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The Chevalley–Shephard–Todd theorem for finite linearly reductive group schemes

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We generalize the classical Chevalley–Shephard–Todd theorem to the case of finite linearly reductive group schemes. As an application, we prove that every scheme X which is étale-locally the quotient of a smooth scheme by a finite linearly reductive group scheme is the coarse space of a smooth tame Artin stack (as defined by Abramovich, Olsson, and Vistoli), whose stacky structure is supported on the singular locus of X.

1. Introduction

Given a field k and an action of a finite (abstract) group G on a k-vector space V, we obtain a linear action of G on the polynomial ring $k[V]$. A central theme in invariant theory is determining when certain nice properties of a ring with G-action are inherited by its invariants. In particular, it is natural to ask when $k[V]^G$ is polynomial. If G acts faithfully on V, we say $g \in G$ is a *pseudoreflection* (with respect to the action of G on V) if V^g is a hyperplane. The classical Chevalley–Shephard–Todd theorem states:

Theorem 1.1 [Bourbaki 1968, §5, Theorem 4]. *If $G \to \mathrm{Aut}_k(V)$ is a faithful representation of a finite group and the order of G is not divisible by the characteristic of k, then $k[V]^G$ is polynomial if and only if G is generated by pseudoreflections.*

In this paper we generalize this theorem to the case of finite linearly reductive group schemes. To do so, we first need a notion of pseudoreflection in this setting.

Definition 1.2. Let k be a field and V a finite-dimensional k-vector space with a faithful action of a finite linearly reductive group scheme G over $\mathrm{Spec}\, k$. We say that a subgroup scheme N of G is a *pseudoreflection* if V^N has codimension 1 in V. We define the *subgroup scheme generated by pseudoreflections* to be the intersection of the subgroup schemes which contain all of the pseudoreflections of G. We say G is *generated by pseudoreflections* if G is the subgroup scheme generated by pseudoreflections.

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Over algebraically closed fields, Theorem 1.1 generalizes to

Theorem 1.3. *Let k be an algebraically closed field and V a finite-dimensional k-vector space with a faithful action of a finite linearly reductive group scheme G over* $\operatorname{Spec} k$. *Then $k[V]^G$ is polynomial if and only if G is generated by pseudoreflections.*

A more technical version of this theorem holds over fields which are not algebraically closed; however, the "only if" direction does not hold for finite linearly reductive group schemes in general (see Example 2.4). We instead prove the "only if" direction for the smaller class of stable group schemes, which we now define (see Proposition 2.2 for examples). Over an algebraically closed field, the class of stable group schemes coincides with that of finite linearly reductive group schemes. Recall from [Abramovich et al. 2008, Definition 2.9] that G is called *well-split* if it is isomorphic to a semidirect product $\Delta \rtimes Q$, where Δ is a finite diagonalizable group scheme and Q is a finite constant tame group scheme; here, tame means that the degree is prime to the characteristic.

Definition 1.4. A group scheme G over a field k is called *stable* if the following two conditions hold:

(a) for all finite field extensions K/k, every subgroup scheme of G_K descends to a subgroup scheme of G;

(b) there exists a finite Galois extension K/k such that G_K is well-split.

Remark 1.5. If G is a finite linearly reductive group scheme over a perfect field k, then [Abramovich et al. 2008, Lemma 2.11] shows that condition (b) is automatically satisfied.

Theorem 1.3 is a special case of the following generalization of the Chevalley–Shephard–Todd theorem. This is the first main result of this paper.

Theorem 1.6. *Let k be a field and V a finite-dimensional k-vector space with a faithful action of a finite linearly reductive group scheme G over* $\operatorname{Spec} k$. *If G is generated by pseudoreflections, then $k[V]^G$ is polynomial. The converse holds if G is stable.*

We also prove a version of this theorem for an action of a finite linearly reductive group scheme on a smooth scheme.

Definition 1.7. Given a smooth affine scheme U over $\operatorname{Spec} k$ with a faithful action of a finite linearly reductive group scheme G which fixes a field-valued point $x \in U(K)$, we say a subgroup scheme N of G is a *pseudoreflection at x* if N_K is a pseudoreflection with respect to the induced action of G_K on the cotangent space at x. We define what it means for G to be generated by pseudoreflections at x in the same manner as in Definition 1.2.

Theorem 1.6 then has the following corollary.

Corollary 1.8. *Let k be a field and let U be a smooth affine k-scheme with a faithful action by a finite linearly reductive group scheme G over* $\operatorname{Spec} k$. *Let $x \in U(K)$, where K/k is a finite separable field extension, and suppose x is fixed by G. If G is generated by pseudoreflections at x, then U/G is smooth at the image of x. The converse holds if G is stable.*

The second main result of this paper is this:

Theorem 1.9. *Let k be a field and let U be a smooth affine k-scheme with a faithful action by a stable group scheme G over* $\operatorname{Spec} k$. *Suppose K/k is a finite separable field extension and G fixes a point $x \in U(K)$. Let $M = U/G$, let M^0 be the smooth locus of M, and let $U^0 = U \times_M M^0$. If G has no pseudoreflections at x, then after possibly shrinking M to a smaller Zariski neighborhood of the image of x, we have that U^0 is a G-torsor over M^0.*

In the classical case, Theorem 1.9 follows directly from Corollary 1.8 and the purity of the branch locus theorem [Grothendieck and Raynaud 1971, X.3.1]. For us, however, a little more work is needed since G is not necessarily étale.

As an application of Theorem 1.9, we generalize the well-known result (see, for example, [Vistoli 1989, (2.9)] or [Fantechi et al. 2007, Remark 4.9]) that schemes with quotient singularities prime to the characteristic are coarse spaces of smooth Deligne–Mumford stacks. We say a scheme has *linearly reductive singularities* if it is étale-locally the quotient of a smooth scheme by a finite linearly reductive group scheme. We show that every such scheme M is the coarse space of a smooth tame Artin stack (in the sense of [Abramovich et al. 2008]) whose stacky structure is supported at the singular locus of M. More precisely,

Theorem 1.10. *Let k be a perfect field and M a k-scheme with linearly reductive singularities. Then it is the coarse space of a smooth tame stack $\mathfrak{X}$ over k such that f^0 in the diagram*

$$\begin{tikzcd} \mathfrak{X}^0 \arrow[r, "j^0"] \arrow[d, "f^0"'] & \mathfrak{X} \arrow[d, "f"] \\ M^0 \arrow[r, "j"'] & M \end{tikzcd}$$

is an isomorphism, where j is the inclusion of the smooth locus of M and $\mathfrak{X}^0 = M^0 \times_M \mathfrak{X}$.

This paper is organized as follows. In Section 2, we prove the "if" direction of Theorem 1.6 and reduce the proof of the "only if" direction to the special case of Theorem 1.9 in which $U = \mathbb{V}^\vee(V)$ for some k-vector space V with G-action (see the section on notation below). This special case is proved in Section 3. The key input for the proof is [Iwanari 2009, Theorem 2.3], which we reinterpret in the language of pseudoreflections. We finish the section by proving Corollary 1.8. In Section 4, we use Corollary 1.8 to complete the proof of Theorem 1.9. In Section 5, we prove Theorem 1.10.

Notation. Throughout this paper, k is a field and $S = \operatorname{Spec} k$. If V is a k-vector space with an action of a group scheme G, then we denote by $\mathbb{V}^\vee(V)$, or simply $\mathbb{V}^\vee$ if V is understood, the scheme $\operatorname{Spec} k[V]$ whose G-action is given by the dual representation on functor points. Said another way, if $G = \operatorname{Spec} A$ is affine and its action on V is given by the coaction map $\sigma : V \longrightarrow V \otimes_k A$, then the coaction map $k[V] \longrightarrow k[V] \otimes_k A$ defining the G-action on $\mathbb{V}^\vee$ is given by $\sum a_i v_i \mapsto \sum a_i \sigma(v_i)$.

All Artin stacks $\mathfrak{X}$ in this paper are assumed to have finite diagonal, so that if $\mathfrak{X}$ is locally of finite presentation, it has a coarse space by [Conrad 2004, Theorem 1.1] (see also [Keel and Mori 1997]). Given a locally finitely presented scheme U with an action of a finite flat group scheme G, we denote by U/G the coarse space of the stack $[U/G]$.

If R is a ring and $\mathcal{I}$ an ideal of R, then we denote by $V(\mathcal{I})$ the closed subscheme of $\operatorname{Spec} R$ defined by $\mathcal{I}$.

2. Linear actions on polynomial rings

The "if" direction of Theorem 1.6. Our first goal is to prove the "if" direction of Theorem 1.6. We begin with examples of stable group schemes and with some basic results about the subgroup scheme generated by pseudoreflections.

Lemma 2.1. *Suppose k is perfect and G is a finite linearly reductive group scheme over S. If the identity component Δ of G is diagonalizable and G/Δ is constant, then there exists a finite linearly reductive group scheme $\tilde{G}$ over $\mathbb{Z}$ such that $\tilde{G}_k = G$. If H is a closed subgroup scheme of G, then there exists a closed subgroup scheme $\tilde{H}$ of $\tilde{G}$ whose pullback to k is H. If H is normal in G, then $\tilde{H}$ is normal in $\tilde{G}$.*

Proof. Let $Q = G/\Delta$. Since k is perfect, the connected-étale sequence

$$1 \longrightarrow \Delta \longrightarrow G \longrightarrow Q \longrightarrow 1$$

is functorially split (see [Tate 1997, 3.7 (IV)]). Since Δ is diagonalizable, it is of the form $\operatorname{Spec} k[A]$, where A is a finitely generated abelian group. Note that as a scheme $G = \Delta \times_k Q$ and that its group scheme structure is given by a homomorphism:

$$\epsilon : Q \longrightarrow \mathcal{A}ut(\Delta) = \operatorname{Aut}(A).$$

We can therefore let $\tilde{G} = \operatorname{Spec} \mathbb{Z}[A] \times_{\mathbb{Z}} Q$ with group scheme structure induced by ϵ.

Now let H be a closed subgroup scheme of G. Letting $\Delta' = H \cap \Delta$ and $Q' = H/\Delta'$, we have a commutative diagram

$$\begin{tikzcd} 1 \arrow[r] & \Delta \arrow[r] & G \arrow[r] & Q \arrow[r] & 1 \\ 1 \arrow[r] & \Delta' \arrow[r] \arrow[u, "\varphi"] & H \arrow[r] \arrow[u] & Q' \arrow[r] \arrow[u, "\psi"'] & 1 \end{tikzcd}$$

with exact rows. Since Δ is connected, we see Δ' is the connected component of the identity of H. Therefore, the bottom row of the above diagram is the connected-étale sequence of H, and so

$$H = \Delta' \rtimes Q',$$

as k is perfect. Since Δ' is diagonalizable and Q' is constant, we can define $\tilde{H}$ in the same way we defined $\tilde{G}$.

We now show that $\tilde{H}$ is a closed subgroup scheme of $\tilde{G}$. Let $*$ denote the action of Q (resp. Q') on Δ (resp. Δ'). Since the splitting of the connected-étale sequence of a finite group scheme over a perfect field is functorial, we see that for all $q' \in Q'$ and local sections δ' of Δ',

$$\psi(q') * \varphi(\delta') = \varphi(q' * \delta').$$

We therefore obtain a closed immersion from $\tilde{H}$ to $\tilde{G}$ whose pullback to k is the morphism from H to G.

Lastly, we show that if H is normal in G, then $\tilde{H}$ is normal in $\tilde{G}$. Let $\Delta' = \operatorname{Spec} k[A']$, where A' is a finitely generated abelian group. Showing that $\tilde{H}$ is normal in $\tilde{G}$ is equivalent to showing that Q' is normal in Q, and for all local sections $\delta \in \Delta$, $\delta' \in \Delta'$, $q \in Q$, and $q' \in Q'$, we have

$$q * (\delta^{-1}\delta' \cdot (q'^{-1} * \delta)) \in \Delta'.$$

We know that Q' is normal in Q as H is normal in G. To check the latter statement about local sections, note that it can be reformulated as follows: for every $q \in Q$ and $q' \in Q$, the homomorphism

$$\begin{aligned} &A \to A \times A' \\ &a \mapsto (q * (a^{-1} \cdot q'^{-1} * a), q * \bar{a}) \end{aligned}$$

factors through A'; here $\bar{a}$ denotes the image of a under the projection from A to A'. Since this statement makes no reference to the base scheme, it can be checked over k, where the normality of H in G yields the desired factorization. □

Proposition 2.2. *Let G be a finite group scheme over S. Consider the following conditions*:

(1) *G is diagonalizable.*

(2) *G is a constant group scheme.*

(3) *k is perfect, the identity component Δ of G is diagonalizable, and G/Δ is constant.*

If any of them holds, then G is stable.

Proof. It is clear that finite diagonalizable group schemes and finite constant group schemes are stable, so we consider the last case. Let $Q = G/\Delta$. Since k is perfect, the connected-étale sequence

$$1 \longrightarrow \Delta \longrightarrow G \longrightarrow Q \longrightarrow 1$$

is functorially split. Let K/k be a finite extension and let H be a subgroup scheme of G_K. Letting $\Delta' = H \cap \Delta_K$ and $Q' = H/\Delta'$, we have a commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & \Delta_K \arrow[r] & G_K \arrow[r] & Q_K \arrow[r] & 1 \\
1 \arrow[r] & \Delta' \arrow[r] \arrow[u] & H \arrow[r] \arrow[u] & Q' \arrow[r] \arrow[u] & 1
\end{tikzcd}$$

with exact rows. Since Δ is connected and has a k-point, [Grothendieck 1967, 4.5.14] shows that Δ is geometrically connected. In particular, Δ_K is the connected component of the identity of G_K, and so Δ' is the connected component of the identity of H. Therefore, the bottom row of the above diagram is the connected-étale sequence of H. The proposition then follows from Lemma 2.1. □

Lemma 2.3. *Let V be a finite-dimensional k-vector space with a faithful action of a stable group scheme G over S, and let H be the subgroup scheme generated by pseudoreflections. If K/k is an algebraic extension of fields, then a subgroup scheme of G_K is a pseudoreflection if and only if it descends to a pseudoreflection. Furthermore, H_K is the subgroup scheme of G_K generated by pseudoreflections.*

Proof. Note first that if P is a subgroup scheme of G_K, then there exists a subgroup scheme P_0 of G such that $(P_0)_K = P$. If K/k is a finite extension, this follows from the fact that G is stable. If K/k is an infinite extension, by a standard limit argument, there exists a finite extension L/k and a subgroup scheme P_1 of G_L such that $(P_1)_K = P$. We then obtain our desired P_0 as L/k is a finite extension. The first claim of the proposition then follows from the fact that

$$(V_K)^{N_K} = (V^N)_K$$

for any subgroup scheme N of G. The second claim follows from the fact that if P' and P'' are subgroup schemes of G, then P'_K contains P''_K if and only if P' contains P''. □

We remark that even in characteristic zero, Lemma 2.3 is false for general finite linearly reductive group schemes G, as the following example shows. Note that this example also shows that the "only if" direction of Theorem 1.6 and of Corollary 1.8 is false for general finite linearly reductive group schemes.

Example 2.4. Let k be a field contained in $\mathbb{R}$ or let $k = \mathbb{F}_p$ for p congruent to 3 mod 4. Let $K = k(i)$, where $i^2 = -1$, and let G be the locally constant group scheme over $\operatorname{Spec} k$ whose pullback to $\operatorname{Spec} K$ is $\mathbb{Z}/2 \times \mathbb{Z}/2$ with the Galois action that switches the two $\mathbb{Z}/2$ factors. Let g_1 and g_2 be the generators of the two $\mathbb{Z}/2$ factors and consider the action

$$\rho : G_K \longrightarrow \operatorname{Aut}_K(K^2)$$

on the K-vector space K^2 given by

$$\rho(g_1) : (a, b) \mapsto (-bi, ai), \qquad \rho(g_2) : (a, b) \mapsto (bi, -ai).$$

Then ρ is Galois-equivariant and hence comes from an action of G on k^2. Note that $\mathbb{Z}/2 \times 1$ and $1 \times \mathbb{Z}/2$ are both pseudoreflections of G_K, as the subspaces which they fix are $K \cdot (1, i)$ and $K \cdot (1, -i)$, respectively. Since G_K is not a pseudoreflection, it follows that there are no Galois-invariant pseudoreflections of G_K, and hence, the subgroup scheme generated by pseudoreflections of G is trivial; the subgroup scheme generated by pseudoreflections of G_K, however, is G_K.

Corollary 2.5. *If V is a finite-dimensional k-vector space with a faithful action of a stable group scheme G over S, then the subgroup scheme generated by pseudoreflections is normal in G.*

Proof. We denote by H the subgroup scheme generated by pseudoreflections. Let T be an S-scheme and let $g \in G(T)$. We must show the subgroup schemes H_T and gH_Tg^{-1} of G_T are equal. To do so, it suffices to check this on stalks and so we can assume $T = \operatorname{Spec} R$, where R is strictly Henselian. By [Abramovich et al. 2008, Lemma 2.17], we need only show that these two group schemes are equal over the closed fiber of T, so we can further assume that $R = K$ is a field. Since G is finite over S, the residue fields of G are finite extensions of k. We can therefore assume that K/k is a finite field extension.

By Lemma 2.3, we know that H_K is the subgroup scheme of G_K generated by pseudoreflections. Note that if N' is a pseudoreflection of G_K, then $gN'g^{-1}$ is as well since

$$V_K^{gN'g^{-1}} = g(V_K^{N'}).$$

As a result, $gH_Kg^{-1} = H_K$, which completes the proof. □

Lemma 2.6. *Given a finite-dimensional k-vector space V with a faithful action of a finite linearly reductive group scheme G over S, let $\{N_i\}$ denote the set of* *pseudoreflections of G and let H be the subgroup scheme generated by pseudoreflections. Then*

$$k[V]^H = \bigcap_i k[V]^{N_i}.$$

Proof. Let $R = \bigcap_i k[V]^{N_i}$. Consider the functor

$$\begin{aligned} F : (k\text{-alg}) &\to (\text{Groups}) \\ A &\mapsto \{g \in G(A) \mid g(m) = m \text{ for all } m \in R \otimes_k A\}. \end{aligned}$$

Since each $k[V]^{N_i}$ is finitely generated, we see R is as well. Let $r_1, \ldots, r_n$ be a finite set of generators for R. We see then that F is representable by the intersection of the stabilizers G_{r_j}, and so is a closed subgroup scheme of G. Since F contains every pseudoreflection, we see $H \subset F$. We therefore have the containments

$$R \subset k[V]^F \subset k[V]^H \subset \bigcap_i k[V]^{N_i},$$

from which the lemma follows. □

If N is any subgroup scheme of G, it is linearly reductive by [Abramovich et al. 2008, Proposition 2.7]. It follows that

$$V \simeq V^N \oplus V/V^N$$

as N-representations. If N is a pseudoreflection, then $\dim_k V/V^N = 1$. Let v be a generator of the 1-dimensional subspace V/V^N and let $\sigma : V \to V \otimes_k B$ be the coaction map, where $N = \operatorname{Spec} B$. Then via the above isomorphism, σ is given by

$$\begin{aligned} V^N \oplus V/V^N &\to (V^N \otimes_k B) \oplus (V/V^N \otimes_k B) \\ (w, w') &\mapsto (w \otimes 1, w' \otimes b) \end{aligned}$$

for some $b \in B$. It follows that there is a k-linear map $h : V \to B$ such that for all $w \in V$,

$$\sigma(w) - (w \otimes 1) = v \otimes h(w).$$

If we continue to denote by σ the induced coaction map $k[V] \longrightarrow k[V] \otimes_k B$, we see that h extends to a $k[V]^N$-module homomorphism $k[V] \longrightarrow k[V] \otimes_k B$, which we continue to denote by h, such that for all $f \in k[V]$,

$$\sigma(f) - (f \otimes 1) = (v \otimes 1) \cdot h(f).$$

We are now ready to prove the "if" direction of Theorem 1.6. Our proof is only a slight variant of the proof of the classical Chevalley–Shephard–Todd theorem presented in [Smith 1985].

Proof of "if" direction of Theorem 1.6. Let $R = k[V]^G$. By Lemma 2.6, we know that the intersection of the $k[V]^N$ is R, where N runs through the pseudoreflections of G. By the proposition on [Smith 1985, page 225], to show R is polynomial, we need only show that $k[V]$ is a free R-module. By graded Nakayama, the projective dimension of $k[V]$ is the smallest integer i such that $\mathrm{Tor}^R_{i+1}(k, k[V]) = 0$, where k is viewed as an R-module via the augmentation map

$$\epsilon : k[V]^G \longrightarrow k[V] \longrightarrow k,$$

sending all positively graded elements to 0. So we must show $\mathrm{Tor}^R_1(k, k[V]) = 0$.

Tensoring the short exact sequence defined by ϵ with $k[V]$, we obtain a long exact sequence

$$0 \longrightarrow \mathrm{Tor}^R_1(k, k[V]) \longrightarrow \ker\epsilon \otimes_R k[V] \xrightarrow{\phi} R \otimes_R k[V] \xrightarrow{\epsilon\otimes 1} k \otimes_R k[V] \longrightarrow 0.$$

To show $\mathrm{Tor}^R_1(k, k[V]) = 0$, we must prove that ϕ is injective. We in fact show

$$\phi \otimes 1 : \ker\epsilon \otimes_R k[V] \otimes_k C \longrightarrow k[V] \otimes_k C$$

is injective for all finite-dimensional k-algebras C. If this is not the case, then the set

$$\{\xi \mid C \text{ is a finite-dimensional } k\text{-algebra}, 0 \neq \xi \in \ker\epsilon \otimes_R k[V] \otimes_k C, (\phi\otimes 1)(\xi) = 0\}$$

is nonempty and we can choose an element ξ of minimal degree, where $\ker\epsilon$ is given its natural grading as a submodule of $k[V]$ and the elements of C are defined to be of degree 0. We begin by showing $\xi \in \ker\epsilon \otimes_R R \otimes_k C$. That is, we show ξ is fixed by all pseudoreflections.

Let $N = \operatorname{Spec} B$ be a pseudoreflection. Let $\sigma : k[V] \longrightarrow k[V] \otimes B$ be the coaction map. As explained above, we get a $k[V]^N$-module homomorphism $h : k[V] \longrightarrow k[V] \otimes B$. Note that this morphism has degree -1. Since

$$(1 \otimes \sigma \otimes 1)(\xi) - \xi \otimes 1 = (1 \otimes h \otimes 1)(\xi) \cdot (1 \otimes v \otimes 1 \otimes 1),$$

the commutativity of

$$\begin{tikzcd} \ker\epsilon \otimes k[V] \otimes B \otimes C \arrow[r, "\phi\otimes 1\otimes 1"] & k[V] \otimes B \otimes C \\ \ker\epsilon \otimes k[V] \otimes C \arrow[r, "\phi\otimes 1"] \arrow[u, "1\otimes\sigma\otimes 1"] & k[V] \otimes C \arrow[u, "\sigma\otimes 1"'] \end{tikzcd}$$

implies

$$(\phi \otimes 1 \otimes 1)(1 \otimes h \otimes 1)(\xi) \cdot (v \otimes 1 \otimes 1) = 0.$$

It follows that $(1 \otimes h \otimes 1)(\xi)$ is killed by $\phi \otimes 1 \otimes 1$. Since h has degree -1, our assumption on ξ shows that $(1\otimes h\otimes 1)(\xi) = 0$. We therefore have $(1\otimes\sigma\otimes 1)(\xi) = \xi \otimes 1$, which proves that ξ is N-invariant.

Since G is linearly reductive, we have a section of the inclusion $k[V]^G \hookrightarrow k[V]$. We therefore, also obtain a section s of the inclusion $j : R \hookrightarrow k[V]$. Let $\psi : \ker\epsilon \otimes_R R \longrightarrow R$ be the canonical map, and consider the diagram

$$\begin{tikzcd}
\ker\epsilon \otimes k[V] \otimes C \arrow[r, "\phi\otimes 1"] \arrow[d, shift left, "1\otimes s\otimes 1"] & k[V] \otimes C \arrow[d, shift left, "s\otimes 1"] \\
\ker\epsilon \otimes R \otimes C \arrow[r, "\psi\otimes 1"] \arrow[u, shift left, "1\otimes j\otimes 1"] & R \otimes C. \arrow[u, shift left, "j\otimes 1"]
\end{tikzcd}$$

We see that

$$(j\otimes 1)(\psi\otimes 1)(1\otimes s\otimes 1)(\xi) = (\phi\otimes 1)(1\otimes j\otimes 1)(1\otimes s\otimes 1)(\xi) = (\phi\otimes 1)(\xi) = 0.$$

But $j\otimes 1$ and $\psi\otimes 1$ are injective, so $(1\otimes s\otimes 1)(\xi) = 0$. Since $\xi \in \ker\epsilon \otimes_R R \otimes_k C$, it follows that $\xi = 0$, which is a contradiction. □

Reducing the "only if" direction of Theorem 1.6 to a case of Theorem 1.9. Now that we have proved the "if" direction of Theorem 1.6, we work toward reducing the "only if" direction to the special case of Theorem 1.9 where $U = \mathbb{V}^\vee$. The main step in this reduction is showing that if G acts faithfully on V, and H denotes the subgroup scheme generated by pseudoreflections, then the action of G/H on $\mathbb{V}^\vee/H$ has no pseudoreflections at the origin. In the classical case, the proof of this statement relies on the fact that G has no pseudoreflections if and only if $\mathbb{V}^\vee \to \mathbb{V}^\vee/G$ is étale in codimension 1. As the following example illustrates, this relation between pseudoreflections and ramification no longer holds in our case.

Example 2.7. Let k be a field of characteristic 2 and $G = \mu_2$. We define an action of G on $V = kx \oplus ky$ as follows: for every k-scheme T and every section $\zeta \in G(T)$, let ζ act on $V \otimes_k \mathcal{O}_T$ by sending x to ζx and y to ζy. Then $\pi : \mathbb{V}^\vee \to \mathbb{V}^\vee/G$ is a G-torsor away from the one singular point in $\mathbb{V}^\vee/G$. Hence, π is ramified at every height 1 prime, but G has no pseudoreflections.

We must therefore take a different approach to showing that the action of G/H on $\mathbb{V}^\vee/H$ has no pseudoreflections at the origin. Our strategy is to reduce to the classical case by lifting to characteristic 0. This is carried out after some preliminary lemmas.

Lemma 2.8. *Let G be a finite group scheme which acts faithfully on an affine scheme U. If H is a normal subgroup scheme of G, then the action of G/H on U/H is faithful.*

Proof. Let $\mathfrak{X} = [U/H]$ and let $\pi : U \to U/H$ be the natural map. We must show that if G' is a subgroup scheme of G such that G'/H acts trivially on U/H, then $G' = H$. Replacing G by G', we can assume $G' = G$.

Since G acts faithfully on U, there is a nonempty open substack of $\mathfrak{X}$ which is isomorphic to its coarse space. That is, we have a nonempty open subscheme V of U/H over which π is an H-torsor. Let $P = V \times_{U/H} U$. Since G acts on P over V, we obtain a morphism

$$s : G \longrightarrow \mathcal{A}ut(P) = H.$$

Note that s is a section of the closed immersion $H \to G$, so $H = G$. □

Lemma 2.9. *Let G be a finite flat linearly reductive group scheme over a complete discrete valuation ring R with residue field k. If G acts linearly on $\mathbb{A}^n_R$ and $\mathbb{A}^n_k/G_k$ is isomorphic to $\mathbb{A}^n_k$, then $\mathbb{A}^n_R/G$ is isomorphic to $\mathbb{A}^n_R$.*

Proof. Let $\mathfrak{m}$ be the maximal ideal of R and let $\mathbb{A}^n_R/G = \operatorname{Spec} A$. Since $\mathbb{A}^n_R$ is flat over R, it follows that $\mathbb{A}^n_R/G$ is as well (see, for example, [Alper 2008, Theorem 4.16(ix)]). Since G is linearly reductive,

$$\operatorname{Spec} k \times_R \mathbb{A}^n_R/G = \mathbb{A}^n_k/G_k.$$

Choose an isomorphism

$$\varphi_0 : k[x_1, \ldots, x_n] \longrightarrow A \otimes_R k$$

and let $r_i \in R$ be an arbitrary lift of $\varphi_0(x_i)$. By Nakayama's lemma, the morphism

$$\varphi : R[x_1, \ldots, x_n] \longrightarrow A$$

sending x_i to r_i is surjective. As R is complete, to show φ is an isomorphism, we need only show that the base change φ_m of φ to $R/\mathfrak{m}^{\ell+1}$ is an isomorphism for every ℓ. This follows from the fact that φ_0 is an isomorphism and $A \otimes_R R/\mathfrak{m}^\ell$ is flat over $R/\mathfrak{m}^\ell$. □

Proposition 2.10. *Let G be a finite linearly reductive group scheme over S with a faithful action on a finite-dimensional k-vector space V. Let $U = \mathbb{V}^\vee$ and let H be the subgroup scheme of G generated by pseudoreflections. Then the induced action of G/H on $U/H \simeq \mathbb{A}^n_k$ has no pseudoreflections at the origin.*

Proof. By the "if" direction of Theorem 1.6, we have $k[V]^H = k[W]$ for some subvector space W of $k[V]$. The proof of [Neusel 2007, Proposition 6.19] shows that the degrees of the homogeneous generators of $k[V]^H$ are determined. As a result, the action of G/H on $k[W]$ is linear. Lemma 2.8 further tells us that this action is faithful.

Assume that the subgroup scheme H'' of G/H generated by pseudoreflections is nontrivial. Then $H'' = H'/H$ where H' is a normal subgroup scheme of G which properly contains H. To prove G/H has no pseudoreflections at the origin, it suffices by Lemma 2.3 to replace k by its algebraic closure. By [Abramovich et al. 2008, Lemma 2.11], we see then that G is the semidirect product of its identity component, which is diagonalizable, and a finite constant tame group scheme. The same is true for H and H'.

Let R be a complete discrete valuation ring whose residue field is k and whose fraction field K is of characteristic 0. Lemma 2.1 shows that there exist finite flat linearly reductive group schemes $\tilde{G}$, $\tilde{H}$, and $\tilde{H}'$ over R whose base changes to k are G, H, and H', respectively. Furthermore, $\tilde{H}'$ and $\tilde{H}$ are normal closed subgroup schemes of $\tilde{G}$, and $\tilde{H}$ is a proper subgroup scheme of $\tilde{H}'$. In characteristic 0, every finite flat group scheme is locally constant, so after replacing R by a finite extension, we can further assume that $\tilde{G}_K$, $\tilde{H}_K$, and $\tilde{H}'_K$ are constant group schemes.

