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Involutions, weights and p-local structure

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We prove that for an odd prime p, a finite group G with no element of order $2p$ has a p-block of defect zero if it has a non-Abelian Sylow p-subgroup or more than one conjugacy class of involutions. For $p=2$, we prove similar results using elements of order 3 in place of involutions. We also illustrate (for an arbitrary prime p) that certain pairs (Q, y), with a p-regular element y and Q a maximal y-invariant p-subgroup, give rise to p-blocks of defect zero of $N_G(Q)/Q$, and we give lower bounds for the number of such blocks which arise. This relates to the weight conjecture of J. L. Alperin.

Introduction

Involutions have played a crucial role in finite group theory for many decades. They also figure prominently in representation theory, both ordinary and modular. Examples of the former include their occurrence in finite reflection groups, and an example of the latter is that in characteristic 2, J. Murray proved in [2006] that the projective summands of the (characteristic 2) permutation module (under conjugation action) on the solutions of $x^2 = 1$ in G are (in bijection with) the real 2-blocks of defect zero.

Involutions also influence representation theory in odd characteristic. It was proved by Brauer and Fowler in [1955] that when p is an odd prime, G has a p-block of defect zero if there is an involution $t \in G$ that neither inverts nor centralizes any nontrivial p-element of G. This result was extended by T. Wada [1977], who proved that if there are r mutually nonconjugate involutions of G that neither invert nor centralize any nontrivial p-element of G, then G has at least r distinct p-blocks of defect zero. We prove here that when $p = 2$, elements of order 3 can play a role analogous to that played when p is odd by involutions in the results above: We prove that the number of 2-blocks of defect zero of G is at least as great as the number of conjugacy classes of elements of order 3 that normalize no nontrivial 2-subgroup of G.

We also point out here that results of this nature can be combined with local group-theoretic analysis to prove that if p is an odd prime and G is a group without

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elements of order $2p$, then G has a p-block of defect zero if it has more than one conjugacy class of involutions (we prove a more precise result without using the classification of finite simple groups, which could be sharpened even further by using that classification).

In a different direction, the celebrated weight conjecture of J. L. Alperin (in its nonblockwise version) defines (for a fixed prime p) a *weight* of G (up to conjugacy) as a pair (Q, S), where Q is a p-subgroup of G and S is an absolutely simple projective $N_G(Q)/Q$ module in characteristic p. Alperin's weight conjecture then asserts that the number of nonconjugate weights of G for p should be the number of conjugacy classes of p-regular elements of G (which is also the number of isomorphism types of absolutely simple modules for G in characteristic p). At present, there seems to be no reason to expect a natural bijection between weights and p-regular conjugacy classes, or between weights and characteristic p simple modules for G (though it is impossible to preclude the possibility that one or the other might emerge in future). Relatively few purely group-theoretic criteria are known to date that place nonconjectural bounds on the number of weights. We give some group-theoretic conditions of this nature that place lower bounds on the number of weights, using sharpenings of results of Brauer and Fowler [1955], Tsushima [1977] and Wada [1977], going somewhat further than my results in [Robinson 1983], and incorporating the result about 2-blocks of defect zero and elements of order 3 that normalize no nontrivial 2-subgroup of G.

A naïve attempt at associating p-regular classes with weights of G might be to consider a p-regular element y and a maximal y-invariant p-subgroup Q. Then y normalizes no nontrivial p-subgroup of $N_G(Q)/Q$ and it might be hoped that a p-block of defect zero of $N_G(Q)/Q$ could be naturally associated to y (or yQ). More ambitiously, it might be hoped that weights could be parametrized in terms of conjugacy classes of pairs (Q, y), where y is a p-regular element of G and Q is a maximal y-invariant p-subgroup of G.

However, there are usually more conjugacy classes of such pairs (Q, y) than there are simple modules. The number of conjugacy classes of such pairs (Q, y) is equal to the number of simple modules precisely when $C_G(y)$ transitively permutes the maximal y-invariant p-subgroups of G for each p-regular $y \in G$. In general, this need not be the case. For example, when $p = 3$ and $G \cong \mathrm{PSL}(2, 11)$ we may take y to be an involution. There is a Sylow 3-subgroup Q of G that is centralized by y, and there is another Sylow 3-subgroup R of G whose nonidentity elements are inverted by y. Clearly Q and R are not conjugate via an element of $C_G(y)$.

