p}_E^{-d})\} \cap \mathsf{Her}^0_{2n}(E),\end{aligned}$$

where $\mathfrak{p}_E$ is the maximal ideal of $\mathcal{O}_E$. Then $\mathsf{Her}^0_{2n}(E) = \mathcal{H} \amalg \mathcal{H}'$,

$$\cdots \subsetneqq \mathcal{H}'_{-1} \subsetneqq \mathcal{H}'_0 \subsetneqq \mathcal{H}'_1 \subsetneqq \cdots,$$

$\bigcap_d \mathcal{H}'_d = \mathcal{H}'$, and $\bigcup_d \mathcal{H}'_d = \mathsf{Her}^0_{2n}(E)$. We say that a test function $\Phi \in \mathcal{S}(V^{2n})$ is *of discriminant d* if

$$\{T(x) \mid x \in \mathsf{Supp}(\Phi)\} \cap \mathcal{H}'_d = \varnothing.$$

We denote by $\mathcal{S}(V^{2n})_d$ the space of such functions, and set

$$\mathcal{S}(V^{2n})_{\mathsf{reg},d} = \mathcal{S}(V^{2n})_{\mathsf{reg}} \cap \mathcal{S}(V^{2n})_d.$$

Lemma 2.3. *For any $d \in \mathbb{Z}$, $\mathcal{S}(V^{2n})_{\mathsf{reg},d}$ is not empty.*

Proof. Fix any d; in fact, we only need to prove that there exists $T \notin \mathcal{H}'_d$ such that $\det T \neq 0$. Then $(T + \mathsf{Her}_{2n}(\mathfrak{p}_E^{-d})) \cap \mathcal{H}' = \varnothing$. Any test function with support whose elements have moment matrices contained in $(T + \mathsf{Her}_{2n}(\mathfrak{p}_E^{-d})) \cap \mathsf{Her}^0_{2n}(E)$, which is open, will be in $\mathcal{S}(V^{2n})_{\mathsf{reg},d}$. Now we want to find such a T. Take any $T_1 \in \mathcal{H}$ with $\det T_1 \neq 0$. Since $\mathcal{H}$ is open, we can find a neighborhood $T_1 + \mathsf{Her}_{2n}(\mathfrak{p}_E^{\nu}) \subset \mathcal{H}$ for a $\nu \in \mathbb{Z}$. If $\nu \leq -d$, then we are done. Otherwise, let ϖ be the uniformizer of F. Then $\varpi^{-\nu-d}(T_1 + \mathsf{Her}_{2n}(\mathfrak{p}_E^{\nu})) \subset \mathcal{H}$. But

$$\varpi^{-\nu-d}(T_1 + \mathsf{Her}_{2n}(\mathfrak{p}_E^{\nu})) = (\varpi^{-\nu-d}T_1 + \mathsf{Her}_{2n}(\mathfrak{p}_E^{-d})).$$

Hence $T = \varpi^{-\nu-d}T_1$ will serve for our purpose. □

Since ψ is nontrivial, we can define its *discriminant* d_ψ to be the largest integer d such that the character ψ_T is trivial on $N(\mathcal{O}_F) \cong \mathsf{Her}_{2n}(\mathcal{O}_E)$ for all $T \in \mathsf{Her}_{2n}(\mathfrak{p}_E^{-d})$. We need to mention that this is not the conductor of a p-adic additive character. But the difference between them only depends on n and the ramification of E/F. The main result of this section is:

Proposition 2.4. *Let $d \geq d_\psi$ be an integer. Given $\Phi \in \mathcal{S}(V^{2n})_{\mathsf{reg},d}$, we have $W_T(s, e, \Phi) \equiv 0$ for $T \in \mathcal{H}'$ nonsingular.*

Proof. For $\Re(s) > n$,

$$W_T(s, e, \Phi) = \int_N \omega_\chi(wn)\Phi(0)\lambda(wn)^s \psi_T(n)^{-1}\, dn$$

is absolutely convergent. Hence it equals

$$\begin{aligned}\int_{\mathrm{Her}_{2n}(E)}\biggl(\int_{V^{2n}}\psi(\mathrm{tr}\, bT(x))\Phi(x)\,dx\biggr)&\lambda(wn(b))^s\psi(-\mathrm{tr}\, Tb)\,db\\ &=\int_{V^{2n}}\Phi(x)\,dx\int_{\mathrm{Her}_{2n}(E)}\lambda(wn(b))^s\psi\bigl(\mathrm{tr}(T(x)-T)b\bigr)\,db\\ &=\int_{V^{2n}}\Phi(x)\,dx\int_{\mathrm{Her}_{2n}(E)}\lambda(wn(b))^s\psi_{T(x)-T}(n(b))\,db. \qquad (2\text{-}12)\end{aligned}$$

Since $\lambda(wn(b)n(b_1))=\lambda(wn(b))$ for $b_1\in\mathrm{Her}_{2n}(\mathcal{O}_E)$,

$$\begin{aligned}(2\text{-}12)=\int_{V^{2n}}\Phi(x)\,dx\int_{\mathrm{Her}_{2n}(E)/\mathrm{Her}_{2n}(\mathcal{O}_E)}&\lambda(wn(b))^s\psi_{T(x)-T}(n(b))\,db\\ &\times\int_{\mathrm{Her}_{2n}(\mathcal{O}_E)}\psi_{T(x)-T}(n(b_1))\,db_1,\end{aligned}$$

in which the last integral is zero for all $x\in\mathrm{Supp}(\Phi)$ by our assumption on Φ. Hence $W_T(s,e,\Phi)\equiv 0$ after continuation. In particular, $W'_T(0,e,\Phi)=0$. ☐

In conclusion, if S is a finite subset of Σ_f with $|S|\geq 2$, $\Phi=\bigotimes\Phi_v\in\mathcal{S}(\mathbb{V}^{2n})$ with $\Phi_v\in\mathcal{S}(\mathbb{V}_v^{2n})_{\mathrm{reg}}$ for $v\in S$, and $\Phi_v\in\mathcal{S}(\mathbb{V}_v^{2n})_{\mathrm{reg},d_v}$ for $v\in S$ nonsplit with $d_v\geq d_{\psi_v}$, then

$$E'(0,g,\Phi)=\sum_{v\notin S}E_v(0,g,\Phi) \qquad (2\text{-}13)$$

for $g\in e_S H''(\mathbb{A}_F^S)$.

2D. *Density of test functions.* In the previous two sections, we have made particular choices of test functions to simplify the formula of the analytic kernel functions. But for our proof of the main theorem, arbitrary choices will not suffice. In this section, we will show that there are "sufficiently many" test functions satisfying these particular choices we have made in the sense of Proposition 2.8. We follow the ideas in [Yuan et al. 2011].

We keep the notation from the previous two sections. In particular, v will be a place in S and will be suppressed from the notation. Recall that we have an H''-intertwining map $\mathcal{S}(V^{2n})\twoheadrightarrow\mathcal{S}(V^{2n})_H\cong R(V,\chi)\hookrightarrow I_{2n}(0,\chi)$ through the Weil representation ω''_χ. Hence we have an $H'\times H'$ admissible representation on $\mathcal{S}(V^{2n})$ through the embedding ι defined in Section 2A.

Lemma 2.5. *If v is nonsplit, then for any $d\in\mathbb{Z}$ we have*

$$\mathcal{S}(V^{2n})_{\mathrm{reg}}=\omega''_\chi(m(F^\times\mathbf{1}_{2n}))\mathcal{S}(V^{2n})_{\mathrm{reg},d}.$$

Proof. Fix $d \in \mathbb{Z}$. For any function $\Phi \in \mathscr{S}(V^{2n})_{\mathrm{reg}}$, $\mathsf{Supp}(\Phi)$ is a compact subset of $\mathscr{H}$. Since $\mathsf{Her}^0_{2n}(E)\backslash\mathscr{H}'_d$ is open and

$$\bigcup_d (\mathsf{Her}^0_{2n}(E)\backslash\mathscr{H}'_d) = \mathsf{Her}^0_{2n}(E)\backslash\bigcap_d \mathscr{H}'_d = \mathsf{Her}^0_{2n}(E)\backslash\mathscr{H}' = \mathscr{H},$$

$(\mathsf{Her}^0_{2n}(E)\backslash\mathscr{H}'_d)_{d\in\mathbb{Z}}$ is an open covering of $\mathsf{Supp}(\Phi)$, and hence has a finite subcover. Then there exists $d_0 \in \mathbb{Z}$ such that $\mathsf{Supp}(\Phi)\cap\mathscr{H}'_{d_0} = \varnothing$. If $d_0 \geq d$, then we are done. Otherwise, consider $\Phi' = \omega''_\chi(m(\varpi^{d_0-d}\mathbf{1}_{2n}))\Phi$; then $\mathsf{Supp}(\Phi')\cap\mathscr{H}'_d = \varnothing$. The lemma follows. $\square$

In the rest of this section, let $n = 1$. Then $H' = \mathsf{U}(W_1)$.

Lemma 2.6. *Let π be an irreducible admissible representation of H' which is not of dimension* 1 *and $A : \mathscr{S}(V) \to \pi$ a surjective H'-intertwining map, where H' acts on $\mathscr{S}(V)$ through a Weil representation ω. Then for any ϕ with $A(\phi) \neq 0$, there is $\phi' \in \mathscr{S}(V)_{\mathrm{reg}}$ such that $A(\phi') \neq 0$ and $\mathsf{Supp}(\phi') \subset \mathsf{Supp}(\phi)$.*

Proof. Let $f = A(\phi)$. If there exists $n \in N$ such that $\pi(n)f \neq f$, then

$$A(\omega(n)\phi - \phi) = \pi(n)f - f \neq 0$$

but

$$\omega(n)\phi(x) - \phi(x) = \big(\psi(bT(x)) - 1\big)\phi(x),$$

where $n = n(b)$. We see that $\phi' = \omega(n)\phi - \phi \in \mathscr{S}(V)_{\mathrm{reg}}$ and $\mathsf{Supp}(\phi') \subset \mathsf{Supp}(\phi)$. If $\pi(n)f = f$ for any $n \in N$, then f will be fixed by an open subgroup of H' containing N since π is smooth. But any such subgroup will contain $\mathsf{SU}(W_1)$, hence π factors through $H'/\mathsf{SU}(W_1) = \mathsf{U}(W_1)/\mathsf{SU}(W_1) \cong E^{\times,1}$, which contradicts the assumption on π. $\square$

Lemma 2.7. *If π_1 and π_2 are two irreducible admissible representations of H' which are not of dimension* 1, *then for any surjective $H' \times H'$-intertwining map $B : \mathscr{S}(V)\otimes\mathscr{S}(V) = \mathscr{S}(V^2) \to \pi_1 \boxtimes \pi_2$ where $H' \times H'$ acts on $\mathscr{S}(V)\otimes\mathscr{S}(V)$ by a pair of Weil representations $\omega_1 \boxtimes \omega_2$, there is an element $\Phi = \phi_1 \otimes \phi_2 \in \mathscr{S}(V^2)_{\mathrm{reg}}$ such that $B(\Phi) \neq 0$.*

Proof. Let $\Phi' \in \mathscr{S}(V^2)$ be such that $B(\Phi') \neq 0$. Write $\Phi' = \sum \phi_{i,1}\otimes\phi_{i,2}$ as an element in $\mathscr{S}(V)\otimes\mathscr{S}(V)$. Hence we can assume that there is $\phi_1 \otimes \phi_2$ such that $B(\phi_1\otimes\phi_2) \neq 0$. By Lemma 2.6, we can also assume that $\phi_1 \in \mathscr{S}(V)_{\mathrm{reg}}$. For $x \in \mathsf{Supp}(\phi_1)$, let V_x be the subspace of V generated by x and V^x its orthogonal complement. Both V_x and V^x are nondegenerate hermitian spaces of dimension 1. As an H'-representation, $\mathscr{S}(V) = \mathscr{S}(V_x)\otimes\mathscr{S}(V^x)$. Now write $\phi_2 = \sum \phi_{i,x}\otimes\phi^x_i$ according to this decomposition. We can assume there is a $\phi_x \otimes \phi^x$ such that $B(\phi_1 \otimes (\phi_x \otimes \phi^x)) \neq 0$, since as an H'-representation, $\mathscr{S}(V^x)$ is generated by the subspace $\mathscr{S}(V^x)_{\mathrm{reg}}$. We can then write

$$\phi_x \otimes \phi^x = \sum \omega_2(g_j)(\omega_2^{-1}(g_j)\phi_x \otimes \phi_j^x),$$

with $\phi_j^x \in \mathscr{S}(V^x)_{\mathrm{reg}}$. So we can further assume that $B(\phi_1 \otimes (\phi_x \otimes \phi^x)) \neq 0$ with $\phi^x \in \mathscr{S}(V^x)_{\mathrm{reg}}$, that is, $\mathrm{Supp}(\phi_x \otimes \phi^x) \cap V_x = \varnothing$. Applying Lemma 2.6 again, we can further assume there exists $\phi_2^{(x)} \in \mathscr{S}(V)_{\mathrm{reg}}$ such that $\mathrm{Supp}(\phi_2^{(x)}) \subset \mathrm{Supp}(\phi_x \otimes \phi^x)$ and $B(\phi_1 \otimes \phi_2^{(x)}) \neq 0$. The condition that $\mathrm{Supp}(\phi_2) \cap V^x = \varnothing$ is open for x. Hence we can find a neighborhood U_x of x such that $\phi_1|_{U_x} \otimes \phi_2^{(x)} \in \mathscr{S}(V^2)_{\mathrm{reg}}$. Since $\mathrm{Supp}(\phi_1)$ is compact, we can find Φ of this kind such that $B(\Phi) \neq 0$. □

Recall the zeta integrals introduced in Section 2A. For $\Phi \in \mathscr{S}(V^{2n})$, we write $Z^*(s, \chi, f, f^\vee, \Phi) = Z^*(\chi, f, f^\vee, \varphi_{\Phi,s})$. Combining Lemmas 2.5 and 2.7, we have:

Proposition 2.8. *Let $n = 1$, $v \in \Sigma_f$, π be an irreducible cuspidal automorphic representation of H', and $V_v = V(\pi_v, \chi_v)$. For any $d \in \mathbb{Z}$, we can find $f_v \in \pi_v$, $f_v^\vee \in \pi_v^\vee$, and $\phi_{1,v} \otimes \phi_{2,v} \in \mathscr{S}(V_v^2)_{\mathrm{reg},d}$ (resp. $\mathscr{S}(V_v^2)_{\mathrm{reg}}$) if v is nonsplit (resp. split) in E, such that the (normalized) zeta integral $Z^*(0, \chi_v, f_v, f_v^\vee, \phi_{1,v} \otimes \phi_{2,v}) \neq 0$.*

3. Geometric kernel functions

3A. ***Néron–Tate height pairing on curves.*** In this section, we will review the general theory of the Néron–Tate height pairing on curves over number fields and some related facts.

Height pairing of cohomologically trivial cycles. Let E be any number field, not necessarily CM, and let M be a connected smooth projective curve over E, not necessarily geometrically connected. Let $\mathrm{CH}^1(M)^0_{\mathbb{C}}$ be the group of *cohomologically trivial cycles* which is the kernel of the map

$$\deg : \mathrm{CH}^1(M)_{\mathbb{C}} \longrightarrow \mathrm{H}^2_{\text{ét}}(M_{E^{\mathrm{ac}}}, \mathbb{Z}_\ell(1))^{\Gamma_E} \otimes_{\mathbb{Z}_\ell} \mathbb{C} \cong \mathbb{C}$$

for any fixed rational prime number ℓ. Let $\mathcal{M}$ be a regular model of M, that is, a regular scheme, flat and projective over $\mathrm{Spec}\, \mathcal{O}_E$ with generic fiber $\mathcal{M}_E \cong M$.

Recall that an *arithmetic $\mathbb{C}$-divisor* is a datum $(\mathcal{Z}, g_{\iota^\circ})$, where $\mathcal{Z} \in \mathrm{Z}^1(\mathcal{M})_{\mathbb{C}}$ is a usual divisor and g_{ι° is a Green's function (that is, a Green's (0,0)-form of logarithmic type [Soulé 1992, II.2]) for the divisor $\mathcal{Z}_{\iota^\circ}(\mathbb{C})$ on $\mathcal{M}_{\iota^\circ}(\mathbb{C})$ for each $\iota^\circ : E \hookrightarrow \mathbb{C}$. We denote by $\widehat{\mathrm{Z}}^1_{\mathbb{C}}(\mathcal{M})$ the group of arithmetic $\mathbb{C}$-divisors. For a nonzero rational function f on $\mathcal{M}$, we define the associated *principal arithmetic divisor* to be

$$\widehat{\mathrm{div}}(f) = (\mathrm{div}(f), -\log|f_{\iota^\circ,\mathbb{C}}|^2).$$

The quotient of $\widehat{\mathrm{Z}}^1_{\mathbb{C}}(\mathcal{M})$ divided by the $\mathbb{C}$-subspace generated by the principal arithmetic divisors is the *arithmetic Chow group*, denoted by $\widehat{\mathrm{CH}}^1_{\mathbb{C}}(\mathcal{M})$. Inside $\widehat{\mathrm{CH}}^1_{\mathbb{C}}(\mathcal{M})$, there is a subspace $\mathrm{CH}^1_{\mathrm{fin}}(\mathcal{M})_{\mathbb{C}}$ which is $\mathbb{C}$-generated by $(\mathcal{Z}, 0)$ with $\mathcal{Z}$ supported on special fibers. Let $\mathsf{CH}^1_{\mathsf{fin}}(\mathcal{M})^{\perp}_{\mathbb{C}} \subset \widehat{\mathsf{CH}}^1_{\mathbb{C}}(\mathcal{M})$ be the orthogonal complement under the $\mathbb{C}$-bilinear pairing

$$\langle\,\cdot\,,\,\cdot\,\rangle_{\mathsf{GS}} : \widehat{\mathsf{CH}}^1_{\mathbb{C}}(\mathcal{M}) \times \widehat{\mathsf{CH}}^1_{\mathbb{C}}(\mathcal{M}) \to \mathbb{C}.$$

Recall that an arithmetic divisor $(\mathcal{Z}, g_{\iota^\circ})$ is *flat* if we have the following equality in the space $\mathsf{D}^{1,1}(M_{\iota^\circ}(\mathbb{C}))$ of $(1, 1)$-currents:

$$dd^c[g_{\iota^\circ}] + \delta_{\mathcal{Z}_{\iota^\circ}(\mathbb{C})} = 0$$

for any ι°, where $d^c = (4\pi i)^{-1}(\partial - \bar{\partial})$, $[-]$ is the associated current, and δ is the Dirac current. Flatness is well-defined in $\widehat{\mathsf{CH}}^1_{\mathbb{C}}(\mathcal{M})$. Now we introduce the subgroup $\widehat{\mathsf{CH}}^1_{\mathbb{C}}(\mathcal{M})^0$ of $\widehat{\mathsf{CH}}^1_{\mathbb{C}}(\mathcal{M})$ consisting of elements (represented by) $(\mathcal{Z}, g_{\iota^\circ})$ such that $(\mathcal{Z}, g_{\iota^\circ}) \in \mathsf{CH}^1_{\mathsf{fin}}(\mathcal{M})^{\perp}_{\mathbb{C}}$, $\mathcal{Z}_E \in \mathsf{CH}^1(M)^0_{\mathbb{C}}$, and $(\mathcal{Z}, g_\iota)$ is flat. Hence we have a natural map

$$\begin{aligned} p_{\mathcal{M}} : \widehat{\mathsf{CH}}^1_{\mathbb{C}}(\mathcal{M})^0 &\to \mathsf{CH}^1(M)^0_{\mathbb{C}} \\ (\mathcal{Z}, g_\iota) &\mapsto \mathcal{Z}_E \end{aligned} \tag{3-1}$$

which is surjective. Now we can define the *Néron–Tate height pairing*:

$$\begin{aligned} \langle\,\cdot\,,\,\cdot\,\rangle_{\mathsf{NT}} : \mathsf{CH}^1(M)^0_{\mathbb{C}} \times \mathsf{CH}^1(M)^0_{\mathbb{C}} &\to \mathbb{C} \\ (Z_1, Z_2) &\mapsto \langle (\mathcal{Z}_1, g_{1,\iota^\circ}), (\mathcal{Z}_2, g_{2,\iota^\circ})\rangle_{\mathsf{GS}}, \end{aligned} \tag{3-2}$$

where $(\mathcal{Z}_i, g_{i,\iota^\circ})$ $(i = 1, 2)$ is any preimage of Z_i under $p_{\mathcal{M}}$. It is easy to see that this is independent of the choices of preimages and also the regular model $\mathcal{M}$.

Modification of the height pairing. Practically, the cycles we are interested are not automatically cohomologically trivial. We need to make some modifications with respect to some auxiliary data. This is quite easy if we are working over a curve. Let $\widehat{\mathsf{Pic}}(M)$ be the abelian group of isomorphism classes of *hermitian line bundles* on M. Recall that a hermitian line bundle is $\overline{\mathcal{L}} = (\mathcal{L}, \|\cdot\|_{\iota^\circ})$, where $\mathcal{L} \in \mathsf{Pic}(M)$ and $\|\cdot\|_{\iota^\circ}$ is a (smooth) hermitian metric on the holomorphic line bundle $\mathcal{L}_{\iota^\circ,\mathbb{C}}$. We assume that $\deg c_1(\mathcal{L}) \neq 0$. For any $Z \in \mathsf{CH}^1(M)$, the divisor

$$Z^0_{\mathcal{L}} = Z - \frac{\deg Z}{\deg c_1(\mathcal{L})} c_1(\mathcal{L}) \in \mathsf{CH}^1(M)^0_{\mathbb{C}}.$$

Now we define the modified height pairing with respect to $\overline{\mathcal{L}}$:

$$\langle Z_1, Z_2\rangle_{\overline{\mathcal{L}}} := \langle Z^0_{1,\mathcal{L}}, Z^0_{2,\mathcal{L}}\rangle_{\mathsf{NT}}$$

for any $Z_i \in \mathsf{CH}^1(M)_{\mathbb{C}}$ $(i = 1, 2)$. In particular, we need to choose a suitable Green's function on Z_i when computing via (3-2). We say that the Green's function g_{ι° of Z is $\overline{\mathcal{L}}$*-admissible* if the following equalities between $(1, 1)$-currents hold:

$$dd^c[g_{\iota^\circ}] + \delta_{Z_{\iota^\circ}(\mathbb{C})} = \frac{\deg Z}{\deg c_1(\mathcal{L})}[c_1(\mathcal{L}_{\iota^\circ,\mathbb{C}}, \|\cdot\|_{\iota^\circ})],$$

$$\int_{M_{\iota^\circ}(\mathbb{C})} g_{\iota^\circ}\cdot c_1(\mathscr{L}_{\iota^\circ,\mathbb{C}},\|\cdot\|_{\iota^\circ})=0,$$

where $c_1(\mathscr{L}_{\iota^\circ,\mathbb{C}},\|\cdot\|_{\iota^\circ})\in \mathsf{A}^{1,1}(M_{\iota^\circ}(\mathbb{C}))$ is the *Chern form* associated to the hermitian holomorphic line bundle $(\mathscr{L}_{\iota^\circ,\mathbb{C}},\|\cdot\|_{\iota^\circ})$ which is a $(1,1)$-form.

3B. ***Decomposition of geometric kernel functions.*** In this section, we will apply the above section to the special case of Shimura curves of unitary groups. We fix until the end of this paper an additive character $\psi: F\backslash\mathbb{A}_F\to\mathbb{C}$ such that ψ_ι is the standard character $\phi^0_\infty: t\mapsto e^{2\pi i t}$ $(t\in F_\iota=\mathbb{R})$ for any $\iota\in\Sigma_\infty$.

Shimura curves of unitary groups. We review the setup of [Liu 2011, Section 3A] in the particular case $m=2$ and $r=1$. Hence $\mathbb{V}$ is a totally positive-definite hermitian space over $\mathbb{A}_E$ of rank 2. Let $\mathbb{H}=\mathrm{Res}_{\mathbb{A}_F/\mathbb{A}}\mathrm{U}(\mathbb{V})$ be the unitary group which is a reductive group over $\mathbb{A}$ and $\mathbb{H}^{\mathrm{der}}=\mathrm{Res}_{\mathbb{A}_F/\mathbb{A}}\mathrm{SU}(\mathbb{V})$ its derived subgroup. Let $\mathbb{T}\cong\mathrm{Res}_{\mathbb{A}_F/\mathbb{A}}\mathbb{A}_E^{\times,1}$ be the maximal abelian quotient of $\mathbb{H}$ which is also isomorphic to its center. Let $T\cong\mathrm{Res}_{F/\mathbb{Q}}E^{\times,1}$ be the unique (up to isomorphism) $\mathbb{Q}$-torus such that $T\times_{\mathbb{Q}}\mathbb{A}\cong\mathbb{T}$. Then T has the property that $T(\mathbb{Q})$ is discrete in $T(\mathbb{A}_f)$. For any open compact subgroup K of $\mathbb{H}(\mathbb{A}_f)$, which we always assume *to be contained in the principal congruence subgroup for* $N\ge 3$, there is a Shimura curve $\mathrm{Sh}_K(\mathbb{H})$ that is smooth over the reflex field E. For any embedding $\iota^\circ: E\hookrightarrow\mathbb{C}$ over $\iota\in\Sigma_\infty$, we have the following ι°-adic uniformization:

$$\mathrm{Sh}_K(\mathbb{H})^{\mathrm{an}}_{\iota^\circ}\cong H^{(\iota)}(\mathbb{Q})\backslash(\mathfrak{D}^{(\iota^\circ)}\times\mathbb{H}(\mathbb{A}_f)/K).$$

We briefly explain the notation above. Let $V^{(\iota)}$ be the *nearby E-hermitian space* of $\mathbb{V}$ at ι, that is, $V^{(\iota)}$ is the unique E-hermitian space (up to isometry) such that $V^{(\iota)}_v\cong\mathbb{V}_v$ for $v\neq\iota$ but $V^{(\iota)}_\iota$ is of signature $(1,1)$ and $H^{(\iota)}=\mathrm{Res}_{F/\mathbb{Q}}\mathrm{U}(V^{(\iota)})$. We identify $H^{(\iota)}(\mathbb{A}_f)$ and $\mathbb{H}(\mathbb{A}_f)$ through the corresponding hermitian spaces. Let $\mathfrak{D}^{(\iota^\circ)}$ be the hermitian domain consisting of all negative $\mathbb{C}$-lines in $V^{(\iota)}_\iota$ whose complex structure is given by the action of $F_\iota\otimes_F E$, which is isomorphic to $\mathbb{C}$ via ι°. The group $H^{(\iota)}(\mathbb{Q})$ diagonally acts on $\mathfrak{D}^{(\iota^\circ)}$ and $\mathbb{H}(\mathbb{A}_f)/K$ via the obvious way. In fact, $\mathfrak{D}^{(\iota^\circ)}$ is canonically identified with the $H^\iota(\mathbb{R})$-conjugacy class of the Hodge map $h^{(\iota)}:\mathbb{S}=\mathrm{Res}_{\mathbb{C}/\mathbb{R}}\boldsymbol{G}_{\mathrm{m},\mathbb{C}}\to H^{(\iota)}_{\mathbb{R}}\cong\mathrm{U}(1,1)_{\mathbb{R}}\times\mathrm{U}(2,0)^{d-1}_{\mathbb{R}}$ given by

$$h^{(\iota)}(z)=\left(\begin{pmatrix}1&\\&\bar z/z\end{pmatrix},\mathbf{1}_2,\dots,\mathbf{1}_2\right).$$

The Shimura curve $\mathrm{Sh}_K(\mathbb{H})$ is nonproper if and only if $F=\mathbb{Q}$ and $\Sigma(\mathbb{V})=\{\infty\}$. In this case, we can add cusps to make it proper. We denote by M_K the compactified (resp. original) Shimura curve if $\mathrm{Sh}_K(\mathbb{H})$ is nonproper (resp. proper) and by M the projective system of $(M_K)_K$ with respect to the projection $\pi^{K'}_K: M_{K'}\to M_K$. On each M_K, we have a Hodge bundle $\mathscr{L}_K\in\mathrm{Pic}(M_K)_{\mathbb{Q}}$ which is ample. They are compatible under pull-backs of $\pi_K^{K'}$, hence define an element

$$\mathscr{L} \in \mathsf{Pic}(M)_{\mathbb{Q}} := \varinjlim_K \mathsf{Pic}(M_K)_{\mathbb{Q}}.$$

Now we briefly recall the construction of *Kudla's special cycles* $Z(x)_K$ and *generating series* $Z_\phi(g)$ (see, for example, [Kudla 1997]). Here, we will make a consistent formulation as in [Liu 2011, Section 3A]. We say $x \in \mathbb{V}$ is *admissible* if $(x, x) \in E$ is totally positive definite. For x admissible, we define $Z(x)_K$ in the following manner: under the uniformization at some ι, it is represented by the points $(z, h_1 h) \in \mathscr{D}^{(\iota^\circ)} \times \mathbb{H}(\mathbb{A}_f)$ where $h \in \mathbb{H}^{\mathsf{der}}(\mathbb{A}_f)$ as in [Liu 2011, Lemma 3.1] (with respect to ι), $z \perp hx$, and h_1 fixes hx. The cycle (actually a divisor) is in fact defined over E and hence independent of ι. We define $Z(x)_K = c_1(\mathscr{L}_K^\vee)$ when $x = 0$ and 0 for all other x. In any case, $Z(x)_K$ defines an element in $\mathsf{CH}^1(M_K)_{\mathbb{Q}}$ which only depends on the class $K\langle x\rangle$.

As in [Liu 2011, Section 3A], we define a subspace $\mathscr{S}(\mathbb{V}_\iota)^{\mathsf{U}_\iota} \subset \mathscr{S}(\mathbb{V}_\iota)$ which consists of functions of the form

$$P(T(x))e^{-2\pi T(x)},$$

where P is a polynomial function on $\mathsf{Her}_1(\mathbb{C}) = \mathbb{R}$. It is a $(\mathsf{Lie} H_{r,\iota}, \mathscr{K}_{r,\iota})$-module which is generated by the Gaussian

$$\phi^0_\infty(x) = e^{-2\pi T(x)}.$$

Let

$$\mathscr{S}(\mathbb{V})^{\mathsf{U}_\infty} = \Big(\bigotimes_{\iota\in\Sigma_\infty} \mathscr{S}(\mathbb{V}_\iota)^{\mathsf{U}_\iota}\Big) \otimes \mathscr{S}(\mathbb{V}_f), \quad \mathscr{S}(\mathbb{V})^{\mathsf{U}_\infty K} = \Big(\bigotimes_{\iota\in\Sigma_\infty} \mathscr{S}(\mathbb{V}_\iota)^{\mathsf{U}_\iota}\Big) \otimes \mathscr{S}(\mathbb{V}_f)^K$$

for an open compact subgroup K of $\mathbb{H}(\mathbb{A}_f)$. Recall that we have the Weil representation $\omega_\chi = \omega_{\chi,\psi}$ of $H'(\mathbb{A}_F)$ on $\mathscr{S}(\mathbb{V})$ (see Section 2A) with $\chi : E^\times \backslash \mathbb{A}_E^\times$ such that $\chi\big|_{\mathbb{A}_F^\times} = 1$. Associated to this χ, we define a sequence $\mathfrak{k}^\chi = (\mathfrak{k}_\iota^\chi)_\iota \in \mathbb{Z}^{\Sigma_\infty}$ determined by $\chi_\iota(z) = z^{\mathfrak{k}_\iota^\chi}$ for $z \in E_\iota^{1,\times} = \mathbb{C}^{\times,1}$. In particular, $\mathfrak{k}_\iota^\chi$ are all even.

Let us recall the definition of Kudla's generating series and define the (modified) compactified one as in [Liu 2011, Section 3C]. They are

$$Z_\phi(g) = \sum_{x\in K\backslash\mathbb{V}_f} \omega_\chi(g)\phi(T(x), x) Z(x)_K,$$

$$Z_\phi^\sim(g) = \begin{cases} Z_\phi(g) & \text{if } \mathsf{Sh}(\mathbb{H})_K \text{ is proper,} \\ Z_\phi(g) + W_0\big(\tfrac{1}{2}, g, \phi\big) c_1(\mathscr{L}_K^\vee) & \text{if not,} \end{cases}$$

as series with values in $\mathsf{CH}^1(M_K)_{\mathbb{C}}$ for $\phi \in \mathscr{S}(\mathbb{V})^{\mathsf{U}_\infty K}$ and $g \in H'(\mathbb{A}_F)$, where $W_0(s, g, \phi) = \prod_v W_0(s, g_v, \phi_v)$ which is holomorphic at $s = \frac{1}{2}$. Here for $\phi = \phi_\infty \phi_f$, we denote $\phi(T(x), x) = \phi_\infty(y)\phi_f(x)$ for any $y \in \mathbb{V}_\infty$ with $T(y) = T(x)$ which does not depend on the choice of y. This makes sense since $Z(x)_K \neq 0$ only for x

admissible or equal to 0 and hence $T(x)$ is totally positive definite or 0. It is easy to see that $Z_\phi(g)$ and $Z^{\sim}_\phi(g)$ are compatible under pull-backs of $\pi^{K'}_K$, and hence define series with values in $\mathsf{CH}^1(M)_{\mathbb{C}} := \varinjlim_K \mathsf{CH}^1(M_K)_{\mathbb{C}}$. Readers may view the modification in the nonproper case as an analogy of the classical Eisenstein series $G_2(\tau)$ (which is not a modular form!). It becomes modular if we add a term $-\pi/\Im\tau$ at the price of being nonholomorphic (see, for example, [Diamond and Shurman 2005, p. 18]).

Now we apply the argument of the previous section to the curve M_K. The cycles whose heights we want to compute are the generating series $Z^{\sim}_\phi(g)$ which are not necessarily cohomologically trivial. We use the dual of the Hodge bundle $\mathscr{L} = (\mathscr{L}_K)_K \in \mathsf{Pic}(M)$ to modify as in the above section. The metric of $\mathscr{L}_{\iota^\circ,\mathbb{C}}$ for some $\iota^\circ \in \Sigma^\circ_\infty$ over $\iota \in \Sigma_\infty$ is the one descended from the H'_ι-invariant metric

$$\|v\|_{\iota^\circ} = \tfrac{1}{2}(v, v)_\iota$$

for $v \in V^{(\iota)}_\iota$ and the hermitian form $(\,\cdot\,,\cdot\,)_\iota$ of $V^{(\iota)}$ at ι. We denote by $\overline{\mathscr{L}} = (\overline{\mathscr{L}}_K)_K \in \widehat{\mathsf{Pic}}(M)$ the corresponding metrized line bundle. Since $\mathscr{L}$ is ample, $\deg c_1(\mathscr{L}_K) \neq 0$. For $\phi \in \mathscr{S}(\mathbb{V})^{\mathsf{U}_\infty K}$ and $g \in H'(\mathbb{A}_F)$, we define *the arithmetic theta series* as

$$\Theta_\phi(g) = Z^{\sim}_\phi(g) - \frac{\deg Z^{\sim}_\phi(g)}{\deg c_1(\mathscr{L}^\vee_{K'})} c_1(\mathscr{L}^\vee_{K'})$$

on any curve $M_{K'}$ with $K' \subset K$. The ratio

$$D(g, \phi) := \frac{\deg Z^{\sim}_\phi(g)}{\deg c_1(\mathscr{L}^\vee_{K'})}$$

is independent of the choice of K'.

Definition 3.1. The series $\Theta_\phi(g)$ is called the *arithmetic theta series*. It is a $\mathsf{CH}^1(M)^0_{\mathbb{C}}$-valued automorphic form of $H'(\mathbb{A}_F)$ by Corollary 3.3.

Degree of the generating series. In this subsection, we will compute the degree function $D(g, \phi)$. From $\phi \in \mathscr{S}(\mathbb{V})^{\mathsf{U}_\infty K}$, which is decomposable, we can form an Eisenstein series

$$E(s, g, \phi) = \sum_{\gamma \in P'(F)\backslash H'(F)} \omega_\chi(\gamma g)\phi(0)\lambda_{P'}(\gamma g)^{s-1/2}$$

on $H'(\mathbb{A}_F)$, which is absolutely convergent if $\Re(s) > \frac{1}{2}$ and has a meromorphic continuation to the entire complex plane. We take Tamagawa measures (with respect to ψ) dh on $\mathbb{H}(\mathbb{A})$, $d\bar{h}$ on $\mathbb{A}^{\times,1}_E = \mathbb{H}/\mathbb{H}^{\mathrm{der}}(\mathbb{A})$, and dh_x on $\mathbb{H}(\mathbb{A})_x$ which is the stabilizer of $x \in \mathbb{V}$. Now for any $v \in \Sigma$, let $b \in F^\times_v$ such that $\Omega_b := \{x \in \mathbb{V}_v \mid T(x) = b\} \neq \varnothing$. Then the local Whittaker integral $W_b(s, e, \phi_v)$ has a holomorphic continuation to the entire complex plane and $W_b\big(\tfrac{1}{2}, e, \phi_v\big)$ is not identically zero. Hence we have an N_v-intertwining map

$$\mathscr{S}(\mathbb{V}_v) \to \mathbb{C}_{N_v,\phi_b}, \quad \phi_v \mapsto W_b\big(\tfrac{1}{2}, e, \phi_v\big).$$

On the other hand, by [Rallis 1987, Lemma 4.2] for v finite and [Rallis 1987, Lemma 4.2] and [Kudla and Rallis 1994, Proposition 2.10] for v infinite (see also [Ichino 2004, Proposition 6.2]), we have

$$W_b\big(\tfrac{1}{2}, e, \phi_v\big) = \gamma_{\mathbb{V}_v} \int_{\Omega_b} \phi_v(x)\, d\mu_{v,b}(x) \tag{3-3}$$

for the quotient measure $d\mu_{v,b} = dh_v/dh_{v,x}$ on Ω_b for any $x \in \Omega_b$.

Proposition 3.2. *The Eisenstein series $E(s, g, \phi)$ is holomorphic at $s = \frac{1}{2}$ and $D(g, \phi) = E(s, g, \phi)\big|_{s=\frac{1}{2}}$.*

Proof. We can assume that ϕ is decomposable. For $b \in F^\times$, let

$$D_b(g, \phi) = \frac{1}{\deg c_1(\mathscr{L}_{K'}^\vee)} \sum_{\substack{x \in K'\backslash \mathbb{V}_f \\ T(x)=b}} \omega_\chi(g)\phi(b, x) \deg Z(x)_{K'}$$

be the b-th Fourier coefficient of $D(g, \phi)$. Now we compute the degree of $Z(x)_{K'}$ when $T(x) = b$ is totally positive. Without lost of generality, let us assume x is contained in the image of some (rational) nearby hermitian space $V^{(\iota)} \hookrightarrow \mathbb{V}_f$ and K' is sufficiently small. The isomorphism $\det : H_x^{(\iota)} \to E^{\times,1}$ induces a surjective map $H_x^{(\iota)}\backslash \mathbb{H}(\mathbb{A}_f)_x/(K' \cap \mathbb{H}(\mathbb{A}_f)_x) \to E^{\times,1}\backslash \mathbb{A}_{E,f}^{\times,1}/\det K'$. Hence

$$\deg Z(x)_{K'} = \left| \frac{\det K'}{K' \cap \mathbb{H}(\mathbb{A}_f)_x} \right| = \frac{\mathrm{vol}(\det K', d\bar{h}_f)}{\mathrm{vol}(K' \cap \mathbb{H}(\mathbb{A}_f)_x, dh_{f,x})}.$$

When $b \neq 0$ and is not totally positive, $\deg Z(x)_{K'} = 0$ by definition. Hence for b totally positive,

$$\begin{aligned} D_b(g, \phi) &= \frac{1}{\deg c_1(\mathscr{L}_{K'}^\vee)} \sum_{\substack{x \in K'\backslash \mathbb{V}_f \\ T(x)=b}} \omega_\chi(g)\phi(b, x) \frac{\mathrm{vol}(\det K')}{\mathrm{vol}(K' \cap \mathbb{H}(\mathbb{A}_f)_x)} \\ &= \frac{\omega_\chi(g_\infty)\phi_\infty(b)\,\mathrm{vol}(\det K')}{\deg c_1(\mathscr{L}_{K'}^\vee)\,\mathrm{vol}(K')} \int_{\substack{x \in \mathbb{V}_f \\ T(x)=b}} \omega_\chi(g)\phi(x)\, d\mu_b(x) \\ &= \frac{\omega_\chi(g_\infty)\phi_\infty(b)\,\mathrm{vol}(\det K')}{\deg c_1(\mathscr{L}_{K'}^\vee)\,\mathrm{vol}(K')} \prod_{v \in \Sigma_f} \int_{\Omega_b} \omega_\chi(g_v)\phi_v(x)\, d\mu_{v,b}(x) \end{aligned} \tag{3-4}$$

and $D_b(g, \phi) = 0$ otherwise.