Let $\mathfrak{m}$ denote the maximal ideal of R and let $R_\ell = R/\mathfrak{m}^\ell$. Let $\tilde{G}_\ell$, $\tilde{H}_\ell$, and $\tilde{H}'_\ell$ denote the base changes of $\tilde{G}$, $\tilde{H}$, and $\tilde{H}'$ to R_ℓ. Choosing a basis for V, we can identify U with $\mathbb{A}^n_k$. The G-action on U is then given by a group scheme homomorphism $\varphi_0 : G \longrightarrow \mathrm{GL}_{n,k}$. By [Grothendieck 1970, Exposé III 2.3], given a deformation $\varphi_\ell : \tilde{G}_\ell \longrightarrow \mathrm{GL}_{n,R_\ell}$ of φ_0, the obstruction to deforming φ_ℓ to a homomorphism $\varphi_{\ell+1} : \tilde{G}_{\ell+1} \longrightarrow \mathrm{GL}_{n,R_{\ell+1}}$ lies in

$$H^2(\tilde{G}_\ell, \mathrm{Lie}(\mathrm{GL}_n) \otimes \mathfrak{m}^\ell/\mathfrak{m}^{\ell+1}),$$

which vanishes as $\tilde{G}_\ell$ is linearly reductive. We therefore obtain a faithful action of $\tilde{G}$ on $\mathbb{A}^n_R$ lifting the action of G on U.

By Lemma 2.9, we see that $\mathbb{A}^n_K/\tilde{H}_K$ and $\mathbb{A}^n_K/\tilde{H}'_K$ are polynomial. The classical Chevalley–Shephard–Todd theorem then shows that there is a pseudoreflection $\tilde{N}_K$ of $\tilde{G}_K$ which is contained in $\tilde{H}'_K$ but not contained in $\tilde{H}_K$. Note that this is not yet a contradiction as it is not clear that $\tilde{H}_K$ is the subgroup scheme of $\tilde{G}_K$ generated by pseudoreflections. Let $\tilde{N}$ be the closure of $\tilde{N}_K$ in $\tilde{G}$. Since $\tilde{G}$ is a finite flat linearly reductive group scheme over R, we see that $\tilde{N}$ is as well. Since $\tilde{N}_K$ is a pseudoreflection, there exists some $v = \sum_i a_i x_i \in K[x_1, \dots, x_n]$ such that $\tilde{N}_K$ acts trivially on $K[x_1, \dots, x_n]/v$. After scaling the a_i, we can assume $a_1 \in R^*$ and all $a_i \in R$. Consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & vK[x_1, \dots, x_n] \arrow[r] & K[x_1, \dots, x_n] \arrow[r] & K[x_1, \dots, x_n]/v \arrow[r] & 0 \\
0 \arrow[r] & vR[x_1, \dots, x_n] \arrow[r] \arrow[u] & R[x_1, \dots, x_n] \arrow[r] \arrow[u] & R[x_1, \dots, x_n]/v \arrow[r] \arrow[u, "\psi"] & 0
\end{tikzcd}$$

of $\tilde{N}$-comodules. Since the left square is Cartesian, we see that ψ is injective. It follows that the action of $\tilde{N}$ on the hyperplane defined by v in $\mathbb{A}^n_R$ is trivial. Reducing mod $\mathfrak{m}$, we see that $\tilde{N}_k$ is a pseudoreflection of G. Furthermore, $\tilde{N}_k$ is not contained in H, which is a contradiction. □

Using Lemma 2.8 and Proposition 2.10, we now prove the "only if" direction of Theorem 1.6, assuming the special case of Theorem 1.9 in which $U = \mathbb{V}^\vee$.

Proof of "only if" direction of Theorem 1.6. Let H be the subgroup scheme generated by pseudoreflections. By the "if" direction, $k[V]^H$ is polynomial and as explained in the proof of Proposition 2.10, the G/H-action on $k[V]^H$ is linear. Since G/H acts faithfully on U/H without pseudoreflections at the origin by Lemma 2.8 and Proposition 2.10, and since $M = U/G$ is smooth by assumption, Theorem 1.9 implies that U/H is a G/H-torsor over U/G after potentially shrinking U/G. Since the origin of U/H is a fixed point, we conclude that $G = H$. □

3. Theorem 1.9 for linear actions on polynomial rings

In Section 2, we reduced the proof of the "only if" direction of Theorem 1.6 to this statement:

Proposition 3.1. *Let G be a stable group scheme over S which acts faithfully on a finite-dimensional k-vector space V. Then Theorem 1.9 holds when $U = \mathbb{V}^\vee$ and x is the origin.*

The proof of this proposition is given in two steps. First we handle the case when G is diagonalizable, and then we use that for the general case (see page 15).

Reinterpreting a result of Iwanari. The key to proving Proposition 3.1 for diagonalizable G is provided by [Iwanari 2009, Theorem 3.3 and Proposition 3.4] after we reinterpret them in the language of pseudoreflections. We refer the reader to [Iwanari 2009, pages 4–6] for the basic definitions concerning monoids. We recall the following definition given as [Iwanari 2009, Definition 2.5].

Definition 3.2. An injective morphism $i : P \to F$ from a simplicially toric sharp monoid to a free monoid is called a *minimal free resolution* if i is close and if for all injective close morphisms $i' : P \to F'$ to a free monoid F' of the same rank as F, there is a unique morphism $j : F \to F'$ such that $i' = ji$.

Given a faithful action of a finite diagonalizable group scheme Δ over S on a k-vector space V of dimension n, we can decompose V as a direct sum of one-dimensional Δ-representations. Therefore, after choosing an appropriate basis, we have an identification of $k[V]$ with $k[\mathbb{N}^n]$ and can assume that the Δ-action on $U = \mathbb{V}^\vee$ is induced from a morphism of monoids

$$\pi : F = \mathbb{N}^n \longrightarrow A,$$

where A is the finite abelian group such that Δ is the Cartier dual $D(A)$ of A. We see then that

$$U/\Delta = \operatorname{Spec} k[P],$$

where P is the submonoid $\{p \mid \pi(p) = 0\}$ of F. Note that P is simplicially toric sharp, that $i : P \to F$ is close, and that $A = F^{gp}/i(P^{gp})$.

We now give the relationship between minimal free resolutions and pseudoreflections.

Proposition 3.3. *With notation as above, $i : P \to F$ is a minimal free resolution if and only if the action of Δ on V has no pseudoreflections.*

Proof. If i is not a minimal free resolution, then without loss of generality, $i = ji'$, where $i' : P \to F$ is close and injective, and $j : F \to F$ is given by

$$j(a_1, a_2, \dots, a_n) = (ma_1, a_2, \dots, a_n),$$

with $m \neq 1$. We have then a short exact sequence

$$0 \longrightarrow F^{gp}/i'(P^{gp}) \longrightarrow F^{gp}/i(P^{gp}) \longrightarrow F^{gp}/(m, 1, \dots, 1)(F^{gp}) \longrightarrow 0.$$

Let N be the Cartier dual of $F^{gp}/(m, 1, \dots, 1)(F^{gp})$, which is a subgroup scheme of Δ. Letting $\{x_i\}$ be the standard basis of F, we see that

$$k[F]^N = k[x_1^m, x_2, \dots, x_n],$$

and so V^N, which is the degree 1 part of $k[F]^N$, has codimension 1 in V. Therefore, N is a pseudoreflection.

Conversely, suppose N is a pseudoreflection. Since N is a subgroup scheme of Δ, it is diagonalizable as well. Let $N = \operatorname{Spec} k[B]$, where B is a finite abelian group, and let $\psi : A \to B$ be the induced map. We see that

$$V^N = \bigoplus_{i \neq j} kx_i$$

for some j. Without loss of generality, $j = 1$. It follows then that

$$\{f \in F \mid \psi\pi(f) = 0\} = (m, 1, \dots, 1)F$$

for some m dividing $|B|$. Since the Δ action on V is assumed to be faithful, we see, in fact, that $m = |B|$. Therefore, i factors through $\cdot(m, 1, \dots, 1) : F \longrightarrow F$, which shows that i is not a minimal free resolution. □

Having reinterpreted minimal free resolutions, the proof of Proposition 3.1 for diagonalizable group schemes G follows easily from Iwanari's work.

Proposition 3.4. *Let $G = \Delta$ be a finite diagonalizable group scheme over S which acts faithfully on a finite-dimensional k-vector space V. Then Theorem 1.9 holds when $U = \mathbb{V}^\vee$ and x is the origin. In this case it is not necessary to shrink M to a smaller Zariski neighborhood of the image of x.*

Proof. Let F and P be as above, and let $\mathfrak{X} = [U/\Delta]$. By Proposition 3.3, the morphism $i : P \to F$ is a minimal free resolution. [Iwanari 2009, Theorem 3.3 (1) and Proposition 3.4] then show that the natural morphism $\mathfrak{X} \times_M M^0 \to M^0$ is an isomorphism. Since $\mathfrak{X} \times_M M^0 = [U^0/\Delta]$, we see U^0 is a Δ-torsor over M^0. □

Finishing the proof. The goal of this subsection is to prove Proposition 3.1. The main result used in the proof of this proposition, as well as in the proof of Theorem 1.9, is the following.

Proposition 3.5. *Let notation and hypotheses be as in Theorem 1.9. Let $X = U/\Delta$ and $G = \Delta \rtimes Q$, where Δ is diagonalizable and Q is constant and tame. If in addition to assuming that G acts without pseudoreflections at x, we assume that Δ is local and that the base change of U to X^{sm} is a Δ-torsor over X^{sm}, then after possibly shrinking M to a smaller Zariski neighborhood of the image of x, the quotient map $f : X \to M$ is unramified in codimension* 1.

Proof. Let g be the quotient map $U \to X$. For every $q \in Q$, consider the Cartesian diagram

$$\begin{tikzcd} Z_q \arrow[r] \arrow[d] & U \arrow[d, "\Delta"] \\ U \arrow[r, "\Gamma_q"] & U \times U, \end{tikzcd}$$

where $\Gamma_q(u) = (u, qu)$. We see that Z_q is a closed subscheme of U and that $Z_q(T)$ is the set of $u \in U(T)$ which are fixed by q. Let Z be the closed subset of U which is the union of the Z_q for $q \neq 1$. Since the action of G on U is faithful, Z is not all of U. Let Z' be the union of the codimension 1 components of Z. Since fg is finite, we see that $fg(Z')$ is a closed subset of M. Moreover, $fg(Z')$ does not contain the image of x, as G is assumed to act without pseudoreflections at x. By shrinking M to $M - fg(Z')$, we can assume that no nontrivial $q \in Q$ acts trivially on a divisor of U.

Let $U = \operatorname{Spec} R$. The morphism f is unramified in codimension 1 if and only if the (traditional) inertia groups of all height 1 primes $\mathfrak{p}$ of R^Δ are trivial. So, we must show that if $q \in Q$ acts trivially on $V(\mathfrak{p})$, then $q = 1$. Since g is finite, and hence integral, the going-up theorem shows that

$$\mathfrak{p}R = \mathfrak{P}_1^{e_1} + \cdots + \mathfrak{P}_n^{e_n},$$

where the $\mathfrak{P}_i$ are height 1 primes and the e_i are positive integers. Note that X is normal and so the complement of X^{sm} in X has codimension at least 2. As a result,

$$h : U \times_X \operatorname{Spec} \mathcal{O}_{X,\mathfrak{p}} \longrightarrow \operatorname{Spec} \mathcal{O}_{X,\mathfrak{p}}$$

is a Δ-torsor. Since Δ is local, h is a homeomorphism of topological spaces, so there is exactly one prime $\mathfrak{P}$ lying over $\mathfrak{p}$. We see then that $U \times_X V(\mathfrak{p}) = V(\mathfrak{P}^e)$ for some e.

Let $V(\mathfrak{p})^0$ be the intersection of $V(\mathfrak{p})$ with X^{sm}, and let $Z^0 = U \times_X V(\mathfrak{p})^0$. Then Z^0 is a Δ-torsor over $V(\mathfrak{p})^0$. Since q acts trivially on $V(\mathfrak{p})$, we obtain an action of

q on Z^0 over $V(\mathfrak{p})^0$, and hence a group scheme homomorphism

$$\varphi : Q'_{V(\mathfrak{p})^0} \longrightarrow \mathcal{A}ut(Z^0/V(\mathfrak{p})^0) = \Delta_{V(\mathfrak{p})^0},$$

where Q' denotes the subgroup of Q generated by q. Since $V(\mathfrak{p})^0$ is reduced, we see that φ factors through the reduction of $\Delta_{V(\mathfrak{p})^0}$, which is the trivial group scheme. Therefore, q acts trivially on Z^0.

Since the complement of X^{sm} in X has codimension at least 2, and since g factors as a flat map $U \to [U/\Delta]$ followed by a coarse space map $[U/\Delta] \to X$, both of which are codimension-preserving (see [Fantechi et al. 2007, Definition 4.2 and Remark 4.3]), we see that the complement of Z^0 in $V(\mathfrak{P}^e)$ has codimension at least 2. Note that if Y is a normal scheme and W is an open subscheme of Y whose complement has codimension at least 2, then any morphism from W to an affine scheme Z extends uniquely to a morphism from Y to Z. Since the action of q on $V(\mathfrak{P}^e)$ restricts to a trivial action on Z^0, the action of q on $V(\mathfrak{P}^e)$ is trivial. Therefore, q acts trivially on a divisor of U, and so $q = 1$. $\square$

Proof of Proposition 3.1. Let k'/k be a finite Galois extension such that $G_{k'} \simeq \Delta \rtimes Q$, where Δ is diagonalizable and Q is constant and tame. Let $S' = \operatorname{Spec} k'$ and consider the diagram

$$\begin{tikzcd} U' \arrow[r] \arrow[d] & U \arrow[d] \\ M' \arrow[r] \arrow[d] & M \arrow[d] \\ S' \arrow[r] & S, \end{tikzcd}$$

where the squares are Cartesian. We denote by x' the induced k'-rational point of U'. Since Δ is the product of a local diagonalizable group scheme and a locally constant diagonalizable group scheme, replacing k' by a further extension if necessary, we can assume that Δ is local.

Since G is stable, $G_{k'}$ has no pseudoreflections at x'. It follows then from Proposition 3.5 that there exists an open neighborhood W' of x' such that $U' \times_{M'} W' \longrightarrow W'$ is unramified in codimension 1. Since k'/k is a finite Galois extension, replacing W' by the intersection of the $\tau(W')$ as τ ranges over the elements of $\mathrm{Gal}(k'/k)$, we can assume W' is Galois-invariant. Hence, $W' = W \times_M M'$ for some open subset W of M. We shrink M to W.

To check that U^0 is a G-torsor over M^0, we can look étale-locally. We can therefore assume $S = S'$. Let $X = U/\Delta$, and let $g : U \to X$ and $f : X \to M$ be the quotient maps. We denote by X^0 the fiber product $X \times_M M^0$ and by f^0 the induced morphism $X^0 \to M^0$.

By Proposition 3.4, we know that the base change of U to X^{sm} is a Δ-torsor over X^{sm}. Since f is unramified in codimension 1, we see that f^0 is as well. Since M^0

is smooth and X^0 is normal, the purity of the branch locus theorem [Grothendieck and Raynaud 1971, X.3.1] implies that f^0 is étale, and hence a Q-torsor. Since X^0 is étale over M^0, it is smooth. As a result, U^0 is a Δ-torsor over X^0 from which it follows that U^0 is a G-torsor over M^0. $\square$

This finishes the proof of Proposition 3.1, and hence also of Theorem 1.6. We conclude this section by proving Corollary 1.8.

Proof of Corollary 1.8. Let $U = \operatorname{Spec} R$ and $M = U/G$. We denote by y the image of x. Since G being generated by pseudoreflections at x implies that G_K is generated by pseudoreflections at x for arbitrary finite linearly reductive group schemes G, and since smoothness of M at y can be checked étale-locally, we can assume that x is k-rational. Let $V = \mathfrak{m}_x/\mathfrak{m}_x^2$ be the cotangent space of x. As G is linearly reductive, there is a G-equivariant section of $\mathfrak{m}_x \to V$. This yields a G-equivariant map $\operatorname{Sym}^\bullet(V) \to R$, which induces an isomorphism $k[[V]] \longrightarrow \hat{\mathcal{O}}_{U,x}$ of G-representations. That is, complete locally, we have linearized the G-action. Since $\hat{\mathcal{O}}_{M,y} = k[[V]]^G$, the corollary follows from Theorem 1.6, as M is smooth at y if and only if $\hat{\mathcal{O}}_{M,y}$ is a formal power series ring over k. $\square$

4. Actions on smooth schemes

Having proved Theorem 1.9 for polynomial rings with linear actions, we now turn to the general case. We begin with two preliminary lemmas and a technical proposition.

Lemma 4.1. *Let U be a smooth affine scheme over S with an action of a finite diagonalizable group scheme Δ. Then there is a closed subscheme Z of U on which Δ acts trivially, with the property that every closed subscheme Y on which Δ acts trivially factors through Z. Furthermore, the construction of Z commutes with flat base change on U/Δ.*

Proof. Let $U = \operatorname{Spec} R$ and $\Delta = \operatorname{Spec} k[A]$, where A is a finite abelian group written additively. The Δ-action on U yields an A-grading

$$R = \bigoplus_{a \in A} R_a.$$

We see that if $\mathcal{J}$ is an ideal of R, then Δ acts trivially on $Y = \operatorname{Spec} R/\mathcal{J}$ if and only if $\mathcal{J}$ contains the R_a for $a \neq 0$. Letting $\mathcal{I}$ be the ideal generated by the R_a for $a \neq 0$, we see that $\operatorname{Spec} R/\mathcal{I}$ is our desired Z.

We now show that the formation of Z commutes with flat base change. Note that

$$U/\Delta = \operatorname{Spec} R_0.$$

Let R_0' be a flat R_0-algebra and let $R' = R_0' \otimes_{R_0} R$. The induced Δ-action on $\operatorname{Spec} R'$ corresponds to the A-grading

$$R' = \bigoplus_{a \in A} (R_0' \otimes_{R_0} R_a).$$

Since R_0' is flat over R_0, we see that $\mathcal{I} \otimes_{R_0} R_0'$ is an ideal of R', and one easily shows that it is the ideal generated by the $R_0' \otimes_{R_0} R_a$ for $a \neq 0$. □

Recall that if G is a group scheme over a base scheme B which acts on a B-scheme U, and if $y : T \to U$ is a morphism of B-schemes, then the stabilizer group scheme G_y is defined by the Cartesian diagram

$$\begin{tikzcd} G_y \arrow[r] \arrow[d] & G \times_B U \arrow[d, "\varphi"] \\ T \arrow[r, "y \times y"] & U \times_B U, \end{tikzcd}$$

where $\varphi(g, u) = (gu, u)$. If U is separated over B, then G_y is a closed subgroup scheme of G_T.

Lemma 4.2. *Let B be a scheme and G a finite flat group scheme over B. If G acts on a B-scheme U, then $U \to U/G$ is a G-torsor if and only if the stabilizer group schemes G_y are trivial for all closed points y of U.*

Proof. The "only if" direction is clear. To prove the "if" direction, it suffices to show that the stabilizer group schemes G_y are trivial for all scheme-valued points $y : T \to U$. This is equivalent to showing that the universal stabilizer G_u is trivial, where $u : U \to U$ is the identity map. Since G_u is a finite group scheme over U, it is given by a coherent sheaf $\mathcal{F}$ on U. The support of $\mathcal{F}$ is a closed subset, and so to prove G_u is trivial, it suffices to check this on stalks of closed points. Nakayama's lemma then shows that we need only check the triviality of G_u on closed fibers. That is, we need only check that the G_y are trivial for closed points y of U. □

Proposition 4.3. *Let U be a smooth affine scheme over S with a faithful action of a stable group scheme G fixing a k-rational point x. If N has a pseudoreflection at x, then there is an étale neighborhood $T \longrightarrow U/G$ of x and a divisor D of U_T defined by a principal ideal on which N_T acts trivially.*

Proof. Let $M = U/G$ and let y be the image of x in M. As in the proof of Corollary 1.8, we have an isomorphism $k[[V]] \longrightarrow \hat{\mathcal{O}}_{U,x}$ of G-representations, where $V = \mathfrak{m}_x/\mathfrak{m}_x^2$. If N is a pseudoreflection at x, then there is some $v \in V$ such that N acts trivially on the closed subscheme of $\operatorname{Spec} k[[V]]$ defined by the prime ideal generated by v.

Consider the contravariant functor F which sends an M-scheme T to the set of divisors of U_T defined by a principal ideal on which N_T acts trivially. As F is locally of finite presentation and $U \times_M \operatorname{Spec} \hat{\mathbb{O}}_{M,y} = \operatorname{Spec} \hat{\mathbb{O}}_{U,x}$, Artin's approximation theorem [Artin 1969] finishes the proof. $\square$

We are now ready to prove Theorem 1.9. Our method of proof is similar to that of Proposition 3.1; we first prove the theorem in the case that G is diagonalizable and then make use of this case to prove the theorem in general.

Proposition 4.4. *Theorem 1.9 holds when $G = \Delta$ is a finite diagonalizable group scheme.*

Proof. Let $g : U \to M$ be the quotient map. Since any subgroup scheme N of Δ is again finite diagonalizable, Lemma 4.1 shows that for every N, there exists a closed subscheme Z_N of U on which N acts trivially, with the property that every closed subscheme Y on which N acts trivially factors through Z_N. Let Z be the union of the finitely many closed subsets Z_N for $N \neq 1$. Since the action of Δ on U is faithful, Z has codimension at least 1. Let Z' be the union of all irreducible components of Z which have codimension 1. Since Δ acts without pseudoreflections at x, we see $x \notin Z'$. Note that $g(Z')$ is closed as g is proper. Since the construction of Z commutes with flat base change on M and since flat morphisms are codimension-preserving, replacing M with $M - g(Z')$, we can assume that there are no nontrivial subgroup schemes of Δ which fppf locally on M act trivially on a divisor of U.

By Lemma 4.2, to show U^0 is a Δ-torsor over M^0, it suffices to show that for every closed point y of U which maps to M^0, the stabilizer group scheme Δ_y is trivial. Fix such a closed point y and let $T = \operatorname{Spec} k(y)$. Since T is fppf over S, we see from Proposition 4.3 that the closed subgroup scheme Δ_y of Δ_T acts faithfully on U_T without pseudoreflections at the $k(y)$-rational point y' of U_T induced by y. Since y maps to a smooth point of M, it follows that y' maps to a smooth point of M_T. Corollary 1.8 then shows that Δ_y is generated by pseudoreflections. Since Δ_y has no pseudoreflections, it is therefore trivial. $\square$

Proof of Theorem 1.9. If $G = \Delta \rtimes Q$, where Δ is diagonalizable and Q is constant and tame, then letting Z' be as in Proposition 4.4 and letting U, X, f, and g be as in the proof of Proposition 3.1, the proof of Proposition 4.4 shows that after replacing M by $M - fg(Z')$, the base change of U to X^{sm} is a Δ-torsor over X^{sm}. As in the proof of Proposition 3.1, we can then reduce the general case to the case when $G = \Delta \rtimes Q$, where Δ is local diagonalizable and Q is constant tame. The last paragraph of the proof of Proposition 3.1 then shows that U^0 is a G-torsor over M^0. $\square$

5. Schemes with linearly reductive singularities

Let k be a perfect field of characteristic p.

Definition 5.1. We say a scheme M over S has *linearly reductive singularities* if there is an étale cover $\{U_i/G_i \to M\}$, where the U_i are smooth over S and the G_i are linearly reductive group schemes which are finite over S.

Note that if M has linearly reductive singularities, then it is automatically normal and, in fact, Cohen–Macaulay by [Hochster and Roberts 1974, page 115].

Our goal in this section is to prove Theorem 1.10, which generalizes the result that every scheme with quotient singularities prime to the characteristic is the coarse space of a smooth Deligne–Mumford stack. We remark that in the case of quotient singularities, the converse of the analogous theorem is true as well; that is, every scheme which is the coarse space of a smooth Deligne–Mumford stack has quotient singularities. It is not clear, however, that the converse of Theorem 1.10 should hold. We know from [Abramovich et al. 2008, Theorem 3.2] that $\mathfrak{X}$ is étale-locally $[V/G_0]$, where G_0 is a finite flat linearly reductive group scheme over V/G_0, but V need not be smooth and G_0 need not be the base change of a group scheme over S. On the other hand, Proposition 5.2 below shows that $\mathfrak{X}$ is étale-locally $[U/G]$ where U is smooth and G is a group scheme over S, but here G is not finite.

Before proving Theorem 1.10, we begin with a technical proposition followed by a series of lemmas.

Proposition 5.2. *Let $\mathfrak{X}$ be a tame stack over S with coarse space M. Then there exists an étale cover $T \to M$ such that*

$$\mathfrak{X} \times_M T = [U/\mathbb{G}^r_{m,T} \rtimes H],$$

where H is a finite constant tame group scheme and U is affine over T. Furthermore, $\mathbb{G}^r_{m,T} \rtimes H$ is the base change to T of a group scheme $\mathbb{G}^r_{m,S} \rtimes H$ over S, so $\mathfrak{X} \times_M T = [U/\mathbb{G}^r_{m,S} \rtimes H]$.

Proof. [Abramovich et al. 2008, Theorem 3.2] shows that there exists an étale cover $T \to M$ and a finite flat linearly reductive group scheme G_0 over T acting on a finite finitely presented scheme V over T such that

$$\mathfrak{X} \times_M T = [V/G_0].$$

By [Abramovich et al. 2008, Lemma 2.20], after replacing T by a finer étale cover if necessary, we can assume there is a short exact sequence

$$1 \longrightarrow \Delta \longrightarrow G_0 \longrightarrow H \longrightarrow 1,$$

where $\Delta = \operatorname{Spec} \mathbb{O}_T[A]$ is a finite diagonalizable group scheme and H is a finite constant tame group scheme. Since Δ is abelian, the conjugation action of G_0 on Δ passes to an action

$$H \longrightarrow \operatorname{Aut}(\Delta) = \operatorname{Aut}(A).$$

Choosing a surjection $F \to A$ in the category of $\mathbb{Z}[H]$-modules from a free module F yields an H-equivariant morphism $\Delta \hookrightarrow \mathbb{G}^r_{m,T}$. Using the H-action on $\mathbb{G}^r_{m,T}$, we define the group scheme $\mathbb{G}^r_{m,T} \rtimes G_0$ over T. Note that there is an embedding

$$\Delta \hookrightarrow \mathbb{G}^r_{m,T} \rtimes G_0$$

sending δ to (δ, δ^{-1}), which realizes Δ as a normal subgroup scheme of $\mathbb{G}^r_{m,T} \rtimes G_0$. We can therefore define

$$G := (\mathbb{G}^r_{m,T} \rtimes G_0)/\Delta.$$

One checks that there is a commutative diagram

$$\begin{tikzcd} 1 \arrow[r] & \Delta \arrow[r] \arrow[d] & G_0 \arrow[r] \arrow[d] & H \arrow[r] \arrow[d, "id"] & 1 \\ 1 \arrow[r] & \mathbb{G}^r_{m,T} \arrow[r] & G \arrow[r, "\pi"] & H \arrow[r] & 1, \end{tikzcd}$$

where the rows are exact and the vertical arrows are injective.

We show that, étale-locally on T, there is a group scheme-theoretic section of π, so that $G = \mathbb{G}^r_{m,T} \rtimes H$. Let P be the sheaf on T such that for any T-scheme W, $P(W)$ is the set of group scheme-theoretic sections of $\pi_W : G_W \to H_W$. Note that the sheaf $\underline{\mathrm{Hom}}(H, G)$ parametrizing group scheme homomorphisms from H to G is representable since it is a closed subscheme of $G^{\times|H|}$ cut out by suitable equations. We see that P is the equalizer of the two maps

$$\underline{\mathrm{Hom}}(H, G) \underset{p_2}{\overset{p_1}{\rightrightarrows}} H^{\times|H|},$$

where $p_1(\phi) = (\pi\phi(h))_h$ and $p_2(\phi) = (h)_h$. That is, there is a Cartesian diagram

$$\begin{tikzcd} P \arrow[r] \arrow[d] & \underline{\mathrm{Hom}}(H, G) \arrow[d, "(p_1, p_2)"] \\ H^{\times|H|} \arrow[r, "\Delta"] & H^{\times|H|} \times H^{\times|H|}. \end{tikzcd}$$

Since H is separated over T, we see that P is a closed subscheme of $\underline{\mathrm{Hom}}(H, G)$. In particular, it is representable and locally of finite presentation over T. Furthermore, $P \to T$ is surjective as [Abramovich et al. 2008, Lemma 2.16] shows that it has a section fppf locally. To show P has a section étale locally, by [Grothendieck 1967, 17.16.3], it suffices to prove P is smooth over T.

Given a commutative diagram

$$\begin{tikzcd} X_0 = \operatorname{Spec} A/\mathcal{I} \arrow[r] \arrow[d] & P \arrow[d] \\ X = \operatorname{Spec} A \arrow[r] \arrow[ur, dashed] & T, \end{tikzcd}$$

with $\mathscr{I}$ a square zero ideal, we want to find a dotted arrow making the diagram commute. That is, given a group scheme-theoretic section $s_0 : G_{W_0} \to H_{W_0}$ of π_{W_0}, we want to find a group scheme homomorphism $s : G_W \to H_W$ which pulls back to s_0 such that $\pi_W \circ s$ is the identity. Note first that any group scheme homomorphism s which pulls back to s_0 is automatically a section of π_W since H is a finite constant group scheme and $\pi_W \circ s$ pulls back to the identity over W_0. By [Grothendieck 1970, Exposé III 2.3], the obstruction to lifting s_0 to a group scheme homomorphism lies in

$$H^2(H, \mathrm{Lie}(G) \otimes \mathscr{I}),$$

which vanishes as H is linearly reductive. This proves the smoothness of P.

