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Coleman maps and the p-adic regulator

Antonio Lei, David Loeffler and Sarah Livia Zerbes

We study the Coleman maps for a crystalline representation V with non-negative Hodge–Tate weights via Perrin-Riou's p-adic "regulator" or "expanded logarithm" map $\mathscr{L}_V$. Denote by $\mathscr{H}(\Gamma)$ the algebra of $\mathbb{Q}_p$-valued distributions on $\Gamma = Gal(\mathbb{Q}_p(\mu_{p^\infty})/\mathbb{Q}_p)$. Our first result determines the $\mathscr{H}(\Gamma)$-elementary divisors of the quotient of $\mathbb{D}_{\text{cris}}(V)\otimes(\mathbb{B}^+_{\text{rig},\mathbb{Q}_p})^{\psi=0}$ by the $\mathscr{H}(\Gamma)$-submodule generated by $(\varphi^*\mathbb{N}(V))^{\psi=0}$, where $\mathbb{N}(V)$ is the Wach module of V. By comparing the determinant of this map with that of $\mathscr{L}_V$ (which can be computed via Perrin-Riou's explicit reciprocity law), we obtain a precise description of the images of the Coleman maps. In the case when V arises from a modular form, we get some stronger results about the integral Coleman maps, and we can remove many technical assumptions that were required in our previous work in order to reformulate Kato's main conjecture in terms of cotorsion Selmer groups and bounded p-adic L-functions.

1. Introduction

1A. ***Background.*** Let p be an odd prime, and write $\mathbb{Q}_\infty = \mathbb{Q}(\mu_{p^\infty})$. Define the Galois groups $\Gamma = \mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q})$ and $\Gamma_1 = \mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q}(\mu_p))$. Note that $\Gamma \cong \Delta \times \Gamma_1$, where Δ is cyclic of order $p-1$ and $\Gamma_1 \cong \mathbb{Z}_p$. For $H \in \{\Gamma, \Gamma_1\}$, denote by $\Lambda(H)$ the Iwasawa algebra of H, and $\Lambda_{\mathbb{Q}_p}(H) = \Lambda(H) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.

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Let V be a crystalline representation of $\mathcal{G}_{\mathbb{Q}_p}$ of dimension d with non-negative Hodge–Tate weights. (We adopt the convention that the cyclotomic character has Hodge–Tate weight 1, so this condition is equivalent to $\mathrm{Fil}^1 \mathbb{D}_{\mathrm{cris}}(V) = 0$.) We define

$$H^1_{\mathrm{Iw}}(\mathbb{Q}_p, V) := \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \varprojlim_n H^1(\mathbb{Q}(\mu_{p^n}), T),$$

where T is a $\mathcal{G}_{\mathbb{Q}_p}$-stable $\mathbb{Z}_p$-lattice in V. This is a $\Lambda_{\mathbb{Q}_p}(\Gamma)$-module independent of the choice of T. In [Lei et al. 2010], we construct $\Lambda(\Gamma)$-homomorphisms (called the Coleman maps)

$$\underline{\mathrm{Col}}_i : H^1_{\mathrm{Iw}}(\mathbb{Q}_p, V) \longrightarrow \Lambda_{\mathbb{Q}_p}(\Gamma)$$

for $i = 1, \dots, d$, depending on a choice of basis of the Wach module $\mathbb{N}(V)$. In the case when $V = V_f(k-1)$, where $f = \sum a_n q^n$ is a modular eigenform of weight $k \geq 2$ and level coprime to p (we assume that $a_n \in \mathbb{Q}$ for the time being in order to simplify notation) and V_f is the 2-dimensional p-adic representation associated to f by Deligne, these maps have two important applications. Firstly, we can define two p-adic L-functions $L_{p,1}, L_{p,2} \in \Lambda_{\mathbb{Q}_p}(\Gamma)$ on applying the Coleman maps to the localisation of the Kato zeta element as constructed in [Kato 2004]. In the supersingular case, i.e., when $p \mid a_p$, this enables us to obtain a decomposition of the p-adic L-functions defined in [Amice and Vélu 1975], which are not elements of $\Lambda_{\mathbb{Q}_p}(\Gamma)$ but of the distribution algebra $\mathcal{H}(\Gamma)$. More precisely, we show that there exists a 2×2 matrix $\mathcal{M} \in M(2, \mathcal{H}(\Gamma_1))$ depending only on k and a_p such that

$$\begin{pmatrix} L_{p,\alpha} \\ L_{p,\beta} \end{pmatrix} = \mathcal{M} \begin{pmatrix} L_{p,1} \\ L_{p,2} \end{pmatrix}.$$

This generalises the results of [Pollack 2003] (when $a_p = 0$) and [Sprung 2009] (when f corresponds to an elliptic curve over $\mathbb{Q}$ and $p = 3$). Secondly, by modifying the local conditions at p in the definition of the p-Selmer group using the kernels of the maps $\underline{\mathrm{Col}}_i$, we define two new Selmer groups $\mathrm{Sel}^i_p(f/\mathbb{Q}_\infty)$. These are both $\Lambda(\Gamma)$-cotorsion, which is not true of the usual Selmer group in the supersingular case.

Fixing a character η of Δ and restricting to the η-isotypical component, we get maps

$$\underline{\mathrm{Col}}^\eta_i : H^1_{\mathrm{Iw}}(\mathbb{Q}_p, V)^\eta \to \Lambda_{\mathbb{Q}_p}(\Gamma_1).$$

Via the Poitou-Tate exact sequence, we can reformulate Kato's main conjecture (after tensoring with $\mathbb{Q}_p$) as follows:

Conjecture 1.1. *For $i = 1, 2$, and each character η of Δ,*

$$\mathrm{Char}_{\Lambda_{\mathbb{Q}_p}(\Gamma_1)}\big(\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathrm{Sel}^i_p(f/\mathbb{Q}_\infty)^{\eta,\vee}\big) = \mathrm{Char}_{\Lambda_{\mathbb{Q}_p}(\Gamma_1)}\big(\mathrm{Im}(\underline{\mathrm{Col}}^\eta_i)/(L^\eta_{p,i})\big),$$

where $M^\vee$ denotes the Pontryagin dual of a $\Lambda(\Gamma_1)$-module M and $\mathrm{Char}_{\Lambda_{\mathbb{Q}_p}(\Gamma_1)} M$ *denotes the $\Lambda_{\mathbb{Q}_p}(\Gamma_1)$-characteristic ideal of M.*

When $v_p(a_p)$ is sufficiently large, we make use of the basis of $\mathbb{N}(V)$ constructed in [Berger et al. 2004] to show that the first Coleman map is surjective under some additional technical conditions. Therefore, we can rewrite Conjecture 1.1 as follows (see [Lei et al. 2010, Corollary 6.6]):

Theorem 1.2. *Under certain technical conditions, the case $i = 1$ in Conjecture 1.1 is equivalent to the assertion that* $\mathrm{Char}_{\Lambda_{\mathbb{Q}_p}(\Gamma_1)}\big(\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathrm{Sel}^1_p(f/\mathbb{Q}_\infty)^{\eta,\vee}\big)$ *is generated by $L^\eta_{p,1}$.*

(In fact we can show that this equivalence holds integrally, i.e., without tensoring with $\mathbb{Q}_p$.)

1B. *Main results.* In this paper, we extend the above results in several ways. Let V be a crystalline representation of $\mathcal{G}_{\mathbb{Q}_p}$ of dimension d with non-negative Hodge–Tate weights. We make the following assumption:

Assumption 1.3. *The representation V admits at least one non-critical refinement, after a suitable extension of coefficients.*

See Section 1C5 below for the definition of a non-critical refinement. For now, let it suffice to say that this assumption holds for all 2-dimensional representations, and conjecturally for all representations "arising from geometry".

We identify $\Lambda(\Gamma_1)$ with the power series ring $\mathbb{Z}_p[\![X]\!]$, where $X = \gamma - 1$ for a topological generator γ of Γ_1. Denote by $\chi : \mathcal{G}_{\mathbb{Q}_p} \to \mathbb{Z}_p^\times$ the cyclotomic character.

Firstly, we study the structure of $\mathbb{N}_{\mathrm{rig}}(V) := \mathbb{N}(V) \otimes_{\mathbb{B}^+_{\mathbb{Q}_p}} \mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p}$ as a Γ-module. If $\varphi^*\mathbb{N}_{\mathrm{rig}}(V)$ denotes the $\mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p}$-span of $\varphi(\mathbb{N}_{\mathrm{rig}}(V))$, then $(\varphi^*\mathbb{N}_{\mathrm{rig}}(V))^{\psi=0}$ is contained in $(\mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p})^{\psi=0} \otimes_{\mathbb{Q}_p} \mathbb{D}_{\mathrm{cris}}(V)$, and both are free $\mathcal{H}(\Gamma)$-modules of rank equal to $d = \dim_{\mathbb{Q}_p} V$. We determine the elementary divisors of the quotient of these modules:

Theorem A (Theorem 2.10). *The $\mathcal{H}(\Gamma)$-elementary divisors of the quotient*

$$\mathbb{D}_{\mathrm{cris}}(V) \otimes_{\mathbb{Q}_p} \mathcal{H}(\Gamma)/(\varphi^*\mathbb{N}_{\mathrm{rig}}(V))^{\psi=0}$$

are $\mathfrak{n}_{r_1}, \dots, \mathfrak{n}_{r_d}$, where $r_1, \dots, r_d$ are the Hodge–Tate weights of V and

$$\mathfrak{n}_k = \frac{\log(1+X)}{X} \cdots \frac{\log(\chi(\gamma)^{1-k}(1+X))}{X - \chi(\gamma)^{k-1} + 1}.$$

This can be seen as a $\mathcal{H}(\Gamma)$-module analogue of [Berger 2004, Proposition III.2.1], which states that the $\mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p}$-elementary divisors of the quotient

$$(\mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p} \otimes_{\mathbb{Q}_p} \mathbb{D}_{\mathrm{cris}}(V))/\mathbb{N}_{\mathrm{rig}}(V)$$

are $\left(\frac{t}{\pi}\right)^{r_1}, \ldots, \left(\frac{t}{\pi}\right)^{r_d}$. It is striking to note that for any $k \geq 0$, the Mellin transform of $\mathfrak{n}_k$ agrees with $(1+\pi)\varphi\left(\frac{t}{\pi}\right)^k$ up to a unit in $\mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p}$ (see Proposition 1.6).

The second aim of this paper is to use Theorem A to determine the image of the map

$$1-\varphi : \mathbb{N}_{\mathrm{rig}}(V)^{\psi=1} \to (\varphi^*\mathbb{N}_{\mathrm{rig}}(V))^{\psi=0}.$$

To do this, we make use of the following commutative diagram of $\mathcal{H}(\Gamma)$-modules:

$$\begin{tikzcd}
\mathbb{N}(V)^{\psi=1} \arrow[rr, "\cong", "h^1_{\mathrm{Iw},V}"'] \arrow[d, "1-\varphi"] & & H^1_{\mathrm{Iw}}(V) \arrow[dd, "\mathcal{L}_V"] \\
(\varphi^*\mathbb{N}_{\mathrm{rig}}(V))^{\psi=0} \arrow[d, hook] & & \\
\mathbb{D}_{\mathrm{cris}}(V) \otimes_{\mathbb{Q}_p} (\mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p})^{\psi=0} \arrow[rr, "1\otimes\mathfrak{M}^{-1}"] & & \mathbb{D}_{\mathrm{cris}}(V) \otimes_{\mathbb{Q}_p} \mathcal{H}(\Gamma).
\end{tikzcd}$$

Here the map $\mathcal{L}_V$ is Perrin-Riou's "regulator" or "expanded logarithm" map (see [Perrin-Riou 1995]), which is a dual version of the more familiar exponential maps $\Omega_{V,h}$ appearing in [Perrin-Riou 1994]; and

$$\mathfrak{M} : \mathcal{H}(\Gamma) \xrightarrow{\cong} (\mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p})^{\psi=0}$$

denotes the Mellin transform. The commutativity of the diagram is a theorem of Berger [2003, Theorem II.13]. Colmez's proof of the "δ_V-conjecture" (see [Colmez 1998, Theorem IX.4.4]), which is part of Perrin-Riou's explicit reciprocity law, gives a formula for the determinant of the matrix of $\mathcal{L}_V$ (up to units). We can compare this with the determinant of the bottom left-hand map, which follows from Theorem A, to deduce that $1-\varphi : \mathbb{N}_{\mathrm{rig}}(V)^{\psi=1} \to (\varphi^*\mathbb{N}_{\mathrm{rig}}(V))^{\psi=0}$ is surjective up to a small error term:

Theorem B (Corollary 4.13). *Suppose that no eigenvalue of φ on $\mathbb{D}_{\mathrm{cris}}(V)$ lies in $p^{\mathbb{Z}}$. Then for each character η of Δ, there is a short exact sequence of $\mathcal{H}(\Gamma_1)$-modules*

$$0 \longrightarrow \mathbb{N}(V)^{\psi=1,\eta} \xrightarrow{1-\varphi} (\varphi^*\mathbb{N}(V))^{\psi=0,\eta} \xrightarrow{A_\eta} \bigoplus_{i=0}^{r_d-1} (\mathbb{D}_{\mathrm{cris}}(V)/V_{i,\eta})(\chi^i\chi_0^{-i}\eta) \longrightarrow 0.$$

Here $V_{i,\eta}$ is a subspace of $\mathbb{D}_{\mathrm{cris}}(V)$ of the same dimension as $\mathrm{Fil}^{-i}\,\mathbb{D}_{\mathrm{cris}}(V)$, *and the map A_η is the composition of the inclusion of $(\varphi^*\mathbb{N}_{\mathrm{rig}}(V))^{\psi=0}$ in $\mathbb{D}_{\mathrm{cris}}(V) \otimes_{\mathbb{Q}_p} \mathcal{H}(\Gamma)$ with the map $\bigoplus_i(\mathrm{id}\otimes A_{\eta,i})$, where $A_{\eta,i}$ is the natural reduction map $\mathcal{H}(\Gamma) \to \mathbb{Q}_p(\chi^i\chi_0^{-i}\eta)$ obtained by quotienting out by the ideal $(X+1-\chi(\gamma)^i)\cdot e_\eta$.*

Using this we can describe the images of the Coleman maps (for any choice of basis of $\mathbb{N}(V)$):

Theorem C (Corollary 4.15). *Let η be any character of Δ. Then for all $1 \le i \le d$,*

$$\operatorname{Im}(\underline{\mathrm{Col}}_i^\eta) = \prod_{j \in I_i^\eta} (X - \chi(\gamma)^j + 1)\Lambda_{\mathbb{Q}_p}(\Gamma_1)$$

for some $I_i^\eta \subset \{0, \dots, r_d - 1\}$.

As a corollary of the proof, we also obtain a formula for the elementary divisors of the matrix of the map $\mathscr{L}_V$, which can be seen as a refinement of the statement of the $\delta(V)$-conjecture. For $i \in \mathbb{Z}$, define

$$\ell_i = \frac{\log(1+X)}{\log_p(\chi(\gamma))} - i.$$

Theorem D (Theorem 4.16). *The elementary divisors of the $\mathscr{H}(\Gamma)$-module quotient*

$$\frac{\mathscr{H}(\Gamma) \otimes_{\mathbb{Q}_p} \mathbb{D}_{\mathrm{cris}}(V)}{\mathscr{H}(\Gamma) \otimes_{\Lambda_{\mathbb{Q}_p}(\Gamma)} \operatorname{Im}(\mathscr{L}_V)}$$

are $[\lambda_{r_1}; \lambda_{r_2}; \dots; \lambda_{r_d}]$, where $\lambda_k = \ell_0 \ell_1 \dots \ell_{k-1}$.

Suppose now that $V = V_f(k-1)$, where $f = \sum a_n e^{2\pi i n z}$ is a modular form of weight $k \ge 2$ and level prime to p, and V_f is the 2-dimensional p-adic representation associated to f by Deligne. (Thus the Hodge–Tate weights of V_f are 0 and $1-k$, and those of V are 0 and $k-1$.) As we show in Section 1C5, Assumption 1.3 is automatically satisfied in this case, since V is 2-dimensional. In this case, we can refine the results above to study the integral structure of the Coleman maps. Let T_f be a $\mathscr{G}_{\mathbb{Q}_p}$-stable lattice in V_f, and let us assume that the $\mathbb{B}^+_{\mathbb{Q}_p}$-basis of $\mathbb{N}(V_f)$ used to define the Coleman maps is in fact an $\mathbb{A}^+_{\mathbb{Q}_p}$-basis of $\mathbb{N}(T_f)$.

Theorem E (Theorem 5.10). *For $i = 1, 2$ and for each character η of Δ, the image of $H^1_{\mathrm{Iw}}(\mathbb{Q}_p, T_f)^\eta$ under $\underline{\mathrm{Col}}_i^\eta$ is a submodule of finite index of the module*

$$\left(\prod_{j \in I_i^\eta} (X - \chi(\gamma)^j + 1) \right) \Lambda(\Gamma_1)$$

for some subset $I_i^\eta \subset \{0, \dots, k-2\}$. Moreover, for each η the sets I_1^η and I_2^η are disjoint.

This theorem generalises [Kurihara and Pollack 2007, Proposition 1.2], which determines the images of $\left(\underline{\mathrm{Col}}_1^\Delta, \underline{\mathrm{Col}}_2^\Delta\right)$ for elliptic curves with $a_p = 0$. As a consequence of Theorem E, we can rewrite Conjecture 1.1 as below, without making any technical assumptions.