We are nevertheless interested in pairs (Q, y), where y is p-regular and Q is a maximal y-invariant p-subgroup, and we will point out some instances where they give rise to weights.

Lemma 1. (i) *Let Q be a p-subgroup of G and y be a p-regular element of $N_G(Q)$ such that $yQ \in O_{p'}(N_G(Q)/Q)$. Then Q is a maximal y-invariant p-subgroup of G if and only if $C_Q(y) \in \mathrm{Syl}_p(N_G(Q) \cap C_G(y))$.*

(ii) *Suppose that p is odd, and let Q be a p-subgroup of G and y be an involution of $N_G(Q)$. Then Q is a maximal y-invariant p-subgroup of G if and only if yQ neither inverts nor centralizes any element of order p in $N_G(Q)/Q$.*

(iii) *Suppose that $p = 2$, and let Q be a 2-subgroup of G and y be an element of order 3 in $N_G(Q)$. Then Q is a maximal y-invariant 2-subgroup of G if and only if yQ is not contained in any subgroup isomorphic to A_4 of $N_G(Q)/Q$, and yQ does not centralize any involution of $N_G(Q)/Q$.*

Proof. (i) Notice that Q is a maximal y-invariant p-subgroup of G if and only if Q is a maximal y-invariant p-subgroup of $N_G(Q)$, for if $Q < R$ and R is another y-invariant p-subgroup of G, then $Q < N_R(Q)$ and $N_R(Q)$ is y-invariant. Hence we may suppose that $Q \lhd G$, and do so. Set $\overline{G} = G/Q$, and so on. Then $\overline{C_G(y)} = C_{\overline{G}}(\bar{y})$ since y is p-regular and Q is a p-group. Since $\bar{y} \in O_{p'}(\overline{G})$, we see that $\bar{y}$ centralizes any p-subgroup of $\overline{G}$ that it normalizes. Hence Q is a maximal y-invariant p-subgroup of G if and only if $\bar{y}$ normalizes no nontrivial p-subgroup of $\overline{G}$, if and only if $\bar{y}$ centralizes no nontrivial p-subgroup of $\overline{G}$, if and only if $C_Q(y) \in \mathrm{Syl}_p(C_G(y))$.

(ii) Again we may suppose that $Q \lhd G$ and we set $\overline{G} = G/Q$. If $\bar{y}$ inverts or centralizes an element of order p in $\overline{G}$, then Q is clearly not a maximal y-invariant p-subgroup of G. On the other hand, if $\bar{y}$ normalizes a nontrivial p-subgroup of $\overline{G}$, then $\bar{y}$ normalizes a nontrivial Abelian p-subgroup $\overline{A}$, say. We have $\overline{A} = [\overline{A}, \bar{y}] \times C_{\overline{A}}(\bar{y})$, so that $\bar{y}$ must either centralize or invert a nonidentity element of $\overline{A}$.

(iii) The proof of this part is analogous to part (ii), except that in the final step, $\overline{A}$ may be chosen to be elementary Abelian, and $[\overline{A}, \bar{y}]$ is a direct product of $\bar{y}$-invariant Klein 4-groups, each acted on by $\bar{y}$ without nontrivial fixed points. □

Definition. When p is a prime and G is a finite group, a pair (Q, x) is called a *pseudoweight* for G if x is a p-regular element of G, Q is a maximal x-invariant p-subgroup of G, and one or more of the following occurs:

(i) $xQ \in O_{p'}(N_G(Q)/Q)$.

(ii) p is odd and x is an involution.

(iii) $p = 2$ and x has order 3.