To complete the proof of the lemma, let $U := V \times^{G_0} G$ and note that

$$\mathfrak{X} \times_M T = [V/G_0] = [U/G].$$

Since V is finite over T and G is affine over T, it follows that U is affine over T as well. Replacing T by a finer étale cover if necessary, we have

$$\mathfrak{X} \times_M T = [U/\mathbb{G}^r_{m,T} \rtimes H].$$

Lastly, the scheme underlying $\mathbb{G}^r_{m,T} \rtimes H$ is $\mathbb{G}^r_{m,T} \times_T H$ and its group scheme structure is determined by the action $H \to \mathrm{Aut}(\mathbb{G}^r_{m,T})$. Since $\mathrm{Aut}(\mathbb{G}^r_{m,T}) = \mathrm{Aut}(\mathbb{Z}^r)$, we can use this same action to define the semidirect product $\mathbb{G}^r_{m,S} \rtimes H$ and it is clear that this group scheme base changes to $\mathbb{G}^r_{m,T} \rtimes H$. □

Lemma 5.3. *If V is a smooth S-scheme with an action of finite linearly reductive group scheme G_0 over S, then $[V/G_0]$ is smooth over S.*

Proof. Let $\mathfrak{X} = [V/G_0]$. To prove $\mathfrak{X}$ is smooth, it suffices to work étale-locally on S, where, by [Abramovich et al. 2008, Lemma 2.20], we can assume G_0 fits into a short exact sequence

$$1 \longrightarrow \Delta \longrightarrow G_0 \longrightarrow H \longrightarrow 1,$$

where Δ is a finite diagonalizable group scheme and H is a finite constant tame group scheme. Let G be obtained from G_0 as in the proof of Proposition 5.2 and let $U = V \times^{G_0} G$. Since $\mathfrak{X} = [U/G]$, it suffices to show U is smooth over S. The action of G_0 on $V \times G$, given by $g_0 \cdot (v, g) = (vg_0, g_0 g)$, is free as the G_0-action on G is free. As a result, $U = [(V \times G)/G_0]$ and $G/G_0 = [G/G_0]$. Since the projection map $p : V \times G \to G$ is G_0-equivariant, we have a Cartesian diagram

$$\begin{tikzcd} V \times G \arrow[r, "p"] \arrow[d] & G \arrow[d] \\ U \arrow[r, "q"] & G/G_0. \end{tikzcd}$$

Since p is smooth, q is as well. Since $G \to [G/G_0] = G/G_0$ is flat and G is smooth, [Grothendieck 1967, 17.7.7] shows that G/G_0 is smooth, and so U is as well. □

Lemma 5.4. *Let X be a smooth S-scheme and $i : U \hookrightarrow X$ an open subscheme whose complement has codimension at least* 2*. Let P be a G-torsor on U, where $G = \mathbb{G}_m^r \rtimes H$ and H is a finite constant étale group scheme. Then P extends uniquely to a G-torsor on X.*

Proof. The structure map from P to U factors as $P \to P_0 \to U$, where P is a $\mathbb{G}_m^r$-torsor over P_0 and P_0 is an H-torsor over U. Since the complement of U in X has codimension at least 2, we have $\pi_1(U) = \pi_1(X)$ and so P_0 extends uniquely to an H-torsor Q_0 on X. Let $i_0 : P_0 \hookrightarrow Q_0$ be the inclusion map. Since Q_0 is smooth and the complement of P_0 in Q_0 has codimension at least 2, the natural map $\mathrm{Pic}(Q_0) \to \mathrm{Pic}(P_0)$ is an isomorphism. It follows that any line bundle over P_0 can be extended uniquely to a line bundle over Q_0. We can therefore inductively construct a unique lift of P over X. □

Our proof of the following lemma closely follows that of [Fantechi et al. 2007, Theorem 4.6].

Lemma 5.5. *Let $f : \mathcal{Y} \to M$ be an S-morphism from a smooth tame stack $\mathcal{Y}$ to its coarse space which pulls back to an isomorphism over the smooth locus M^0 of M. If $h : \mathcal{X} \to M$ is a dominant, codimension-preserving morphism* (*see [Fantechi et al. 2007, Definition* 4.2*]*) *from a smooth tame stack, then there is a morphism $g : \mathcal{X} \to \mathcal{Y}$, unique up to unique isomorphism, such that $fg = h$.*

Proof. We show that if such a morphism g exists, then it is unique. Suppose g_1 and g_2 are two such morphisms. We see then that $g_1\big|_{h^{-1}(M^0)} = g_2\big|_{h^{-1}(M^0)}$. Since h is dominant and codimension-preserving, $h^{-1}(M^0)$ is open and dense in $\mathcal{X}$. [Fantechi et al. 2007, Proposition 1.2] shows that if $\mathcal{X}$ and $\mathcal{Y}$ are Deligne–Mumford with $\mathcal{X}$ normal and $\mathcal{Y}$ separated, then g_1 and g_2 are uniquely isomorphic. The proof, however, applies equally well to tame stacks since the only key ingredient used about Deligne–Mumford stacks is that they are locally $[U/G]$ where G is a separated group scheme.

By uniqueness, to show the existence of g, we can assume by Proposition 5.2 that $\mathcal{Y} = [U/G]$, where U is smooth and affine, and $G = \mathbb{G}_m^r \rtimes H$, where H is a finite constant tame group scheme. Let $p : V \to \mathcal{X}$ be a smooth cover by a smooth scheme. Since smooth morphisms are dominant and codimension-preserving, uniqueness implies that to show the existence of g, we need only show there is a morphism $g_1 : V \to \mathcal{Y}$ such that $fg_1 = hp$. So, we can assume $\mathcal{X} = V$.

Given a stack $\mathscr{Z}$ over M, let $\mathscr{Z}^0 = M^0 \times_M \mathscr{Z}$. Given a morphism $\pi : \mathscr{Z}_1 \to \mathscr{Z}_2$ of M-stacks, let $\pi^0 : \mathscr{Z}_1^0 \to \mathscr{Z}_2^0$ denote the induced morphism. Since f^0 is an isomorphism, there is a morphism $g^0 : V^0 \to \mathcal{Y}^0$ such that $f^0 g^0 = h^0$. It follows that there is a G-torsor P^0 over V^0 and a G-equivariant map from P^0 to U^0 such that the diagram

$$\begin{tikzcd} P^0 \arrow[r] \arrow[d] & U^0 \arrow[d] \\ V^0 \arrow[r] \arrow[d] & \mathcal{Y}^0 \arrow[dl, "\simeq"] \\ M^0 & \end{tikzcd}$$

commutes and the square is Cartesian. By Lemma 5.4, P^0 extends to a G-torsor P over V.

Note that if X is a normal algebraic space and $i : W \hookrightarrow X$ is an open subalgebraic space whose complement has codimension at least 2, then any morphism from W to an affine scheme Y extends uniquely to a morphism $X \to Y$. As a result, the morphism from P^0 to U^0 extends to a morphism $q : P \to U$. Consider the diagram

$$\begin{tikzcd} G \times P \arrow[r, "id \times q"] \arrow[d] & G \times U \arrow[d] \\ P \arrow[r, "q"] & U, \end{tikzcd}$$

where the vertical arrows are the action maps. Precomposing either of the two maps in the diagram from $G \times P$ to U by the inclusion $G \times P^0 \hookrightarrow G \times P$ yields the same morphism. That is, the two maps from $G \times P$ to U are both extensions of the same map from $G \times P^0$ to the affine scheme U, and hence are equal. This shows that q is G-equivariant, and therefore yields a map $g : V \to \mathcal{Y}$ such that $fg = h$. $\square$

Proof of Theorem 1.10. We begin with the following observation. Suppose U is smooth and affine over S with a faithful action of a finite linearly reductive group scheme G over S. Let y be a closed point of U mapping to $x \in U/G$. After making the étale base change $\operatorname{Spec} k(y) \to S$, we can assume y is a k-rational point. Let G_y be the stabilizer subgroup scheme of G fixing y. Since

$$U/G_y \longrightarrow U/G$$

is étale at y, replacing U/G by an étale cover, we can further assume that G fixes y. Then by Corollary 1.8, we can assume G has no pseudoreflections at y, and, hence, Theorem 1.9 shows that after shrinking U/G about x, we can assume that the base change of U to the smooth locus of U/G is a G-torsor.

We now turn to the proof. Since M has linearly reductive singularities, there is an étale cover $\{U_i/G_i \to M\}$, where U_i is smooth and affine over S and G_i is a finite linearly reductive group scheme over S which acts faithfully on U_i. By the above discussion, replacing this étale cover by a finer étale cover if necessary, we can assume that the base change of U_i to the smooth locus of U_i/G_i is a G_i-torsor. Let $M_i = U_i/G_i$ and $\mathfrak{X}_i = [U_i/G_i]$. We see that the $\mathfrak{X}_i$ are locally the desired stacks, so we need only glue the $\mathfrak{X}_i$. Let $M_{ij} = M_i \times_M M_j$ and let $V_i \to \mathfrak{X}_i$ be a smooth cover. Since M_{ij} is the coarse space of both $\mathfrak{X}_i \times_{M_i} M_{ij}$ and $\mathfrak{X}_j \times_{M_j} M_{ij}$, and since coarse space maps are dominant and codimension-preserving, Lemma 5.5 shows that there is a unique isomorphism of $\mathfrak{X}_i \times_{M_i} M_{ij}$ and $\mathfrak{X}_j \times_{M_j} M_{ij}$. Identifying these two stacks via this isomorphism, let I_{ij} be the fiber product over the stack of $V_i \times_{M_i} M_{ij}$ and $V_j \times_{M_j} M_{ij}$. We see then that we have a morphism $I_{ij} \to U_i \times_M U_j$. This yields a groupoid

$$\coprod I_{ij} \longrightarrow \coprod U_i \times_M U_j,$$

which defines our desired glued stack $\mathfrak{X}$. Note that $\mathfrak{X}$ is smooth and tame by [Abramovich et al. 2008, Theorem 3.2]. □

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The minimal resolution conjecture for points on del Pezzo surfaces

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Mustaţă (1997) stated a generalized version of the minimal resolution conjecture for a set Z of general points in arbitrary projective varieties and he predicted the graded Betti numbers of the minimal free resolution of I_Z. In this paper, we address this conjecture and we prove that it holds for a general set Z of points on any (not necessarily normal) del Pezzo surface $X \subseteq \mathbb{P}^d$ — up to three sporadic cases — whose cardinality $|Z|$ sits into the interval $[P_X(r-1), m(r)]$ or $[n(r), P_X(r)]$, $r \geq 4$, where $P_X(r)$ is the Hilbert polynomial of X, $m(r) := \frac{1}{2}dr^2 + \frac{1}{2}r(2-d)$ and $n(r) := \frac{1}{2}dr^2 + \frac{1}{2}r(d-2)$. As a corollary we prove: (1) Mustaţă's conjecture for a general set of $s \geq 19$ points on any integral cubic surface in $\mathbb{P}^3$; and (2) the ideal generation conjecture and the Cohen–Macaulay type conjecture for a general set of cardinality $s \geq 6d+1$ on a del Pezzo surface $X \subseteq \mathbb{P}^d$.

1. Introduction

Given a general set Z of s distinct points in $\mathbb{P}^n$ it is a long-standing problem in algebraic geometry to find out the exact shape of the minimal free resolution of its saturated ideal I_Z. It is well-known that it has to be of the form

$$0 \to F_n \to \cdots \to F_1 \to I_Z \to 0$$

with

$$F_i \cong R(-r-i)^{b_{i,r}} \oplus R(-r-i+1)^{b_{i,r-1}},$$

where R is the coordinate ring of $\mathbb{P}^n$ and r is the unique nonnegative integer such that $\binom{r+n-1}{n} \leq s < \binom{r+n}{n}$. Moreover,

$$b_{i+1,r-1} - b_{i,r} = \binom{r+i-1}{i}\binom{r+n}{n-i} - s\binom{n}{i}.$$

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The minimal resolution conjecture (MRC, for short) stated in [Lorenzini 1993] says that this resolution has no *ghost* terms, that is, $b_{i+1,r-1}b_{i,r} = 0$ for all i. The MRC is known to hold for $n \leq 4$ [Gaeta 1951; Ballico and Geramita 1986; Walter 1995] and for large values of s for any n [Hirschowitz and Simpson 1996] but it is false in general: Eisenbud, Popescu, Schreyer and Walter showed that it fails for any $n \geq 6$, $n \neq 9$ (see [Eisenbud et al. 2002]).

Besides MRC, two weaker conjectures have been stated concerning the initial and ending terms of the minimal free resolution of an ideal of points: the *ideal generation conjecture* (IGC for short), which says that the minimal number of generators of the ideal of a general set of points will be as small as possible; this conjecture can be translated in terms of the Betti numbers saying that $b_{1,r}b_{2,r-1} = 0$. At the other extreme of the resolution the *Cohen–Macaulay type conjecture* (CMC for short) affirms that the canonical module $\mathrm{Ext}^n_R(R/I_Z, R(-n-1))$ has as few generators as possible, i.e., $b_{n-1,r}b_{n,r-1} = 0$.

Mustaţă [1998] introduced a generalized version of MRC for points in arbitrary projective varieties (see Section 2 for a precise statement). Roughly speaking, it says that given a projective variety $X \subseteq \mathbb{P}^n$, the minimal free resolution of the ideal of any general set of points on X is determined by the resolution of the ideal of X. When $X = \mathbb{P}^n$, this formulation coincides with the original Lorenzini's statement. Giuffrida, Maggioni and Ragusa proved that this generalized conjecture holds for any general set of points when X is a smooth quadric surface in $\mathbb{P}^3$ [Giuffrida et al. 1996]. Casanellas [2009] proved that this conjecture holds for some special cardinalities of sets of general points on a smooth cubic surface. In [Miró-Roig and Pons-Llopis 2012] we showed that it also holds for any general set of at least 19 points on a smooth cubic surface in $\mathbb{P}^3$; Migliore and Patnott have been able to prove this for sets of general distinct points of any cardinality on a cubic surface $X \subseteq \mathbb{P}^3$ given that X is smooth or has at most isolated double points [Migliore and Patnott 2011, Theorem 1]. For the case of nonreduced 0-dimensional schemes see [Miró-Roig and Pons-Llopis 2012].

The goal of this paper it to prove MRC for general points on a del Pezzo surface $(X, \mathcal{O}_X(1))$, i.e., an integral arithmetically Gorenstein (not necessarily normal) surface with a very ample line bundle $\mathcal{O}_X(1)$ such that its dualizing sheaf verifies $\omega_X \cong \mathcal{O}_X(-1)$. This kind of variety has been studied thoroughly by Fujita [1990] in connection with his theory of Δ-genus. He defines the Δ-genus of a polarized variety $(X, \mathcal{O}_X(1))$ of dimension n as $\Delta(X, \mathcal{O}_X(1)) := n + \mathcal{O}_X(1)^n - \mathrm{h}^0(X, \mathcal{O}_X(1))$. In his terminology, del Pezzo varieties are ACM varieties of Δ-genus one.

The main technique used in this paper is the theory of Gorenstein liaison (see Section 2 for a brief account). Roughly speaking, knowing that two sets of points are G-linked will allow to pass from the minimal resolution of the ideal of one of them to the resolution of the other one (mapping cone procedure). Then once MRC is known to hold for a general set of $d+2$ points on a del Pezzo surface $X \subseteq \mathbb{P}^d$ an induction process will provide us with our main theorems (4.2, 4.3 and 4.4).

Let us briefly explain how this paper is organized. In Section 2 we introduce the background and main techniques needed, including general facts on del Pezzo surfaces and basic results on G-liaison. In Section 3 we establish MRC for sets of general points of two specific cardinalities,

$$m(r) := \tfrac{1}{2}dr^2 + \tfrac{1}{2}r(2-d) \quad \text{and} \quad n(r) := \tfrac{1}{2}dr^2 + \tfrac{1}{2}r(d-2),$$

with $r \geq 2$, on a del Pezzo surface $X \subset \mathbb{P}^d$ (up to four sporadic cases). We first establish the result for $m(2)$ points on X, which gives the initial step of our induction (Lemma 3.5). An easy remark gives us that if $n(r)$ general points on X have the expected resolution then $n(r)+1$ general points do as well. Then, using G-liaison, we prove that if $m(r)$ general points on a del Pezzo surface X satisfy MRC then so do $n(r)$ general points (Proposition 3.6). Finally, again using G-liaison, we show that if $n(r)+1$ general points on a del Pezzo surface satisfy MRC then so do $m(r+1)$ (Proposition 3.8). Section 4 contains the main results of this paper: namely that MRC holds on a del Pezzo surface (up to three of the four sporadic cases just mentioned) for general sets of points whose cardinality falls in the intervals $[P_X(r-1), m(r)]$ and $[n(r), P_X(r)]$ for any $r \geq 4 = \operatorname{reg} X + 1$, with $P_X(r)$ the Hilbert polynomial (see Theorem 4.2). As a corollary, we will get that Mustaţă's conjecture holds for any general set of at least 19 points on a cubic surface in $\mathbb{P}^3$ (Theorem 4.4) and the ideal generation conjecture as well as the Cohen–Macaulay type conjecture holds for any general set of at least $6d+1$ points on a del Pezzo surface in $\mathbb{P}^d$ (Theorem 4.3).

2. Preliminaries

We work over an algebraically closed field k of characteristic zero. We set $R = k[X_0, \ldots, X_n]$ and denote the associated projective space by $\mathbb{P}^n := \operatorname{Proj}(R)$. Given closed subschemes $Y \subseteq X \subseteq \mathbb{P}^n$, we denote the ideal sheaf of Y in X by $\mathcal{I}_{Y|X}$ and the homogeneous saturated ideal by $I_{Y|X} := \mathrm{H}^0_*(X, \mathcal{I}_{Y|X})$ (or simply I_Y when $X = \mathbb{P}^n$). We denote by R_X the homogeneous coordinate ring of X, defined as $k[X_0, \ldots, X_n]/I_X$. For any coherent sheaf $\mathscr{E}$ on X we denote the twisted sheaf $\mathscr{E} \otimes \mathcal{O}_X(l)$ by $\mathscr{E}(l)$. As usual, $\mathrm{H}^i(X, \mathscr{E})$ stands for the i-th cohomology group and $\mathrm{h}^i(X, \mathscr{E})$ for its dimension. We use the notation $\mathrm{H}^i_*(\mathscr{E})$ for the graded R-module $\bigoplus_{l\in\mathbb{Z}} \mathrm{H}^i(\mathbb{P}^n_k, \mathscr{E}(l))$ and ω_X will stand for the dualizing sheaf. The Hilbert function and Hilbert polynomial of X are denoted, respectively, by $H_X(t)$ and $P_X(t) \in \mathbb{Q}[t]$. The regularity of X is defined to be that of I_X; i.e., $\operatorname{reg} X \leq m$ if and only if $\mathrm{H}^i(\mathbb{P}^n, \mathcal{I}_X(m-i)) = 0$ for $i \geq 1$. By [Eisenbud 2005, Chapter IV, Theorem 4.2] we know that $P_X(t) = H_X(t)$ for any $t \geq \operatorname{reg} X - 1 + \delta - n$, where δ is the projective dimension of R_X. By a *variety* we mean an integral and proper scheme over k.

Definition 2.1. Let $X \subseteq \mathbb{P}^n$ be a subscheme with minimal graded free resolution

$$F_\bullet : 0 \to F_{n+1} \xrightarrow{d_{n+1}} F_n \to \cdots \to F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} R/I_X \to 0.$$

The graded *Betti numbers* $b_{ij}(X)$ are defined by

$$F_i = \bigoplus_{j\in\mathbb{Z}} R(-i-j)^{b_{ij}(X)}, \quad \text{i.e.,} \quad b_{ij}(X) = \dim_k \operatorname{Tor}^i(R/I_X, k)_{i+j}$$

and the *Betti diagram* of X has in the (i, j)-th position the Betti number $b_{ij}(X)$.

Remark 2.2. The free resolution $F_\bullet$ is minimal if, after choosing basis of F_i, the matrices representing d_i do not have any nonzero scalar.

Mustaţă [1998] predicted the minimal free resolution of a general set of points Z in an arbitrary projective variety X; he proved that the first rows of the Betti diagram of Z coincide with the Betti diagram of X and that there are two extra nontrivial rows at the bottom. He also gave lower bounds for the Betti numbers in these last two rows and the minimal resolution conjecture (MRC) for points on a projective variety states that these lower bounds are attained for a general set of points. Let us recall it.

Theorem 2.3 [Mustaţă 1998]. *Let $X \subseteq \mathbb{P}^n$ be a projective variety with $\dim X \geq 1$ and $\operatorname{reg} X = m$. Let s be an integer with $P_X(r-1) \leq s < P_X(r)$ for some $r \geq m+1$ and let Z be a set of s general points on X. If*

$$0 \to F_n \to F_{n-1} \to \cdots \to F_2 \to F_1 \to R \to R/I_X \to 0$$

is a minimal free R-resolution of R/I_X, then R/I_Z has a minimal free R-resolution of the type

$$\begin{aligned} 0 \to F_n \oplus R(-r-n+1)^{b_{n,r-1}} &\oplus R(-r-n)^{b_{n,r}} \\ &\to \cdots \to F_2 \oplus R(-r-1)^{b_{2,r-1}} \oplus R(-r-2)^{b_{2,r}} \\ &\qquad \to F_1 \oplus R(-r)^{b_{1,r-1}} \oplus R(-r-1)^{b_{1,r}} \to R \to R/I_Z \to 0; \end{aligned}$$

moreover,

$$b_{i+1,r-1}(Z) - b_{i,r}(Z) = \sum_{l=0}^{\dim X - 1} (-1)^l \binom{n-l-1}{i-l} \Delta^{l+1} P_X(r+l) - \binom{n}{i}(s - P_X(r-1)).$$

The *minimal resolution conjecture* (MRC for short) says that $b_{i+1,r-1} \cdot b_{i,r} = 0$ for $i = 1, \ldots, n-1$. Related to it are two weaker conjectures that deal only with a part of the minimal resolution of a general set of points: the *ideal generation conjecture* (IGC for short), which says that the minimal number of generators of the ideal of a general set of points will be as small as possible; this conjecture can be translated in terms of the Betti numbers saying that $b_{1,r}b_{2,r-1} = 0$. At the other extreme of the resolution the *Cohen–Macaulay type conjecture* (CMC for short) affirms that the canonical module $\mathrm{Ext}^n_R(R/I_Z, R(-n-1))$ has as few generators as possible, i.e., $b_{n-1,r}b_{n,r-1}=0$.

One of the main tools used in this paper is Gorenstein liaison theory. We recall its main features, of this theory referring the reader to [Kleppe et al. 2001] for a complete account.

Definition 2.4. A closed subscheme $X \subseteq \mathbb{P}^n$ of dimension r is said to be *Arithmetically Cohen–Macaulay (briefly, ACM)* if its homogeneous coordinate ring R_X is a Cohen–Macaulay ring or, equivalently, $\dim R_X = \operatorname{depth} \ R_X$.

Thanks to the graded version of the Auslander–Buchsbaum formula (for any finitely generated R-module M):

$$\mathrm{pd}(M) = n+1-\operatorname{depth}(M),$$

we deduce that a subscheme $X \subseteq \mathbb{P}^n$ is ACM if and only if the projective dimension of R_X is equal to the codimension of X; i.e.,

$$\mathrm{pd}(R_X) = \operatorname{codim} X. \tag{2-1}$$

Hence, if $X \subseteq \mathbb{P}^n$ is a codimension c ACM subscheme, a graded minimal free R-resolution of I_X is of the form:

$$0 \to F_c \to F_{c-1} \to \cdots \to F_1 \to I_X \to 0$$

where $F_i = \bigoplus_{j\in\mathbb{Z}} R(-j)^{\beta_{i,j}}$, $i=1,\dots,c$.

Definition 2.5. If $X \subseteq \mathbb{P}^n$ is an ACM subscheme then, the rank of the last free R-module in a minimal free R-resolution of I_X is called the *Cohen–Macaulay type* of X.

Definition 2.6. A codimension c subscheme X of $\mathbb{P}^n$ is *arithmetically Gorenstein* (briefly AG) if its homogeneous coordinate ring R_X is a Gorenstein ring or, equivalently, its saturated homogeneous ideal, I_X, has a minimal free graded R-resolution of the following type:

$$0 \to R(-t) \to \bigoplus_{i=1}^{\alpha_{c-1}} R(-n_{c-1,i}) \to \cdots \to \bigoplus_{i=1}^{\alpha_1} R(-n_{1,i}) \to I_X \to 0.$$

In other words, an AG scheme is an ACM scheme with Cohen–Macaulay type 1.

Any zero-dimensional scheme is ACM. For varieties of higher dimension we have the following characterization:

Lemma 2.7. *If* $\dim X \geq 1$, *then* $X \subseteq \mathbb{P}^n$ *is ACM if and only if* $\mathrm{H}^i_*(\mathcal{I}_X)=0$ *for* $1 \leq i \leq \dim X$.

The following remark will be used without further mention throughout the paper:

Remark 2.8. Let $X \subseteq \mathbb{P}^n$ be an ACM variety of dimension ≥ 1 and let $Y \subseteq X$ be any subvariety. Then the saturated ideal $I_{Y|X}$ equals $I_{Y|\mathbb{P}^n}/I_{X|\mathbb{P}^n}$.

Definition 2.9. Two subschemes X_1 and X_2 of $\mathbb{P}^n$ are *directly Gorenstein linked* (*directly G-linked* for short) by an AG scheme $G \subseteq \mathbb{P}^n$ if $I_G \subseteq I_{X_1} \cap I_{X_2}$ and

$$[I_G : I_{X_1}] = I_{X_2}, \quad [I_G : I_{X_2}] = I_{X_1}.$$

We say that X_2 is *residual* to X_1 in G. When G is a complete intersection we talk about a CI-link.

When X_1 and X_2 do not share any component, being directly G-linked by an AG scheme G is equivalent to $G = X_1 \cup X_2$.

Usually it is not easy to find out AG schemes to work with. The following theorem gives a useful way to construct them. Notice that, since we will want to work with varieties that can even be nonnormal, we will have to work in the framework of generalized divisors as introduced in [Hartshorne 1994; 2007]. The only general requirements to be fulfilled in order to work in this context are that the schemes satisfy conditions S_2 and G_1.

Definition 2.10. A subscheme $X \subseteq \mathbb{P}^n$ satisfies the condition G_r if every localization of R_X of dimension $\leq r$ is a Gorenstein ring. Usually this property is quoted as "Gorenstein in codimension $\leq r$", i.e., the non locally Gorenstein locus has codimension $\geq r+1$. In particular, G_0 is generically Gorenstein.

Theorem 2.11 (compare [Kleppe et al. 2001, Lemma 5.4]). *Let $S \subseteq \mathbb{P}^n$ be an ACM scheme satisfying condition G_1. Denote by K_S the canonical divisor and by H_S a general hyperplane section of S. Then any effective divisor in the linear system $|mH_S - K_S|$ is arithmetically Gorenstein.*

The main feature of G-liaison exploited in this paper is that through the mapping cone procedure it is possible to pass from the free resolution of a scheme X_1 to the free resolution of its residual X_2 on an arithmetically Gorenstein scheme. Let us recall how it works [Weibel 1994]:

Lemma 2.12 (mapping cone procedure). *Let*

$$0 \to M \xrightarrow{\alpha} N \to P \to 0$$

be a short exact sequence of finitely generated R-modules and let us consider free resolutions

$$e_\bullet : 0 \to G_{n+1} \xrightarrow{e_{n+1}} G_n \to \cdots \xrightarrow{e_1} G_0 \xrightarrow{e_0} M \to 0$$

and

$$d_\bullet : 0 \to F_{n+1} \xrightarrow{d_{n+1}} F_n \to \cdots \xrightarrow{d_1} F_0 \xrightarrow{d_0} N \to 0.$$

Then the morphism α *lifts to a morphism between the resolutions* $\alpha_\bullet : e_\bullet \to d_\bullet$ *and a* (*not necessarily minimal*) *free resolution for* P *is*

$$0 \to G_{n+1} \xrightarrow{c_{n+2}} G_n \oplus F_{n+1} \xrightarrow{c_{n+1}} \cdots \xrightarrow{c_3} G_1 \oplus F_2 \xrightarrow{c_2} G_0 \oplus F_1 \xrightarrow{c_1} F_0 \xrightarrow{c_0} P \to 0,$$

where

$$c_{i+1} = \begin{pmatrix} -e_i & 0 \\ \alpha_i & d_{i+1} \end{pmatrix} \quad \textit{for } 1 \le i \le n.$$

Lemma 2.13. *Let* $V_1, V_2 \subseteq \mathbb{P}^n$ *be two ACM schemes of codimension* c *directly* G*-linked by an AG scheme* W*. Let the minimal free resolutions of* I_{V_1} *and* I_W *be*

$$0 \to F_c \xrightarrow{d_c} F_{c-1} \xrightarrow{d_{c-1}} \cdots F_1 \xrightarrow{d_1} I_{V_1} \to 0$$

and

$$0 \to R(-t) \xrightarrow{e_c} G_{c-1} \xrightarrow{e_{c-1}} \cdots G_1 \xrightarrow{e_1} I_W \to 0$$

respectively. Then the functor $\operatorname{Hom}(-, R(-t))$ *applied to a free resolution of* I_{V_1}/I_W *gives a* (*not necessarily minimal*) *resolution of* I_{V_2}:

$$0 \to F_1^\vee(-t) \to F_2^\vee(-t) \oplus G_1^\vee(-t) \to \cdots \to F_c^\vee(-t) \oplus G_{c-1}^\vee(-t) \to I_{V_2} \to 0.$$

Let us finish this section introducing the family of varieties that we deal with throughout this paper.

Definition 2.14. A *del Pezzo* surface is a nondegenerate 2-dimensional projective variety $X \subseteq \mathbb{P}^d$ that is locally Gorenstein and such that its canonical sheaf verifies $\omega_X \cong \mathcal{O}_X(-1)$.

As examples of del Pezzo surfaces, we can consider any integral cubic surface in $\mathbb{P}^3$ or any complete intersection of two quadrics in $\mathbb{P}^4$. Notice that there exists a more general definition of del Pezzo surface for which it is only required that ω_X^{-1} is ample. Smooth surfaces with ample anticanonical sheaf are classically classified; see, for example, [Manin 1986, Chapter IV, Theorems 24.3 and 24.4] or [Dolgachev 2010, Corollary 8.1.17].

Any del Pezzo surface $X \subseteq \mathbb{P}^d$ satisfies $\deg(X) = d = \operatorname{codim}(X) + 2$. Recall that given a nondegenerate projective variety $X \subseteq \mathbb{P}^d$ it always holds that $\Delta(X) := \deg X + \dim X - \mathrm{h}^0(\mathcal{O}_X(1)) \ge 0$. It is a classical result the classification of *minimal varieties*, i.e., varieties for which there is equality in the previous expression (see, for instance, [Dolgachev 2010, Theorem 8.1.1]). Moreover, in the setting of his theory of Δ-genus, Fujita has also a satisfactory classification of *quasiminimal varieties*, i.e., varieties X satisfying $\Delta(X) = 1$. In his terminology, del Pezzo surfaces correspond to quasiminimal surfaces with sectional genus one (i.e., the arithmetic genus of a general hyperplane section is one). For more details see [Fujita 1990].

Any del Pezzo surface is ACM [Fujita 1990, (6.4)]. Therefore, according to [Hoa 1993, Theorem 1], the minimal free resolution of the coordinate ring of a del Pezzo surface $X \subseteq \mathbb{P}^d$ has the form:

$$0 \to R(-d) \to R(-d+2)^{\alpha_{d-3}} \to \cdots \to R(-2)^{\alpha_1} \to R \to R_X \to 0 \qquad (2\text{-}2)$$

where

$$\alpha_i = i\binom{d-1}{i+1} - \binom{d-2}{i-1} \text{ for } 1 \le i \le d-3.$$

Notice that X turns out to be AG and, in particular, $\alpha_i = \alpha_{d-2-i}$ for all $i = 1, \dots, d-2$.