Theorem F. *For $i = 1, 2$, Conjecture 1.1 is equivalent to the assertion that for each η the characteristic ideal* $\mathrm{Char}_{\Lambda_{\mathbb{Q}_p}(\Gamma_1)}\big(\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathrm{Sel}^i_p(f/\mathbb{Q}_\infty)^{\eta,\vee}\big)$ *is generated by $L^\eta_{p,i}/\prod_{j\in I^\eta_i}(X-\chi(\gamma)^j+1)$ where I^η_i is as given by Theorem E.*

Finally, we explain in Section 5C how it is possible to choose a basis in such a way that $I^\eta_1 = I^\eta_2 = \varnothing$, i.e., the modules $\Lambda(\Gamma_1)/\operatorname{Im}(\underline{\mathrm{Col}}^\eta_i)$ are pseudo-null for both $i = 1$ and 2.

Remark 1.4. The local results in this paper (Theorems A, B, C and D) hold with representations of $\mathcal{G}_{\mathbb{Q}_p}$ replaced by representations of $\mathcal{G}_F$ for an arbitrary finite unramified extension $F/\mathbb{Q}_p$, with essentially the same proofs. We have chosen to work over $\mathbb{Q}_p$ for the sake of simplicity, since this is all that is needed for applications to modular forms.

In [Loeffler and Zerbes 2012], these methods are applied to the study of the "critical-slope" L-function attached to an ordinary modular form (corresponding to the non-unit Frobenius eigenvalue).

1C. *Setup and notation.*

1C1. *Fontaine rings.* We review the definitions of the Fontaine rings we use in this paper. Details can be found in [Berger 2004] or [Lei et al. 2010].

Throughout this paper, p is an odd prime. If K is a number field or a local field of characteristic 0, then G_K denotes its absolute Galois group and $\mathcal{O}_K$ the ring of integers of K. We write Γ for the Galois group $\mathrm{Gal}(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q})$, which we identify with $\mathbb{Z}_p^\times$ via the cyclotomic character χ. Then $\Gamma \cong \Delta \times \Gamma_1$, where Δ is of order $p-1$ and $\Gamma_1 \cong \mathbb{Z}_p$. We fix a topological generator γ of Γ_1.

We write $\mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p}$ for the ring of power series $f(\pi) \in \mathbb{Q}_p[\![\pi]\!]$ such that $f(X)$ converges everywhere on the open unit p-adic disc. Equip $\mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p}$ with actions of Γ and a Frobenius operator φ by $g.\pi = (\pi+1)^{\chi(g)} - 1$ and $\varphi(\pi) = (\pi+1)^p - 1$. We can then define a left inverse ψ of φ satisfying

$$\varphi \circ \psi(f(\pi)) = \frac{1}{p}\sum_{\zeta^p=1} f(\zeta(1+\pi)-1).$$

Inside $\mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p}$, we have subrings $\mathbb{A}^+_{\mathbb{Q}_p} = \mathbb{Z}_p[\![\pi]\!]$ and $\mathbb{B}^+_{\mathbb{Q}_p} = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathbb{A}^+_{\mathbb{Q}_p}$. Moreover, the actions of φ, ψ and Γ preserve these subrings. Finally, we write $t = \log(1+\pi) \in \mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p}$ and $q = \varphi(\pi)/\pi \in \mathbb{A}^+_{\mathbb{Q}_p}$. A formal power series calculation shows that $g(t) = \chi(g)t$ for $g \in \Gamma$ and $\varphi(t) = pt$.

1C2. *Iwasawa algebras and power series.* Given a finite extension K of $\mathbb{Q}_p$, denote by $\Lambda_{\mathcal{O}_K}(\Gamma)$ (respectively $\Lambda_{\mathcal{O}_K}(\Gamma_1)$) the Iwasawa algebra $\mathbb{Z}_p[\![\Gamma]\!] \otimes_{\mathbb{Z}_p} \mathcal{O}_K$ (respectively $\mathbb{Z}_p[\![\Gamma_1]\!] \otimes_{\mathbb{Z}_p} \mathcal{O}_K$) over $\mathcal{O}_K$. We further write $\Lambda_K(\Gamma) = \mathbb{Q} \otimes \Lambda_{\mathcal{O}_K}(\Gamma)$ and $\Lambda_K(\Gamma_1) = \mathbb{Q} \otimes \Lambda_{\mathcal{O}_K}(\Gamma_1)$. If M is a finitely generated torsion $\Lambda_{\mathcal{O}_K}(\Gamma_1)$-module, we write $\mathrm{Char}_{\Lambda_{\mathcal{O}_K}(\Gamma_1)}(M)$ for its characteristic ideal.

Let $\mathcal{H}(\Gamma)$ be the space of distributions on Γ (the continuous dual of the space of locally analytic functions on Γ), with the ring structure defined by convolution. We may identify this with the space of formal power series

$$\{f \in \mathbb{Q}_p[\Delta][\![X]\!] : f \text{ converges everywhere on the open unit } p\text{-adic disc}\},$$

where X corresponds to $\gamma - 1$. We may identify $\Lambda_{\mathbb{Q}_p}(\Gamma)$ with the subring of $\mathcal{H}(\Gamma)$ consisting of power series with bounded coefficients.

The action of Γ on $\mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p}$ gives an isomorphism of $\mathcal{H}(\Gamma)$ with $(\mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p})^{\psi=0}$, the Mellin transform

$$\begin{aligned}\mathfrak{M} : \mathcal{H}(\Gamma) &\to (\mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p})^{\psi=0}\\ f(\gamma-1) &\mapsto f(\gamma-1)\cdot(\pi+1).\end{aligned}$$

In particular, $\Lambda_{\mathbb{Z}_p}(\Gamma)$ corresponds to $(\mathbb{A}^+_{\mathbb{Q}_p})^{\psi=0}$ under $\mathfrak{M}$. Similarly, we define $\mathcal{H}(\Gamma_1)$ as the subring of $\mathcal{H}(\Gamma)$ defined by power series over $\mathbb{Q}_p$, rather than $\mathbb{Q}_p[\Delta]$. Then, $\mathcal{H}(\Gamma_1)$ corresponds to $(1+\pi)\varphi(\mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p})$ under $\mathfrak{M}$, and $\Lambda_{\mathbb{Z}_p}(\Gamma_1)$ to $(1+\pi)\varphi(\mathbb{A}^+_{\mathbb{Q}_p})$. (See [Perrin-Riou 2001, B.2.8] for more details.)

If d is an integer and S is a $\Lambda_K(\Gamma_1)$-submodule of $K \otimes_{\mathbb{Q}_p} \mathcal{H}(\Gamma_1)^{\oplus d}$ which is free of rank d, we write $\det(S)$ for the determinant of any basis of S, which is well-defined up to multiplication by a unit of $\Lambda_K(\Gamma_1)$. If F is a homomorphism of $\Lambda_K(\Gamma_1)$-modules whose image is a free rank d $\Lambda_K(\Gamma_1)$-submodule of $K \otimes_{\mathbb{Q}_p} \mathcal{H}(\Gamma_1)^{\oplus d}$, we write $\det(F)$ for $\det(\mathrm{Im}(F))$.

For an integer i, define

$$\left.\begin{aligned}\ell_i &= \frac{\log(1+X)}{\log_p(\chi(\gamma))} - i\\ \delta_i &= \frac{\ell_i}{X+1-\chi(\gamma)^i}\end{aligned}\right\} \in \mathcal{H}(\Gamma_1).$$

Note that ℓ_i is independent of the choice of generator γ (hence the choice of normalising factor), but δ_i is not.

Remark 1.5. Note that for any positive integer k, we have

$$\mathfrak{n}_k = a_k \delta_{k-1} \dots \delta_0,$$

where $a_k = \log(\chi(\gamma))^k \in \mathbb{Z}_p$ is nonzero.

The following result slightly refines [Berger 2003, Lemma II.2].

Proposition 1.6. *For any $k \geq 0$, we have*

$$\mathfrak{M}(\ell_{k-1}\dots\ell_0\mathscr{H}(\Gamma)) = (t^k\mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p})^{\psi=0}$$
$$\mathfrak{M}(\delta_{k-1}\dots\delta_0\mathscr{H}(\Gamma)) = \left(\left(\tfrac{t}{\varphi(\pi)}\right)^k \mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p}\right)^{\psi=0}.$$

Proof. One checks easily that ℓ_i acts on $\mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p}$ as the differential operator

$$(1+\pi)t\frac{\mathrm{d}}{\mathrm{d}\pi} - i,$$

and hence

$$\ell_j(t^j f) = t^{j+1}(1+\pi)\frac{\mathrm{d}f}{\mathrm{d}\pi}.$$

Since $(1+\pi)\frac{\mathrm{d}}{\mathrm{d}\pi}$ is an isomorphism on $(\mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p})^{\psi=0}$ (it is the map on distributions dual to the map $f(x) \mapsto xf(x)$ on functions), it follows that each ℓ_j maps $(t^j\mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p})^{\psi=0}$ bijectively onto $(t^{j+1}\mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p})^{\psi=0}$.

To prove a similar statement for the δ_i, we note that

$$(\mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p}/\varphi(\pi)\mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p})^{\psi=0}$$

is isomorphic to $\mathbb{Q}_p[\Delta]$ as a Γ-module. Since t is a uniformiser of the ideal $\varphi(\pi)$, we have

$$(\varphi(\pi)^j\mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p}/\varphi(\pi)^{j+1}\mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p})^{\psi=0} = (t^j\mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p}/t^j\varphi(\pi)\mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p})^{\psi=0} \cong \mathbb{Q}_p[\Delta](j)$$

as a Γ-module. Hence its annihilator is $X+1-\chi(\gamma)^j$. These factors are mutually coprime and coprime to $\delta_0\dots\delta_{k-1}$, and the product is $\ell_0\dots\ell_{k-1}$, so the result follows. □

1C3. *Isotypical components.* Let $\eta : \Delta \to \mathbb{Z}_p^\times$ be a character. We write $e_\eta = (p-1)^{-1}\sum_{\sigma\in\Delta}\eta^{-1}(\sigma)\sigma$. If M is a $\Lambda_E(\Gamma)$-module, its η-isotypical component is given by $M^\eta = e_\eta M$. When $\eta = 1$, we write M^Δ in place of M^η.

We identify $\Lambda_E(\Gamma_1)$ with the power series in $X = \gamma - 1$ with bounded coefficients in E. Given

$$F = \sum_{\sigma\in\Delta, n\geq 0} a_{\sigma,n}\sigma(\gamma-1)^n \in \Lambda(\Gamma),$$

we write $F^\eta = e_\eta F$ for its image in $\Lambda_E(\Gamma)^\eta$. In particular,

$$F^\eta = e_\eta \sum_{n\geq 0}\left(\sum_{\sigma\in\Delta} a_{\sigma,n}\eta(\sigma)\right)(\gamma-1)^n \in e_\eta\Lambda_E(\Gamma_1).$$

Therefore, we can identify F^η with a power series in $X = \gamma - 1$. Under this identification, the value $F^\eta|_{X=\chi(\gamma)^j-1}$ is given by $\chi^j\chi_0^{-j}\eta(F)$ where $\chi_0 = \chi|_\Delta$ for all $j \in \mathbb{Z}$.

1C4. *Crystalline representations.* Let E and F be finite extensions of $\mathbb{Q}_p$. Let V be a crystalline E-linear representation of $G_{\mathbb{Q}_p}$. We denote the Dieudonné module of V by $\mathbb{D}_{\mathrm{cris}}(V)$. If $j \in \mathbb{Z}$, $\mathrm{Fil}^j\, \mathbb{D}_{\mathrm{cris}}(V)$ denotes the jth step in the de Rham filtration of $\mathbb{D}_{\mathrm{cris}}(V)$. We say V is *positive* if $\mathbb{D}_{\mathrm{cris}}(V) = \mathrm{Fil}^0\, \mathbb{D}_{\mathrm{cris}}(V)$ (following the standard, but unfortunate, convention that positive representations are precisely those with non-positive Hodge–Tate weights).

The (φ, Γ)-module of V is denoted by $\mathbb{D}(V)$. As shown by Fontaine (unpublished; for a reference see [Cherbonnier and Colmez 1999, Section II]), we have a canonical isomorphism of $\Lambda_E(\Gamma)$-modules

$$h^1_{\mathrm{Iw},V} : \mathbb{D}(V)^{\psi=1} \to H^1_{\mathrm{Iw}}(\mathbb{Q}_p, V).$$

We write $\exp_{F,V} : F \otimes \mathbb{D}_{\mathrm{cris}}(V) \to H^1(F, V)$ for the Bloch–Kato exponential over F.

For an integer j, $V(j)$ denotes the jth Tate twist of V, i.e., $V(j) = V \otimes Ee_j$ where $G_{\mathbb{Q}_p}$ acts on e_j via χ^j. We have

$$\mathbb{D}_{\mathrm{cris}}(V(j)) = t^{-j}\mathbb{D}_{\mathrm{cris}}(V) \otimes e_j.$$

For any $v \in \mathbb{D}_{\mathrm{cris}}(V)$, $v_j = t^{-j} v \otimes e_j$ denotes its image in $\mathbb{D}_{\mathrm{cris}}(V(j))$.

If $h \geq 1$ is an integer such that $\mathrm{Fil}^{-h}\, \mathbb{D}_{\mathrm{cris}}(V) = \mathbb{D}_{\mathrm{cris}}(V)$, we write $\Omega_{V,h}$ for the Perrin-Riou exponential as defined in [Perrin-Riou 1994].

Let T be an $\mathcal{O}_E$-lattice in V which is stable under $G_{\mathbb{Q}_p}$. We denote the Wach module of V (respectively T) by $\mathbb{N}(V)$ (respectively $\mathbb{N}(T)$), a free module of rank d over $\mathbb{B}^+_{\mathbb{Q}_p}$ (respectively $\mathbb{A}^+_{\mathbb{Q}_p}$). Recall that Γ acts on both of these objects, and there is a map $\varphi : \mathbb{N}(T)[\pi^{-1}] \to \mathbb{N}(T)[\varphi(\pi)^{-1}]$, preserving $\mathbb{N}(T)$ if T is positive (and similarly for V).

For any $j \in \mathbb{Z}$ we can identify $\mathbb{N}(T(j))$ with $\pi^{-j}\mathbb{N}(T) \otimes e_j$, where e_j is as above. Given an R-module M with an action of φ and a submodule N, $\varphi^* N$ denotes the R-submodule of M generated by $\varphi(N)$, e.g., $\varphi^*\mathbb{N}(T)$ denotes the $\mathbb{A}^+_{\mathbb{Q}_p}$-submodule of $\mathbb{N}(T)[\pi^{-1}]$ generated by $\varphi(\mathbb{N}(T))$. Finally, we write

$$\mathbb{N}_{\mathrm{rig}}(V) = \mathbb{N}(V) \otimes_{\mathbb{B}^+_{\mathbb{Q}_p}} \mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p}.$$

The following lemma is implicit in the calculations of [Lei et al. 2010, §3], but for the convenience of the reader we give a separate proof:

Lemma 1.7. *If the Hodge–Tate weights of V are ≥ 0, then we have*

$$\mathbb{N}(T) \subseteq \varphi^*\mathbb{N}(T)$$

and similarly for $\mathbb{N}(V)$.

Proof. It suffices to prove the result for T. Suppose that the Hodge–Tate weights of V are in $[0, m]$. Then $\mathbb{N}(T) = \pi^{-m}\mathbb{N}(T(-m))$. Since $T(-m)$ is positive, φ

preserves $\mathbb{N}(T(-m))$ and $\mathbb{N}(T(-m))/\varphi^*\mathbb{N}(T(-m))$ is killed by q^m [Berger 2004, proof of Theorem III.3.1]. Equivalently, we have

$$q^m \cdot \pi^m \mathbb{N}(T) \subseteq \varphi^*(\pi^m \mathbb{N}(T)) = \varphi(\pi)^m \varphi^* \mathbb{N}(T).$$

Since $q = \varphi(\pi)/\pi$, the result follows. □

1C5. *Refinements of crystalline representations.* Let V be an E-linear crystalline representation of $G_{\mathbb{Q}_p}$ of dimension d, and let $s_1 \le \cdots \le s_d$ be the jumps in the filtration of $\mathbb{D}_{\mathrm{cris}}(V)$, so the Hodge–Tate weights are $-s_i$. If Y is an E-linear subspace of $\mathbb{D}_{\mathrm{cris}}(V)$ of dimension $e \le d$, we say Y is *in general position* (with respect to the Hodge filtration) if the intersections $\mathrm{Fil}^j Y = Y \cap \mathrm{Fil}^j \mathbb{D}_{\mathrm{cris}}(V)$ have the smallest possible dimension; that is,

$$\dim \mathrm{Fil}^j Y = \begin{cases} \dim \mathrm{Fil}^j \mathbb{D}_{\mathrm{cris}}(V) - d + e & \text{if } \dim \mathrm{Fil}^j V \ge d - e, \\ 0 & \text{otherwise.} \end{cases}$$

This is equivalent to the requirement that the jumps of the filtration $\mathrm{Fil}^j Y$ are $s_1, \dots, s_e$.

As in [Bellaïche and Chenevier 2009, §2.4.1], we define a *refinement* of V to be a family $\underline{Y} = (Y_i)_{i=1}^d$ of E-linear subspaces of $\mathbb{D}_{\mathrm{cris}}(V)$ stable under φ, with $0 \subsetneq Y_1 \subsetneq \cdots \subsetneq Y_d = \mathbb{D}_{\mathrm{cris}}(V)$, so $\dim_E Y_i = i$. It is clear that refinements exist if and only if the eigenvalues of φ on $\mathbb{D}_{\mathrm{cris}}(V)$ lie in E.

We say that the refinement is *non-critical* if each of the subspaces Y_i is in general position, or equivalently if $Y_i \cap \mathrm{Fil}^{s_i+1} \mathbb{D}_{\mathrm{cris}}(V) = 0$ for all i.