On the other hand, $E_b(s,g,\phi)$ is holomorphic at $s=\frac{1}{2}$ for $b\neq 0$. For b not totally positive, $E_b(s,g,\phi)\big|_{s=\frac{1}{2}}=0$; otherwise,

$$E_b(s,g,\phi)\big|_{s=\frac{1}{2}}=W_b\big(\tfrac{1}{2},g,\phi\big)=\prod_{v\in\Sigma}W_b\big(\tfrac{1}{2},g_v,\phi_v\big)$$

and so, by (3-3),

$$\begin{aligned}E_b(s,g,\phi)\big|_{s=\frac{1}{2}}&=\prod_{v\in\Sigma}\gamma_{\mathbb{V}_v}\int_{\Omega_b}\omega_\chi(g_v)\phi_v(x)\,d\mu_{v,b}(x)\\&=-\mathrm{vol}(\Omega_\infty)\omega_\chi(g_\infty)\phi_\infty(b)\prod_{v\in\Sigma_f}\int_{\Omega_b}\omega_\chi(g_v)\phi_v(x)\,d\mu_{v,b}(x),\end{aligned}\tag{3-5}$$

where $\mathrm{vol}(\Omega_\infty)=\mathrm{vol}(\Omega_{\infty,b})$ for any totally positive b. Let

$$D=\frac{\mathrm{vol}(\det K')}{\mathrm{vol}(\Omega_\infty)\deg c_1(\mathscr{L}_{K'})\,\mathrm{vol}(K')}.$$

Now we compute the constant term

$$D_0(g,\phi)=\omega_\chi(g)\phi(0)+W_0\big(\tfrac{1}{2},g,\phi\big).$$

On the other hand, the constant term of $E\big(\frac{1}{2},g,\phi\big)$ is

$$E_0\big(\tfrac{1}{2},g,\phi\big)=\omega_\chi(g)\phi(0)+W_0\big(\tfrac{1}{2},g,\phi\big).$$

Here the intertwining term $W_0(\frac{1}{2},g,\phi)$ is nonzero only if $\mathrm{Sh}_K(\mathbb{H})$ is nonproper, that is, $|\Sigma(\mathbb{V})|=1$. Now, if $\mathrm{Sh}_K(\mathbb{H})$ is proper, then we can apply the theorem of modularity of the generating series (see [Liu 2011, Theorem 3.5]) to see that $D(g,\phi)$ is already an automorphic form. Comparing the ratio of the constant term and nonconstant terms, we find that $D=1$. Second, if $\mathrm{Sh}_K(\mathbb{H})$ is not proper, we calculate the degree of the Hodge bundle in the classical way on modular curves and find that $D=1$.

Let $E(g,\phi)=E(s,g,\phi)\big|_{s=\frac{1}{2}}-W_0(\frac{1}{2},g,\phi)$; then

$$\Theta_\phi(g)=Z_\phi(g)-E(g,\phi)c_1(\mathscr{L}_K^\vee).$$

If $|\Sigma(\mathbb{V})|>1$, $W_0(\frac{1}{2},g,\phi)=0$; otherwise, it equals $C(\tilde{\chi}\circ\det)$, where C is a constant and $\tilde{\chi}$ is the descent of χ to $\mathbb{A}_E^{\times,1}$. Precisely, $\tilde{\chi}(x)=\chi(e_x)$ for any $e_x\in\mathbb{A}_E^\times$ such that $x=e_x/e_x^\tau$. In any case, $E(g,\phi)$ is a linear combination of an Eisenstein series and an automorphic character for g in $P'_\infty H'(\mathbb{A}_{f,F})$. □

From this computation, we have the following corollary of the *modularity of the generating series in the compactified case*:

Corollary 3.3. *For any linear functional $\ell\in\mathrm{CH}^1(M)^*_{\mathbb{C}}$, $\ell(Z^{\sim}_\phi)(g)$ (and hence $\ell(\Theta_\phi)(g)$ as well) is absolutely convergent and an automorphic form of $H'(\mathbb{A}_F)$.*

Proof. We only need to prove that $Z^{\sim}_{\phi}(\gamma g) = Z^{\sim}_{\phi}(g) \in \mathrm{Pic}(M)_{\mathbb{C}}$ for any $\gamma \in H'(\mathbb{Q})$. But by [Liu 2011, Theorem 3.5] and the fact that the Hodge bundle is supported on the cusps, $Z^{\sim}_{\phi}(\gamma g) = Z^{\sim}_{\phi}(g)$ in $\mathrm{CH}^1(\mathrm{Sh}_K(\mathbb{H}))_{\mathbb{C}}$, so their difference must be supported on the set of cusps. By a theorem due to Manin [1972] and Drinfeld [1973], which posits that any two cusps are the same in $\mathrm{CH}^1(M_K)_{\mathbb{C}}$, we have an exact sequence

$$\mathbb{C} \to \mathrm{CH}^1(M_K)_{\mathbb{C}} \to \mathrm{CH}^1(\mathrm{Sh}_K(\mathbb{H}))_{\mathbb{C}} \to 0.$$

Hence we only need to prove that $\deg Z^{\sim}_{\phi}(\gamma g) = \deg Z^{\sim}_{\phi}(g)$, which is true by the above proposition. □

Definition 3.4. Let π be an irreducible cuspidal automorphic representation of $H'(\mathbb{A}_F)$. For any cusp form $f \in \pi$ and $\phi \in \mathcal{S}(\mathbb{V})^{\mathsf{U}_\infty}$, the integral

$$\Theta^f_\phi = \int_{H'(F)\backslash H'(\mathbb{A}_F)} f(g)\Theta_\phi(g)\,dg \in \mathrm{CH}^1(M)^0_{\mathbb{C}}$$

is called the *arithmetic theta lifting* of f which is a divisor on (compactified) Shimura curves. The original idea of this construction comes from Kudla [2003, Section 8; 2006, Section 9.1]. He constructed the arithmetic theta series as an Arakelov divisor on a certain integral model of a Shimura curve.

Geometric kernel functions. For $\Phi = \sum \phi_{i,1} \otimes \phi_{i,2}$ with $\phi_{i,j} \in \mathcal{S}(\mathbb{V})^{\mathsf{U}_\infty K}$, we define the *geometric kernel function* associated to the test function Φ to be

$$\mathbb{E}(g_1, g_2; \Phi) := \mathrm{vol}(K') \sum \langle \Theta_{\phi_{i,1}}(g_1), \Theta_{\phi_{i,2}}(g_2) \rangle^{K'}_{\mathrm{NT}},$$

where the measure giving $\mathrm{vol}(K')$ is defined in [Liu 2011, Theorem 4.20], and the superscript K' means that we are taking the Néron–Tate height pairing on the curve $M_{K'}$ for some $K' \subset K$. The function is independent of what K' we choose. By Corollary 3.3, $\mathbb{E}(g_1, g_2; \Phi) \in \mathcal{A}(H' \times H')$, the space of automorphic forms of the group $H'(\mathbb{A}_F) \times H'(\mathbb{A}_F)$. Now let us just work over M_K and choose a regular model $\mathcal{M}_K$ of it. We fix an arithmetic line bundle $\widehat{\omega}_K$ to extend $\overline{\mathcal{L}}^\vee_K$. Of course, the metric on $\widehat{\omega}_K$ is same as that on $\overline{\mathcal{L}}^\vee_K$.

Now since the map $p_{\mathcal{M}_K}$ (see (3-1)) is surjective, we can fix an inverse linear map $p^{-1}_{\mathcal{M}_K}$ and write

$$\widehat{\Theta}_\phi(g) := p^{-1}_{\mathcal{M}_K}(\Theta_\phi(g)) = ([Z_\phi(g)]^{\mathrm{Zar}}, g_{\iota^\circ}) + (\mathcal{V}_\phi(g), 0) - E(g, \phi)\widehat{\omega}_K,$$

where g_{ι° is an $\overline{\mathcal{L}}_K$-admissible Green's function of $Z_\phi(g)$ and $\mathcal{V}_\phi(g)$ is the sum of (finitely many) vertical components supported on special fibers. We also write simply

$$\widehat{Z}_\phi(g) = ([Z_\phi(g)]^{\mathrm{Zar}}, g_{\iota^\circ}) + (\mathcal{V}_\phi(g), 0).$$

Then we have for $\phi_i \in \mathcal{S}(\mathbb{V})^{\mathsf{U}_\infty K}$ $(i = 1, 2)$,

$$\begin{aligned}
&\mathbb{E}(g_1, g_2; \phi_1 \otimes \phi_2) \\
&\quad = \mathrm{vol}(K)\langle \Theta_{\phi_1}(g_1), \Theta_{\phi_2}(g_2)\rangle^K_{\mathsf{NT}} \\
&\quad = -\mathrm{vol}(K)\langle \widehat{\Theta}_{\phi_1}(g_1), \widehat{\Theta}_{\phi_2}(g_2)\rangle_{\mathsf{GS}} \\
&\quad = -\mathrm{vol}(K)\langle \widehat{Z}_{\phi_1}(g_1) - E(g_1, \phi_1)\widehat{\omega}_K, \widehat{Z}_{\phi_2}(g_2) - E(g_2, \phi_2)\widehat{\omega}_K\rangle_{\mathsf{GS}} \\
&\quad = -\mathrm{vol}(K)\langle \widehat{Z}_{\phi_1}(g_1), \widehat{Z}_{\phi_2}(g_2)\rangle_{\mathsf{GS}} + E(g_1, \phi_1)\,\mathrm{vol}(K)\langle \widehat{\omega}_K, \widehat{\Theta}_{\phi_2}(g_2)\rangle_{\mathsf{GS}} \\
&\qquad\qquad + E(g_2, \phi_2)\,\mathrm{vol}(K)\langle \widehat{\Theta}_{\phi_1}(g_1), \widehat{\omega}_K\rangle_{\mathsf{GS}} \\
&\qquad\qquad + E(g_1, \phi_1)E(g_2, \phi_2)\,\mathrm{vol}(K)\langle \widehat{\omega}_K, \widehat{\omega}_K\rangle_{\mathsf{GS}},
\end{aligned} \tag{3-6}$$

where the Gillet–Soulé pairing are taken on the model $\mathcal{M}_K$. By Corollary 3.3, $A(g, \phi) := \mathrm{vol}(K)\langle \widehat{\omega}_K, \widehat{\Theta}_\phi(g)\rangle_{\mathsf{GS}}$ is an automorphic form of H' which may depend on K, and also on the model $\mathcal{M}_K$ since we do not require any canonicality of $p^{-1}_{\mathcal{M}_K}$. Let $C = \mathrm{vol}(K)\langle \widehat{\omega}_K, \widehat{\omega}_K\rangle_{\mathsf{GS}}$. Then

$$\begin{aligned}
(3\text{-}6) = -\mathrm{vol}(K)\langle \widehat{Z}_{\phi_1}(g_1), \widehat{Z}_{\phi_2}(g_2)\rangle_{\mathsf{GS}} + E(g_1, \phi_1)A(g_2, \phi_2) \\
+ A(g_1, \phi_1)E(g_2, \phi_2) + C E(g_1, \phi_1)E(g_2, \phi_2).
\end{aligned} \tag{3-7}$$

Now we assume that ϕ_1 and ϕ_2 are decomposable and $\phi_{1,v} \otimes \phi_{2,v} \in \mathcal{S}(\mathbb{V}_v^2)_{\mathsf{reg}}$ for some $v \in \Sigma_f$. Then $Z_{\phi_1}(g_1)$ and $Z_{\phi_2}(g_2)$ will not intersect on the generic fiber if $g_i \in P'_v H'(\mathbb{A}_F^v)$ $(i = 1, 2)$. Then

$$\langle \widehat{Z}_{\phi_1}(g_1), \widehat{Z}_{\phi_2}(g_2)\rangle_{\mathsf{GS}} = \sum_{v^\circ \in \Sigma^\circ} \langle \widehat{Z}_{\phi_1}(g_1), \widehat{Z}_{\phi_2}(g_2)\rangle_{v^\circ}, \tag{3-8}$$

where the intersection $\langle\,\cdot\,,\,\cdot\,\rangle_{v^\circ}$ is taken on the local model $\mathcal{M}_{K;\mathfrak{p}^\circ} := \mathcal{M}_K \times_{\mathcal{O}_E} \mathcal{O}_{E_{\mathfrak{p}^\circ}}$ if $v^\circ = \mathfrak{p}^\circ$ is finite and $M_{K,\iota^\circ}(\mathbb{C})$ if $v^\circ = \iota^\circ$ is infinite. Combining (3-7) and (3-8), we have for such ϕ_i and g_i $(i = 1, 2)$,

$$\begin{aligned}
\mathbb{E}(g_1, g_2; \phi_1 \otimes \phi_2) = -\mathrm{vol}(K) \sum_{v^\circ \in \Sigma^\circ} \langle \widehat{Z}_{\phi_1}(g_1), \widehat{Z}_{\phi_2}(g_2)\rangle_{v^\circ} + E(g_1, \phi_1)A(g_2, \phi_2) \\
+ A(g_1, \phi_1)E(g_2, \phi_2) + C E(g_1, \phi_1)E(g_2, \phi_2).
\end{aligned} \tag{3-9}$$

4. Comparison at finite places: good reduction

4A. ***Nonarchimedean Whittaker integrals.*** In this section, we calculate certain Whittaker integrals $W_T(s, g, \Phi)$ and their derivatives (at $s = 0$) at a nonarchimedean place when T has rank 2.

Let $F/\mathbb{Q}_p$ be a finite extension and E/F a quadratic extension with $\mathrm{Gal}(E/F) = \{1, \tau\}$. We fix a uniformizer ϖ of F and let q be the cardinality of the residue field of F. Let V^+ (resp. V^-) be the two-dimensional E-hermitian space with $\epsilon(V^+) = 1$ (resp. $\epsilon(V^-) = -1$); it is unique up to isometry. Set $H^\pm = \mathsf{U}(V^\pm)$, and let $\Lambda^\pm$

be a maximal $\mathcal{O}_E$-lattice in $V^{\pm}$ where the hermitian form takes values in $\mathcal{O}_E$. Let $\phi^{0\pm} \in \mathcal{S}(V^{\pm})$ (resp. $\Phi^{0\pm} \in \mathcal{S}((V^{\pm})^2)$) be the characteristic function of $\Lambda^{\pm}$ (resp. $(\Lambda^{\pm})^2$), and let $K_0^{\pm}$ be the stabilizer of $\Lambda^{\pm}$ in $H^{\pm}$ which is a maximal compact subgroup. Recall that we have local reductive groups $H' \cong H_1$, $H'' \cong H_2$, $P, \dots$.

Now, we assume that E/F is unramified and $p > 2$. Let ψ be the unramified character of F. For nonsingular $T \in \mathrm{Her}_2(E)$, we consider the Whittaker integral,

$$W_T(s, g, \Phi^{0+}) = \int_{\mathrm{Her}_2(E)} \varphi_{\Phi^{0+},s}(wn(u)g)\psi_T(n(u))^{-1}\, du, \tag{4-1}$$

for $\Re(s) > 1$, where du is the self-dual measure with respect to ψ. We write $g = n(b)m(a)k$ under the Iwasawa decomposition of H''. Then

$$\begin{aligned}(4\text{-}1) &= \int_{\mathrm{Her}_2(E)} \omega''_{\mathbf{1}}(wn(u)n(b)m(a)k)\Phi^{0+}(0)\lambda_P(wn(u)n(b)m(a)k)^s\psi(-\mathrm{tr}\, Tu)\, du\\ &= \psi(\mathrm{tr}\, Tb)\int_{\mathrm{Her}_2(E)} \lambda_P(wn(u)m(a))^s\psi(-\mathrm{tr}\, Tu)\, du\\ &= \psi(\mathrm{tr}\, Tb)|\det a|_E^{1-s} W_{{}^t a^\tau Ta}(s, e, \Phi^{0+}).\end{aligned}$$

Hence we only need to consider the integral $W_T(s, e, \Phi^{0+})$. If $T \notin \mathrm{Her}_2(\mathcal{O}_E)$, then $W_T(s, e, \Phi^{0+})$ is identically 0. For $T \in \mathrm{Her}_2(\mathcal{O}_E)$, it is known (see [Kudla 1997, Appendix], for example) that for $r \in \mathbb{Z}$ and $r > 1$, $W_T(r, e, \Phi^{0+}) = \gamma_{V^+}\alpha_F(\mathbf{1}_{2+r}, T)$ where γ_{V^+} is the Weil constant and α_F is the classical representation density (for hermitian matrices). From [Hironaka 1999], we see that for $r \geq 0$

$$\alpha_F(\mathbf{1}_{2+r}, T) = P_F(\mathbf{1}_2, T; (-q)^{-r})$$

for a polynomial $P_F(\mathbf{1}_2, T; X) \in \mathbb{Q}[X]$. By analytic continuation, we see that

$$W_T(s, e, \Phi^{0+}) = \gamma_{V^+} P_F(\mathbf{1}_2, T; (-q)^{-s}).$$

If $\mathrm{ord}(\det T)$ is odd, that is, if T cannot be represented by V^+, then we know that $W_T(0, e, \Phi^{0+}) = P_F(\mathbf{1}_2, T; 1) = 0$. Taking the derivative at $s = 0$, we have

$$W'_T(0, e, \Phi^{0+}) = -\gamma_{V^+} \log q \cdot \frac{d}{dX} P_F(\mathbf{1}_2, T; X)\Big|_{X=1}.$$

Proposition 4.1 [Hironaka 1999]. *If T is $\mathrm{GL}_2(\mathcal{O}_E)$-equivalent to $\mathrm{diag}[\varpi^a, \varpi^b]$ with $0 \leq a < b$, then*

$$P_F(\mathbf{1}_2, T; X) = (1 + q^{-1}X)(1 - q^{-2}X) \sum_{l=0}^{a} (qX)^l \left(\sum_{k=0}^{a+b-2l} (-X)^k \right).$$

Corollary 4.2. *If $a + b$ is odd, then*

$$W'_T(0, e, \Phi^{0+}) = \gamma_{V^+} b_2(0)^{-1} \log q \cdot \frac{1}{2} \sum_{l=0}^{a} q^l(a + b - 2l + 1).$$

4B. ***Integral models.*** We now introduce the integral model of the Shimura curves M_K with full-level structure at $\mathfrak{p}$ and integral special subschemes. First, we fix some notation for Sections 4 and 5.

- For any rational prime p, we fix an isomorphism $\iota_{(p)} : \mathbb{C} \xrightarrow{\sim} \mathbb{C}_p$ once for all.
- Let ι_i $(i = 1, \dots, d)$ be all the embeddings of F into $\mathbb{C}$, and let ι_i° and $\iota_i^\bullet$ be the embeddings of E into $\mathbb{C}$ that induce ι_i.
- For $\mathfrak{p}$ a finite place of F, let $\mathfrak{p}^\circ$ be that of E over $\mathfrak{p}$ if $\mathfrak{p}$ is nonsplit and $\mathfrak{p}^\circ$ and $\mathfrak{p}^\bullet$ be those of E if $\mathfrak{p}$ is split. We fix a uniformizer ϖ of $F_\mathfrak{p}$.
- For a number field F, $T_F = \mathrm{Res}_{F/\mathbb{Q}} \boldsymbol{G}_{\mathrm{m},F}$ and $T^1_{F'} = \mathrm{Res}_{F/\mathbb{Q}} F'^{\times,1}$ for any quadratic extension F'/F where $F'^{\times,1} = \ker[\mathrm{Nm} : F'^\times \to F^\times]$. If F is totally real, then F^+ is the set of all totally positive elements.
- For any finite extension $L/\mathbb{Q}_p$ of local fields with ring of integers $\mathbb{O}_L$ and maximal ideal $\mathfrak{q} \subset \mathbb{O}_L$, let U_L^s be the subgroup of $\mathbb{O}_L^\times$ congruent to 1 mod $\mathfrak{q}^s$. Denote by L^{nr} the maximal unramified extension of L and $\hat{L}$ its completion whose ring of integers is $\mathbb{O}_{\hat{L}}$. Let L^s be the finite extension of $L^{\mathrm{nr}} = L^0$ corresponding to U_L^s through local class field theory and $\hat{L}^s$ its completion.
- Let $\mathbb{F}$ be the algebraic closure of $\mathbb{F}_p$.

If the Shimura curve $\mathrm{Sh}_K(\mathbb{H})$ is nonproper (then $F = \mathbb{Q}$), we add cusps to make it proper — this holds also for integral models (see Remark 4.10 and [Katz and Mazur 1985] for details). In what follows, we will not pay any attention to these cusps since they will not affect our later computations. In the first two subsections, we recall some results in [Carayol 1986] which are useful for us.

Change of Shimura data. Let $\mathfrak{p} = \mathfrak{p}_1, \mathfrak{p}_2, \dots, \mathfrak{p}_r$ $(1 \le r \le d)$ be all places of F dividing p and $\mathfrak{p}^\circ$ the place of E above $\mathfrak{p}$. We assume that the embedding

$$\iota_{(p)} \circ \iota_1^\circ : E \hookrightarrow \mathbb{C}_p$$

induces the place $\mathfrak{p}^\circ$. As before, we suppress ι_1 for the nearby objects. Hence, we have the hermitian space V over E of dimension two, whose signature is $(1, 1)$ at ι_1 and $(2, 0)$ elsewhere, the unitary group H, and the Shimura curve $M_K = \mathrm{Sh}_K(H, X)$ for a sufficiently small open compact subgroup $K \subset H(\mathbb{A}_f)$, which is a smooth projective curve defined over $\iota_1^\circ(E)$. Here X is the conjugacy class of the Hodge map $h : \mathbb{S} \to H_\mathbb{R}$ defined by

$$z = x + iy \mapsto \left[\begin{pmatrix} x & y \\ -y & x \end{pmatrix}^{-1} \times z, 1, \dots, 1\right] \in H(\mathbb{R}) \subset (\mathrm{GL}_2(\mathbb{R}) \times_{\mathbb{R}^\times} \mathbb{C}^\times) \times (\mathbb{H}^\times \times_{\mathbb{R}^\times} \mathbb{C}^\times)^{d-1},$$

where we identify $T_E(\mathbb{R})$ with $(\mathbb{C}^\times)^d$ via $(\iota_1^\circ, \dots, \iota_d^\circ)$. We denote by $\nu : H \to T_E^1$ the determinant map. We have the zero-dimensional Shimura variety

$$L_K = \mathrm{Sh}_{\nu(K)}(T_E^1, \nu(X))$$

and a smooth morphism (also denoted by) $\nu : M_K \to L_K$ between $\iota_1^\circ(E)$-schemes such that the fiber of each geometric point is connected.

Now let us define a subgroup $K_{\mathfrak{p},n}$ of $\mathrm{U}(V_\mathfrak{p})$. Recalling the notation in the last section and assuming that $V_\mathfrak{p} \cong V^\pm$, we have the lattice $\Lambda^\pm$ (if $\mathfrak{p}$ is split, we only have the positive one). For any integer $n \geq 0$, we define $K_{\mathfrak{p},n}$ to be the subgroup of $K_0^\pm$ whose elements act on $\Lambda^\pm/\varpi^n\Lambda^\pm$ trivially. Then $K_{\mathfrak{p},0} = K_0^\pm$ is a maximal compact subgroup. For $K = K_{\mathfrak{p},n} \times K^\mathfrak{p}$, we write $M_{n,K^\mathfrak{p}}$ (resp. $L_{n,K^\mathfrak{p}}$) for M_K (resp. L_K).

Since the Shimura datum (H, X) is not of PEL type when $F \neq \mathbb{Q}$ (even when $F = \mathbb{Q}$, we still need another PEL datum for later computation, see Remark 4.10), we need a variation in order to obtain the moduli interpretation, and hence the integral model. This is analogous to the case considered in [Carayol 1986; Zhang 2001a; 2001b; Yuan et al. 2011] and we refer for the detailed proof of various facts to [Carayol 1986]. We choose a negative number $\lambda \in \mathbb{Q}$ such that the extension $\mathbb{Q}(\sqrt{\lambda})$ is split at p and the CM extension $F^\dagger = F(\sqrt{\lambda})/F$ with $\mathrm{Gal}(F^\dagger/F) = \{1, \tau^\dagger\}$ is not isomorphic to E/F. We fix a square root of λ in $\mathbb{C}$ with positive imaginary part, say λ', and a square root of λ in $\mathbb{Q}_p$, λ_p. Let ι_i^1 (resp. ι_i^2) be the embeddings of $F^\dagger$ into $\mathbb{C}$ above ι_i ($i = 1, \dots, d$) which sends $\sqrt{\lambda}$ to λ' (resp. $-\lambda'$). Since p is split in $\mathbb{Q}(\sqrt{\lambda})$, each $\mathfrak{p}_i$ ($i = 1, \dots, r$) is split in $F^\dagger$; we denote by $\mathfrak{p}_i^1$ (resp. $\mathfrak{p}_i^2$) the place above $\mathfrak{p}_i$ that sends $\sqrt{\lambda}$ to λ_p (resp. $-\lambda_p$) and assume that $\iota_{(p)} \circ \iota_1^1$ induces $\mathfrak{p}_1^1$.

By the Hasse principle, we see that there is a unique quaternion algebra B over F, up to isomorphism, such that B, as an F-quadratic space (of dimension 4), is isometric to V viewed as an F-quadratic space with the quadratic form $\frac{1}{2}\mathrm{Tr}_{E/F}(\,\cdot\,,\,\cdot\,)$ where $(\,\cdot\,,\,\cdot\,)$ is the hermitian form on V. More precisely, for v finite, $B_v = B \otimes_F F_v$ is nonsplit if and only if v is nonsplit and $V_v \cong V^-$. Also, $B_{\iota_1}(\mathbb{R}) \cong \mathrm{Mat}_2(\mathbb{R})$ and $B_{\iota_i}(\mathbb{R}) \cong \mathbb{H}$ for $i > 1$. We identify the two quadratic spaces B and V through a fixed isometry; hence V has both left and right multiplication by B. We fix an embedding $E \hookrightarrow B$ through which the action of E induced from the left multiplication of B coincides with the E-vector space structure of V. Let $G = \mathrm{Res}_{F/\mathbb{Q}} B^\times$ with center $T \cong T_F$ and

$$G^\dagger := G \times_T T_{F^\dagger} \xrightarrow{\nu^\dagger} T \times T^1_{F^\dagger},$$

where $\nu^\dagger$ sends $(g \times z)$ to $(\mathrm{Nm}g \cdot zz^{\tau^\dagger}, z/z^{\tau^\dagger})$. Consider the subtorus $T^\dagger = \boldsymbol{G}_{\mathrm{m},\mathbb{Q}} \times T^1_{F^\dagger}$ and let $H^\dagger$ be the preimage of $T^\dagger$ under $\nu^\dagger$. We define the Hodge map $h^\dagger : \mathbb{S} \to H^\dagger_\mathbb{R} \subset G_\mathbb{R} \times_{T_\mathbb{R}} T_{F^\dagger,\mathbb{R}}$ by

$$z = x + iy \mapsto \left[\begin{pmatrix} x & y \\ -y & x \end{pmatrix}^{-1} \times 1, \mathbf{1}_2 \times z^{-1}, \dots, \mathbf{1}_2 \times z^{-1}\right]$$

and let $X^\dagger$ be the $H^\dagger(\mathbb{R})$ conjugacy class of $h^\dagger$, where we identify $T_{F^\dagger}(\mathbb{R})$ with $(\mathbb{C}^\times)^d$ through $(\iota_1^1, \dots, \iota_d^1)$. So we have the Shimura curve $M^\dagger_{K^\dagger} := \mathrm{Sh}_{K^\dagger}(H^\dagger, X^\dagger)$

which is defined over $\iota_1^1(F^\dagger)$ for an open compact subgroup $K^\dagger$ of $H^\dagger(\mathbb{A}_f)$. Similarly, we have the smooth morphism $\nu^\dagger : M^\dagger_{K^\dagger} \to L^\dagger_{K^\dagger}$. Moreover $h^\dagger(i)$ defines a complex structure on V_{ι_1}; hence V_{ι_1} becomes a complex hermitian space of dimension 2 which is isometric to its original complex hermitian space structure from the E-hermitian space V. Then $X^\dagger$ can be identified with the set of negative definite complex lines in V_{ι_1}; and hence $X^\dagger$ is isomorphic to X as hermitian symmetric domains.

As in [Carayol 1986, Section 2.2], we can view $H^\dagger$ as a group of symplectic similitude. In fact, let $B^\dagger = B \otimes_F F^\dagger$ and let $b \mapsto \bar{b}$ be the involution of the second kind on $B^\dagger$ which is the tensor product of the canonical involution of B and the conjugation of $F^\dagger$. Consider the $\mathbb{Q}$-vector space $V^\dagger$ underlying $B^\dagger$. We define a symplectic form by

$$\psi^\dagger(v, w) := \mathrm{Tr}_{F^\dagger/\mathbb{Q}}(\sqrt{\lambda}\,\mathrm{Tr}_{B^\dagger/F^\dagger}(v\bar{w}))$$

for $v, w \in B^\dagger$. Then $H^\dagger$ can be identified with the group of $B^\dagger$-linear symplectic similitude of $(V^\dagger, \psi^\dagger)$ with the left action given by $h.v = vh^{-1}$ and hence $H^\dagger(\mathbb{Q}_p)$ can be identified with the group $\mathbb{Q}_p^\times \times \prod_{i=1}^r B_{\mathfrak{p}_i}^\times$. For any open compact subgroups $K_p^{\dagger,\mathfrak{p}}$ of $\prod_{i=2}^r B_{\mathfrak{p}_i}^\times$ and $K^{\dagger,p}$ of $H^\dagger(\mathbb{A}_f^p)$, we simply write $M^\dagger_{0,K_p^{\dagger,\mathfrak{p}},K^{\dagger,p}}$ for $M^\dagger_{K^\dagger}$ where $K^\dagger = \mathbb{Z}_p^\times \times \mathbb{O}_{B_\mathfrak{p}}^\times \times K_p^{\dagger,\mathfrak{p}} \times K^{\dagger,p}$ and similarly for $L^\dagger_{0,K_p^{\dagger,\mathfrak{p}},K^{\dagger,p}}$.

Through $\iota_{(p)}$, we can take the base change of the curve M_K to $E_{\mathfrak{p}^\circ}$ and the curve $M^\dagger_{K^\dagger}$ to $F^\dagger_{\mathfrak{p}_1^\dagger} \cong F_\mathfrak{p}$ embedded in $E_{\mathfrak{p}^\circ}$, which are denoted by $M_{K;\mathfrak{p}^\circ}$ and $M^\dagger_{K^\dagger;\mathfrak{p}}$, and similarly for L and $L^\dagger$. Since H and $H^\dagger$ have the same derived subgroup, which is also the derived subgroup of G, we have

Proposition 4.3 [Carayol 1986, Section 4]. *Let $K^\mathfrak{p} \subset H_f^\mathfrak{p} := \mathrm{U}(V \otimes_F \mathbb{A}_{f,F}^\mathfrak{p})$ be an open compact subgroup which is decomposable and sufficiently small. Then there is an open compact subgroup $K_p^{\dagger,\mathfrak{p}} \times K^{\dagger,p} \subset \prod_{i=2}^r B_{\mathfrak{p}_i}^\times \times H^\dagger(\mathbb{A}_f^p)$, such that the geometric neutral components $M^\circ_{0,K^\mathfrak{p};\mathfrak{p}^\circ}$ and $M^{\dagger,\circ}_{0,K_p^{\dagger,\mathfrak{p}},K^{\dagger,p};\mathfrak{p}}$ are defined and isomorphic over $E^{\mathrm{nr}}_{\mathfrak{p}^\circ}$.*

Moduli interpretations and integral models. From the Hodge map defined above, we have a Hodge filtration $0 \subset \mathrm{Fil}^0(V^\dagger_\mathbb{C}) = (V^\dagger_\mathbb{C})^{0,-1} \subset V^\dagger_\mathbb{C}$ and define $t^\dagger(b) = \mathrm{tr}(b;\, V^\dagger_\mathbb{C}/\mathrm{Fil}^0(V^\dagger_\mathbb{C})) \in \iota_1^1(F^\dagger)$ for $b \in B^\dagger$. For sufficiently small $K^\dagger$, the curve $M^\dagger_{K^\dagger}$ represents the following moduli functor (see [Kottwitz 1992]) on the category of locally noetherian schemes over $\iota_1^1(F^\dagger)$: for such a scheme S, $M^\dagger_{K^\dagger}(S)$ is the set of equivalence classes of quadruples $(A, \theta, i, \bar{\eta})$, where

- A is an abelian scheme over S of dimension $4d$;
- $\theta : A \to A^\vee$ is a polarization;
- $i : B^\dagger \hookrightarrow \mathrm{End}^0(A)$ is a monomorphism of $\mathbb{Q}$-algebras such that, for all $b \in B^\dagger$, we have $\mathrm{tr}(i(b); \mathrm{Lie}_S(A)) = t^\dagger(b)$ and $\theta \circ i(b) = i(\bar{b})^\vee \circ \theta$;

- $\bar{\eta}$ is a $K^{\dagger}$-level structure; that is, for a chosen geometric point s on each connected component of S, $\bar{\eta}$ is a $\pi_1(S,s)$-invariant $K^{\dagger}$-orbit of $B^{\dagger}\otimes\mathbb{A}_f$-linear symplectic similitude $\eta : V^{\dagger}\otimes\mathbb{A}_f \to \mathrm{H}_1^{\text{ét}}(A_s,\mathbb{A}_f)$, where the pairing on the latter space is the θ-Weil pairing.

Here in the third condition, we view $t^{\dagger}(b)$ as a constant section of $\mathbb{O}_S$ via the structure map $\iota_1^1(F^{\dagger})\to\mathbb{O}_S$. This convention is also applied to other trace conditions appearing later. The two quadruples $(A,\theta,i,\bar{\eta})$ and $(A',\theta',i',\bar{\eta}')$ are equivalent if there is an isogeny $A\to A'$ which takes θ to a $\mathbb{Q}^{\times}$-multiple of θ', i to i', and $\bar{\eta}$ to $\bar{\eta}'$.

Now taking the base change through $\iota_{(p)}$, we obtain the functor $M^{\dagger}_{K^{\dagger};\mathfrak{p}}$ over the completion $[\iota_1^1(F^{\dagger})]^{\wedge}_{\iota_{(p)}}\cong F_{\mathfrak{p}}$. For any $(A,\theta,i,\bar{\eta})\in M^{\dagger}_{K^{\dagger};\mathfrak{p}}(S,s)$, $\mathrm{Lie}_S(A)$ is a $B^{\dagger}_p=B^{\dagger}\otimes_{\mathbb{Q}}\mathbb{Q}_p$-module. Since the algebra $B^{\dagger}_p=B\otimes_F(F^{\dagger}\otimes\mathbb{Q}_p)$ decomposes as

$$B^{\dagger}_p=B^1_1\oplus B^1_2\oplus\cdots\oplus B^1_r\oplus B^2_1\oplus B^2_2\oplus\cdots\oplus B^2_r,$$

where $B^j_i=B^{\dagger}\otimes_F F^{\dagger}_{\mathfrak{p}^j_i}$ is isomorphic to $B_{\mathfrak{p}_i}$ as $F_{\mathfrak{p}_i}$-algebra, the $B^{\dagger}_p$-module $\mathrm{Lie}_S(A)$ decomposes as

$$\mathrm{Lie}_S(A)\;=\;\bigoplus_{i=1}^r\mathrm{Lie}^1_i(A)\;\oplus\;\bigoplus_{i=1}^r\mathrm{Lie}^2_i(A),$$

while

$$A_{p^{\infty}}\;=\;\bigoplus_{i=1}^r(A_{p^{\infty}})^1_i\;\oplus\;\bigoplus_{i=1}^r(A_{p^{\infty}})^2_i,$$

for the associated p-divisible group. Since the involution $b\mapsto\bar{b}$ on $B^{\dagger}_p$ changes the factors B^1_i and B^2_i, by computing the trace we see that the condition that $\mathrm{tr}(i(b);\mathrm{Lie}_S(A))=t^{\dagger}(b)$ is equivalent to

$$\mathrm{tr}(b\in B^2_1;\mathrm{Lie}^2_1(A))=\mathrm{Tr}_{B^2_1/F_{\mathfrak{p}}}(b)\text{ and }\mathrm{Lie}^2_i(A)=0\text{ for }i\geq 2.\tag{4-2}$$

Fix a maximal order $\Lambda^2_i=\mathbb{O}_{B_{\mathfrak{p}_i}}$ of B^2_i for each $i=1,\dots,r$ and let Λ^1_i be the dual of Λ^2_i. Then

$$\Lambda_p\;=\;\bigoplus_{i=1}^r\Lambda^1_i\;\oplus\;\bigoplus_{i=1}^r\Lambda^2_i\;\subset\;\bigoplus_{i=1}^r(V^{\dagger}_p)^1_i\;\oplus\;\bigoplus_{i=1}^r(V^{\dagger}_p)^2_i\;=\;V^{\dagger}_p\;=\;V^{\dagger}\otimes\mathbb{Q}_p$$

is a $\mathbb{Z}_p$-lattice in $V^{\dagger}_p$ and self-dual under $\psi^{\dagger}$. There is a unique maximal $\mathbb{Z}_{(p)}$-order $\mathbb{O}^{\dagger}\subset B^{\dagger}$ such that $\bar{\mathbb{O}}^{\dagger}=\mathbb{O}^{\dagger}$ and $\mathbb{O}^{\dagger}_{\mathfrak{p}^2_i}=\mathbb{O}_{B_{\mathfrak{p}_i}}$ acting on Λ^2_i where $\mathbb{O}^{\dagger}_{\mathfrak{p}^2_i}$ is the image of $\mathbb{O}^{\dagger}\otimes_{\mathbb{Z}_{(p)}}\mathbb{Z}_p\subset B^{\dagger}\otimes_{\mathbb{Q}}\mathbb{Q}_p=B^{\dagger}_p$ in the B^2_i component. Then the functor $M^{\dagger}_{0,K^{\dagger,\mathfrak{p}}_p,K^{\dagger,p};\mathfrak{p}}$ is isomorphic to the following moduli functor in the category of locally noetherian schemes over $F_{\mathfrak{p}}$: for such a scheme S, $M^{\dagger}_{0,K^{\dagger,\mathfrak{p}}_p,K^{\dagger,p};\mathfrak{p}}(S)$ is the set of equivalence classes of quintuples $(A,\theta,i,\bar{\eta}^p,\bar{\eta}^{\mathfrak{p}}_p)$ where

- A is an abelian scheme over S of dimension $4d$;
- $\theta : A \to A^\vee$ is a prime-to-p polarization;
- $i : \mathbb{O}^\dagger \hookrightarrow \mathsf{End}(A) \otimes \mathbb{Z}_{(p)}$ is a monomorphism such that (4-2) is satisfied and $\theta \circ i(b) = i(\bar{b})^\vee \circ \theta$ for all $b \in \mathbb{O}^\dagger$;
- $\bar{\eta}^p$ is a $K^{\dagger,p}$-level structure, that is, a $\pi_1(S, s)$-invariant $K^{\dagger,p}$-class of $B^\dagger \otimes \mathbb{A}_f^p$-linear symplectic similitudes $\eta^p : V^\dagger \otimes \mathbb{A}_f^p \to \mathsf{H}_1^{\text{ét}}(A_s, \mathbb{A}_f^p)$;
- $\bar{\eta}_p^{\mathfrak{p}}$ is a $K_p^{\dagger,p}$-level structure, that is, a $\pi_1(S, s)$-invariant $K_p^{\dagger,\mathfrak{p}}$-class of isomorphisms of $\mathbb{O}^\dagger$-modules $\eta_p^{\mathfrak{p}} : \bigoplus_{i=2}^r \Lambda_i^2 \to \bigoplus_{i=2}^r \mathsf{H}_1^{\text{ét}}(A_s, \mathbb{Z}_p)_i^2$.