3. The minimal resolution conjecture for sets of $m(r)$ and $n(r)$ general points

The goal of this section is to prove MRC for general sets of points of two specific cardinalities

$$m(r) := \tfrac{1}{2}dr^2 + \tfrac{1}{2}r(2-d) \quad \text{and} \quad n(r) := \tfrac{1}{2}dr^2 + \tfrac{1}{2}r(d-2)$$

on a del Pezzo surface X. Since the structure of our proof requires that X contains at least a line L and moreover that the elements of the linear system $|L + rH|$ satisfy condition G_1 in order to apply the theory of generalized divisors, we need to exclude the four cases $X \cong \mathbb{P}^2$, $X \cong \mathbb{P}^1 \times \mathbb{P}^1$, $X \cong \mathrm{F}_2 := \mathbb{P}(\mathbb{O}_{\mathbb{P}^1} \oplus \mathbb{O}_{\mathbb{P}^1}(-2))$, and X the *Bordello surface*, a complete intersection of two quadrics on $\mathbb{P}^4$ with a double line. Therefore, in this section $X \subseteq \mathbb{P}^d$ will stand for any del Pezzo surface as was defined in Definition 2.14 except the four aforementioned sporadic cases. The Hilbert polynomial and the regularity of X can be easily computed using (2-2): $P_X(r) = \frac{1}{2}d(r^2+r)+1$ and $\operatorname{reg} X = 3$. Notice that

$$P_X(r-1) < m(r) < n(r) < P_X(r). \qquad (3\text{-}1)$$

We also set the following notation.

(i) L is any line on X.

(ii) H denotes a general hyperplane section of X.

(iii) If C is a curve on X, H_C will be a general hyperplane section of C and K_C the canonical divisor on C.

The strategy for finding the minimal free resolution for a general set of points with cardinality $n(r)$ or $m(r)$, for $r \ge 2$, is as follows. First we establish the result for $m(2) = d+2$ points, which gives the starting point for our induction process. Then, using G-liaison, we prove that if $m(r)$ general points on any del Pezzo surface satisfy MRC then so do $n(r)$ general points. Next we observe that if $n(r)$ general points on X have the expected minimal free resolution then $n(r)+1$

general points do as well. And, finally, we show that if $n(r)+1$ general points on a del Pezzo surface satisfy MRC then so do $m(r+1)$.

We will prove the result via a series of lemmas and propositions. Since the shape of the minimal free resolution of the homogeneous ideal $I_{X|\mathbb{P}^3}$ of a del Pezzo surface of degree 3 is slightly different from that of a del Pezzo surface of degree $d \geq 4$ we need to consider the two cases separately in some arguments. We will give complete proofs in the case of degree $d \geq 4$. The concrete arguments on the case of degree 3 are analogous but much easier to write down and will therefore be left to the reader.

Lemma 3.1. *Let $X \subseteq \mathbb{P}^d$ be any del Pezzo surface of degree $d \geq 4$ and take $C \in |(r+\epsilon)H|$, $r \geq 2$, $\epsilon \in \{0,1\}$. Then, any effective divisor G in the linear system $|rH_C|$ is AG and it has a minimal free resolution of the following form:*

$$0 \to R(-2r-d-\epsilon) \to R(-2r-d+2-\epsilon)^{\alpha_{d-3}} \oplus R(-r-d)^{2-\epsilon} \oplus R(-r-d-1)^{\epsilon}$$
$$\to \cdots \to M_i \to R(-2r-\epsilon) \oplus R(-r-2)^{(2-\epsilon)\alpha_1} \oplus R(-r-3)^{\epsilon\alpha_1}$$
$$\to M_1 := R(-r)^{2-\epsilon} \oplus R(-r-1)^{\epsilon} \to I_{G|X} \to 0,$$

where $M_i := R(-2r-i+1-\epsilon)^{\alpha_{i-2}} \oplus R(-r-i)^{(2-\epsilon)\alpha_{i-1}} \oplus R(-r-i-1)^{\epsilon\alpha_{i-1}}$ for $i = 3, \ldots, d-2$ and $\alpha_i = i\binom{d-1}{i+1} - \binom{d-2}{i-1}$ for $1 \leq i \leq d-3$.

Proof. A curve C in $|(r+\epsilon)H|$ has saturated ideal $I_{C|X} = \mathrm{H}^0_*(\mathcal{O}_X(-r-\epsilon))$. From (2-2) we have

$$0 \to \mathcal{O}_{\mathbb{P}^d}(-d) \to \mathcal{O}_{\mathbb{P}^d}(-d+2)^{\alpha_{d-3}} \to \cdots \to \mathcal{O}_{\mathbb{P}^d}(-2)^{\alpha_1} \to \mathcal{O}_{\mathbb{P}^d} \to \mathcal{O}_X \to 0, \quad (3\text{-}2)$$

with $\alpha_i = i\binom{d-1}{i+1} - \binom{d-2}{i-1}$ for $1 \leq i \leq d-3$. Twisting (3-2) with $\mathcal{O}_{\mathbb{P}^d}(-r-\epsilon)$ and taking global sections we get the minimal graded free resolution of $I_{C|X}$:

$$0 \to R(-r-d-\epsilon) \to \cdots \to R(-r-(i+\epsilon))^{\alpha_{i-1}}$$
$$\to \cdots \to R(-r-2-\epsilon)^{\alpha_1} \to R(-r-\epsilon) \to I_{C|X} \to 0.$$

Now we can apply the horseshoe lemma to the exact sequence

$$0 \to I_{X|\mathbb{P}^d} \to I_{C|\mathbb{P}^d} \to I_{C|X} \to 0$$

to obtain the minimal free resolution of $I_{C|\mathbb{P}^d}$:

$$0 \to R(-r-d-\epsilon) \to R(-r-d+2-\epsilon)^{\alpha_{d-3}} \oplus R(-d) \to \cdots$$
$$\to T_i := R(-r-i-\epsilon)^{\alpha_{i-1}} \oplus R(-(i+1))^{\alpha_i} \to \cdots$$
$$\to R(-r-\epsilon) \oplus R(-2)^{\alpha_1} \to I_{C|\mathbb{P}^d} \to 0.$$

This sequence shows that $C \subseteq \mathbb{P}^d$ is an arithmetically Gorenstein variety with canonical module $K_C := Ext_R^{d-1}(R/I_C, R(-d-1)) = R_C(r-1+\epsilon)$. Therefore

$I_{G|C} = H^0_*(\mathbb{O}_C(-r)) = K_C(-2r+1-\epsilon)$. We apply $\mathrm{Hom}(-, R(-d-1))$ to the previous sequence and we get a graded minimal free resolution of K_C:

$$0 \to R(-d-1) \to R(r-d-1+\epsilon) \oplus R(-d+1)^{\alpha_{d-3}} \to \cdots \\ \to T_i' \to \cdots \to R(-1) \oplus R(r-3+\epsilon)^{\alpha_1} \to R(r-1+\epsilon) \to K_C \to 0,$$

where $T_i' := T^\vee_{d-i}(-d-1) = R(r-i-\epsilon)^{\alpha_{i-1}} \oplus R(-i)^{\alpha_{i-2}}$ for $i = 3, \ldots, d-2$. If we twist the previous sequence by $-2r+1-\epsilon$ we get the minimal resolution of $I_{G|C}$:

$$0 \to R(-2r-d-\epsilon) \to R(-r-d) \oplus R(-2r-d+2-\epsilon)^{\alpha_{d-3}} \to \cdots \to T_i'(-2r+1-\epsilon) \\ \to \cdots \to R(-2r-\epsilon) \oplus R(-r-2)^{\alpha_1} \to R(-r) \to I_{G|C} \to 0.$$

Finally, we can apply the horseshoe lemma to the short exact sequence

$$0 \to I_{C|X} \to I_{G|X} \to I_{G|C} \to 0$$

to recover the resolution of $I_{G|X}$:

$$0 \to R(-2r-d-\epsilon) \to R(-2r-d+2-\epsilon)^{\alpha_1} \oplus R(-r-d)^{2-\epsilon} \oplus R(-r-d-1)^{\epsilon} \\ \to \cdots \to M_i \to \cdots \to R(-2r-\epsilon) \oplus R(-r-2)^{(2-\epsilon)\alpha_1} \oplus R(-r-3)^{\epsilon\alpha_1} \\ \to R(-r)^{2-\epsilon} \oplus R(-r-1)^{\epsilon} \to I_{G|X} \to 0,$$

where $M_i := R(-2r-i+1-\epsilon)^{\alpha_{i-2}} \oplus R(-r-i)^{(2-\epsilon)\alpha_{i-1}} \oplus R(-r-i-1)^{\epsilon\alpha_{i-1}}$ for $i = 3, \ldots, d-2$. □

Lemma 3.2. *Let $X \subseteq \mathbb{P}^3$ be a del Pezzo surface of degree 3 and take $C \in |(r+\epsilon)H|$, $r \geq 2$, $\epsilon \in \{0, 1\}$. Then, any effective divisor G in the linear system $|rH_C|$ is AG and it has a minimal free resolution of the form*

$$0 \to R(-2r-3-\epsilon) \to R(-2r-\epsilon) \oplus R(-r-3)^{2-\epsilon} \oplus R(-r-4)^{\epsilon} \\ \to R(-r)^{2-\epsilon} \oplus R(-r-1)^{\epsilon} \to I_{G|X} \to 0.$$

Proof. This is completely analogous to Lemma 3.1. See also [Casanellas 2009, Proposition 3.5]. □

Lemma 3.3. *Let $X \subseteq \mathbb{P}^d$ be a del Pezzo surface and let $L \subseteq X$ be a line on it. Take $C \in |L + rH|$, $r \geq 2$, and let G be any effective divisor in the linear system $|2rH_C - K_C|$. Then, G is arithmetically Gorenstein and the minimal free resolution of $I_{G|C}$ has the form*

$$0 \to R(-2r-d-1) \to R(-2r-d+1)^{\alpha_1} \oplus R(-r-d)^{d-1} \to \cdots \\ \to R(-2r-i)^{\alpha_{d-i}} \oplus R(-r-i-1)^{\binom{d-1}{d-i}+\alpha_{d-i-1}} \to \cdots \\ \to R(-2r-1) \oplus R(-r-3)^{\binom{d-1}{d-2}+\alpha_{d-3}} \to R(-r-1) \oplus R(-r-2) \to I_{G|C} \to 0,$$

with $\alpha_i = i\binom{d-1}{i+1} - \binom{d-2}{i-1}$ *for* $1 \le i \le d-3$.

Proof. Let $L \subseteq X$ be any line. Its ideal as a subvariety of $\mathbb{P}^d$ has a resolution

$$0 \to R(-d+1) \to \cdots \to R(-i)^{\binom{d-1}{i}} \to \cdots \to R(-1)^{d-1} \to I_{L|\mathbb{P}^d} \to 0.$$

Using the mapping cone procedure for the exact sequence $0 \to I_{X|\mathbb{P}^d} \to I_{L|\mathbb{P}^d} \to I_{L|X} \to 0$ we get

$$0 \to R(-d) \oplus R(-d+1) \to \cdots \to R(-i)^{\binom{d-1}{i}+\alpha_{i-1}} \to \cdots \to R(-1)^{d-1} \to I_{L|X} \to 0$$

with $\alpha_i = i\binom{d-1}{i+1} - \binom{d-2}{i-1}$ for $1 \le i \le d-3$. Therefore, $C \in |L + rH|$ has the minimal graded free resolution

$$\begin{aligned} 0 \to R(-r-d) \oplus R(-r-d+1) \to \cdots \to R(-r-i)^{\binom{d-1}{i}+\alpha_{i-1}} \\ \to \cdots \to R(-r-1)^{d-1} \to I_{C|X} \to 0. \end{aligned} \tag{3-3}$$

Now the horseshoe lemma applied to $0 \to I_{X|\mathbb{P}^d} \to I_{C|\mathbb{P}^d} \to I_{C|X} \to 0$ gives us

$$\begin{aligned} 0 \to R(-r-d) \oplus R(-r-d+1) \to R(-r-d+2)^{\binom{d-1}{d-2}+\alpha_{d-3}} \oplus R(-d) \\ \to \cdots \to R(-r-i)^{\binom{d-1}{i}+\alpha_{i-1}} \oplus R(-(i+1))^{\alpha_i} \to \cdots \to \\ R(-r-1)^{d-1} \oplus R(-2)^{\alpha_1} \to I_{C|\mathbb{P}^d} \to 0. \end{aligned}$$

Since C is ACM we can apply $\mathrm{Hom}(-, R(-d-1))$ to get a resolution of K_C:

$$\begin{aligned} 0 \to R(-d-1) \to R(-d+1)^{\alpha_1} \oplus R(r-d)^{d-1} \\ \to \cdots \to R(r-i-1)^{\binom{d-1}{d-i}+\alpha_{d-i-1}} \oplus R(-i)^{\alpha_{d-i}} \to \cdots \to \\ R(r-3)^{\binom{d-1}{d-2}+\alpha_{d-3}} \oplus R(-1) \to R(r-1) \oplus R(r-2) \to K_C \to 0. \end{aligned}$$

Now, since $G \in |2rH_C - K_C|$ we have

$$\begin{aligned} 0 \to R(-2r-d-1) \to R(-2r-d+1)^{\alpha_1} \oplus R(-r-d)^{d-1} \\ \to \cdots \to R(-2r-i)^{\alpha_{d-i}} \oplus R(-r-i-1)^{\binom{d-1}{d-i}+\alpha_{d-i-1}} \to \cdots \to \\ R(-2r-1) \oplus R(-r-3)^{\binom{d-1}{d-2}+\alpha_{d-3}} \to R(-r-1) \oplus R(-r-2) \to I_{G|C} \to 0. \end{aligned}$$

□

Lemma 3.4. *Let $X \subseteq \mathbb{P}^3$ be an integral cubic surface and let $L \subseteq X$ be a line on it. Take $C \in |L + rH|$, $r \ge 2$, and let G be any effective divisor in the linear system $|2rH_C - K_C|$. Then, G is arithmetically Gorenstein and the minimal free resolution of $I_{G|C}$ has the following form*:

$$0 \to R(-2r-4) \to R(-2r-1) \oplus R(-r-3)^2 \to R(-r-1) \oplus R(-r-2) \to I_{G|C} \to 0$$

Proof. This is completely analogous to Lemma 3.3. □

Lemma 3.5. *A general set Z of $m(2) = d+2$ points on any del Pezzo surface $X \subseteq \mathbb{P}^d$ has a minimal free resolution of the type*

$$0 \to R(-d-2) \to R(-d)^{\gamma_{d-1}} \to \cdots \to R(-3)^{\gamma_2} \to R(-2)^{2d-1} \to I_{Z|X} \to 0,$$

with

$$\gamma_i = \sum_{l=0}^{1} (-1)^l \binom{d-l-1}{i-l} \Delta^{l+1} H_X(2+l) - \binom{d}{i}(m(2) - H_X(1)).$$

Proof. A general set Z of $d+2$ points on X is in linearly general position (i.e., any subset of Z of $d+1$ points spans $\mathbb{P}^d$). It is well-known that such a Z is AG with minimal free resolution

$$0 \to R(-d-2) \to R(-d)^{\rho_{d-1}} \to R(-d+1)^{\rho_{d-2}} \\ \to \cdots \to R(-3)^{\rho_2} \to R(-2)^{\rho_1} \to I_{Z|\mathbb{P}^d} \to 0,$$

where $\rho_i = i\binom{d+1}{i+1} - \binom{d}{i-1}$ for $1 \le i \le d-1$. We now apply the mapping cone procedure to $0 \to I_X \to I_Z \to I_{Z|X} \to 0$ to obtain a free resolution of $I_{Z|X}$:

$$0 \to R(-d-2) \to R(-d)^{\rho_{d-1}+1} \to R(-d+1)^{\rho_{d-2}} \to R(-d+2)^{\rho_{d-3}-\alpha_{d-3}} \\ \to \cdots \to R(-3)^{\rho_2-\alpha_2} \to R(-2)^{\rho_1-\alpha_1} \to I_{Z|X} \to 0,$$

with $\alpha_i = i\binom{d-1}{i+1} - \binom{d-2}{i-1}$ for $1 \le i \le d-3$. Since there are no ghost terms on the previous exact sequence, it is minimal and the coefficients are forced to be given by the formula from the statement. □

Once we have fixed the starting point of the induction we can deal with the different steps of the procedure.

Fix an integer $r \ge 2$ and let $Z_{m(r)}$ and $Z_{n(r)}$ be general sets of points on X of cardinality $m(r)$ and $n(r)$ respectively. We show that they are directly G-linked by an effective divisor G linearly equivalent to rH_C, where C is a curve in the linear system $|rH_X|$. Two issues need to be checked. First, we must show that $\mathrm{h}^0(\mathcal{O}_X(r)) > m(r)$, to guarantee the existence of a curve C in the linear system $|rH_X|$ such that $Z_{m(r)}$ lies on C. Secondly, we need to verify that $n(r) > p_a(C)$, to be able to apply Riemann–Roch Theorem for (singular) curves, which assures that there exists an effective divisor $Z_{n(r)}$ of degree $n(r)$ such that $Z_{m(r)} + Z_{n(r)}$ is linearly equivalent to a divisor rH_C. Notice that, thanks to [Eisenbud 2005, Chapter IV, Theorem 4.2], $P_X(r) = H_X(r) = \mathrm{h}^0(\mathcal{O}_X(r))$ for any $r \ge 1$.

Regarding the first issue, we have $\mathrm{h}^0(\mathcal{O}_X(r)) = P_X(r) > m(r)$ by construction and by (3-1).

Regarding the second issue, consider the exact sequence

$$0 \to \mathcal{O}_X(-r) \to \mathcal{O}_X \to \mathcal{O}_C \to 0.$$

Applying the functor of global sections we have

$$0 = H^1(X, \mathcal{O}_X) \to H^1(C, \mathcal{O}_C) \to H^2(X, \mathcal{O}_X(-r)) \to H^2(X, \mathcal{O}_X) = 0$$

and therefore $p_a(C) = \mathrm{h}^1(\mathcal{O}_C) = \mathrm{h}^2(\mathcal{O}_X(-r)) = \mathrm{h}^0(\mathcal{O}_X(r-1))$, where the last equality holds by Serre duality and taking into account that $\omega_X \cong \mathcal{O}_X(-1)$. Then, since

$$n(r) = dr^2 - m(r) > P_X(r-1) = \mathrm{h}^0(\mathcal{O}_X(r-1)) = p_a(C),$$

we are done.

Since this construction can also be performed starting from a general set $Z_{n(r)}$ of $n(r)$ points we see that a general set of $m(r)$ points is G-linked to a general set of $n(r)$ points and vice versa. This yields:

Proposition 3.6. *Fix $r \geq 2$ and assume that the ideal $I_{Z_{m(r)}|X}$ of $m(r)$ general points on a del Pezzo surface $X \subseteq \mathbb{P}^d$ has the minimal free resolution*

$$0 \to R(-r-d)^{r-1} \to R(-r-d+2)^{\gamma_{d-1,r-1}} \to \cdots \to R(-r-1)^{\gamma_{2,r-1}} \to R(-r)^{(d-1)r+1} \to I_{Z_{m(r)}|X} \to 0,$$

with

$$\gamma_{i,r-1} = \sum_{l=0}^{1} (-1)^l \binom{d-l-1}{i-l} \Delta^{l+1} P_X(r+l) - \binom{d}{i}(m(r) - P_X(r-1)).$$

Then the ideal $I_{Z_{n(r)}|X}$ of $n(r)$ general points has the minimal free resolution

$$0 \to R(-r-d)^{(d-1)r-1} \to R(-r-d+1)^{\beta_{d-1,r}} \to \cdots \to R(-r-2)^{\beta_{2,r}} \to R(-r)^{r+1} \to I_{Z_{n(r)}|X} \to 0,$$

with

$$\beta_{i,r} = \sum_{l=0}^{1} (-1)^{l+1} \binom{d-l-1}{i-l} \Delta^{l+1} P_X(r+l) + \binom{d}{i}(n(r) - P_X(r-1)).$$

Conversely, if $n(r)$ general points on a del Pezzo surface $X \subseteq \mathbb{P}^d$ have the expected resolution then $m(r)$ general points do as well.

Proof. As mentioned before, we give the complete proof in the case $d \geq 4$. The case $d = 3$ is completely analogous using Lemma 3.2 instead of Lemma 3.1. So suppose that $d \geq 4$. We will check that if $m(r)$ general points have the expected resolution then so do $n(r)$ and we leave to the reader the converse (which is proved analogously). By the preceding discussion, $m(r)$ and $n(r)$ general points on X are G-linked by $G \in |rH_C|$, where C is a curve in the linear system $|rH|$. Thanks to Lemma 3.1 we know the resolution of $I_{G|X}$ and hence we can apply the mapping cone procedure to the commutative diagram

$$
\begin{tikzcd}
 & R(-2r-d) \arrow[r] \arrow[d] & R(-r-d)^{r-1} \arrow[d] & & \\
 & R(-2r-d+2)^{\alpha_1}\oplus R(-r-d)^2 \arrow[r] \arrow[d] & R(-r-d+2)^{\gamma_{d-1}} \arrow[d] & & \\
 & \vdots \arrow[d] & \vdots \arrow[d] & & \\
 & R(-2r-i+1)^{\alpha_{d-i}}\oplus R(-r-i)^{2\alpha_{i-1}} \arrow[r] \arrow[d] & R(-r-i+1)^{\gamma_i} \arrow[d] & & \\
 & \vdots \arrow[d] & \vdots \arrow[d] & & \\
 & R(-2r)\oplus R(-r-2)^{2\alpha_1} \arrow[r] \arrow[d] & R(-r-1)^{\gamma_2} \arrow[d] & & \\
 & R(-r)^2 \arrow[r] \arrow[d] & R(-r)^{(d-1)r+1} \arrow[d] & & \\
0 \arrow[r] & I_{G|X} \arrow[r] \arrow[d] & I_{Z_{m(r)}|X} \arrow[r] \arrow[d] & I_{Z_{m(r)}|G} \arrow[r] & 0. \\
 & 0 & 0 & &
\end{tikzcd}
$$

Since $I_{G|X} \subseteq I_{Z_{m(r)}|X}$, we can take as part of the generators of $I_{Z_{m(r)}|X}$ the generators of $I_{G|X}$ and therefore the matrix defining the first horizontal map contains nonzero scalar entries. So the repeated elements can be split off. Therefore we get

$$
\begin{aligned}
0 \to R(-r-d)^{(d-1)r-1} &\to R(-r-d+1)^{\beta_{d-1,r}} \\
&\to \cdots \to R(-2)^{\alpha_1}\oplus R(-r)^{r+1} \to I_{Z_{n(r)}|\mathbb{P}^d} \to 0.
\end{aligned}
$$

The mapping cone procedure applied to the exact sequence $0 \to I_X \to I_{Z_{n(r)}} \to I_{Z_{n(r)}|X} \to 0$ then gives the desired minimal resolution for $I_{Z_{n(r)}|X}$. $\square$

Lemma 3.7. *Let $X \subset \mathbb{P}^d$ be any del Pezzo surface. Fix $r \geq 2$ and assume that the ideal $I_{Z_{n(r)}|X}$ of a set $Z_{n(r)}$ of $n(r)$ general points on $X \subseteq \mathbb{P}^d$ has the expected minimal free graded resolution then a set of $n(r)+1$ general points do as well.*

Proof. Since $I_{Z_{n(r)}|X}$ has the expected minimal free resolution, we know that $I_{Z_{n(r)}|X}$ is generated by $r+1$ forms of degree r. Moreover, we know that there are no linear relations among them. We take a general point $p \in X$ and set $Z := Z_{n(r)} \cup \{p\}$. Since $I_{Z|X} \subset I_{Z_{n(r)}|X}$, we can take the r generators of $I_{Z|X}$ in degree r to be a subset of the generators of $I_{Z_{n(r)}|X}$ in degree r; in particular, they do not have linear syzygies.

We must add d generators of degree $r+1$ in order to get a minimal system of generators of $I_{Z|X}$. Hence the first module in the minimal free resolution of $I_{Z|X}$ is $R(-r)^r \oplus R(-r-1)^d$ which forces the remaining part of the resolution. $\square$

Proposition 3.8. *Let $X \subseteq \mathbb{P}^d$ be a del Pezzo surface. Fix $r \geq 2$ and assume that the ideal $I_{Z_{p(r)}|X}$ of $p(r) := n(r)+1$ general points on X has the minimal free resolution*

$$0 \to R(-r-d)^{(d-1)r} \to R(-r-d+1)^{\delta_{d-1,r}} \to \cdots \to R(-r-2)^{\delta_{2,r}} \to R(-r)^r \oplus R(-r-1)^d \to I_{Z_{p(r)}|X} \to 0,$$

with

$$\delta_{i,r} = \sum_{l=0}^{1}(-1)^{l+1}\binom{d-l-1}{i-l}\Delta^{l+1}H_X(r+l) + \binom{d}{i}(p(r) - H_X(r-1)).$$

Then the ideal $I_{Z_{m(r+1)}|X}$ of $m(r+1)$ general points has the minimal free resolution

$$0 \to R(-r-d-1)^r \to R(-r-d+1)^{\gamma_{d-1,r}} \to \cdots \to R(-r-2)^{\gamma_{2,r}} \to R(-r-1)^{(d-1)r+d} \to I_{Z_{m(r+1)}|X} \to 0,$$

with

$$\gamma_{i,r} = \sum_{l=0}^{1}(-1)^{l}\binom{d-l-1}{i-l}\Delta^{l+1}H_X(r+1+l) - \binom{d}{i}(m(r+1) - H_X(r)).$$

Proof. Let $Z_{p(r)}$ be a set of $p(r)$ general points with resolution as in the statement. Let us consider the linear system $|L+rH|$. Since $\dim|L+rH| \geq \dim|rH| = h^0(\mathbb{O}_X(r)) - 1 = P_X(r) - 1 > p(r)$, we can find a curve $C \in |L+rH|$ passing through these $p(r)$ points. As it is shown in [Pons-Liopis 2011, Chapter II] we can suppose that C verifies condition G_1. Notice that $\deg(C) = 1 + rd$ and $p_a(C) = d\binom{r}{2} + r$. Since $p_a(C) < m(r+1)$ we can find an effective divisor $Z_{m(r+1)}$ of degree $m(r+1)$ such that $Z_{p(r)}$ and $Z_{m(r+1)}$ are G-linked by a divisor of degree $p(r) + m(r+1) = dr^2 + dr + 2 = deg(2rH_C - K_C)$. This will allowed us to find the resolution of $I_{m(r+1)|X}$. First of all, let us find the minimal free resolution of the ideal $I_{p(r)|C}$ from the exact sequence $0 \to I_{C|X} \to I_{p(r)|X} \to I_{p(r)|C} \to 0$ through the mapping cone procedure, with the resolution of $I_{C|X}$ as it was found in (3-3). It turns out to be

$$0 \to R(-r-d)^{(d-1)r+1} \to R(-r-d+1)^{c_{d-1,r}} \to \cdots \to R(-r-2)^{c_{2,r}} \to R(-r)^r \oplus R(-r-1) \to I_{p(r)|C} \to 0.$$

Since we have already found out the minimal free resolution of $I_{G|C}$ (Lemma 3.3) we can use the mapping cone procedure applied to the sequence $0 \to I_{G|C} \to I_{p(r)|C} \to I_{p(r)|G} \to 0$ to get

$$\begin{aligned} 0 \to R(-2r-d-1) \to R(-r-d)^{(d-1)r+d} \oplus R(-2r-d+1)^{\alpha_1} \\ \to \cdots \to R(-r-i)^{d_{i,r}} \oplus R(-2r-i+1)^{\alpha_{d-i+1}} \to \cdots \to \\ R(-r-2)^{d_{2,r}} \to R(-r)^r \to I_{Z_{p(r)}|G} \to 0. \end{aligned}$$

(If $d=3$ we have instead $0 \to R(-2r-4) \to R(-r-3)^{2r+2} \oplus R(-2r-1) \to R(-r-2)^{d_{2,r}} \to R(-r)^r \to I_{Z_{p(r)}|G} \to 0$.)

Finally we obtain the minimal free resolution of $I_{m(r+1)|\mathbb{P}^d}$:

$$\begin{aligned} 0 \to R(-r-d-1)^r \to R(-r-d+1)^{\gamma_{d-1,r}} \to R(-r-d+2)^{\gamma_{d-2,r}} \oplus R(-d) \\ \to \cdots \to R(-r-i)^{\gamma_{i,r}} \oplus R(-i)^{\alpha_i} \to \cdots \to \\ R(-r-1)^{(d-1)r+d} \oplus R(-2)^{\alpha_1} \to I_{Z_{m(r+1)}|\mathbb{P}^d} \to 0 \end{aligned}$$

($0 \to R(-r-4)^r \to R(-r-2)^{\gamma_{2,r}} \to R(-r-1)^{2r+3} \oplus R(-3) \to I_{Z_{m(r+1)}|\mathbb{P}^3} \to 0$ if $d=3$) from which it is straightforward to recover the predicted resolution of $I_{Z_{m(r+1)|X}}$. □

We are now ready to prove the main theorem of this section:

Theorem 3.9. *Let $X \subseteq \mathbb{P}^d$ be a del Pezzo surface. We have*:

(1) *Let $Z_{n(r)} \subseteq X$ be a general set of $n(r)$ points, $r \geq 2$. Then the minimal graded free resolution of $I_{Z_{n(r)}|X}$ has the form*

$$\begin{aligned} 0 \to R(-r-d)^{(d-1)r-1} \to R(-r-d+1)^{\beta_{d-1,r}} \to R(-r-d+2)^{\beta_{d-2,r}} \\ \to \cdots \to R(-r-2)^{\beta_{2,r}} \to R(-r)^{r+1} \to I_{Z_{n(r)}|X} \to 0, \end{aligned}$$

with

$$\beta_{i,r} = \sum_{l=0}^{1} (-1)^{l+1} \binom{n-l-1}{i-l} \Delta^{l+1} H_X(r+l) + \binom{n}{i} (n(r) - H_X(r-1)).$$

(2) *Let $Z_{m(r)} \subseteq X$ be a general set of $m(r)$ points, $r \geq 2$. Then its minimal graded free resolution has the form*

$$\begin{aligned} 0 \to R(-r-d)^{r-1} \to R(-r-d+2)^{\gamma_{d-1,r-1}} \\ \to \cdots \to R(-r-1)^{\gamma_{2,r-1}} \to R(-r)^{(d-1)r+1} \to I_{Z_{m(r)}|X} \to 0, \end{aligned}$$

with

$$\gamma_{i,r-1} = \sum_{l=0}^{1} (-1)^{l} \binom{n-l-1}{i-l} \Delta^{l+1} P_X(r+l) - \binom{n}{i} (m(r) - P_X(r-1)).$$

In particular, Mustață's conjecture works for $n(r)$ and $m(r)$, $r \geq 4$, general points on a del Pezzo surface $X \subseteq \mathbb{P}^d$.