(If the Hodge–Tate weights of V are distinct, as Bellaïche and Chenevier assume, then this is equivalent to the assertion that $\mathbb{D}_{\mathrm{cris}}(V) = Y_i \oplus \mathrm{Fil}^{s_i+1} \mathbb{D}_{\mathrm{cris}}(V)$ for each i, which coincides with Definition 2.4.5 of [op. cit.]).

Proposition 1.8. *If the eigenvalues of Frobenius on $\mathbb{D}_{\mathrm{cris}}(V)$ lie in E, and either $d = 2$ or φ acts semisimply on $\mathbb{D}_{\mathrm{cris}}(V)$, then there exists a non-critical refinement of V.*

Proof. As noted in [Bellaïche and Chenevier 2009, Remark 2.4.6(iii)], the case where φ acts semisimply is obvious: any basis of eigenvectors of $\mathbb{D}_{\mathrm{cris}}(V)$ defines $d!$ refinements, one for each ordering of the basis vectors, and it is easy to see that we can choose an ordering such that the resulting refinement is non-critical. Hence let us assume that V is 2-dimensional and φ acts non-semisimply on $\mathbb{D}_{\mathrm{cris}}(V)$. Thus $\mathbb{D}_{\mathrm{cris}}(V)$ has a basis (e_1, e_2) such that $\varphi(e_1) = \alpha e_1$ and $\varphi(e_2) = e_1 + \alpha(e_2)$, for some $\alpha \in E^\times$. By twisting, we may assume that the jumps in the Hodge filtration are 0 and s with $s \ge 0$. Let N be the valuation of α; the Newton and Hodge numbers of $\mathbb{D}_{\mathrm{cris}}(V)$ are $t_H = s$ and $t_N = 2N$, so we have $s = 2N$ by weak admissibility.

The unique possible refinement is given by $Y_1 = E e_1$, and this is non-critical unless $s > 0$ and $\mathrm{Fil}^1 \mathbb{D}_{\mathrm{cris}}(V) = Y_1$. If this is the case, then the Newton and Hodge numbers of Y_1 are respectively $t_H(Y_1) = s$ and $t_N(Y_1) = N$, and since $s = 2N > N$ this contradicts the weak admissibility of $\mathbb{D}_{\mathrm{cris}}(V)$. $\square$

Remark 1.9. (1) It is shown in [Milne 1994] that the Tate conjecture implies the semisimplicity of φ on the crystalline cohomology groups of any smooth projective variety over $\mathbb{F}_p$ (or, more generally, on the crystalline realisation of any motive over $\mathbb{F}_p$); so the hypotheses of the proposition conjecturally hold for all crystalline representations "arising from geometry".

(2) For representations of dimension ≥ 3 with non-semisimple Frobenius there may be no non-critical refinements, as the following counterexample shows. Let $D = \mathbb{Q}_p^3$ with its standard basis e_1, e_2, e_3, and let $\varphi : D \to D$ be given by the matrix

$$\begin{pmatrix} \alpha & 1 & 0 \\ 0 & \alpha & 1 \\ 0 & 0 & \alpha \end{pmatrix},$$

where $\alpha \in \mathbb{Z}_p$ has valuation 1. We define a filtration on D with jumps $\{0, 1, 2\}$ by $\mathrm{Fil}^0(D) = D$, $\mathrm{Fil}^1 D = \mathbb{Q}_p e_1 + \mathbb{Q}_p e_3$, $\mathrm{Fil}^2 D = \mathbb{Q}_p e_3$, $\mathrm{Fil}^3 D = 0$. Then the only φ-stable submodules are $Y_0 = 0$, $Y_1 = \mathbb{Q}_p e_1$, and $Y_2 = \mathbb{Q}_p e_1 + \mathbb{Q}_p e_2$ and $Y_3 = D$. The Hodge and Newton numbers are given by

i	$t_H(Y_i)$	$t_N(Y_i)$
1	1	1
2	1	2
3	3	3

so D is a weakly admissible filtered φ-module; and the unique refinement of D is $(Y_i)_{i=0,\dots,3}$, but Y_1 is not in general position.

1C6. *Modular forms.* Let $f(z) = \sum a_n e^{2\pi i n z}$ be a normalised new eigenform of weight $k \geq 2$, level N and nebentypus ε. Write $F_f = \mathbb{Q}(a_n : n \geq 1)$ for its coefficient field. Let

$$\bar{f}(z) = \sum \bar{a}_n e^{2\pi i n z}$$

be the dual form to f, which also has coefficients in F_f. We assume that $p \nmid N$ and a_p is not a p-adic unit, so f is supersingular at p.

Remark 1.10. We make this assumption in order to save ourselves from doing the same calculations twice in Section 5; they easily generalise to the ordinary case.

We fix a prime of F_f above p. We denote the completion of F_f at this prime by E and fix a uniformiser ϖ_E. We write V_f for the 2-dimensional E-linear representation of $G_{\mathbb{Q}}$ associated to f from [Deligne 1971]. We fix an $\mathcal{O}_E$-lattice T_f

stable under $G_{\mathbb{Q}}$, which determines a lattice $T_{\bar{f}}$ of $V_{\bar{f}}$. When restricted to $G_{\mathbb{Q}_p}$, V_f is crystalline and its de Rham filtration is given by

$$\dim_E \operatorname{Fil}^i \mathbb{D}_{\mathrm{cris}}(V_f) = \begin{cases} 2 & \text{if } i \le 0, \\ 1 & \text{if } 1 \le i \le k-1, \\ 0 & \text{if } i \ge k. \end{cases}$$

The action of φ on $\mathbb{D}_{\mathrm{cris}}(V_f)$ satisfies $\varphi^2 - a_p\varphi + \varepsilon(p)p^{k-1} = 0$. Let us choose a "good basis" ν_1, ν_2 of $\mathbb{D}_{\mathrm{cris}}(V_f)$ as in [Lei et al. 2010, §3.3]; that is, ν_1 spans $\operatorname{Fil}^1 \mathbb{D}_{\mathrm{cris}}(V_f)$ and $\nu_2 = p^{1-k}\varphi(\nu_1)$. We also choose a basis $\bar{\nu}_1, \bar{\nu}_2$ of $\mathbb{D}_{\mathrm{cris}}(V_{\bar{f}})$ in the same way. The isomorphism $V_{\bar{f}} = V_f^*(1-k)$ gives a pairing $\mathbb{D}_{\mathrm{cris}}(V_f) \times \mathbb{D}_{\mathrm{cris}}(V_{\bar{f}}) \to \mathbb{D}_{\mathrm{cris}}(E(1-k)) = E \cdot t^{k-1}e_{1-k} \cong E$. As noted in [Lei et al. 2010, §3.4], we have $[\nu_1, \bar{\nu}_1] = [\nu_2, \bar{\nu}_2] = 0$ and $[\nu_2, \bar{\nu}_1] = -[\nu_1, \bar{\nu}_2]$, and (by scaling) we may assume without loss of generality that $[\nu_1, \bar{\nu}_2] = 1$.

Unless otherwise stated, we always assume that the eigenvalues of φ on $\mathbb{D}_{\mathrm{cris}}(V_f)$ are not integral powers of p and the nebentypus of f is trivial. Our assumption on the eigenvalues of φ allows us to define the Perrin-Riou pairing

$$\mathscr{L}_i = \mathscr{L}_{1,(1+\pi)\otimes\nu_{i,1}} : H^1_{\mathrm{Iw}}(\mathbb{Q}_p, V_{\bar{f}}(k-1)) \to \mathscr{H}(\Gamma)$$

for $i = 1, 2$ where we have identified $V_f(1)^*(1)$ with $V_{\bar{f}}(k-1)$ (see [Lei 2011, Section 3.2] or [Lei et al. 2010, Section 3.3] for details).

1C7. *Adequate rings and elementary divisors.* Let R be a commutative integral domain with identity, such that the following conditions hold:

- All finitely generated ideals in R are principal (i.e., R is a Bézout domain).
- R is *adequate*; i.e., for any $a, b \in R$ with $a \neq 0$, we may write $a = a_1a_2$, where $(a_1, b) = (1)$ and $(d, b) \neq (1)$ for every non-unit divisor d of a_2.

Then R is an elementary divisor ring. That is, let $M \subseteq N$ be finitely generated R-modules such that $N \cong R^d$. Then there exists a R-basis $n_1, \dots, n_d$ of N and $r_1, \dots, r_d \in R$ (unique up to units of R) such that $r_1 \mid \cdots \mid r_d$ and $r_1n_1, \dots, r_en_e$, where e is the largest integer such that $r_e \neq 0$, form a R-basis of M. In particular, we have $\det(M) = r_1 \dots r_d$. In this case, we write $[N : M] = [N : M]_R = [r_1; \cdots; r_d]$. When $d = 1$, we simply write $[N : M] = r_1$.

If Q is an arbitrary finitely presented R-module, then we may write Q as a quotient N/M where N is a free module of finite rank and M is a finitely generated submodule of N, so the elementary divisors $[N : M]_R$ are defined. It is easy to check that these are independent of the choice of presentation of Q, and we define these to be the elementary divisors of Q.

As explained in [Berger 2002, §4.2], $\mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p}$ is an adequate Bézout domain and hence an elementary divisor ring. The same is true of $E \otimes_{\mathbb{Q}_p} \mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p}$ for any finite extension E of $\mathbb{Q}_p$, and of $\mathcal{H}(\Gamma_1)$ (which is isomorphic to $\mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p}$ as an abstract ring).

We will need the following lemma; see [Lang 2002, Lemma III.7.6].

Lemma 1.11. *Let R be an adequate Bézout domain, M a finitely presented R-module, and N a submodule of M. Suppose that there is some $a \in R$ such that $N \cong R/a$ and $aM = 0$. Then $M \cong N \oplus M/N$.*

Proof. Let $q_1, \dots, q_r$ be a set of generators for M/N, with annihilators a_i, giving an isomorphism $M/N \cong \oplus_{i=1}^r R/a_i$. Since $aM = 0$, each a_i divides a. Let p_i be an arbitrary lift of q_i; then $a_i p_i \in N$, so $a_i p_i = b_i p_0$ where p_0 is a generator of N and $b_i \in R/aR$. Since $aM = 0$, we have $0 = (a/a_i)a_i p_i = (a/a_i)b_i p_0$.

Then we must have $(a/a_i)b_i \in aR$, so $ab_i \in aa_i R$. Since R is an integral domain, we must have $a_i \mid b_i$, and we may write $b_i = a_i c_i$. Thus $p_i' = p_i - c_i p_0$ is a lift of p_i such that $a_i p_i' = a_i p_i - a_i c_i p_0 = a_i p_i - b_i p_0 = 0$. It follows that the subgroup generated by the p_i' maps bijectively to M/N, giving the required splitting. □

A straightforward induction gives the following generalisation:

Corollary 1.12. *If M is an R-module with a filtration by submodules $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_d = M$, and there are elements $a_1, \dots, a_d \in R$ such that for each $i = 1, \dots, d$ we have $M_i/M_{i-1} \cong R/a_i$ and $a_i M \subseteq M_{i-1}$, then $M \cong \bigoplus_{i=1}^d R/a_i$.*

The ring $\mathcal{H}(\Gamma)$ is not a domain; but it is equal to the direct sum of its subrings $e_\eta \mathcal{H}(\Gamma)$, where e_η is the idempotent in $\mathbb{Q}_p[\Delta]$ corresponding to the character $\eta : \Delta \to \mathbb{Q}_p^\times$ as above. Each of these subrings is isomorphic to $\mathcal{H}(\Gamma_1)$, and hence admits a theory of elementary divisors. If M is a submodule of $\mathcal{H}(\Gamma)^{\oplus d}$, we define the ith elementary divisor of M to be $\sum_\eta e_\eta a_i^\eta$, where a_i^η is the ith elementary divisor of the submodule $M^\eta = e_\eta M \subseteq e_\eta \mathcal{H}(\Gamma)$ considered as a $\mathcal{H}(\Gamma_1)$-module. In practice we shall only apply this in situations where M has the form $\mathbb{Q}_p[\Gamma] \otimes_{\mathbb{Q}_p} M'$ for an $\mathcal{H}(\Gamma_1)$-module M, in which case the isotypical components M^η all have the same elementary divisors.

2. Refinements of crystalline representations and $\mathcal{H}(\Gamma)$-structure

In this section, we will prove Theorem A. We will do this by working with a certain filtration of the module $\mathbb{N}_{\mathrm{rig}}(V)$, which is a (φ, Γ)-module over $\mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p}$; the steps in this filtration are (φ, Γ)-modules over $\mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p}$, but they are not necessarily of the form $\mathbb{N}_{\mathrm{rig}}(W)$ for any representation W, so we begin by systematically developing a theory of such modules. Our approach is very much influenced by the description of the theory of (φ, Γ)-modules over the Robba ring $\mathbb{B}^\dagger_{\mathrm{rig},\mathbb{Q}_p}$ given in [Bellaïche and Chenevier 2009, §2.2].

2A. ***Some properties of (φ, Γ)-modules over $\mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p}$.*** We define a (φ, Γ)-module over $\mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p}$ to be a free $\mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p}$-module $\mathcal{N}$ of finite rank, endowed with semilinear commuting actions of φ and Γ, such that the quotient $\mathcal{N}/\varphi^*(\mathcal{N})$ is annihilated by some power of q (where $q = \varphi(\pi)/\pi$ as above). We define

$$\mathbb{D}_{\mathrm{cris}}(\mathcal{N}) = \mathcal{N}^{\Gamma}.$$

We equip $\mathbb{D}_{\mathrm{cris}}(\mathcal{N})$ with the filtration defined by

$$\mathrm{Fil}^i\, \mathbb{D}_{\mathrm{cris}}(\mathcal{N}) = \{v \in \mathbb{D}_{\mathrm{cris}}(\mathcal{N}) : \varphi(v) \in q^i \mathcal{N}\}.$$

Let $K_n = \mathbb{Q}_p(\mu_{p^n})$ and $K_\infty = \bigcup_n K_n$. We define

$$\mathbb{D}^{(n)}_{\mathrm{dR}}(\mathcal{N}) = \left(K_\infty \otimes_{K_n} K_n[\![t]\!] \otimes_{\mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p}} \mathcal{N}\right)^{\Gamma},$$

where the tensor product is via the embedding $\mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p} \hookrightarrow K_n[\![t]\!]$ arising from the fact that

$$K_n \cong \mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p}/\varphi^{n-1}(q)$$

and t is a uniformiser of the prime ideal $\varphi^{n-1}(q)$. We endow $K_n[\![t]\!]$ with the obvious semilinear action of Γ, for which this homomorphism is Γ-equivariant, and the t-adic filtration. Then $\mathbb{D}^{(n)}_{\mathrm{dR}}(\mathcal{N})$ is a filtered $\mathbb{Q}_p$-vector space, of dimension $\le d$ where d is the $\mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p}$-rank of $\mathcal{N}$ (since $K_\infty((t))^\Gamma = \mathbb{Q}_p$ [Bellaïche and Chenevier 2009, §2.2.7]); the operator φ gives an isomorphism of filtered $\mathbb{Q}_p$-vector spaces

$$\mathbb{D}^{(n)}_{\mathrm{dR}}(\mathcal{N}) \xrightarrow{\cong} \mathbb{D}^{(n+1)}_{\mathrm{dR}}(\mathcal{N})$$

for each n, and an embedding of filtered $\mathbb{Q}_p$-vector spaces

$$\mathbb{D}_{\mathrm{cris}}(\mathcal{N}) \hookrightarrow \mathbb{D}^{(1)}_{\mathrm{dR}}(\mathcal{N}).$$

We say that $\mathcal{N}$ is *crystalline* if $\dim_{\mathbb{Q}_p} \mathbb{D}_{\mathrm{cris}}(\mathcal{N}) = d$, and we say it is *de Rham* if $\dim_{\mathbb{Q}_p} \mathbb{D}^{(n)}_{\mathrm{dR}}(\mathcal{N}) = d$ (for some, and hence all, $n \ge 1$). If $\mathcal{N}$ is de Rham, we define the *Hodge–Tate weights* of $\mathcal{N}$ to be the integers r such that $\mathrm{Fil}^{-r}\, \mathbb{D}^{(n)}_{\mathrm{dR}}(\mathcal{N}) \neq \mathrm{Fil}^{1-r}\, \mathbb{D}^{(n)}_{\mathrm{dR}}(\mathcal{N})$ (with multiplicities given by the size of the jump in dimension). Note that these are necessarily ≤ 0, which is unfortunate but necessary for compatibility with the usual definition in the case of Galois representations.

Finally, we define $\mathbb{D}^{(n)}_{\mathrm{Sen}}(\mathcal{N}) = K_\infty \otimes_{K_n} \mathcal{N}/\varphi^{n-1}(q)\mathcal{N}$. This is a K_∞-vector space of dimension d, with a semilinear action of Γ. As above, the φ operator gives isomorphisms $\mathbb{D}^{(n)}_{\mathrm{Sen}}(\mathcal{N}) \to \mathbb{D}^{(n+1)}_{\mathrm{Sen}}(\mathcal{N})$, of K_∞-vector spaces with semilinear Γ-action. (So both $\mathbb{D}_{\mathrm{Sen}}(\mathcal{N})$ and $\mathbb{D}_{\mathrm{dR}}(\mathcal{N})$ are independent of n as abstract objects; we retain the n in the notation when we are interested in the relation between these spaces and the original module $\mathcal{N}$.)