The two quintuples $(A, \theta, i, \bar{\eta}^p, \bar{\eta}_p^{\mathfrak{p}})$ and $(A', \theta', i', (\bar{\eta}^p)', (\bar{\eta}_p^{\mathfrak{p}})')$ are equivalent if there is a prime-to-p isogeny $A \to A'$ such that it carries θ to a $\mathbb{Z}_{(p)}^\times$-multiple of θ', i to i', $\bar{\eta}^p$ to $(\bar{\eta}^p)'$, and $\bar{\eta}_p^{\mathfrak{p}}$ to $(\bar{\eta}_p^{\mathfrak{p}})'$.

We are going to extend this moduli functor to $\mathbb{O}_{F_{\mathfrak{p}}}$ to get an integral model of $M^\dagger_{0,K_p^{\dagger,\mathfrak{p}},K^{\dagger,p};\mathfrak{p}}$. Now let us consider an abelian scheme (A, θ, i) which is a part of the datum defined just above, but for A over an $\mathbb{O}_{F_{\mathfrak{p}}}$-scheme S. Through θ, we see that $(A_{p^\infty})_i^1$ and $(A_{p^\infty})_i^2$ are Cartier dual to each other. We replace (4-2) by

$$\mathrm{tr}(b \in \mathbb{O}_{B_{\mathfrak{p}}} \subset B_1^2; \mathsf{Lie}_1^2(A)) = \mathsf{Tr}_{B_1^2/F_{\mathfrak{p}}}(b) \in \mathbb{O}_{F_{\mathfrak{p}}} \text{ and } \mathsf{Lie}_i^2(A) = 0 \text{ for } i \geq 2. \quad (4\text{-}3)$$

This means that the p-divisible group $(A_{p^\infty})_i^2$ is étale for $i \geq 2$. We let $T_p A = \varprojlim_n A[p^n](S)$. Then $(T_p A)_i^2$ is isomorphic to Λ_i^2 as $\mathbb{O}^\dagger$-modules if S is simply connected.

Now we define a moduli functor $\mathcal{M}^\dagger_{0,K_p^{\dagger,\mathfrak{p}},K^{\dagger,p}}$ on the category of locally noetherian schemes over $\mathbb{O}_{F_{\mathfrak{p}}}$: for such a scheme S, $\mathcal{M}^\dagger_{0,K_p^{\dagger,\mathfrak{p}},K^{\dagger,p}}(S)$ is the set of equivalence classes of quintuple $(A, \theta, i, \bar{\eta}^p, \bar{\eta}_p^{\mathfrak{p}})$ where

- (A, θ, i) is as in the last moduli problem but satisfies (4-3);
- $\bar{\eta}^p$ is a $K^{\dagger,p}$-level structure, that is, a $\pi_1(S, s)$-invariant $K^{\dagger,p}$-class of $B^\dagger \otimes \mathbb{A}_f^p$-linear symplectic similitudes $\eta^p : V^\dagger \otimes \mathbb{A}_f^p \to \mathsf{H}_1^{\text{ét}}(A_s, \mathbb{A}_f^p)$;
- $\bar{\eta}_p^{\mathfrak{p}}$ is a $K_p^{\dagger,p}$-level structure, that is, a $\pi_1(S, s)$-invariant $K_p^{\dagger,\mathfrak{p}}$-class of isomorphisms of $\mathbb{O}^\dagger$-modules $\eta_p^{\mathfrak{p}} : \bigoplus_{i=2}^r \Lambda_i^2 \to \bigoplus_{i=2}^r (T_p A_s)_i^2$.

The two quintuples $(A, \theta, i, \bar{\eta}^p, \bar{\eta}_p^{\mathfrak{p}})$ and $(A', \theta', i', (\bar{\eta}^p)', (\bar{\eta}_p^{\mathfrak{p}})')$ are equivalent if there exists a prime-to-p isogeny $A \to A'$ satisfying the same requirements as in the last moduli problem. For sufficiently small $K_p^{\dagger,\mathfrak{p}} \times K^{\dagger,p}$, this moduli functor is represented by a regular scheme (also denoted by) $\mathcal{M}^\dagger_{0,K_p^{\dagger,\mathfrak{p}},K^{\dagger,p}}$, which is flat and projective over $\mathbb{O}_{F_{\mathfrak{p}}}$. Using Proposition 4.3, we get a regular scheme $\mathcal{M}_{0,K^{\mathfrak{p}}}$ flat and projective over $\mathbb{O}_{E_{\mathfrak{p}^\circ}}$ whose generic fiber is $M_{0,K^{\mathfrak{p}};\mathfrak{p}^\circ}$. Here, we also need to use the fact that $\mathcal{M}^\dagger_{0,K_p^{\dagger,\mathfrak{p}},K^{\dagger,p}}$ is stable for $K^{\dagger,p}$ small and the results in [Deligne and Mumford 1969, Section 1] to make the descent argument. By construction, the neutral components of $\mathcal{M}^\dagger_{0,K_p^{\dagger,\mathfrak{p}},K^{\dagger,p}} \times_{\mathbb{O}_{F_{\mathfrak{p}}}} \mathbb{O}_{E^{\mathrm{nr}}_{\mathfrak{p}^\circ}}$ and $\mathcal{M}_{0,K^{\mathfrak{p}}} \times_{\mathbb{O}_{E_{\mathfrak{p}^\circ}}} \mathbb{O}_{E^{\mathrm{nr}}_{\mathfrak{p}^\circ}}$ are isomorphic.

We shall write $(\mathscr{A}, \theta, i)$ for (a part of the datum of) the universal object over $\mathcal{M}^{\dagger}_{0,K^{\dagger,\mathfrak{p}}_p,K^{\dagger,p}}$. We also denote by $\mathscr{X}^{\dagger} = (\mathscr{A}_{p^\infty})^2_1 \to \mathcal{M}^{\dagger}_{0,K^{\dagger,\mathfrak{p}}_p,K^{\dagger,p}}$ the universal p-divisible group with $\mathbb{O}_{B_{\mathfrak{p}}}$-action and an action by $\prod_{i=2}^r B^{\times}_{\mathfrak{p}_i} \times H^{\dagger}(\mathbb{A}^p_f)$ compatible with that on the underlying scheme $\mathcal{M}^{\dagger}_{0,K^{\dagger,\mathfrak{p}}_p,K^{\dagger,p}}$. We also have a p-divisible group $\mathscr{X} \to \mathcal{M}_{0,K^{\mathfrak{p}}}$ with an action by $H^{\mathfrak{p}}_f$ compatible with that on $\mathcal{M}_{0,K^{\mathfrak{p}}}$.

Remark 4.4. In fact, when $\mathfrak{p}|2$ and $B_{\mathfrak{p}}$ is division, the condition (4-3) is not enough. One needs to pose that $(A_{p^\infty})^2_1$ is special (see [Boutot and Carayol 1991, Section II.2]) at all geometric points of characteristic p.

Now consider the case where $\epsilon(\mathbb{V}_{\mathfrak{p}}) = 1$, that is, $V_{\mathfrak{p}} \cong V^+$, and $B_{\mathfrak{p}} \cong \mathsf{Mat}_2(F_{\mathfrak{p}})$ or $\mathsf{U}(V_{\mathfrak{p}})$ is quasisplit. At the outset, we introduce some notation for the Morita equivalence. Let R be a (commutative) ring (with 1) and M a (left) R-module (or p-divisible group according to the context). Let $m > 0$ be any integer. We denote by $M^{\sharp} = M^m$ (arranged in a column) the left $\mathsf{Mat}_m(R)$-module in the natural way. Conversely, for any left $\mathsf{Mat}_m(R)$-module N, we denote by $N^{\flat} = eN$ the (left) R-module, where $e = \mathsf{diag}[1, 0, \dots, 0] \in \mathsf{Mat}_m(R)$ and the action is given by $r.(en) = (e \times \mathsf{diag}[r, \dots, r]).n$ for $r \in R$ and $n \in N$. It is easy to see that the functors $(-)^{\sharp}$ and $(-)^{\flat}$ are a pair of equivalences between corresponding categories.

We identify $\Lambda^2_1 = \mathbb{O}_{B_{\mathfrak{p}}}$ with $\mathsf{Mat}_2(\mathbb{O}_{F_{\mathfrak{p}}})$ and hence $\mathbb{O}^{\times}_{B_{\mathfrak{p}}}$ with $\mathsf{GL}_2(\mathbb{O}_{F_{\mathfrak{p}}})$. Using Morita equivalence, we easily see that in the moduli problem $\mathcal{M}^{\dagger}_{0,K^{\dagger,\mathfrak{p}}_p,K^{\dagger,p}}$, we can replace the first condition in (4-3) by the following:

$$\mathrm{tr}(b \in \mathbb{O}_{F_{\mathfrak{p}}}; \mathsf{Lie}^2_1(A)^{\flat}) = b. \tag{4-4}$$

Consider a geometric point $s : \mathsf{Spec}\, \mathbb{F} \to \mathcal{M}^{\dagger}_{0,K^{\dagger,\mathfrak{p}}_p,K^{\dagger,p}}$ of characteristic p and let $\hat{\mathbb{O}}_{(s)}$ be the completion of the henselization of the local ring at s. By the Serre–Tate theorem, it is the universal deformation ring of $(\mathscr{A}_s, \theta_s, i_s)$ which is the same as that of $(\mathscr{A}_{s,p_\infty}, \theta_s, i_s)$. By the conditions in the moduli problem and the Morita equivalence, we see that this is the same deformation ring of the p-divisible group $\mathscr{X}^{\dagger,\flat}_s = (\mathscr{A}_{s,p_\infty})^{2,\flat}_1$ which is an $\mathbb{O}_{F_{\mathfrak{p}}}$-module of dimension 1 and height 2. Hence $\hat{\mathbb{O}}_{(s)} \cong \mathbb{O}_{\hat{F}_{\mathfrak{p}}}[[t]]$. We have

Proposition 4.5 [Carayol 1986, Section 6]. *The scheme $\mathcal{M}^{\dagger}_{0,K^{\dagger,\mathfrak{p}}_p,K^{\dagger,p}}$ (resp. $\mathcal{M}_{0,K^{\mathfrak{p}}}$) is smooth and projective over $\mathbb{O}_{F_{\mathfrak{p}}}$ (resp. $\mathbb{O}_{E_{\mathfrak{p}^\circ}}$).*

For a geometric point s of characteristic p on $\mathcal{M}^{\dagger}_{0,K^{\dagger,\mathfrak{p}}_p,K^{\dagger,p}}$ (resp. $\mathcal{M}_{0,K^{\mathfrak{p}}}$), there are two cases. We say s is *ordinary* if the formal part of $\mathscr{X}^{\dagger}_s$ (resp. $\mathscr{X}_s$) is of height 1; *supersingular* if $\mathscr{X}^{\dagger}_s$ (resp. $\mathscr{X}_s$) is formal. We denote by $[\mathcal{M}^{\dagger}_{0,K^{\dagger,\mathfrak{p}}_p,K^{\dagger,p}}]_{\mathsf{s.s.}}$, (resp. $[\mathcal{M}_{0,K^{\mathfrak{p}}}]_{\mathsf{s.s.}}$) the supersingular locus of the scheme $\mathcal{M}^{\dagger}_{0,K^{\dagger,\mathfrak{p}}_p,K^{\dagger,p}}$ (resp. $\mathcal{M}_{0,K^{\mathfrak{p}}}$).

A basic abelian scheme. To give the moduli interpretation of the special cycles, we need first to construct a basic abelian scheme. We fix an imaginary element μ in E, that is, $\mu^{\tau} = -\mu$ and $\mu \neq 0$. Since we only care about the place $\mathfrak{p}$, we identify the following two isomorphic commutative diagrams to ease our notation:

$$
\begin{tikzcd}
\iota_1^\circ(E) \arrow[rr, hook] & & {[\iota_1^\circ(E)]^\wedge_{\iota_{(p)}}} & \\
& \iota_1^1(F^\dagger) \arrow[rr, hook] & & {[\iota_1^1(F^\dagger)]^\wedge_{\iota_{(p)}}}, \arrow[ul, hook] \\
\iota_1(F) \arrow[uu, hook] \arrow[ur, hook] \arrow[rr, hook] & & {[\iota_1(F)]^\wedge_{\iota_{(p)}}} \arrow[uu, hook] \arrow[ur, equal] &
\end{tikzcd}
\qquad
\begin{tikzcd}
E \arrow[rr, hook] & & E_{\mathfrak{p}^\circ} & \\
& F^\dagger \arrow[rr, hook] & & F_{\mathfrak{p}}. \arrow[ul, hook] \\
F \arrow[uu, hook] \arrow[ur, hook] \arrow[rr, hook] & & F_{\mathfrak{p}} \arrow[uu, hook] \arrow[ur, equal] &
\end{tikzcd}
$$

where $[-]^\wedge_{\iota_{(p)}}$ means the completion in $\mathbb{C}_p$ through $\iota_{(p)}$.

Now let $E^\dagger = E \otimes_F F^\dagger$, which is a CM field of degree $4d$ and a subalgebra of $B^\dagger$ extending the fixed embedding $E \hookrightarrow B$. We also denote by $e \mapsto \bar{e}$ the canonical involution of the second kind of $E^\dagger$ such that the subfield fixed by this involution is totally real. The maps $\iota_i^\circ \otimes \iota_i^j : E \otimes_F F^\dagger \to \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \to \mathbb{C}$ and $\iota_i^\bullet \otimes \iota_i^j : E \otimes_F F^\dagger \to \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \to \mathbb{C}$ ($i = 1, \dots, d$, $j = 1, 2$) give $4d$ different complex embeddings of $E^\dagger$ into $\mathbb{C}$, where $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \to \mathbb{C}$ is the usual multiplication. We choose a CM type $\Phi = \{\iota_1^\circ \otimes \iota_1^1, \iota_1^\circ \otimes \iota_1^2; \iota_i^\circ \otimes \iota_i^1, \iota_i^\bullet \otimes \iota_i^1 \mid i = 2, \dots, d\}$ of $E^\dagger$. Then Φ determines a Hodge map $h^\ddagger : \mathbb{S} \to T^\ddagger_{\mathbb{R}}$ where $T^\ddagger$ is the subtorus of $\mathsf{Res}_{E^\dagger/\mathbb{Q}} \boldsymbol{G}_{\mathsf{m},E^\dagger}$ consisting of elements e such that $e\bar{e} \in \boldsymbol{G}_{\mathsf{m},\mathbb{Q}}$. The Shimura varieties $M^\ddagger_{K^\ddagger} = \mathsf{Sh}_{K^\ddagger}(T^\ddagger, \{h^\ddagger\})$ basically parametrize abelian varieties over $E^{\dagger,\Phi}$ with CM by $E^\dagger$ of type Φ. It is finite and projective over $\mathsf{Spec}\, E^{\dagger,\Phi}$ where $E^{\dagger,\Phi}$ is the reflex field of $(E^\dagger, \Phi)$.

To make this more precise, let $V^\ddagger$ be the $\mathbb{Q}$-vector space underlying $E^\dagger$. Define a symplectic form

$$\psi^\ddagger(v, w) := \mathsf{Tr}_{F^\dagger/\mathbb{Q}}(\sqrt{\lambda}\,\mathsf{Tr}_{E^\dagger/F^\dagger}(v\bar{w}))$$

for $v, w \in E^\dagger$. Then $T^\ddagger$ can be identified with the group of $E^\dagger$-linear symplectic similitude of $(V^\ddagger, \psi^\ddagger)$ and $T^\ddagger(\mathbb{Q}_p)$ can be identified with $\mathbb{Q}_p^\times \times \prod_{i=1}^r E_{\mathfrak{p}_i}^\times$. The Hodge map $h^\ddagger$ induces a filtration $0 \subset \mathsf{Fil}^0(V^\ddagger_{\mathbb{C}}) \subset V^\ddagger_{\mathbb{C}}$ such that $t^\ddagger(e) = \mathrm{tr}(e; V^\ddagger_{\mathbb{C}}/\mathsf{Fil}^0(V^\ddagger_{\mathbb{C}})) = \sum_{\iota \in \Phi} \iota(e)$ for $e \in E^\dagger$. Since we have identified E (resp. $F^\dagger$) with its embedding through ι_1° (resp. ι_1^1), we now identify $E^\dagger$ with its embedding through $\iota_1^\circ \otimes \iota_1^1$, that is, with $\iota_1^\circ(E).\iota_1^1(F^\dagger) \subset \mathbb{C}$.

Lemma 4.6. *The reflex field $E^{\dagger,\Phi}$ is $E^\dagger$.*

Proof. By definition, $E^{\dagger,\Phi}$ is the field generated by the numbers $t^\ddagger(e)$ for all $e \in E^\dagger$. Let $e = (x + y\mu) \times (x' + y'\lambda')$ be an element in $E^\dagger$ with $x, y, x', y' \in F$. Then

$$
\begin{aligned}
t^\ddagger(e) &= (x + y\mu)(2x') + \sum_{i=2}^{d} 2\iota_i(x)(\iota_i(x') + \iota_i(y')\lambda') \\
&= 2\mathsf{Tr}_{F/\mathbb{Q}}(xx') + 2\mathsf{Tr}_{F/\mathbb{Q}}(xy')\lambda' + 2yx'\mu - 2xy'\lambda'.
\end{aligned}
$$

Hence $E^{\dagger,\Phi} = E^\dagger$. $\square$

As before, the algebra $E^\dagger_p = E^\dagger \otimes_{\mathbb{Q}} \mathbb{Q}_p$ has a decomposition

$$E^\dagger_p = \bigoplus_{i=1}^r E^1_i \oplus \bigoplus_{i=1}^r E^2_i \cong \bigoplus_{i=1}^r E_{\mathfrak{p}_i} \oplus \bigoplus_{i=1}^r E_{\mathfrak{p}_i},$$

which is also true for its modules. Let π^1_1 be the projection of $E^\dagger_p$ to the first factor E^1_1. The additive map $t^\ddagger$ extends to a map $t^\ddagger_p : E^\dagger_p \to E^\dagger_p$. From the calculation in the above lemma, we find that, for $(e^j_i) = (e^1_1, \dots, e^1_r; e^2_1, \dots, e^2_r) \in E^\dagger_p$,

$$\pi^1_1 \circ t^\ddagger_p((e^j_i)) = \sum_{i=1}^r \mathsf{Tr}_{E_{\mathfrak{p}_i}/\mathbb{Q}_p}(e^1_i) + e^1_1 + e^2_1 - \mathsf{Tr}_{E_{\mathfrak{p}_1}/F_{\mathfrak{p}_1}}(e^1_1). \tag{4-5}$$

Let $\mathbb{O}^\ddagger \subset \mathbb{O}^\dagger$ be the unique maximal $\mathbb{Z}_{(p)}$-order in $E^\dagger$ such that $\bar{\mathbb{O}}^\ddagger = \mathbb{O}^\ddagger$ and the ring of integers is $\mathbb{O}^\ddagger_{\mathfrak{p}^2_i} = \mathbb{O}_{E_{\mathfrak{p}_i}}$, where $\mathbb{O}^\ddagger_{\mathfrak{p}^2_i}$ is the image of $\mathbb{O}^\ddagger \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_p$ in the E^2_i component. For any abelian variety A over an $E_{\mathfrak{p}^\circ}$-scheme S with an action by $\mathbb{O}^\ddagger$, $\mathsf{Lie}_S(A)$ is an $E^\dagger_p$-module, and hence decomposes as the direct sum of $\mathsf{Lie}^j_i(A)$ ($i = 1, \dots, r$, $j = 1, 2$). In view of (4-3) and (4-5), we pose the trace condition as

$$\mathrm{tr}(e \in \mathbb{O}_{E_\mathfrak{p}} \subset E^2_1; \mathsf{Lie}^2_1(A)) = e_{\mathfrak{p}^\circ} \in E_{\mathfrak{p}^\circ} \text{ and } \mathsf{Lie}^2_i(A) = 0 \text{ for } i \geq 2, \tag{4-6}$$

where $e_{\mathfrak{p}^\circ}$ is just e if $E_\mathfrak{p} = E_{\mathfrak{p}^\circ}$ is a field or the component in $E_{\mathfrak{p}^\circ}$ if $E_\mathfrak{p} = E_{\mathfrak{p}^\circ} \oplus E_{\mathfrak{p}^\bullet}$ is split.

Let

$$K^\ddagger = \mathbb{Z}^\times_p \times \prod_{i=1}^r \mathbb{O}^\times_{E_{\mathfrak{p}_i}} \times K^{\ddagger,p}$$

be an open compact subgroup of $T^\ddagger(\mathbb{A}_f)$. Set $M^\ddagger_{00,K^{\ddagger,p}} = M^\ddagger_{K^\ddagger}$, and let $M^\ddagger_{00,K^{\ddagger,p};\mathfrak{p}^\circ}$ be the base change under $\iota_{(p)} \circ (\iota^\circ_1 \otimes \iota^1_1) : E^\dagger \hookrightarrow E_{\mathfrak{p}^\circ}$. Then for sufficiently small $K^{\ddagger,p}$, $M^\ddagger_{00,K^{\ddagger,p}}$ represents the following moduli functor (due to (4-6)) on the category of locally noetherian schemes over $E_{\mathfrak{p}^\circ}$: for such a scheme S, $M^\ddagger_{00,K^{\ddagger,p}}(S)$ is the set of equivalence classes of quadruples $(A, \vartheta, j, \bar{\eta}^p)$ where

- A is an abelian scheme over S of dimension $2d$;
- $\vartheta : A \to A^\vee$ is a prime-to-p polarization;
- $j : \mathbb{O}^\ddagger \hookrightarrow \mathsf{End}(A) \otimes \mathbb{Z}_{(p)}$ is a monomorphism of $\mathbb{Z}_{(p)}$-algebras such that (4-6) is satisfied and $\vartheta \circ j(e) = j(\bar{e})^\vee \circ \vartheta$ for all $e \in \mathbb{O}^\ddagger$;
- $\bar{\eta}^p$ is a $K^{\ddagger,p}$-structure, that is, a $\pi_1(S, s)$-invariant $K^{\ddagger,p}$-class of $E^\dagger \otimes \mathbb{A}^p_f$-linear symplectic similitude $\eta^p : V^\ddagger \otimes \mathbb{A}^p_f \to \mathsf{H}^{\text{ét}}_1(A_s, \mathbb{A}^p_f)$.

The notion of equivalence is similarly defined as before. Moreover, we can extend this moduli functor to be over $\mathbb{O}_{E_{\mathfrak{p}^\circ}}$. We omit the detailed definition. One can similarly prove that the extended one, say $\mathcal{M}^\ddagger_{00,K^{\ddagger,p}}$, is finite, projective, smooth over $\mathbb{O}_{E_{\mathfrak{p}^\circ}}$, and connected, and hence is isomorphic to $\mathsf{Spec}\, \mathbb{O}_{E^\natural}$ for some finite unramified extension of local fields $E^\natural/E_{\mathfrak{p}^\circ}$. Fix an embedding $\iota^\natural : E^\natural \hookrightarrow E^{\mathrm{nr}}_{\mathfrak{p}^\circ}$ and let $(\mathscr{E}, \vartheta, j)$ be the universal object over $\mathscr{M}^\ddagger_{00,K^{\ddagger,p}} \times_{\mathbb{O}_{E^\natural},\iota^\natural} \mathbb{O}_{E^{\mathrm{nr}}_{\mathfrak{p}^\circ}} \cong \mathrm{Spec}\, \mathbb{O}_{E^{\mathrm{nr}}_{\mathfrak{p}^\circ}}$ and $\mathscr{Y} = (\mathscr{E}_{p^\infty})^2_1$. Fix a geometric point $s : \mathbb{O}_{E^{\mathrm{nr}}_{\mathfrak{p}^\circ}} \hookrightarrow \mathbb{C}$ of characteristic zero and an $\mathbb{O}^\ddagger$-generator $\boldsymbol{x}$ of $\mathrm{H}^{\mathrm{B}}_1(\mathscr{E}_s, \mathbb{Z}_{(p)})$ where $\mathrm{H}^{\mathrm{B}}_1$ is the first Betti homology. We call the quadruple $(\mathscr{E}, \vartheta, j; \boldsymbol{x})$ a *basic unitary datum*.

For any $x \in V$ which is positive definite, we have a subscheme $Z(x)_K$ on M_K which is a special cycle as in Section 3B. Let us consider the curve $M_{0,K^{\mathfrak{p}};\mathfrak{p}^\circ}$ and its subscheme $Z(x)_{0,\mathfrak{p}^\circ}$, which is the base change of $Z(x)_{K_{\mathfrak{p},0}K^{\mathfrak{p}}}$ in $M_{0,K^{\mathfrak{p}}}$. We denote by $Z(x)^\circ_{0,\mathfrak{p}^\circ}$ the neutral component of $Z(x)_{0,\mathfrak{p}^\circ} \times_{E^\circ_{\mathfrak{p}}} E^{\mathrm{nr}}_{\mathfrak{p}^\circ}$. By Proposition 4.3, we have an irreducible closed subscheme $Z(x)^\dagger_{0,\mathfrak{p}}$ on $M^{\dagger,\circ}_{0,K^{\dagger,\mathfrak{p}}_p,K^{\dagger,p};\mathfrak{p}}$ by passing between the neutral components, which is an element in $\mathrm{CH}^1(M^\dagger_{0,K^{\dagger,\mathfrak{p}}_p,K^{\dagger,p};\mathfrak{p}} \times_{F_{\mathfrak{p}}} E^{\mathrm{nr}}_{\mathfrak{p}^\circ})$. For $K^{\mathfrak{p}}$ sufficiently small (which is independent of x), $Z(x)^\dagger_{0,\mathfrak{p},o}(\mathbb{C})$ is represented by (z, e) for any complex geometric point $o : E^{\mathrm{nr}}_{\mathfrak{p}^\circ} \hookrightarrow \mathbb{C}$ where $z \perp x$ and $e \in H^\dagger(\mathbb{A}_f)$ is the identity element. Actually, $Z(x)^\dagger_{0,\mathfrak{p}}$ is defined for any $x \in V^\dagger - \{0\}$ if we view $x = x \otimes 1 \in V^\dagger$ and extend by $Z(ax)^\dagger_{0,\mathfrak{p}} = Z(x)^\dagger_{0,\mathfrak{p}}$ for $a \in F^{\dagger,\times}$. Let $[Z(x)^\dagger_{0,\mathfrak{p}}]^{\mathrm{Zar}}$ be the Zariski closure of $Z(x)^\dagger_{0,\mathfrak{p}}$ in $\mathscr{M}^\dagger_{0,K^{\dagger,\mathfrak{p}}_p,K^{\dagger,p}} \times_{\mathbb{O}_{F_{\mathfrak{p}}}} \mathbb{O}_{E^{\mathrm{nr}}_{\mathfrak{p}^\circ}}$.

We fix a basic unitary datum $(\mathscr{E}, \vartheta, j; \boldsymbol{x})$. Since $\mathrm{Spec}\, \mathbb{O}_{E^{\mathrm{nr}}_{\mathfrak{p}^\circ}}$ is simply connected, we can extend $\boldsymbol{x}$ to a section $\boldsymbol{x}^p$ of the lisse $\mathbb{A}^p_f$-sheaf $\mathrm{H}^{\text{ét}}_1(\mathscr{E}, \mathbb{A}^p_f)$ over $\mathrm{Spec}\, \mathbb{O}_{E^{\mathrm{nr}}_{\mathfrak{p}^\circ}}$, hence a section $\boldsymbol{x}^p_S$ of $\mathrm{H}^{\text{ét}}_1(\mathscr{E} \times_{\mathrm{Spec}\, \mathbb{O}_{E^{\mathrm{nr}}_{\mathfrak{p}^\circ}}} S, \mathbb{A}^p_f)$ for any $\mathbb{O}_{E^{\mathrm{nr}}_{\mathfrak{p}^\circ}}$-scheme S. If S is an $E^{\mathrm{nr}}_{\mathfrak{p}^\circ}$-scheme, we have a section $\boldsymbol{x}_S$ of $\mathrm{H}^{\text{ét}}_1(\mathscr{E} \times_{\mathrm{Spec}\, E^{\mathrm{nr}}_{\mathfrak{p}^\circ}} S, \mathbb{A}_f)$. In particular,

$$(\boldsymbol{x}_{\mathfrak{p},S}, \boldsymbol{x}^{\mathfrak{p}}_{p,S}) \in (T_p(\mathscr{E} \times_{\mathrm{Spec}\, \mathbb{O}_{E^{\mathrm{nr}}_{\mathfrak{p}^\circ}}} S))^2_1 \oplus \bigoplus_{i=2}^{r} (T_p(\mathscr{E} \times_{\mathrm{Spec}\, \mathbb{O}_{E^{\mathrm{nr}}_{\mathfrak{p}^\circ}}} S))^2_i.$$

Let $\boldsymbol{E} = \mathscr{E} \times_{\mathbb{O}^{\mathrm{nr}}_{\mathfrak{p}^\circ}} \mathbb{F}$ be the special fiber. Since $(\boldsymbol{E}_{p^\infty})^2_i$ is étale for $i \geq 2$, $(T_p\boldsymbol{E})^2_i$ is canonically isomorphic to $(\mathrm{H}^{\mathrm{B}}_1(\mathscr{E}_s, \mathbb{Z}_p))^2_i$. Hence $\boldsymbol{x}$ canonically determines an element $\boldsymbol{x}^{\mathfrak{p}}_p \in \bigoplus_{i=2}^r (T_p\boldsymbol{E})^2_i$. For any $\mathbb{F}$-scheme S, we have element $\boldsymbol{x}^{\mathfrak{p}}_{p,S}$.

Proposition 4.7. *Let $x \in V^\dagger \cap \Lambda^2_1$ such that $Z(x)^\dagger_{0,\mathfrak{p}}$ is nonempty and let S be an $\mathbb{O}_{E^{\mathrm{nr}}_{\mathfrak{p}^\circ}}$-scheme. For any morphism $S \to [Z(x)^\dagger_{0,\mathfrak{p}}]^{\mathrm{Zar}} \hookrightarrow \mathscr{M}^\dagger_{0,K^{\dagger,\mathfrak{p}}_p,K^{\dagger,p}} \times_{\mathbb{O}_{F_{\mathfrak{p}}}} \mathbb{O}_{E^{\mathrm{nr}}_{\mathfrak{p}^\circ}}$ inducing the quintuple $(A, \theta, i_A, \bar{\eta}^p, \bar{\eta}^{\mathfrak{p}}_p)$, there is a quasihomomorphism $\varrho_A : \mathscr{E} \times_{\mathrm{Spec}\, \mathbb{O}_{E^{\mathrm{nr}}_{\mathfrak{p}^\circ}}} S \to A$ satisfying the following conditions*:

- *For any $e \in \mathbb{O}^\ddagger$, the following diagram commutes*:

$$\begin{CD} \mathscr{E} \times_{\mathrm{Spec}\, \mathbb{O}_{E^{\mathrm{nr}}_{\mathfrak{p}^\circ}}} S @>{\varrho_A}>> A \\ @V{j(e)}VV @VV{i_A(e)}V \\ \mathscr{E} \times_{\mathrm{Spec}\, \mathbb{O}_{E^{\mathrm{nr}}_{\mathfrak{p}^\circ}}} S @>{\varrho_A}>> A \end{CD} \tag{4-7}$$

- *ϱ_A induces a homomorphism from $\mathscr{Y} \times_{\mathrm{Spec}\, \mathbb{O}_{E^{\mathrm{nr}}_{\mathfrak{p}^\circ}}} S$ to $(A_{p^\infty})^2_1$.*
- *For any geometric point $s \in S$, the map $\varrho_{A_s,*} : \mathrm{H}^{\text{ét}}_1(\mathscr{E}_s, \mathbb{A}^p_f) \to \mathrm{H}^{\text{ét}}_1(A_s, \mathbb{A}^p_f)$ sends $\boldsymbol{x}^p_s$ into $\bar{\eta}^p(x)$.*

- *For any geometric point $s \in S$, the map*

$$\varrho_{A_s,*} : \bigoplus_{i=2}^{r} (T_p \mathscr{E}_s)_i^2 \to \bigoplus_{i=2}^{r} (T_p A_s)_i^2 \otimes_{\mathbb{O}_{F_{\mathfrak{p}_i}}} F_{\mathfrak{p}_i}$$

sends $\boldsymbol{x}_{p,s}^{\mathfrak{p}}$ into $\bar{\eta}_p^{\mathfrak{p}}(x)$.

Proof. Let $o : E_{\mathfrak{p}^\circ}^{\mathrm{nr}} \hookrightarrow \mathbb{C}$ be any embedding such that it gives a complex geometric point of $[Z(x)_{0,\mathfrak{p}}^{\dagger}]^{\mathrm{Zar}}$ which corresponds to the quintuple $(A_o, \theta_o, i_{A_o}, (\bar{\eta}^p)_o, (\bar{\eta}_p^{\mathfrak{p}})_o)$, and also to the point represented by (z, e). This means that we can find a symplectic similitude $\gamma : \mathrm{H}_1^{\mathrm{B}}(A_o, \mathbb{Q}) \to V^{\dagger}$ such that $\gamma^{-1}(x) \in \mathrm{Lie}(A_o)$ is an i-eigenvector of $i_{A_o}(h^{\dagger}(i))$ (see [Kottwitz 1992, Section 8] for the complex points of Shimura varieties of PEL type). Consider the operator $i_{A_o}(h^{\dagger}(i)) - i$ acting on $\mathrm{Lie}(A_o)$ which is by zero on the $\mathbb{Z}_{(p)}$-sublattice $\Upsilon_{(p)} = i_{A_o}(\mathbb{O}^{\ddagger})(\gamma^{-1}(x))$ of rank $4d$, and hence is also by zero on the $\mathbb{R}$-subspace it generates, which is $\Upsilon_{\mathbb{R}} = i_{A_o}(E^{\dagger} \otimes_{\mathbb{Q}} \mathbb{R})(\gamma^{-1}(x))$. But since, on this space, multiplying i is the same as $i_{A_o}(h^{\dagger}(i))$, we see that $\Upsilon_{\mathbb{R}}/\Upsilon_{(p)}$ defines a complex subtorus (up to prime-to-p isogeny) E of dimension $2d$ and a quasihomomorphism $\varrho'_{A_o} : E \to A_o$, hence E is a complex abelian variety. It is easy to see that E is isogenous to $\mathscr{E} \times_{\mathbb{O}_{E_{\mathfrak{p}^\circ}^{\mathrm{nr}}},o} \mathbb{C}$, hence we can find a unique quasihomomorphism $\varrho_E : \mathscr{E} \times_{\mathbb{O}_{E_{\mathfrak{p}^\circ}^{\mathrm{nr}}},o} \mathbb{C} \to E$ sending $\boldsymbol{x}$ to $\gamma^{-1}x \in \Upsilon_{(p)} = \mathrm{H}_1^{\mathrm{B}}(E, \mathbb{Z}_{(p)})$, and $\varrho_{A_o} = \varrho_E \circ \varrho'_{A_o}$ satisfies the properties in the proposition. Since o is arbitrary, it is not difficult to see that ϱ_{A_o} descends to a quasihomomorphism $\varrho_A : \mathscr{E} \times_{\mathbb{O}_{E_{\mathfrak{p}^\circ}^{\mathrm{nr}}}} E_{\mathfrak{p}^\circ}^{\mathrm{nr}} \to A$ satisfying all properties but where $(A, \theta, i_A, \bar{\eta}^p, \bar{\eta}_p^{\mathfrak{p}})$ is the quintuple on $Z(x)_{0,\mathfrak{p}}^{\dagger}$. If we denote by A' the corresponding abelian scheme over $[Z(x)_{0,\mathfrak{p}}^{\dagger}]^{\mathrm{Zar}}$, the quasihomomorphism ϱ_A uniquely extends to $\varrho_{A'} : \mathscr{E} \to A'$, satisfying (4-7). The other properties follow from the comparison theorems (for homology) and the assumption that $x \in \Lambda_1^2$. For general S, we only need to pull back $\varrho_{A'}$. □

Integral special subschemes in the quasisplit case. We now assume that $\mathfrak{p}$ is nonsplit in E. For any element $t \in F^{\dagger,\times} \cap \mathbb{O}_{F_{\mathfrak{p}}}$, we define a functor $\mathscr{Z}(t)_{0,\mathfrak{p}}^{\dagger}$ in the following way. For any $\mathbb{O}_{E_{\mathfrak{p}^\circ}^{\mathrm{nr}}}$-scheme S, $\mathscr{Z}(t)_{0,\mathfrak{p}}^{\dagger}(S)$ is a sextuple $(A, \theta, i_A, \bar{\eta}^p, \bar{\eta}_p^{\mathfrak{p}}; \varrho_A)$ where $(A, \theta, i_A, \bar{\eta}^p, \bar{\eta}_p^{\mathfrak{p}})$ is an element in $\mathscr{M}_{0,K_p^{\dagger,\mathfrak{p}},K^{\dagger,p}}^{\dagger}(S)$ and $\varrho_A : \mathscr{E} \times_{\mathrm{Spec}\,\mathbb{O}_{E_{\mathfrak{p}^\circ}^{\mathrm{nr}}}} S \to A$ is a nontrivial quasihomomorphism such that

- we have a commutative diagram like (4-7);
- ϱ induces a homomorphism from $\mathscr{Y} \times_{\mathrm{Spec}\,\mathbb{O}_{E_{\mathfrak{p}^\circ}^{\mathrm{nr}}}} S$ to $(A_{p^\infty})_1^2$; and
- the nontrivial quasihomomorphism $\vartheta^{-1} \circ \varrho_A^{\vee} \circ \theta \circ \varrho_A$ induces $j(t) \times_{\mathrm{Spec}\,\mathbb{F}} S_{\mathrm{spe}} : \boldsymbol{E} \times_{\mathrm{Spec}\,\mathbb{F}} S_{\mathrm{spe}} \to \boldsymbol{E} \times_{\mathrm{Spec}\,\mathbb{F}} S_{\mathrm{spe}}$ where $S_{\mathrm{spe}} = S \times_{\mathbb{O}_{E_{\mathfrak{p}^\circ}^{\mathrm{nr}}}} \mathbb{F}$.