Proof. Lemma 3.5 establishes the result for a set of $m(2)$ general points, the starting point of our induction process. Therefore, the result about the resolution of $I_{Z_{n(r)}|X}$ and $I_{Z_{m(r)}|X}$ follows using Lemma 3.7, Propositions 3.6 and 3.8 and applying induction. $\square$

4. Main theorem

In this last section, we are going to prove that MRC holds for a general set of points Z on a del Pezzo surface (excluding three of the four sporadic cases pointed out at the beginning of the previous section: $X \cong \mathbb{P}^1 \times \mathbb{P}^1$, $X \cong \mathrm{F}_2 := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2))$, and X the Bordello surface) when the cardinality of Z falls in intervals of the form $[P_X(r-1), m(r)]$ or $[n(r), P_X(r)]$, $r \geq 4$. So for the rest of the paper X will denote any del Pezzo surface excluding these three particular surfaces. We will use the fact that we already know that $n(r)$ and $m(r)$ general points on a del Pezzo surface satisfy MRC together with the following lemma which controls how the bottom lines of the Betti diagram of a set of general points on a projective variety change when we add another general point. This lemma will turn out to be a cornerstone in our proof of MRC for del Pezzo surfaces:

Lemma 4.1. *Let $X \subseteq \mathbb{P}^n$ be a projective variety with $\dim X \geq 2$, $\operatorname{reg} X = m$ and with Hilbert polynomial P_X. Let s be an integer with $P_X(r-1) \leq s < P_X(r)$ for some $r \geq m+1$, let Z be a set of s general points on X and let $P \in X \setminus Z$ be a general point. We have*

(i) $b_{i,r-1}(Z) \geq b_{i,r-1}(Z \cup P)$ *for every i.*

(ii) $b_{i,r}(Z) \leq b_{i,r}(Z \cup P)$ *for every i.*

Proof. See [Mustaţă 1998, Proposition 1.7.]. $\square$

We are now ready to state the main result of this paper:

Theorem 4.2. *Let $X \subseteq \mathbb{P}^d$ be a del Pezzo surface. Let r satisfy $r \geq \operatorname{reg} X + 1 = 4$. Then for a general set of points Z on X such that $P_X(r-1) \leq |Z| \leq m(r)$ or $n(r) \leq |Z| \leq P_X(r)$ the minimal resolution conjecture is true.*

Proof. First of all we want to point out that the result was already known in the cases $|Z| = P_X(r-1)$ and $|Z| = P_X(r) - 1$ [Mustaţă 1998, Examples 1 and 2].

On the other hand, the results about Ulrich bundles proved in [Pons-Liopis 2011, Chapter II] and Serre's correspondence allows us to deal with the case of $X \cong \mathbb{P}^2$. So let X be any other del Pezzo surface. Let Z' be a general set of points of cardinality $|Z'| = n(r)$ and add general points to Z' in order to get a set of points Z of cardinality $n(r) \leq |Z| \leq P_X(r)$. By Theorem 3.9 we have that $b_{i,r-1}(Z') = 0$ for all $i = 2, \dots, d$. Therefore we can apply Lemma 4.1 to deduce that $b_{i,r-1}(Z) = 0$ for all $i = 2, \dots, d$. Thus, by semicontinuity, MRC holds for a general set of $|Z|$ points.

Now if $|Z| \leq m(r)$, we can add general points to Z in order to have a general set Z' including Z and such that $|Z'| = m(r)$. Again from the previous Theorem we have that $b_{i,r}(Z') = 0$ for all $i = 1, \ldots, d-1$. So we can use again Lemma 4.1 to deduce that $b_{i,r}(Z) = 0$ for all $i = 1, \ldots, d-1$ and therefore MRC holds for Z. □

As a consequence of Theorem 3.9 we will prove that the number of generators of the ideal of a general set of points on a del Pezzo surface is as small as possible and so it is the number of generators of its canonical module as well. In fact, we have:

Theorem 4.3. *Let $X \subseteq \mathbb{P}^d$ be a del Pezzo surface. Then for a general set of points Z on X such that $|Z| \geq P_X(3)$ the Cohen–Macaulay type conjecture and the ideal generation conjecture are true.*

Proof. Let Z be a general set of points on our del Pezzo surface X. If it is the case that $n(r) \leq |Z| \leq m(r+1)$ the result has been proved on the previous theorem. So we can assume that $m(r) < |Z| < n(r)$ for some r. We know that MRC holds for a general set $|Z'|$ of $n(r)$ points on X, $Z \subseteq Z'$ and in particular $b_{1,r}(Z') = 0$. Applying Lemma 4.1 inductively we see that $b_{1,r}(Z) = 0$. Analogously, since MRC holds for a general set Z'' of $m(r)$ points, $b_{d,r-1}(Z'') = 0$ with $Z'' \subseteq Z$. Applying once again the same lemma we see that $b_{d,r-1}(Z) = 0$. □

In the particular case of the cubic surface, since the minimal free resolution of its points has length three, we recover one of the main results of [Miró-Roig and Pons-Llopis 2012] (see also [Migliore and Patnott 2011; Casanellas 2009]):

Theorem 4.4. *Let $X \subseteq \mathbb{P}^3$ be an integral cubic surface* (*i.e.*, *a del Pezzo surface of degree three*). *Then the minimal resolution conjecture holds for a general set of points on X of cardinality $\geq P_X(3) = 19$.*

Proof. By Theorem 4.3 we know that any set Z of general points on X verify the Cohen–Macaulay type conjecture and the ideal generation conjecture. But since the codimension is three there is no further term on the resolution left to consider so the general MRC also holds. □

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Short before we finished writing this paper we were informed that Migliore and Patnott were able to prove the minimal resolution conjecture for sets of general distinct points of any cardinality on a cubic surface $X \subseteq \mathbb{P}^3$ given that X is smooth or it has at most isolated double points [Migliore and Patnott 2011]. We are very grateful to J. Migliore for sending their paper to us, and for the helpful discussions we have had.

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L-series of Artin stacks over finite fields

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We develop the notion of stratifiability in the context of derived categories and the six operations for stacks. Then we reprove the Lefschetz trace formula for stacks, and give the meromorphic continuation of L-series (in particular, zeta functions) of $\mathbb{F}_q$-stacks. We also give an upper bound for the weights of the cohomology groups of stacks, and an "independence of ℓ" result for a certain class of quotient stacks.

1. Introduction

In topology there is the famous Lefschetz–Hopf trace formula, which roughly says that if $f : X \to X$ is an endomorphism of a compact connected oriented space X with isolated fixed points, then the number of fixed points of f, counted with multiplicity, is equal to the alternating sum of the traces of f^* on the singular cohomology groups $H^i(X, \mathbb{Q})$.

There is also a trace formula in algebraic geometry, for schemes over finite fields, due to Grothendieck. It says that if X_0 is a scheme over $\mathbb{F}_q$, separated and of finite type, and F_q is the q-geometric Frobenius map, then

$$\#X_0(\mathbb{F}_q) = \sum_{i=0}^{2 \dim X_0} (-1)^i \operatorname{Tr}(F_q, H^i_c(X, \overline{\mathbb{Q}}_\ell)),$$

where $H^i_c(X, \overline{\mathbb{Q}}_\ell)$ is the ℓ-adic cohomology with compact support. In fact he proved the trace formula for an arbitrary constructible sheaf, see [Grothendieck 1965, Verdier 1967, Deligne 1977].

Behrend conjectured the trace formula for smooth algebraic stacks over $\mathbb{F}_q$ in his thesis and [Behrend 1993], and proved it in [Behrend 2003]. However, he used ordinary cohomology and arithmetic Frobenius (rather than compact support cohomology and geometric Frobenius) to prove the "dual statement," probably because at that time the theory of dualizing complexes of algebraic stacks, as well as compact support cohomology groups of stacks, were not developed. Later Laszlo and Olsson [2008a; 2008b] developed the theory of the six operations for

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algebraic stacks, which makes it possible to reprove the trace formula, and remove the smoothness assumption in Behrend's result. Also we will work with a fixed isomorphism of fields $\iota : \overline{\mathbb{Q}}_\ell \xrightarrow{\sim} \mathbb{C}$, namely we will work with *ι-mixed* complexes, rather than *mixed* ones, and this is a more general setting (see Remark 2.8.1).

Once we have the trace formula, we get a factorization of the zeta function into a possibly infinite product of L-factors, and from this one can deduce the meromorphic continuation of the zeta functions, generalizing a result of Behrend [1993, 3.2.4]. Furthermore, to locate the zeros and poles of the zeta functions, we give a result on the weights of cohomology groups of stacks.

We briefly mention the technical issues. As pointed out in [Behrend 2003], a big difference between schemes and stacks is the following. If

$$f : X_0 \to Y_0$$

is a morphism of $\mathbb{F}_q$-schemes of finite type, and $K_0 \in D_c^b(X_0, \overline{\mathbb{Q}}_\ell)$, then f_*K_0 and $f_!K_0$ are also bounded complexes. Since often we are mainly interested in the simplest case when K_0 is a sheaf concentrated in degree 0 (for instance, the constant sheaf $\overline{\mathbb{Q}}_\ell$), and D_c^b is stable under f_* and $f_!$, it is enough to consider D_c^b only. But for a morphism

$$f : \mathscr{X}_0 \to \mathscr{Y}_0$$

of $\mathbb{F}_q$-algebraic stacks of finite type, f_* and $f_!$ do not necessarily preserve boundedness. For instance, the cohomology ring $H^*(B\mathbb{G}_m, \overline{\mathbb{Q}}_\ell)$ is the polynomial ring $\overline{\mathbb{Q}}_\ell[T]$ with $\deg(T) = 2$. So for stacks we have to consider unbounded complexes, even if we are only interested in the constant sheaf $\overline{\mathbb{Q}}_\ell$. In order to define the trace of the Frobenius on cohomology groups, we need to consider the convergence of the complex series of the traces. This leads to the notion of an ι-convergent complex of sheaves (see Definition 4.1).

Another issue is the following. In the scheme case one considers bounded complexes, and for any bounded complex K_0 on a scheme X_0, there exists a stratification of X_0 that "trivializes the complex K_0" (that is, the restrictions of all cohomology sheaves $\mathscr{H}^i K_0$ to each stratum are lisse). But in the stack case we have to consider unbounded complexes, and in general there might be no stratification of the stack that trivializes every cohomology sheaf. This leads to the notion of a stratifiable complex of sheaves (see Definition 3.1). We need the stratifiability condition to control the dimensions of cohomology groups (see Lemma 3.10). All bounded complexes are stratifiable by Lemma 3.4 (v).

Also we will have to impose the condition of ι-mixedness, due to unboundedness. For bounded complexes on schemes, the trace formula can be proved without using this assumption, although the conjecture of Deligne [1980, 1.2.9] that all sheaves are ι-mixed is proved by Laurent Lafforgue, see Remark 2.8.1.

We briefly introduce the main results of this paper.

Fixed point formula.

Theorem 1.1. *Let $\mathscr{X}_0$ be an Artin stack of finite type over $\mathbb{F}_q$, and let $[\mathscr{X}_0(\mathbb{F}_q)]$ be the set of isomorphism classes of the groupoid of $\mathbb{F}_q$-points of $\mathscr{X}_0$. Then the series*

$$\sum_{n\in\mathbb{Z}}(-1)^n \operatorname{Tr}(F_q, H_c^n(\mathscr{X}, \overline{\mathbb{Q}}_\ell)),$$

regarded as a complex series via ι, is absolutely convergent, and its limit is "the number of $\mathbb{F}_q$-points of $\mathscr{X}_0$", namely

$$\#\mathscr{X}_0(\mathbb{F}_q) \;:=\sum_{x\in[\mathscr{X}_0(\mathbb{F}_q)]}\frac{1}{\#\operatorname{Aut}_x \mathbb{F}_q}.$$

Here F_q denotes the q-geometric Frobenius. To generalize, one wants to impose some condition (P) on complexes $K_0 \in D_c^-(\mathscr{X}_0, \overline{\mathbb{Q}}_\ell)$ such that:

(1) The condition (P) is preserved by $f_!$.

(2) If a complex K_0 satisfies (P), then the "naive local terms" are well-defined.

(3) Trace formula holds in this case.

The condition (P) on K_0 turns out to be a combination of three parts: ι-convergence (which implies (2) for K_0), ι-mixedness and stratifiability (which, together with the first part, implies (2) for $f_!K_0$). See Theorem 4.2 for the general statement. These conditions are due to Behrend [2003].

Meromorphic continuation. The rationality in Weil conjecture was first proved by Dwork, namely the zeta function $Z(X_0, t)$ of every variety X_0 over $\mathbb{F}_q$ is a rational function in t. Later, this was reproved using the fixed point formula [Grothendieck 1965, Illusie 1977]. Following Behrend [1993, 3.2.3], we define the zeta functions of stacks as follows.

Definition 1.2. *For an $\mathbb{F}_q$-algebraic stack $\mathscr{X}_0$ of finite type, define the* zeta function

$$Z(\mathscr{X}_0, t) = \exp\left(\sum_{v\geq 1}\frac{t^v}{v}\sum_{x\in[\mathscr{X}_0(\mathbb{F}_{q^v})]}\frac{1}{\#\operatorname{Aut}_x \mathbb{F}_{q^v}}\right),$$

as a formal power series in the variable t.

Notice that in general, the zeta function is not rational (see Section 7). Behrend [1993, 3.2.4, 3.2.5] proved that if $\mathscr{X}_0$ is a smooth algebraic stack, and it is a quotient of an algebraic space by a linear algebraic group, then its zeta function $Z(\mathscr{X}_0, t)$ is a meromorphic function in the complex t-plane; if $\mathscr{X}_0$ is a smooth Deligne–Mumford stack, then $Z(\mathscr{X}_0, t)$ is a rational function. These results can be generalized as follows.

Theorem 1.3. *For every $\mathbb{F}_q$-algebraic stack $\mathscr{X}_0$ of finite type, its zeta function $Z(\mathscr{X}_0, t)$ defines a meromorphic function in the whole complex t-plane. If $\mathscr{X}_0$ is Deligne–Mumford, then $Z(\mathscr{X}_0, t)$ is a rational function.*

See Proposition 7.3.1 and Theorem 8.1 for the general statement.

A theorem of weights. One of the main results in [Deligne 1980] is that if X_0 is an $\mathbb{F}_q$-scheme, separated and of finite type, and $\mathscr{F}_0$ is an ι-mixed sheaf on X_0 of punctual ι-weights $\le w \in \mathbb{R}$, then for every n, the punctual ι-weights of $H^n_c(X, \mathscr{F})$ are $\le w + n$. The cohomology groups are zero unless $0 \le n \le 2 \dim X_0$. We will see in Remark 7.2.1 that the upper bound $w + n$ for the punctual ι-weights does not work in general for algebraic stacks. We will give an upper bound that applies to all algebraic stacks. Deligne's upper bound of weights still applies to stacks for which all the automorphism groups are affine.

Theorem 1.4. *Let $\mathscr{X}_0$ be an $\mathbb{F}_q$-algebraic stack of finite type, and let $\mathscr{F}_0$ be an ι-mixed sheaf on $\mathscr{X}_0$, with punctual ι-weights $\le w$, for some $w \in \mathbb{R}$. Then the ι-weights of $H^n_c(\mathscr{X}, \mathscr{F})$ are $\le \dim \mathscr{X}_0 + \frac{n}{2} + w$, and they are congruent mod $\mathbb{Z}$ to weights that appear in $\mathscr{F}_0$. If $n > 2 \dim \mathscr{X}_0$, then $H^n_c(\mathscr{X}, -) = 0$ on sheaves. If for all points $\overline{x} \in \mathscr{X}(\mathbb{F})$ in the support of $\mathscr{F}$, the automorphism group schemes $\mathrm{Aut}_{\overline{x}}$ are affine, then the ι-weights of $H^n_c(\mathscr{X}, \mathscr{F})$ are $\le n + w$.*

Organization. In Section 2 we review some preliminaries on derived categories of ℓ-adic sheaves on algebraic stacks over $\mathbb{F}_q$ and ι-mixed complexes, and show that ι-mixedness is stable under the six operations.

In Section 3 we develop the notion of stratifiable complexes in the context of Laszlo and Olsson's ℓ-adic derived categories, and prove its stability under the six operations.

In Section 4 we discuss convergent complexes, and show that they are preserved by $f_!$. In Section 5 we prove the trace formula for stacks. These two theorems are stated and proved in [Behrend 2003] in terms of ordinary cohomology and arithmetic Frobenius, and the proof we give here uses geometric Frobenius.

In Section 6 we discuss convergence of infinite products of formal power series, which will be used in the proof of the meromorphic continuation. In Section 7 we give some examples of zeta functions of stacks, and give the functional equation of the zeta functions and independence of ℓ of Frobenius eigenvalues for proper varieties with quotient singularities in Proposition 7.3.2.

In Section 8 and Section 9, we prove the meromorphic continuation and the weight theorem respectively. Finally in Section 10 we discuss "independence of ℓ" for stacks, and prove Proposition 10.5 that for the quotient stack $[X_0/G_0]$, where X_0 is a proper smooth variety and G_0 is a linear algebraic group acting on X_0, the Frobenius eigenvalues on its cohomology groups are independent of ℓ.

Notation and conventions.

1.5.1. We fix a prime power $q = p^a$ and an algebraic closure $\mathbb{F}$ of the finite field $\mathbb{F}_q$ with q elements. Let F or F_q be the q-geometric Frobenius, namely the q-th root automorphism on $\mathbb{F}$. Let ℓ be a prime number, $\ell \neq p$, and fix an isomorphism of fields $\overline{\mathbb{Q}}_\ell \xrightarrow{\iota} \mathbb{C}$. For simplicity, let $|\alpha|$ denote the complex absolute value $|\iota\alpha|$, for $\alpha \in \overline{\mathbb{Q}}_\ell$.

1.5.2. In this paper, by an Artin stack (or an algebraic stack) over a base scheme S, we mean an S-algebraic stack in the sense of M. Artin [Laumon and Moret-Bailly 2000, 4.1] *of finite type*. When we want the more general setting of Artin stacks *locally of finite type*, we will mention that explicitly.

1.5.3. Objects over $\mathbb{F}_q$ will be denoted with an index $_0$. For instance, $\mathscr{X}_0$ will denote an $\mathbb{F}_q$-Artin stack; if $\mathscr{F}_0$ is a lisse-étale sheaf (or more generally, a Weil sheaf 2.4) on $\mathscr{X}_0$, then $\mathscr{F}$ denotes its inverse image on $\mathscr{X} := \mathscr{X}_0 \otimes_{\mathbb{F}_q} \mathbb{F}$.

1.5.4. For a field k, let $\mathrm{Gal}\, k$ denote its absolute Galois group $\mathrm{Gal}(k^{\mathrm{sep}}/k)$. By a variety over k we mean a separated reduced k-scheme of finite type. Let $W(\mathbb{F}_q)$ be the *Weil group* $F_q^{\mathbb{Z}}$ of $\mathbb{F}_q$.

1.5.5. For a profinite group H, by $\overline{\mathbb{Q}}_\ell$-representations of H we always mean finite-dimensional continuous representations (see [Deligne 1980], 1.1.6), and denote by $\mathrm{Rep}_{\overline{\mathbb{Q}}_\ell} H$ the category of such representations.

1.5.6. For a scheme X, we denote by $|X|$ the set of its closed points. For a category $\mathscr{C}$ we write $[\mathscr{C}]$ for the collection of isomorphism classes of objects in $\mathscr{C}$. For example, if $v \geq 1$ is an integer, then $[\mathscr{X}_0(\mathbb{F}_{q^v})]$ denotes the set of isomorphism classes of $\mathbb{F}_{q^v}$-points of the stack $\mathscr{X}_0$. This is a finite set.

For $x \in \mathscr{X}_0(\mathbb{F}_{q^v})$ we will write $k(x)$ for the field $\mathbb{F}_{q^v}$. For an $\mathbb{F}_q$-scheme X_0 (always of finite type) and $x \in |X_0|$, we denote by $k(x)$ the residue field of x. In both cases, let $d(x)$ be the degree of the field extension $[k(x) : \mathbb{F}_q]$, and let $N(x) = q^{d(x)} = \# k(x)$. Also in both cases let $x : \mathrm{Spec}\, \mathbb{F}_{q^v} \to \mathscr{X}_0$ (or X_0) be the natural map ($v = d(x)$), although in the second case the map is defined only up to an automorphism in $\mathrm{Gal}(k(x)/\mathbb{F}_q)$. Given a $K_0 \in D_c(\mathscr{X}_0, \overline{\mathbb{Q}}_\ell)$ (see Section 2), the pullback $x^*K_0 \in D_c(\mathrm{Spec}\, k(x), \overline{\mathbb{Q}}_\ell) = D_c(\mathrm{Rep}_{\overline{\mathbb{Q}}_\ell} \mathrm{Gal}\, k(x))$ gives a complex $K_{\overline{x}}$ of representations of $\mathrm{Gal}\, k(x)$, and we let F_x be the geometric Frobenius generator $F_{q^{d(x)}}$ of this group, called “the local Frobenius”.

1.5.7. Let V be a finite dimensional $\overline{\mathbb{Q}}_\ell$-vector space and φ an endomorphism of V. For a function $f : \overline{\mathbb{Q}}_\ell \to \mathbb{C}$, we denote by $\sum_{V,\varphi} f(\alpha)$ the sum of values of f in α, with α ranging over all the eigenvalues of φ on V with multiplicities. For instance, $\sum_{V,\varphi} \alpha = \mathrm{Tr}(\varphi, V)$.

A 0×0-matrix has trace 0 and determinant 1. For $K \in D_c^b(\overline{\mathbb{Q}}_\ell)$ and an endomorphism φ of K, we define, following Deligne [1977],

$$\operatorname{Tr}(\varphi, K) := \sum_{n \in \mathbb{Z}} (-1)^n \operatorname{Tr}(H^n(\varphi), H^n(K)), \quad \text{and}$$

$$\det(1 - \varphi t, K) := \prod_{n \in \mathbb{Z}} \det(1 - H^n(\varphi)t, H^n(K))^{(-1)^n}.$$

For unbounded complexes K we use similar notations if the series (respectively the infinite product) converges (converges term by term; see Definition 6.2).

1.5.8. For a map $f : X \to Y$ and a sheaf $\mathscr{F}$ on Y, we sometimes write $H^n(X, \mathscr{F})$ for $H^n(X, f^*\mathscr{F})$. We write $H^n(\mathscr{X})$ for $H^n(\mathscr{X}, \overline{\mathbb{Q}}_\ell)$, and $h^n(\mathscr{X}, \mathscr{F})$ for $\dim H^n(\mathscr{X}, \mathscr{F})$, and similarly for $H_c^n(\mathscr{X})$ and $h_c^n(\mathscr{X}, \mathscr{F})$.

1.5.9. For an $\mathbb{F}_q$-algebraic stack $\mathscr{X}_0$ and a Weil complex K_0 on $\mathscr{X}_0$, by $R\Gamma(\mathscr{X}_0, K_0)$ (respectively $R\Gamma_c(\mathscr{X}_0, K_0)$) we mean Ra_*K_0 (respectively $Ra_!K_0$), where the morphism $a : \mathscr{X}_0 \to \operatorname{Spec} \mathbb{F}_q$ is the structural map.

The derived functors Rf_*, $Rf_!$, Lf^* and $Rf^!$ are usually abbreviated as f_*, $f_!$, f^* and $f^!$. But we reserve $\otimes$, $\mathscr{H}om$ and *Hom* for the ordinary sheaf tensor product, sheaf Hom and Hom group, respectively, and use $\otimes^L$, $R\mathscr{H}om$ and *RHom* for their derived functors.

2. Derived category of ℓ-adic sheaves and mixedness

We briefly review the definition in [Laszlo and Olsson 2008a; 2008b] for derived category of ℓ-adic sheaves on stacks. Then we show that ι-mixedness is stable under the six operations. As a consequence of Lafforgue's result from Remark 2.8.1, this is automatic, but we still want to give a much more elementary argument. The proof works for *mixed* complexes as well, see Remark 2.12. One can also generalize the structure theorem of ι-mixed sheaves in [Deligne 1980] to algebraic stacks as in Remark 2.7.1.

2.1. Let Λ be a complete discrete valuation ring with maximal ideal $\mathfrak{m}$ and residual characteristic ℓ. Let $\Lambda_n = \Lambda/\mathfrak{m}^{n+1}$, and let $\Lambda_\bullet$ be the pro-ring $(\Lambda_n)_n$. We take the base scheme S to be a scheme that satisfies the following condition:

(LO): S is noetherian, affine, excellent, finite-dimensional, in which ℓ is invertible, and all S-schemes of finite type have finite ℓ-cohomological dimension.

We denote by $\mathscr{X}, \mathscr{Y}, \ldots$ Artin stacks locally of finite type over S.

Consider the ringed topos $\mathscr{A} = \mathscr{A}(\mathscr{X}) := \operatorname{Mod}(\mathscr{X}^{\mathbb{N}}_{\text{lis-ét}}, \Lambda_\bullet)$ of projective systems $(M_n)_n$ of $\operatorname{Ab}(\mathscr{X}_{\text{lis-ét}})$ such that M_n is a Λ_n-module for each n, and the transition maps are Λ-linear. An object $M \in \mathscr{A}$ is said to be *AR-null*, if there exists an integer $r > 0$ such that for every integer n, the composed map $M_{n+r} \to M_n$ is the zero map. A complex K in $\mathscr{A}$ is called *AR-null*, if all cohomology systems $\mathscr{H}^i(K)$ are AR-null; it is called *almost AR-null*, if for every U in Lis-ét$(\mathscr{X})$ (assumed to be of finite type over S), the restriction of $\mathscr{H}^i(K)$ to Ét(U) is AR-null. Let $\mathscr{D}(\mathscr{A})$ be the ordinary derived category of $\mathscr{A}$. See [Laumon and Moret-Bailly 2000, 18.1.4] for the definition of constructible sheaves on $\mathscr{X}_{\text{lis-ét}}$.

Definition 2.2. *An object $M = (M_n)_n \in \mathscr{A}$ is* adic *if all the M_n's are constructible, and for every n, the natural map*

$$\Lambda_n \otimes_{\Lambda_{n+1}} M_{n+1} \to M_n$$

is an isomorphism. It is called almost adic *if all the M_n's are constructible, and for every object U in* Lis-ét$(\mathscr{X})$, *the restriction $M|_U$ is AR-adic, that is, there exists an adic $N_U \in \mathrm{Mod}(U_{\text{ét}}^{\mathbb{N}}, \Lambda_\bullet)$ and a morphism $N_U \to M|_U$ with AR-null kernel and cokernel.*

A complex K in $\mathscr{A}$ is a λ-complex *if $\mathscr{H}^i(K) \in \mathscr{A}$ are almost adic for all i. Let $\mathscr{D}_c(\mathscr{A})$ be the full triangulated subcategory of $\mathscr{D}(\mathscr{A})$ consisting of λ-complexes, and let $D_c(\mathscr{X}, \Lambda)$ be the quotient of $\mathscr{D}_c(\mathscr{A})$ by the thick full subcategory of those which are almost AR-null. This is called the* derived category of Λ-adic sheaves on $\mathscr{X}$.

Remark 2.2.1. (i) $D_c(\mathscr{X}, \Lambda)$ is a triangulated category with a natural t-structure, and its heart is the quotient of the category of almost adic systems in $\mathscr{A}$ by the thick full subcategory of almost AR-null systems. One can use this t-structure to define the subcategories $D_c^\dagger(\mathscr{X}, \Lambda)$ for $\dagger = \pm, b$.

If $\mathscr{X}/S$ is of finite type (in particular, quasi-compact), it is clear that $K \in \mathscr{D}_{\text{cart}}(\mathscr{A})$ is AR-null if it is almost AR-null. Also if $M \in \mathscr{A}$ is almost adic, the adic system N_U and the map $N_U \to M|_U$ in the definition above are unique up to unique isomorphism, for each U, so by [Laumon and Moret-Bailly 2000, 12.2.1] they give an adic system N of Cartesian sheaves on $\mathscr{X}_{\text{lis-ét}}$, and an AR-isomorphism $N \to M$. This shows that an almost adic system is AR-adic, and it is clear [Illusie 1977, p. 234] that the natural functor

$$\Lambda\text{-Sh}(\mathscr{X}) \to \text{heart } D_c(\mathscr{X}, \Lambda)$$

is an equivalence of categories, where Λ-Sh$(\mathscr{X})$ denotes the category of Λ-adic (in particular, constructible) systems.

(ii) $D_c(\mathscr{X}, \Lambda)$ is different from the ordinary derived category of $\mathrm{Mod}(\mathscr{X}_{\text{lis-ét}}, \Lambda)$ with constructible cohomology; the latter can be denoted by $\mathscr{D}_c(\mathscr{X}, \Lambda)$. Here $\mathrm{Mod}(\mathscr{X}_{\text{lis-ét}}, \Lambda)$ denotes the abelian category of $\Lambda_{\mathscr{X}}$-modules, not adic sheaves Λ-Sh$(\mathscr{X})$.

(iii) When $S = \operatorname{Spec} k$ for k a finite field or an algebraically closed field, and $\mathscr{X} = X$ is a separated S-scheme, [Laszlo and Olsson 2008b, 3.1.6] gives a natural equivalence of triangulated categories between $D_c^b(X, \Lambda)$ and Deligne's definition $\mathscr{D}_c^b(X, \Lambda)$ in [Deligne 1980, 1.1.2].

2.3. Let $\pi : \mathscr{X}^{\mathbb{N}}_{\text{lis-ét}} \to \mathscr{X}_{\text{lis-ét}}$ be the morphism of topoi where π^{-1} takes a sheaf F to the constant projective system $(F)_n$, and π_* takes a projective system to the inverse limit. This morphism induces a morphism of ringed topoi

$$(\pi^*, \pi_*) : (\mathscr{X}^{\mathbb{N}}_{\text{lis-ét}}, \Lambda_\bullet) \to (\mathscr{X}_{\text{lis-ét}}, \Lambda).$$

The functor $R\pi_* : \mathscr{D}_c(\mathscr{A}) \to \mathscr{D}(\mathscr{X}, \Lambda)$ vanishes on almost AR-null objects [Laszlo and Olsson 2008b, 2.2.2], hence factors through $D_c(\mathscr{X}, \Lambda)$. In [ibid., 3.0.8], the normalization functor is defined to be

$$K \mapsto \widehat{K} := L\pi^* R\pi_* K : \ D_c(\mathscr{X}, \Lambda) \to \mathscr{D}(\mathscr{A}).$$

This functor plays an important role in defining the six operations (ibid.). For instance:

• For $F \in D_c^-(\mathscr{X}, \Lambda)$ and $G \in D_c^+(\mathscr{X}, \Lambda)$, $R\mathscr{H}om(F, G)$ is defined to be the image of $R\mathscr{H}om_{\Lambda_\bullet}(\widehat{F}, \widehat{G})$ in $D_c(\mathscr{X}, \Lambda)$.