Proposition 2.1. *Let $j \geq 0$, and suppose $\mathcal{N}$ is de Rham. Then there is an isomorphism of $\mathbb{Q}_p$-vector spaces*

$$\operatorname{Fil}^j \mathbb{D}_{\mathrm{dR}}(\mathcal{N}) / \operatorname{Fil}^{j+1} \mathbb{D}_{\mathrm{dR}}(\mathcal{N}) \xrightarrow{\cong} \mathbb{D}_{\mathrm{Sen}}(\mathcal{N})^{\Gamma=\chi^{-j}}.$$

Proof. Let us fix an $n \geq 1$ and let θ be the reduction map $K_n[\![t]\!] \to K_n$. Then θ induces a map

$$\mathbb{D}_{\mathrm{dR}}^{(n)}(\mathcal{N}) \to \mathbb{D}_{\mathrm{Sen}}^{(n)}(\mathcal{N})^{\Gamma}$$

with kernel $\operatorname{Fil}^1 \mathbb{D}_{\mathrm{dR}}(\mathcal{N})$ and whose image is a $\mathbb{Q}_p$-linear subspace $S_0 \subseteq \mathbb{D}_{\mathrm{Sen}}(\mathcal{N})^{\Gamma}$. Similarly, we find that $\theta \circ t^{-j}$ gives an injection

$$\operatorname{Fil}^j \mathbb{D}_{\mathrm{dR}}(\mathcal{N}) / \operatorname{Fil}^{j+1} \mathbb{D}_{\mathrm{dR}}(\mathcal{N}) \to \mathbb{D}_{\mathrm{Sen}}(\mathcal{N})^{\Gamma=\chi^{-j}},$$

whose image is a $\mathbb{Q}_p$-linear subspace S_j.

Since $\bigoplus_{j=0}^{\infty} S_j$ has dimension d, it suffices to show that

$$\dim_{\mathbb{Q}_p} \bigoplus_{j=0}^{\infty} \mathbb{D}_{\mathrm{Sen}}(\mathcal{N})^{\Gamma=\chi^{-j}} \leq d.$$

This follows from the fact that it is a subspace of $\left(K_\infty((t)) \otimes_{K_\infty} \mathbb{D}_{\mathrm{Sen}}(\mathcal{N})\right)^{\Gamma}$, and (as remarked above) $K_\infty((t)))$ is a field, with $K_\infty((t))^{\Gamma} = \mathbb{Q}_p$. $\square$

Corollary 2.2. *If $\mathcal{N}$ is crystalline, then the map*

$$\mathbb{D}_{\mathrm{cris}}(\mathcal{N}) = \mathcal{N}^{\Gamma} \xrightarrow{\varphi^n} (\mathcal{N}/\varphi^{n-1}(q)^r \mathcal{N})^{\Gamma}$$

is surjective for all $r \geq 1$ and $n \geq 1$, with kernel $\operatorname{Fil}^r \mathbb{D}_{\mathrm{cris}}(\mathcal{N})$.

Proof. Let us define $\mathcal{N}^{(n)} = K_\infty \otimes_{K_n} K_n[\![t]\!] \otimes_{\mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p}} \mathcal{N}$, so $(\mathcal{N}^{(n)})^{\Gamma} = \mathbb{D}_{\mathrm{dR}}(\mathcal{N})$. By hypothesis the map $\varphi^n : \mathbb{D}_{\mathrm{cris}}(\mathcal{N}) \to \mathbb{D}_{\mathrm{dR}}^{(n)}(\mathcal{N})$ is an isomorphism of filtered vector spaces, and the filtration on $\mathbb{D}_{\mathrm{dR}}(\mathcal{N})$ is defined by the t-adic filtration of $\mathcal{N}^{(n)}$, so it suffices to show that reduction modulo t^r gives a surjection

$$(\mathcal{N}^{(n)})^{\Gamma} \to (\mathcal{N}^{(n)}/t^r \mathcal{N}^{(n)})^{\Gamma}.$$

We show that for each j, the map $(t^j \mathcal{N}^{(n)})^{\Gamma} \to (t^j \mathcal{N}^{(n)}/t^{j+1}\mathcal{N}^{(n)})^{\Gamma}$ is surjective. Multiplication by t^{-j} gives an isomorphism

$$(t^j \mathcal{N}^{(n)}/t^{j+1}\mathcal{N}^{(n)})^{\Gamma} \to (\mathcal{N}^{(n)}/t\mathcal{N}^{(n)})^{\Gamma=\chi^{-j}};$$

but $\mathcal{N}^{(n)}/t\mathcal{N}^{(n)} = \mathbb{D}_{\mathrm{Sen}}^{(n)}(\mathcal{N})$, and by the preceding proposition we know that $\theta \circ t^{-j}$ gives an isomorphism from $\operatorname{Fil}^j \mathbb{D}_{\mathrm{dR}}(\mathcal{N}) / \operatorname{Fil}^{j+1} \mathbb{D}_{\mathrm{dR}}(\mathcal{N})$ to $\mathbb{D}_{\mathrm{Sen}}^{(n)}(\mathcal{N})^{\Gamma=\chi^{-j}}$. So the map $(\mathcal{N}^{(n)})^{\Gamma} \to (\mathcal{N}^{(n)}/t^r \mathcal{N}^{(n)})^{\Gamma}$ is a morphism of filtered vector spaces for which the associated map of graded modules is surjective. Since the domain and codomain are finite-dimensional and their filtrations are separated, the original map is itself surjective. □

Let us write $\mathcal{M} = \mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p} \otimes_{\mathbb{Q}_p} \mathbb{D}_{\mathrm{cris}}(\mathcal{N}) \subseteq \mathcal{N}$.

Proposition 2.3. *If $\mathcal{N}$ is crystalline and Γ acts trivially on $\mathcal{N}/\pi\mathcal{N}$, then the elementary divisors of $\mathcal{N}/\mathcal{M}$ are*

$$\left(\tfrac{t}{\pi}\right)^{s_1}, \dots, \left(\tfrac{t}{\pi}\right)^{s_d},$$

where $-s_1 \geq \cdots \geq -s_d$ are the Hodge–Tate weights of $\mathcal{N}$.

Proof. This follows exactly as in [Berger 2004, Proposition III.2.1]. □

2B. *Quotients of* (φ, Γ)*-modules.* We now let $\mathcal{N}$ be a (φ, Γ)-module over $\mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p}$, as above. We assume that $\mathcal{N}$ is crystalline and Γ acts trivially on $\mathcal{N}/\pi\mathcal{N}$, and investigate the properties of a certain class of (φ, Γ)-modules obtained as quotients of $\mathcal{N}$. We continue to write $\mathcal{M} = \mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p} \otimes_{\mathbb{Q}_p} \mathbb{D}_{\mathrm{cris}}(\mathcal{N}) \subseteq \mathcal{N}$.

Let Y be a φ-stable E-linear subspace of $\mathbb{D}_{\mathrm{cris}}(\mathcal{N})$. We set

$$\mathcal{Y} = \mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p} \otimes_{\mathbb{Q}_p} Y \subseteq \mathcal{M}.$$

and

$$\mathcal{X} = \mathcal{N} \cap \mathcal{Y}\left[\left(\tfrac{t}{\pi}\right)^{-1}\right] = \left\{x \in \mathcal{N} : \left(\tfrac{t}{\pi}\right)^m x \in \mathcal{Y} \text{ for some } m\right\} \subseteq \mathcal{N}.$$

Proposition 2.4. *The spaces Y, $\mathcal{Y}$, $\mathcal{X}$ have the following properties*:

(a) *$\mathcal{X}$ is a $\mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p}$-submodule of $\mathcal{N}$ stable under φ and Γ*;

(b) *$\mathcal{X} = \{x \in \mathcal{N} : ax \in \mathcal{Y}$ for some nonzero $a \in \mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p}\}$ (the saturation of $\mathcal{Y}$)*;

(c) *$\mathcal{X}$ is free of rank $\dim_{\mathbb{Q}_p} Y$ as an $\mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p}$-module*;

(d) *$Y = \mathcal{X} \cap \mathbb{D}_{\mathrm{cris}}(\mathcal{N})$ and $\mathcal{Y} = \mathcal{X} \cap \mathcal{M}$*;

(e) *$\mathcal{X}$ and $\mathcal{W} = \mathcal{N}/\mathcal{X}$ are (φ, Γ)-modules over $\mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p}$.*

Proof. Part (a) is immediate from the definition.

For (b), suppose $x \in \mathcal{N}$ and there is some nonzero $a \in \mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p}$ such that $ax \in \mathcal{Y}$. By Proposition 2.3, we can find m such that $\left(\tfrac{t}{\pi}\right)^m x \in \mathcal{M}$, and $a\left(\tfrac{t}{\pi}\right)^m x \in \mathcal{Y}$. Since $\mathcal{Y}$ is clearly saturated in $\mathcal{M}$, we deduce that $\left(\tfrac{t}{\pi}\right)^m x \in \mathcal{Y}$, and hence $x \in \mathcal{X}$ as required.

For part (c), we note that $\mathcal{X}$ is a closed submodule of $\mathcal{N}$, since it is the intersection of the closed submodules $\left(\tfrac{t}{\pi}\right)^{-N}\mathcal{Y}$ and $\mathcal{N}$ of $\left(\tfrac{t}{\pi}\right)^{-N}\mathcal{N}$, for any sufficiently large N. (It suffices to take N larger than s_d, where $-s_d$ is the lowest Hodge–Tate weight of $\mathcal{N}$.) Hence $\mathcal{X}$ is also a free module, of finite rank. As $\mathcal{X}\left[\left(\tfrac{t}{\pi}\right)^{-1}\right]$ is clearly free of rank $\dim_{\mathbb{Q}_p} Y$ as a $\mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p}\left[\left(\tfrac{t}{\pi}\right)^{-1}\right]$-module, the rank of $\mathcal{X}$ over $\mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p}$ must also be equal to $\dim_{\mathbb{Q}_p} Y$.

For part (d), it is clear that $\mathcal{Y} \subseteq \mathcal{X} \cap \mathcal{M}$; and this inclusion is an equality, since $\mathcal{M}/\mathcal{Y}$ is torsion-free and $\mathcal{X}/\mathcal{Y}$ is torsion. Since $\mathcal{Y} \cap \mathbb{D}_{\mathrm{cris}}(\mathcal{N}) = Y$, the statement follows.

For the final statement (e), since $\mathscr{X}$ and $\mathscr{W} = \mathscr{N}/\mathscr{X}$ are both free $\mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p}$-modules with semilinear actions of φ and Γ, it suffices to check that the modules $\mathscr{X}/\varphi^*\mathscr{X}$ and $\mathscr{W}/\varphi^*\mathscr{W}$ are annihilated by a power of q. Since $\mathscr{X}$ is saturated in $\mathscr{N}$, and $\mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p}$ is an elementary divisor ring, we can find a $\mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p}$-basis $n_1, \dots, n_d$ of $\mathscr{N}$ such that $n_1, \dots, n_r$ is a basis of $\mathscr{X}$ and the images of $n_{r+1}, \dots, n_d$ are a basis of $\mathscr{W}$, where $r = \dim_{\mathbb{Q}_p} Y$. Since $\mathscr{X}$ is φ-stable, the matrix of φ in this basis is of the form $\left(\begin{smallmatrix} A & B \\ 0 & C \end{smallmatrix}\right)$. Hence we have $\det(\varphi^*\mathscr{N}) = \det(A)\det(C)$. As $\mathscr{N}/\varphi^*\mathscr{N}$ is annihilated by a power of q, $\det(\varphi^*\mathscr{N})$ is a power of q, and thus the same is true of $\det(A)$ and $\det(C)$. Since A and C are the matrices of φ on $\mathscr{X}$ and $\mathscr{W}$ in the bases described above, the modules $\mathscr{X}/\varphi^*\mathscr{X}$ and $\mathscr{W}/\varphi^*\mathscr{W}$ are also annihilated by a power of q, as required. $\square$

Let $W = \mathbb{D}_{\mathrm{cris}}(\mathscr{N})/Y$, and (as above) let $\mathscr{W} = \mathscr{N}/\mathscr{X}$. The natural map $W \hookrightarrow \mathbb{D}_{\mathrm{cris}}(\mathscr{W})$ is injective, by part (d) of the preceding proposition; hence it is also surjective, for reasons of dimension. Thus $\mathscr{W}$ is a crystalline (φ, Γ)-module and $\mathbb{D}_{\mathrm{cris}}(\mathscr{W}) = W$.

Proposition 2.5. *The quotient filtration* $\mathrm{Fil}^\bullet W$ *induced on* W *by the filtration of* $\mathbb{D}_{\mathrm{cris}}(N)$ *agrees with the filtation* $\underline{\mathrm{Fil}}$ *given by*

$$\underline{\mathrm{Fil}}^r\, W = \{w \in W : \varphi(w) \in q^r \mathscr{W}\}.$$

Proof. It is clear from the definition that $\mathrm{Fil}^r W \subseteq \underline{\mathrm{Fil}}^r\, W$.

Conversely, let $y \in \mathbb{D}_{\mathrm{cris}}(N)$ such that $[y] \in \underline{\mathrm{Fil}}^r\, W$, so we can write $\varphi(y) = q^r y' + z$ for some $y' \in N$ and $z \in \mathscr{X}$. Then

$$z \bmod q^r\mathscr{X} \in (\mathscr{X}/q^r\mathscr{X})^\Gamma.$$

Applying Corollary 2.2 to $\mathscr{X}$, we find that z is congruent modulo q^r to an element of $\mathscr{X}^\Gamma = Y$. $\square$

The final result we will need about these quotients is the following slightly fiddly lemma. Let us suppose that the jumps in the filtration of $\mathbb{D}_{\mathrm{cris}}(\mathscr{N})$, with multiplicity, are $s_1 \le s_2 \cdots \le s_d$ (i.e., the Hodge–Tate weights of $\mathscr{N}$ are $-s_i$). We say that the φ-stable subspace Y is *in general position* (with respect to the Hodge filtration of $\mathbb{D}_{\mathrm{cris}}(\mathscr{N})$) if the jumps in the filtration $\mathrm{Fil}^\bullet Y$ are $s_1, \dots, s_j$, where $j = \dim_{\mathbb{Q}_p} Y$.

Lemma 2.6. *If* Y *is in general position, then for any* $m \ge s_d$, *we have*

$$\left(\tfrac{t}{\pi}\right)^{m - s_{(j+1)}} \mathscr{M} \subseteq \left(\tfrac{t}{\pi}\right)^m \mathscr{N} + \mathscr{Y}.$$

Proof. As remarked above, the quotient module $\mathscr{W} = \mathscr{N}/\mathscr{X}$ is a crystalline (φ, Γ)-module over $\mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p}$ of rank $d - j$, with Γ acting trivially modulo π. By Proposition 2.5, the Hodge–Tate weights of $\mathscr{W}$ are exactly $\{-s_{(j+1)}, \dots, -s_d\}$; hence its Γ-invariants lie in $\left(\frac{t}{\pi}\right)^{s_{(j+1)}} \mathscr{W}$. This is equivalent to $\mathscr{M} \subseteq \left(\frac{t}{\pi}\right)^{s_{(j+1)}} \mathscr{N} + \mathscr{X}$. Multiplying by $(\frac{t}{\pi})^{m-s_{(j+1)}}$, we see that

$$\left(\tfrac{t}{\pi}\right)^{m-s_{(j+1)}} \mathcal{M} \subseteq \left(\tfrac{t}{\pi}\right)^{m} \mathcal{N} + \left(\tfrac{t}{\pi}\right)^{m-s_{(j+1)}} \mathcal{X}.$$

Since both $\left(\frac{t}{\pi}\right)^{m-s_{(j+1)}} \mathcal{M}$ and $\left(\frac{t}{\pi}\right)^{m} \mathcal{N}$ are manifestly contained in $\mathcal{M}$, we may replace the last term with its intersection with $\mathcal{M}$, which is clearly contained in $\mathcal{X} \cap \mathcal{M} = \mathcal{Y}$. □

2C. ***Application to crystalline representations.*** Let V be a d-dimensional crystalline representation of $G_{\mathbb{Q}_p}$ with Hodge–Tate weights $\{-s_1, \dots, -s_d\}$, where $0 \le s_1 \le \dots \le s_d$ (so V is positive in the sense of Section 1C4 above). As above, we define $\mathbb{N}_{\mathrm{rig}}(V) = \mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p} \otimes_{\mathbb{B}^+_{\mathbb{Q}_p}} \mathbb{N}(V)$, where $\mathbb{N}(V)$ is the Wach module of V as constructed in [Berger 2004]. Then $\mathbb{N}_{\mathrm{rig}}(V)$ is a crystalline (φ, Γ)-module over $\mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p}$ with Γ acting trivially modulo π, and $\mathbb{D}_{\mathrm{cris}}(V)$ is isomorphic (as a filtered φ-module over $\mathbb{Q}_p$) to $\mathbb{D}_{\mathrm{cris}}(\mathbb{N}_{\mathrm{rig}}(V))$ as defined in the previous section [Berger 2004, Theorems II.2.2 and III.4.4].

If V is in fact an E-linear representation, for E some finite extension of $\mathbb{Q}_p$, then $\mathbb{N}_{\mathrm{rig}}(V)$ is naturally an $E \otimes_{\mathbb{Q}_p} \mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p}$-module, and $\mathbb{D}_{\mathrm{cris}}(V)$ is a filtered E-vector space. If we choose an E-linear φ-stable subspace, then all of the above constructions commute with the additional E-linear structure.

We shall suppose that V admits a non-critical refinement, and fix a choice of such a refinement $\underline{Y}$. Applying the above theory to each of the subspaces Y_i, we obtain $E \otimes_{\mathbb{Q}_p} \mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p}$-submodules $\mathcal{Y}_i = \mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p} \otimes_{\mathbb{Q}_p} Y_i \subseteq \mathcal{M}$ and $\mathcal{X}_i = \mathcal{Y}_i^{\mathrm{sat}}$ of $\mathbb{N}_{\mathrm{rig}}(V)$.