These properties cut out a subscheme, which is still denoted by $\mathscr{Z}(t)_{0,\mathfrak{p}}^{\dagger}$, of $\mathscr{M}_{0,K_p^{\dagger,\mathfrak{p}},K^{\dagger,p}}^{\dagger}$. By the positivity property of the Rosati involution, one easily sees that it is nonempty if and only if t is totally positive. Let o be an $E_{\mathfrak{p}^\circ}^{\mathrm{nr}}$-point of the generic fiber $\mathfrak{X}(t)^{\dagger}_{0,\mathfrak{p}}$ such that it is equal to some $Z(x)^{\dagger}_{0,\mathfrak{p}}$. We write $T(o)=t$ to indicate their relation. If $\mathfrak{X}(o)^{\dagger}_{0,\mathfrak{p}}$ is the unique irreducible component of $\mathfrak{X}(t)^{\dagger}_{0,\mathfrak{p}}$ containing o, then it is a closed subscheme of $\mathcal{M}^{\dagger}_{0,K_p^{\dagger,\mathfrak{p}},K^{\dagger,p}}\times_{\mathbb{O}_{F_\mathfrak{p}}}\mathbb{O}_{E^{\mathrm{nr}}_{\mathfrak{p}^\circ}}$. By construction, o corresponds to a special cycle $Z(x)^{\dagger}_{0,\mathfrak{p}}$. Hence we have the following identity between sets:

$$\bigcup_{t\in F^{\dagger,+}\cap\mathbb{O}_{F_\mathfrak{p}}}\left\{\text{generic point of }\mathfrak{X}(o)^{\dagger}_{0,\mathfrak{p}}\;\middle|\;T(o)=t\right\}=\left\{Z(x)^{\dagger}_{0,\mathfrak{p}}\;\middle|\;x\in V^{\dagger}\cap\Lambda_1^2-\{0\}\right\}$$
$$=\left\{Z(x)^{\circ}_{0,\mathfrak{p}^\circ}\;\middle|\;x\in V\cap\Lambda_\mathfrak{p}-\{0\}\right\},$$

where $\Lambda_\mathfrak{p}=\Lambda^{\pm}$ is the self-dual lattice of $V_\mathfrak{p}$. Hence, again by Proposition 4.3, we obtain an *integral special subscheme* $\mathfrak{X}(x)^{\circ}_{0,\mathfrak{p}^\circ}$ of $\mathcal{M}^{\circ}_{0,K^\mathfrak{p}}$ such that its generic fiber is $Z(x)^{\circ}_{0,\mathfrak{p}^\circ}$.

From now on, we assume further that $\epsilon(\mathbb{V}_p)=1$.

Proposition 4.8. *The special fiber* $[\mathfrak{X}(o)^{\dagger}_{0,\mathfrak{p}}]_{\mathsf{spe}}$ (*resp.*, $[\mathfrak{X}(x)^{\circ}_{0,\mathfrak{p}^\circ}]_{\mathsf{spe}}$) *of* $\mathfrak{X}(o)^{\dagger}_{0,\mathfrak{p}}$ (*resp.* $\mathfrak{X}(x)^{\circ}_{0,\mathfrak{p}^\circ}$) *lies in the supersingular locus* $[\mathcal{M}^{\dagger}_{0,K_p^{\dagger,\mathfrak{p}},K^{\dagger,p}}]_{\mathsf{s.s.}}$ (*resp.* $[\mathcal{M}_{0,K^\mathfrak{p}}]_{\mathsf{s.s.}}$).

Proof. We only need to prove this for $[\mathfrak{X}(o)^{\dagger}_{0,\mathfrak{p}}]_{\mathsf{spe}}$. Let $s=(A,\theta,i_A,\bar\eta^p,\bar\eta^{\mathfrak{p}}_p;\varrho_A)$ be an $\mathbb{F}$-point of $\mathfrak{X}(o)^{\dagger}_{0,\mathfrak{p}}$. Then we have a nontrivial homomorphism between $\mathbb{O}_{F_\mathfrak{p}}$-modules $\varrho_{A,*}:\boldsymbol{Y}\to(A_{p^\infty})_1^2=(\mathscr{X}_s^{\dagger})\cong(\mathscr{X}_s^{\dagger,\flat})^{\oplus 2}$. Hence there is at least one projection $(\mathscr{X}_s^{\dagger,\flat})^{\oplus 2}\to\mathscr{X}_s^{\dagger,\flat}$ whose composition with $\varrho_{A,*}$ is nonzero. We know that both $\boldsymbol{Y}$ and $\mathscr{X}_s^{\dagger,\flat}$ are $\mathbb{O}_{F_\mathfrak{p}}$-modules of dimension 1 and height 2. But since we have assumed that $\mathfrak{p}$ is nonsplit in E, $\boldsymbol{Y}$ is formal, which implies that $\mathscr{X}_s^{\dagger}$ is also formal; that is, s is located in the supersingular locus. □

We need to study the supersingular loci of $\mathcal{M}^{\dagger}_{0,K_p^{\dagger,\mathfrak{p}},K^{\dagger,p}}$ and $\mathcal{M}_{0,K^\mathfrak{p}}$. We fix an integral special subscheme $\mathfrak{X}(o)^{\dagger}_{0,\mathfrak{p}}$ with $T(o)=1$ and an $\mathbb{F}$-point s of it. We set $\boldsymbol{A}=\mathscr{A}_s$ and $\boldsymbol{X}=\mathscr{X}_s^{\dagger}$, which is a formal $\mathbb{O}_{F_\mathfrak{p}}$-module of dimension 2 and height 4 over $\mathbb{F}$ with an action by $\mathrm{GL}_2(\mathbb{O}_{F_\mathfrak{p}})$. In fact, the isomorphism class of $\boldsymbol{X}$ is independent of the choice of o and s. We denote by $(\boldsymbol{A}^0,\theta^0,i_{\boldsymbol{A}^0})$ the (unique) isogeny class of the abelian variety with polarization and endomorphism $(\boldsymbol{A},\theta,i_{\boldsymbol{A}})$. Let $\breve{B}$ be the division algebra over F obtained from B by changing Hasse invariants at ι_1 and $\mathfrak{p}$, hence both B_{ι_1} and $B_\mathfrak{p}$ are division algebras. Let $\breve{B}^{\dagger}=\breve{B}\otimes_F F^{\dagger}$, $\breve{G}=\mathrm{Res}_{F/\mathbb{Q}}\breve{B}^{\times}$ with center T, and

$$\breve{G}^{\dagger}:=\breve{G}\times_T T_{F^\dagger}\xrightarrow{\breve\nu^{\dagger}}T\times T_{F^\dagger},$$

where $\breve\nu^{\dagger}(\breve g\times z)=(\mathrm{Nm}\breve g\cdot zz^{\tau^\dagger},z/z^{\tau^\dagger})$. Let $\breve{H}^{\dagger}$ be the preimage of $T^{\dagger}$ under $\breve\nu^{\dagger}$. Then we have $\mathrm{End}(\boldsymbol{A}^0,i_{\boldsymbol{A}^0})\cong\breve{B}^{\dagger}$ as an $F^{\dagger}$-algebra and $\mathrm{Aut}(\boldsymbol{A}^0,\theta^0,i_{\boldsymbol{A}^0})\cong\breve{H}^{\dagger}(\mathbb{Q})$ (see [Carayol 1986, Section 11]). We can also choose the isomorphism such that the involution of the second kind on $\breve{B}^{\dagger}$ induced by θ is the tensor product of the canonical involution of $\breve{B}$ and the conjugation of $F^{\dagger}$. In what follows, we identity

$\mathsf{End}(\boldsymbol{A}^0, i_{\boldsymbol{A}^0})$ with $\breve{B}^\dagger$ and $\mathsf{Aut}(\boldsymbol{A}^0, \theta^0, i_{\boldsymbol{A}^0})$ with $\breve{H}^\dagger(\mathbb{Q})$. We also identify $\breve{H}^\dagger(\mathbb{A}_f^p)$ with $H^\dagger(\mathbb{A}_f^p)$ through the level structure $(\bar{\eta}^p, \bar{\eta}_p^{\mathfrak{p}})$ of $\boldsymbol{A}$.

Let $S^\dagger = [\mathcal{M}^\dagger_{0,K_p^{\dagger,\mathfrak{p}},K^{\dagger,p}}]_{\mathsf{s.s.}}(\mathbb{F})$ be the set of supersingular points in the special fiber together with a p-divisible group $\mathscr{X}^\dagger\big|_{S^\dagger}$ on it. The group $H^\dagger(\mathbb{A}_f)$ acts on $\mathscr{X}^\dagger\big|_{S^\dagger}$ which is compatible with its action on $S^\dagger$. It is easy to see that the action factors through $\mathbb{Z}_p^\times \times \mathsf{SL}_2(\mathbb{O}_{F_\mathfrak{p}})$. Hence its normalizer $\mathsf{SL}_2(F_\mathfrak{p})$ acts trivially on $S^\dagger$ and the action factors through $F_\mathfrak{p}^\times \times \prod_{i=2}^r B_{\mathfrak{p}_i}^\times \times H^\dagger(\mathbb{A}_f^p)$.

By Honda–Tate theory, it is proved in [Carayol 1986, Section 11.3] that

$$F_\mathfrak{p}^\times \times \prod_{i=2}^r B_{\mathfrak{p}_i}^\times \times H^\dagger(\mathbb{A}_f^p)$$

acts transitively on $S^\dagger$ and its stabilizer at s is $\breve{H}^\dagger(\mathbb{Q}) \times (\mathbb{O}_{F_\mathfrak{p}}^\times \times K_p^{\dagger,\mathfrak{p}} \times K^{\dagger,p})$. Hence we have

$$S^\dagger \cong \breve{H}^\dagger(\mathbb{Q}) \Big\backslash \Big(\mathbb{Z} \times \prod_{i=2}^r B_{\mathfrak{p}_i}^\times / K_p^{\dagger,\mathfrak{p}} \times \breve{H}^\dagger(\mathbb{A}_f^p)/K^{\dagger,p} \Big).$$

Similarly for $S = [\mathcal{M}_{0,K^\mathfrak{p}}]_{\mathsf{s.s.}}(\mathbb{F})$, we have a p-divisible group $\mathscr{X}\big|_S$ with an action by $H(\mathbb{A}_f)$ which is compatible with that on S. The action factors through $\mathsf{SL}_2(\mathbb{O}_{F_\mathfrak{p}}) \subset H(\mathbb{A}_f)$. Hence its normalizer $\mathsf{SL}_2(F_\mathfrak{p})$ acts trivially on S and the action factors through $E_{\mathfrak{p}^\circ}^{\times,1} \times H_f^\mathfrak{p}$. Let Z be the center of H. Then the stabilizer at s is $\breve{H}(\mathbb{Q}) \times (E_{\mathfrak{p}^\circ}^{\times,1} \times K^\mathfrak{p})$, where $\breve{H} = Z \cdot \breve{H}^{\dagger,\mathsf{der}}$. Hence we have

$$S \cong \breve{H}(\mathbb{Q}) \backslash \breve{H}_f^\mathfrak{p} / K^\mathfrak{p},$$

where $\breve{H}_f^\mathfrak{p} = H_f^\mathfrak{p}$.

Moreover, if we denote by $[\mathcal{M}^\dagger_{0,K_p^{\dagger,\mathfrak{p}},K^{\dagger,p}}]_s^\wedge$ (resp. $[\mathcal{M}_{0,K^\mathfrak{p}}]_s^\wedge$) the formal completion at the point s, then we have

$$[\mathcal{M}^\dagger_{0,K_p^{\dagger,\mathfrak{p}},K^{\dagger,p}}]_s^\wedge \cong \mathcal{N}, \qquad [\mathcal{M}_{0,K^\mathfrak{p}}]_s^\wedge \cong \mathcal{N}',$$

where $\mathcal{N} = \mathsf{Spf}\, R_{F_\mathfrak{p},2}$ with $R_{F_\mathfrak{p},2} = \mathbb{O}_{\hat{F}_\mathfrak{p}}[\![t]\!]$ and $\mathcal{N}' = \mathcal{N} \times_{\mathbb{O}_{\hat{F}_\mathfrak{p}}} \mathbb{O}_{\hat{E}_{\mathfrak{p}^\circ}}$. Hence we have the following p-adic uniformization of the formal completion at the supersingular locus:

$$[\mathcal{M}^\dagger_{0,K_p^{\dagger,\mathfrak{p}},K^{\dagger,p}}]_{\mathsf{s.s.}}^\wedge \times_{\mathbb{O}_{F_\mathfrak{p}}} \mathbb{O}_{\hat{F}_\mathfrak{p}} \cong \breve{H}^\dagger(\mathbb{Q}) \Big\backslash \Big(\mathcal{N} \times \mathbb{Z} \times \prod_{i=2}^r B_{\mathfrak{p}_i}^\times / K_p^{\dagger,\mathfrak{p}} \times \breve{H}^\dagger(\mathbb{A}_f^p)/K^{\dagger,p} \Big)$$

and

$$[\mathcal{M}_{0,K^\mathfrak{p}}]_{\mathsf{s.s.}}^\wedge \times_{\mathbb{O}_{E_{\mathfrak{p}^\circ}}} \mathbb{O}_{\hat{E}_{\mathfrak{p}^\circ}} \cong \breve{H}(\mathbb{Q}) \backslash \mathcal{N}' \times \breve{H}_f^\mathfrak{p} / K^\mathfrak{p},$$

where $\breve{H}^\dagger(\mathbb{Q})$ (resp. $\breve{H}(\mathbb{Q})$) acts on $\mathcal{N}$ (resp. $\mathcal{N}'$) through the $\mathfrak{p}$-component, which is trivial on the center. Such uniformization is a special case of that considered in [Rapoport and Zink 1996].

Next we must determine the formal completion $[\mathcal{Z}(o)^{\dagger}_{0,\mathfrak{p}}]^{\wedge}_{\mathsf{spe}}$ and $[\mathcal{Z}(x)^{\circ}_{0,\mathfrak{p}^\circ}]^{\wedge}_{\mathsf{spe}}$ of the integral special subschemes. We consider the $\mathbb{Z}_{(p)}$-module

$$\breve{\Lambda} = \mathsf{End}((\boldsymbol{E}, j), (\boldsymbol{A}, i_{\boldsymbol{A}})) := \{\varrho : \boldsymbol{E} \to \boldsymbol{A} \mid \varrho \circ j(e) = i_{\boldsymbol{A}}(e) \circ \varrho \text{ for all } e \in \mathbb{O}^{\ddagger}\},$$

which has an action by $\mathbb{O}^{\ddagger}$ through $e.\breve{x} = \breve{x} \circ j(e)$ for $e \in \mathbb{O}^{\ddagger}$ and $\breve{x} \in \breve{\Lambda}$. Hence $\breve{\Lambda}_{\mathbb{Q}} = \breve{\Lambda} \otimes \mathbb{Q}$ has an $E^{\dagger}$-vector space structure which is of dimension 2. The group $\breve{H}^{\dagger}(\mathbb{Q})$ acts on $\breve{\Lambda}_{\mathbb{Q}}$ through $\breve{h}.\breve{x} = \breve{h} \circ \breve{x}$. Hence we have an action of $\breve{H}(\mathbb{Q})$ on $\breve{\Lambda}_{\mathbb{Q}}$. Let $\breve{V}$ be the $\mathbb{Q}$-vector subspace of $\breve{\Lambda}_{\mathbb{Q}}$ generated by $\breve{H}(\mathbb{Q}).\breve{x}_0$ which is an E-vector space of dimension 2, where $\breve{x}_0 = \varrho_{\boldsymbol{A}}$. We define an E-hermitian form $(\,\cdot\,,\cdot\,)'$ on it in the following way:

$$(\breve{x}, \breve{y})' = j^{-1} \circ \vartheta^{-1} \circ \breve{y}^{\vee} \circ \theta \circ \breve{x} \in E, \qquad \breve{x}, \breve{y} \in \breve{V} \subset \breve{\Lambda}_{\mathbb{Q}}. \tag{4-8}$$

By the following lemma, $\breve{H}$ is the unitary group of $(\breve{V}, (\,\cdot\,,\cdot\,)')$ (and the hermitian form really takes values in E). In particular $\breve{V}$ is totally positive definite and is isometric to the nearby hermitian space $V^{(\mathfrak{p})}$ of $\mathbb{V}$.

Lemma 4.9. *For any $\breve{b} \in \breve{B}^{\times,1} = \breve{H}^{\dagger,\mathsf{der}}(\mathbb{Q})$, we have $(\breve{b}.\breve{x}_0, \breve{b}.\breve{x}_0)' = 1$, where $\breve{B}^{\times,1}$ is the subgroup of all norm-*1 *elements.*

Proof. By definition,

$$\begin{aligned}(\breve{b}.\breve{x}_0, \breve{b}.\breve{x}_0)' &= j^{-1} \circ \vartheta^{-1} \circ (\breve{b} \circ \breve{x}_0)^{\vee} \circ \theta \circ (\breve{b}.\breve{x}_0) = j^{-1} \circ \vartheta^{-1} \circ \breve{x}_0^{\vee} \circ (\breve{b}^{\vee} \circ \theta \circ \breve{b}) \circ \breve{x}_0 \\ &= j^{-1} \circ \vartheta^{-1} \circ \breve{x}_0^{\vee} \circ (\bar{\breve{b}}^{\vee,-1} \circ \theta \circ \breve{b}) \circ \breve{x}_0 \\ &= j^{-1} \circ \vartheta^{-1} \circ \breve{x}_0^{\vee} \circ \theta \circ \breve{x}_0 = (\breve{x}_0, \breve{x}_0)' = 1. \end{aligned}$$

□

The level structure $(\bar{\eta}^p, \bar{\eta}^{\mathfrak{p}}_p)$ of $\boldsymbol{A}$ gives a $K^{\mathfrak{p}}$-class of isometries

$$V \otimes_F \mathbb{A}^{\mathfrak{p}}_{f,F} \to \breve{V} \otimes_F \mathbb{A}^{\mathfrak{p}}_{f,F}$$

by sending x to $\breve{x} \in \breve{V} \otimes_F \mathbb{A}^{\mathfrak{p}}_{f,F}$ such that $\breve{x}_*(\boldsymbol{x}^p_s) \in \bar{\eta}^p(x)$ and $\breve{x}_*(\boldsymbol{x}^{\mathfrak{p}}_{p,s}) \in \bar{\eta}^{\mathfrak{p}}_p(x)$. We identify these two spaces through any isometry in this class. For the place $\mathfrak{p}$, we let $\breve{\Lambda}_{\mathfrak{p}} = \mathsf{End}((\boldsymbol{Y}, j), (\boldsymbol{X}, i_{\boldsymbol{X}}))$. Then $\breve{\Lambda}_{\mathfrak{p}}$ is a maximal lattice in $\breve{V}_{\mathfrak{p}}$ such that the hermitian form restricted to it takes values in $\mathbb{O}_{E_{\mathfrak{p}^\circ}}$.

Let $\pi : \mathcal{N}' \to [\mathcal{M}^{\dagger}_{0,K^{\dagger,\mathfrak{p}}_p,K^{\dagger,p}}]^{\wedge}_{\mathsf{s.s.}} \times_{\mathbb{O}_{F_{\mathfrak{p}}}} \mathbb{O}_{\hat{E}_{\mathfrak{p}^\circ}}$ be the natural projection map. Then the base change $\pi^{-1}([\mathcal{Z}(o)^{\dagger}_{0,\mathfrak{p}}]^{\wedge}_{s})$ is $\mathcal{Z}(\breve{x}_0) \times_{\mathbb{O}_{\hat{F}_{\mathfrak{p}}}} \mathbb{O}_{\hat{E}_{\mathfrak{p}^\circ}}$. Here, $\mathcal{Z}(\breve{x}_0)$ is a cycle on $\mathcal{N}$ where the endomorphism $\breve{x}_0 \in \breve{\Lambda}_{\mathfrak{p}}$ deforms. In the next section, we will define and discuss in detail the cycle $\mathcal{Z}(\breve{x})$ for any $\breve{x} \in \breve{\Lambda}_{\mathfrak{p}} - \{0\}$.

For any $\breve{h} \in \breve{H}^{\mathfrak{p}}_f / K^{\mathfrak{p}}$, we denote by $\mathcal{Z}(\breve{x}, \breve{h})$ the cycle of $[\mathcal{M}_{0,K^{\mathfrak{p}}}]^{\wedge}_{\mathsf{s.s.}} \times_{\mathbb{O}_{E_{\mathfrak{p}^\circ}}} \mathbb{O}_{\hat{E}_{\mathfrak{p}^\circ}}$ represented by $(\mathcal{Z}(\breve{x}) \times_{\mathbb{O}_{\hat{F}_{\mathfrak{p}}}} \mathbb{O}_{\hat{E}_{\mathfrak{p}^\circ}}, \breve{h})$. For any $h \in H^{\mathfrak{p}}_f / K^{\mathfrak{p}}$, we denote by $\mathcal{Z}(x, h)$ the cycle of $\mathcal{M}_{0,K^{\mathfrak{p}}} \times_{\mathbb{O}_{E_{\mathfrak{p}^\circ}}} \mathbb{O}_{\hat{E}_{\mathfrak{p}^\circ}}$ which is the translation of $\mathcal{Z}(x)^{\circ}_{0,\mathfrak{p}^\circ}$ by the Hecke operator of h. Since $\breve{\Lambda}_{\mathbb{Q}} \cap \breve{\Lambda}_{\mathfrak{p}} = (\breve{H}^{\dagger}(\mathbb{Q}) \cap \mathsf{GL}(\breve{\Lambda}_{\mathfrak{p}})).(\breve{V} \cap \breve{\Lambda}_{\mathfrak{p}})$, we have the following identity between sets:

$$\begin{aligned}\big\{\mathscr{Z}(\breve{x},\breve{h}) \;\big|\; \breve{x}\in \breve{H}(\mathbb{Q})\backslash(\breve{V}\cap\breve{\Lambda}_{\mathfrak{p}}-\{0\}),\, \breve{h}\in \breve{H}_{\breve{x}}(\mathbb{Q})\backslash \breve{H}^{\mathfrak{p}}_{f}/K^{\mathfrak{p}}\big\}\\ =\big\{[\mathscr{Z}(x,h)]^{\wedge}_{\mathsf{spe}} \;\big|\; x\in H(\mathbb{Q})\backslash(V\cap\Lambda_{\mathfrak{p}}-\{0\}),\, h\in H_x(\mathbb{Q})\backslash H^{\mathfrak{p}}_{f}/K^{\mathfrak{p}}\big\}.\end{aligned} \tag{4-9}$$

Remark 4.10. In the case $F=\mathbb{Q}$, the Shimura datum (H,X) is of PEL type. But for our later computation, we still want to change the datum. Recall that from the hermitian space V, we get a unique quaternion algebra B over $\mathbb{Q}$ which is indefinite. Now the group $H^{\dagger}=B^{\times}$ and $M^{\dagger}_{K^{\dagger}}:=\mathsf{Sh}_{K^{\dagger}}(H^{\dagger},X^{\dagger})$ is just the usual Shimura curve over $\mathbb{Q}$ (the modular curve if B is the matrix algebra). In this case, we don't introduce the auxiliary imaginary field $\mathbb{Q}(\sqrt{\lambda})$ anymore. We still have Proposition 4.3.

For moduli interpretations, it is well known that $M^{\dagger}_{K^{\dagger}}$ parametrizes abelian surfaces with $\mathbb{O}_B$ action and $K^{\dagger}$-level structure. But notice that for local decomposition, we only have one term which is $B_p=B_1^2$. If B_p is a matrix algebra for some rational prime p we still have the object $(\mathscr{A}_{p^{\infty}})^{2,\flat}_1$, which is just $(\mathscr{A}_{p^{\infty}})^{\flat}$, and Proposition 4.5 holds. In particular, when $M^{\dagger}_{K^{\dagger}}$ is the modular curve, we use the results in [Katz and Mazur 1985] to construct the compactified integral models.

The basic abelian scheme in this case is just the elliptic curve over $\mathbb{O}_{E^{\mathrm{nr}}_{p^{\circ}}}$ with generic fiber of CM type $(E,\iota^{\bullet})$. Moreover, we have a similar but simpler version of Propositions 4.7 and 4.8. For various kinds of special cycles, we can define similar notions and their relation (4-9) still holds.

4C. ***Local intersection numbers.*** In this section, we study the formal scheme $\mathcal{N}$ and its formal special subschemes $\mathscr{Z}(\breve{x})$. Then we compute certain intersection numbers of these formal special subschemes. In fact, the case we consider is essentially the same one as in [Kudla and Rapoport 2011] with the signature $(1,1)$, only with mild modifications. We keep assuming that $\epsilon(\mathbb{V}_{\mathfrak{p}})=1$ and $\mathfrak{p}$ is nonsplit in E.

Formal special subschemes. Let $(\boldsymbol{X},i_{\boldsymbol{X}})$ be as in the last section. Then $\boldsymbol{X}^{\flat}$ is a formal $\mathbb{O}_{F_{\mathfrak{p}}}$-module of dimension 1 and height 2. We define a moduli functor $\mathcal{N}$ on $\mathfrak{Nilp}_{\mathbb{O}_{\hat{F}_{\mathfrak{p}}}}$, the category of $\mathbb{O}_{\hat{F}_{\mathfrak{p}}}$-schemes where ϖ is locally nilpotent. For any $S\in\mathrm{Obj}\,\mathfrak{Nilp}_{\mathbb{O}_{\hat{F}_{\mathfrak{p}}}}$, $\mathcal{N}(S)$ is the set of equivalence classes of the couples (G,ρ_G) where

- G is an $\mathbb{O}_{F_{\mathfrak{p}}}$-module of dimension 1 and height 2 over S, and
- $\rho_G : G\times_S S_{\mathsf{spe}}\to \boldsymbol{X}^{\flat}\times_{\mathbb{F}} S_{\mathsf{spe}}$ is a quasiisogeny of height 0 (isomorphism actually).

Two couples (G,ρ_G), $(G',\rho_{G'})$ are equivalent if there is an isomorphism $G'\to G$ sending ρ_G to $\rho_{G'}$. Then $\mathcal{N}$ is represented by a formal scheme of finite type over $\mathsf{Spf}\,\mathbb{O}_{\hat{F}_{\mathfrak{p}}}$ of relative dimension 1 which is just $\mathsf{Spf}\,\mathbb{O}_{\hat{F}_{\mathfrak{p}}}[\![t]\!]$.

Recall that we have a two-dimensional $E_{\mathfrak{p}^\circ}$-hermitian space $\breve{V}_{\mathfrak{p}} = \mathsf{Hom}((\boldsymbol{Y}, j), (\boldsymbol{X}, i_{\boldsymbol{X}})) \otimes \mathbb{Q} \cong V^-$. For any $\breve{x} \in \breve{\Lambda}_{\mathfrak{p}} - \{0\}$, we define a subfunctor $\mathcal{Z}(\breve{x})$ of $\mathcal{N}$ as follows: for any $S \in \mathsf{Obj}\, \mathfrak{Nilp}_{\mathbb{O}_{\hat{F}_{\mathfrak{p}}}}$, $\mathcal{Z}(\breve{x})(S)$ is the set of equivalence classes of (G, ρ_G) such that the composed homomorphism

$$\mathcal{Y} \times_{\mathbb{O}_{\hat{F}_{\mathfrak{p}}}} S_{\mathsf{spe}} = \boldsymbol{Y} \times_{\mathbb{F}} S_{\mathsf{spe}} \xrightarrow{\breve{x}} \boldsymbol{X} \times_{\mathbb{F}} S_{\mathsf{spe}} \xrightarrow{(\rho_G^\sharp)^{-1}} G^\sharp \times_S S_{\mathsf{spe}}$$

extends to a homomorphism $\mathcal{Y} \times_{\mathbb{O}_{\hat{F}_{\mathfrak{p}}}} S \to G^\sharp$. Then $\mathcal{Z}(\breve{x})$ is represented by a closed formal subscheme of $\mathcal{N}$ (in fact, one can show that it is a relative divisor as in [Kudla and Rapoport 2011, Proposition 3.5]). For linearly independent $\vec{x} = (\breve{x}_1, \breve{x}_2) \in (\breve{\Lambda}_{\mathfrak{p}} - \{0\})^2$, two formal special subschemes $\mathcal{Z}(\breve{x}_1)$ and $\mathcal{Z}(\breve{x}_2)$ intersect properly at the unique closed point in $\mathcal{N}_{\mathsf{red}}$. Assuming that $\vec{y} = \vec{x} g$ for some $g \in \mathsf{GL}_2(\mathbb{O}_{E_{\mathfrak{p}^\circ}})$ such that $\vec{y} = (\breve{y}_1, \breve{y}_2)$ has the moment matrix $T(\vec{y}) = \mathsf{diag}[\varpi^a, \varpi^b]$, then of $a, b \geq 0$, one is odd and the other is even.

Intersection numbers. Now we assume that $\mathfrak{p} \nmid 2$ *is unramified in* E. Let $\overline{\boldsymbol{Y}}$ be the unique (up to isomorphism) formal $\mathbb{O}_{F_{\mathfrak{p}}}$-module of dimension 1 and height 2 over $\mathbb{F}$ but with action by $\mathbb{O}_{E_{\mathfrak{p}^\circ}}$ such that the induced character of $\mathbb{O}_{E_{\mathfrak{p}^\circ}}$ on $\mathsf{Lie}(\overline{\boldsymbol{Y}})$ is the one twisted by τ from that on $\mathsf{Lie}(\boldsymbol{Y})$. By [Kudla and Rapoport 2011, Lemma 4.2], there is an isomorphism

$$\rho_{\boldsymbol{X}} : \boldsymbol{Y} \times \overline{\boldsymbol{Y}} \to \boldsymbol{X}$$

in $\mathsf{Hom}_{\mathbb{O}_{E_{\mathfrak{p}^\circ}}}(\boldsymbol{Y} \times \overline{\boldsymbol{Y}}, \boldsymbol{X})$ such that, as elements of $\mathsf{Hom}_{\mathbb{O}_{E_{\mathfrak{p}^\circ}}}(\boldsymbol{Y}, \boldsymbol{Y} \times \overline{\boldsymbol{Y}})$,

$$\rho_{\boldsymbol{X}}^{-1} \circ \breve{y}_i = \begin{cases} \mathsf{inc}_i \circ \Pi^a, & i = 1, \\ \mathsf{inc}_i \circ \Pi^b, & i = 2, \end{cases}$$

where inc_i denotes the inclusion into the i-th factor of the product and Π is a fixed uniformizer of $\mathsf{End}(\boldsymbol{Y})$. We identify $\boldsymbol{Y} \times \overline{\boldsymbol{Y}}$ with $\boldsymbol{X}$, and hence $\breve{\Lambda}_{\mathfrak{p}}$ with $\mathsf{Hom}_{\mathbb{O}_{E_{\mathfrak{p}^\circ}}}(\boldsymbol{Y}, \boldsymbol{Y} \times \overline{\boldsymbol{Y}})$. If we denote by $\mathsf{Def}(\boldsymbol{X}^\flat; \vec{x})$ the subring of $\mathsf{Def}(\boldsymbol{X}^\flat) = \mathbb{O}_{\hat{F}_{\mathfrak{p}}}[\![t]\!]$ deforming $\vec{x}$, then

$$\mathcal{Z}(\breve{x}_1) \cdot \mathcal{Z}(\breve{x}_2) = \mathsf{length}_{\mathbb{O}_{\hat{F}_{\mathfrak{p}}}} \mathsf{Def}(\boldsymbol{X}^\flat; \vec{x}) = \mathsf{length}_{\mathbb{O}_{\hat{F}_{\mathfrak{p}}}} \mathsf{Def}(\boldsymbol{X}^\flat; \vec{y}) = \mathcal{Z}(\breve{y}_1) \cdot \mathcal{Z}(\breve{y}_2).$$

Let F_s be a quasicanonical lifting of level s which is an $\mathbb{O}_{F_{\mathfrak{p}}}$-module over $\mathbb{O}_{\hat{F}_{\mathfrak{p}}^s}$, unique up to the Galois action (see [Gross 1986]). Hence it defines a morphism $\mathsf{Spf}\, \mathbb{O}_{\hat{F}_{\mathfrak{p}}^s} \to \mathcal{N}$ which is a closed immersion. Let $\mathcal{Z}_s$ be the divisor on $\mathcal{N}$ defined by the image, which is independent of the choice of F_s. We have the following proposition generalizing [Kudla and Rapoport 2011, Proposition 8.1].

Proposition 4.11. *As divisors on* $\mathcal{N}$,

$$\mathcal{Z}(\breve{y}_1) = \sum_{\substack{s=0 \\ \text{even}}}^{a} \mathcal{Z}_s, \qquad \mathcal{Z}(\breve{y}_2) = \sum_{\substack{s=1 \\ \text{odd}}}^{b} \mathcal{Z}_s.$$

Proof. The original proof of [Kudla and Rapoport 2011, Proposition 8.1] again works for one direction:

$$\sum_{\substack{s=0\\ \text{even}}}^{a} \mathfrak{Z}_s \leq \mathfrak{Z}(\breve{y}_1), \qquad \sum_{\substack{s=1\\ \text{odd}}}^{b} \mathfrak{Z}_s \leq \mathfrak{Z}(\breve{y}_2).$$

To prove the other side, we only need to prove that the intersection multiplicities of both sides with the special fiber $\mathcal{N}_{\mathsf{spe}} = \mathsf{Spf}\ \mathbb{F}[[t]]$ are the same. For the left side we have

$$\sum_{\substack{s=0\\ \text{even}}}^{a} \mathfrak{Z}_s \cdot \mathcal{N}_{\mathsf{spe}} = \sum_{\substack{s=0\\ \text{even}}}^{a} [\mathbb{O}_{F_{\mathfrak{p}}}^{\times} : U_{F_{\mathfrak{p}}}^{s}] = \frac{q^{a+1}-1}{q-1},$$

$$\sum_{\substack{s=1\\ \text{odd}}}^{b} \mathfrak{Z}_s \cdot \mathcal{N}_{\mathsf{spe}} = \sum_{\substack{s=1\\ \text{odd}}}^{b} [\mathbb{O}_{F_{\mathfrak{p}}}^{\times} : U_{F_{\mathfrak{p}}}^{s}] = \frac{q^{b+1}-1}{q-1},$$

where q is the cardinality of the residue field of $F_{\mathfrak{p}}$. The assertion follows from the following proposition, which is a generalization of [Kudla and Rapoport 2011, Proposition 8.2]. □

Proposition 4.12. *For* $\breve{y} \in \mathrm{Hom}_{\mathbb{O}_{E_{\mathfrak{p}^\circ}}}(\boldsymbol{Y}, \boldsymbol{X})$, *the intersection multiplicity*

$$\mathfrak{Z}(\breve{y}) \cdot \mathcal{N}_{\mathsf{spe}} = \frac{q^{v+1}-1}{q-1},$$

where $v \geqslant 0$ *is the valuation of* $(\breve{y}, \breve{y})'$.

Proof. We generalize the proof of [Kudla and Rapoport 2011, Proposition 8.2] to the case $F_{\mathfrak{p}} \neq \mathbb{Q}_p$ again by using the theory of windows and displays of p-divisible groups [Zink 2001; 2002]. In the proof, we simply write $F = F_{\mathfrak{p}}$, $E = E_{\mathfrak{p}^\circ}$ and let e and f be the ramification index and extension degree of residue fields of $F/\mathbb{Q}_p$, respectively. Then $q = p^f$. Let $R = \mathbb{F}[[t]]$ and $A = W[[t]]$ where $W = W(\mathbb{F})$. We extend the Frobenius automorphism σ on W to A by setting $\sigma(t) = t^p$. For any $s \geq 1$, we set $R_s = R/t^s$ and $A_s = A/t^s$. Then A (resp. A_s) is a frame of R (resp. R_s). The category of formal p-divisible groups over R is equivalent to the category of pairs (M, α), consisting of a free A-module of finite rank and an A-linear injective homomorphism $\alpha : M \to M^{(\sigma)} := A \otimes_{A,\sigma} M$, such that $\mathrm{coker}(\alpha)$ is a free R-module.

First, we treat the case $f = 1$. Consider the p-divisible group $\boldsymbol{Y}$ over $\mathbb{F}$ of dimension 1 and (absolute) height $2e$ with action by $\mathbb{O}_E$. It corresponds to the pair (N, β), where N is the $\mathbb{Z}/2$-graded free $\mathbb{O}_{\hat{F}} = \mathbb{O}_F \otimes_{\mathbb{Z}_p} W$-module of rank 2 (which is a free W-module of rank $2e$) with $N_i = \mathbb{O}_{\hat{F}} \cdot n_i$ $(i = 1, 2)$ and $\beta(n_0) = \varpi \otimes n_1$, $\beta(n_1) = 1 \otimes n_0$. We extend $\mathbb{O}_F$-linearly the Frobenius automorphism on W to $\mathbb{O}_{\hat{F}}$.

Similarly as in the proof of [Kudla and Rapoport 2011, Proposition 8.2], the p-divisible group $\boldsymbol{X}$ over $\mathbb{F}$ corresponds to (M, α) there and its universal deformation is (M, α_t). The only difference is that we should replace p by ϖ. The proof there works exactly in this case.

Now we treat the general case and assume that $f \geq 2$. Consider the p-divisible group $\boldsymbol{Y}$ over $\mathbb{F}$. It corresponds to the pair (N, β), where N is a $\mathbb{Z}/2$-graded free $\mathbb{O}_F \otimes_{\mathbb{Z}_p} W$-module of rank 2. Since

$$\mathbb{O}_F \otimes_{\mathbb{Z}_p} W = \bigoplus_{j=0}^{f-1} \mathbb{O}_F \otimes_{W(k),\sigma^j} W =: \bigoplus_{j=0}^{f-1} \mathbb{O}_{\hat{F}}^{(\sigma^j)},$$

where k is the residue field of F, we can write

$$N = \left(\bigoplus_{j=0}^{f-1} \mathbb{O}_{\hat{F}}^{(\sigma^j)} e_{0,j}\right) \oplus \left(\bigoplus_{j=0}^{f-1} \mathbb{O}_{\hat{F}}^{(\sigma^j)} e_{1,j}\right)$$

and $\beta(e_{i,j}) = e_{i,j+1}$ for $i = 1, 2$ and $0 \leqslant j < f-1$; $\beta(e_{0,f-1}) = e_{1,0}$ and $\beta(e_{1,f-1}) = \varpi e_{0,0}$. Similarly, the p-divisible group $\overline{\boldsymbol{Y}}$ corresponds to $(\overline{N}, \bar{\beta})$ where we write

$$\overline{N} = \left(\bigoplus_{j=0}^{f-1} \mathbb{O}_{\hat{F}}^{(\sigma^j)} \bar{e}_{0,j}\right) \oplus \left(\bigoplus_{j=0}^{f-1} \mathbb{O}_{\hat{F}}^{(\sigma^j)} \bar{e}_{1,j}\right)$$

and $\bar{\beta}(\bar{e}_{i,j}) = \bar{e}_{i,j+1}$ for $i = 0, 1$ and $0 \leqslant j < f-1$; $\bar{\beta}(\bar{e}_{1,f-1}) = \bar{e}_{0,0}$ and $\bar{\beta}(\bar{e}_{0,f-1}) = \varpi \bar{e}_{1,0}$. Then we extend them to $\mathbb{F}[\![t]\!]$ by scalars, still denoted by N and $\overline{N}$.