• For $F, G \in D_c^-(\mathscr{X}, \Lambda)$, the derived tensor product $F \otimes^L G$ is defined to be the image of $\widehat{F} \otimes^L_{\Lambda_\bullet} \widehat{G}$.

• For a morphism $f : \mathscr{X} \to \mathscr{Y}$ and $F \in D_c^+(\mathscr{X}, \Lambda)$, the derived direct image $f_* F$ is defined to be the image of $f_*^{\mathbb{N}} \widehat{F}$.

Let E_λ be a finite extension of $\mathbb{Q}_\ell$ with ring of integers $\mathbb{O}_\lambda$. Following Laszlo and Olsson [2008b] we define $D_c(\mathscr{X}, E_\lambda)$ to be the quotient of $D_c(\mathscr{X}, \mathbb{O}_\lambda)$ by the full subcategory consisting of complexes K such that, for every integer i, there exists an integer $n_i \geq 1$ such that $\mathscr{H}^i(K)$ is annihilated by λ^{n_i}. Then we define

$$D_c(\mathscr{X}, \overline{\mathbb{Q}}_\ell) = 2\text{-colim}_{E_\lambda} D_c(\mathscr{X}, E_\lambda),$$

where E_λ ranges over all finite subextensions of $\overline{\mathbb{Q}}_\ell / \mathbb{Q}_\ell$, and the transition functors are

$$E_{\lambda'} \otimes_{E_\lambda} - : D_c(\mathscr{X}, E_\lambda) \to D_c(\mathscr{X}, E_{\lambda'})$$

for $E_\lambda \subset E_{\lambda'}$.

2.4. From now on in this section, $S = \operatorname{Spec} \mathbb{F}_q$. We recall some notions of weights and mixedness from [Deligne 1980], generalized to $\mathbb{F}_q$-algebraic stacks.

2.4.1. Frobenius endomorphism. For an $\mathbb{F}_q$-scheme X_0, let $F_{X_0} : X_0 \to X_0$ be the morphism that is identity on the underlying topological space and q-th power on the structure sheaf $\mathbb{O}_{X_0}$; this is an $\mathbb{F}_q$-morphism. Let $F_X : X \to X$ be the induced $\mathbb{F}$-morphism $F_{X_0} \times \mathrm{id}_{\mathbb{F}}$ on $X = X_0 \otimes \mathbb{F}$.

By functoriality of the maps F_{X_0}, we can extend it to stacks. For an $\mathbb{F}_q$-algebraic stack $\mathscr{X}_0$, define $F_{\mathscr{X}_0} : \mathscr{X}_0 \to \mathscr{X}_0$ to be such that for every $\mathbb{F}_q$-scheme X_0, the map

$$F_{\mathscr{X}_0}(X_0) : \mathscr{X}_0(X_0) \to \mathscr{X}_0(X_0)$$

sends x to $x \circ F_{X_0}$. We also define $F_{\mathscr{X}} : \mathscr{X} \to \mathscr{X}$ to be $F_{\mathscr{X}_0} \times \mathrm{id}_{\mathbb{F}}$. This morphism is a universal homeomorphism, hence $F_{\mathscr{X}}^*$ and $F_{\mathscr{X}*}$ are quasi-inverse to each other, and $F_{\mathscr{X}}^* \simeq F_{\mathscr{X}}^!$, $F_{\mathscr{X}*} \simeq F_{\mathscr{X}!}$.

2.4.2. Weil complexes. A *Weil complex* K_0 *on* $\mathscr{X}_0$ is a pair (K, φ), where K is in $D_c(\mathscr{X}, \overline{\mathbb{Q}}_\ell)$ and $\varphi : F_{\mathscr{X}}^* K \to K$ is an isomorphism. A morphism of Weil complexes on $\mathscr{X}_0$ is a morphism of complexes on $\mathscr{X}$ commuting with φ. We also call K_0 a *Weil sheaf* if K is a sheaf. Let $W(\mathscr{X}_0, \overline{\mathbb{Q}}_\ell)$ be the category of Weil complexes on $\mathscr{X}_0$; it is a triangulated category with the standard t-structure, and its core is the category of Weil sheaves. There is a natural fully faithful triangulated functor

$$D_c(\mathscr{X}_0, \overline{\mathbb{Q}}_\ell) \to W(\mathscr{X}_0, \overline{\mathbb{Q}}_\ell).$$

The usual six operations are well-defined on Weil complexes.

• Verdier duality. The Weil complex structure on $D_{\mathscr{X}} K$ is given by the inverse of the isomorphism

$$D_{\mathscr{X}} K \xrightarrow{D\varphi} D_{\mathscr{X}} F_{\mathscr{X}}^* K \xrightarrow{\sim} F_{\mathscr{X}}^* D_{\mathscr{X}} K.$$

• Tensor product. Let K_0 and L_0 be two Weil complexes such that $K \otimes^L L$ (which is $K \otimes L$ since they are of $\overline{\mathbb{Q}}_\ell$-coefficients) is constructible. This is the case when they are both bounded above. The Weil complex structure on $K \otimes L$ is given by

$$F_{\mathscr{X}}^*(K \otimes L) \xrightarrow{\sim} F_{\mathscr{X}}^* K \otimes F_{\mathscr{X}}^* L \xrightarrow{\varphi_K \otimes \varphi_L} K \otimes L.$$

• Pullback. This is clear:

$$F_{\mathscr{X}}^* f^* K \xrightarrow{\sim} f^* F_{\mathscr{Y}}^* K \xrightarrow{f^*\varphi} f^* K.$$

Here $f : \mathscr{X}_0 \to \mathscr{Y}_0$ is an $\mathbb{F}_q$-morphism and (K, φ) is a Weil complex on $\mathscr{Y}_0$.

• Pushforward. Let $f : \mathscr{X}_0 \to \mathscr{Y}_0$ and $K_0 \in W^+(\mathscr{X}_0, \overline{\mathbb{Q}}_\ell)$. The Weil complex structure on $f_* K$ is given by

$$F_{\mathscr{Y}}^* f_* K \longrightarrow f_* F_{\mathscr{X}}^* K \xrightarrow{f^*\varphi} f_* K,$$

where the first arrow is an isomorphism, because it is adjoint to

$$f_* K \to F_{\mathscr{Y}*} f_* F_{\mathscr{X}}^* K \simeq f_* F_{\mathscr{X}*} F_{\mathscr{X}}^* K$$

obtained by applying f_* to the adjunction morphism $K \to F_{\mathscr{X}*} F_{\mathscr{X}}^* K$, which is an isomorphism.

• The remaining cases $f^!$, $f_!$ and $R\mathcal{H}om$ follow from the previous cases.

In this article, when discussing stacks over $\mathbb{F}_q$, by a "sheaf" or "complex of sheaves," we usually mean a "Weil sheaf" or "Weil complex," whereas a "lisse-étale sheaf or complex" will be an ordinary constructible $\overline{\mathbb{Q}}_\ell$-sheaf or complex on the lisse-étale site of $\mathscr{X}_0$.

For $x \in \mathscr{X}_0(\mathbb{F}_{q^v})$, it is a fixed point of $F^v_{\mathscr{X}}$, hence there is a "local Frobenius automorphism" $F_x : K_{\overline{x}} \to K_{\overline{x}}$ for every Weil complex K_0, defined to be

$$K_{\overline{x}} \simeq K_{F_{\mathscr{X}}(\overline{x})} = (F^*_{\mathscr{X}} K)_{\overline{x}} \xrightarrow{\varphi} K_{\overline{x}}.$$

2.4.3. ι-Weights and ι-mixedness. Recall that ι is a fixed isomorphism $\overline{\mathbb{Q}}_\ell \to \mathbb{C}$. For $\alpha \in \overline{\mathbb{Q}}^*_\ell$, let $w_q(\alpha) := 2\log_q |\iota\alpha|$, called the ι-weight of α relative to q. For a real number β, a sheaf $\mathscr{F}_0$ on $\mathscr{X}_0$ is said to be *punctually ι-pure of weight β*, if for every integer $v \geq 1$ and every $x \in \mathscr{X}_0(\mathbb{F}_{q^v})$, and every eigenvalue α of F_x acting on $\mathscr{F}_{\overline{x}}$, we have $w_{N(x)}(\alpha) = \beta$. We say $\mathscr{F}_0$ is *ι-mixed* if it has a finite filtration with successive quotients punctually ι-pure, and the weights of these quotients are called the punctual ι-weights of $\mathscr{F}_0$. A complex $K_0 \in W(\mathscr{X}_0, \overline{\mathbb{Q}}_\ell)$ is said to be ι-mixed if all the cohomology sheaves $\mathscr{H}^i K_0$ are ι-mixed. Let $W_m(\mathscr{X}_0, \overline{\mathbb{Q}}_\ell)$ (respectively $D_m(\mathscr{X}_0, \overline{\mathbb{Q}}_\ell)$) be the full subcategory of ι-mixed complexes in $W(\mathscr{X}_0, \overline{\mathbb{Q}}_\ell)$ (respectively $D_c(\mathscr{X}_0, \overline{\mathbb{Q}}_\ell)$).

One can also define *punctually pure sheaves, mixed sheaves* and *mixed complexes* for algebraic stacks.

2.4.4. Twists. For $b \in \overline{\mathbb{Q}}^*_\ell$, let $\overline{\mathbb{Q}}^{(b)}_\ell$ be the Weil sheaf on Spec $\mathbb{F}_q$ of rank one, where F acts by multiplication by b. This is an étale sheaf if and only if b is an ℓ-adic unit [Deligne 1980, 1.2.7]. For an algebraic stack $\mathscr{X}_0/\mathbb{F}_q$, we also denote by $\overline{\mathbb{Q}}^{(b)}_\ell$ the inverse image on $\mathscr{X}_0$ of the above Weil sheaf via the structural map. If $\mathscr{F}_0$ is a sheaf on $\mathscr{X}_0$, we denote by $\mathscr{F}^{(b)}_0$ the tensor product $\mathscr{F}_0 \otimes \overline{\mathbb{Q}}^{(b)}_\ell$, and say that $\mathscr{F}^{(b)}_0$ is deduced from $\mathscr{F}_0$ by a generalized Tate twist. Note that the operation $\mathscr{F}_0 \mapsto \mathscr{F}^{(b)}_0$ adds the weights by $w_q(b)$. For an integer d, the usual Tate twist $\overline{\mathbb{Q}}_\ell(d)$ is $\overline{\mathbb{Q}}^{(q^{-d})}_\ell$. We denote by $\langle d \rangle$ the operation $(d)[2d]$ on complexes of sheaves, where $[2d]$ means shifting $2d$ to the left. Note that ι-mixedness is stable under the operation $\langle d \rangle$.

Lemma 2.5. *Let $\mathscr{X}_0$ be an $\mathbb{F}_q$-algebraic stack.*

(i) *If $\mathscr{F}_0$ is an ι-mixed sheaf on $\mathscr{X}_0$, then so is every subquotient of $\mathscr{F}_0$.*

(ii) *If $0 \to \mathscr{F}'_0 \to \mathscr{F}_0 \to \mathscr{F}''_0 \to 0$ is an exact sequence of sheaves on $\mathscr{X}_0$, and $\mathscr{F}'_0$ and $\mathscr{F}''_0$ are ι-mixed, then so is $\mathscr{F}_0$.*

(iii) *The full subcategory $W_m(\mathscr{X}_0, \overline{\mathbb{Q}}_\ell)$ (respectively $D_m(\mathscr{X}_0, \overline{\mathbb{Q}}_\ell)$) of $W(\mathscr{X}_0, \overline{\mathbb{Q}}_\ell)$ (respectively $D_c(\mathscr{X}_0, \overline{\mathbb{Q}}_\ell)$) is a triangulated subcategory with induced standard t-structure.*

(iv) *If f is a morphism of $\mathbb{F}_q$-algebraic stacks, then f^* on complexes of sheaves preserves ι-mixedness.*

(v) *If $j : \mathscr{U}_0 \hookrightarrow \mathscr{X}_0$ is an open immersion and $i : \mathscr{Z}_0 \hookrightarrow \mathscr{X}_0$ is its complement, then $K_0 \in W(\mathscr{X}_0, \overline{\mathbb{Q}}_\ell)$ is ι-mixed if and only if j^*K_0 and i^*K_0 are ι-mixed.*

Proof. (i) If $\mathscr{F}_0$ is punctually ι-pure of weight β, then so is every subquotient of it. Now suppose $\mathscr{F}_0$ is ι-mixed and $\mathscr{F}'_0$ is a subsheaf of $\mathscr{F}_0$. Let W be a finite filtration

$$0 \subset \cdots \subset \mathscr{F}_0^{i-1} \subset \mathscr{F}_0^i \subset \cdots \subset \mathscr{F}_0$$

of $\mathscr{F}_0$ such that $\mathrm{Gr}_i^W \mathscr{F}_0 := \mathscr{F}_0^i/\mathscr{F}_0^{i-1}$ is punctually ι-pure for every i. Let W' be the induced filtration $W \cap \mathscr{F}'_0$ of $\mathscr{F}'_0$. Then $\mathrm{Gr}_i^{W'} \mathscr{F}'_0$ is the image of

$$\mathscr{F}_0^i \cap \mathscr{F}'_0 \subset \mathscr{F}_0^i \twoheadrightarrow \mathrm{Gr}_i^W \mathscr{F}_0,$$

so it is punctually ι-pure. Let $\mathscr{F}''_0 = \mathscr{F}_0/\mathscr{F}'_0$ be a quotient of $\mathscr{F}_0$, and let W'' be the induced filtration of $\mathscr{F}''_0$, namely $(\mathscr{F}''_0)^i := \mathscr{F}_0^i/(\mathscr{F}_0^i \cap \mathscr{F}'_0)$. Then

$$\mathrm{Gr}_i^{W''} \mathscr{F}''_0 = \mathscr{F}_0^i/(\mathscr{F}_0^{i-1} + \mathscr{F}_0^i \cap \mathscr{F}'_0),$$

which is a quotient of $\mathscr{F}_0^i/\mathscr{F}_0^{i-1} = \mathrm{Gr}_i^W \mathscr{F}_0$, so it is punctually ι-pure. Hence every subquotient of $\mathscr{F}_0$ is ι-mixed.

(ii) Let W' and W'' be finite filtrations of $\mathscr{F}'_0$ and $\mathscr{F}''_0$ respectively, such that $\mathrm{Gr}_i^{W'} \mathscr{F}'_0$ and $\mathrm{Gr}_i^{W''} \mathscr{F}''_0$ are punctually ι-pure for every i. Then W' can be regarded as a finite filtration of $\mathscr{F}_0$ such that every member of the filtration is contained in $\mathscr{F}'_0$, and W'' can be regarded as a finite filtration of $\mathscr{F}_0$ such that every member contains $\mathscr{F}'_0$. Putting these two filtrations together, we get the desired filtration for $\mathscr{F}_0$.

(iii) Being a triangulated subcategory means [Deligne 1977, p. 271] that if the sequence $K'_0 \to K_0 \to K''_0 \to K'_0[1]$ is an exact triangle in $W(\mathscr{X}_0, \overline{\mathbb{Q}}_\ell)$, and two of the three complexes are ι-mixed, then so is the third. By the rotation axiom of a triangulated category, we can assume K'_0 and K''_0 are ι-mixed. We have the exact sequence

$$\cdots \longrightarrow \mathscr{H}^n K'_0 \longrightarrow \mathscr{H}^n K_0 \longrightarrow \mathscr{H}^n K''_0 \longrightarrow \cdots,$$

and by (i) and (ii) we see that $\mathscr{H}^n K_0$ is ι-mixed.

$W_m(\mathscr{X}_0, \overline{\mathbb{Q}}_\ell)$ has the induced t-structure because if K_0 is ι-mixed, then its truncations $\tau_{\leq n} K_0$ and $\tau_{\geq n} K_0$ are ι-mixed.

(iv) On sheaves, f^* preserves stalks, so it is exact and preserves punctual ι-purity on sheaves. Let $f : \mathscr{X}_0 \to \mathscr{Y}_0$. Given an ι-mixed sheaf $\mathscr{F}_0$ on $\mathscr{Y}_0$, let W be a finite filtration of $\mathscr{F}_0$ such that each $\mathrm{Gr}_i^W \mathscr{F}_0$ is punctually ι-pure. Then f^*W gives a finite filtration of $f^*\mathscr{F}_0$ and each $\mathrm{Gr}_i^{f^*W} f^*\mathscr{F}_0 = f^* \mathrm{Gr}_i^W \mathscr{F}_0$ is punctually ι-pure. So the sheaf $f^*\mathscr{F}_0$ is ι-mixed. For an ι-mixed complex K_0 on $\mathscr{Y}_0$, note that $\mathscr{H}^n(f^*K_0) = f^*\mathscr{H}^n(K_0)$, hence f^*K_0 is ι-mixed.

(v) One direction follows from (iv). For the other direction, note that $j_!$ and i_* are exact and preserve punctual ι-purity on sheaves. If $\mathscr{F}_0$ is an ι-mixed sheaf on $\mathscr{U}_0$, with a finite filtration W such that each $\mathrm{Gr}_i^W \mathscr{F}_0$ is punctually ι-pure, then for the induced filtration $j_!W$ of $j_!\mathscr{F}_0$, we see that $\mathrm{Gr}_i^{j_!W} j_!\mathscr{F}_0 = j_! \mathrm{Gr}_i^W \mathscr{F}_0$ is punctually ι-pure, so $j_!\mathscr{F}_0$ is ι-mixed. For an ι-mixed complex K_0 on $\mathscr{U}_0$, we use that $\mathscr{H}^n(j_!K_0) = j_!\mathscr{H}^n(K_0)$. Similarly i_* also preserves ι-mixedness on complexes. Finally the result follows from (iii) and the exact triangle

$$j_!j^*K_0 \longrightarrow K_0 \longrightarrow i_*i^*K_0 \longrightarrow . \qquad \square$$

To show that ι-mixedness is stable under the six operations, we need to show that ι-mixedness of complexes on stacks can be checked locally on their presentations. To descend a filtration on a presentation to the stack, we generalize the structure theorem of ι-mixed sheaves to algebraic spaces. Recall the following theorem of Deligne [1980, 3.4.1].

Theorem 2.6. *Let $\mathscr{F}_0$ be an ι-mixed sheaf on a scheme X_0 over $\mathbb{F}_q$.*

(i) *$\mathscr{F}_0$ has a unique decomposition $\mathscr{F}_0 = \bigoplus_{b\in\mathbb{R}/\mathbb{Z}} \mathscr{F}_0(b)$, called the* decomposition according to the weights mod $\mathbb{Z}$, *such that the punctual ι-weights of $\mathscr{F}_0(b)$ are all in the coset b. This decomposition, in which almost all the $\mathscr{F}_0(b)$ are zero, is functorial in $\mathscr{F}_0$. Note that each $\mathscr{F}_0(b)$ is deduced by twist from an ι-mixed sheaf with integer punctual weights.*

(ii) *If the punctual weights of $\mathscr{F}_0$ are integers and $\mathscr{F}_0$ is lisse, $\mathscr{F}_0$ has a unique finite increasing filtration W by lisse subsheaves, called the* filtration by punctual weights, *such that $\mathrm{Gr}_i^W \mathscr{F}_0$ is punctually ι-pure of weight i. This filtration is functorial in $\mathscr{F}_0$. More precisely, any morphism between ι-mixed lisse sheaves of integer punctual weights is strictly compatible with their filtrations by punctual weights.*

(iii) *If $\mathscr{F}_0$ is lisse and punctually ι-pure, and X_0 is normal, then the sheaf $\mathscr{F}$ on X is semisimple.*

Remark 2.6.1. (i) If $\mathscr{C}$ is an abelian category and $\mathscr{D}$ is an abelian full subcategory of $\mathscr{C}$, and C is an object in $\mathscr{D}$, then every direct summand of C in $\mathscr{C}$ lies in $\mathscr{D}$ (or isomorphic to some object in $\mathscr{D}$). This is because the kernel of the composition

$$A \oplus B \xrightarrow{pr_A} A \xhookrightarrow{i_A} A \oplus B$$

is B. So direct summands of a lisse sheaf are lisse. If $\mathscr{F}_0$ in Theorem 2.6 (i) is lisse, then each $\mathscr{F}_0(b)$ is lisse.

(ii) If the $\overline{\mathbb{Q}}_\ell$-sheaf $\mathscr{F}_0$ is defined over some finite subextension E_λ of $\overline{\mathbb{Q}}_\ell/\mathbb{Q}_\ell$, then its decomposition in Theorem 2.6 (i) and filtration in Theorem 2.6 (ii) are defined over E_λ. This is because the E_λ-action commutes with the Galois action.

(iii) Deligne [1980] made the assumption that all schemes are separated, at least in order to use Nagata compactification to define $f_!$. After the work of Laszlo and Olsson [2008a, 2008b], one can remove this assumption, and many results in [Deligne 1980], for instance this one and (3.3.1), remain valid. For [Deligne 1980, 3.4.1] one can take a cover of a not necessarily separated scheme X_0 by open affines (which are separated), and use the functoriality to glue the decomposition or filtration on intersections.

Lemma 2.7. *Let X_0 be an $\mathbb{F}_q$-algebraic space, and $\mathscr{F}_0$ an ι-mixed sheaf on X_0.*

(i) *$\mathscr{F}_0$ has a unique decomposition $\mathscr{F}_0 = \bigoplus_{b\in\mathbb{R}/\mathbb{Z}} \mathscr{F}_0(b)$, the* decomposition according to the weights mod $\mathbb{Z}$, *with the same property as in* Theorem 2.6 (i). *This decomposition is functorial in $\mathscr{F}_0$.*

(ii) *If the punctual ι-weights of $\mathscr{F}_0$ are integers and $\mathscr{F}_0$ is lisse, $\mathscr{F}_0$ has a unique finite increasing filtration W by lisse subsheaves, called the* filtration by punctual weights, *with the same property as in* Theorem 2.6 (ii). *This filtration is functorial in $\mathscr{F}_0$.*

Proof. Let $P : X_0' \to X_0$ be an étale presentation, and let $\mathscr{F}_0' = P^*\mathscr{F}_0$, which is also ι-mixed by Lemma 2.5 (iv). Let X_0'' be the fiber product

$$\begin{tikzcd} X_0'' = X_0' \times_{X_0} X_0' \arrow[r, "p_1"] \arrow[d, "p_2"'] & X_0' \arrow[d, "P"] \\ X_0' \arrow[r, "P"'] & X_0. \end{tikzcd}$$

Then X_0'' is an $\mathbb{F}_q$-scheme of finite type.

(i) Applying Theorem 2.6 (i) to $\mathscr{F}_0'$ we get a decomposition $\mathscr{F}_0' = \bigoplus_{b\in\mathbb{R}/\mathbb{Z}} \mathscr{F}_0'(b)$. For $j = 1, 2$, applying p_j^* we get a decomposition

$$p_j^*\mathscr{F}_0' = \bigoplus_{b\in\mathbb{R}/\mathbb{Z}} p_j^*\mathscr{F}_0'(b).$$

Since p_j^* preserves weights, by the uniqueness in Theorem 2.6 (i), this decomposition is the decomposition of $p_j^*\mathscr{F}_0'$ according to the weights mod $\mathbb{Z}$. By the functoriality in Theorem 2.6 (i), the canonical isomorphism $\mu : p_1^*\mathscr{F}_0' \to p_2^*\mathscr{F}_0'$ takes the form $\bigoplus_{b\in\mathbb{R}/\mathbb{Z}} \mu_b$, where $\mu_b : p_1^*\mathscr{F}_0'(b) \to p_2^*\mathscr{F}_0'(b)$ is an isomorphism satisfying cocycle condition as μ does. Therefore the decomposition $\mathscr{F}_0' = \bigoplus_{b\in\mathbb{R}/\mathbb{Z}} \mathscr{F}_0'(b)$ descends to a decomposition $\mathscr{F}_0 = \bigoplus_{b\in\mathbb{R}/\mathbb{Z}} \mathscr{F}_0(b)$. We still need to show each direct summand $\mathscr{F}_0(b)$ is ι-mixed.

Fix a coset b and consider the summand $\mathscr{F}_0(b)$. Twisting it appropriately, we can assume that its inverse image $\mathscr{F}_0'(b)$ is ι-mixed with integer punctual ι-weights. By Lemma 2.5 (v) and noetherian induction, we can shrink X_0 to a nonempty open subspace and assume $\mathscr{F}_0(b)$ is lisse. Then $\mathscr{F}'_0(b)$ is also lisse, and applying Theorem 2.6 (ii) we get a finite increasing filtration W' of $\mathscr{F}'_0(b)$ by lisse subsheaves $\mathscr{F}'_0(b)^i$, such that each $\mathrm{Gr}_i^{W'}\mathscr{F}'_0(b)$ is punctually ι-pure of weight i. Pulling back this filtration via p_j, we get a finite increasing filtration p_j^*W' of $p_j^*\mathscr{F}'_0(b)$, and since

$$\mathrm{Gr}_i^{p_j^*W'}p_j^*\mathscr{F}'_0(b)=p_j^*\,\mathrm{Gr}_i^{W'}\mathscr{F}'_0(b)$$

is punctually ι-pure of weight i, it is the filtration by punctual weights given by Theorem 2.6 (ii), hence it is functorial. It follows that the canonical isomorphism $\mu_b : p_1^*\mathscr{F}'_0(b)\to p_2^*\mathscr{F}'_0(b)$ maps $p_1^*\mathscr{F}'_0(b)^i$ isomorphically onto $p_2^*\mathscr{F}'_0(b)^i$, satisfying cocycle condition. Therefore the filtration W' of $\mathscr{F}'_0(b)$ descends to a filtration W of $\mathscr{F}_0(b)$, and $P^*\,\mathrm{Gr}_i^W\mathscr{F}_0(b)=\mathrm{Gr}_i^{W'}\mathscr{F}'_0(b)$ is punctually ι-pure of weight i. Note that P is surjective, so every point $x\in X_0(\mathbb{F}_{q^v})$ can be lifted to a point $x'\in X'_0(\mathbb{F}_{q^{nv}})$ after some base extension $\mathbb{F}_{q^{nv}}$ of $\mathbb{F}_{q^v}$. This shows $\mathrm{Gr}_i^W\mathscr{F}_0(b)$ is punctually ι-pure of weight i, therefore $\mathscr{F}_0(b)$ is ι-mixed. This proves the existence of the decomposition in (i).

For uniqueness, let $\mathscr{F}_0=\bigoplus\widetilde{\mathscr{F}}_0(b)$ be another decomposition with the desired property. Then their restrictions to X'_0 are both equal to the decomposition of $\mathscr{F}'_0$, which is unique Theorem 2.6 (i), so they are both obtained by descending this decomposition, and so they are isomorphic, that is, for every coset b there exists an isomorphism making the diagram commute:

$$\begin{tikzcd} F_0(b) \arrow[rr, "\sim"] \arrow[dr, hook] & & \widetilde{\mathscr{F}}_0(b) \arrow[dl, hook'] \\ & \mathscr{F}_0. & \end{tikzcd}$$

For functoriality, let $\mathscr{G}_0=\bigoplus\mathscr{G}_0(b)$ be another ι-mixed sheaf with decomposition on X_0, and let $\varphi:\mathscr{F}_0\to\mathscr{G}_0$ be a morphism of sheaves. Pulling φ back via P we get a morphism $\varphi':\mathscr{F}'_0\to\mathscr{G}'_0$ on X'_0, and the diagram

$$\begin{tikzcd} p_1^*\mathscr{F}'_0 \arrow[r, "\mu_{\mathscr{F}_0}"] \arrow[d, "p_1^*\varphi'"'] & p_2^*\mathscr{F}'_0 \arrow[d, "p_2^*\varphi'"] \\ p_1^*\mathscr{G}'_0 \arrow[r, "\mu_{\mathscr{G}_0}"'] & p_2^*\mathscr{G}'_0 \end{tikzcd}$$

commutes. By Theorem 2.6 (i) $\varphi'=\bigoplus\varphi'(b)$ for morphisms $\varphi'(b):\mathscr{F}'_0(b)\to\mathscr{G}'_0(b)$, and the diagram

$$\begin{tikzcd} p_1^*\mathscr{F}_0'(b) \arrow[r, "\mathrm{can}"] \arrow[d, "p_1^*\varphi'"'] & p_2^*\mathscr{F}_0'(b) \arrow[d, "p_2^*\varphi'"] \\ p_1^*\mathscr{G}_0'(b) \arrow[r, "\mathrm{can}"'] & p_2^*\mathscr{G}_0'(b) \end{tikzcd}$$

commutes for each b. It follows that the morphisms $\varphi'(b)$ descend to morphisms $\varphi(b) : \mathscr{F}_0(b) \to \mathscr{G}_0(b)$ such that $\varphi = \bigoplus \varphi(b)$.

(ii) The proof is similar to part (i). Applying Theorem 2.6 (ii) to $\mathscr{F}_0'$ on X_0' we get a finite increasing filtration W' of $\mathscr{F}_0'$ by lisse subsheaves $\mathscr{F}_0'^i$ with desired property. Pulling back this filtration via $p_j : X_0'' \to X_0'$ we get the filtration by punctual weights of $p_j^*\mathscr{F}_0'$. By functoriality in Theorem 2.6 (ii), the canonical isomorphism $\mu : p_1^*\mathscr{F}_0' \to p_2^*\mathscr{F}_0'$ maps $p_1^*\mathscr{F}_0'^i$ isomorphically onto $p_2^*\mathscr{F}_0'^i$ satisfying cocycle condition, therefore the filtration W' descends to a finite increasing filtration W of $\mathscr{F}_0$ by certain subsheaves $\mathscr{F}_0^i$. By Olsson [2007, 9.1] they are lisse subsheaves.

For uniqueness, if $\widetilde{W}$ is another filtration of $\mathscr{F}_0$ by certain subsheaves $\widetilde{\mathscr{F}}_0^i$ with desired property, then their restrictions to X_0' are both equal to the filtration W' by punctual weights, which is unique Theorem 2.6 (ii), so they are both obtained by descending this filtration W', and therefore they are isomorphic.