Let us consider the representation $V(m)$, for some $m \ge s_d$. This has non-negative Hodge–Tate weights $\{m - s_i\}_{i=1,\dots,d}$. If e_m denotes a basis of $\mathbb{Q}_p(m)$, then we have

$$\mathbb{D}_{\mathrm{cris}}(V(m)) = \{t^{-m} x \otimes e_m : x \in \mathbb{D}_{\mathrm{cris}}(V)\},$$
$$\mathbb{N}_{\mathrm{rig}}(V(m)) = \{\pi^{-m} y \otimes e_m : y \in \mathbb{N}_{\mathrm{rig}}(V)\}.$$

We define $\mathcal{A}_i = \{\pi^{-m} y \otimes e_m : y \in \mathcal{X}_i\}$ and $\mathcal{B}_i = \{t^{-m} x \otimes e_m : x \in \mathcal{Y}_i\}$.

Proposition 2.7. *For each $i = 1, \dots, d$,*

(a) $\left(\frac{t}{\pi}\right)^{m-s_i} \mathcal{B}_i \supseteq \mathcal{A}_i \supseteq \left(\frac{t}{\pi}\right)^{m-s_1} \mathcal{B}_i$;

(b) *$\mathcal{B}_i$ is the saturation of $\mathcal{A}_i$ in $\mathcal{B}_d = \mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p} \otimes \mathbb{D}_{\mathrm{cris}}(V(m))$;*

(c) *The inclusion $\mathcal{A}_d \hookrightarrow \mathcal{B}_d$ identifies $\mathcal{A}_d/\mathcal{A}_{i-1}$ with a submodule of $\mathcal{B}_d/\mathcal{B}_{i-1}$ and the quotient is annihilated by $\left(\frac{t}{\pi}\right)^{m-s_i}$.*

Proof. The chain of inclusions in (a) is equivalent to $\left(\frac{t}{\pi}\right)^{s_1} \mathcal{X}_i \supseteq \mathcal{Y}_i \supseteq \left(\frac{t}{\pi}\right)^{s_i} \mathcal{X}_i$, and this is a consequence of Proposition 2.3 since the Hodge–Tate weights of $\mathcal{X}_i$ are $\{-s_1, \dots, -s_i\}$. Moreover, $\mathcal{B}_i$ is manifestly saturated in $\mathcal{B}_d$ (being the base extension of a subspace of $\mathbb{D}_{\mathrm{cris}}(V(m))$), and together with (a), this proves (b). For part (c), we note that $\mathcal{A}_d \cap \mathcal{B}_{i-1} = \mathcal{A}_{i-1}$, so the given map is well-defined and injective; to show that the annihilator is as claimed, we must check that

$$\left(\tfrac{t}{\pi}\right)^{m-s_i}\mathcal{B}_d \subseteq \mathcal{B}_{i-1} + \mathcal{A}_d,$$

which is equivalent to Lemma 2.6. □

We now pass from the "additive" to the "multiplicative" situation. Let us define $\widetilde{\mathcal{A}}_i = \bigoplus_{s=1}^{p-1}(1+\pi)^s\varphi(\mathcal{A}_i)$, and similarly for $\widetilde{\mathcal{B}}_i$. Note that these are Γ-stable, since Γ and φ commute. Moreover, the action of Γ on $\widetilde{\mathcal{B}}_d$ clearly extends to an action of the ring $\mathcal{H}(\Gamma)$, which is continuous with respect to the Fréchet topologies of $\mathcal{H}(\Gamma)$ and $\widetilde{\mathcal{B}}_d$. As the submodules $\widetilde{\mathcal{B}}_i$ and $\widetilde{\mathcal{A}}_i$ are all clearly closed and Γ-invariant, they also inherit a Fréchet topology and a continuous action of $\mathcal{H}(\Gamma)$.

Remark 2.8. Note that we can define an operator $\psi : \mathcal{B}_d \to \mathcal{B}_d$ which is φ^{-1} on $\mathbb{D}_{\mathrm{cris}}(V)$ and is $\mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p}$-semilinear (for the usual definition of ψ acting on $\mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p}$). Then $\widetilde{\mathcal{A}}_i = (\varphi^*\mathcal{A}_i)^{\psi=0}$, where $\varphi^*\mathcal{A}_i$ is the $\mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p}$-submodule of $\mathcal{B}_i$ generated by $\varphi(\mathcal{A}_i)$. Clearly we have $\varphi^*(\mathcal{B}_i) = \mathcal{B}_i$ for all i, and $\widetilde{\mathcal{B}}_i = (\varphi^*\mathcal{B}_i)^{\psi=0} = \mathcal{B}_i^{\psi=0}$.

Lemma 2.9. *For each $i = 1, \dots, d$, these spaces have the following properties*:

(a) $\widetilde{\mathcal{A}}_i \subseteq \widetilde{\mathcal{B}}_i$.

(b) $\widetilde{\mathcal{A}}_d \cap \widetilde{\mathcal{B}}_i = \widetilde{\mathcal{A}}_i$.

(c) *The quotient $\widetilde{\mathcal{B}}_d/(\widetilde{\mathcal{B}}_{i-1} + \widetilde{\mathcal{A}}_d)$ is annihilated by $\mathfrak{n}_{m-s_i}$.*

(d) *The quotient $\widetilde{\mathcal{B}}_i/(\widetilde{\mathcal{B}}_{i-1} + \widetilde{\mathcal{A}}_i)$ is cyclic as a $\mathcal{H}(\Gamma)$-module*; *it is generated by $(1+\pi)\varphi(v_i)$, and its annihilator is $\mathfrak{n}_{m-s_i}$.*

Proof. Parts (a) and (b) are clear from the corresponding statements for the spaces $\mathcal{A}_i$ and $\mathcal{B}_i$. For part (c), we note that $\mathcal{B}_d/\mathcal{B}_{i-1}$ is isomorphic as a (φ, Γ)-module over $\mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p}$ to the tensor product

$$\mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p} \otimes_{\mathbb{Q}_p} (Y_d/Y_{i-1})$$

with Γ acting trivially on the latter factor and the φ-action multiplied by p^{-m}. By Proposition 1.6, we have

$$\mathfrak{n}_k \cdot (\mathcal{B}_d/\mathcal{B}_{i-1})^{\psi=0} = \left(\left(\frac{t}{\varphi(\pi)}\right)^k \mathcal{B}_d/\mathcal{B}_{i-1}\right)^{\psi=0}. \qquad (*)$$

Since $\mathcal{B}_d/(\mathcal{B}_{i-1}+\mathcal{A}_d)$ is annihilated by $\left(\tfrac{t}{\pi}\right)^{m-s_i}$, we deduce that $\mathcal{B}_d/(\mathcal{B}_{i-1}+\varphi^*\mathcal{A}_d)$ is annihilated by

$$\left(\frac{t}{\varphi(\pi)}\right)^{m-s_i}.$$

Hence, by (*), $\widetilde{\mathcal{B}}_d/(\widetilde{\mathcal{B}}_{i-1} + \widetilde{\mathcal{A}}_d)$ is annihilated by the ideal $\mathfrak{n}_{m-s_i}$ of $\mathcal{H}(\Gamma_1)$.

Similarly, $\mathcal{B}_i/(\mathcal{B}_{i-1}+\mathcal{A}_i)$ has the single elementary divisor

$$\left(\frac{t}{\pi}\right)^{m-s_i}$$

(by applying Proposition 2.3 to $\mathcal{X}_i/\mathcal{X}_{i-1}$, which is a (φ,Γ)-module over $\mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p}$ by Proposition 2.4(e)). Hence we deduce that $\mathfrak{n}_{m-s_i}$ is the exact annihilator of the corresponding $\mathcal{H}(\Gamma)$-module $\widetilde{\mathcal{B}}_i/(\widetilde{\mathcal{B}}_{i-1}+\widetilde{\mathcal{A}}_i)$. □

We are now in a position to complete the proof of Theorem A.

Theorem 2.10 (Theorem A). *Let W be any E-linear crystalline representation of $G_{\mathbb{Q}_p}$ with non-negative Hodge–Tate weights $r_1 \le \cdots \le r_d$. Suppose that there exists a finite extension F of E such that $V \otimes_E F$ admits a non-critical refinement. Then the $E \otimes \mathcal{H}(\Gamma)$-elementary divisors of the quotient*

$$(\mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p})^{\psi=0} \otimes \mathbb{D}_{\mathrm{cris}}(W)/(\varphi^*\mathbb{N}_{\mathrm{rig}}(W))^{\psi=0}$$

are $[\mathfrak{n}_{r_1}; \dots; \mathfrak{n}_{r_d}]$.

Proof. Let us choose an m such that $V = W(-m)$ is positive. Then the Hodge–Tate weights of V are $-s_1 \ge \cdots \ge -s_d$, where $s_i = m - r_{d+1-i} \ge 0$. Suppose first that V admits a non-critical refinement. Choosing such a refinement, we may argue as above to deduce that the $E \otimes \mathcal{H}(\Gamma)$-module

$$M = (\mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p})^{\psi=0} \otimes \mathbb{D}_{\mathrm{cris}}(W)/(\varphi^*\mathbb{N}_{\mathrm{rig}}(W))^{\psi=0} = \mathcal{B}_d/\mathcal{A}_d$$

has a filtration by $E \otimes_{\mathbb{Q}_p} \mathcal{H}(\Gamma)$-modules $M_i = \mathcal{B}_i/\mathcal{A}_i$ where M_i/M_{i-1} is cyclic with annihilator $\mathfrak{n}_{m-s_i}$, and $\mathfrak{n}_{m-s_i}$ annihilates M/M_{i-1}. So for each character η of Δ, the module M^η is an $\mathcal{H}(\Gamma_1)$-module of the type covered by Corollary 1.12. This gives the result in this special case.

If V only admits a non-critical refinement after extending scalars to an extension F/E, then we may consider the representation $V \otimes_E F$ and apply the above argument to this representation. It is clear that if M is any $E \otimes \mathcal{H}(\Gamma)$-module, then the elementary divisors of $F \otimes_E M$ as a $F \otimes \mathcal{H}(\Gamma)$-module are the base extensions of the elementary divisors of M; by uniqueness, this gives the proposition. □

We now briefly explain how $\varphi^*\mathbb{N}_{\mathrm{rig}}(V)$ is related to the Wach module $\mathbb{N}(V)$ considered in our earlier work. Note that $\mathcal{H}(\Gamma)$ and $\varphi(\mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p})$ are both Fréchet–Stein algebras in the sense of [Schneider and Teitelbaum 2003] (by Theorem 5.1 of that reference); hence any finite-rank free module over either one of these algebras has a canonical topology, and a submodule of such a module is finitely generated if and only if it is closed in this topology (Corollary 3.4(ii) of [op. cit.]). Moreover, $(\mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p})^{\psi=0} = \bigoplus_{i=1}^{p-1}(1+\pi)^i\varphi(\mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p})$ is a free module over $\varphi(\mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p})$ of rank $p-1$.

Proposition 2.11. *There is an isomorphism*

$$(\varphi^*\mathbb{N}_{\mathrm{rig}}(V))^{\psi=0} \cong \mathcal{H}(\Gamma) \otimes_{\Lambda_{\mathbb{Q}_p}(\Gamma)} (\varphi^*\mathbb{N}(V))^{\psi=0}.$$

Proof. We first note that $(\varphi^*\mathbb{N}_{\mathrm{rig}}(V))^{\psi=0} = \bigoplus_{i=1}^{p-1}(1+\pi)^i\varphi(\mathbb{N}_{\mathrm{rig}}(V))$ is a finitely generated $\varphi(\mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p})$-submodule of $\mathbb{D}_{\mathrm{cris}}(V) \otimes_{\mathbb{Q}_p} (\mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p})^{\psi=0}$. Hence it is closed in the canonical Fréchet topology of the latter space. It is also Γ-stable. Since the Mellin transform is a topological isomorphism between $(\mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p})^{\psi=0}$ and $\mathcal{H}(\Gamma)$, we see that $(\varphi^*\mathbb{N}_{\mathrm{rig}}(V))^{\psi=0}$ is a closed Γ-stable subspace of a finite-rank free $\mathcal{H}(\Gamma)$-module; hence the action of Γ extends to a (continuous) action of $\mathcal{H}(\Gamma)$. So there is a natural embedding of $\mathcal{H}(\Gamma) \otimes_{\Lambda_{\mathbb{Q}_p}(\Gamma)} (\varphi^*\mathbb{N}(V))^{\psi=0}$ into $(\varphi^*\mathbb{N}_{\mathrm{rig}}(V))^{\psi=0}$.

The image of this embedding is a $\mathcal{H}(\Gamma)$-submodule, which is finitely generated, since $(\varphi^*\mathbb{N}(V))^{\psi=0}$ is finitely generated as a Λ_E-module [Lei et al. 2010, Theorem 3.5]. So it is closed. On the other hand, the image contains $(\varphi^*\mathbb{N}(V))^{\psi=0}$. Since we evidently have

$$(\varphi^*\mathbb{N}_{\mathrm{rig}}(V))^{\psi=0} = \bigoplus_{i=1}^{p-1}(1+\pi)^i\varphi(\mathbb{N}_{\mathrm{rig}}(V))$$

and $\varphi(\mathbb{N}_{\mathrm{rig}}(V)) = \varphi(\mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p}) \otimes_{\varphi(\mathbb{B}^+_{\mathbb{Q}_p})} \varphi(\mathbb{N}(V))$, it follows that

$$\begin{aligned}(\varphi^*\mathbb{N}_{\mathrm{rig}}(V))^{\psi=0} &= \varphi(\mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p}) \otimes_{\varphi(\mathbb{B}^+_{\mathbb{Q}_p})} \left(\bigoplus_{i=1}^{p-1}(1+\pi)^i\varphi(\mathbb{N}(V))\right)\\ &= \varphi(\mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p}) \otimes_{\varphi(\mathbb{B}^+_{\mathbb{Q}_p})} (\varphi^*\mathbb{N}(V))^{\psi=0}.\end{aligned}$$

Since $\varphi(\mathbb{B}^+_{\mathbb{Q}_p})$ is dense in $\varphi(\mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p})$, it follows now that $(\varphi^*\mathbb{N}(V))^{\psi=0}$ is dense in $(\varphi^*\mathbb{N}_{\mathrm{rig}}(V))^{\psi=0}$. Thus the image of $\mathcal{H}(\Gamma) \otimes_{\Lambda_{\mathbb{Q}_p}(\Gamma)} (\varphi^*\mathbb{N}(V))^{\psi=0}$ in $(\varphi^*\mathbb{N}_{\mathrm{rig}}(V))^{\psi=0}$ is both dense and closed; hence it is everything. □

We recall the following result from our previous work:

Theorem 2.12 ([Lei et al. 2010, Lemma 3.15]). $(\varphi^*\mathbb{N}(V))^{\psi=0}$ *is a free* $\Lambda_E(\Gamma)$-*module of rank* d. *More specifically, for any basis* $\nu_1, \dots, \nu_d$ *of* $\mathbb{D}_{\mathrm{cris}}(V)$, *there exists a* $E \otimes \mathbb{B}^+_{\mathbb{Q}_p}$-*basis* $n_1, \dots, n_d$ *of* $\mathbb{N}(V)$ *such that* $n_i = \nu_i \bmod \pi$ *and* $(1+\pi)\varphi(n_1), \dots, (1+\pi)\varphi(n_d)$ *form a* $\Lambda_E(\Gamma)$-*basis of* $(\varphi^*\mathbb{N}(V))^{\psi=0}$.

Combining this with Proposition 2.11, the following corollary is immediate:

Corollary 2.13. $(\varphi^*\mathbb{N}_{\mathrm{rig}}(V))^{\psi=0}$ *is a free* $E \otimes \mathcal{H}(\Gamma)$-*module of rank* d. *More specifically, for any basis* $\nu_1, \dots, \nu_d$ *of* $\mathbb{D}_{\mathrm{cris}}(V)$, *there exists a* $E \otimes \mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p}$-*basis* $n_1, \dots, n_d$ *of* $\mathbb{N}_{\mathrm{rig}}(V)$ *such that* $n_i = \nu_i \bmod \pi$ *and* $(1+\pi)\varphi(n_1), \dots, (1+\pi)\varphi(n_d)$ *form a* $E \otimes \mathcal{H}(\Gamma)$-*basis of* $(\varphi^*\mathbb{N}_{\mathrm{rig}}(V))^{\psi=0}$.

Remark 2.14. We conjecture that for *any* $E\otimes\mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p}$-basis $m_1,\dots,m_d$ of $\mathbb{N}_{\mathrm{rig}}(V)$, the vectors $(1+\pi)\varphi(m_i)$ are a $E\otimes\mathcal{H}(\Gamma)$-basis of $(\varphi^*\mathbb{N}_{\mathrm{rig}}(V))^{\psi=0}$, and similarly for $\mathbb{N}(V)$; but we do not know a proof of this statement.

3. The construction of Coleman maps

3A. ***Coleman maps and the Perrin-Riou p-adic regulator.*** Let E be a finite extension of $\mathbb{Q}_p$. Let V be a d-dimensional E-linear representation of $G_{\mathbb{Q}_p}$ with non-negative Hodge–Tate weights $r_1\le r_2\le\cdots\le r_d$. We assume that V has no quotient isomorphic to the trivial representation. Let T be a $\mathcal{G}_{\mathbb{Q}_p}$-stable $\mathcal{O}_E$-lattice in V. Under these assumptions, there is a canonical isomorphism of $\Lambda_{\mathcal{O}_E}(\Gamma)$-modules

$$h^1_{\mathrm{Iw}}:\mathbb{N}(T)^{\psi=1}\xrightarrow{\cong}H^1_{\mathrm{Iw}}(\mathbb{Q}_p,T).$$

by [Berger 2003, Theorem A.3]. Moreover, since the Hodge–Tate weights of V are non-negative, we have $\mathbb{N}(T)\subseteq\varphi^*\mathbb{N}(T)$ by Lemma 1.7. Hence there is a well-defined map $1-\varphi:\mathbb{N}(T)\to\varphi^*\mathbb{N}(T)$, which maps $\mathbb{N}(T)^{\psi=1}$ to $(\varphi^*\mathbb{N}(T))^{\psi=0}$.