The p-divisible group $\boldsymbol{X}$ corresponds the direct sum $(M, \alpha) := (N, \beta) \oplus (\overline{N}, \bar{\beta})$. Under the basis $\{e_{0,0}, \bar{e}_{1,0}, \ldots, e_{0,f-1}, \bar{e}_{1,f-1}; e_{1,0}, \bar{e}_{0,0}, \ldots, e_{1,f-1}, \bar{e}_{0,f-1}\}$, the matrix of α is

$$\alpha = \begin{pmatrix} & & & & & & \varpi & \\ & & & & & & & \varpi \\ 1 & & & & & & & \\ & \ddots & & & & & & \\ & & 1 & & & & & \\ & & & 1 & & & & \\ & & & & \ddots & & & \\ & & & & & 1 & & \end{pmatrix}.$$

Let (M, α_t) corresponds to the universal deformation of (X, i_X) over $\mathbb{F}[[t]]$. We can write in the same basis

$$\alpha_t = \begin{pmatrix} 1 & & & & & t & & \\ & 1 & & & -t & & & \\ & & 1 & & & & & \\ & & & \ddots & & & & \\ & & & & 1 & & & \\ & & & & & 1 & & \\ & & & & & & \ddots & \\ & & & & & & & 1 \end{pmatrix} \cdot \alpha.$$

Explicitly, we have

$$\begin{aligned} \alpha_t(e_{i,j}) &= e_{i,j+1}, \quad i = 0, 1 \text{ and } j = 0, \dots, f-2, \\ \alpha_t(e_{0,f-1}) &= e_{1,0} - t\bar{e}_{1,0}, \qquad \alpha_t(e_{1,f-1}) = \varpi e_{0,0}, \\ \alpha_t(\bar{e}_{i,j}) &= \bar{e}_{i,j+1}, \quad i = 0, 1 \text{ and } j = 0, \dots, f-2, \\ \alpha_t(\bar{e}_{1,f-1}) &= \bar{e}_{0,0} + te_{0,0}, \qquad \alpha_t(\bar{e}_{0,f-1}) = \varpi \bar{e}_{1,0}. \end{aligned}$$

Now we denote by $\sigma^k(\alpha) : M^{(\sigma^k)} \to M^{(\sigma^{k+1})}$ the induced homomorphism for $k \geqslant 0$. Then, formally, we have

$$\begin{aligned} &\sigma^k(\alpha)^{-1}(e_{i,j}) = e_{i,j-1}, \quad i = 1, 2 \text{ and } j = 1, \dots, f-1, \\ &\sigma^k(\alpha)^{-1}(e_{0,0}) = \frac{1}{\varpi} e_{1,f-1}, \qquad \sigma^k(\alpha)^{-1}(e_{1,0}) = e_{0,f-1} + \frac{t^{p^k}}{\varpi} \bar{e}_{0,f-1}, \\ &\sigma^k(\alpha)^{-1}(\bar{e}_{i,j}) = \bar{e}_{i,j-1}, \quad i = 1, 2 \text{ and } j = 1, \dots, f-1, \\ &\sigma^k(\alpha)^{-1}(\bar{e}_{1,0}) = \frac{1}{\varpi} \bar{e}_{0,f-1}, \qquad \sigma^k(\alpha)^{-1}(\bar{e}_{0,0}) = \bar{e}_{1,f-1} - \frac{t^{p^k}}{\varpi} e_{1,f-1}. \end{aligned}$$

Now let $\check{y}$ correspond to the graded A_1-linear homomorphism $\gamma : N \otimes_A A_1 \to M$. Then the length $\ell = \mathfrak{X}(\check{y}) \cdot \mathcal{N}_{\mathsf{spe}}$ of the deformation space of γ is the maximal number a such that there exists a diagram of the form

$$\begin{tikzcd} N \arrow[r, "\beta"] \arrow[d, "\tilde{\gamma}"'] & N^{(\sigma)} \arrow[d, "\tilde{\gamma}^{(\sigma)}"] \\ M \arrow[r, "\alpha_t"] & M^{(\sigma)} \end{tikzcd}$$

that commutes modulo t^a, where $\tilde{\gamma}$ lifts γ.

<u>Case i</u>: *$v = 2r$ is even.* We may assume that $\gamma = \varpi^r \mathsf{inc}_1$, represented by the $4f \times 2f$ matrix

$$X(0)=\begin{pmatrix} \varpi^r & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & \varpi^r & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \varpi^r \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

If $r=0$, in order to lift γ mod t^p, we search for a $4f\times 2f$ matrix $X(1)$ with entries in A_p such that $X(1)\equiv X(0)$ in A_1 and satisfies

$$\alpha_t\circ X(1)=\sigma(X(1))\circ\beta.$$

But $\sigma(X(1))=\sigma(X(0))=X(0)$. Hence we need to find the largest $a\leqslant p$ such that $\alpha_t^{-1}\circ X(0)\circ\beta$ has integral entries $\bmod t^a$. But the entry at the place $(e_{0,f-1},\bar{e}_{0,f-1})$ is t/ϖ, so the largest a is just 1. Hence when $v=r=0$, the proposition holds.

If $r>0$, we first show that we can lift γ mod $t^{q^{2r}}$. By induction, we introduce $X(k)$ for $k\geq 1$ by requiring that $X(k+1)\equiv X(k)$ in A_{p^k} and $\alpha_t\circ X(k+1)=\sigma(X(k+1))\circ\beta$. But $\sigma(X(k+1))=\sigma(X(k))$; hence formally we should have $X(k+1)=\alpha_t^{-1}\circ\sigma(X(k))\circ\beta$. We need to show that

$$X(2rf)=\alpha_t^{-1}\circ\sigma(\alpha_t)^{-1}\circ\cdots\circ\sigma^{2rf-1}(\alpha_t)^{-1}\circ X(0)\circ\beta^{2rf}$$

has integral entries. Let $x_{i,j;i',j'}$ (resp. $\bar{x}_{i,j;i',j'}$) be the entry of $X(2rf)$ mod ϖ at the place $(e_{i,j},e_{i',j'})$ (resp. $(e_{i,j},\bar{e}_{i',j'})$). Then among all these terms, the only nonzero terms are

$$\bar{x}_{0,j;0,j}=(-1)^{r-1}t^{p^{f-1-j}(q^{2r-2}+q^{2r-3}+\cdots+1)},\qquad j=0,\ldots,f-1,$$
$$x_{1,j;1,j}=(-1)^{r}t^{p^{f-1-j}(q^{2r-1}+q^{2r-2}+\cdots+1)},\qquad j=0,\ldots,f-1,$$

which shows that we can lift γ mod $t^{q^{2r}}$. Next, we consider the lift of γ mod $t^{pq^{2r}}$, that is, the matrix

$$X(2rf+1)=\alpha_t^{-1}\circ\sigma(X(2rf))\circ\beta.$$

It has exactly one entry which is not integral: the place of $(e_{0,f-1},\bar{e}_{0,f-1})$ whose nonintegral part is

$$\frac{t}{\varpi}(-1)^r t^{p\cdot p^{f-1}(q^{2r-1}+q^{2r-2}+\cdots+1)}=\frac{(-1)^r}{\varpi}t^{q^{2r}+q^{2r-1}+\cdots+1}.$$

It turns out that the length $\ell=\mathfrak{L}(\check{y})\cdot\mathcal{N}_{\mathsf{spe}}$ is exactly

$$\frac{q^{2r+1}-1}{q-1}=\frac{q^{v+1}-1}{q-1}.$$

Case ii: $v = 2s+1$ *is even.* We may assume that We may assume that $\gamma = \varpi^s \mathsf{inc}_2 \circ \Pi$, where Π is the endomorphism of $\boldsymbol{Y}$ determined by $\Pi(e_{0,j}) = e_{0,j}$ and $\Pi(e_{1,j}) = \varpi e_{0,j}$ for $j = 0, \dots, f-1$. Then γ is represented by the $4f \times 2f$ matrix

$$\left(\begin{array}{cccccc} & & & 0 & \cdots & 0 \\ & & & \varpi^{s+1} & \cdots & 0 \\ & & & \vdots & \ddots & \vdots \\ & & & 0 & \cdots & 0 \\ & & & 0 & \cdots & \varpi^{s+1} \\ 0 & \cdots & 0 & & & \\ \varpi^{s} & \cdots & 0 & & & \\ \vdots & \ddots & \vdots & & & \\ 0 & \cdots & 0 & & & \\ 0 & \cdots & \varpi^{s} & & & \end{array}\right).$$

Similarly, we introduce the notation $Y(k)$ for $k \geq 0$. We first show that γ lifts mod $t^{q^{2s+1}}$, that is, the matrix

$$Y((2s+1)f) = \alpha_t^{-1} \circ \sigma(\alpha_t)^{-1} \circ \cdots \circ \sigma^{(2s+1)f-1}(\alpha_t)^{-1} \circ Y(0) \circ \beta^{(2s+1)f}$$

has integral entries. Let $y_{i,j;i',j'}$ (resp. $\bar{y}_{i,j;i',j'}$) be the entry of $Y((2s+1)f)$ mod ϖ at the place $(e_{i,j}, e_{i',j'})$ (resp. $(e_{i,j}, \bar{e}_{i',j'})$). Then among all these terms, the only nonzero terms are

$$\begin{aligned} \bar{y}_{0,j;0,j} &= (-1)^s t^{p^{f-1-j}(q^{2s-1}+q^{2s-2}+\cdots+1)}, & j &= 0, \dots, f-1, \\ y_{1,j;1,j} &= (-1)^{s+1} t^{p^{f-1-j}(q^{2s}+q^{2s-1}+\cdots+1)}, & j &= 0, \dots, f-1, \end{aligned}$$

which shows that we can lift γ mod $t^{q^{2s+1}}$. Next we consider the lift of γ mod $t^{pq^{2s+1}}$, that is, the matrix

$$Y((2s+1)f+1) = \alpha_t^{-1} \circ \sigma\big(Y((2s+1)f)\big) \circ \beta.$$

It has exactly one entry which is not integral: the place of $(e_{0,f-1}, \bar{e}_{0,f-1})$ whose nonintegral part is

$$\frac{t}{\varpi}(-1)^{s+1} t^{p \cdot p^{f-1}(q^{2s}+q^{2s-1}+\cdots+1)} = \frac{(-1)^{s+1}}{\varpi} t^{q^{2s+1}+q^{2s}+\cdots+1}.$$

It turns out that the length $\ell = \mathfrak{L}(\breve{y}) \cdot \mathcal{N}_{\mathsf{spe}}$ is exactly

$$\frac{q^{2s+2}-1}{q-1} = \frac{q^{v+1}-1}{q-1}.$$

This proves the proposition. □

The results in [Görtz and Rapoport 2007] used in the proof of [Kudla and Rapoport 2011, Proposition 8.4] also work for $F_{\mathfrak{p}}$, not just $\mathbb{Q}_p$. For $0 < s \le b$ odd we have

$$\mathfrak{X}(\breve{y}_1)\cdot\mathfrak{X}_s = \begin{cases} \dfrac{q^{a+1}-1}{q-1}, & a < s,\\ \dfrac{q^s-1}{q-1} + \dfrac{1}{2}(a+1-s)[\mathcal{O}_{F_{\mathfrak{p}}}^{\times} : U_{F_{\mathfrak{p}}}^{s}], & a \ge s. \end{cases}$$

By summing over s, we get the following *local arithmetic Siegel–Weil formula* at a good finite place:

Theorem 4.13. *Let $\breve{x}_i \in \breve{\Lambda}_{\mathfrak{p}} - \{0\}$, for $i = 1, 2$, be linearly independent. Then the intersection multiplicity $\mathfrak{X}(\breve{x}_1)\cdot\mathfrak{X}(\breve{x}_2)$ only depends on the $\mathsf{GL}_2(\mathcal{O}_{E_{\mathfrak{p}^\circ}})$-equivalence class of the moment matrix $T = T(\vec{x})$. Moreover, if we set $T \sim \mathsf{diag}[\varpi^a, \varpi^b]$ with $0 \le a < b$, then*

$$H_{\mathfrak{p}}(T) := \mathfrak{X}(\breve{x}_1)\cdot\mathfrak{X}(\breve{x}_2) = \frac{1}{2}\sum_{l=0}^{a} q^l(a+b+1-2l),$$

where q is the cardinality of the residue field of $F_{\mathfrak{p}}$.

4D. ***Comparison at good places.*** In this section, we will consider the local height pairing $\langle \widehat{Z}_{\phi_1}(g_1), \widehat{Z}_{\phi_2}(g_2)\rangle_{v^\circ}$ at a finite place v° of E which is good. Recall that we have a Shimura curve M_K and we assume that K is sufficiently small and decomposable. We also assume that ϕ_i $(i = 1, 2)$ are decomposable.

Let $\mathcal{S} \subset \Sigma_f$ be a finite subset with $|\mathcal{S}| \ge 2$ such that for any finite place $\mathfrak{p} \notin \mathcal{S}$, we have

- $\mathfrak{p} \nmid 2$, $\mathfrak{p}$ is unramified or split in E;
- $\epsilon(\mathbb{V}_{\mathfrak{p}}) = 1$;
- $\phi_{i,\mathfrak{p}} = \phi_{\mathfrak{p}}^0$ $(i = 1, 2)$ are the characteristic functions of a self-dual lattice $\Lambda_{\mathfrak{p}} = \Lambda^+ \subset \mathbb{V}_{\mathfrak{p}}$;
- $K_{\mathfrak{p}}$ is the subgroup of $\mathsf{U}(\mathbb{V}_{\mathfrak{p}})$ stabilizing $\Lambda_{\mathfrak{p}}$, that is, $K_{\mathfrak{p}}$ is a hyperspecial maximal compact subgroup;
- χ and ψ are unramified at $\mathfrak{p}$.

Fix a place $\mathfrak{p} \notin \mathcal{S}$ such that $\mathfrak{p}$ is nonsplit in E. We have the generating series

$$\begin{aligned} Z_{\phi_i}(g_i) &= \sum_{x_i \in K\backslash\mathbb{V}} \omega_\chi(g_i)\phi_i(x_i)Z(x_i)_K \\ &= \sum_{x_i \in H(\mathbb{Q})\backslash V}\ \sum_{h_i \in H_{x_i}(\mathbb{A}_f)\backslash H(\mathbb{A}_f)/K} \omega_\chi(g_i)\phi_i(h_i^{-1}x_i)Z(h_i^{-1}x_i)_K, \end{aligned} \tag{4-10}$$

where $V=V^{(\iota_1)}$ is the nearby (coherent) hermitian space as in Section 4B. Write $g_{i,\mathfrak{p}}=n(b_{i,\mathfrak{p}})m(a_{i,\mathfrak{p}})k_{i,\mathfrak{p}}$ in the Iwasawa decomposition and choose any number $e_i\in E^\times$ such that $\mathrm{val}_{\mathfrak{p}}(e_i)=\mathrm{val}_{\mathfrak{p}}(a_{i,\mathfrak{p}})$. Let $\tilde{g}_i=m(e_i)^{-1}g_i$; then

$$\begin{aligned}
(4\text{-}10) &= \sum_{x_i\in H(\mathbb{Q})\backslash V}\ \sum_{h_i\in H_{x_i}(\mathbb{A}_f)\backslash H(\mathbb{A}_f)/K} \omega_\chi(\tilde{g}_i)\phi_i(h_i^{-1}x_ie_i)Z(h_i^{-1}x_i)_K\\
&= \sum_{x_i\in H(\mathbb{Q})\backslash V}\ \sum_{h_i\in H_{x_i}(\mathbb{A}_f)\backslash H(\mathbb{A}_f)/K} \omega_\chi(\tilde{g}_i)\phi_i(h_i^{-1}x_ie_i)Z(h_i^{-1}x_ie_i)_K\\
&= \sum_{x_i\in H(\mathbb{Q})\backslash V}\ \sum_{h_i\in H_{x_i}(\mathbb{A}_f)\backslash H(\mathbb{A}_f)/K} \omega_\chi(\tilde{g}_i)\phi_i(h_i^{-1}x_i)Z(h_i^{-1}x_i)_K\\
&= \sum_{x_i\in H(\mathbb{Q})\backslash V}\ \sum_{h_i\in H_{x_i}(\mathbb{A}_f)\backslash H(\mathbb{A}_f)/K} \psi_{\mathfrak{p}}(\tilde{b}_{i,\mathfrak{p}}T(x_i))\phi^0_{\mathfrak{p}}\otimes(\omega_\chi(\tilde{g}^{\mathfrak{p}}_i)\phi^{\mathfrak{p}}_i)(h_i^{-1}x_i)Z(h_i^{-1}x_i)_K
\end{aligned}$$

since in the Iwasawa decomposition $\tilde{g}_{i,\mathfrak{p}}=n(\tilde{b}_{i,\mathfrak{p}})m(\tilde{a}_{i,\mathfrak{p}})k_{i,\mathfrak{p}}$, $\tilde{a}_{i,\mathfrak{p}}\in\mathbb{O}_{E_{\mathfrak{p}^\circ}}$. In what follows, we assume that $\phi_{1,v}\otimes\phi_{2,v}\in\mathcal{S}(\mathbb{V}_v^2)_{\mathrm{reg}}$ for at least one $v\in\mathcal{S}$ and $g_i\in P'_vH'(\mathbb{A}_F^v)$. Then if $Z(h_i^{-1}x_i)_K$ appears in the generating series $Z_{\phi_i}(g_i)$, we must have $x_i\in V-\{0\}$ and $h_{i,\mathfrak{p}}^{-1}x_i\in\Lambda_{\mathfrak{p}}$. Hence by the last part of Section 4B, we can extend $Z(h_i^{-1}x_i)_K$ to a union of integral special subschemes $\mathfrak{X}(x_i,h_i)$ on the smooth model $\mathcal{M}_{0,K^{\mathfrak{p}}}$. We define

$$\begin{aligned}
&\mathfrak{X}_{\phi_i}(g_i)\\
&\quad= \sum_{x_i\in H(\mathbb{Q})\backslash V}\ \sum_{h_i\in H_{x_i}(\mathbb{Q})\backslash H(\mathbb{A}_f)/K} \psi_{\mathfrak{p}}(\tilde{b}_{i,\mathfrak{p}}T(x_i))\phi^0_{\mathfrak{p}}\otimes(\omega_\chi(\tilde{g}^{\mathfrak{p}}_i)\phi^{\mathfrak{p}}_i)(h_i^{-1}x_i)\mathfrak{X}(x_i,h_i),
\end{aligned}$$

which is a cycle of $\mathcal{M}_{0,K^{\mathfrak{p}}}$ extending the generating series $Z_{\phi_i}(g_i)$.

If $\mathfrak{p}\notin\mathcal{S}$ is split in E, we just take $\mathfrak{X}_{\phi_i}(g_i)$ to be the Zariski closure of $Z_{\phi_i}(g_i)$ in $\mathcal{M}_{0,K^{\mathfrak{p}}}$.

We now state the main theorem of this section. Here vol is the same volume as in [Liu 2011, Theorem 4.20].

Theorem 4.14. *Suppose that $\phi_{1,v}\otimes\phi_{2,v}\in\mathcal{S}(\mathbb{V}_v^2)_{\mathrm{reg}}$ for at least one $v\in\mathcal{S}$ and $g_i\in P'_vH'(\mathbb{A}_F^v)$. Let $\mathfrak{p}$ be a finite place not in $\mathcal{S}$, let $\mathcal{M}_{K;\mathfrak{p}^\circ}=\mathcal{M}_{0,K^{\mathfrak{p}}}$ be the smooth local model introduced in Section 4B, and let $\mathfrak{X}_{\phi_i}(g_i)$ be the cycle introduced above.*

(1) *For $\mathfrak{p}$ nonsplit in E,*

$$E_{\mathfrak{p}}(0,\iota(g_1,g_2^\vee),\phi_1\otimes\phi_2)=-\mathrm{vol}(K)\langle\widehat{Z}_{\phi_1}(g_1),\widehat{Z}_{\phi_2}(g_2)\rangle_{\mathfrak{p}^\circ},$$

where, by definition,

$$\langle\widehat{Z}_{\phi_1}(g_1),\widehat{Z}_{\phi_2}(g_2)\rangle_{\mathfrak{p}^\circ}=\log q^2(\mathfrak{X}_{\phi_1}(g_1)\cdot\mathfrak{X}_{\phi_2}(g_2))$$

(2) $\mathfrak{X}_{\phi_1}(g_1)\cdot\mathfrak{X}_{\phi_2}(g_2)=0$ *for $\mathfrak{p}$ split in E.*

Combining this with [Liu 2011, Theorem 4.20], we have:

Corollary 4.15. *Assume that* $\phi_i=\phi^0_\infty\phi_{i,f}$ *satisfy* $\phi_{1,v}\otimes\phi_{2,v}\in\mathcal{S}(\mathbb{V}_v^2)_{\mathrm{reg}}$ *for all* $v\in\mathcal{S}$ *and* $\phi_{1,v}\otimes\phi_{2,v}\in\mathcal{S}(\mathbb{V}_v^2)_{\mathrm{reg},d_v}$ *with* $d_v\geq d_{\psi_v}$ *for nonsplit* $v\in\mathcal{S}$ (see Section 2C for the notation). *Assume further that* $g_i\in e_{\mathcal{S}}H'(\mathbb{A}_F^{\mathcal{S}})$ $(i=1,2)$, *and that the local model* $\mathcal{M}_{K;\mathfrak{p}^\circ}$ *is* $\mathcal{M}_{0,K^{\mathfrak{p}}}$ *for all finite places* $\mathfrak{p}^\circ|\mathfrak{p}\notin\mathcal{S}$. *Then*

$$E'(0,\iota(g_1,g_2^\vee),\phi_1\otimes\phi_2)=-\mathrm{vol}(K)\sum_{\substack{v^\circ|v\\ v\notin\mathcal{S}}}\langle\widehat{Z}_{\phi_1}(g_1),\widehat{Z}_{\phi_2}(g_2)\rangle_{v^\circ}.$$

(Here the Green's functions used in archimedean places are those defined in [Liu 2011, Theorem 4.20], not the admissible Green's functions defined in Section 3B.)

Proof of Theorem 4.14. (1) Since the special fiber of $\mathfrak{Z}_{\phi_i}(g_i)$ locates in the supersingular locus, we have

$$\mathfrak{Z}_{\phi_1}(g_1)\cdot\mathfrak{Z}_{\phi_2}(g_2)=[\mathfrak{Z}_{\phi_1}(g_1)]^\wedge_{\mathsf{spe}}\cdot[\mathfrak{Z}_{\phi_2}(g_2)]^\wedge_{\mathsf{spe}}. \tag{4-11}$$

But

$$[\mathfrak{Z}_{\phi_i}(g_i)]^\wedge_{\mathsf{spe}}=\sum_{\breve{x}_i\in\breve{H}(\mathbb{Q})\backslash\breve{V}}\ \sum_{\breve{h}_i\in\breve{H}_{\breve{x}_i}(\mathbb{Q})\backslash\breve{H}_f^{\mathfrak{p}}/K^{\mathfrak{p}}}\psi_{\mathfrak{p}}(\tilde{b}_{i,\mathfrak{p}}T(\breve{x}_i))\breve{\phi}^0_{\mathfrak{p}}\otimes\omega_\chi(\tilde{g}_i^{\mathfrak{p}})\phi_i^{\mathfrak{p}}(\breve{h}_i^{-1}\breve{x}_i)\mathfrak{Z}(\breve{x}_i,\breve{h}_i), \tag{4-12}$$

where $\breve{\phi}^0_{\mathfrak{p}}$ is the characteristic function of $\breve{\Lambda}_{\mathfrak{p}}$. For any $t_i\in F\cap\mathcal{O}_{F_{\mathfrak{p}}}$ which is totally positive, we fix an element $\breve{x}_{t_i}\in\breve{V}\cap\breve{\Lambda}_{\mathfrak{p}}$ with $T(\breve{x}_{t_i})=t_i$. Then

$$\begin{aligned}\text{(4-12)}&=\sum_{t_i}\psi_{\mathfrak{p}}(\tilde{b}_{i,\mathfrak{p}}t_i)\sum_{\breve{h}_i\in\breve{H}_{\breve{x}_i}(\mathbb{Q})\backslash\breve{H}_f^{\mathfrak{p}}/K^{\mathfrak{p}}}\omega_\chi(\tilde{g}_i^{\mathfrak{p}})\phi_i^{\mathfrak{p}}(\breve{h}_i^{-1}\breve{x}_i)\mathfrak{Z}(\breve{x}_i,\breve{h}_i)\\&=\sum_{\breve{x}_i\in\breve{V}\cap\breve{\Lambda}_{\mathfrak{p}}-\{0\}}\psi_{\mathfrak{p}}(\tilde{b}_{i,\mathfrak{p}}T(\breve{x}_i))\sum_{\breve{h}_i\in\breve{H}(\mathbb{Q})\backslash\breve{H}_f^{\mathfrak{p}}/K^{\mathfrak{p}}}\omega_\chi(\tilde{g}_i^{\mathfrak{p}})\phi_i^{\mathfrak{p}}(\breve{h}_i^{-1}\breve{x}_i)\mathfrak{Z}(\breve{x}_i,\breve{h}_i).\end{aligned}$$

Two formal cycles $\mathfrak{Z}(\breve{x}_1,\breve{h}_1)$ and $\mathfrak{Z}(\breve{x}_2,\breve{h}_2)$ intersect only if $\breve{h}_1$ and $\breve{h}_2$ are in the same double coset of $\breve{H}(\mathbb{Q})\backslash\breve{H}_f^{\mathfrak{p}}/K^{\mathfrak{p}}$. Hence

$$\begin{aligned}\text{(4-11)}&=\sum_{\vec{x}=(x_1,x_2)}\psi_{\mathfrak{p}}(\tilde{b}_{\mathfrak{p}}T(\vec{x}))\sum_{\breve{h}\in\breve{H}(\mathbb{Q})\backslash\breve{H}_f^{\mathfrak{p}}/K^{\mathfrak{p}}}\omega''_\chi(\iota(\tilde{g}_1^{\mathfrak{p}},\tilde{g}_2^{\mathfrak{p},\vee}))\phi_1^{\mathfrak{p}}\otimes\phi_2^{\mathfrak{p}}(\breve{h}^{-1}\vec{x})\mathfrak{Z}(\breve{x}_1)\cdot\mathfrak{Z}(\breve{x}_2)\\&=\sum_{T\in\mathrm{GL}_2(E_{\mathfrak{p}^\circ})\cap\mathrm{Her}_2(\mathcal{O}_{E_{\mathfrak{p}^\circ}})}\psi_{\mathfrak{p}}(\tilde{b}_{\mathfrak{p}}T(\vec{x}))\\&\qquad\times\sum_{\breve{h}\in\breve{H}_f^{\mathfrak{p}}/K^{\mathfrak{p}}}\omega''_\chi(\iota(\tilde{g}_1^{\mathfrak{p}},\tilde{g}_2^{\mathfrak{p},\vee}))\phi_1^{\mathfrak{p}}\otimes\phi_2^{\mathfrak{p}}(\breve{h}^{-1}\vec{x})H_{\mathfrak{p}}(T),\end{aligned}\tag{4-13}$$

where $\tilde{b}_{\mathfrak{p}} = \mathrm{diag}[\tilde{b}_{1,\mathfrak{p}}, \tilde{b}_{2,\mathfrak{p}}]$. By Theorem 4.13, Corollary 4.2 and following the same steps in the proof of [Liu 2011, Theorem 4.20], we get

$$-\mathrm{vol}(K)\log q^2(\mathcal{Z}_{\phi_1}(g_1)\cdot\mathcal{Z}_{\phi_2}(g_2)) = E_{\mathfrak{p}}(0, \iota(\tilde{g}_1, \tilde{g}_2^\vee), \phi_1\otimes\phi_2). \tag{4-14}$$

By definition,

$$\begin{aligned}
&(4\text{-}14)\\
&= \sum_{\mathrm{Diff}(T,\mathbb{V})=\{\mathfrak{p}\}} W'_T(0, \iota(\tilde{g}_{1,\mathfrak{p}}, \tilde{g}_{2,\mathfrak{p}}^\vee), \phi_{1,\mathfrak{p}}\otimes\phi_{2,\mathfrak{p}}) \prod_{v\neq\mathfrak{p}} W_T(0, \iota(\tilde{g}_{1,v}, \tilde{g}_{2,v}^\vee), \phi_{1,v}\otimes\phi_{2,v})\\
&= \sum_{\mathrm{Diff}(T,\mathbb{V})=\{\mathfrak{p}\}} W'_{e^\tau Te}(0, \iota(g_{1,\mathfrak{p}}, g_{2,\mathfrak{p}}^\vee), \phi_{1,\mathfrak{p}}\otimes\phi_{2,\mathfrak{p}})\\
&\qquad\qquad \times \prod_{v\neq\mathfrak{p}} W_{e^\tau Te}(0, \iota(g_{1,v}, g_{2,v}^\vee), \phi_{1,v}\otimes\phi_{2,v})\\
&= \sum_{\mathrm{Diff}(T,\mathbb{V})=\{\mathfrak{p}\}} W'_T(0, \iota(g_{1,\mathfrak{p}}, g_{2,\mathfrak{p}}^\vee), \phi_{1,\mathfrak{p}}\otimes\phi_{2,\mathfrak{p}}) \prod_{v\neq\mathfrak{p}} W_T(0, \iota(g_{1,v}, g_{2,v}^\vee), \phi_{1,v}\otimes\phi_{2,v})\\
&= E_{\mathfrak{p}}(0, \iota(g_1, g_2^\vee), \phi_1\otimes\phi_2),
\end{aligned}$$

where $e = \mathrm{diag}[e_1, e_2] \in \mathrm{GL}_2(E)$.

(2) We will prove this in a more general case in Lemma 5.1. □

5. Comparison at finite places: bad reduction

In this section, we discuss the contribution of the local height pairing at a finite place in $\mathcal{S}$. There are three cases we need to consider: the split case (that is, $\mathrm{U}(\mathbb{V}_{\mathfrak{p}})$ is split), the quasisplit case (that is, $\mathrm{U}(\mathbb{V}_{\mathfrak{p}})$ is quasisplit but not split) and the nonsplit case (that is, $\mathrm{U}(\mathbb{V}_{\mathfrak{p}})$ is not quasisplit).

5A. ***Split case.*** We first discuss the contribution of the local height pairing at a split (finite) place in $\mathcal{S}$.

We fix a prime $\mathfrak{p}\in\mathcal{S}$ which is split in E and any $\mathfrak{p}^\circ\in\Sigma_f^\circ$ over $\mathfrak{p}$. What we want to consider is the height pairing $\langle\widehat{Z}_{\phi_1}(g_1), \widehat{Z}_{\phi_2}(g_2)\rangle_{\mathfrak{p}^\circ}$ on a certain model $\mathcal{M}_{K;\mathfrak{p}^\circ}$, for $\phi_{1,\mathcal{S}}\otimes\phi_{2,\mathcal{S}}\in\mathcal{S}(\mathbb{V}_{\mathcal{S}}^2)_{\mathrm{reg}}$ and $g_i\in e_{\mathcal{S}}H'(\mathbb{A}_F^{\mathcal{S}})$. We assume that $K = K_{\mathfrak{p}}K^{\mathfrak{p}}$ with $K^{\mathfrak{p}}$ sufficiently small and $K_{\mathfrak{p}} = K_{\mathfrak{p},n}$ for $n\geq 0$, hence $M_K = M_{n,K^{\mathfrak{p}}}$. In Section 4B, we constructed a smooth integral model $\mathcal{M}_{0,K^{\mathfrak{p}}}$ for $M_{0,K^{\mathfrak{p}};\mathfrak{p}^\circ}$, a p-divisible group $\mathcal{X}\to\mathcal{M}_{0,K^{\mathfrak{p}}}$, and hence $\mathcal{X}^\flat\to\mathcal{M}_{0,K^{\mathfrak{p}}}$, which is an $\mathcal{O}_{F_{\mathfrak{p}}}$-module of dimension 1 and height 2. A Drinfeld ϖ^n-structure for an $\mathcal{O}_{F_{\mathfrak{p}}}$-module X of height 2 over an $\mathcal{O}_{F_{\mathfrak{p}}}$-scheme S is an $\mathcal{O}_{F_{\mathfrak{p}}}$-homomorphism,

$$\alpha_n : (\mathcal{O}_{F_{\mathfrak{p}}}/\varpi^n\mathcal{O}_{F_{\mathfrak{p}}})^2 \to X[\varpi^n](S),$$

such that the image forms a full set of sections of $X[\varpi^n]$ in the sense of [Katz and Mazur 1985, Section 1.8]. Let $\mathcal{M}_{n,K^{\mathfrak{p}}} = \mathcal{M}_{0,K^{\mathfrak{p}}}(n)$ be the universal scheme over $\mathcal{M}_{0,K^{\mathfrak{p}}}$ of the Drinfeld ϖ^n-structure α_n (see [Harris and Taylor 2001, Lemma II.2.1]). Then $\mathcal{M}_{n,K^{\mathfrak{p}}}$ is regular, finite over $\mathcal{M}_{0,K^{\mathfrak{p}}}$, and its generic fiber is $M_{n,K^{\mathfrak{p}};\mathfrak{p}^\circ}$. We compute the intersection number after a base change $M'_{n,K^{\mathfrak{p}};\mathfrak{p}^\circ} = M_{n,K^{\mathfrak{p}};\mathfrak{p}^\circ} \times_{F_{\mathfrak{p}}} F^n_{\mathfrak{p}}$. Then $\mathcal{M}'_{n,K^{\mathfrak{p}}}$, the normalization of $\mathcal{M}_{n,K^{\mathfrak{p}}} \times_{\mathcal{O}_{F_{\mathfrak{p}}}} \mathcal{O}_{F^n_{\mathfrak{p}}}$, is still regular and its generic fiber is $M'_{n,K^{\mathfrak{p}};\mathfrak{p}^\circ}$. We denote by $[\mathcal{M}'_{n,K^{\mathfrak{p}}}]_{\mathsf{ord}}$ the ordinary locus of the special fiber $[\mathcal{M}'_{n,K^{\mathfrak{p}}}]_{\mathsf{spe}} = \mathcal{M}'_{n,K^{\mathfrak{p}}} \times_{\mathcal{O}_{F^n_{\mathfrak{p}}}} \mathbb{F}$ which is also the smooth locus. The set of connected components of $[\mathcal{M}'_{n,K^{\mathfrak{p}}}]_{\mathsf{spe}}$ canonically corresponds to the set of geometric connected component of $M_{n,K^{\mathfrak{p}}}$, and hence to $E^{\times,1}\backslash\mathbb{A}_E^{\times,1}/\nu(K)$. The set of irreducible components on each connected component of $[\mathcal{M}'_{n,K^{\mathfrak{p}}}]_{\mathsf{spe}}$, that is, the *Igusa curves*, corresponds to the set $\boldsymbol{P}(V_{\mathfrak{p}})/K_{\mathfrak{p},n}$, where $\boldsymbol{P}(V_{\mathfrak{p}})$ is the set of all $E_{\mathfrak{p}} \cong F_{\mathfrak{p}} \oplus F_{\mathfrak{p}}$-lines in $V_{\mathfrak{p}}$ where $\mathsf{U}(V_{\mathfrak{p}})$ acts from right by $l.h = h^{-1}l$ for $l \in \boldsymbol{P}(V_{\mathfrak{p}})$ and $h \in \mathsf{U}(V_{\mathfrak{p}})$. Together, the set of irreducible components of $[\mathcal{M}'_{n,K^{\mathfrak{p}}}]_{\mathsf{spe}}$ is

$$\mathsf{Ig}_{n,K^{\mathfrak{p}}} := \boldsymbol{P}(V_{\mathfrak{p}})/K_{\mathfrak{p},n} \times (E^{\times,1}\backslash\mathbb{A}_E^{\times,1}/\nu(K)).$$

Now we consider the special cycles. We use the same notation for the base change of special cycles $Z(x)_K$ and the generating series $Z_{\phi_i}(g_i)$ on $M'_{n,K^{\mathfrak{p}};\mathfrak{p}^\circ}$. As before, we denote by $\mathcal{Z}(x)_K$ (resp. $\mathcal{Z}_{\phi_i}(g_i)$) the Zariski closure of $Z(x)_K$ (resp. $Z_{\phi_i}(g_i)$) in $\mathcal{M}'_{n,K^{\mathfrak{p}}}$. Since $\mathfrak{p}$ is split in E, the special fiber $[\mathcal{Z}_{\phi_i}(g_i)]_{\mathsf{spe}} \subset [\mathcal{M}'_{n,K^{\mathfrak{p}}}]_{\mathsf{ord}}$. Let $\boldsymbol{P}(V)$ be the set of E-lines in V. Then the set of geometric special points of $M_{n,K^{\mathfrak{p}}}$ (also of $M_{n,K^{\mathfrak{p}};\mathfrak{p}^\circ}$, $M'_{n,K^{\mathfrak{p}};\mathfrak{p}^\circ}$) is

$$\mathsf{Sp}_K := H(\mathbb{Q})\backslash\boldsymbol{P}(V) \times H(\mathbb{A}_f)/K = \coprod_{l\in H(\mathbb{Q})\backslash\boldsymbol{P}(V)} H_l(\mathbb{Q})\backslash H(\mathbb{A}_f)/K$$

and the set $[\mathcal{M}'_{n,K^{\mathfrak{p}}}]_{\mathsf{ord}}(\mathbb{F})$ is

$$\coprod_{l\in H(\mathbb{Q})\backslash\boldsymbol{P}(V)} H_l(\mathbb{Q})\backslash\big((N_l\backslash\mathsf{U}(V_{\mathfrak{p}})/K_{\mathfrak{p},n}) \times H_f^{\mathfrak{p}}/K^{\mathfrak{p}}\big),$$

where $N_l \subset \mathsf{U}(V_{\mathfrak{p}})$ is the unipotent subgroup of the parabolic subgroup fixing l. The reduction map

$$\mathsf{Sp}_K \to [\mathcal{M}'_{n,K^{\mathfrak{p}}}]_{\mathsf{ord}}(\mathbb{F}) \to \mathsf{Ig}_{n,K^{\mathfrak{p}}} \tag{5-1}$$

is given by $(l,h) \mapsto (l, h_{\mathfrak{p}}, h^{\mathfrak{p}}) \mapsto (h_{\mathfrak{p}}^{-1}l, \nu(h_{\mathfrak{p}}h^{\mathfrak{p}}))$ (see [Zhang 2001b, Section 5.4] for a discussion).

We compute the local height pairing on the model $\mathcal{M}'_{n,K^{\mathfrak{p}}}$. We write $\widehat{Z}_{\phi_i}(g_i) = \mathcal{Z}_{\phi_i}(g_i) + \mathcal{V}_{\phi_i}(g_i)$ for some cycle $\mathcal{V}_{\phi_i}(g_i)$ supported on the special fiber as in

Section 3B. Let $\omega_{\mathfrak{p}^\circ}$ be the base change of ω_K to $\mathcal{M}'_{n,K^{\mathfrak{p}}}$. We have

$$
\begin{aligned}
&(\log q)^{-1}\langle \widehat{Z}_{\phi_1}(g_1), \widehat{Z}_{\phi_2}(g_2)\rangle_{\mathfrak{p}^\circ} \\
&\quad = (\mathcal{X}_{\phi_1}(g_1)+\mathcal{V}_{\phi_1}(g_1))\cdot(\mathcal{X}_{\phi_2}(g_2)+\mathcal{V}_{\phi_2}(g_2)-E(g_2,\phi_2)\omega_{\mathfrak{p}^\circ}+E(g_2,\phi_2)\omega_{\mathfrak{p}^\circ}) \\
&\quad = \mathcal{X}_{\phi_1}(g_1)\cdot(\mathcal{X}_{\phi_2}(g_2)+\mathcal{V}_{\phi_2}(g_2)-E(g_2,\phi_2)\omega_{\mathfrak{p}^\circ}) \\
&\qquad\qquad\qquad\qquad + E(g_2,\phi_2)(\mathcal{X}_{\phi_1}(g_1)+\mathcal{V}_{\phi_1}(g_1))\cdot\omega_{\mathfrak{p}^\circ} \\
&\quad = \mathcal{X}_{\phi_1}(g_1)\cdot\mathcal{X}_{\phi_2}(g_2)+\mathcal{X}_{\phi_1}(g_1)\cdot\mathcal{V}_{\phi_2}(g_2)+E(g_2,\phi_2)\mathcal{V}_{\phi_1}(g_1)\cdot\omega_{\mathfrak{p}^\circ}, \qquad (5\text{-}2)
\end{aligned}
$$

where q is the cardinality of the residue field of $F_{\mathfrak{p}}$.