Remark. It is easy to check that $(Q, 1)$ is a pseudoweight for G if and only if $Q \in \mathrm{Syl}_p(G)$, so there is a unique conjugacy class of pseudoweights with second component 1_G. When Q is a Sylow p-subgroup of G, notice that the number of nonconjugate pseudoweights with first component Q is the number of conjugacy classes of p-regular elements of $N_G(Q)$, since $N_G(Q)/Q$ is a p'-group. If p is odd, every involution occurs as the second component of at least one pseudoweight, since whenever t is an involution, there is at least one maximal t-invariant p-subgroup of G (which may be trivial). Similarly, if $p = 2$, then every element of order 3 occurs as the second component of at least one pseudoweight.

Before our first result, we recall some results of [Murray 1999; Robinson 1983]. Let P be a Sylow p-subgroup of G. In [Robinson 1983], it is proved that the number of p-blocks of defect zero is the rank of a matrix S with entries in GF(p) defined as follows: The rows and columns of S are indexed by the conjugacy classes of p-regular elements y of G such that $C_G(y)$ is a p'-group. The (i, j)-entry of S is s_{ij}, which is the residue (mod p) of $|\Omega_{ij}|/|P|$, where Ω_{ij} is the set of $(u, v) \in C_i \times C_j$ such that $u^{-1}v \in P$, where C_i is the i-th conjugacy class of p-regular elements of p-defect zero. This is refined by [Murray 1999, 6.3], which shows that Ω_{ij} may be replaced by $\tilde{\Omega}_{ij}$, which is obtained by only counting ordered pairs (u, v) such that $u^{-1}v$ is an element of P of order at most p, and we may use $\tilde{S}$ in place of S, where $\tilde{s}_{ij}$ is the residue (mod p) of $|\tilde{\Omega}_{ij}|/|P|$. We will see that, when $p = 2$, this refinement is advantageous.

Theorem 2. *For each p-subgroup Q of G, the number of conjugacy classes of weights of G with first component conjugate to Q is greater than or equal to the number of conjugacy classes of pseudoweights of G with first component conjugate to Q.*

Proof. First note that G permutes its pseudoweights by conjugation. For each p-subgroup of G, the G-conjugate pseudoweights with first component Q correspond bijectively to the $N_G(Q)/Q$-conjugacy classes of pseudoweights with trivial first component, since there is a bijection between p-regular conjugacy classes of $N = N_G(Q)$ and p-regular conjugacy classes of N/Q. Hence it suffices to prove that the number of p-blocks of defect zero is at least the number of conjugacy classes of pseudoweights with trivial first component.

Let $(1, x_1), \dots, (1, x_d)$ be representatives for the conjugacy classes of pseudoweights of G with trivial first component. Then no x_i normalizes any nontrivial p-subgroup of G.

Let us label so that $x_i \in C_i$ for $1 \leq i \leq d$. We show that the first $d \times d$ minor of $\tilde{S}$ is an invertible diagonal matrix, so that $\tilde{S}$ has rank at least d. For if $1 \leq i, j \leq d$, and u is conjugate to x_i and v is conjugate to x_j with $u^{-1}v \in P$ of order at most p, then $u^{-1}v$ is p-regular (if u or v is in $O_{p'}(G)$ this is clear). If p is odd and u and v are both involutions that invert no element of order p, then $u^{-1}v$ must be p-regular. If $p = 2$ and u and v are both elements of order 3 that normalize no nontrivial 2-subgroup of G and $u^{-1}v$ is an involution, then $\langle u, v\rangle \cong A_4$ and u is conjugate to v within $\langle u, v\rangle$, a contradiction. Hence $u^{-1}v$ is p-regular in all cases, (so is the identity, as P is a p-group). Thus $\tilde{s}_{ij} = 0$ for $i \neq j$ and $1 \leq i, j \leq d$. Also, $\tilde{s}_{ii}$ is the residue (mod p) of $|C_i|/|P|$ for $1 \leq i \leq d$. Thus $\tilde{s}_{ii} \neq 0$ for $1 \leq i \leq d$, as required to complete the proof. □

Because of its analogy with the result of Brauer and Fowler [1955] mentioned previously, we single out for special mention this:

Corollary 3. *Let G be a finite group of order divisible by* 6. *If G contains an element of order* 3 *that normalizes no nontrivial* 2*-subgroup of G, then G has a* 2*-block of defect zero. More precisely, the number of* 2*-blocks of defect zero is greater than or equal to the number of conjugacy class of elements of order* 3 *of G that normalize no nontrivial* 2*-subgroup of G.*

We now combine some local-group theoretic analysis with the block-theoretic results we have used.