As we recalled above, [Lei et al. 2010, Theorem 3.5] (due to Laurent Berger) shows that for some basis $n_1,\dots,n_d$ of $\mathbb{N}(T)$ as an $\mathcal{O}_E\otimes\mathbb{A}^+_{\mathbb{Q}_p}$-module, the vectors $(1+\pi)\varphi(n_1),\dots,(1+\pi)\varphi(n_d)$ form a basis of $(\varphi^*\mathbb{N}(T))^{\psi=0}$ as a $\Lambda_{\mathcal{O}_E}(\Gamma)$-module. This basis gives an isomorphism

$$\mathfrak{J}:(\varphi^*\mathbb{N}(T))^{\psi=0}\xrightarrow{\cong}\Lambda_{\mathcal{O}_E}(\Gamma)^{\oplus d}$$

(the *Iwasawa transform*), and we define the Coleman map

$$\underline{\mathrm{Col}}=(\underline{\mathrm{Col}}_i)_{i=1}^d:\mathbb{N}(T)^{\psi=1}\to\Lambda_{\mathcal{O}_E}(\Gamma)^{\oplus d}$$

as the composition $\mathfrak{J}\circ(1-\varphi)$.

Remark 3.1. This direct definition of the Coleman map is equivalent to that given in our earlier work, but applies to any representation with non-negative Hodge–Tate weights, rather than starting with a positive representation and twisting by the sum of its Hodge–Tate weights as in [Lei et al. 2010].

Let $\nu_1,\dots,\nu_d$ be a basis of $\mathbb{D}_{\mathrm{cris}}(V)$, so $(1+\pi)\otimes\nu_1,\dots,(1+\pi)\otimes\nu_d$ are a basis of $(\mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p})^{\psi=0}\otimes\mathbb{D}_{\mathrm{cris}}(V)$ as an $\mathcal{H}(\Gamma)$-module; and let $n_1,\dots,n_d$ be a basis of $\mathbb{N}(V)$ lifting $\nu_1,\dots,\nu_d$ as in Theorem 2.12. Then there exists a unique $d\times d$ matrix $\underline{M}$ with entries in $\mathcal{H}(\Gamma)$ such that

$$\begin{pmatrix}(1+\pi)\varphi(n_1)\\ \vdots\\ (1+\pi)\varphi(n_d)\end{pmatrix}=\underline{M}\cdot\begin{pmatrix}(1+\pi)\otimes\nu_1\\ \vdots\\ (1+\pi)\otimes\nu_d\end{pmatrix}. \tag{1}$$

In fact $\underline{M}$ is defined over $\mathcal{H}(\Gamma_1)$, since the n_i lie in $(1+\pi)\varphi(\mathbb{N}(V)) \subseteq (1+\pi)\varphi(\mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p}) \otimes \mathbb{D}_{\mathrm{cris}}(V)$. By Theorem 2.10, we know that the elementary divisors of $\underline{M}$ are $\mathfrak{n}_{r_1}, \ldots, \mathfrak{n}_{r_d}$.

Corollary 3.2. *Up to a unit*, $\det(\underline{M})$ *is equal to* $\prod_{i=1}^{d} \mathfrak{n}_{r_i}$.

We can write the Coleman map $\underline{\mathrm{Col}}$ in terms of $\underline{M}$ as follows:

Lemma 3.3. *For* $x \in \mathbb{N}(T)^{\psi=1}$, *we have*

$$(1-\varphi)(x) = \underline{\mathrm{Col}}(x) \cdot \underline{M} \cdot \begin{pmatrix} (1+\pi)\otimes \nu_1 \\ \vdots \\ (1+\pi)\otimes \nu_d \end{pmatrix}.$$

Proof. We have by definition

$$(1-\varphi)x = \underline{\mathrm{Col}}(x) \cdot \begin{pmatrix} (1+\pi)\varphi(n_1) \\ \vdots \\ (1+\pi)\varphi(n_d) \end{pmatrix}.$$

Therefore, we are done on combining this with (1). □

Definition 3.4. The *Perrin-Riou p-adic regulator* $\mathcal{L}_V$ for V is defined to be the $\Lambda_E(\Gamma)$-homomorphism

$$\left(\mathfrak{M}^{-1} \otimes 1\right) \circ (1-\varphi) \circ \left(h^1_{\mathrm{Iw},V}\right)^{-1} : H^1_{\mathrm{Iw}}(\mathbb{Q}_p, V) \longrightarrow \mathcal{H}(\Gamma) \otimes \mathbb{D}_{\mathrm{cris}}(V).$$

Using the isomorphism $h^1_{\mathrm{Iw},V} : \mathbb{N}(V)^{\psi=1} \to H^1_{\mathrm{Iw}}(\mathbb{Q}_p, V)$, we can thus rewrite Lemma 3.3 as

$$\mathcal{L}_V(z) = \left(\underline{\mathrm{Col}} \circ (h^1_{\mathrm{Iw},V})^{-1}\right)(z) \cdot \underline{M} \cdot \begin{pmatrix} \nu_1 \\ \vdots \\ \nu_d \end{pmatrix}. \tag{2}$$

4. Images of the Coleman maps

Let η be a character on Δ. In this section, we study the image of $\underline{\mathrm{Col}}^{\eta}(\mathbb{N}(V)^{\psi=1})$ as a subset of $\Lambda_E(\Gamma_1)^{\oplus d}$ for a crystalline representation V of dimension d with non-negative Hodge–Tate weights. We then consider the projection of this image, giving a description of $\mathrm{Im}(\underline{\mathrm{Col}}_i^{\eta})$ for $i = 1, \ldots, d$.

4A. ***Preliminary results on*** $\mathbf{\Lambda_E(\Gamma_1)}$***-modules.*** Recall that we identify $\Lambda_E(\Gamma_1)$ with the power series ring $E \otimes \mathcal{O}_E[[X]]$ by identifying $\gamma - 1$ with X. Therefore, if $F \in \Lambda_E(\Gamma_1)$ and x is an element of the maximal ideal of E, $F|_{X=x} \in E$.

Lemma 4.1. *Let V be an E-subspace of E^d with codimension n. For a fixed element x of the maximal ideal of E, we define the $\Lambda_E(\Gamma_1)$-module*

$$S = \left\{(F_1, \dots, F_d) \in \Lambda_E(\Gamma_1)^{\oplus d} : (F_1(x), \dots, F_d(x)) \in V\right\}.$$

Then, S is free of rank d over $\Lambda_E(\Gamma_1)$ and $\det(S) = (X - x)^n$.

Proof. Let $v_1, \dots, v_d$ be a basis of E such that $\sum_{i=1}^d e_i v_i \in V$ if and only if $e_i = 0$ for all $i > d - n$. On multiplying elementary matrices in $\mathrm{GL}_d(E)$ if necessary, we may assume that S is of the form

$$\begin{aligned} S &= \left\{(F_1, \dots, F_d) \in \Lambda_E(\Gamma_1)^{\oplus d} : F_{d-n+1}(x) = \dots = F_d(x) = 0\right\} \\ &= \Lambda_E(\Gamma_1)^{\oplus(d-n)} \oplus \left((X - x)\Lambda_E(\Gamma_1)\right)^{\oplus n}, \end{aligned}$$

so we are done. □

Proposition 4.2. *Let $I = \{x_0, \dots, x_m\}$ be a subset of the maximal ideal of E. For each $i = 0, \dots, m$, let V_i be an E-subspace of $E^{\oplus d}$ with codimension n_i. Define*

$$S = \left\{(F_1, \dots, F_d) \in \Lambda_E(\Gamma_1)^{\oplus d} : \left(F_1(x_i), \dots, F_d(x_i)\right) \in V_i, i = 0, \dots, m\right\},$$

then S is free of rank d over $\Lambda_E(\Gamma_1)$, and $\det(S) = \prod_{i=0}^m (X - x_i)^{n_i}$.

Proof. We prove the result by induction on m. The case $m = 0$ is just Lemma 4.1.

Assume that $m > 0$ and let

$$S' = \left\{(F_1, \dots, F_d) \in \Lambda_E(\Gamma_1)^{\oplus d} : (F_1(x_i), \dots, F_d(x_i)) \in V_i, i = 0, \dots, m - 1\right\}.$$

By induction, S' is free of rank d over $\Lambda_E(\Gamma_1)$ and $\det(S') = \prod_{i=0}^{m-1}(X - x_i)^{n_i}$. Let $F^{(i)} = \left(F_1^{(i)}, \dots, F_d^{(i)}\right)$, $i = 1, \dots, d$, be a $\Lambda_E(\Gamma_1)$-basis of S'. Write $\mathcal{F}_m$ for the $d \times d$ matrix with entries $F_j^{(i)}(x_m)$. As $X - x_m$ does not divide $\det(F_j^{(i)})$, we have $\mathcal{F}_m \in \mathrm{GL}_d(E)$. Define

$$S'' = \left\{(G_1, \dots, G_d) \in \Lambda_E(\Gamma_1)^{\oplus d} : (G_1(x_m), \dots, G_d(x_m)) \in V_m \mathcal{F}_m^{-1}\right\}.$$

By Lemma 4.1, S'' is free of rank d over $\Lambda_E(\Gamma_1)$ and $\det(S'') = (X - x_m)^{n_m}$. Say, $(G_1^{(k)}, \dots, G_d^{(k)})$, $k = 1, \dots, d$, is a basis.

For $(G_1, \dots, G_d) \in \Lambda_E(\Gamma_1)^{\oplus d}$, we have $\sum_{i=1}^d G_i F^{(i)} \in S' \subset S$ by definition. It is easy to check that $\sum_{i=1}^d G_i F^{(i)} \in S$ if and only if $(G_1, \dots, G_d) \in S''$. Therefore, a basis for S is given by the row vectors of $(G_i^{(k)})(F_j^{(i)})$ and $\det(S) = \det(S')\det(S'')$. Hence, we are done. □

Lemma 4.3. *If S is a $\Lambda_E(\Gamma_1)$-module as in the statement of Proposition 4.2, then the image of a projection from S into $\Lambda_E(\Gamma_1)$ is of the form $\prod_{i \in J}(X - x_i)\Lambda_E(\Gamma_1)$ where J is some subset of* $\{0, \dots, m\}$.

Proof. We consider the first projection $\mathrm{pr}_1 : (F_1, \dots, F_d) \mapsto F_1$. Let

$$J = \{i \in [0, m] : (e_1, \dots, e_d) \in V_i \Rightarrow e_1 = 0\}.$$

It is clear that $\mathrm{Im}(\mathrm{pr}_1) \subset \prod_{i \in J}(X - x_i)\Lambda_E(\Gamma_1)$. It remains to show that

$$\prod_{i \in J}(X - x_i) \in \mathrm{Im}(\mathrm{pr}_1).$$

By definition, for each $i \notin J$, there exist $e_k^{(i)} \in E$, $k = 2, \dots, d$, such that

$$\left(\prod_{j \in J}(x_i - x_j), e_2^{(i)}, \dots, e_d^{(i)}\right) \in V_i.$$

Similarly, take any $(0, e_2^{(i)}, \dots, e_d^{(i)}) \in V_i$ for $i \in J$. There exist polynomials F_k over E such that $F_k(x_i) = e_j^{(i)}$ for $k = 2, \dots, d$ and $i = 0, \dots, m$. It is then clear that

$$\left(\prod_{i \in J}(X - x_i), F_2, \dots, F_d\right) \in S.$$

Hence we are done. □

4B. ***On the image of the Perrin-Riou p-adic regulator.*** with Hodge–Tate weights $-r_d \le -r_{d-1} \le \cdots \le -r_1 \le 0$. As in Section 3A, fix bases $n_1, \dots, n_d$ and $\nu_1, \dots, \nu_d$ of $\mathbb{N}(\mathcal{T})$ and $\mathbb{D}_{\mathrm{cris}}(\mathcal{V})$, respectively, such that $\nu_i = n_i \mod \pi$. For the rest of this paper, we make the following assumption.

$\mathcal{G}_{\mathbb{Q}_p}$ with non-negative Hodge–Tate weights $m - r_1 \ge \cdots \ge m - r_d \ge 0$. Moreover, our assumption on the eigenvalues of φ implies that V has no quotient isomorphic to E.

Let V be a d-dimensional E-linear crystalline representation of $\mathcal{G}_{\mathbb{Q}_p}$ with non-negative Hodge–Tate weights $r_1 \le \cdots \le r_d$.

Definition 4.4. For an integer $i \ge 0$, we write

$$n_i = \dim_E \mathrm{Fil}^{-i}\, \mathbb{D}_{\mathrm{cris}}(V) = \#\{j : r_j \le i\}.$$

We make the following assumption:

Assumption 4.5. *The eigenvalues of φ on $\mathbb{D}_{\mathrm{cris}}(V)$ are not integer powers of p.*

Recall from [Perrin-Riou 1994] that we have the exponential map

$$\Omega_{V,r_d} : (\mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p})^{\psi=0} \otimes \mathbb{D}_{\mathrm{cris}}(V) \to \mathcal{H}(\Gamma) \otimes H^1_{\mathrm{Iw}}(\mathbb{Q}_p, V).$$

The Perrin-Riou p-adic regulator is related to Ω_{V,r_d} via the following equation.

Theorem 4.6. *As maps on $H^1_{\mathrm{Iw}}(\mathbb{Q}_p, V)$, we have*

$$\mathcal{L}_V = \left(\mathfrak{M}^{-1} \otimes 1\right) \left(\prod_{i=0}^{r_d-1} \ell_i\right) \left(\Omega_{V,r_d}\right)^{-1}.$$

Proof. By definition, this is the same as saying

$$(1-\varphi) \circ \left(h^1_{\mathrm{Iw},V}\right)^{-1} = \left(\prod_{i=0}^{r_d-1} \ell_i\right) \left(\Omega_{V,r_d}\right)^{-1},$$

which is just a rewrite of [Berger 2003, Theorem II.13]. □

Corollary 4.7. *We have*

$$\det(\mathcal{L}_V) = \prod_{i=0}^{r_d-1} (\ell_i)^{d-n_i}.$$

Proof. The $\delta(V)$-conjecture (see [Perrin-Riou 1994, Conjecture 3.4.7]) predicts that

$$\det(\Omega_{V,r_d}) = \prod_{i \leq r_d-1} (\ell_i)^{n_i}.$$

As pointed out in Proposition 3.6.7 of the same work, this conjecture is a consequence of Perrin-Riou's explicit reciprocity law, labeled "Conjecture (Réc)" in [op. cit.], and proved in [Colmez 1998, théorème IX.4.5]. Therefore, Theorem 4.6 implies that

$$\det(\mathcal{L}_V) = \left(\prod_{i=0}^{r_d-1} (\ell_i)^d\right)\left(\prod_{i \leq r_d-1} (\ell_i)^{-n_i}\right),$$

which finishes the proof, since $n_i = 0$ for $i < 0$. □

Let $z \in H^1_{\mathrm{Iw}}(\mathbb{Q}_p, V)$. Then $\mathcal{L}_V(z) \in \mathcal{H}(\Gamma) \otimes_{\mathbb{Q}_p} \mathbb{D}_{\mathrm{cris}}(V)$, so we can apply to $\mathcal{L}_V(z)$ any character on Γ to obtain an element in $\mathbb{D}_{\mathrm{cris}}(V)$. The following proposition studies elements obtained in this way when we choose characters of a specific kind. Recall that we denote by χ the cyclotomic character, and by χ_0 the restriction of χ to Δ.

Proposition 4.8. *Let $z \in H^1_{\mathrm{Iw}}(\mathbb{Q}_p, V)$. Then for any integer $0 \leq i \leq r_d - 1$ and any Dirichlet character δ of conductor $p^n > 1$, we have*

$$(1-\varphi)^{-1} \left(1 - p^{-1}\varphi^{-1}\right) \chi^i (\mathcal{L}_V(z) \otimes t^i e_{-i}) \in \mathrm{Fil}^0\, \mathbb{D}_{\mathrm{cris}}(V(-i)); \tag{3}$$

and

$$\varphi^{-n} \left(\chi^i \delta(\mathcal{L}_V(z) \otimes t^i e_{-i})\right) \in \mathbb{Q}_{p,n} \otimes \mathrm{Fil}^0\, \mathbb{D}_{\mathrm{cris}}(V(-i)). \tag{4}$$

Proof. We write $[\ ,\ ]$ for the pairing

$$\mathbb{D}_{\mathrm{cris}}(V(-i))\times\mathbb{D}_{\mathrm{cris}}(V^*(1+i))\longrightarrow\mathbb{D}_{\mathrm{cris}}(E(1))=E\cdot t^{-1}e_1.$$

The orthogonal complement of $\mathrm{Fil}^0\,\mathbb{D}_{\mathrm{cris}}(V(-i))$ under $[\ ,\ ]$ is $\mathrm{Fil}^0(V^*(1+i))$. Let $x\in\mathrm{Fil}^0\,\mathbb{D}_{\mathrm{cris}}(V^*(1+i))$ and $x'=(1-\varphi)(1-p^{-1}\varphi^{-1})^{-1}x$, and write x'_{-i} for $x'\otimes t^ie_{-i}$. Then

$$\begin{aligned}\left[(1-\varphi)^{-1}\left(1-p^{-1}\varphi^{-1}\right)\chi^i(\mathscr{L}_V(z)\otimes t^ie_{-i}),x\right]&=\left[\chi^i(\mathscr{L}_V(z)\otimes t^ie_{-i}),x'\right]\\&=\chi^i[\mathscr{L}_V(z),x'_{-i}],\end{aligned}$$

where the first equality follows from the observation that $1-\varphi$ and $1-p^{-1}\varphi^{-1}$ are adjoint to each other under the pairing $[\ ,\ ]$.