For functoriality, let $\mathscr{G}_0$ be another lisse ι-mixed sheaf with integer punctual ι-weights, and let V be its filtration by punctual weights, and let $\varphi : \mathscr{F}_0 \to \mathscr{G}_0$ be a morphism. Pulling φ back via P we get a morphism $\varphi' : \mathscr{F}_0' \to \mathscr{G}_0'$ on X_0', and the diagram

$$\begin{tikzcd} p_1^*\mathscr{F}_0' \arrow[r, "\mu_{\mathscr{F}_0}"] \arrow[d, "p_1^*\varphi'"'] & p_2^*\mathscr{F}_0' \arrow[d, "p_2^*\varphi'"] \\ p_1^*\mathscr{G}_0' \arrow[r, "\mu_{\mathscr{G}_0}"'] & p_2^*\mathscr{G}_0' \end{tikzcd}$$

commutes. By Theorem 2.6 (ii) we have $\varphi'(\mathscr{F}_0'^i) \subset \mathscr{G}_0'^i$, and the diagram

$$\begin{tikzcd} p_1^*\mathscr{F}_0'^i \arrow[r, "\mu_{\mathscr{F}_0}"] \arrow[d, "p_1^*\varphi'"'] & p_2^*\mathscr{F}_0'^i \arrow[d, "p_2^*\varphi'"] \\ p_1^*\mathscr{G}_0'^i \arrow[r, "\mu_{\mathscr{G}_0}"'] & p_2^*\mathscr{G}_0'^i \end{tikzcd}$$

commutes for each i. Let $\varphi'^i : \mathscr{F}_0'^i \to \mathscr{G}_0'^i$ be the restriction of φ'. Then they descend to morphisms $\varphi^i : \mathscr{F}_0^i \to \mathscr{G}_0^i$, which are restrictions of φ. □

Remark 2.7.1. One can prove a similar structure theorem of ι-mixed sheaves on algebraic stacks over $\mathbb{F}_q$: the proof of Lemma 2.7 carries over verbatim to the case of algebraic stacks, except that for a presentation $X_0' \to \mathscr{X}_0$, the fiber product $X_0'' = X_0' \times_{\mathscr{X}_0} X_0'$ may not be a scheme, so we use the case for algebraic spaces and replace every "2.6" in the proof by "2.7". It turns out that Theorem 2.6 (iii) also holds for algebraic stacks, as a consequence of the proof of Theorem 1.4. As we will not use these results in this paper, we do not give the proof in detail here, but refer to [Sun 2012, 2.1].

Proposition 2.8. *Let $\mathscr{X}_0$ be an $\mathbb{F}_q$-algebraic stack, and let $P : X_0 \to \mathscr{X}_0$ be a presentation (that is, a smooth surjection with X_0 a scheme). Then a complex $K_0 \in W(\mathscr{X}_0, \overline{\mathbb{Q}}_\ell)$ is ι-mixed if and only if P^*K_0 (respectively $P^!K_0$) is ι-mixed.*

Proof. We consider P^*K_0 first. The "only if" part follows from Lemma 2.5 (iv). For the "if" part, since P^* is exact on sheaves and so $\mathscr{H}^i(P^*K_0) = P^*\mathscr{H}^i(K_0)$, we reduce to the case when $K_0 = \mathscr{F}_0$ is a sheaf. So we assume the sheaf $\mathscr{F}_0' := P^*\mathscr{F}_0$ on X_0 is ι-mixed, and want to show $\mathscr{F}_0$ is also ι-mixed. The proof is similar to the argument in Lemma 2.7.

Let X_0'' be the fiber product

$$\begin{tikzcd} X_0'' = X_0 \times_{\mathscr{X}_0} X_0 \arrow[r, "p_1"] \arrow[d, "p_2"'] & X_0 \arrow[d, "P"] \\ X_0 \arrow[r, "P"'] & \mathscr{X}_0. \end{tikzcd}$$

Then X_0'' is an algebraic space of finite type. Applying Theorem 2.6 (i) to $\mathscr{F}_0'$ we get a decomposition $\mathscr{F}_0' = \bigoplus_{b \in \mathbb{R}/\mathbb{Z}} \mathscr{F}_0'(b)$. For $j = 1, 2$, applying p_j^* we get a decomposition

$$p_j^*\mathscr{F}_0' = \bigoplus_{b \in \mathbb{R}/\mathbb{Z}} p_j^*\mathscr{F}_0'(b),$$

which is the decomposition of $p_j^*\mathscr{F}_0'$ according to the weights mod $\mathbb{Z}$. By the functoriality in Lemma 2.7 (i), the canonical isomorphism $\mu : p_1^*\mathscr{F}_0' \to p_2^*\mathscr{F}_0'$ takes the form $\bigoplus_{b \in \mathbb{R}/\mathbb{Z}} \mu_b$, where $\mu_b : p_1^*\mathscr{F}_0'(b) \to p_2^*\mathscr{F}_0'(b)$ is an isomorphism satisfying cocycle condition as μ does. Therefore the decomposition of $\mathscr{F}_0'$ descends to a decomposition $\mathscr{F}_0 = \bigoplus_{b \in \mathbb{R}/\mathbb{Z}} \mathscr{F}_0(b)$. The ι-weights of the local Frobenius eigenvalues of $\mathscr{F}_0(b)$ at each point of $\mathscr{X}_0$ are in the coset b. Next we show that $\mathscr{F}_0(b)$'s are ι-mixed.

Replacing $\mathscr{F}_0$ by a direct summand $\mathscr{F}_0(b)$ and then twisting it appropriately, we may assume its inverse image $\mathscr{F}_0'$ is ι-mixed with integer punctual ι-weights. By Lemma 2.5 (v) we can shrink $\mathscr{X}_0$ to a nonempty open substack and assume $\mathscr{F}_0$ is lisse. Then $\mathscr{F}_0'$ is also lisse, and applying Theorem 2.6 (ii) we get a finite increasing filtration W' of $\mathscr{F}_0'$ by lisse subsheaves $\mathscr{F}_0'^{i}$, such that each $\mathrm{Gr}_i^{W'}\mathscr{F}_0'$ is punctually

ι-pure of weight i. Pulling back this filtration via p_j, we get a finite increasing filtration p_j^*W' of $p_j^*\mathscr{F}'_0$, and since

$$\mathrm{Gr}_i^{p_j^*W'} p_j^*\mathscr{F}'_0 = p_j^* \mathrm{Gr}_i^{W'} \mathscr{F}'_0$$

is punctually ι-pure of weight i, it is the filtration by punctual weights given by Lemma 2.7 (ii). By functoriality, the canonical isomorphism $\mu : p_1^*\mathscr{F}'_0 \to p_2^*\mathscr{F}'_0$ maps $p_1^*\mathscr{F}'^i_0$ isomorphically onto $p_2^*\mathscr{F}'^i_0$, satisfying cocycle condition. Therefore the filtration W' of $\mathscr{F}'_0$ descends to a filtration W of $\mathscr{F}_0$, and $P^* \mathrm{Gr}_i^W \mathscr{F}_0 = \mathrm{Gr}_i^{W'} \mathscr{F}'_0$ is punctually ι-pure of weight i. Since π is surjective, $\mathrm{Gr}_i^W \mathscr{F}_0$ is also punctually ι-pure of weight i, therefore $\mathscr{F}_0$ is ι-mixed.

Next we consider $P^!K_0$. We know that P is smooth of relative dimension d, for some function $d : \pi_0(X_0) \to \mathbb{N}$. Let X_0^0 be a connected component of X_0. Since $\pi_0(X_0)$ is finite, X_0^0 is both open and closed in X_0, so

$$f : X_0^0 \xrightarrow{j} X_0 \xrightarrow{P} \mathscr{X}_0$$

is smooth of relative dimension $d(X_0^0)$. Then P^*K_0 is ι-mixed if and only if $f^*K_0 = j^*P^*K_0$ is ι-mixed for the inclusion j of every connected component, if and only if $f^!K_0 = f^*K_0\langle d(X_0^0)\rangle$ is ι-mixed, if and only if $P^!K_0$ is ι-mixed, since $f^! = j^!P^! = j^*P^!$. $\square$

Remark 2.8.1. As a consequence of Lafforgue's theorem on the Langlands correspondence for function fields and a Ramanujan–Petersson type of result, one deduces that all complexes on any $\mathbb{F}_q$-algebraic stack is ι-mixed, for any ι. To see this, by Proposition 2.8 and Lemma 2.5 (ii,v), we reduce to the case of an irreducible lisse sheaf on a smooth (in particular, normal) $\mathbb{F}_q$-scheme. By [Deligne 1980, 1.3.6] we reduce to the case where the determinant of the lisse sheaf has finite order, and Lafforgue's result [Laumon 2002, 1.3] applies. In the following, when we want to emphasize the assumption of ι-mixedness, we will still write $W_m(\mathscr{X}_0, \overline{\mathbb{Q}}_\ell)''$, although it equals the full category $W(\mathscr{X}_0, \overline{\mathbb{Q}}_\ell)$.

Next we show the stability of ι-mixedness, first for a few operations on complexes on algebraic spaces, and then for all the six operations on stacks. Denote by $D_{\mathscr{X}_0}$ or just D the dualizing functor $R\mathscr{H}om(-, K_{\mathscr{X}_0})$, where $K_{\mathscr{X}_0}$ is a dualizing complex on $\mathscr{X}_0$ [Laszlo and Olsson 2008b, §7].

2.9. Recall [Kiehl and Weissauer 2001, II 12.2] that, for $\mathbb{F}_q$-schemes and bounded complexes of sheaves on them, the operations f_*, $f_!$, f^*, $f^!$, D and $-\otimes^L-$ all preserve ι-mixedness. Since we are working with $\overline{\mathbb{Q}}_\ell$-coefficients, $\otimes^L = \otimes$.

Lemma 2.10. *Let $f : X_0 \to Y_0$ be a morphism of $\mathbb{F}_q$-algebraic spaces. Then the operations $-\otimes-$, D_{X_0}, f_* and $f_!$ all preserve ι-mixedness, namely, they induce* *functors*

$$\begin{aligned}
-\otimes- &: W_m^-(X_0,\overline{\mathbb{Q}}_\ell)\times W_m^-(X_0,\overline{\mathbb{Q}}_\ell)\longrightarrow W_m^-(X_0,\overline{\mathbb{Q}}_\ell),\\
D &: W_m(X_0,\overline{\mathbb{Q}}_\ell)\longrightarrow W_m(X_0,\overline{\mathbb{Q}}_\ell)^{\mathrm{op}},\\
f_* &: W_m^+(X_0,\overline{\mathbb{Q}}_\ell)\longrightarrow W_m^+(Y_0,\overline{\mathbb{Q}}_\ell),\\
f_! &: W_m^-(X_0,\overline{\mathbb{Q}}_\ell)\longrightarrow W_m^-(Y_0,\overline{\mathbb{Q}}_\ell).
\end{aligned}$$

Proof. We will reduce to the case of unbounded complexes on schemes, and then prove the scheme case. Let $P: X_0'\to X_0$ be an étale presentation.

Reduction for $\otimes$. For objects K_0 and L_0 in $W_m^-(X_0,\overline{\mathbb{Q}}_\ell)$, we have that

$$P^*(K_0\otimes L_0)=(P^*K_0)\otimes(P^*L_0),$$

and the reduction follows from Proposition 2.8.

Reduction for D. For $K_0\in W_m(X_0,\overline{\mathbb{Q}}_\ell)$, we have $P^*DK_0=DP^!K_0$, so the reduction follows from Proposition 2.8.

Reduction for f_* and $f_!$. By definition [Laszlo and Olsson 2008b, 9.1] we have $f_*=Df_!D$, so it suffices to prove the case for $f_!$. Let $K_0\in W_m^-(X_0,\overline{\mathbb{Q}}_\ell)$, and let $P': Y_0'\to Y_0$ and $X_0'\to X_0\times_{Y_0}Y_0'$ be étale presentations:

$$\begin{tikzcd}
X_0' \arrow[r] \arrow[rr, bend left, "g"] \arrow[dr, "P"'] & (X_0)_{Y_0'} \arrow[r, "f'"'] \arrow[d, "h"] & Y_0' \arrow[d, "P'"] \\
& X_0 \arrow[r, "f"] & Y_0.
\end{tikzcd}$$

By smooth base change [Laszlo and Olsson 2008b, 12.1] we get $P'^*f_!K_0=f'_!h^*K_0$. Replacing f by f' we can assume Y_0 is a scheme. Let $j: U_0\to X_0$ be an open dense subscheme [Knutson 1971, II 6.7], with complement $i: Z_0\to X_0$. Applying $f_!$ to the exact triangle

$$j_!j^*K_0\longrightarrow K_0\longrightarrow i_*i^*K_0\longrightarrow$$

we get

$$(fj)_!j^*K_0\longrightarrow f_!K_0\longrightarrow (fi)_!i^*K_0\longrightarrow$$

By Lemma 2.5 (iii) and noetherian induction, we can replace X_0 by U_0, and reduce to the case where f is a morphism between schemes.

This finishes the reduction to the case of unbounded complexes on schemes, and now we prove this case.

For the Verdier dual D_{X_0}, since the dualizing complex K_{X_0} has finite quasi-injective dimension, for every $K_0\in W_m(X_0,\overline{\mathbb{Q}}_\ell)$ and every integer i, there exist integers a and b such that

$$\mathscr{H}^i(D_{X_0}K_0) \simeq \mathscr{H}^i(D_{X_0}\tau_{[a,b]}K_0),$$

and by 2.9, we see that $D_{X_0}K_0$ is ι-mixed.

Next we prove the case of $\otimes$. For K_0 and $L_0 \in W_m^-(X_0, \overline{\mathbb{Q}}_\ell)$, we have

$$\mathscr{H}^r(K_0 \otimes L_0) = \bigoplus_{i+j=r} \mathscr{H}^i(K_0) \otimes \mathscr{H}^j(L_0).$$

The result follows from 2.9.

Finally we prove the case of f_* and $f_!$. Let $K_0 \in W_m^+(X_0, \overline{\mathbb{Q}}_\ell)$. Then we have the spectral sequence

$$E_2^{ij} = R^i f_*(\mathscr{H}^j K_0) \Longrightarrow R^{i+j} f_* K_0,$$

and the result follows from 2.9 and Lemma 2.5 (i,ii). The case for $f_! = Df_*D$ also follows. □

Finally we prove the main result of this section. This generalizes [Behrend 2003, 6.3.7].

Theorem 2.11. *Let $f : \mathscr{X}_0 \to \mathscr{Y}_0$ be a morphism of $\mathbb{F}_q$-algebraic stacks. Then the operations f_*, $f_!$, f^*, $f^!$, $D_{\mathscr{X}_0}$, $-\otimes-$ and $R\mathscr{H}om(-,-)$ all preserve ι-mixedness, namely, they induce functors*

$$f_* : W_m^+(\mathscr{X}_0, \overline{\mathbb{Q}}_\ell) \longrightarrow W_m^+(\mathscr{Y}_0, \overline{\mathbb{Q}}_\ell), \qquad f_! : W_m^-(\mathscr{X}_0, \overline{\mathbb{Q}}_\ell) \longrightarrow W_m^-(\mathscr{Y}_0, \overline{\mathbb{Q}}_\ell),$$
$$f^* : W_m(\mathscr{Y}_0, \overline{\mathbb{Q}}_\ell) \longrightarrow W_m(\mathscr{X}_0, \overline{\mathbb{Q}}_\ell), \qquad f^! : W_m(\mathscr{Y}_0, \overline{\mathbb{Q}}_\ell) \longrightarrow W_m(\mathscr{X}_0, \overline{\mathbb{Q}}_\ell),$$
$$R\mathscr{H}om(-,-) : W_m^-(\mathscr{X}_0, \overline{\mathbb{Q}}_\ell)^{\mathrm{op}} \times W_m^+(\mathscr{X}_0, \overline{\mathbb{Q}}_\ell) \longrightarrow W_m^+(\mathscr{X}_0, \overline{\mathbb{Q}}_\ell),$$
$$\otimes : W_m^-(\mathscr{X}_0, \overline{\mathbb{Q}}_\ell) \times W_m^-(\mathscr{X}_0, \overline{\mathbb{Q}}_\ell) \longrightarrow W_m^-(\mathscr{X}_0, \overline{\mathbb{Q}}_\ell),$$
$$D : W_m(\mathscr{X}_0, \overline{\mathbb{Q}}_\ell) \longrightarrow W_m(\mathscr{X}_0, \overline{\mathbb{Q}}_\ell)^{\mathrm{op}}.$$

Proof. Recall from [Laszlo and Olsson 2008b, 9.1] that we have $f_! := Df_*D$ and $f^! := Df^*D$. By [ibid., 6.0.12, 7.3.1], for $K_0 \in W^-(\mathscr{X}_0, \overline{\mathbb{Q}}_\ell)$ and $L_0 \in W^+(\mathscr{X}_0, \overline{\mathbb{Q}}_\ell)$, we have

$$\begin{aligned} D(K_0 \otimes DL_0) = R\mathscr{H}om(K_0 \otimes DL_0, K_{\mathscr{X}_0}) &= R\mathscr{H}om(K_0, R\mathscr{H}om(DL_0, K_{\mathscr{X}_0})) \\ &= R\mathscr{H}om(K_0, DDL_0) = R\mathscr{H}om(K_0, L_0). \end{aligned}$$

Therefore it suffices to prove the result for f_*, f^*, D and $-\otimes-$. The case of f^* is proved in Lemma 2.5 (iv).

For D: Let $P : X_0 \to \mathscr{X}_0$ be a presentation. Since $P^*D = DP^!$, the result follows from Proposition 2.8 and Lemma 2.10.

For $\otimes$: Since we have $P^*(K_0 \otimes L_0) = P^*K_0 \otimes P^*L_0$, the result follows from Proposition 2.8 and Lemma 2.10.

For f_* and $f_!$: We will start with $f_!$, in order to use smooth base change to reduce to the case when $\mathscr{Y}_0$ is a scheme, and then turn to f_* in order to use cohomological descent.

Let $K_0 \in W_m^-(\mathscr{X}_0, \overline{\mathbb{Q}}_\ell)$, and let $P : Y_0 \to \mathscr{Y}_0$ be a presentation and the following diagram be 2-Cartesian:

$$\begin{tikzcd} (\mathscr{X}_0)_{Y_0} \arrow[r, "f'"] \arrow[d, "P'"'] & Y_0 \arrow[d, "P"] \\ \mathscr{X}_0 \arrow[r, "f"] & \mathscr{Y}_0 \end{tikzcd}$$

We have [ibid., 12.1] that $P^* f_! K_0 = f'_! P'^* K_0$, so by 2.8 we can assume $\mathscr{Y}_0 = Y_0$ is a scheme.

Now we switch to f_*, where $f : \mathscr{X}_0 \to Y_0$, and $K_0 \in W_m^+(\mathscr{X}_0, \overline{\mathbb{Q}}_\ell)$. Let $X_0 \to \mathscr{X}_0$ be a presentation. Then it gives a strictly simplicial smooth hypercover $X_{0,\bullet}$ of $\mathscr{X}_0$:

$$X_{0,n} := \underbrace{X_0 \times_{\mathscr{X}_0} \cdots \times_{\mathscr{X}_0} X_0}_{n+1 \text{ factors}},$$

where each $X_{0,n}$ is an $\mathbb{F}_q$-algebraic space of finite type. Let $f_n : X_{0,n} \to Y_0$ be the restriction of f to $X_{0,n}$. Then we have the spectral sequence [Laszlo and Olsson 2008b, 10.0.9]

$$E_1^{ij} = R^j f_{i*}(K_0|_{X_{0,i}}) \Longrightarrow R^{i+j} f_* K_0.$$

Since f_i's are morphisms of algebraic spaces, the result follows from Lemma 2.10 and Lemma 2.5 (i, ii). □

Remark 2.12. In fact, we can take the dualizing complex $K_{\mathscr{X}_0}$ to be *mixed*, and results in this section hold (and can be proved verbatim) for *mixed* complexes. In particular, mixedness is preserved by the six operations and the Verdier dualizing functor for stacks (if we take a mixed dualizing complex).

3. Stratifiable complexes

In this section, we use the same notations and hypotheses in 2.1. For the purpose of this article, it suffices to take S to be Spec k for an algebraically closed field k of characteristic not equal to ℓ, but we want to work in the general setting (namely, that of any scheme that satisfies (LO)) for future applications; for instance, when proving the generic base change. Let $\mathscr{X}, \mathscr{Y}, \ldots$ be S-algebraic stacks of finite type. By "sheaves" we mean "lisse-étale sheaves". "Locally constant constructible" is abbreviated as "lcc". A *stratification* $\mathscr{S}$ of an S-algebraic stack $\mathscr{X}$ is a finite set of disjoint locally closed substacks that cover $\mathscr{X}$. If $\mathscr{F}$ is a lcc $(\Lambda_n)_{\mathscr{X}}$-module, a *decomposition series* of $\mathscr{F}$ is a filtration by lcc $\Lambda_{\mathscr{X}}$-subsheaves, such that the successive quotients are simple $\Lambda_{\mathscr{X}}$-modules. Note that the filtration is always finite, and the simple successive quotients, which are $(\Lambda_0)_{\mathscr{X}}$-modules, are independent (up to order) of the decomposition series chosen. They are called the *Jordan–Hölder components of* $\mathscr{F}$.

Definition 3.1. (i) *A complex* $K = (K_n)_n \in \mathscr{D}_c(\mathscr{A})$ *is said to be* stratifiable *if there exists a pair* $(\mathscr{S}, \mathscr{L})$, *where* $\mathscr{S}$ *is a stratification of* $\mathscr{X}$ *and* $\mathscr{L}$ *is a function that assigns to every stratum* $\mathscr{U} \in \mathscr{S}$ *a finite set* $\mathscr{L}(\mathscr{U})$ *of isomorphism classes of simple* (*that is, irreducible*) *lcc* Λ_0*-modules on* $\mathscr{U}_{\text{lis-ét}}$, *such that for each pair* (i, n) *of integers the restriction of the sheaf* $\mathscr{H}^i(K_n) \in \mathrm{Mod}_c(\mathscr{X}_{\text{lis-ét}}, \Lambda_n)$ *to each stratum* $\mathscr{U} \in \mathscr{S}$ *is lcc, with Jordan–Hölder components* (*as a* $\Lambda_{\mathscr{U}}$*-module*) *contained in* $\mathscr{L}(\mathscr{U})$. *We say that the pair* $(\mathscr{S}, \mathscr{L})$ trivializes K (*or* K *is* $(\mathscr{S}, \mathscr{L})$-stratifiable), *and denote the full subcategory of* $(\mathscr{S}, \mathscr{L})$*-stratifiable complexes by* $\mathscr{D}_{\mathscr{S},\mathscr{L}}(\mathscr{A})$. *The full subcategory of stratifiable complexes in* $\mathscr{D}_c(\mathscr{A})$ *is denoted by* $\mathscr{D}_c^{\text{stra}}(\mathscr{A})$.

(ii) *Let* $D_c^{\text{stra}}(\mathscr{X}, \Lambda)$ *be the essential image of* $\mathscr{D}_c^{\text{stra}}(\mathscr{A})$ *in* $D_c(\mathscr{X}, \Lambda)$, *and we call the objects of* $D_c^{\text{stra}}(\mathscr{X}, \Lambda)$ stratifiable complexes of sheaves.

(iii) *Let* E_λ *be a finite extension of* $\mathbb{Q}_\ell$ *with ring of integers* $\mathscr{O}_\lambda$. *Then the definition above applies to* $\Lambda = \mathscr{O}_\lambda$. *Let* $D_c^{\text{stra}}(\mathscr{X}, E_\lambda)$ *be the essential image of* $D_c^{\text{stra}}(\mathscr{X}, \mathscr{O}_\lambda)$ *in* $D_c(\mathscr{X}, E_\lambda)$. *Finally we define*

$$D_c^{\text{stra}}(\mathscr{X}, \overline{\mathbb{Q}}_\ell) = 2\text{-colim}_{E_\lambda} D_c^{\text{stra}}(\mathscr{X}, E_\lambda).$$

Remark 3.1.1. (i) This notion is due to Beilinson, Bernstein and Deligne [1982], and Behrend [2003] used it to define his derived category for stacks. Many results in this section are borrowed from [Behrend 2003], but reformulated and reproved in terms of the derived categories defined in [Laszlo and Olsson 2008b].

(ii) Let $\mathscr{F}$ be a Λ_n-sheaf trivialized by a pair $(\mathscr{S}, \mathscr{L})$, and let $\mathscr{G}$ be a subquotient sheaf of $\mathscr{F}$. Then $\mathscr{G}$ is not necessarily trivialized by $(\mathscr{S}, \mathscr{L})$. But if $\mathscr{G}$ is lcc on each stratum in $\mathscr{S}$, then it is necessarily trivialized by $(\mathscr{S}, \mathscr{L})$.

3.2. We say that the pair $(\mathscr{S}', \mathscr{L}')$ *refines the pair* $(\mathscr{S}, \mathscr{L})$, if $\mathscr{S}'$ refines $\mathscr{S}$, and for every $\mathscr{V} \in \mathscr{S}'$, $\mathscr{U} \in \mathscr{S}$ and $L \in \mathscr{L}(\mathscr{U})$, such that $\mathscr{V} \subset \mathscr{U}$, the restriction $L|_{\mathscr{V}}$ is trivialized by $\mathscr{L}'(\mathscr{V})$. Given a pair $(\mathscr{S}, \mathscr{L})$ and a refined stratification $\mathscr{S}'$ of $\mathscr{S}$, there is a canonical way to define $\mathscr{L}'$ such that $(\mathscr{S}', \mathscr{L}')$ refines $(\mathscr{S}, \mathscr{L})$: for every $\mathscr{V} \in \mathscr{S}'$, we take $\mathscr{L}'(\mathscr{V})$ to be the set of isomorphism classes of Jordan–Hölder components of the lcc sheaves $L|_{\mathscr{V}}$ for $L \in \mathscr{L}(\mathscr{U})$, where $\mathscr{U}$ ranges over all strata in $\mathscr{S}$ that contains $\mathscr{V}$. It is clear that the set of all pairs $(\mathscr{S}, \mathscr{L})$ form a filtered direct system.

A pair $(\mathscr{S}, \mathscr{L})$ is said to be *tensor closed* if for every $\mathscr{U} \in \mathscr{S}$ and $L, M \in \mathscr{L}(\mathscr{U})$, the sheaf tensor product $L \otimes_{\Lambda_0} M$ has Jordan–Hölder components in $\mathscr{L}(\mathscr{U})$.

For a pair $(\mathscr{S}, \mathscr{L})$, a *tensor closed hull* of this pair is a tensor closed refinement.

Lemma 3.3. *Every pair* $(\mathscr{S}, \mathscr{L})$ *can be refined to a tensor closed pair* $(\mathscr{S}', \mathscr{L}')$.

Proof. First we show that for a lcc sheaf of sets $\mathcal{F}$ on $\mathcal{X}_{\text{lis-ét}}$, there exists a finite étale morphism $f : \mathcal{Y} \to \mathcal{X}$ of algebraic S-stacks such that $f^{-1}\mathcal{F}$ is constant. Consider the total space $[\mathcal{F}]$ of the sheaf $\mathcal{F}$. Precisely, this is the category fibered in groupoids over (Aff/S) with the underlying category described as follows. Its objects are triples $(U \in \mathrm{obj}(\mathrm{Aff}/S), u \in \mathrm{obj}\ \mathcal{X}(U), s \in (u^{-1}\mathcal{F})(U))$, and morphisms from (U, u, s) to (V, v, t) are pairs $(f : U \to V, \alpha : vf \Rightarrow u)$ such that t is mapped to s under the identification $\alpha : f^{-1}v^{-1}\mathcal{F} \cong u^{-1}\mathcal{F}$. The map $(U, u, s) \mapsto (U, u)$ gives a map $g : [\mathcal{F}] \to \mathcal{X}$, which is representable finite étale (because it is so locally). The pullback sheaf $g^{-1}\mathcal{F}$ on $[\mathcal{F}]$ has a global section, so the total space breaks up into two parts, one part being mapped isomorphically onto the base $[\mathcal{F}]$. By induction on the degree of g we are done.

Next we show that for a fixed representable finite étale morphism $\mathcal{Y} \to \mathcal{X}$, there are only finitely many isomorphism classes of simple lcc Λ_0-sheaves on $\mathcal{X}$ that become constant when pulled back to $\mathcal{Y}$. We can assume that both $\mathcal{X}$ and $\mathcal{Y}$ are connected. By the following Lemma 3.3.1, we reduce to the case where $\mathcal{Y} \to \mathcal{X}$ is Galois with group G, for some finite group G. Then simple lcc Λ_0-sheaves on $\mathcal{X}$ that become constant on $\mathcal{Y}$ correspond to simple left $\Lambda_0[G]$-modules, which are cyclic and hence isomorphic to $\Lambda_0[G]/I$ for left maximal ideals I of $\Lambda_0[G]$. There are only finitely many such ideals since $\Lambda_0[G]$ is a finite set.

Also note that a lcc subsheaf of a constant constructible sheaf on a connected stack is also constant. Let L be a lcc subsheaf on $\mathcal{X}$ of the constant sheaf associated to a finite set M. Consider their total spaces. We have an inclusion of substacks $i : [L] \hookrightarrow \coprod_{m \in M} \mathcal{X}_m$, where each part $\mathcal{X}_m$ is identified with $\mathcal{X}$. Then $i^{-1}(\mathcal{X}_m) \to \mathcal{X}_m$ is finite étale, and is the inclusion of a substack, hence is either an equivalence or the inclusion of the empty substack, since $\mathcal{X}$ is connected. It is clear that L is also constant, associated to the subset of those $m \in M$ for which $i^{-1}(\mathcal{X}_m) \neq \varnothing$.

Finally we prove the lemma. Refining $\mathcal{S}$ if necessary, we assume all strata are connected stacks. For each stratum $\mathcal{U} \in \mathcal{S}$, let $\mathcal{Y} \to \mathcal{U}$ be a representable finite étale morphism, such that all sheaves in $\mathcal{L}(\mathcal{U})$ become constant on $\mathcal{Y}$. Then define $\mathcal{L}'(\mathcal{U})$ to be the set of isomorphism classes of simple lcc Λ_0-sheaves on $\mathcal{U}_{\text{lis-ét}}$ which become constant on $\mathcal{Y}$. For any L and $M \in \mathcal{L}'(\mathcal{U})$, since all lcc subsheaves of $L \otimes_{\Lambda_0} M$ are constant on $\mathcal{Y}$, we see that $L \otimes_{\Lambda_0} M$ has Jordan–Hölder components in $\mathcal{L}'(\mathcal{U})$ and hence $(\mathcal{S}, \mathcal{L}')$ is a tensor closed refinement of $(\mathcal{S}, \mathcal{L})$. ☐

Lemma 3.3.1. *Let $\mathcal{Y} \to \mathcal{X}$ be a representable finite étale morphism between connected S-algebraic stacks. Then there exists a morphism $\mathcal{Z} \to \mathcal{Y}$, such that $\mathcal{Z}$ is* Galois *over $\mathcal{X}$, that is, it is a G-torsor for some finite group G.*

Proof. Assume $\mathcal{X}$ is nonempty, and take a geometric point $\overline{x} \to \mathcal{X}$. Let $\mathcal{C}$ be the category $\mathrm{FÉt}(\mathcal{X})$ of representable finite étale morphisms to $\mathcal{X}$, and let

$$F : \mathcal{C} \to \mathrm{FSet}$$

be the fiber functor to the category of finite sets, namely $F(\mathscr{Y}) = Hom_{\mathscr{X}}(\overline{x}, \mathscr{Y})$. Note that this Hom, which is *a priori* a category, is a finite set, since $\mathscr{Y} \to \mathscr{X}$ is representable and finite. Then one can verify that $(\mathscr{C}, F : \mathscr{C} \to \mathrm{FSet})$ satisfies the axioms of Galois formalism in [Grothendieck and Raynaud 1971, Exp. V, 4], and use the consequence g) on p. 121 in (loc. cit.) For the reader's convenience, we follow Olsson's suggestion and explain the proof briefly. Basically, we will verify certain axioms of (G1)–(G6), and deduce the conclusion as in (loc. cit.).