Now we discuss the above three height pairings respectively. First, we have

Lemma 5.1. *Under the weaker hypotheses that $\phi_{1,v}\otimes\phi_{2,v}\in\mathcal{S}(\mathbb{V}_v^2)_{\mathrm{reg}}$ and $g_i\in P'_vH'(\mathbb{A}_F^v)$ for some finite place v other than $\mathfrak{p}$, $\mathcal{X}_{\phi_1}(g_1)$ and $\mathcal{X}_{\phi_2}(g_2)$ do not intersect.*

Proof. This is clear from the first arrow of the reduction map (5-1). □

Second, we define a function $\nu(\,\cdot\,;\phi_2,g_2)$ on $\mathbb{V}_{\mathfrak{p}}-\{0\}$ in the following way. For any $x\in\mathbb{V}_{\mathfrak{p}}-\{0\}$, write $\bar{x}$ for the line in $\boldsymbol{P}(\mathbb{V}_{\mathfrak{p}})$ containing x. Then $\nu(x;\phi_2,g_2)$ is the coefficient of the geometric irreducible component represented by $(\bar{x},1)$ in $\mathrm{lg}_{n,K^{\mathfrak{p}}}$ appearing in $\mathcal{V}_{\phi_2}(g_2)$. It is a locally constant function and

$$\nu(\,\cdot\,,\phi_{1,\mathfrak{p}};\phi_2,g_2)=\frac{\mathrm{vol}(\det K)}{\mathrm{vol}(K)}\phi_{1,\mathfrak{p}}\otimes\nu(\,\cdot\,;\phi_2,g_2)$$

extends to a function in $\mathcal{S}(\mathbb{V}_{\mathfrak{p}})$ such that $\nu(0,\phi_{1,\mathfrak{p}};\phi_2,g_2)=0$ since $\phi_{1,\mathfrak{p}}(0)=0$. Then the intersection number

$$
\begin{aligned}
\mathcal{X}_{\phi_1}(g_1)\cdot\mathcal{V}_{\phi_2}(g_2) &= \sum_{x\in K\backslash\mathbb{V}_f}\omega_\chi(g_1)\phi_1(x)\mathcal{X}(x)_K\cdot\mathcal{V}_{\phi_2}(g_2) \\
&= \sum_{\substack{x\in K\backslash\mathbb{V}_f\\ T(x)\in F^+}}\frac{\mathrm{vol}(K)}{\mathrm{vol}(K\cap\mathbb{H}(\mathbb{A}_f)_x)}\nu(\,\cdot\,,\phi_{1,\mathfrak{p}};\phi_2,g_2)\otimes(\omega_\chi(g_1^{\mathfrak{p}})\phi_1^{\mathfrak{p}})(x) \qquad (5\text{-}3)
\end{aligned}
$$

since $g_1\in e_{\mathfrak{p}}H'(\mathbb{A}^{\mathfrak{p}})$.

On the other hand, we let

$$
\begin{aligned}
&E(s,g,\nu(\,\cdot\,,\phi_{1,\mathfrak{p}};\phi_2,g_2)\otimes\phi_1^{\mathfrak{p}}) \\
&\qquad\qquad = \sum_{\gamma\in P'(F)\backslash H'(F)}\omega_\chi(\gamma g)(\nu(\,\cdot\,,\phi_{1,\mathfrak{p}};\phi_2,g_2)\otimes\phi_1^{\mathfrak{p}})(0)\lambda_{P'}(\gamma g)^{s-\frac{1}{2}}
\end{aligned}
$$

be an Eisenstein series which is holomorphic at $s=\frac{1}{2}$. Then we have

$$(5\text{-}3)=E(s,g,\nu(\,\cdot\,,\phi_{1,\mathfrak{p}};\phi_2,g_2)\otimes\phi_1^{\mathfrak{p}})\big|_{s=\frac{1}{2}}-W_0\big(\tfrac{1}{2},g,\nu(\,\cdot\,,\phi_{1,\mathfrak{p}};\phi_2,g_2)\otimes\phi_1^{\mathfrak{p}}\big)$$

by the standard Siegel–Weil argument and the argument in Proposition 3.2. For simplicity, we let

$$\begin{aligned}&E_{(\mathfrak{p}^\circ)}(g,\phi_1;\phi_2,g_2)\\&=\log q\big[E(s,g,\nu(\,\cdot\,,\phi_{1,\mathfrak{p}};\phi_2,g_2)\otimes\phi_1^{\mathfrak{p}})\big|_{s=\frac12}-W_0\big(\tfrac12,g,\nu(\,\cdot\,,\phi_{1,\mathfrak{p}};\phi_2,g_2)\otimes\phi_1^{\mathfrak{p}}\big)\big].\end{aligned}$$

Finally, we let

$$A_{(\mathfrak{p}^\circ)}(g_1,\phi_1)=\log q\,\mathscr{V}_{\phi_1}(g_1)\cdot\omega_{\mathfrak{p}^\circ}.$$

Then in summary, we have

Proposition 5.2. *For $\phi_{1,\mathcal{S}}\otimes\phi_{2,\mathcal{S}}\in\mathscr{S}(\mathbb{V}_{\mathcal{S}}^2)_{\mathsf{reg}}$ and $g_i\in e_{\mathcal{S}}H'(\mathbb{A}_F^{\mathcal{S}})$ $(i=1,2)$,*

$$\langle\widehat{Z}_{\phi_1}(g_1),\widehat{Z}_{\phi_2}(g_2)\rangle_{\mathfrak{p}^\circ}=E_{(\mathfrak{p}^\circ)}(g_1,\phi_1;g_2,\phi_2)+A_{(\mathfrak{p}^\circ)}(g_1,\phi_1)E(g_2,\phi_2).$$

5B. ***Quasisplit case.*** In this section, we discuss the contribution of the local height pairing at a nonsplit (finite) place $\mathfrak{p}$ in $\mathcal{S}$ such that $\epsilon(\mathbb{V}_{\mathfrak{p}})=1$.

We fix such a $\mathfrak{p}$ and denote by $\mathfrak{p}^\circ$ the unique place of E over $\mathfrak{p}$ as usual. As before, we need to consider the height pairing $\langle\widehat{Z}_{\phi_1}(g_1),\widehat{Z}_{\phi_2}(g_2)\rangle_{\mathfrak{p}^\circ}$ on a certain model $\mathcal{M}_{K;\mathfrak{p}^\circ}$, for $\phi_{1,\mathfrak{p}}\otimes\phi_{2,\mathfrak{p}}\in\mathscr{S}(\mathbb{V}_{\mathfrak{p}}^2)_{\mathsf{reg}}$ and $g_i\in e_{\mathfrak{p}}H'(\mathbb{A}_F^{\mathfrak{p}})$. We assume that $K=K_{\mathfrak{p}}K^{\mathfrak{p}}$ with $K^{\mathfrak{p}}$ sufficiently small and $K_{\mathfrak{p}}=K_{\mathfrak{p},n}$ for $n\ge 0$. In Section 4B, we have fixed an isometry between $\mathbb{V}_{\mathfrak{p}}$ and $\mathsf{Mat}_2(F_{\mathfrak{p}})$ sending $\Lambda_{\mathfrak{p}}$ to $\mathsf{Mat}_2(\mathbb{O}_{F_{\mathfrak{p}}})$. Hence $\mathbb{V}_{\mathfrak{p}}$ has an action by $\mathsf{GL}_2(F_{\mathfrak{p}})$ by both left and right multiplication where the latter is $E_{\mathfrak{p}^\circ}$-linear. We write $\mathbb{V}_{\mathfrak{p}}^\flat$ with respect to the left multiplication and $\mathsf{GL}_2(F_{\mathfrak{p}})$ acts on $\mathscr{S}(\mathbb{V}_{\mathfrak{p}})$ via $(g.\phi)(x)=\phi(xg)$. By enlarging n (to be an least 1), we assume that $\phi_{i,\mathfrak{p}}$ is not only invariant under $K_{\mathfrak{p},n}$ but also $\mathbf{1}_2+\varpi^n\mathsf{GL}_2(\mathbb{O}_{F_{\mathfrak{p}}})$.

We let $\mathcal{M}_{n,K^{\mathfrak{p}}}$ be the normalization of $\mathcal{M}_{0,K^{\mathfrak{p}}}$ in $M_{n,K^{\mathfrak{p}};\mathfrak{p}^\circ}$ which is regular and finite over $\mathcal{M}_{0,K^{\mathfrak{p}}}$. We take a base change $M'_{n,K^{\mathfrak{p}};\mathfrak{p}^\circ}=M_{n,K^{\mathfrak{p}};\mathfrak{p}^\circ}\times_{E_{\mathfrak{p}^\circ}}E_{\mathfrak{p}^\circ}^{(n)}$, where $E_{\mathfrak{p}^\circ}^{(n)}=E_{\mathfrak{p}^\circ}F_{\mathfrak{p}}^n$ and $\hat{E}_{\mathfrak{p}^\circ}^{(n)}=E_{\mathfrak{p}^\circ}\hat{F}_{\mathfrak{p}}^n$. Let $\mathcal{M}'_{n,K^{\mathfrak{p}}}$ be the normalization of $\mathcal{M}_{n,K^{\mathfrak{p}}}\times_{\mathbb{O}_{E_{\mathfrak{p}^\circ}}}\mathbb{O}_{E_{\mathfrak{p}^\circ}^{(n)}}$, which in turn is a regular model of $M'_{n,K^{\mathfrak{p}};\mathfrak{p}^\circ}$. Then the set of supersingular points is

$$S_n=[\mathcal{M}'_{n,K^{\mathfrak{p}}}]_{\mathsf{s.s.}}(\mathbb{F})\cong\breve{H}(\mathbb{Q})\backslash(E_{\mathfrak{p}^\circ}^{\times,1}/\nu(K_{\mathfrak{p},n})\times\breve{H}_f^{\mathfrak{p}}/K^{\mathfrak{p}}),$$

where $\breve{H}(\mathbb{Q})$ acts on the first factor by multiplying the determinant. For any point $s\in S_n$, the completion $[\mathcal{M}'_{n,K^{\mathfrak{p}}}]_s^\wedge$ at the point s is isomorphic to a formal scheme $\mathcal{N}'_n$ over $\mathsf{Spf}\ \mathbb{O}_{\hat{E}_{\mathfrak{p}^\circ}^{(n)}}$. It can be constructed in the following way. We have a p-divisible group $\mathscr{X}^{\mathsf{univ}}\to\mathcal{N}\cong\mathsf{Spf}\ R_{F_{\mathfrak{p}},2}$. Let $R_{F_{\mathfrak{p}},2,n}$ (see [Harris and Taylor 2001, Lemma II.2.2]) be such that $\mathsf{Spec}\ R_{F_{\mathfrak{p}},2,n}=(\mathsf{Spec}\ R_{F_{\mathfrak{p}},2})(n)$ is the universal scheme of the Drinfeld ϖ^n-structure for $\mathscr{X}^{\mathsf{univ},\flat}$ which is even defined over $\mathsf{Spec}\ R_{F_{\mathfrak{p}},2}$. Let $\mathsf{Spec}\ R'_n$ be the normalization of $\mathsf{Spec}\ R_{F_{\mathfrak{p}},2}\otimes_{\mathbb{O}_{\hat{F}_{\mathfrak{p}}}}\mathbb{O}_{\hat{E}_{\mathfrak{p}^\circ}^{(n)}}$ in any connected component of $\mathsf{Spec}\ R_{F_{\mathfrak{p}},2,n}\otimes_{\mathbb{O}_{\hat{F}_{\mathfrak{p}}}}\hat{E}_{\mathfrak{p}^\circ}^{(n)}$. Then $\mathcal{N}'_n\cong\mathsf{Spf}\ R'_n$ and is finite over $\mathcal{N}'\times_{\mathbb{O}_{\hat{E}_{\mathfrak{p}^\circ}^{\mathsf{nr}}}}\mathbb{O}_{\hat{E}_{\mathfrak{p}^\circ}^{(n)}}$. The generic fiber $\mathcal{N}'_{n,\eta}:=\mathsf{Spec}\ R'_n\otimes\hat{E}_{\mathfrak{p}^\circ}^{(n)}$ is Galois over $\mathcal{N}'_\eta:=\mathsf{Spec}\ R_{F_{\mathfrak{p}},2}\otimes\hat{E}_{\mathfrak{p}^\circ}^{(n)}$

with Galois group $\mathsf{SL}_2(\mathcal{O}_{F_\mathfrak{p}}/\varpi^n\mathcal{O}_{F_\mathfrak{p}})$. Moreover, it inherits a universal p-divisible $\mathscr{X}' \to \mathsf{Spec}\ R'_n$, a universal Drinfeld ϖ^n-structure

$$\alpha'_{n,\eta} : \Lambda_\mathfrak{p}^\flat/\varpi^n\Lambda_\mathfrak{p}^\flat \to \mathscr{X}'^\flat_\eta[\varpi^n](\mathcal{N}'_{n,\eta})$$

for $\mathscr{X}'^\flat_\eta$. In particular, we have the uniformization

$$[\mathcal{M}'_{n,K^\mathfrak{p}}]^\wedge_{\mathsf{s.s.}} \cong \breve{H}(\mathbb{Q})\backslash(\mathcal{N}'_n \times E_{\mathfrak{p}^\circ}^{\times,1}/\nu(K_{\mathfrak{p},n}) \times \breve{H}_f^\mathfrak{p}/K^\mathfrak{p}).$$

In Section 4B, we construct the (irreducible) integral special subscheme $\mathfrak{Z}(x, h')$ for $x \in V \cap \Lambda_\mathfrak{p} - \{0\}$, $h' \in H_f^\mathfrak{p}/K^\mathfrak{p}$. We still write $\mathfrak{Z}(x, h')$ for its base change under the map $\mathcal{M}'_{n,K^\mathfrak{p}} \to \mathcal{M}_{0,K^\mathfrak{p}} \times_{\mathcal{O}_{E_{\mathfrak{p}^\circ}}} \mathcal{O}_{E^{(n)}_{\mathfrak{p}^\circ}}$. But now it is not irreducible anymore. We write $\mathfrak{Z}(x, h)$ with $h \in K_{\mathfrak{p},0}/K_{\mathfrak{p},n} \times H_f^\mathfrak{p}/K^\mathfrak{p}$ and $h^\mathfrak{p} = h'$, for all its irreducible components such that its complex geometric fiber (point) is represented by (z, h) with $z \perp x$. Each $\mathfrak{Z}(x, h)$ is defined over $\mathcal{O}_{E^{(n)}_{\mathfrak{p}^\circ}}$, is geometrically irreducible, and $[\mathfrak{Z}(x, h)]_{\mathsf{spe}} \subset [\mathcal{M}'_{n,K^\mathfrak{p}}]_{\mathsf{s.s.}}$.

We have a p-divisible group $\mathscr{X}'|_{\mathfrak{Z}(\breve{x})} \to \mathfrak{Z}(\breve{x})$ where we use the same notation for the pull-back of $\mathfrak{Z}(\breve{x})$ from $\mathsf{Spf}\ R_{F_\mathfrak{p},2}$ to $\mathcal{N}'_n$. Consider $\mathscr{X}'_\eta|_{\mathfrak{Z}(\breve{x})_\eta} \to \mathfrak{Z}(\breve{x})_\eta$ and

$$\alpha'_{n,\eta} : \Lambda_\mathfrak{p}^\flat/\varpi^n\Lambda_\mathfrak{p}^\flat \xrightarrow{\sim} \mathscr{X}'^\flat_\eta[\varpi^n](\mathfrak{Z}(\breve{x})_{\eta,0}),$$

where $\mathfrak{Z}(\breve{x})_{\eta,0}$ is some connected component of $\mathfrak{Z}(\breve{x})_\eta$. By the definition of $\mathfrak{Z}(\breve{x})$, the element $\breve{x} \in \mathsf{Hom}((\boldsymbol{Y}, j), (\boldsymbol{X}, i_{\boldsymbol{X}}))$ canonically induces a homomorphism $\varrho_{\breve{x}} : \mathscr{Y} \times_{\mathcal{O}_{\hat{E}^{\mathsf{nr}}_{\mathfrak{p}^\circ}}} \mathfrak{Z}(\breve{x}) \to \mathscr{X}'|_{\mathfrak{Z}(\breve{x})}$, hence $\varrho_{\breve{x}} : \mathscr{Y}_\eta \times_{\hat{E}^{\mathsf{nr}}_{\mathfrak{p}^\circ}} \mathfrak{Z}(\breve{x})_\eta \to \mathscr{X}'_\eta|_{\mathfrak{Z}(\breve{x})_\eta}$. In particular, we have an element

$$\varrho_{\breve{x},*}(\boldsymbol{x}_\mathfrak{p}) \in T_\mathfrak{p}\big(\mathscr{X}'_\eta|_{\mathfrak{Z}(\breve{x})_\eta}\big) = \varprojlim_n \mathscr{X}'_\eta[\varpi^n](\mathfrak{Z}(\breve{x})_\eta) = \big(\varprojlim_n \mathscr{X}'^\flat_\eta[\varpi^n](\mathfrak{Z}(\breve{x})_\eta)\big)^\sharp.$$

For each connected component $\mathfrak{Z}(\breve{x})_{\eta,0}$, $\alpha'_{n,\eta}$ extends to a $(\mathbf{1}_2 + \varpi^n\mathsf{GL}_2(\mathcal{O}_{F_\mathfrak{p}}))$-class of isomorphisms $\bar{\eta}_\mathfrak{p} : \Lambda_\mathfrak{p} \xrightarrow{\sim} T_\mathfrak{p}(\mathscr{X}'_{\eta,0})$, where $\mathscr{X}'_{\eta,0}$ is the restriction of $\mathscr{X}'_\eta$ to $\mathfrak{Z}(\breve{x})_{\eta,0}$. Let $x = \bar{\eta}_\mathfrak{p}^{-1}(\varrho_{\breve{x},*}(\boldsymbol{x}_\mathfrak{p}))$, which is well-defined in $\Lambda_\mathfrak{p}/(\mathbf{1}_2 + \varpi^n\mathsf{GL}_2(\mathcal{O}_{F_\mathfrak{p}}))$. By construction, we have the following property:

$$(x, x)/(\breve{x}, \breve{x})' \in 1 + \varpi^n\mathcal{O}_{F_\mathfrak{p}} \text{ and } x \in \varpi^m\Lambda_\mathfrak{p} \Leftrightarrow \breve{x} \in \varpi^m\breve{\Lambda}_\mathfrak{p} \text{ for all } m \geq 0. \tag{5-4}$$

We denote by $\mathfrak{Z}(\breve{x}, x)$ the union of all irreducible components of $\mathfrak{Z}(\breve{x})$ containing $\mathfrak{Z}(\breve{x})_{\eta,0}$ whose $\bar{\eta}_\mathfrak{p}^{-1}(\varrho_{\breve{x},*}(\boldsymbol{x}_\mathfrak{p})) = x$. It is nonempty only when (5-4) is satisfied. Hence for a fixed $\breve{x}$, the number of x such that $\mathfrak{Z}(\breve{x}, x)$ is nonempty is at most $|\mathsf{SL}_2(\mathcal{O}_{F_\mathfrak{p}}/\varpi^n\mathcal{O}_{F_\mathfrak{p}})|$. Now for any $\breve{h} \in E_{\mathfrak{p}^\circ}^{\times,1}/\nu(K_{\mathfrak{p},n}) \times \breve{H}_f^\mathfrak{p}/K^\mathfrak{p}$, we let $\mathfrak{Z}(\breve{x}, x, \breve{h})$ be the cycle of $[\mathcal{M}'_{n,K^\mathfrak{p}}]^\wedge_{\mathsf{s.s.}}$ represented by $(\mathfrak{Z}(\breve{x}, x), \breve{h})$. Then we have the following identity between sets:

$$\begin{aligned}&\big\{\mathfrak{Z}(\breve{x}, x, \breve{h}) \,\big|\, \breve{x} \in \breve{H}(\mathbb{Q})\backslash(\breve{\Lambda}_\mathfrak{p} - \{0\}), \breve{h} \in \breve{H}_{\breve{x}}(\mathbb{Q})\backslash(E_{\mathfrak{p}^\circ}^{\times,1}/\nu(K_{\mathfrak{p},n}) \times \breve{H}_f^\mathfrak{p}/K^\mathfrak{p})\big\}\\ &= \big\{[\mathfrak{Z}(x, h)]^\wedge_{\mathsf{spe}} \,\big|\, x \in H(\mathbb{Q})\backslash(V \cap \Lambda_\mathfrak{p} - \{0\}), h \in H_x(\mathbb{Q})\backslash(K_{\mathfrak{p},0}/K_{\mathfrak{p},n} \times H_f^\mathfrak{p}/K^\mathfrak{p})\big\}.\end{aligned} \tag{5-5}$$

Now we can consider the height pairing. Pick an element $e \in E^\times$ such that $-\mathrm{val}_{\mathfrak{p}}(e)$ is sufficiently large. Then

$$\begin{aligned} Z_{\phi_i}(g_i) &= \sum_{x_i \in K\backslash \mathbb{V}_f} \omega_\chi(g_i)\phi_i(x_i)Z(x_i)_K = \sum_{x_i \in K\backslash \mathbb{V}_f} \omega_\chi(g_i)\phi_i(x_i e)Z(x_i e)_K \\ &= \sum_{x_i \in K\backslash \mathbb{V}_f} \omega_\chi(g_i)\phi_i(x_i e)Z(x_i)_K. \end{aligned}$$

Hence we can assume that $\phi_{i,\mathfrak{p}}$ is supported on $\Lambda_{\mathfrak{p}}$. Recall that we assume $\phi_{1,\mathfrak{p}} \otimes \phi_{2,\mathfrak{p}} \in \mathcal{S}(\mathbb{V}_{\mathfrak{p}}^2)_{\mathrm{reg}}^{K_{\mathfrak{p},n}}$, $g_i \in e_{\mathfrak{p}} H'(\mathbb{A}_F^{\mathfrak{p}})$, and $\phi_{i,\mathfrak{p}}$ is also invariant under $\mathbf{1}_2 + \varpi^n \mathrm{GL}_2(\mathbb{O}_{F_{\mathfrak{p}}})$. We define

$$\mathfrak{X}_{\phi_i}(g_i) = \sum_{x_i \in H(\mathbb{Q})\backslash V} \sum_{h_i \in H_{x_i}(\mathbb{Q})\backslash H(\mathbb{A}_f)/K} \phi_{\mathfrak{p}} \otimes (\omega_\chi(g_i)\phi_i^{\mathfrak{p}})(h_i^{-1}x_i)\mathfrak{X}(x_i, h_i)$$

which is a cycle of $\mathcal{M}'_{n,K^{\mathfrak{p}}}$ extending the generating series $Z_{\phi_i}(g_i)$ on $M'_{n,K^{\mathfrak{p}};\mathfrak{p}^\circ}$. The special fiber $[\mathfrak{X}_{\phi_i}(g_i)]_{\mathrm{spe}} \subset [\mathcal{M}'_{n,K^{\mathfrak{p}}}]_{\mathrm{s.s.}}$ and

$$\begin{aligned} [\mathfrak{X}_{\phi_i}(g_i)]^\wedge_{\mathrm{spe}} = \sum_{\breve{x}_i \in \breve{V} \cap \breve{\Lambda}_{\mathfrak{p}} - \{0\}, x_i} \sum_{\breve{h}_i \in \breve{H}(\mathbb{Q})\backslash (E_{\mathfrak{p}^\circ}^{\times;1}/\nu(K_{\mathfrak{p},n}) \times \breve{H}_f^{\mathfrak{p}}/K^{\mathfrak{p}})} \phi_{i,\mathfrak{p}}(x_i) \\ \times (\omega_\chi(g_i)\phi_i^{\mathfrak{p}})(\breve{h}_i^{-1}\breve{x}_i)\mathfrak{X}(\breve{x}_i, x_i, \breve{h}_i). \end{aligned}$$

We have a similar decomposition as in (5-2) but the first term is not zero anymore. First, we consider

$$\begin{aligned} &\mathfrak{X}_{\phi_1}(g_1) \cdot \mathfrak{X}_{\phi_2}(g_2) = [\mathfrak{X}_{\phi_1}(g_1)]^\wedge_{\mathrm{spe}} \cdot [\mathfrak{X}_{\phi_2}(g_2)]^\wedge_{\mathrm{spe}} \\ &\quad = \sum_{\breve{x}_1, \breve{x}_2 \in \breve{V} \cap \breve{\Lambda}_{\mathfrak{p}} - \{0\}} \sum_{x_1, x_2} \sum_{\breve{h} \in \breve{H}(\mathbb{Q})\backslash (E_{\mathfrak{p}^\circ}^{\times;1}/\nu(K_{\mathfrak{p},n}) \times \breve{H}_f^{\mathfrak{p}}/K^{\mathfrak{p}})} \phi_{1,\mathfrak{p}}(x_1)\phi_{2,\mathfrak{p}}(x_2) \\ &\qquad \big(\omega''_\chi(\iota(g_1, g_2^\vee))\phi_1^{\mathfrak{p}} \otimes \phi_2^{\mathfrak{p}}(\breve{h}^{-1}(\breve{x}_1, \breve{x}_2))\big)\mathfrak{X}(\breve{x}_1, x_1) \cdot \mathfrak{X}(\breve{x}_2, x_2). \end{aligned} \tag{5-6}$$

Now the key point is to analyze the last intersection number. We have:

Lemma 5.3. *We can extend*

$$\mu(\breve{x}_1, \breve{x}_2; \phi_{1,\mathfrak{p}}, \phi_{2,\mathfrak{p}}) := \sum_{x_1, x_2 \in \Lambda_{\mathfrak{p}}/(\mathbf{1}_2 + \varpi^n \mathrm{GL}_2(\mathbb{O}_{F_{\mathfrak{p}}}))} \phi_{1,\mathfrak{p}}(x_1)\phi_{2,\mathfrak{p}}(x_2)\mathfrak{X}(\breve{x}_1, x_1) \cdot \mathfrak{X}(\breve{x}_2, x_2)$$

to a function in $\mathcal{S}(\breve{V}_{\mathfrak{p}}^2)$.

Proof. First we note that $\mathfrak{X}(\breve{x}_1, x_1)$ and $\mathfrak{X}(\breve{x}_2, x_2)$ intersect properly unless $\breve{x}_1$ and $\breve{x}_2$ are $E_{\mathfrak{p}^\circ}$-colinear and x_1 and x_2 are also $E_{\mathfrak{p}^\circ}$-colinear. But in this case, $\phi_{1,\mathfrak{p}}(x_1)\phi_{2,\mathfrak{p}}(x_2) = 0$ by our regularity assumption. For $(\breve{x}_1, \breve{x}_2) \notin (\breve{\Lambda}_{\mathfrak{p}} - \{0\})^2$, we let $\mu(\breve{x}_1, \breve{x}_2; \phi_{1,\mathfrak{p}}, \phi_{2,\mathfrak{p}}) = 0$. Hence $\mu(\breve{x}_1, \breve{x}_2; \phi_{1,\mathfrak{p}}, \phi_{2,\mathfrak{p}})$ is now a function on $\breve{V}_{\mathfrak{p}}^2$ which is compactly supported. We only need to prove that it is locally constant. We have several cases.

- If one $\breve{x}_i$, say $\breve{x}_1$, is not in $\breve{\Lambda}_{\mathfrak{p}}$, then μ is locally zero at $(\breve{x}_1, \breve{x}_2)$.
- If, say, $\breve{x}_1 = 0$, then since $\phi_{1,\mathfrak{p}}$ vanishes on a neighborhood of 0 and (5-4), μ is also locally zero.
- If both $\breve{x}_i$ are in $\Lambda_{\mathfrak{p}} - \{0\}$, but are not $E_{\mathfrak{p}^\circ}$-colinear, choose a neighborhood U such that any $(x_1', x_2') \in U$ is still not $E_{\mathfrak{p}^\circ}$-colinear; $(x_i', x_i')'/(\breve{x}_i, \breve{x}_i)' \in 1 + \varpi^n \mathcal{O}_{F_{\mathfrak{p}}}$; (x_1', x_2') and $(\breve{x}_1, \breve{x}_2)$ span the same $\mathcal{O}_{E_{\mathfrak{p}^\circ}}$-sublattice in $\breve{\Lambda}_{\mathfrak{p}}$. Then μ is locally constant on U.
- If both $\breve{x}_i$ are in $\Lambda_{\mathfrak{p}} - \{0\}$ and $E_{\mathfrak{p}^\circ}$-colinear, we choose U as above. Then for x_1, x_2 not $E_{\mathfrak{p}^\circ}$-colinear, $\mathfrak{X}(\breve{x}_1, x_1) \cdot \mathfrak{X}(\breve{x}_2, x_2)$ is locally constant on U, and hence μ is also. □

By the lemma,

$$(5\text{-}6) = \sum_{\vec{x} \in \breve{V}^2} \sum_{\breve{h} \in \breve{H}(\mathbb{Q}) \backslash (E_{\mathfrak{p}^\circ}^{\times,1}/\nu(K_{\mathfrak{p},n}) \times \breve{H}_f^{\mathfrak{p}}/K^{\mathfrak{p}})} \big(\omega_\chi''(\iota(g_1, g_2^\vee))\phi_1^{\mathfrak{p}} \otimes \phi_2^{\mathfrak{p}}(\breve{h}^{-1}\vec{x})\big)\mu(\vec{x}; \phi_{1,\mathfrak{p}}, \phi_{2,\mathfrak{p}}).$$

Since the set $\breve{H}(\mathbb{Q}) \backslash (E_{\mathfrak{p}^\circ}^{\times,1}/\nu(K_{\mathfrak{p},n}) \times \breve{H}_f^{\mathfrak{p}}/K^{\mathfrak{p}})$ is finite, we let

$$\theta_{(\mathfrak{p}^\circ)}^{\mathsf{hor}}(\,\cdot\,; \phi_1, \phi_2) = \log q \sum_{\vec{x} \in \breve{V}^2} \omega_\chi''(\,\cdot\,)\Phi^{\mathsf{hor}}(\vec{x})$$

be the theta series for the Schwartz function

$$\Phi^{\mathsf{hor}} = \sum_{\breve{h} \in \breve{H}(\mathbb{Q}) \backslash (E_{\mathfrak{p}^\circ}^{\times,1}/\nu(K_{\mathfrak{p},n}) \times \breve{H}_f^{\mathfrak{p}}/K^{\mathfrak{p}})} \mu(\,\cdot\,, \cdot\,; \phi_{1,\mathfrak{p}}, \phi_{2,\mathfrak{p}}) \otimes (\omega_\chi''(\breve{h})\phi_1^{\mathfrak{p}} \otimes \phi_2^{\mathfrak{p}}).$$

Then we have

Lemma 5.4. *For $g_i \in e_{\mathfrak{p}} H'(\mathbb{A}_F^{\mathfrak{p}})$,*

$$\log q\, \mathfrak{Z}_{\phi_1}(g_1) \cdot \mathfrak{Z}_{\phi_2}(g_2) = \theta_{(\mathfrak{p}^\circ)}^{\mathsf{hor}}(\iota(g_1, g_2^\vee); \phi_1, \phi_2),$$

where q is the cardinality of the residue field of $E_{\mathfrak{p}^\circ}$.

Now we consider the second term, $\mathfrak{Z}_{\phi_1}(g_1) \cdot \mathcal{V}_{\phi_2}(g_2)$. For any

$$\breve{h}_1 \in \breve{H}(\mathbb{Q}) \backslash (E_{\mathfrak{p}^\circ}^{\times,1}/\nu(K_{\mathfrak{p},n}) \times \breve{H}_f^{\mathfrak{p}}/K^{\mathfrak{p}}),$$

we write $s(\breve{h}_1)$ for the corresponding supersingular point in S_n. Then

$$\begin{aligned}\mathfrak{Z}_{\phi_1}(g_1) \cdot \mathcal{V}_{\phi_2}(g_2) &= [\mathfrak{Z}_{\phi_1}(g_1)]^\wedge_{\mathsf{spe}} \cdot [\mathcal{V}_{\phi_2}(g_2)]^\wedge_{\mathsf{s.s.}} \\ &= \sum_{\breve{x}_1 \in \breve{V} \cap \breve{\Lambda}_{\mathfrak{p}} - \{0\}, x_1} \sum_{\breve{h}_1 \in \breve{H}(\mathbb{Q}) \backslash (E_{\mathfrak{p}^\circ}^{\times,1}/\nu(K_{\mathfrak{p},n}) \times \breve{H}_f^{\mathfrak{p}}/K^{\mathfrak{p}})} \phi_{1,\mathfrak{p}}(x_1)(\omega_\chi(g_1)\phi_1^{\mathfrak{p}})(\breve{h}_1^{-1}\breve{x}_1) \\ &\qquad \mathfrak{X}(\breve{x}_1, x_1, \breve{h}_1) \cdot [\mathcal{V}_{\phi_2}(g_2)]^\wedge_{s(\breve{h}_1)}.\end{aligned} \tag{5-7}$$

Lemma 5.5. *For any $\breve{h}_1$, we can extend*

$$\nu(\breve{x}_1,\phi_{1,\mathfrak{p}},\breve{h}_1;\phi_2,g_2):=\sum_{x_1\in\Lambda_{\mathfrak{p}}/(\mathbf{1}_2+\varpi^n\mathrm{GL}_2(\mathbb{O}_{F_{\mathfrak{p}}}))}\phi_{1,\mathfrak{p}}(x_1)\mathfrak{X}(\breve{x}_1,x_1,\breve{h}_1)\cdot[\mathcal{V}_{\phi_2}(g_2)]^{\wedge}_{s(\breve{h}_1)}$$

to a function in $\mathcal{S}(\breve{V}_{\mathfrak{p}})$.

Proof. The proof is similar to that of Lemma 5.3. In fact, for $\breve{x}_1\in\breve{\Lambda}_{\mathfrak{p}}-\{0\}$, let U be a neighborhood such that for any $x_1'\in U$, $(x_i',x_i')'/(\breve{x}_i,\breve{x}_i)'\in 1+\varpi^n\mathbb{O}_{F_{\mathfrak{p}}}$. Then $\mathfrak{X}(\,\cdot\,,x_1,\breve{h}_1)\cdot[\mathcal{V}_{\phi_2}(g_2)]^{\wedge}_{s(\breve{h}_1)}$ is locally constant on U. □

The lemma implies that

(5-7) =
$$\sum_{\breve{x}_1\in\breve{V}}\ \sum_{\breve{h}_1\in\breve{H}(\mathbb{Q})\backslash(E^{\times,1}_{\mathfrak{p}^\circ}/\nu(K_{\mathfrak{p},n})\times\breve{H}^{\mathfrak{p}}_f/K^{\mathfrak{p}})}\phi_{1,\mathfrak{p}}(x_1)(\omega_\chi(g_1)\phi_1^{\mathfrak{p}})(\breve{h}_1^{-1}\breve{x}_1)\nu(\breve{x}_1,\phi_{1,\mathfrak{p}},\breve{h}_1;\phi_2,g_2).$$

Let $\theta^{\mathrm{ver}}_{(\mathfrak{p}^\circ)}(\,\cdot\,,\phi_1;g_2,\phi_2)=\sum_{\breve{x}_1\in\breve{V}}\omega_\chi(\,\cdot\,)\phi^{\mathrm{ver}}(\breve{x}_1)$ be the theta series for the Schwartz function

$$\phi^{\mathrm{ver}}=\sum_{\breve{h}_1\in\breve{H}(\mathbb{Q})\backslash(E^{\times,1}_{\mathfrak{p}^\circ}/\nu(K_{\mathfrak{p},n})\times\breve{H}^{\mathfrak{p}}_f/K^{\mathfrak{p}})}\nu(\,\cdot\,,\phi_{1,\mathfrak{p}},\breve{h}_1;\phi_2,g_2)\otimes\omega_\chi(\breve{h}_1)\phi_1^{\mathfrak{p}}.$$

Then we have:

Lemma 5.6. *For $g_i\in e_{\mathfrak{p}}H'(\mathbb{A}_F^{\mathfrak{p}})$,*

$$\log q\,\mathfrak{X}_{\phi_1}(g_1)\cdot\mathcal{V}_{\phi_2}(g_2)=\theta^{\mathrm{ver}}_{(\mathfrak{p}^\circ)}(g_1,\phi_1;g_2,\phi_2)$$

is a theta series for $g_1\in e_{\mathfrak{p}}H'(\mathbb{A}_F^{\mathfrak{p}})$.

Finally, we let

$$A_{(\mathfrak{p}^\circ)}(g_1,\phi_1)=\log q\,\mathcal{V}_{\phi_1}(g_1)\cdot\omega_{\mathfrak{p}^\circ}.$$

Then, in summary, we have:

Proposition 5.7. *For $\phi_{1,\mathcal{S}}\otimes\phi_{2,\mathcal{S}}\in\mathcal{S}(\mathbb{V}^2_{\mathcal{S}})_{\mathrm{reg}}$ and $g_i\in e_{\mathcal{S}}H'(\mathbb{A}_F^{\mathcal{S}})$ $(i=1,2)$,*

$$\begin{aligned}&\langle\widehat{Z}_{\phi_1}(g_1),\widehat{Z}_{\phi_2}(g_2)\rangle_{\mathfrak{p}^\circ}\\&\quad=\theta^{\mathrm{hor}}_{(\mathfrak{p}^\circ)}(\iota(g_1,g_2^\vee);\phi_1,\phi_2)+\theta^{\mathrm{ver}}_{(\mathfrak{p}^\circ)}(g_1,\phi_1;g_2,\phi_2)+A_{(\mathfrak{p}^\circ)}(g_1,\phi_1)E(g_2,\phi_2).\end{aligned}$$

5C. ***Nonsplit case.*** In this section, we discuss the contribution of the local height pairing at a nonsplit (finite) place $\mathfrak{p}$ in $\mathcal{S}$ such that $\epsilon(\mathbb{V}_{\mathfrak{p}})=-1$.