We extend $[\ ,\ ]$ to a pairing on

$$\mathscr{H}(\Gamma)\otimes_{\mathbb{Q}_p}\mathbb{D}_{\mathrm{cris}}(V)\times\mathscr{H}(\Gamma)\otimes_{\mathbb{Q}_p}\mathbb{D}_{\mathrm{cris}}(V^*(1))\longrightarrow E\otimes_{\mathbb{Q}_p}\mathscr{H}(\Gamma)$$

in the natural way. By Perrin-Riou's explicit reciprocity law (see previous proof) and Theorem 4.6, we have

$$\left[\mathscr{L}_V(z),x'_{-i}\right]=(-1)^{r_d-1}\left\langle\left(\prod_{j=0}^{r_d-1}\ell_j\right)z,\Omega_{V^*(1),1-r_d}((1+\pi)\otimes x'_{-i})\right\rangle\tag{5}$$

where $\langle\ ,\ \rangle$ denotes the pairing

$$\left(\mathscr{H}(\Gamma)\otimes H^1_{\mathrm{Iw}}(\mathbb{Q}_p,V)\right)\times\left(\mathscr{H}(\Gamma)\otimes H^1_{\mathrm{Iw}}(\mathbb{Q}_p,V^*(1))\right)\longrightarrow E\otimes\mathscr{H}(\Gamma)$$

as defined in [Perrin-Riou 1994, § 3.6]. By [Perrin-Riou 1994, Lemme 3.6.1(i)], the right-hand side of (5) in fact equals

$$\left\langle z,\left(\prod_{j=0}^{r_d-1}\ell_{-j}\right)\Omega_{V^*(1),1-r_d}((1+\pi)\otimes x'_{-i})\right\rangle=\left\langle z,\Omega_{V^*(1),1}((1+\pi)\otimes x'_{-i})\right\rangle.\tag{6}$$

By an abuse of notation, we let Tw denote the twist map on the H^1_{Iw}'s as well as the map on $\mathscr{H}(\Gamma)$ that sends any $g\in\Gamma$ to $\chi(g)g$. We have

$$\langle\mathrm{Tw}^{-i}(x),\mathrm{Tw}^i(y)\rangle=\mathrm{Tw}^i\langle x,y\rangle$$

for any x and y by [Perrin-Riou 1994, Lemme 3.6.1(ii)]. Therefore, by combining (5) and (6), $\chi^i[\mathscr{L}_V(z),x'_{-i}]$ is equal to the projection of

$$\left\langle\mathrm{Tw}^{-i}(z),\mathrm{Tw}^i\left(\Omega_{V^*(1),1}((1+\pi)\otimes x'_{-i})\right)\right\rangle$$

into E. The projection of $\mathrm{Tw}^i\left(\Omega_{V^*(1),1}((1+\pi)\otimes x'_{-i})\right)$ into $H^1(\mathbb{Q}_p,V^*(1+i))$ at the origin is equal to a scalar multiple of

$$\exp_{\mathbb{Q}_p,V^*(1+i)}\left((1-p^{-1}\varphi^{-1})(1-\varphi)^{-1}(x')\right)$$

(see for example [Lei et al. 2010, Proposition 3.19]). But

$$(1-p^{-1}\varphi^{-1})(1-\varphi)^{-1}(x') = x \in \mathrm{Fil}^0\,\mathbb{D}_{\mathrm{cris}}(V^*(1+i))$$

by definition. Therefore, as $\exp_{\mathbb{Q}_p,V^*(1+i)}$ vanishes on $\mathrm{Fil}^0\,\mathbb{D}_{\mathrm{cris}}(V^*(1+i))$ by construction, it follows that

$$\exp_{\mathbb{Q}_p,V^*(1+i)}\big((1-\varphi)^{-1}(1-p^{-1}\varphi^{-1})(x')\big) = 0$$

and hence that

$$\big[(1-\varphi)^{-1}\left(1-p^{-1}\varphi^{-1}\right)\chi^i(\mathscr{L}_V(z)\otimes t^i e_{-i}), x\big] = \chi^i\big[\mathscr{L}_V(z), x'_{-i}\big] = 0.$$

This implies (3), and (4) can be proved similarly. □

For any character η of Δ and an integer $0 \le i \le r_d - 1$, define

$$V_{i,\eta} = \begin{cases} (1-p^i\varphi)(1-p^{-1-i}\varphi^{-1})^{-1}\,\mathrm{Fil}^{-i}\,\mathbb{D}_{\mathrm{cris}}(V) & \text{if } \chi_0^i = \eta, \\ \varphi\big(\mathrm{Fil}^{-i}\,\mathbb{D}_{\mathrm{cris}}(V)\big) & \text{otherwise.} \end{cases}$$

Note that $V_{i,\eta}$ is a subspace of $\mathbb{D}_{\mathrm{cris}}(V)$ of the same dimension as $\mathrm{Fil}^{-i}\,\mathbb{D}_{\mathrm{cris}}(V)$.

Corollary 4.9. *If η is a character on Δ, then*

$$\big\{\chi^i\chi_0^{-i}\eta(e_\eta\mathscr{L}_V(z)) : z \in H^1_{\mathrm{Iw}}(\mathbb{Q}_p, V)\big\} \subset V_{i,\eta}.$$

Proof. Note that $\mathrm{Fil}^{-i}\,\mathbb{D}_{\mathrm{cris}}(V) = \mathrm{Fil}^0\,\mathbb{D}_{\mathrm{cris}}(V(-i))\otimes t^{-i}e_i$. Therefore, if $\chi_0^i = \eta$, the result follows from (3) and the fact that $\varphi(t^i e_{-i}) = p^i t^i e_{-i}$. Assume otherwise. Since $\chi^i\chi_0^{-i}\eta|_\Delta = \eta$, we have $\chi^i\chi_0^{-i}\eta(e_\eta\mathscr{L}_V(z)) = \chi^i\chi_0^{-i}\eta(\mathscr{L}_V(z))$. Hence, (4) implies that

$$\varphi^{-1}\left(\chi^i\chi_0^{-i}\eta(e_\eta\mathscr{L}_V(z)\otimes t^i e_{-i})\right) \in \mathbb{Q}_p(\mu_p)\otimes\mathrm{Fil}^0\,\mathbb{D}_{\mathrm{cris}}(V(-i)).$$

But

$$\chi^i\chi_0^{-i}\eta(e_\eta\mathscr{L}_V(z)\otimes t^i e_{-i}) = \mathscr{L}_V(z)^\eta|_{X=\chi(\gamma)^i-1}\otimes t^i e_{-i}$$

in fact lies inside $\mathbb{D}_{\mathrm{cris}}(V(-i))$. Hence,

$$\varphi^{-1}\left(\chi^i\chi_0^{-i}\eta(e_\eta\mathscr{L}_V(z)\otimes t^i e_{-i})\right) \in \mathrm{Fil}^0\,\mathbb{D}_{\mathrm{cris}}(V(-i)) = \mathrm{Fil}^{-i}\,\mathbb{D}_{\mathrm{cris}}(V)\otimes t^i e_{-i}$$

and we are done on applying φ to both sides. □

Corollary 4.10. *If η is a character on Δ, then*

$$\big\{\mathscr{L}_V(z)^\eta|_{X=\chi(\gamma)^i-1} : z \in H^1_{\mathrm{Iw}}(\mathbb{Q}_p, V)\big\} \subset V_{i,\eta}.$$

Proof. This is immediate from Corollary 4.9, since

$$\mathscr{L}_V(z)^\eta|_{X=\chi(\gamma)^i-1} = \chi^i\chi_0^{-i}\eta(e_\eta\mathscr{L}_V(z)).$$

□

4C. ***Images of the Coleman maps.*** We now fix a character $\eta : \Delta \to \mathbb{Z}_p^\times$. Let $\nu_1, \dots, \nu_d$ be a basis of $\mathbb{D}_{\mathrm{cris}}(V)$ and $n_1, \dots, n_d$ a basis of $\mathbb{N}(V)$ lifting $\nu_1, \dots, \nu_d$ as in Theorem 2.12. We consider the image of the Coleman map defined with respect to this basis as in Section 3.

Proposition 4.11. *The image of the map*

$$\underline{\mathrm{Col}}^\eta : \mathbb{N}(V)^{\psi=1} \longrightarrow \Lambda_E(\Gamma_1)^{\oplus d}$$

lies inside a $\Lambda_E(\Gamma_1)$-submodule S as described in the statement of Proposition 4.2 with

$$I = \{x_i = \chi(\gamma)^i - 1 : 0 \le i \le r_d - 1\} \quad \text{and} \quad V_i = V_{i,\eta},$$

which is an E-vector space of the same (co-)dimension as $\mathrm{Fil}^{-i}\,\mathbb{D}_{\mathrm{cris}}(V)$.

Proof. Recall from (2) that

$$\mathscr{L}_V = \left(\underline{\mathrm{Col}} \circ h^1_{\mathrm{Iw},V}\right) \underline{M} \begin{pmatrix} \nu_1 \\ \vdots \\ \nu_d \end{pmatrix}$$

where $\underline{M}$ is as defined in (1). Note that $\underline{M}^\eta = \underline{M}$ for any character η of Δ, since $\underline{M}$ is defined over $\mathscr{H}(\Gamma_1)$. Moreover, Corollary 3.2 implies that $X - \chi(\gamma)^i + 1$ does not divide $\det(\underline{M})$, so $\underline{M}|_{X=\chi(\gamma)^i-1} \in \mathrm{GL}_d(E)$. Therefore, we are done by Corollary 4.10. □

Theorem 4.12. *Equality holds in Proposition 4.11.*

Proof. Write S for the basis matrix of the $\Lambda_E(\Gamma_1)$-submodule of $\Lambda_E(\Gamma_1)^{\oplus d}$ described in the statement of Proposition 4.11. Then, Proposition 4.2 says that

$$\det(S) = \prod_{i=0}^{r_d-1} (X - \chi(\gamma)^i + 1)^{d-n_i}.$$

But

$$\det(\underline{M}) = \prod_{j=1}^{d} \left(\prod_{i=0}^{r_j-1} \frac{\ell_i}{X - \chi(\gamma)^i + 1} \right) = \prod_{i=0}^{r_d-1} \left(\frac{\ell_i}{X - \chi(\gamma)^i + 1} \right)^{d-n_i},$$

since $n_i = \#\{j : r_j \le i\}$, as noted above. Hence, Corollary 4.7 implies that

$$\det(\mathscr{L}_V) = \det(\underline{M}) \det(S)$$

and we are done. □

We can summarize the above results via the following short exact sequence:

Corollary 4.13. *Suppose that no eigenvalue of φ on $\mathbb{D}_{\mathrm{cris}}(V)$ lies in $p^{\mathbb{Z}}$. Then for each character η of Δ, there is a short exact sequence of $\mathcal{H}(\Gamma_1)$-modules*

$$0 \longrightarrow \mathbb{N}(V)^{\psi=1,\eta} \xrightarrow{1-\varphi} (\varphi^*\mathbb{N}(V))^{\psi=0,\eta} \xrightarrow{A_\eta} \bigoplus_{i=0}^{r_d-1}(\mathbb{D}_{\mathrm{cris}}(V)/V_{i,\eta})(\chi^i\chi_0^{-i}\eta) \longrightarrow 0.$$

Here the map A_η equals $\bigoplus_i(1\otimes A_{\eta,i})$, where $A_{\eta,i}$ is the natural reduction map $\mathcal{H}(\Gamma)\to\mathbb{Q}_p(\chi^i\chi_0^{-i}\eta)$ obtained by quotienting out by the ideal $(X+1-\chi(\gamma)^i)\cdot e_\eta$.

Remark 4.14. The short exact sequence in Corollary 4.13 can be seen as an analogue of Perrin-Riou's exact sequence (see [Perrin-Riou 1994, §2.2])

$$\begin{aligned} 0 \longrightarrow \bigoplus_{i=0}^{r_d} t^i\mathbb{D}_{\mathrm{cris}}(V)^{\varphi=p^{-i}} &\longrightarrow \left(\mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p}\otimes\mathbb{D}_{\mathrm{cris}}(V)\right)^{\psi=1} \\ &\xrightarrow{\varphi-1} (\mathbb{B}^+_{\mathrm{rig},\mathbb{Q}_p})^{\psi=0}\otimes\mathbb{D}_{\mathrm{cris}}(V) \longrightarrow \bigoplus_{i=0}^{r_d}\left(\frac{\mathbb{D}_{\mathrm{cris}}(V)}{1-p^i\varphi}\right)(i) \longrightarrow 0. \end{aligned}$$

In particular, the injectivity of the first map in our sequence follows from Perrin-Riou's sequence, whose first term vanishes in view of our assumption on V.

We can now prove Theorem C.

Corollary 4.15. *For $i=1,\dots,d$, we have*

$$\mathrm{Im}(\underline{\mathrm{Col}}_i^\eta) = \prod_{j\in I_i^\eta}(X-\chi(\gamma)^j+1)\Lambda_E(\Gamma_1)$$

for some $I_i^\eta\subset\{0,\dots,r_d-1\}$.

Proof. This follows immediately from Lemma 4.3. □

We can also use this argument to determine the elementary divisors of the cokernel of the map $\mathcal{L}_V$, refining the result of Corollary 4.7.

Theorem 4.16. *The elementary divisors of the $\mathcal{H}(\Gamma)$-module quotient*

$$\frac{\mathcal{H}(\Gamma)\otimes_{\mathbb{Q}_p}\mathbb{D}_{\mathrm{cris}}(V)}{\mathcal{H}(\Gamma)\otimes_{\Lambda_{\mathbb{Q}_p}(\Gamma)}\mathrm{Im}(\mathcal{L}_V)}$$

are $[\lambda_{r_1};\dots;\lambda_{r_d}]$, where $\lambda_k=\ell_0\ell_1\dots\ell_{k-1}$.

Proof. We know that the matrix of $\mathcal{L}_V$ is equal to $M\cdot S$, where M and S have elementary divisors that are coprime. Hence the elementary divisors of the product matrix are the products of the elementary divisors, which gives the above formula. □

5. The Coleman maps for modular forms

In this section, we fix a modular form f as in Section 1C6. We pick bases n_1, n_2 of $\mathbb{N}(T_{\bar{f}})$ and $\bar{v}_1, \bar{v}_2$ of $\mathbb{D}_{\mathrm{cris}}(V_{\bar{f}})$ as in [Lei et al. 2010, Section 3.3]. Let $V = V_{\bar{f}}(k-1)$, which has Hodge–Tate weights 0 and $k-1$. We consider the Coleman maps $\underline{\mathrm{Col}}_1^\eta$ and $\underline{\mathrm{Col}}_2^\eta$ defined on $\mathbb{N}(V)^{\psi=1}$ where η is a fixed character on Δ. As a special case for Theorem 4.12 and Corollary 4.15, we have the following result.

Proposition 5.1. *There exist* 1*-dimensional E-subspaces V_i of E^2 for $0 \le i < k-1$ such that*

$$\mathrm{Im}(\underline{\mathrm{Col}}^\eta) = \{(F, G) \in \Lambda_E(\Gamma_1) : \big(F(\chi^i(\gamma)-1), G(\chi^i(\gamma)-1)\big) \in V_i\}.$$

Moreover, for $l = 1, 2$, we have

$$\mathrm{Im}(\underline{\mathrm{Col}}_l^\eta) = \prod_{j \in I_l^\eta} (X - \chi^j(\gamma) + 1)\Lambda_E(\Gamma_1)$$

for some $I_l^\eta \subset \{0, \dots, k-2\}$ with I_1^η and I_2^η disjoint.

Proof. For $0 \le j \le k-2$, $\mathrm{Fil}^{-j}\,\mathbb{D}_{\mathrm{cris}}(V)$ is of dimension 1 over E. Hence the first part of the proposition by Theorem 4.12. The second part of the proposition follows by putting

$$I_1^\eta = \{i : V_i = 0 \oplus E\} \qquad \text{and} \qquad I_2^\eta = \{i : V_i = E \oplus 0\}. \qquad \square$$

Remark 5.2. Note that the second part of the proposition is a slightly stronger version of Corollary 4.15.

Corollary 5.3. *In particular, there exist nonzero elements $r_i \in E$ for $i \in I_3^\eta := \{0, \dots, k-2\} \setminus (I_1^\eta \cup I_2^\eta)$ such that*

$$\mathrm{Im}(\underline{\mathrm{Col}}^\eta) = \left\{ (F, G) \in \Lambda_E(\Gamma_1) \;\middle|\; \begin{array}{l} F(u^i - 1) = 0 \ \ \text{if } i \in I_1^\eta \\ G(u^i - 1) = 0 \ \ \text{if } i \in I_2^\eta \\ F(u^i - 1) = r_i G(u^i - 1) \ \ \text{if } i \in I_3^\eta \end{array} \right\}$$

where $u = \chi(\gamma)$.

The aim of this section is to study the set above in more detail.