Theorem 4. *Let G be a finite group of even order that contains no element of order $2p$ for some odd prime p. Then either G has a p-block of defect zero or else G has Abelian Sylow p-subgroups and a unique conjugacy class of involutions. Furthermore, if G has no p-block of defect zero, and has Sylow p-subgroups of rank at least* 3, *then either $G/O_{\{2,p\}'}(G)$ has a normal Sylow p-subgroup or else G has a strongly p-embedded subgroup.*

Proof. Suppose that G has no p-block of defect zero. Set $\pi = \{2, p\}$. To prove the theorem, it suffices to consider the case that $O_{\pi'}(G) = 1$. By the result of Brauer and Fowler mentioned earlier, every involution of G inverts an element of order p, as G has no element of order $2p$. Also, since G contains no element of order $2p$, no section of G is isomorphic to SL(2, p), so that, by a theorem of Glauberman [1968], $N = N_G(ZJ(P))$ controls strong fusion in G for $P \in \mathrm{Syl}_p(G)$. Thus N must have even order, as some element of order p is conjugate to its inverse in G.

Since G contains no element of order $2p$, the Sylow 2-subgroups of N must be cyclic or generalized quaternion, since if there were a Klein 4-subgroup, V say, of N, then each involution of V would invert every element of $ZJ(P)$, which is a contradiction since the product of any two involutions that invert all of $ZJ(P)$ centralizes $ZJ(P)$. Hence N has a unique conjugacy class of involutions and, by the Brauer–Suzuki theorem, $N = O_{2'}(N)C_N(t)$ for t any involution of N. Thus $O_{2'}(N)$ contains P as $C_N(t)$ is a p'-group. We may suppose that P is t-invariant, so that P is Abelian as t acts without nontrivial fixed-points on P. We wish to prove that G has a unique conjugacy class of involutions. Let u be an involution of G. Then, replacing u by a conjugate if necessary, we may suppose that u inverts an element h of order p in P. Then $N_G(\langle h\rangle) = C_G(h)N_N(\langle h\rangle)$ so that u is conjugate within $N_G(\langle h\rangle)$ to an involution of $N_N(\langle h\rangle)$ since $C_G(h)$ has odd order. In particular, u is conjugate in G to an involution of N. This completes the proof of the first claim, as N has one conjugacy class of involutions.

For the second claim, set $A = \Omega_1(P)$, and suppose that $|A| \geq p^3$. For each $a \in A^{\#}$, we know that $C_G(a)$ has odd order by hypothesis, and so is solvable. Thus $C_G(a) = C_N(a)O_{p'}(C_G(a))$ for each such a. If $O_{p'}(C_G(a)) = 1$ for each such a, then either N is strongly p-embedded in G or else $P \lhd G$ (for if $O_p(G) \neq 1$, then $G = O_{2'}(G)C_G(t)$ for t an involution, and $O_{2'}(G)$ has a normal Sylow p-subgroup since $O_{\pi'}(G) = 1$). Otherwise, by the solvable signalizer functor theorem [Glauberman 1976],

$$\theta(A) = \langle O_{p'}(C_G(a)) : a \in A^{\#} \rangle$$

is a solvable π'-group. Then $M = N_G(\theta(A)) < G$. Now

$$N = N_G(P) \leq N_G(A) \leq M.$$

Also, for each $a \in A^{\#}$, we have $C_G(a) = C_N(a)O_{p'}(C_G(a)) \leq M$. For each nontrivial subgroup B of P, we have

$$N_G(B) \leq N_G(\Omega_1(B)) = C_G(\Omega_1(B))N_N(\Omega_1(B)) \leq M.$$

Thus M is strongly p-embedded in this case. □

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