We fix such a $\mathfrak{p}$ and denote by $\mathfrak{p}^\circ$ the unique place of E over $\mathfrak{p}$ as usual. As before, we need to consider the height pairing $\langle\widehat{Z}_{\phi_1}(g_1),\widehat{Z}_{\phi_2}(g_2)\rangle_{\mathfrak{p}^\circ}$ on a certain model $\mathcal{M}_{K;\mathfrak{p}^\circ}$, for $\phi_{1,\mathfrak{p}}\otimes\phi_{2,\mathfrak{p}}\in\mathcal{S}(\mathbb{V}^2_{\mathfrak{p}})_{\mathrm{reg}}$ and $g_i\in e_{\mathfrak{p}}H'(\mathbb{A}_F^{\mathfrak{p}})$. We assume that $K=K_{\mathfrak{p}}K^{\mathfrak{p}}$ with $K^{\mathfrak{p}}$ sufficiently small and $K_{\mathfrak{p}}=K_{\mathfrak{p},n}$ for $n\ge 0$. In Section 4B, we have fixed an isometry between $\mathbb{V}_\mathfrak{p}$ and $B_\mathfrak{p}$, the division quaternion algebra over $F_\mathfrak{p}$, sending $\Lambda_\mathfrak{p}$ to $\mathcal{O}_{B_\mathfrak{p}}$, the maximal order. Hence $\mathbb{V}_\mathfrak{p}$ has an action of $B_\mathfrak{p}$ by both left and right multiplication where the latter is $E_{\mathfrak{p}^\circ}$-linear. Also, $B_\mathfrak{p}$ acts on $\mathcal{S}(\mathbb{V}_\mathfrak{p})$ via $(g.\phi)(x)=\phi(xg)$. By enlarging n (to be an least 1), we assume that $\phi_{i,\mathfrak{p}}$ is not only invariant under $K_{\mathfrak{p},n}$ but also $\mathbf{1}_2+\varpi^n\mathcal{O}_{B_\mathfrak{p}}$. Moreover, we assume that $\phi_{i,\mathfrak{p}}$ is supported on $\Lambda_\mathfrak{p}$.

We need to choose some model for $M_{n,K^\mathfrak{p};\mathfrak{p}^\circ}$. If $n=0$, we have already constructed a regular model $\mathcal{M}_{0,K^\mathfrak{p}}$ which is flat and projective, but not smooth over $\mathcal{O}_{E_{\mathfrak{p}^\circ}}$. Similar to the quasisplit case in Section 4B, we let $\breve{B}$ be the quaternion algebra over F by changing the Hasse invariant at ι_1 and $\mathfrak{p}$ from which we construct $\breve{B}^\dagger$ and an algebraic group $\breve{H}^\dagger$ over $\mathbb{Q}$. We let $\breve{H}=Z\cdot\breve{H}^{\dagger,\mathrm{der}}$. Then we have the following variant of the *Cherednik–Drinfeld uniformization theorem* (see [Boutot and Carayol 1991, Chapitre III]):

$$\begin{array}{ccc} [\mathcal{X}]^\wedge_{\mathsf{spe}}\times_{\mathcal{O}_{E_{\mathfrak{p}^\circ}}}\mathcal{O}_{\hat{E}_{\mathfrak{p}^\circ}} & \xrightarrow{\ \sim\ } & \breve{H}(\mathbb{Q})\backslash\mathcal{X}'\times\breve{H}^\mathfrak{p}_f/K^\mathfrak{p} \\ \downarrow & & \downarrow \\ [\mathcal{M}_{0,K^\mathfrak{p}}]^\wedge_{\mathsf{spe}}\times_{\mathcal{O}_{E_{\mathfrak{p}^\circ}}}\mathcal{O}_{\hat{E}_{\mathfrak{p}^\circ}} & \xrightarrow{\ \sim\ } & \breve{H}(\mathbb{Q})\backslash\Omega'\times\breve{H}^\mathfrak{p}_f/K^\mathfrak{p}, \end{array}$$

where $\Omega'=\Omega\times_{\mathcal{O}_{F_\mathfrak{p}}}\mathcal{O}_{\hat{E}_{\mathfrak{p}^\circ}}$ (resp. $\mathcal{X}'=\mathcal{X}^{\mathsf{univ}}\times_{\mathcal{O}_{F_\mathfrak{p}}}\mathcal{O}_{\hat{E}_{\mathfrak{p}^\circ}}$) and Ω (resp. $\mathcal{X}^{\mathsf{univ}}$) is the *formal Drinfeld upper half plane* over $\mathcal{O}_{F_\mathfrak{p}}$ (resp. universal $\mathcal{O}_{B_\mathfrak{p}}$-module over Ω).

For general $n\geq 1$, we construct an integral model of the base change $M'_{n,K^\mathfrak{p};\mathfrak{p}^\circ}=M_{n,K^\mathfrak{p};\mathfrak{p}^\circ}\times_{E_{\mathfrak{p}^\circ}}E^{(n)}_{\mathfrak{p}^\circ}$ as follows. Let $\Omega^{\mathsf{rig}}_n=\mathcal{X}^{\mathsf{univ,rig}}[\varpi^n]-\mathcal{X}^{\mathsf{univ,rig}}[\varpi^{n-1}]$ be the étale covering over Ω^{rig} with Galois group $(\mathcal{O}_{B_\mathfrak{p}}/\varpi^n\mathcal{O}_{B_\mathfrak{p}})^\times$. Consider $\Omega^{\mathsf{rig}}_n\times_{F_\mathfrak{p}}\hat{E}^{(n)}_{\mathfrak{p}^\circ}$, it has $(\mathcal{O}_{F_\mathfrak{p}}/\varpi^n\mathcal{O}_{F_\mathfrak{p}})^\times$ connected components. Pick any connected component Σ_n which is étale over $\Omega^{\mathsf{rig}}\times_{F_\mathfrak{p}}\hat{E}^{(n)}_{\mathfrak{p}^\circ}$ with Galois group $(\mathcal{O}_{B_\mathfrak{p}}/\varpi^n\mathcal{O}_{B_\mathfrak{p}})^{\times,1}$. Then it is easy to see that

$$M'^{\mathsf{rig}}_{n,K^\mathfrak{p};\mathfrak{p}^\circ}\cong\breve{H}(\mathbb{Q})\backslash(\Sigma_n\times E^{\times,1}_{\mathfrak{p}^\circ}/\nu(K_{\mathfrak{p},n})\times\breve{H}^\mathfrak{p}_f/K^\mathfrak{p}),$$

where $\breve{H}(\mathbb{Q})$ acts on Σ_n through the $\mathfrak{p}$-component modulo center and acts on $E^{\times,1}_{\mathfrak{p}^\circ}/\nu(K_{\mathfrak{p},n})$ via the determinant map.

Let Ω_n be the normalization of $\Omega'\times_{\mathcal{O}_{\hat{E}_{\mathfrak{p}^\circ}}}\mathcal{O}_{\hat{E}^{(n)}_{\mathfrak{p}^\circ}}$ in Σ_n. It is not regular but has double points; we blow up these points to get a regular formal scheme Ω'_n and (for sufficiently small $K^\mathfrak{p}$)

$$\breve{H}(\mathbb{Q})\backslash(\Omega'_n\times E^{\times,1}_{\mathfrak{p}^\circ}/\nu(K_{\mathfrak{p},n})\times\breve{H}^\mathfrak{p}_f/K^\mathfrak{p})$$

is regular, flat, and projective over $\mathcal{O}_{\hat{E}^{(n)}_{\mathfrak{p}^\circ}}$, where $\breve{H}(\mathbb{Q})$ acts on Ω'_n by the universal property of normalization and blowing-up of double points. By Grothendieck's existence theorem, we have a regular scheme $\mathcal{M}'_{n,K^\mathfrak{p}}$ that is flat and projective over $\mathsf{Spec}\,\mathcal{O}_{E^{(n)}_{\mathfrak{p}^\circ}}$, and a morphism $\pi_n:\mathcal{M}'_{n,K^\mathfrak{p}}\to\mathcal{M}'_{0,K^\mathfrak{p}}\times_{\mathcal{O}_{E_{\mathfrak{p}^\circ}}}\mathcal{O}_{E^{(n)}_{\mathfrak{p}^\circ}}$ such that the following diagram commutes:

$$\begin{tikzcd}
[\mathcal{M}'_{n,K^{\mathfrak{p}}}]^{\wedge}_{\mathsf{spe}} \arrow[r, "\sim"] \arrow[d, "\pi_n^{\wedge}"] & \breve{H}(\mathbb{Q})\backslash(\Omega'_n \times E^{\times,1}_{\mathfrak{p}^{\circ}}/\nu(K_{\mathfrak{p},n}) \times \breve{H}^{\mathfrak{p}}_f/K^{\mathfrak{p}}) \arrow[d] \\
[\mathcal{M}'_{0,K^{\mathfrak{p}}} \times_{\mathbb{O}_{E_{\mathfrak{p}^{\circ}}}} \mathbb{O}_{E^{(n)}_{\mathfrak{p}^{\circ}}}]^{\wedge}_{\mathsf{spe}} \arrow[r, "\sim"] & (\breve{H}(\mathbb{Q})\backslash\Omega' \times \breve{H}^{\mathfrak{p}}_f/K^{\mathfrak{p}}) \times_{\mathbb{O}_{\hat{E}_{\mathfrak{p}^{\circ}}}} \mathbb{O}_{\hat{E}^{(n)}_{\mathfrak{p}^{\circ}}}.
\end{tikzcd}$$

Now let us define the integral special subschemes on these models. We recall the integral special subschemes $\mathfrak{Z}(o)^{\dagger}_{\mathfrak{p}}$ (resp. $\mathfrak{Z}(x)^{\circ}_{0,\mathfrak{p}^{\circ}}$) on $\mathcal{M}^{\dagger}_{0,K^{\dagger,\mathfrak{p}}_p,K^{\dagger,p}}$ (resp. $\mathcal{M}_{0,K^{\mathfrak{p}}}$) defined in Section 4B. Similar to the quasisplit case, we fix an integral special subscheme $\mathfrak{Z}(o)^{\dagger}_{\mathfrak{p}}$ with $T(o)=1$. Let s be the unique geometric point in the Zariski closure of the generic fiber of $\mathfrak{Z}(o)^{\dagger}_{\mathfrak{p}}$. We set $\boldsymbol{A}=\mathcal{A}_s$ and $\boldsymbol{X}=\mathcal{X}^{\dagger}_s$, a special formal $\mathbb{O}_{B_{\mathfrak{p}}}$-module of height 4. The isogeny class of $\boldsymbol{A}$ is independent of o. We denote by $(\boldsymbol{A}^0, \theta^0, i_{\boldsymbol{A}^0})$ the corresponding abelian variety up to isogeny. Then we have $\mathsf{End}(\boldsymbol{A}^0, i_{\boldsymbol{A}^0}) \cong \breve{B}^{\dagger}$ as an $F^{\dagger}$-algebra and $\mathsf{Aut}(\boldsymbol{A}^0, \theta^0, i_{\boldsymbol{A}^0}) \cong \breve{H}^{\dagger}(\mathbb{Q})$. We define $\breve{\Lambda}=\mathsf{Hom}((\boldsymbol{E}, j), (\boldsymbol{A}, i_{\boldsymbol{A}}))$ and $\breve{\Lambda}_{\mathbb{Q}}=\breve{\Lambda}\otimes\mathbb{Q}$. Let $\breve{V}\subset\breve{\Lambda}_{\mathbb{Q}}$ be the sub-E-vector space generated by $\breve{H}(\mathbb{Q}).x_0$ where $x_0=\varrho_{\boldsymbol{A}}$. One can define a hermitian form $(\,\cdot\,,\,\cdot\,)'$ on $\breve{V}$ as in (4-8) such that $(\breve{V}, (\,\cdot\,,\,\cdot\,)')$ is isometric to the nearby hermitian space $V^{(\mathfrak{p})}$ of $\mathbb{V}$ and has the unitary group $\breve{H}$. The level structure $(\bar{\eta}^p, \bar{\eta}^{\mathfrak{p}}_p)$ of $\boldsymbol{A}$ gives a $K^{\mathfrak{p}}$-class of isometries $V\otimes_F\mathbb{A}^{\mathfrak{p}}_{f,F}\to\breve{V}\otimes_F\mathbb{A}^{\mathfrak{p}}_{f,F}$. We identify $V\otimes_F\mathbb{A}^{\mathfrak{p}}_{f,F}$ with $\breve{V}\otimes_F\mathbb{A}^{\mathfrak{p}}_{f,F}$ via a fixed isometry in this class. For the place $\mathfrak{p}$, we let $\breve{\Lambda}_{\mathfrak{p}}=\mathsf{Hom}((\boldsymbol{Y}, j), (\boldsymbol{X}, i_{\boldsymbol{X}}))$, which is a self-dual lattice in $\breve{V}_{\mathfrak{p}}$. We are going to define a formal special subscheme $\mathfrak{Z}(\breve{x})$ on $\hat{\Omega}:=\Omega\times_{\mathbb{O}_{F_{\mathfrak{p}}}}\mathbb{O}_{\hat{F}_{\mathfrak{p}}}$.

Let us first recall the moduli problem represented by $\hat{\Omega}$. For any element $S\in\mathsf{Obj}\,\mathfrak{Nilp}_{\mathbb{O}_{\hat{F}_{\mathfrak{p}}}}$, $\hat{\Omega}(S)$ is the set of equivalence classes of couples (Φ, ρ_{Φ}) where

- Φ is a special formal $\mathbb{O}_{B_{\mathfrak{p}}}$-module of height 4 over S and
- $\rho_{\Phi}:\Phi\times_S S_{\mathsf{spe}}\to\boldsymbol{X}\times_{\mathbb{F}}S_{\mathsf{spe}}$ is a quasiisogeny of height 0.

Two couples (Φ, ρ_{Φ}) and $(\Phi', \rho_{\Phi'})$ are equivalent if there is an isomorphism $\Phi'\to\Phi$ sending ρ_{Φ} to $\rho_{\Phi'}$. For any $\breve{x}\in\breve{\Lambda}_{\mathfrak{p}}$, we define a subfunctor $\mathfrak{Z}(\breve{x})$ as follows: for any $S\in\mathsf{Obj}\,\mathfrak{Nilp}_{\mathbb{O}_{\hat{F}_{\mathfrak{p}}}}$, $\mathfrak{Z}(\breve{x})(S)$ is the set of equivalence classes (Φ, ρ_{Φ}) such that the composed quasihomomorphism

$$\mathcal{Y}\times_{\mathbb{O}_{\hat{F}_{\mathfrak{p}}}}S_{\mathsf{spe}}=\boldsymbol{Y}\times_{\mathbb{F}}S_{\mathsf{spe}}\xrightarrow{\breve{x}}\boldsymbol{X}\times_{\mathbb{F}}S_{\mathsf{spe}}\xrightarrow{\rho_{\Phi}^{-1}}\Phi\times_S S_{\mathsf{spe}}$$

extends to a homomorphism $\mathcal{Y}\times_{\mathbb{O}_{\hat{F}_{\mathfrak{p}^{\circ}}}}S\to\Phi$.

Now we proceed exactly as in Section 5B. We use the same notation for the pullback of $\mathfrak{Z}(x)^{\circ}_{0,\mathfrak{p}^{\circ}}$ to the scheme $\mathcal{M}'_{n,K^{\mathfrak{p}}}$ and define $\mathfrak{Z}(x,h)$ for $x\in V\cap\Lambda_{\mathfrak{p}}-\{0\}$, $h\in K_{\mathfrak{p},0}/K_{\mathfrak{p},n}\times H^{\mathfrak{p}}_f/K^{\mathfrak{p}}$. We also have the formal special subscheme $\mathfrak{Z}(\breve{x}, x, h)$

for $\breve{x} \in \breve{\Lambda}_{\mathfrak{p}} - \{0\}$, $\breve{h} \in E_{\mathfrak{p}^\circ}^{\times,1}/\nu(K_{\mathfrak{p},n}) \times \breve{H}_f^{\mathfrak{p}}/K^{\mathfrak{p}}$, and x satisfying a similar relation as in (5-4). The identity of sets (5-5) still holds. There is only one difference: when $n \geq 1$, we only keep the irreducible component which is not supported on the special fiber when defining $\mathfrak{X}(\breve{x}, x, h)$.

Now we can consider the height pairing. We define

$$\mathfrak{X}_{\phi_i}(g_i) = \sum_{x_i \in H(\mathbb{Q})\backslash V} \sum_{h_i \in H_{x_i}(\mathbb{Q})\backslash H(\mathbb{A}_f)/K} \phi_{\mathfrak{p}} \otimes (\omega_\chi(g_i)\phi_i^{\mathfrak{p}})(h_i^{-1}x_i)\mathfrak{X}(x_i, h_i),$$

which is a cycle of $\mathcal{M}'_{n,K^{\mathfrak{p}}}$ extending the generating series $Z_{\phi_i}(g_i)$ on $M'_{n,K^{\mathfrak{p}};\mathfrak{p}^\circ}$. We have

$$[\mathfrak{X}_{\phi_i}(g_i)]^\wedge_{\mathsf{spe}} = \sum_{\breve{x}_i \in \breve{V} \cap \breve{\Lambda}_{\mathfrak{p}} - \{0\}, x_i} \sum_{\breve{h}_i \in \breve{H}(\mathbb{Q})\backslash (E_{\mathfrak{p}^\circ}^{\times,1}/\nu(K_{\mathfrak{p},n}) \times \breve{H}_f^{\mathfrak{p}}/K^{\mathfrak{p}})} \phi_{i,\mathfrak{p}}(x_i)(\omega_\chi(g_i)\phi_i^{\mathfrak{p}})(\breve{h}_i^{-1}\breve{x}_i)\mathfrak{X}(\breve{x}_i, x_i, \breve{h}_i).$$

We have a similar decomposition as in (5-2) but the first term is not zero anymore. First, we consider

$$\begin{aligned}\mathfrak{X}_{\phi_1}(g_1) \cdot \mathfrak{X}_{\phi_2}(g_2) &= [\mathfrak{X}_{\phi_1}(g_1)]^\wedge_{\mathsf{spe}} \cdot [\mathfrak{X}_{\phi_2}(g_2)]^\wedge_{\mathsf{spe}} \\ &= \sum_{\breve{x}_1, \breve{x}_2 \in \breve{V} \cap \breve{\Lambda}_{\mathfrak{p}} - \{0\}} \sum_{x_1, x_2} \sum_{\breve{h} \in \breve{H}(\mathbb{Q})\backslash (E_{\mathfrak{p}^\circ}^{\times,1}/\nu(K_{\mathfrak{p},n}) \times \breve{H}_f^{\mathfrak{p}}/K^{\mathfrak{p}})} \phi_{1,\mathfrak{p}}(x_1)\phi_{2,\mathfrak{p}}(x_2) \\ &\quad \times \big(\omega''_\chi(\iota(g_1, g_2^\vee))\phi_1^{\mathfrak{p}} \otimes \phi_2^{\mathfrak{p}}(\breve{h}^{-1}(\breve{x}_1, \breve{x}_2))\big)\mathfrak{X}(\breve{x}_1, x_1, \breve{h}) \cdot \mathfrak{X}(\breve{x}_2, x_2, \breve{h}). \quad (5\text{-}8)\end{aligned}$$

We have the following lemma, whose proof is similar to that of Lemma 5.3.

Lemma 5.8. *We can extend*

$$\mu(\breve{x}_1, \breve{x}_2; \breve{h}, \phi_{1,\mathfrak{p}}, \phi_{2,\mathfrak{p}}) := \sum_{x_1, x_2 \in \Lambda_{\mathfrak{p}}/(\mathbf{1}_2 + \varpi^n \mathrm{GL}_2(\mathbb{O}_{F_{\mathfrak{p}}}))} \phi_{1,\mathfrak{p}}(x_1)\phi_{2,\mathfrak{p}}(x_2)\mathfrak{X}(\breve{x}_1, x_1, \breve{h}) \cdot \mathfrak{X}(\breve{x}_2, x_2, \breve{h})$$

to a function in $\mathcal{S}(\breve{V}_{\mathfrak{p}}^2)$.

By the lemma,

$$(5\text{-}6) = \sum_{\vec{x} \in \breve{V}^2} \sum_{\breve{h} \in \breve{H}(\mathbb{Q})\backslash (E_{\mathfrak{p}^\circ}^{\times,1}/\nu(K_{\mathfrak{p},n}) \times \breve{H}_f^{\mathfrak{p}}/K^{\mathfrak{p}})} \big(\omega''_\chi(\iota(g_1, g_2^\vee))\phi_1^{\mathfrak{p}} \otimes \phi_2^{\mathfrak{p}}(\breve{h}^{-1}\vec{x})\big)\mu(\vec{x}; \breve{h}, \phi_{1,\mathfrak{p}}, \phi_{2,\mathfrak{p}}).$$

Since the set $\breve{H}(\mathbb{Q})\backslash (E_{\mathfrak{p}^\circ}^{\times,1}/\nu(K_{\mathfrak{p},n}) \times \breve{H}_f^{\mathfrak{p}}/K^{\mathfrak{p}})$ is finite, we let

$$\theta^{\mathsf{hor}}_{(\mathfrak{p}^\circ)}(\,\cdot\,; \phi_1, \phi_2) = \log q \sum_{\vec{x} \in \breve{V}^2} \omega''_\chi(\,\cdot\,)\Phi^{\mathsf{hor}}(\vec{x})$$

be the theta series for the Schwartz function

$$\Phi^{\mathrm{hor}} = \sum_{\breve{h}\in \breve{H}(\mathbb{Q})\backslash(E^{\times,1}_{\mathfrak{p}^\circ}/\nu(K_{\mathfrak{p},n})\times \breve{H}^{\mathfrak{p}}_f/K^{\mathfrak{p}})} \mu(\,\cdot\,,\cdot\,;\breve{h},\phi_{1,\mathfrak{p}},\phi_{2,\mathfrak{p}})\otimes(\omega''_\chi(\breve{h})\phi_1^{\mathfrak{p}}\otimes\phi_2^{\mathfrak{p}}).$$

Lemma 5.9. *For $g_i\in e_{\mathfrak{p}}H'(\mathbb{A}_F^{\mathfrak{p}})$,*

$$\log q\,\mathfrak{X}_{\phi_1}(g_1)\cdot\mathfrak{X}_{\phi_2}(g_2)=\theta^{\mathrm{hor}}_{(\mathfrak{p}^\circ)}(\iota(g_1,g_2^\vee);\phi_1,\phi_2),$$

where q is the cardinality of the residue field of $E_{\mathfrak{p}^\circ}$.

Now we consider the second term, $\mathfrak{X}_{\phi_1}(g_1)\cdot\mathcal{V}_{\phi_2}(g_2)$. For any

$$\breve{h}_1\in\breve{H}(\mathbb{Q})\backslash(E^{\times,1}_{\mathfrak{p}^\circ}/\nu(K_{\mathfrak{p},n})\times\breve{H}^{\mathfrak{p}}_f/K^{\mathfrak{p}}),$$

we write $s(\breve{h}_1)$ for the corresponding connected component of $[\mathcal{M}'_{n,K^{\mathfrak{p}}}]_{\mathsf{spe}}$. Then

$$\begin{aligned}\mathfrak{X}_{\phi_1}(g_1)\cdot\mathcal{V}_{\phi_2}(g_2)&=[\mathfrak{X}_{\phi_1}(g_1)]^\wedge_{\mathsf{spe}}\cdot[\mathcal{V}_{\phi_2}(g_2)]^\wedge_{\mathsf{spe}}\\&=\sum_{\breve{x}_1\in\breve{V}\cap\breve{\Lambda}_{\mathfrak{p}}-\{0\},x_1}\ \sum_{\breve{h}_1\in\breve{H}(\mathbb{Q})\backslash(E^{\times,1}_{\mathfrak{p}^\circ}/\nu(K_{\mathfrak{p},n})\times\breve{H}^{\mathfrak{p}}_f/K^{\mathfrak{p}})}\phi_{1,\mathfrak{p}}(x_1)(\omega_\chi(g_1)\phi_1^{\mathfrak{p}})(\breve{h}_1^{-1}\breve{x}_1)\\&\qquad\qquad\mathfrak{X}(\breve{x}_1,x_1,\breve{h}_1)\cdot[\mathcal{V}_{\phi_2}(g_2)]^\wedge_{s(\breve{h}_1)}.\end{aligned}\tag{5-9}$$

Lemma 5.10. *For any $\breve{h}_1$, we can extend*

$$\nu(\breve{x}_1,\phi_{1,\mathfrak{p}},\breve{h}_1;\phi_2,g_2):=\sum_{x_1\in\Lambda_{\mathfrak{p}}/(\mathbf{1}_2+\varpi^n\mathrm{GL}_2(\mathbb{O}_{F_{\mathfrak{p}}}))}\phi_{1,\mathfrak{p}}(x_1)\mathfrak{X}(\breve{x}_1,x_1,\breve{h}_1)\cdot[\mathcal{V}_{\phi_2}(g_2)]^\wedge_{s(\breve{h}_1)}$$

to a function in $\mathcal{S}(\breve{V}_{\mathfrak{p}})$.

This lemma implies that (5-9) equals

$$\sum_{\breve{x}_1\in\breve{V}}\ \sum_{\breve{h}_1\in\breve{H}(\mathbb{Q})\backslash(E^{\times,1}_{\mathfrak{p}^\circ}/\nu(K_{\mathfrak{p},n})\times\breve{H}^{\mathfrak{p}}_f/K^{\mathfrak{p}})}\phi_{1,\mathfrak{p}}(x_1)\times(\omega_\chi(g_1)\phi_1^{\mathfrak{p}})(\breve{h}_1^{-1}\breve{x}_1)\nu(\breve{x}_1,\phi_{1,\mathfrak{p}},\breve{h}_1;\phi_2,g_2).$$

Let

$$\theta^{\mathrm{ver}}_{(\mathfrak{p}^\circ)}(\,\cdot\,,\phi_1;g_2,\phi_2)=\sum_{\breve{x}_1\in\breve{V}}\omega_\chi(\,\cdot\,)\phi^{\mathrm{ver}}(\breve{x}_1)$$

be the theta series for the Schwartz function

$$\phi^{\mathrm{ver}}=\sum_{\breve{h}_1\in\breve{H}(\mathbb{Q})\backslash(E^{\times,1}_{\mathfrak{p}^\circ}/\nu(K_{\mathfrak{p},n})\times\breve{H}^{\mathfrak{p}}_f/K^{\mathfrak{p}})}\nu(\,\cdot\,,\phi_{1,\mathfrak{p}},\breve{h}_1;\phi_2,g_2)\otimes\omega_\chi(\breve{h})\phi_1^{\mathfrak{p}}.$$

Lemma 5.11. *For $g_i\in e_{\mathfrak{p}}H'(\mathbb{A}_F^{\mathfrak{p}})$,*

$$\log q\,\mathfrak{X}_{\phi_1}(g_1)\cdot\mathcal{V}_{\phi_2}(g_2)=\theta^{\mathrm{ver}}_{(\mathfrak{p}^\circ)}(g_1,\phi_1;g_2,\phi_2)$$

is a theta series for $g_1\in e_{\mathfrak{p}}H'(\mathbb{A}_F^{\mathfrak{p}})$.

Finally, we let

$$A_{(\mathfrak{p}^\circ)}(g_1,\phi_1)=\log q\,\mathscr{V}_{\phi_1}(g_1)\cdot\omega_{\mathfrak{p}^\circ}.$$

Then, in summary, we have:

Proposition 5.12. *For $\phi_{1,\mathcal{S}}\otimes\phi_{2,\mathcal{S}}\in\mathscr{S}(\mathbb{V}_{\mathcal{S}}^2)_{\mathsf{reg}}$ and $g_i\in e_{\mathcal{S}}H'(\mathbb{A}_F^{\mathcal{S}})$ $(i=1,2)$,*

$$\begin{aligned}\langle\widehat{Z}_{\phi_1}(g_1),\widehat{Z}_{\phi_2}(g_2)\rangle_{\mathfrak{p}^\circ}\\ =\theta^{\mathsf{hor}}_{(\mathfrak{p}^\circ)}(\iota(g_1,g_2^\vee);\phi_1,\phi_2)+\theta^{\mathsf{ver}}_{(\mathfrak{p}^\circ)}(g_1,\phi_1;g_2,\phi_2)+A_{(\mathfrak{p}^\circ)}(g_1,\phi_1)E(g_2,\phi_2).\end{aligned}$$

6. An arithmetic inner product formula

6A. ***Holomorphic projection.*** In this section, we calculate the holomorphic projection of the analytic kernel function $E'(0,\iota(g_1,g_2^\vee),\phi_1\otimes\phi_2)$ and its relation with the geometric kernel function when $n=1$. We follow the general theory for the GL_2 case in [Gross and Zagier 1986; Zhang 2001a; 2001b; Yuan et al. 2011].

Holomorphic projection in general. Let $\mathfrak{k}=(\mathfrak{k}_\iota)_\iota\in\mathbb{Z}^{\Sigma_\infty}$ be a sequence of integers. We denote by $\mathscr{A}_0(H')\subset\mathscr{A}(H')$ the subspace of cuspidal automorphic forms of $H'=\mathsf{U}(W_1)$ and by $\mathscr{A}_0^{\mathfrak{k}}(H')\subset\mathscr{A}_0(H')$ those ones whose archimedean component is in a discrete series representation of weight $(1+\mathfrak{k},1-\mathfrak{k})$. Let Z' be the center of H', and hence isomorphic to $E^{\times,1}$ as an F-torus. From $\mathfrak{k}$, we define a character $\zeta^{\mathfrak{k}}$ of Z'_∞ by $\zeta_\iota^{\mathfrak{k}}(z)=z^{2\mathfrak{k}_\iota}$. Let $\mathscr{A}(H',\zeta^{\mathfrak{k}})$ be the subspace of $\mathscr{A}(H')$ consisting of all forms which have the archimedean central character $\zeta^{\mathfrak{k}}$. It is obvious that $\mathscr{A}_0^{\mathfrak{k}}(H')\subset\mathscr{A}(H',\zeta^{\mathfrak{k}})$. For any element in $\mathscr{A}_0^{\mathfrak{k}}(H')$ and any $t\in F^+$, the t-th archimedean Whittaker function (with respect to the standard ψ_∞^0) is $W_t^{\mathfrak{k}}$, where

$$W_t^{\mathfrak{k}}(n(b)m(a)[k_1,k_2])=\prod_{\iota\in\Sigma_\infty}e^{2\pi it(b_\iota+ia_\iota\bar{a}_\iota)}(a_\iota\bar{a}_\iota)k_{1,\iota}^{1+\mathfrak{k}_\iota}k_{2,\iota}^{1-\mathfrak{k}_\iota}$$

for all $a=(a_\iota)\in E_\infty^\times$, $b=(b_\iota)\in F_\infty$ and $[k_1,k_2]=([k_{1,\iota},k_{2,\iota}])$ in the standard maximal compact subgroup $\mathscr{K}'_\infty$.

We let $\mathscr{A}_0^{\mathfrak{k}}(H'\times H')$ (resp. $\mathscr{A}(H'\times H',\zeta^{\mathfrak{k}})$) to be the subspace of $\mathscr{A}(H'\times H')$ consisting of functions F such that $F(\cdot,g_2)$ and $F(g_1,\cdot)$ are both in $\mathscr{A}_0^{\mathfrak{k}}(H')$ (resp. $\mathscr{A}(H',\zeta^{\mathfrak{k}})$) for all $g_1,g_2\in H'(\mathbb{A}_F)$. For any form $F\in\mathscr{A}(H'\times H',\zeta^{\mathfrak{k}})$ and any $f_1\otimes f_2\in\mathscr{A}_0^{\mathfrak{k}}(H'\times H')$, we can define the usual Petersson inner product as

$$(F,f_1\otimes f_2)_{H'}=\int_{[H'(F)\backslash H'(\mathbb{A}_F)]^2}F(g_1,g_2)\overline{f_1(g_1)f_2(g_2)}\,dg_1\,dg_2.$$

Definition 6.1. The *holomorphic projection* Pr is a linear map from $\mathscr{A}(H'\times H',\zeta^{\mathfrak{k}})$ to $\mathscr{A}_0^{\mathfrak{k}}(H'\times H')$, such that $\mathsf{Pr}(F)$ is the unique form in $\mathscr{A}_0^{\mathfrak{k}}(H'\times H')$ satisfying $(\mathsf{Pr}(F),f_1\otimes f_2)_{H'}=(F,f_1\otimes f_2)_{H'}$ for any $f_1\otimes f_2\in\mathscr{A}_0^{\mathfrak{k}}(H'\times H')$.

For any automorphic form $F \in \mathscr{A}(H' \times H', \zeta^{\mathfrak{k}})$, we define the Whittaker function for a nontrivial character ψ of $F \backslash \mathbb{A}_F$ to be

$$F_{\psi,s}(g_1, g_2) = (4\pi)^{2d} W^{\mathfrak{k}}(g_{1,\infty}) W^{\mathfrak{k}}(g_{2,\infty}) \int_{[Z'(F_\infty)N'(F_\infty)\backslash H'(F_\infty)]^2} \lambda_{P'}(h_1)^s \lambda_{P'}(h_2)^s \times F_\psi(g_{1,f}h_1, g_{2,f}h_2)\overline{W^{\mathfrak{k}}(h_1)W^{\mathfrak{k}}(h_2)}\, dh_1\, dh_2, \tag{6-1}$$

where $W^{\mathfrak{k}} = W_t^{\mathfrak{k}}$ such that $\psi_\infty(x) = \psi_\infty^0(tx)$ and $d = [F : \mathbb{Q}]$ as before.

Proposition 6.2. *Let $F \in \mathscr{A}(H' \times H', \zeta^{\mathfrak{k}})$ be a form with asymptotic behavior*

$$F(m(a_1)g_1, m(a_2)g_2) = O_{g_1,g_2}(|a_1a_2|_{\mathbb{A}_E}^{1-\epsilon})$$

as $a_i \in \mathbb{A}_E^\times$ and $|a_1a_2|_{\mathbb{A}_E} \to \infty$ for some $\epsilon > 0$. Then the holomorphic projection $\mathsf{Pr}(F)$ *has the Whittaker function*

$$\mathsf{Pr}(F)_\psi(g_1, g_2) = \lim_{s\to 0} F_{\psi,s}(g_1, g_2).$$

Proof. First, we can decompose $F = \sum_s F_{\zeta_{s,1},\zeta_{s,2}}$ as a finite sum of element $F_{\zeta_{s,1},\zeta_{s,2}} \in \mathscr{A}(H' \times H', \zeta^{\mathfrak{k}})$ which has central character $\zeta_{s,1} \boxtimes \zeta_{s,2}$ such that $(\zeta_{s,1}, \zeta_{s,2})$ are distinct pairs. One can easily show that if F satisfies the asymptotic behavior in the proposition, so does each $F_{\zeta_{s,1},\zeta_{s,2}}$.

Now consider any Whittaker function $W^i(g_i) = W^{\mathfrak{k}}(g_{i,\infty})W_f^i(g_{i,f})$ $(i = 1, 2)$ of $H'(\mathbb{A}_F)$ with central character ζ_i such that $W_f^i(g_{i,f})$ is compactly supported modulo $Z'(\mathbb{A}_{f,F})N'(\mathbb{A}_{f,F})$, we define the Poincaré series as

$$P_{W^i}(g_i) = \lim_{s\to 0^+} \sum_{\gamma \in Z'(F)N'(F)\backslash H'(F)} W^i(\gamma g_i)\lambda_{P'}(\gamma_\infty g_{i,\infty})^s.$$

If (ζ_1, ζ_2) doesn't appear in $\{(\zeta_{s,1}, \zeta_{s,2})\}$, then the Petersson inner product

$$(F, P_{W^1} \otimes P_{W^2})_{H'}$$

is automatically zero; hence we only need to consider the case where it appears. Then, assuming that F has the asymptotic behavior as in the proposition, we have, after choosing suitable quotient measures dg_1 and dg_2,

$$\begin{aligned}
(F, P_{W^1} \otimes P_{W^2})_{H'} \\
&= \int F_{\zeta_1,\zeta_2}(g_1, g_2)\overline{P_{W^1}(g_1)P_{W^2}(g_2)}\,dg_1\,dg_2 \\
&= \lim_{s\to 0^+} \int F_{\zeta_1,\zeta_2}(g_1, g_2)\overline{W^1(g_1)W^2(g_2)}\lambda_{P'}(g_{1,\infty})^s\lambda_{P'}(g_{2,\infty})^s\,dg_1\,dg_2 \\
&= \lim_{s\to 0^+} \int (F_{\zeta_1,\zeta_2})_\psi(g_1, g_2)\overline{W^1(g_1)W^2(g_2)}\lambda_{P'}(g_{1,\infty})^s\lambda_{P'}(g_{2,\infty})^s\,dg_1\,dg_2,
\end{aligned}$$

where the first two integrals are taken over $[Z'(\mathbb{A}_F)H'(F)\backslash H'(\mathbb{A}_F)]^2$ and that last over $[Z'(\mathbb{A}_F)N'(\mathbb{A}_F)\backslash H'(\mathbb{A}_F)]^2$.

Since $(\mathsf{Pr}(F), P_{W^1}\otimes P_{W^2})_{H'}$ is equal to $(F, P_{W^1}\otimes P_{W^2})_{H'}$, its value is

$$\int_{[Z'(F_\infty)N'(F_\infty)\backslash H'(F_\infty)]^2} W^{\mathfrak{k}}(g_1)W^{\mathfrak{k}}(g_2)\overline{W^{\mathfrak{k}}(g_1)W^{\mathfrak{k}}(g_2)}\,dg_1\,dg_2\times$$
$$\int_{[Z'(\mathbb{A}_{f,F})N'(\mathbb{A}_{f,F})\backslash H'(\mathbb{A}_{f,F})]^2} (\mathsf{Pr}(F)_{\zeta_1,\zeta_2})_\psi(g_{1,f},g_{2,f})\overline{W^1_f(g_{1,f})W^2_f(g_{2,f})}\,dg_{1,f}dg_{2,f}.$$

The first factor equals $(4\pi)^{-2d}$. Therefore

$$(\mathsf{Pr}(F), P_{W^1}\otimes P_{W^2})_{H'}=(4\pi)^{-2d}\times$$
$$\int_{[Z'(\mathbb{A}_{f,F})N'(\mathbb{A}_{f,F})\backslash H'(\mathbb{A}_{f,F})]^2} (\mathsf{Pr}(F)_{\zeta_1,\zeta_2})_\psi(g_{1,f},g_{2,f})\overline{W^1_f(g_{1,f})W^2_f(g_{2,f})}\,dg_{1,f}dg_{2,f}.$$

Since this holds for all possible W^1_f, W^2_f, and (ζ_1,ζ_2), we conclude that

$$\mathsf{Pr}(F)_\psi(g_1,g_2)=\lim_{s\to 0}F_{\psi,s}(g_1,g_2). \qquad \square$$

Now suppose F does not satisfy an asymptotic behavior as in Proposition 6.2.