5A. ***Some explicit linear relations.*** Recall from [Lei et al. 2010, proof of Proposition 3.22] that the maps $\mathscr{L}_1$ and $\mathscr{L}_2$ as defined in Section 1C6 satisfy

$$\mathscr{L}_V(z) = -\mathscr{L}_2(z)\bar{v}_{1,k-1} + \mathscr{L}_1(z)\bar{v}_{2,k-1}$$

for any $z \in H^1_{\mathrm{Iw}}(\mathbb{Q}_p, V)$. Therefore, Corollary 4.9 says that $\mathscr{L}_1(z)$ and $\mathscr{L}_2(z)$ satisfy some linear relations when evaluated at $\chi^j\delta$ for $0 \le j \le k-2$ and δ some character on Δ. We now make these relations explicit. First we recall that we have:

Lemma 5.4. *Let $j, n \geq 0$ be integers and $i \in \{1, 2\}$. For $z \in H^1_{\mathrm{Iw}}(\mathbb{Q}_p, V)$, we write $z_{-j,n}$ for the image of z under*

$$H^1_{\mathrm{Iw}}(\mathbb{Q}_p, V) \to H^1_{\mathrm{Iw}}(\mathbb{Q}_p, V(-j)) \to H^1(\mathbb{Q}_{p,n}, V(-j)) \tag{7}$$

where the first map is the twist map $(-1)^j \operatorname{Tw}_j$ and the second map is the projection. Then, we have

$$\chi^j(\mathscr{L}_i(z)) = j!\big[(1-\varphi)^{-1}(1-p^{-1}\varphi^{-1})\nu_{i,j+1}, \exp^*_{\mathbb{Q}_p,V(j)}(z_{-j,0})\big]. \tag{8}$$

If δ is a character of G_n which does not factor through G_{n-1} with $n \geq 1$, then

$$\chi^j\delta(\mathscr{L}_i(z)) = \frac{j!}{\tau(\delta^{-1})} \sum_{\sigma\in G_n} \delta^{-1}(\sigma)\big[\varphi^{-n}(\nu_{i,j+1}), \exp^*_{\mathbb{Q}_{p,1},V(j)}(z^\sigma_{-j,n})\big] \tag{9}$$

where τ denotes the Gauss sum.

Proof. See, for example, [Lei 2011, Lemma 3.5 and (4)]. □

Lemma 5.5. *If $0 \leq j \leq k-2$ and δ is a nontrivial character on Δ, then*

$$\chi^j\delta(\mathscr{L}_2(z)) = 0.$$

Proof. On putting $n = 1$ in (9), we have

$$\chi^j\delta(\mathscr{L}_2(z)) = \frac{j!}{\tau(\delta^{-1})} \sum_{\sigma\in\Delta} \delta^{-1}(\sigma)\big[\varphi^{-1}(\nu_{2,j+1}), \exp^*_{\mathbb{Q}_{p,1},V(j)}(z^\sigma_{-j,1})\big].$$

But $\nu_2 = p^{1-k}\varphi(\nu_1)$, so

$$\varphi^{-1}(\nu_{2,j+1}) \in E \cdot \nu_{1,j+1} = \operatorname{Fil}^0 \mathbb{D}_{\mathrm{cris}}(V_f(j+1)).$$

Therefore, we have

$$[\varphi^{-1}(\nu_{2,j+1}), \exp^*_{\mathbb{Q}_{p,1},V(j)}(z^\sigma)] = 0$$

for all $\sigma \in \Delta$ and we are done. □

Lemma 5.6. *If $\varphi^2 + a\varphi + b = 0$, then*

$$(1-\varphi)^{-1}(1-p^{-1}\varphi^{-1}) = \frac{(1+a+pb)\varphi + a(1+a+pb) + b(p-1)}{pb(1+a+b)}.$$

Proof. We can write $\varphi^2 + a\varphi + b = 0$, $\varphi^2 - 1 + a(\varphi-1) = -1-a-b$, and $(1-\varphi)(\varphi+1+a) = 1+a+b$. Therefore,

$$(1-\varphi)^{-1} = \frac{\varphi+1+a}{1+a+b}.$$

Similarly, we have

$$\varphi^{-1} = -\frac{\varphi+a}{b}.$$

The result then follows from explicit calculation. □

Corollary 5.7. *For* $0 \le j \le k-2$, *we have*

$$(-a_p + p^{j+1} + p^{k-1-j})\chi^j(\mathscr{L}_2(z)) = (p-1)\chi^j(\mathscr{L}_1(z)).$$

Proof. On $\mathbb{D}_{\mathrm{cris}}(V_{\bar{f}}(k-1-j))$, φ satisfies

$$\varphi^2 - a_p p^{-k+1+j}\varphi + p^{-k+1+2j} = 0,$$

as we assume $\varepsilon(p) = 1$. Let

$$\begin{aligned} u &= 1 - a_p p^{-k+1+j} + p^{-k+2+2j}, \\ u' &= -a_p p^{-k+1+j} u + p^{-k+1+2j}(p-1). \end{aligned}$$

Then, Proposition 4.8 and Lemma 5.6 imply that

$$(u\varphi + u')\chi^j(-\mathscr{L}_2(z)\bar{v}_{1,k-1-j} + \mathscr{L}_1(z)\bar{v}_{2,k-1-j})$$

lies in $\mathrm{Fil}^0\,\mathbb{D}_{\mathrm{cris}}(V_{\bar{f}}(k-1-j)) = E v_{1,k-1-j}$. On writing this expression as a linear combination of $v_{1,k-1-j}$ and $v_{2,k-1-j}$, the coefficient of the latter turns out to be

$$-p^j u \chi^j(\mathscr{L}_2(z)) + (u' + a_p p^{-k+1+j} u)\chi^r(\mathscr{L}_1(z)),$$

which must be zero, hence the result. $\square$

Remark 5.8. The coefficient $-a_p + p^{j+1} + p^{k-1-j}$ is nonzero by the Weil bound.

Recall from [Lei et al. 2010, (32)] that we have

$$\big(-\mathscr{L}_2\ \ \mathscr{L}_1\big) = \big(\underline{\mathrm{Col}} \circ h^1_{\mathrm{Iw},V}\big)\,\underline{M}.$$

By [Lei et al. 2010, proof of Proposition 3.28 and Theorem 5.4], we have

$$\underline{M}|_{X=0} = A_\varphi^T = \begin{pmatrix} 0 & p^{k-1} \\ -1 & a_p \end{pmatrix}.$$

Therefore, the relations for $j = 0$ are given by

$$\begin{aligned} (-a_p + 1 + p^{k-2})\,\underline{\mathrm{Col}}_2(x)^\Delta|_{X=0} &= p^{k-2}(p-1)\,\underline{\mathrm{Col}}_1(x)^\Delta|_{X=0} && \text{if } \eta = 1, \\ \underline{\mathrm{Col}}_2(x)^\eta|_{X=0} &= 0 && \text{if } \eta \ne 1. \end{aligned}$$

In particular, for the case $k = 2$, we have the following analogue of [Kurihara and Pollack 2007, Proposition 1.2].

Proposition 5.9. *If* $k = 2$, *the trivial isotypical component of the Coleman maps give a short exact sequence*

$$0 \longrightarrow H^1_{\mathrm{Iw}}(\mathbb{Q}_p, V) \xrightarrow{\underline{\mathrm{Col}}^\Delta} \Lambda_E(\Gamma_1) \oplus \Lambda_E(\Gamma_1) \xrightarrow{\rho} \mathbb{Q}_p \longrightarrow 0,$$

where ρ *is defined by*

$$\rho\big(g(X), h(X)\big) = (2 - a_p)g(0) - (p-1)h(0).$$

5B. ***Integral structure of the images.*** We now describe the integral structure of $\mathrm{Im}(\underline{\mathrm{Col}}_i)^\eta$. Under the notation of Corollary 5.3, we define

$$X_i^\eta = \prod_{j \in I_i^\eta} (X - \chi(\gamma)^j + 1).$$

Then, we have:

Theorem 5.10. *For $i = 1, 2$, let X_i^η be as defined above. Then*

$$\underline{\mathrm{Col}}_i\big(\mathbb{D}(T_{\bar f}(k-1))^{\psi=1}\big)^\eta \subset X_i^\eta \Lambda_{\mathbb{O}_E}(\Gamma_1).$$

Moreover, $X_i^\eta \Lambda_{\mathbb{O}_E}(\Gamma_1)/\,\underline{\mathrm{Col}}_i\big(\mathbb{D}(T_{\bar f}(k-1))^{\psi=1}\big)^\eta$ is pseudo-null.

Proof. Let

$$X_k = \prod_{j=0}^{k-2} (X - \chi(\gamma)^j + 1).$$

Note that the proof of Proposition 4.11 in [Lei et al. 2010] is true integrally. We therefore have

$$\big(\varphi^{k-1}(\pi)\varphi^*\mathbb{N}(T_{\bar f}(k-1))\big)^{\psi=0} \subset (1-\varphi)\mathbb{N}(T_{\bar f}(k-1))^{\psi=1}.$$

This implies that $X_k \in \mathrm{Im}(\underline{\mathrm{Col}}_i)$ for $i = 1, 2$. Hence, we have the following inclusions:

$$X_k \Lambda_{\mathbb{O}_E}(\Gamma_1) \subset \underline{\mathrm{Col}}_i\left(\mathbb{D}(T_{\bar f}(k-1))^{\psi=1}\right)^\eta \subset X_i^\eta \Lambda_{\mathbb{O}_E}(\Gamma_1)$$

for $i = 1, 2$. Since X_k is not divisible by ϖ_E, the quotient

$$X_i^\eta \Lambda_{\mathbb{O}_E}(\Gamma_1)/X_k \Lambda_{\mathbb{O}_E}(\Gamma_1)$$

is a free $\mathbb{O}_E$-module of finite rank. Moreover, for a coset representative, x say, it follows from Corollary 4.15 that there exists an integer n such that

$$\varpi_E^n x \in \underline{\mathrm{Col}}_i\big(\mathbb{D}(T_{\bar f}(k-1))^{\psi=1}\big)^\eta.$$

Therefore, $\underline{\mathrm{Col}}_i\big(\mathbb{D}(T_{\bar f}(k-1))^{\psi=1}\big)^\eta$ is of finite index in $X_i^\eta \Lambda_{\mathbb{O}_E}(\Gamma_1)$. □

5C. ***Surjectivity via a change of basis.*** Unfortunately, we do not have an explicit description of the sets I_i^η given by Corollary 5.3. However, this can be resolved by choosing a different basis:

Proposition 5.11. *Let S be a subset of $\Lambda_E(\Gamma_1)^{\oplus 2}$ as defined in Corollary 5.3. Then, there exists $A \in \mathrm{GL}(2, \mathbb{O}_E)$ such that $SA = S'$ for some S' which is of the form*

$$\big\{(F, G) \in \Lambda_E(\Gamma_1)^{\oplus 2} : F(u^i - 1) = r_i' G(u^i - 1), 0 \le i \le k-2\big\}$$

for some nonzero elements $r_i' \in E$.

Proof. Let $e_1, e_2 \in \mathbb{O}_E$ be nonzero elements such that $e_1e_2 \neq 1$, then

$$\begin{pmatrix} 1 & e_2 \\ e_1 & 1 \end{pmatrix} \in \mathrm{GL}(2, \mathbb{O}_E).$$

Let $(F, G) \in \Lambda_E(\Gamma_1)^{\oplus 2}$, we write $\begin{pmatrix} F' & G' \end{pmatrix} = \begin{pmatrix} F & G \end{pmatrix} A = \begin{pmatrix} F + e_1 G & G + e_2 F \end{pmatrix}$. We have

$$F(u^i - 1) = 0 \iff F'(u^i - 1) = e_1 G'(u^i - 1);$$
$$G(u^i - 1) = 0 \iff G'(u^i - 1) = e_2 F'(u^i - 1);$$
$$F(u^i - 1) = r_i G(u^i - 1) \iff (e_2 r_i + 1) F'(u^i - 1) = (e_1 + r_i) G'(u^i - 1).$$

Therefore, we are done on choosing $e_2 \neq -r_i^{-1}$ and $e_1 \neq -r_i$ for all $i \in I_3^{\eta}$. □

Remark 5.12. In the construction of the Coleman maps, replacing $\binom{n_1}{n_2}$ by $A\binom{n_1}{n_2}$, where $A \in \mathrm{GL}(2, \mathbb{O}_E)$, is equivalent to replacing $\underline{M}$ by $A\underline{M}$.

Therefore, on multiplying $\underline{M}$ by an appropriate matrix in $\mathrm{GL}(2, \mathbb{O}_E)$ on the left, we can make both Coleman maps surjective (though we cannot assume $\underline{M}|_{X=0} = A_{\varphi}^{T}$ any more). By Proposition 5.11, we deduce

Theorem 5.13. *There exists a basis of $\mathbb{N}(T_{\bar{f}})$ such that the corresponding Coleman maps have the following properties*:

$$\Lambda_{\mathbb{O}_E}(\Gamma_1)/\,\underline{\mathrm{Col}}_i(\mathbb{D}(T_{\bar{f}}(k-1))^{\psi=1})^{\eta}$$

is pseudo-null for $i = 1, 2$.

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References

[Amice and Vélu 1975] Y. Amice and J. Vélu, "Distributions p-adiques associées aux séries de Hecke", pp. 119–131 in *Journées Arithmétiques de Bordeaux* (Bordeaux, 1974), Astérisque **24-25**, Soc. Math. France, Paris, 1975. MR 51 #12709 Zbl 0332.14010

[Bellaïche and Chenevier 2009] J. Bellaïche and G. Chenevier, *Families of Galois representations and Selmer groups*, Astérisque **324**, 2009. MR 2011m:11105 Zbl 1192.11035

[Berger 2002] L. Berger, "Représentations p-adiques et équations différentielles", *Invent. Math.* **148**:2 (2002), 219–284. MR 2004a:14022 Zbl 1113.14016

[Berger 2003] L. Berger, "Bloch and Kato's exponential map: three explicit formulas", pp. 99–129 in *Kazuya Kato's fiftieth birthday*, edited by S. Bloch et al., Documenta Mathematica, Bielefeld, 2003. Collection appeared as an unnumbered extra volume of *Doc. Math.* (2003). MR 2005f:11268 Zbl 1064.11077

[Berger 2004] L. Berger, "Limites de représentations cristallines", *Compos. Math.* **140**:6 (2004), 1473–1498. MR 2098398 (2006c:11138) Zbl 1071.11067

[Berger et al. 2004] L. Berger, H. Li, and H. J. Zhu, "Construction of some families of 2-dimensional crystalline representations", *Math. Ann.* **329**:2 (2004), 365–377. MR 2005k:11104 Zbl 1085.11028

[Cherbonnier and Colmez 1999] F. Cherbonnier and P. Colmez, "Théorie d'Iwasawa des représentations p-adiques d'un corps local", *J. Amer. Math. Soc.* **12**:1 (1999), 241–268. MR 99g:11141 Zbl 0933.11056

[Colmez 1998] P. Colmez, "Théorie d'Iwasawa des représentations de de Rham d'un corps local", *Ann. of Math.* (2) **148**:2 (1998), 485–571. MR 2000f:11077 Zbl 0928.11045

[Deligne 1971] P. Deligne, "Formes modulaires et représentations ℓ-adiques", pp. 139–172 (exposé 355) in *Séminaire Bourbaki, 1968/69*, Lecture Notes in Mathematics **179**, Springer, Berlin, 1971. Zbl 0206.49901

[Kato 2004] K. Kato, "p-adic Hodge theory and values of zeta functions of modular forms", pp. 117–290 in *Cohomologies p-adiques et applications arithmétiques, III*, edited by P. Berthelot et al., Astérisque **295**, Société Mathématique de France, Paris, 2004. MR 2006b:11051 Zbl 1142.11336

[Kurihara and Pollack 2007] M. Kurihara and R. Pollack, "Two p-adic L-functions and rational points on elliptic curves with supersingular reduction", pp. 300–332 in *L-functions and Galois representations* (Durham, 2004), edited by D. Burns et al., London Math. Soc. Lecture Note Ser. **320**, Cambridge Univ. Press, 2007. MR 2009g:11069 Zbl 1148.11029

[Lang 2002] S. Lang, *Algebra*, 3rd ed., Graduate Texts in Mathematics **211**, Springer, New York, 2002. MR 2003e:00003 Zbl 0984.00001

[Lei 2011] A. Lei, "Iwasawa theory for modular forms at supersingular primes", *Compos. Math.* **147**:3 (2011), 803–838. MR 2012e:11186 Zbl 1234.11148

[Lei et al. 2010] A. Lei, D. Loeffler, and S. L. Zerbes, "Wach modules and Iwasawa theory for modular forms", *Asian J. Math.* **14**:4 (2010), 475–528. MR 2774276 Zbl 05878707

[Loeffler and Zerbes 2012] D. Loeffler and S. L. Zerbes, "Wach modules and critical slope p-adic L-functions", *J. Reine Angew. Math.* (2012).

[Milne 1994] J. S. Milne, "Motives over finite fields", pp. 401–459 in *Motives* (Seattle, 1991), edited by U. Jannsen et al., Proc. Sympos. Pure Math. **55**, Amer. Math. Soc., Providence, RI, 1994. MR 95g:11053 Zbl 0811.14018

[Perrin-Riou 1994] B. Perrin-Riou, "Théorie d'Iwasawa des représentations p-adiques sur un corps local", *Invent. Math.* **115**:1 (1994), 81–161. MR 95c:11082 Zbl 0838.11071

[Perrin-Riou 1995] B. Perrin-Riou, *Fonctions L p-adiques des représentations p-adiques*, Astérisque **229**, Soc. Mat. de France, Paris, 1995. MR 96e:11062 Zbl 0845.11040

[Perrin-Riou 2001] B. Perrin-Riou, *Théorie d'Iwasawa des représentations p-adiques semi-stables*, Mém. Soc. Math. Fr. (N.S.) **84**, Soc. Mat. de France, Paris, 2001. MR 2003c:11144 Zbl 1031.11064

[Pollack 2003] R. Pollack, "On the p-adic L-function of a modular form at a supersingular prime", *Duke Math. J.* **118**:3 (2003), 523–558. MR 2004e:11050 Zbl 1074.11061

[Schneider and Teitelbaum 2003] P. Schneider and J. Teitelbaum, "Algebras of p-adic distributions and admissible representations", *Invent. Math.* **153**:1 (2003), 145–196. MR 2004g:22015 Zbl 1028.11070

[Sprung 2009] F. I. Sprung, "Iwasawa theory for elliptic curves at supersingular primes: beyond the case $a_p = 0$", preprint, 2009. arXiv 0903.3419v1

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