Definition 6.3. For any $F\in\mathscr{A}(H'\times H',\zeta^{\mathfrak{k}})$, we let

$$\widetilde{\mathsf{Pr}}(F)_\psi(g_1,g_2)=\operatorname*{const}_{s\to 0}F_{\psi,s}(g_1,g_2),$$

where $\mathsf{const}_{s\to 0}$ denotes the constant term at $s=0$ after the meromorphic continuation (around 0). We define the *quasiholomorphic projection* of F to be

$$\widetilde{\mathsf{Pr}}(F)(g_1,g_2)=\sum_\psi \widetilde{\mathsf{Pr}}(F)_\psi(g_1,g_2),$$

where the sum is taken over all nontrivial characters of $F\backslash\mathbb{A}_F$. The above proposition just says that for F satisfying that asymptotic behavior, we have $\widetilde{\mathsf{Pr}}(F)=\mathsf{Pr}(F)$. In fact, the definition can apply to more general functions just in

$$L^2(N'(F)\backslash H'(\mathbb{A}_F),\zeta^{\mathfrak{k}})\otimes L^2(N'(F)\backslash H'(\mathbb{A}_F),\zeta^{\mathfrak{k}}).$$

Holomorphic projection of the analytic kernel function. Now we want to apply the above theory to the particular form $E'(0,\iota(g_1,g_2^\vee),\phi_1\otimes\phi_2))\in\mathscr{A}(H'\times H',\chi^\circ)$ for $\phi_\infty=\phi^\vee_\infty=\phi^0_\infty$, where $\chi^\circ:=\chi\big|_{Z'_\infty}=\zeta^{\mathfrak{k}\chi/2}$. Unfortunately, this form does not have the asymptotic behavior stated in Proposition 6.2. To find its holomorphic projection, we introduce the following function:

$$F(s;g_1,g_2;\phi_1,\phi_2)=E\big(s+\tfrac{1}{2},g_1,\phi_1\big)E\big(s+\tfrac{1}{2},g_2,\phi_2\big)\in\mathscr{A}(H'\times H',\chi^\circ),$$

where we use the Weil representation $\omega_{\chi,\psi}$ in both Eisenstein series on $H'(\mathbb{A}_F)$. This function is holomorphic at $s=0$. We claim that

Proposition 6.4. *The difference* $E'(0,\iota(g_1,g_2^\vee),\phi_1\otimes\phi_2)-F'(0;g_1,g_2;\phi_1,\phi_2)$ *has the asymptotic behavior stated in Proposition 6.2.*

Proof. Since it is symmetric in g_1 and g_2, without lost of generality, we prove the asymptotic behavior for g_1. Consider the Fourier expansion

$$E\big(s,\iota(m(a_1)g_1,g_2^\vee),\phi_1\otimes\phi_2\big)=\sum_{T\in\mathrm{Her}_2(E)}E_T\big(s,\iota(m(a_1)g_1,g_2^\vee),\phi_1\otimes\phi_2\big)$$

in which all terms except those with

$$T=\begin{pmatrix}0&0\\0&d_2\end{pmatrix}$$

in the summation are bounded as $|a_1|_{\mathbb{A}_E}\to\infty$. Hence we only need to consider those T. Before we compute these terms, we recall some matrices representing elements in Weyl groups:

$$w_2=\begin{pmatrix}&\mathbf{1}_2\\-\mathbf{1}_2&\end{pmatrix},\qquad w_{2,1}=\begin{pmatrix}1&&&\\&&&1\\&&1&\\&-1&&\end{pmatrix},\qquad w_1=\begin{pmatrix}&1\\-1&\end{pmatrix}.$$

<u>Case I</u>: $d_2\in F^\times$. We have

$$\begin{aligned}&E_T(s,\iota(g_1,g_2^\vee),\phi_1\otimes\phi_2)\\&=\int_{\mathrm{Her}_2(\mathbb{A}_E)}\omega_\chi(w_2n(b)\iota(g_1,g_2^\vee))\phi_1\otimes\phi_2(0)\lambda_P(w_2n(b)\iota(g_1,g_2^\vee))^s\psi(\mathrm{tr}\,Tb)^{-1}db\\&\quad+\int_{\mathrm{Her}_2(\mathbb{A}_E)}\omega_\chi(w_{2,1}n(b)\iota(g_1,g_2^\vee))\phi_1\otimes\phi_2(0)\lambda_P(w_{2,1}n(b)\iota(g_1,g_2^\vee))^s\psi(\mathrm{tr}\,Tb)^{-1}db\\&\qquad+\text{terms that are bounded,}\end{aligned}$$

where the first term is just $W_T(s,\iota(g_1,g_2^\vee),\phi_1\otimes\phi_2)$ and the second term equals

$$\begin{aligned}W_{1,d_2}(s;g_1,g_2^\vee;\phi_1,\phi_2):=&\int_{\mathbb{A}_F}\omega_\chi\left(w_{2,1}n\begin{pmatrix}0&0\\0&b_2\end{pmatrix}\iota(g_1,g_2^\vee)\right)\phi_1\otimes\phi_2(0)\lambda_{P'}(g_1)^s\\&\qquad\times\lambda_{P'}(w_1n(b_2)g_2)^s\psi(d_2b_2)\,db_2\\&=\omega_\chi(g_1)\phi_1(0)\lambda_{P'}(g_1)^s\times W_{-d_2}\big(s+\tfrac12,g_2,\phi_2\big).\end{aligned}$$

<u>Case II</u>: $d_2=0$. We have, apart from the terms $W_T(s,\iota(g_1,g_2^\vee),\phi_1\otimes\phi_2)$ and $W_{1,0}(s;g_1,g_2^\vee;\phi_1,\phi_2)$, another term,

$$W_{2,0}(s;g_1,g_2^\vee;\phi_1,\phi_2):=\omega_\chi(g_1)\phi_1(0)\lambda_{P'}(g_1)^s\times\omega_\chi(g_2^\vee)\phi_2(0)\lambda_{P'}(g_2)^s.$$

Now the term $W_T(s,\iota(m(a_1)g_1,g_2^\vee),\phi_1\otimes\phi_2)$ has asymptotic behavior $O_{g_1,g_2}(1)$ as $|a_1|_{\mathbb{A}_E}\to\infty$, hence we don't need to consider it. What is left is

$$E_{00}(s,\iota(g_1,g_2^\vee),\phi_1\otimes\phi_2):=W_{2,0}(s;g_1,g_2^\vee;\phi_1,\phi_2)+\sum_{d_2\in F}W_{1,d_2}(s;g_1,g_2^\vee;\phi_1,\phi_2)$$
$$=\omega_\chi(g_1)\phi_1(0)\lambda_{P'}(g_1)^s\times E\big(s+\tfrac{1}{2},g_2,\phi_2\big).$$

It turns out that our form $F(s;g_1,g_2;\phi_1,\phi_2)$ is just the Eisenstein series (in g_1) of the section $E_{00}(s,\iota(g_1,g_2^\vee),\phi_1\otimes\phi_2)$, namely

$$F(s;g_1,g_2;\phi_1,\phi_2)=\sum_{\gamma\in P'(F)\backslash H'(F)}E_{00}(s,\iota(\gamma g_1,g_2^\vee),\phi_1\otimes\phi_2).$$

Viewing it as a function in g_1, we have the Fourier expansion

$$F(s;g_1,g_2;\phi_1,\phi_2)=\sum_{d_1\in F}E_{d_1}\big(s+\tfrac{1}{2},g_1,\phi_1\big)\times E\big(s+\tfrac{1}{2},g_2,\phi_2\big).$$

For all $d_1\in F^\times$, the term $E_{d_1}\big(s+\frac{1}{2},m(a_1)g_1,\phi_1\big)$ decays exponentially as $|a_1|_{\mathbb{A}_E}$ goes to infinity. Hence we only need to consider the term

$$\begin{aligned}
&F_0(s;g_1,g_2;\phi_1,\phi_2)\\
&\quad=E_0\big(s+\tfrac{1}{2},g_1,\phi_1\big)E\big(s+\tfrac{1}{2},g_2,\phi_2\big)\\
&\quad=\big(\omega_\chi(g_1)\phi_1(0)\lambda_{P'}(g_1)^s+W_0\big(s+\tfrac{1}{2},g_1,\phi_1\big)\big)E\big(s+\tfrac{1}{2},g_2,\phi_2\big)\\
&\quad=E_{00}(s,\iota(g_1,g_2^\vee),\phi_1\otimes\phi_2)+W_0\big(s+\tfrac{1}{2},g_1,\phi_1\big)E\big(s+\tfrac{1}{2},g_2,\phi_2\big).
\end{aligned}$$

Now the proposition is equivalent to showing that

$$\begin{aligned}
&E'_{00}(0,\iota(\gamma g_1,g_2^\vee),\phi_1\otimes\phi_2)-F'_0(0;g_1,g_2;\phi_1,\phi_2)\\
&\quad=-\frac{d}{ds}\Big|_{s=0}\big(W_0\big(s+\tfrac{1}{2},g_1,\phi_1\big)E\big(s+\tfrac{1}{2},g_2,\phi_2\big)\big)\\
&\quad=-W_0\big(\tfrac{1}{2},g_1,\phi_1\big)E'\big(\tfrac{1}{2},g_2,\phi_2\big)-W'_0\big(\tfrac{1}{2},g_1,\phi_1\big)E\big(\tfrac{1}{2},g_2,\phi_2\big)
\end{aligned}$$

has the asymptotic behavior (in g_1). This is true since

$$W_0\big(\tfrac{1}{2},m(a_1)g_1,\phi_1\big)=O_{g_1,g_2}(1),\quad W'_0\big(\tfrac{1}{2},m(a_1)g_1,\phi_1\big)=O_{g_1,g_2}(\log|a_1|_{\mathbb{A}_E}).\quad\square$$

By the proposition, we have

$$\begin{aligned}
&\mathrm{Pr}\big(E'(0,\iota(g_1,g_2^\vee),\phi_1\otimes\phi_2)\big) \qquad (6\text{-}2)\\
&\quad=\mathrm{Pr}\big(E'(0,\iota(g_1,g_2^\vee),\phi_1\otimes\phi_2)-F'(0;g_1,g_2;\phi_1,\phi_2)\big)+\mathrm{Pr}(F'(0;g_1,g_2;\phi_1,\phi_2))\\
&\quad=\widetilde{\mathrm{Pr}}\big(E'(0,\iota(g_1,g_2^\vee),\phi_1\otimes\phi_2)-F'(0;g_1,g_2;\phi_1,\phi_2)\big)+\mathrm{Pr}(F'(0;g_1,g_2;\phi_1,\phi_2)).
\end{aligned}$$

Since

$$F'(0;g_1,g_2;\phi_1,\phi_2)=E'\big(\tfrac{1}{2},g_1,\phi_1\big)E\big(\tfrac{1}{2},g_2,\phi_2\big)+E\big(\tfrac{1}{2},g_1,\phi_1\big)E'\big(\tfrac{1}{2},g_2,\phi_2\big),$$

its holomorphic projection $\mathsf{Pr}(F'(0; g_1, g_2; \phi_1, \phi_2)) = 0$. Then

$$\begin{aligned}\text{(6-2)} &= \widetilde{\mathsf{Pr}}(E'(0, \iota(g_1, g_2^\vee), \phi_1 \otimes \phi_2) - F'(0; g_1, g_2; \phi_1, \phi_2)) \\ &= \widetilde{\mathsf{Pr}}\big(E'(0, \iota(g_1, g_2^\vee), \phi_1 \otimes \phi_2)\big) - \widetilde{\mathsf{Pr}}(F'(0; g_1, g_2; \phi_1, \phi_2)) \\ &= \widetilde{\mathsf{Pr}}\big(E'(0, \iota(g_1, g_2^\vee), \phi_1 \otimes \phi_2)\big) - \widetilde{\mathsf{Pr}}\big(E'\big(\tfrac{1}{2}, g_1, \phi_1\big)E\big(\tfrac{1}{2}, g_2, \phi_2\big)\big) \\ &\qquad - \widetilde{\mathsf{Pr}}\big(E\big(\tfrac{1}{2}, g_1, \phi_1\big)E'\big(\tfrac{1}{2}, g_2, \phi_2\big)\big).\end{aligned}$$

It is easy to see that

$$\widetilde{\mathsf{Pr}}\big(E'\big(\tfrac{1}{2}, g_1, \phi_1\big)E\big(\tfrac{1}{2}, g_2, \phi_2\big)\big) = \widetilde{\mathsf{Pr}}\big(E'\big(\tfrac{1}{2}, g_1, \phi_1\big)\big)E_*\big(\tfrac{1}{2}, g_2, \phi_2\big),$$

where

$$E_*\big(\tfrac{1}{2}, g_i, \phi_i\big) = \sum_{d_i \in F^\times} W_{d_i}\big(\tfrac{1}{2}, g_i, \phi_i\big).$$

In particular, if $\phi_{1,v} \otimes \phi_{2,v} \in \mathscr{S}(\mathbb{V}_v^2)_{\mathsf{reg}}$ for at least one finite place v, then given g_1 and g_2 in $P_v' H'(\mathbb{A}_F^v)$, each $E_*(\frac{1}{2}, g_i, \phi_i)$, $i = 1, 2$, is a linear combination of an Eisenstein series and an automorphic character. In summary:

Proposition 6.5. *The holomorphic projection of the analytic kernel function is*

$$\begin{aligned}\mathsf{Pr}\big(E'(0, \iota(g_1, g_2^\vee), \phi_1 \otimes \phi_2)\big) &= \widetilde{\mathsf{Pr}}\big(E'(0, \iota(g_1, g_2^\vee), \phi_1 \otimes \phi_2)\big) \\ &\quad - \widetilde{\mathsf{Pr}}\big(E'\big(\tfrac{1}{2}, g_1, \phi_1\big)\big)E_*\big(\tfrac{1}{2}, g_2, \phi_2\big) - E_*\big(\tfrac{1}{2}, g_1, \phi_1\big)\widetilde{\mathsf{Pr}}\big(E'\big(\tfrac{1}{2}, g_2, \phi_2\big)\big).\end{aligned}$$

Quasiholomorphic projection of the analytic kernel function. Now we are going to compute the quasiholomorphic projection of $E'(0, \iota(g_1, g_2^\vee), \phi_1 \otimes \phi_2)$ under the following assumption:

(REG) $\phi_i = \phi_\infty^0 \phi_{i,f}$ *with* $\phi_{1,v} \otimes \phi_{2,v} \in \mathscr{S}(\mathbb{V}_v^2)_{\mathsf{reg}}$ *for all* $v \in \mathcal{S}$ *and* $\phi_{1,v} \otimes \phi_{2,v} \in \mathscr{S}(\mathbb{V}_v^2)_{\mathsf{reg}, d_v}$ *for* $v \in \mathcal{S}$ *nonsplit with* $d_v \geq d_{\psi_v}$*;* $g_i \in e_{\mathcal{S}} H'(\mathbb{A}_F^{\mathcal{S}})$.

Recall from (2-13) that

$$E'(0, \iota(g_1, g_2^\vee), \phi_1 \otimes \phi_2) = \sum_{v \notin \mathcal{S}} E_v'(0, \iota(g_1, g_2^\vee), \phi_1 \otimes \phi_2).$$

It is clear that when we apply $\widetilde{\mathsf{Pr}}$ to the above expression, nothing will change except the terms $E_\iota(0, \iota(g_1, g_2^\vee), \phi_1 \otimes \phi_2)$ for $\iota \in \Sigma_\infty$. Now we just fix one $\iota \in \Sigma_\infty$ and consider, by [Liu 2011, Theorem 4.20],

$$-2\,\mathsf{vol}(K)\langle (Z_{\phi_1}(g_1), \Xi_{\phi_1}(g_1)_{\iota^\circ}), (Z_{\phi_2}(g_2), \Xi_{\phi_2}(g_2)_{\iota^\circ})\rangle_{M_K},$$

which is (after forgetting the constant $-2\,\mathsf{vol}(K)$), by definition, the integration over the (complex) Shimura curve of

$$\left(\sum_{x_1\neq 0}\sum_{h_1\in H(\mathbb{Q})\backslash H(\mathbb{A}_f)/K}\omega_\chi(g_1)\phi_i(T(x_1),h_1^{-1}x_1)\,\Xi_{x_1a_1,h_1}\right) * \left(\sum_{x_2\neq 0}\sum_{h_2\in H(\mathbb{Q})\backslash H(\mathbb{A}_f)/K}\omega_\chi(g_2)\phi_2(T(x_2),h_2^{-2}x_2)\,\Xi_{x_2a_2,h_2}\right).$$

See [Liu 2011, Section 4C] for the notation. Since this expression and the process of taking $\widetilde{\mathsf{Pr}}$ is symmetric in g_1 and g_2, let us just do the first variable and hence omit the subscript 1 in the following calculation. Of course we only need to do this for the neutral component, hence we consider the current

$$\sum_{x\neq 0}\omega_\chi(g)\phi(T(x),x)\,\Xi_{xa}.$$

It is clear that for $T(x)=t\notin F^+$,

$$\widetilde{\mathsf{Pr}}(\omega_\chi(-)\phi(t,x)\,\Xi_{xa})_{\psi_{t'}}(g)=0$$

for any $t'\in F^+$ (which is just the set of all totally positive numbers in F), hence these terms vanish after applying $\widetilde{\mathsf{Pr}}$. For those x such that $T(x)=t\in F^+$, the corresponding term will contribute to the t-th Fourier coefficient in the quasiholomorphic projection. Namely,

$$\widetilde{\mathsf{Pr}}\left(\sum_{x\neq 0}\omega_\chi(g)\phi(T(x),x)\,\Xi_{xa}\right)=\sum_{t\in F^+}\widetilde{\mathsf{Pr}}\left(\sum_{T(x)=t}\omega_\chi(-)\phi(t,x)\,\Xi_{xa}\right)_{\psi_t}(g),$$

where, similar to (6-1),

$$\widetilde{\mathsf{Pr}}\left(\sum_{T(x)=t}\omega_\chi(-)\phi(T(x),x)\,\Xi_{xa}\right)_{\psi_t}(g)=\operatorname*{const}_{s\to 0}(4\pi t)W_t^{\mathfrak{k}\chi/2}(g_\iota) \times\sum_{T(x)=t}\int_{Z'(\mathbb{R})N'(\mathbb{R})\backslash H'(\mathbb{R})}\lambda_{P'}(h)^s\omega_\chi(g^\iota h)\phi(t,x)\,\Xi_{xa}\,dh, \tag{6-3}$$

where we identify F_ι with $\mathbb{R}$ in the domain of the integral and a is such that $h=n(b)m(a)k$ of h in the Iwasawa decomposition. Making the substitution $y=a\bar{a}$, we have

$$\begin{aligned}(6\text{-}3)&=\operatorname*{const}_{s\to 0}(4\pi t)W_t^{\mathfrak{k}\chi/2}(g_\iota)\sum_{T(x)=t}\omega_\chi(g^\iota)\phi^\iota(t,x)\int_0^\infty\Xi_{\sqrt{y}\cdot x}\,y^se^{-4\pi ty}\,dy\\ &=\operatorname*{const}_{s\to 0}(4\pi t)\sum_{T(x)=t}\omega_\chi(g)\phi(t,x)\int_0^\infty\Xi_{\sqrt{y}\cdot x}\,y^se^{-4\pi ty}\,dy.\end{aligned}\tag{6-4}$$

If we let $\delta_x(z)=R(x,z)/2t=-(x_z,x_z)/(x,x)$, then

$$
\begin{aligned}
\text{(6-4)} &= \operatorname*{const}_{s\to 0}(4\pi t)\sum_{T(x)=t}\omega_\chi(g)\phi(t,x)\int_0^\infty\biggl(\int_1^\infty \frac{e^{-4\pi t y u\delta_x(z)}}{u}\,du\biggr)y^s e^{-4\pi t y}\,dy\\
&= \operatorname*{const}_{s\to 0}(4\pi t)\sum_{T(x)=t}\omega_\chi(g)\phi(t,x)t^{-1-s}\int_0^\infty\int_1^\infty \frac{e^{-4\pi y u\delta_x(z)}}{u}y^s e^{-4\pi t y}\,du\,dy\\
&= \operatorname*{const}_{s\to 0}(4\pi t)\sum_{T(x)=t}\omega_\chi(g)\phi(t,x)t^{-1-s}\int_1^\infty \frac{1}{u}\biggl(\int_0^\infty e^{-4\pi y(1+u\delta_x(z))}\biggr)\,du\\
&= \operatorname*{const}_{s\to 0}(4\pi t)\sum_{T(x)=t}\omega_\chi(g)\phi(t,x)\frac{\Gamma(1+s)}{(4\pi t)^{1+s}}\int_1^\infty \frac{du}{u(1+u\delta_x(z))^{1+s}}\\
&= \operatorname*{const}_{s\to 1}\sum_{T(x)=t}\omega_\chi(g)\phi(t,x)\int_1^\infty \frac{du}{u(1+u\delta_x(z))^{s}}.
\end{aligned}
\tag{6-5}
$$

Admissible Green's function. As in [Gross and Zagier 1986], we introduce the Legendre function of the second type:

$$
Q_{s-1}(t)=\int_0^\infty (t+\sqrt{t^2-1}\cosh u)^{-s}\,du,\qquad t>1,\ s>0.
$$

Then the admissible Green's function attached to the divisor $\sum_{T(x)=t}\omega_\chi(g)\phi(t,x)Z_x$ (on the neutral component) is

$$
\Xi_\phi^{\mathrm{adm}}(g)_t=\operatorname*{const}_{s\to 1}2\sum_{T(x)=t}\omega_\chi(g)\phi(t,x)Q_{s-1}(1+2\delta_x(z)).
$$

By a result of Gross and Zagier, we have

$$
\int_1^\infty \frac{du}{u(1+uc)^s}=2Q_{s-1}(1+2c)+O(c^{-s-1}),\qquad c\to+\infty.
$$

Combining (6-5), Corollary 4.15, and Proposition 6.5, we have:

Proposition 6.6. *Under the assumptions (REG) for $\phi_1\otimes\phi_2$ and g_i, we have*

$$
\begin{aligned}
\mathrm{Pr}\bigl(E'(0,\iota(g_1,g_2^\vee),\phi_1\otimes\phi_2)\bigr) &= -\mathrm{vol}(K)\sum_{\substack{v^\circ\mid v\\ v\notin\mathcal{S}}}\langle\widehat{Z}_{\phi_1}(g_1),\widehat{Z}_{\phi_2}(g_2)\rangle_{v^\circ}\\
&\quad-\widetilde{\mathrm{Pr}}\bigl(E'(\tfrac12,g_1,\phi_1)\bigr)E_*(\tfrac12,g_2,\phi_2)-E_*(\tfrac12,g_1,\phi_1)\widetilde{\mathrm{Pr}}\bigl(E'(\tfrac12,g_2,\phi_2)\bigr),
\end{aligned}
$$

where at the archimedean places we are using admissible Green's functions.

6B. ***Uniqueness of local invariant functionals.*** We now fix a place $v\in\Sigma$ and suppress it from the notation. We prove that the space $\mathrm{Hom}_{H'\times H'}(I_2(0,\chi),\pi^\vee\boxtimes\chi\pi)$ is of dimension 1, following [Harris et al. 1996].

From χ, we have defined an automorphic character π_χ of H' in the following way. Given $g \in H'$, $\det g \in E^{\times,1}$; hence we can write $\det g = e_g/e_g^\tau$ for some $e_g \in E^\times$, by Hilbert's Theorem 90. Define $\pi_\chi(g) = \chi(e_g)$, which is well-defined since $\chi|_{F^\times} = 1$.

Proposition 6.7. *For an irreducible admissible representation $\pi \not\cong \pi_\chi^{-1}$ of H', we have* $\dim \mathrm{Hom}_{H'\times H'}(I_2(0,\chi), \pi^\vee \boxtimes \chi\pi) = 1$ *and $L(s,\pi,\chi)$ is holomorphic at $s = \frac{1}{2}$.*

First, we have a double coset decomposition

$$H'' = P\gamma_0\iota(H'\times H') \sqcup P\iota(H'\times H') =: \Omega_0 \sqcup \Omega_1$$

with Ω_0 open and Ω_1 closed. Hence we have a filtration $I_2(0,\chi) \supset I_2^{(0)}(0,\chi)$ where

$$I_2^{(0)}(0,\chi) = \{\varphi \in I_2(0,\chi) \mid \mathrm{Supp}\varphi \subset \Omega_0\},$$

which is invariant under the action of $H'\times H'$ by right translation through ι. As $H'\times H'$ representations, we have $Q_2^{(0)}(0,\chi) = I_2^{(0)}(0,\chi)$ and $Q_2^{(1)}(0,\chi) = I_2(0,\chi)/I_2^{(0)}(0,\chi)$. We have an $H'\times H'$ intertwining operator

$$\begin{aligned} Q_2^{(0)}(0,\chi) &\xrightarrow{\sim} \mathscr{S}(H')(1\otimes\chi) \\ \varphi &\longmapsto \Psi(g) = \varphi(\gamma_0\iota(g,\mathbf{1}_2)), \end{aligned}$$

where $\mathscr{S}(H')$ is the space of Schwartz functions on H', since

$$\varphi(\gamma_0\iota(g,\mathbf{1}_2)\iota(g_1,g_2)) = \varphi(\gamma_0\iota(g_2,g_2)\iota(g_2^{-1}gg_1,\mathbf{1}_2)) = \chi(\det g_2)\varphi(\gamma_0\iota(g_2^{-1}gg_1,\mathbf{1}_2)).$$

There is on $\mathscr{S}(H')\otimes(\pi\boxtimes\pi^\vee)$ a unique $H'\times H'$-invariant functional (up to a constant) given by

$$\Psi\otimes(f\otimes f^\vee) \mapsto \int_{H'} \langle \pi(g)f, f^\vee\rangle \Psi(g)\,dg.$$

But,

$$\begin{aligned} \mathrm{Hom}_{H'\times H'}(\mathscr{S}(H')\otimes(\pi\boxtimes\pi^\vee),\mathbb{C}) &= \mathrm{Hom}_{H'\times H'}(\mathscr{S}(H'),\pi^\vee\boxtimes\pi) \\ &= \mathrm{Hom}_{H'\times H'}(\mathscr{S}(H')\otimes(1\boxtimes\chi),\pi^\vee\boxtimes\chi\pi) \\ &= \mathrm{Hom}_{H'\times H'}(Q_2^{(0)}(0,\chi),\pi^\vee\boxtimes\chi\pi). \end{aligned}$$

For the representation $Q_2^{(1)}(0,\chi)$, we have:

Lemma 6.8. *If $\pi \not\cong \pi_\chi^{-1}$, then* $\mathrm{Hom}_{H'\times H'}(Q_2^{(1)}(0,\chi), \pi^\vee\boxtimes\chi\pi) = 0.$

Proof. We have an intertwining isomorphism $Q_2^{(1)}(0,\chi) \to I_1\left(\frac{1}{2},\chi\right) \boxtimes I_1\left(\frac{1}{2},\chi\right)$ given by

$$\bar{\varphi} \mapsto \left((g_1,g_2) \mapsto \bar{\varphi}(\iota(g_1,g_2))\right),$$

since $\bar{\varphi}(\iota(p_1g_1,p_2g_2)) = \bar{\varphi}(\iota(p_1,p_2)\iota(g_1,g_2)) = \chi(a_1a_2)|a_1a_2|_E\bar{\varphi}(\iota(g_1,g_2))$ where $p_1 = n(b_1)m(a_1)k_1$ and $p_2 = n(b_2)m(a_2)k_2$. Hence

$$\begin{aligned}\mathrm{Hom}_{H'\times H'}(Q_2^{(1)}(0,\chi), \pi^\vee \boxtimes \chi\pi) &= \mathrm{Hom}_{H'\times H'}\left(I_1\left(\tfrac{1}{2},\chi\right) \boxtimes I_1\left(\tfrac{1}{2},\chi\right), \pi^\vee \boxtimes \chi\pi\right)\\ &= \mathrm{Hom}_{H'\times H'}\left(\pi \boxtimes \chi^{-1}\pi^\vee, I_1\left(-\tfrac{1}{2},\chi^{-1}\right) \boxtimes I_1\left(-\tfrac{1}{2},\chi^{-1}\right)\right).\end{aligned}$$

By [Kudla and Sweet 1997, Theorem 1.2] for v finite nonsplit, [Kudla and Sweet 1997, Theorem 1.3] for v finite split, and [Lee 1994, Theorem 6.10 (1-b)] for v infinite, the only irreducible H'-submodule contained in $I_1(-\frac{1}{2},\chi^{-1})$ is isomorphic to π_χ^{-1}. The lemma follows by our assumption on π. $\square$

Proof of Proposition 6.7. The normalized zeta integral (2-1) has already defined a nonzero element in $\mathrm{Hom}_{H'\times H'}(I_2(0,\chi), \pi^\vee \boxtimes \chi\pi)$, so the dimension is at least 1. If it is higher than one, we can find a nonzero element in $\mathrm{Hom}_{H'\times H'}(I_2(0,\chi), \pi^\vee \boxtimes \chi\pi)$ whose restriction to $I_2^{(0)}(0,\chi)$ is zero since $\dim \mathrm{Hom}_{H'\times H'}(Q_2^{(0)}(0,\chi), \pi^\vee \boxtimes \chi\pi) = 1$. Then it defines a nonzero element in $\mathrm{Hom}_{H'\times H'}(Q_2^{(1)}(0,\chi), \pi^\vee \boxtimes \chi\pi)$ which is 0 by the above lemma. Hence $\dim \mathrm{Hom}_{H'\times H'}(I_2(0,\chi), \pi^\vee \boxtimes \chi\pi) = 1$.

For the L-factor part, the restriction of the normalized zeta integral to $I_2^{(0)}(0,\chi)$ is nonzero. But the original zeta integral has already been absolutely convergent at $s=0$ if $\varphi \in I_2^{(0)}(0,\chi)$, hence $L(s,\pi,\chi)$ cannot have a pole at $s=\frac{1}{2}$ since $b_2(s)$ is holomorphic and nonzero at $s=0$. $\square$

Remark 6.9. Proposition 6.7 is conjectured to be true for any n, s, χ, and irreducible admissible representation π. This is proved in [Harris et al. 1996] for π supercuspidal — more precisely, for π *not occurring in the boundary at the point* s, which exactly equivalent to the assumption of Lemma 6.8 when $n=1$.

6C. *Final proof.* In this section, we prove the main theorem by combining all the results we have obtained.

We need to compare the (holomorphic projection of the) analytic kernel function and the geometric kernel function defined in Section 3B. First, we still assume (REG) on $\phi_1 \otimes \phi_2$ and g_i. Let

$$\mathfrak{E}(g_1,g_2;\phi_1\otimes\phi_2) = \mathrm{Pr}\left(E'(0,\iota(g_1,g_2^\vee),\phi_1\otimes\phi_2)\right) - \mathbb{E}(g_1,g_2;\phi_1\otimes\phi_2),$$

which is in $\mathscr{A}(H'\times H',\chi^\circ)$. By (3-9), and Propositions 5.2, 5.7, 5.12, and 6.6, we have that the restriction of $\mathfrak{E}(g_1,g_2;\phi_1\otimes\phi_2)$ to the subset $[e_S H'(\mathbb{A}_F^S)]^2$ is equal to the sum of the following terms:

$$\begin{aligned}\mathfrak{E}_{\mathrm{I}}(g_1,g_2;\phi_1\otimes\phi_2)&=-E(g_1,\phi_1)A(g_2,\phi_2)-A(g_1,\phi_1)E(g_2,\phi_2)\\&\qquad\qquad-CE(g_1,\phi_1)E(g_2,\phi_2);\\ \mathfrak{E}_{\mathrm{II}}(g_1,g_2;\phi_1\otimes\phi_2)&=\sum_{\substack{\mathfrak{p}^\circ|\mathfrak{p}\\ \mathfrak{p}\in\mathcal{S}}}A_{(\mathfrak{p}^\circ)}(g_1,\phi_1)E(g_2,\phi_2);\\ \mathfrak{E}_{\mathrm{III}}(g_1,g_2;\phi_1\otimes\phi_2)&=\sum_{\substack{\mathfrak{p}^\circ|\mathfrak{p}\text{ split}\\ \mathfrak{p}\in\mathcal{S}}}E_{(\mathfrak{p}^\circ)}(g_1,\phi_1;g_2,\phi_2);\\ \mathfrak{E}_{\mathrm{IV}}(g_1,g_2;\phi_1\otimes\phi_2)&=\sum_{\substack{\mathfrak{p}^\circ|\mathfrak{p}\text{ nonsplit}\\ \mathfrak{p}\in\mathcal{S}}}\theta^{\mathsf{hor}}_{(\mathfrak{p}^\circ)}(\iota(g_1,g_2^\vee);\phi_1,\phi_2)+\theta^{\mathsf{ver}}_{(\mathfrak{p}^\circ)}(g_1,\phi_1;g_2,\phi_2);\\ \mathfrak{E}_{\mathrm{V}}(g_1,g_2;\phi_1,\phi_2)&=-\widetilde{\mathsf{Pr}}\big(E'\big(\tfrac12,g_1,\phi_1\big)\big)E_*\big(\tfrac12,g_2,\phi_2\big)\\&\qquad\qquad-E_*\big(\tfrac12,g_1,\phi_1\big)\widetilde{\mathsf{Pr}}\big(E'\big(\tfrac12,g_2,\phi_2\big)\big).\end{aligned}$$

Now given any cuspidal automorphic representation π of H' such that π_∞ is a discrete series of weight $(1-\mathfrak{k}^\chi/2,1+\mathfrak{k}^\chi/2)$ and $\epsilon(\pi,\chi)=-1$, for any $f\in\pi$ and $f^\vee\in\pi^\vee$, the integral

$$\int_{[P'_\infty e_{\mathcal{S}}H'(\mathbb{A}^{\mathcal{S}}_{f,F})]^2} f(g_1)f^\vee(g_2^\vee)\chi^{-1}(\det g_2)\mathfrak{E}_?(g_1,g_2;\phi_1\otimes\phi_2)=0$$

for $?=\mathrm{I},\mathrm{II},\mathrm{III},\mathrm{IV},\mathrm{V}$, since each term involves either Eisenstein series, automorphic characters, or theta series when restricted to $e_{\mathcal{S}}H'(\mathbb{A}^{\mathcal{S}}_F)$ which is dense in $H'(F)\backslash H'(\mathbb{A}_F)$! Hence we have

$$\begin{aligned}\int_{[H'(F)\backslash H'(\mathbb{A}_F)]^2} &f(g_1)f^\vee(g_2)\chi^{-1}(\det g_2)E'(0,\iota(g_1,g_2),\phi_1\otimes\phi_2)\\&=\int_{[H'(F)\backslash H'(\mathbb{A}_F)]^2} f(g_1)f^\vee(g_2)\chi^{-1}(\det g_2)\mathbb{E}(g_1,g_2^\vee;\phi_1\otimes\phi_2).\end{aligned}\tag{6-6}$$

Recall our definition of the geometric kernel function, which is

$$\mathbb{E}(g_1,g_2;\phi_1\otimes\phi_2)=\mathsf{vol}(K)\sum\langle\Theta_{\phi_1}(g_1),\Theta_{\phi_2}(g_2)\rangle^K_{\mathsf{NT}},$$

and we are using the Weil representation ω_χ in the formation of both $\Theta_{\phi_i}(g_i)$ $(i=1,2)$. If we now use $\omega_\chi^\vee$ to form the second and, to be consistent with the previous convention, write $\phi=\phi_1$ and $\phi^\vee=\phi_2$, then $\Theta_{\phi_2}(g_2^\vee)=\Theta_{\phi^\vee}(g_2)\chi(\det g_2)$.

Recall our definition of arithmetic theta lifting (with respect to the Weil representation ω_χ) in Section 3B:

$$\Theta^f_\phi=\int_{H'(F)\backslash H(\mathbb{A}_F)} f(g)\Theta_\phi(g)\,dg,\tag{6-7}$$

which is an element in $\mathrm{CH}^1(M)^0_{\mathbb{C}}$ and also $\Theta^{f^\vee}_{\phi^\vee}$ with respect to $\omega^\vee_\chi$, where $M = (M_K)_K$ is the projective system of (compactified) Shimura curves. For any K' under which ϕ and $\phi^\vee$ are invariant, the height pairing $\mathrm{vol}(K')\langle\Theta^f_\phi, \Theta^{f^\vee}_{\phi^\vee}\rangle^{K'}_{\mathrm{NT}}$ calculated on $M_{K'}$ is independent of K', where $\mathrm{vol}(K')$ is defined as before. Hence we can use $\langle\Theta^f_\phi, \Theta^{f^\vee}_{\phi^\vee}\rangle_{\mathrm{NT}}$ to denote this number.

Recall that we have a totally positive-definite incoherent hermitian space $\mathbb{V}(\pi,\chi)$. We now prove our main theorem:

Theorem 6.10 (Arithmetic inner product formula). *Let π and χ be as above and $\mathbb{V}$ any totally positive-definite incoherent hermitian space over $\mathbb{A}_E$ of rank* 2. *Then*

(1) *If $\mathbb{V} \not\cong \mathbb{V}(\pi,\chi)$, then the arithmetic theta lifting $\Theta^f_\phi = 0$ for any $f \in \pi$ and $\phi \in \mathscr{S}(\mathbb{V})^{\mathrm{U}_\infty}$;*

(2) *If $\mathbb{V} \cong \mathbb{V}(\pi,\chi)$, then for any $f \in \pi$, $f^\vee \in \pi^\vee$, and any $\phi, \phi^\vee \in \mathscr{S}(\mathbb{V})^{\mathrm{U}_\infty}$ decomposable, we have*

$$\langle\Theta^f_\phi, \Theta^{f^\vee}_{\phi^\vee}\rangle_{\mathrm{NT}} = \frac{L'(\frac{1}{2},\pi,\chi)}{L_F(2)L(1,\epsilon_{E/F})}\prod_v Z^*(0,\chi_v,f_v,f^\vee_v,\phi_v\otimes\phi^\vee_v),$$

where almost all normalized zeta integrals (see Section 2A) appearing in the product are 1.

Proof. We first prove (2). Recall that in Section 2A, we defined the functional

$$\alpha(f,f^\vee,\phi,\phi^\vee) = \prod_v Z^*(0,\chi_v,f_v,f^\vee_v,\phi_v\otimes\phi^\vee_v)$$

in the space $\bigotimes \mathrm{Hom}_{H'_v\times H'_v}(R(\mathbb{V}_v,\chi_v), \pi^\vee_v\boxtimes\chi_v\pi_v)$ which is nonzero since $\mathbb{V}\cong\mathbb{V}(\pi,\chi)$. By Proposition 2.8 for $v\in\mathcal{S}$ and the fact that π_∞ is a discrete series representation of weight $(1-\mathfrak{k}^\chi/2, 1+\mathfrak{k}^\chi/2)$, we can choose local components f_v and $f^\vee_v$ for all $v\in\Sigma$ and ϕ_v and $\phi^\vee_v$ for $v\in\Sigma_f$ such that $\phi\otimes\phi^\vee$ satisfies the assumption (REG) and $\alpha(f,f^\vee,\phi,\phi^\vee)\neq 0$. On the other hand, the functional

$$\gamma(f,f^\vee,\phi,\phi^\vee) := \langle\Theta^f_\phi, \Theta^{f^\vee}_{\phi^\vee}\rangle_{\mathrm{NT}} = \mathrm{vol}(K)\langle\Theta^f_\phi, \Theta^{f^\vee}_{\phi^\vee}\rangle^K_{\mathrm{NT}}$$

is also in $\bigotimes_v \mathrm{Hom}_{H'_v\times H'_v}(R(\mathbb{V}_v,\chi_v), \pi^\vee_v\boxtimes\chi_v\pi_v)$ whose dimension is 1 according to Proposition 6.7. Hence we know that the ratio γ/α is a constant. By our special choice of f, ϕ, and $\phi^\vee$ and by (6-7), (6-6), and (2-3), we have

$$\frac{\gamma}{\alpha} = \frac{L'(\frac{1}{2},\pi,\chi)}{L_F(2)L(1,\epsilon_{E/F})}.$$

Hence

$$\langle\Theta^f_\phi, \Theta^{f^\vee}_{\phi^\vee}\rangle_{\mathrm{NT}} = \frac{L'(\frac{1}{2},\pi,\chi)}{L_F(2)L(1,\epsilon_{E/F})}\prod_v Z^*(0,\chi_v,f_v,f^\vee_v,\phi_v\otimes\phi^\vee_v)$$

for *any* $f\in\pi$, $f^\vee\in\pi^\vee$, and $\phi,\phi^\vee\in\mathscr{S}(\mathbb{V}_f)^{\mathrm{U}_\infty}$.

For (1), the functional γ is zero since $\mathbb{V} \ncong \mathbb{V}(\pi, \chi)$. If we take $\phi^\vee = \bar{\phi}$ and $f^\vee = \bar{f}$, then $\Theta_\phi^f = 0$ since the Néron–Tate height pairing on curves is positive definite. □

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