First note that $\mathscr{C}$, which is *a priori* a 2-category, is a 1-category. This is because for any 2-commutative diagram

$$\begin{tikzcd} \mathscr{Y} \arrow[rr, "f"] \arrow[dr] & & \mathscr{Z} \arrow[dl] \\ & \mathscr{X} & \end{tikzcd}$$

where $\mathscr{Y}$ and $\mathscr{Z}$ are in $\mathscr{C}$, the morphism f is also representable (and finite étale), so $Hom_{\mathscr{X}}(\mathscr{Y}, \mathscr{Z})$ is discrete. By definition, the functor F preserves fiber-products, and $F(\mathscr{X})$ is a one-point set.

Let $f : \mathscr{Y} \to \mathscr{Z}$ be a morphism in $\mathscr{C}$, then it is finite étale. So if the degree of f is 1, then f is an isomorphism. This implies that the functor F is *conservative*, that is, f is an isomorphism if $F(f)$ is. In particular, f is a monomorphism if and only if $F(f)$ is. This is because f is a monomorphism if and only if $p_1 : \mathscr{Y} \times_{\mathscr{Z}} \mathscr{Y} \to \mathscr{Y}$ is an isomorphism, and F preserves fiber-products.

Since $f : \mathscr{Y} \to \mathscr{Z}$ is finite étale, its image stack $\mathscr{Y}' \subset \mathscr{Z}$ is both open and closed, hence $\mathscr{Y}' \to \mathscr{Z}$ is a monomorphism that is an isomorphism onto a direct summand of $\mathscr{Z}$ (that is, $\mathscr{Z} = \mathscr{Y}' \coprod \mathscr{Y}''$ for some other open and closed substack $\mathscr{Y}'' \subset \mathscr{Z}$). Also, since $\mathscr{Y} \to \mathscr{Y}'$ is epic and finite étale, it is *strictly epic*, that is, for every $\mathscr{Z} \in \mathscr{C}$, the diagram

$$Hom(\mathscr{Y}', \mathscr{Z}) \to Hom(\mathscr{Y}, \mathscr{Z}) \rightrightarrows Hom(\mathscr{Y} \times_{\mathscr{Y}'} \mathscr{Y}, \mathscr{Z})$$

is an equalizer.

Every object $\mathscr{Y}$ in $\mathscr{C}$ is artinian: for a chain of monomorphisms

$$\cdots \to \mathscr{Y}_n \to \cdots \to \mathscr{Y}_2 \to \mathscr{Y}_1 \to \mathscr{Y},$$

we get a chain of injections

$$\cdots \to F(\mathscr{Y}_n) \to \cdots \to F(\mathscr{Y}_1) \to F(\mathscr{Y}),$$

which is stable since $F(\mathscr{Y})$ is a finite set, and so the first chain is also stable since F is conservative.

Since F is left exact and every object in $\mathscr{C}$ is artinian, by [Grothendieck 1960, 3.1] the functor F is *strictly pro-representable*, that is, there exists a projective system $P = \{P_i; i \in I\}$ of objects in $\mathscr{C}$ indexed by a filtered partially ordered set I, with epic transition morphisms $\varphi_{ij} : P_j \to P_i$ $(i \le j)$, such that there is a natural isomorphism of functors

$$F \xrightarrow{\sim} Hom(P, -) := \operatorname{colim}_I Hom(P_i, -).$$

Let $\psi_i : P \to P_i$ be the canonical projection in the category Pro($\mathscr{C}$) of pro-objects of $\mathscr{C}$. We may assume that every epimorphism $P_j \to \mathfrak{X}$ in $\mathscr{C}$ is isomorphic to $P_j \xrightarrow{\varphi_{ij}} P_i$ for some $i \le j$. This is because one can add $P_j \to \mathfrak{X}$ into the projective system P without changing the functor it represents. Also one can show that the P_i's are connected [Grothendieck 1960], and morphisms in $\mathscr{C}$ between connected stacks are strictly epic.

Given $\mathscr{Y} \in \mathscr{C}$, now we show that there exists an object $\mathfrak{X} \to \mathscr{X}$ that is Galois and factors through $\mathscr{Y}$. Since $F(\mathscr{Y})$ is a finite set, there exists an index $j \in I$ such that all maps $P \to \mathscr{Y}$ factors through $P \xrightarrow{\psi_j} P_j$. This means that the canonical map

$$P \to \mathscr{Y}^J := \underbrace{\mathscr{Y} \times_{\mathscr{X}} \cdots \times_{\mathscr{X}} \mathscr{Y}}_{\#J \text{ factors}}, \quad \text{where } J := F(\mathscr{Y}) = Hom_{\mathrm{Pro}(\mathscr{C})}(P, \mathscr{Y})$$

factors as

$$P \xrightarrow{\psi_j} P_j \xrightarrow{A} \mathscr{Y}^J.$$

Let $P_j \to P_i \xrightarrow{B} \mathscr{Y}^J$ be the factorization of A into a composition of an epimorphism and a monomorphism B. We claim that P_i is Galois over $\mathscr{X}$.

Since $F(P_i)$ is a finite set, there exists an index $k \in I$ such that all maps $P \to P_i$ factors through $P \xrightarrow{\psi_k} P_k$. Fix any $v : P_k \to P_i$. To show P_i is Galois, it suffices to show that $\mathrm{Aut}(P_i)$ acts on $F(P_i) = Hom(P_k, P_i)$ transitively, that is, there exists a $\sigma \in \mathrm{Aut}(P_i)$ making the triangle commute:

$$\begin{tikzcd} P_k \arrow[r, "v"] \arrow[dr, "\varphi_{ik}"'] & P_i \arrow[d, "\sigma"] \\ & P_i \end{tikzcd}$$

For every $u \in J = Hom(P_i, \mathscr{Y})$, we have $u \circ v \in Hom(P_k, \mathscr{Y})$, so there exists a $u' \in Hom(P_i, \mathscr{Y})$ making the diagram commute:

$$\begin{tikzcd} P_k \arrow[r, "v"] \arrow[d, "\varphi_{ik}"'] & P_i \arrow[d, "u"] \\ P_i \arrow[r, "u'"'] & \mathscr{Y}. \end{tikzcd}$$

Since v is epic, the function $u \mapsto u' : J \to J$ is injective, hence a bijection. Let $\alpha : \mathscr{Y}^J \to \mathscr{Y}^J$ be the isomorphism induced by the map $u \mapsto u'$. Then the diagram

$$\begin{tikzcd} P_k \arrow[r, "v"] \arrow[dr, "\varphi_{ik}"'] & P_i \arrow[r, "B"] & \mathcal{Y}^J \arrow[d, "\alpha"] \\ & P_i \arrow[r, "B"'] & \mathcal{Y}^J \end{tikzcd}$$

commutes. By the uniqueness of the factorization of the map $P_k \to \mathcal{Y}^J$ into the composition of an epimorphism and a monomorphism, there exists a $\sigma \in \mathrm{Aut}(P_i)$ such that $\sigma \circ v = \varphi_{ik}$. This finishes the proof. $\square$

We give some basic properties of stratifiable complexes.

Lemma 3.4. (i) $\mathcal{D}_c^{\mathrm{stra}}(\mathcal{A})$ *(respectively* $D_c^{\mathrm{stra}}(\mathcal{X}, \Lambda)$*) is a triangulated subcategory of* $\mathcal{D}_c(\mathcal{A})$ *(respectively* $D_c(\mathcal{X}, \Lambda)$*) with the induced standard t-structure.*

(ii) *If $f : \mathcal{X} \to \mathcal{Y}$ is an S-morphism, then* $f^* : \mathcal{D}_c(\mathcal{A}(\mathcal{Y})) \to \mathcal{D}_c(\mathcal{A}(\mathcal{X}))$ *(respectively* $f^* : D_c(\mathcal{Y}, \Lambda) \to D_c(\mathcal{X}, \Lambda)$*) preserves stratifiability.*

(iii) *If $\mathcal{S}$ is a stratification of $\mathcal{X}$, then $K \in \mathcal{D}_c(\mathcal{A}(\mathcal{X}))$ is stratifiable if and only if $K|_V$ is stratifiable for every $V \in \mathcal{S}$.*

(iv) *Let $P : X \to \mathcal{X}$ be a presentation, and let $K = (K_n)_n \in \mathcal{D}_c(\mathcal{A}(\mathcal{X}))$. Then K is stratifiable if and only if P^*K is stratifiable.*

(v) $D_c^{\mathrm{stra}}(\mathcal{X}, \Lambda)$ *contains* $D_c^b(\mathcal{X}, \Lambda)$*, and the heart of* $D_c^{\mathrm{stra}}(\mathcal{X}, \Lambda)$ *is the same as that of* $D_c(\mathcal{X}, \Lambda)$ *in* Remark 2.2.1 (i).

(vi) *Let $K \in \mathcal{D}_c(\mathcal{A})$ be a normalized complex* [Laszlo and Olsson 2008b, 3.0.8]. *Then K is trivialized by a pair $(\mathcal{S}, \mathcal{L})$ if and only if K_0 is trivialized by this pair.*

(vii) *Let* $K \in \mathcal{D}_c^{\mathrm{stra}}(\mathcal{A})$. *Then its Tate twist $K(1)$ is also stratifiable.*

Proof. (i) To show $\mathcal{D}_c^{\mathrm{stra}}(\mathcal{A})$ is a triangulated subcategory, it suffices to show [Deligne 1977, p. 271] that for every exact triangle $K' \to K \to K'' \to K'[1]$ in $\mathcal{D}_c(\mathcal{A})$, if K' and K'' are stratifiable, so also is K.

Using refinement we may assume that K' and K'' are trivialized by the same pair $(\mathcal{S}, \mathcal{L})$. Consider the cohomology sequence of this exact triangle at level n, restricted to a stratum $\mathcal{U} \in \mathcal{S}$. By Olsson [2007, 9.1], to show that a sheaf is lcc on $\mathcal{U}$, one can pass to a presentation U of the stack $\mathcal{U}$. Then by Milne [2008, 20.3] and the five-lemma, we see that the $\mathcal{H}^i(K_n)$'s are lcc on $\mathcal{U}$, with Jordan–Hölder components contained in $\mathcal{L}(\mathcal{U})$. Therefore $\mathcal{D}_c^{\mathrm{stra}}(\mathcal{A})$ (and hence $D_c^{\mathrm{stra}}(\mathcal{X}, \Lambda)$) is a triangulated subcategory.

The t-structure is inherited by $\mathcal{D}_c^{\mathrm{stra}}(\mathcal{A})$ (and hence by $D_c^{\mathrm{stra}}(\mathcal{X}, \Lambda)$) because, if $K \in \mathcal{D}_c(\mathcal{A})$ is stratifiable, so also are its truncations $\tau_{\leq r}K$ and $\tau_{\geq r}K$.

(ii) The functor f^* is exact on the level of sheaves, and takes a lcc sheaf to a lcc sheaf. If $(K_n)_n \in \mathcal{D}_c(\mathcal{A}(\mathcal{Y}))$ is trivialized by $(\mathcal{S}, \mathcal{L})$, then $(f^*K_n)_n$ is trivialized by $(f^*\mathcal{S}, f^*\mathcal{L})$, where $f^*\mathcal{S} = \{f^{-1}(V) | V \in \mathcal{S}\}$ and $(f^*\mathcal{L})(f^{-1}(V))$ is the set of isomorphism classes of Jordan–Hölder components of $f^*L,\ L \in \mathscr{L}(V)$. The case of $D_c(-, \Lambda)$ follows easily.

(iii) The "only if" part follows from (ii). The "if" part is clear: if $(\mathscr{S}_V, \mathscr{L}_V)$ is a pair on V that trivializes $(K_n|_V)_n$, then the pair $(\mathscr{S}_{\mathscr{X}}, \mathscr{L})$ on $\mathscr{X}$, where $\mathscr{S}_{\mathscr{X}} = \cup \mathscr{S}_V$ and $\mathscr{L} = \{\mathscr{L}_V\}_{V \in \mathscr{S}}$, trivializes $(K_n)_n$.

(iv) The "only if" part follows from (ii). For the "if" part, assume P^*K is trivialized by a pair $(\mathscr{S}_X, \mathscr{L}_X)$ on X. Let $U \in \mathscr{S}_X$ be an open stratum, and let $V \subset \mathscr{X}$ be the image of U [Laumon and Moret-Bailly 2000, 3.7]. Recall that for every T in Aff/S, $V(T)$ is the full subcategory of $\mathscr{X}(T)$ consisting of objects x that are locally in the essential image of $U(T)$, that is, such that there exists an étale surjection $T' \to T$ in Aff/S and $u' \in U(T')$, such that the image of u' in $\mathscr{X}(T')$ and $x|_{T'}$ are isomorphic. Then V is an open substack of $\mathscr{X}$ (hence also an algebraic stack) and $P|_U : U \to V$ is a presentation. Replacing $P : X \to \mathscr{X}$ by $P|_U : U \to V$ and using noetherian induction and (iii), we may assume $\mathscr{S}_X = \{X\}$.

It follows from a theorem of Gabber [Illusie et al. 2008] that P_* takes a bounded complex to a bounded complex. In fact, using base change by P, we may assume that $P : Y \to X$ is a morphism from an S-algebraic space Y to an S-scheme X. Let $j : U \to Y$ be an open dense subscheme of Y with complement $i : Z \to Y$. For a bounded complex L of Λ_n-sheaves on Y, we have the exact triangle

$$(Pi)_* i^! L \longrightarrow P_* L \longrightarrow (Pj)_* j^* L \longrightarrow .$$

Gabber's theorem implies that $(Pj)_* j^* L$ is bounded, since $Pj : U \to X$ is a morphism between schemes. Note that the dualizing functor preserves boundedness, so does $i^! = D_Z i^* D_Y$, and therefore we may assume that $(Pi)_* i^! L$ is bounded by noetherian induction. It follows that $P_* L$ is bounded.

Now take a pair $(\mathscr{S}, \mathscr{L})$ on $\mathscr{X}$ that trivializes all $P_* L$'s, for $L \in \mathscr{L}_X$; this is possible since each $P_* L$ is bounded and $\mathscr{L}_X$ is a finite set. We claim that K is trivialized by $(\mathscr{S}, \mathscr{L})$.

For each sheaf $\mathscr{F}$ on $\mathscr{X}$, the natural map $\mathscr{F} \to R^0 P_* P^* \mathscr{F}$ is injective. This follows from the sheaf axioms for the lisse-lisse topology, and the fact that the lisse-étale topos and the lisse-lisse topos are the same. Explicitly, to verify the injectivity on $X_U \to U$, for any $u \in \mathscr{X}(U)$, since the question is étale local on U, one can assume $P : X_U \to U$ has a section $s : U \to X_U$. Then the composition $\mathscr{F}_U \to R^0 P_* P^* \mathscr{F}_U \to R^0 P_* R^0 s_* s^* P^* \mathscr{F}_U = \mathscr{F}_U$ of the two adjunctions is the adjunction for $P \circ s = \mathrm{id}$, so the composite is an isomorphism, and the first map is injective.

We take $\mathscr{F}$ to be the cohomology sheaves $\mathscr{H}^i(K_n)$. Since $P^* \mathscr{H}^i(K_n)$ is an iterated extension of sheaves in $\mathscr{L}_X$, we see that $P_* P^* \mathscr{H}^i(K_n)$, and in particular $R^0 P_* P^* \mathscr{H}^i(K_n)$, are trivialized by $(\mathscr{S}, \mathscr{L})$ by (i). Since $\mathscr{H}^i(K_n)$ is lcc [Olsson

2007, 9.1], by Remark 3.1.1 (ii) we see that $\mathscr{H}^i(K_n)$ (hence K) is trivialized by $(\mathscr{S}, \mathscr{L})$.

(v) By part (i) and Remark 2.2.1 (i) it is enough to show that all adic systems $M = (M_n)_n \in \mathscr{A}$ are stratifiable. By (iv) we may assume $\mathscr{X} = X$ is an S-scheme. Since X is noetherian, there exists a stratification [Illusie 1977, VI, 1.2.6] of X such that M is lisse on each stratum. By (iii) we may assume M is lisse on X.

Let $\mathscr{L}$ be the set of isomorphism classes of Jordan–Hölder components of the Λ_0-sheaf M_0. We claim that $\mathscr{L}$ trivializes M_n for all n. Suppose it trivializes M_{n-1} for some $n \geq 1$. Consider the sub-Λ_n-modules $\lambda M_n \subset M_n[\lambda^n] \subset M_n$, where $M_n[\lambda^n]$ is the kernel of the map $\lambda^n : M_n \to M_n$. Since M is adic, we have exact sequences of Λ_X-modules

$$0 \longrightarrow \lambda M_n \longrightarrow M_n \longrightarrow M_0 \longrightarrow 0,$$

$$0 \longrightarrow M_n[\lambda^n] \longrightarrow M_n \longrightarrow \lambda^n M_n \longrightarrow 0,$$

$$0 \longrightarrow \lambda^n M_n \longrightarrow M_n \longrightarrow M_{n-1} \longrightarrow 0.$$

The natural surjection $M_n/\lambda M_n \to M_n/M_n[\lambda^n]$ implies that $\mathscr{L}$ trivializes $\lambda^n M_n$, and therefore it also trivializes M_n. By induction on n we are done.

Since $D_c^b \subset D_c^{\text{stra}} \subset D_c$, and D_c^b and D_c have the same heart, it is clear that D_c^{stra} has the same heart as them.

(vi) Applying $-\otimes^L_{\Lambda_n} K_n$ to the following exact sequence, viewed as an exact triangle in $\mathscr{D}(\mathscr{X}, \Lambda_n)$

$$0 \longrightarrow \Lambda_{n-1} \xrightarrow{1 \mapsto \lambda} \Lambda_n \longrightarrow \Lambda_0 \longrightarrow 0,$$

we get an exact triangle by Laszlo and Olsson [2008b, 3.0.10]

$$K_{n-1} \longrightarrow K_n \longrightarrow K_0 \longrightarrow .$$

By induction on n and Remark 3.4.1 below, we see that K is trivialized by $(\mathscr{S}, \mathscr{L})$ if K_0 is.

(vii) Let $K = (K_n)_n$. Recall that the Tate twist $K(1)$ is defined to be the system $(K_n(1))_n$, where $K_n(1) = K_n \otimes^L_{\Lambda_n} \Lambda_n(1)$. Note that the sheaf $\Lambda_n(1)$ is a flat Λ_n-module: to show that $- \otimes_{\Lambda_n} \Lambda_n(1)$ preserves injections, one can pass to stalks at geometric points, over which we have a trivialization $\Lambda_n \simeq \Lambda_n(1)$.

Suppose K is $(\mathscr{S}, \mathscr{L})$-stratifiable. Using the isomorphism

$$\mathscr{H}^i(K_n) \otimes_{\Lambda_n} \Lambda_n(1) = \mathscr{H}^i(K_n \otimes^L_{\Lambda_n} \Lambda_n(1)),$$

it suffices to show the existence of a pair $(\mathscr{S}, \mathscr{L}')$ such that for each $\mathscr{U} \in \mathscr{S}$, the Jordan–Hölder components of the lcc sheaves $L \otimes_{\Lambda_n} \Lambda_n(1)$ lie in $\mathscr{L}'(\mathscr{U})$, for all

$L \in \mathscr{L}(\mathscr{U})$. Since L is a Λ_0-module, we have

$$L \otimes_{\Lambda_n} \Lambda_n(1) = (L \otimes_{\Lambda_n} \Lambda_0) \otimes_{\Lambda_n} \Lambda_n(1) = L \otimes_{\Lambda_n} (\Lambda_0 \otimes_{\Lambda_n} \Lambda_n(1)) \\ = L \otimes_{\Lambda_n} \Lambda_0(1) = L \otimes_{\Lambda_0} \Lambda_0(1),$$

and we can take $\mathscr{L}'(\mathscr{U})$ to be a tensor closed hull of $\{\Lambda_0(1), L \in \mathscr{L}(\mathscr{U})\}$. □

Remark 3.4.1. In fact the proof of Lemma 3.4 (i) shows that $\mathfrak{D}_{\mathscr{S},\mathscr{L}}(\mathscr{A})$ is a triangulated subcategory with induced standard t-structure, for each fixed pair $(\mathscr{S}, \mathscr{L})$. Let $D_{\mathscr{S},\mathscr{L}}(\mathscr{X}, \Lambda)$ be the essential image of $\mathfrak{D}_{\mathscr{S},\mathscr{L}}(\mathscr{A})$ in $D_c(\mathscr{X}, \Lambda)$, and this is also a triangulated subcategory with induced standard t-structure.

Also if $E_r^{ij} \Longrightarrow E^n$ is a spectral sequence in the category $\mathscr{A}(\mathscr{X})$, and the E_r^{ij}'s are trivialized by $(\mathscr{S}, \mathscr{L})$ for all i, j, then all the E^n's are trivialized by $(\mathscr{S}, \mathscr{L})$.

We denote by $D_c^{\dagger,\mathrm{stra}}(\mathscr{X}, \Lambda)$, for $\dagger = \pm, b$, the full subcategory of $\dagger$-bounded stratifiable complexes, using the induced t-structure.

The following is a key result for showing the stability of stratifiability under the six operations later. Recall that $M \mapsto \widehat{M} = L\pi^* R\pi_* M$ is the normalization functor, where $\pi : \mathscr{X}^{\mathbb{N}} \to \mathscr{X}$ is the morphism of topoi in [Laszlo and Olsson 2008b, 2.1], mentioned in 2.3.

Proposition 3.5. *For a pair $(\mathscr{S}, \mathscr{L})$ on $\mathscr{X}$, if $M \in \mathfrak{D}_{\mathscr{S},\mathscr{L}}(\mathscr{A})$, then $\widehat{M} \in \mathfrak{D}_{\mathscr{S},\mathscr{L}}(\mathscr{A})$, too. In particular, if $K \in D_c(\mathscr{X}, \Lambda)$, then K is stratifiable if and only if its normalization $\widehat{K} \in \mathfrak{D}_c(\mathscr{A})$ is stratifiable.*

Proof. First, we will reduce to the case where M is essentially bounded (that is, $\mathscr{H}^i M$ is AR-null for $|i| \gg 0$). Let $P : X \to \mathscr{X}$ be a presentation. The ℓ-cohomological dimension of $X_{\text{ét}}$ is finite, by the assumption (LO) on S. Therefore, by Laszlo and Olsson [2008b, 2.1.i], the normalization functor for X has finite cohomological dimension, and the same is true for $\mathscr{X}$ since $P^*\widehat{M} = \widehat{P^*M}$, by [ibid., 2.2.1, 3.0.11]. This implies that for each integer i, there exist integers a and b with $a \leq b$, such that $\mathscr{H}^i(\widehat{M}) = \mathscr{H}^i(\widehat{\tau_{[a,b]}M})$. Since $\tau_{[a,b]}M$ is also trivialized by $(\mathscr{S}, \mathscr{L})$, we may assume $M \in \mathfrak{D}^b_{\mathscr{S},\mathscr{L}}(\mathscr{A}(\mathscr{X}))$.

Since $\widehat{M}$ is normalized, by Lemma 3.4 (vi), it suffices to show that $(\widehat{M})_0$ is trivialized by $(\mathscr{S}, \mathscr{L})$. Using projection formula and the flat resolution of Λ_0

$$0 \longrightarrow \Lambda \xrightarrow{\lambda} \Lambda \xrightarrow{\varepsilon} \Lambda_0 \longrightarrow 0,$$

we have [ibid., p.176]

$$(\widehat{M})_0 = \Lambda_0 \otimes_{\Lambda}^{L} R\pi_* M = R\pi_*(\pi^*\Lambda_0 \otimes_{\Lambda_\bullet}^{L} M),$$

where $\pi^*\Lambda_0$ is the constant projective system defined by Λ_0. Let $C \in \mathfrak{D}(\mathscr{A})$ be the complex of projective systems $\pi^*\Lambda_0 \otimes_{\Lambda_\bullet}^{L} M$; it is a λ-complex, and we have $C_n = \Lambda_0 \otimes_{\Lambda_n}^{L} M_n \in \mathfrak{D}_c(\mathscr{X}, \Lambda_0)$.

Recall [Illusie 1977, V, 3.2.3] that a projective system $(K_n)_n$ ringed by $\Lambda_\bullet$ in an abelian category is AR-adic if and only if

• it satisfies the condition (MLAR) [Illusie 1977, V, 2.1.1], hence (ML), and denote by $(N_n)_n$ the projective system of the universal images of $(K_n)_n$;

• there exists an integer $k \geq 0$ such that the projective system $(L_n)_n := (N_{n+k} \otimes \Lambda_n)_n$ is adic.

Moreover, $(K_n)_n$ is AR-isomorphic to $(L_n)_n$. Now for each i, the projective system $\mathcal{H}^i(C)$ is AR-adic Remark 2.2.1 (i). Let $N^i = (N^i_n)_n$ be the projective system of the universal images of $\mathcal{H}^i(C)$, and choose an integer $k \geq 0$ such that the system $L^i = (L^i_n)_n = (N^i_{n+k} \otimes \Lambda_n)_n$ is adic. Since $N^i_{n+k} \subset \mathcal{H}^i(C_{n+k})$ is annihilated by λ, we have $L^i_n = N^i_{n+k}$, and the transition morphism gives an isomorphism

$$L^i_n \simeq L^i_n \otimes_{\Lambda_n} \Lambda_{n-1} \xrightarrow{\sim} L^i_{n-1}.$$

This means the projective system L^i is the constant system $\pi^* L^i_0$. By Laszlo and Olsson [2008b, 2.2.2] we have $R\pi_* \mathcal{H}^i(C) \simeq R\pi_* L^i$, which is just L^i_0 by [ibid., 2.2.3].

The spectral sequence

$$R^j \pi_* \mathcal{H}^i(C) \Longrightarrow \mathcal{H}^{i+j}((\widehat{M})_0)$$

degenerates to isomorphisms $L^i_0 \simeq \mathcal{H}^i((\widehat{M})_0)$, so we only need to show that L^i_0 is trivialized by $(\mathcal{S}, \mathcal{L})$. Using the periodic Λ_n-flat resolution of Λ_0

$$\cdots \longrightarrow \Lambda_n \xrightarrow{\lambda} \Lambda_n \xrightarrow{\lambda^n} \Lambda_n \xrightarrow{\lambda} \Lambda_n \xrightarrow{\varepsilon} \Lambda_0 \longrightarrow 0,$$

we see that $\Lambda_0 \otimes^L_{\Lambda_n} \mathcal{H}^j(M_n)$ is represented by the complex

$$\cdots \longrightarrow \mathcal{H}^j(M_n) \xrightarrow{\lambda^n} \mathcal{H}^j(M_n) \xrightarrow{\lambda} \mathcal{H}^j(M_n) \longrightarrow 0,$$

so $\mathcal{H}^i(\Lambda_0 \otimes^L_{\Lambda_n} \mathcal{H}^j(M_n))$ are trivialized by $(\mathcal{S}, \mathcal{L})$, for all i, j. Since M is essentially bounded, we have the spectral sequence

$$\mathcal{H}^i(\Lambda_0 \otimes^L_{\Lambda_n} \mathcal{H}^j(M_n)) \Longrightarrow \mathcal{H}^{i+j}(C_n),$$

from which we deduce (by Remark 3.4.1) that the $\mathcal{H}^i(C_n)$'s are trivialized by $(\mathcal{S}, \mathcal{L})$. The universal image N^i_n is the image of $\mathcal{H}^i(C_{n+r}) \to \mathcal{H}^i(C_n)$ for some $r \gg 0$, therefore the N^i_n's (and hence the L^i_n's) are trivialized by $(\mathcal{S}, \mathcal{L})$.

For the second claim, let $K \in D_c(\mathcal{X}, \Lambda)$. Since K is isomorphic to the image of $\widehat{K}$ under the localization $\mathcal{D}_c(\mathcal{A}) \to D_c(\mathcal{X}, \Lambda)$ [Laszlo and Olsson 2008b, 3.0.14], we see that K is stratifiable if $\widehat{K}$ is. Conversely, if K is stratifiable, which means that it is isomorphic to the image of some $M \in \mathcal{D}^{\mathrm{stra}}_c(\mathcal{A})$, then $\widehat{K} = \widehat{M}$ is also stratifiable. $\square$

3.5.1. For $K \in D_c(\mathscr{X}, \Lambda)$, we say that K is $(\mathscr{S}, \mathscr{L})$-*stratifiable* if $\widehat{K}$ is, and then Proposition 3.5 implies that $K \in D_{\mathscr{S},\mathscr{L}}(\mathscr{X}, \Lambda)$ (see Remark 3.4.1) if and only if K is $(\mathscr{S}, \mathscr{L})$-stratifiable.

Corollary 3.6. (i) *If $\mathscr{S}$ is a stratification of $\mathscr{X}$, then $K \in D_c(\mathscr{X}, \Lambda)$ is stratifiable if and only if $K|_V$ is stratifiable for every $V \in \mathscr{S}$.*

(ii) *Let $K \in D_c(\mathscr{X}, \Lambda)$. Then K is stratifiable if and only if its Tate twist $K(1)$ is.*

(iii) *Let $P : X \to \mathscr{X}$ be a presentation, and let $K \in D_c(\mathscr{X}, \Lambda)$. Then K is stratifiable if and only if P^*K (respectively $P^!K$) is stratifiable.*

Proof. (i) The "only if" part follows from Lemma 3.4 (ii). For the "if" part, we first prove the following.

Lemma 3.6.1. *For an S-algebraic stack $\mathscr{X}$ locally of finite type, let*

$$N \xrightarrow{u} M \longrightarrow C \longrightarrow N[1]$$

be an exact triangle in $\mathscr{D}_c(\mathscr{A})$, where N is a normalized complex and C is almost AR-null. Then the morphism u is isomorphic to the natural map $\widehat{M} \to M$.

Proof. Consider the following diagram

$$\begin{tikzcd} \widehat{N} \arrow[r, "\widehat{u}"] \arrow[d, "\simeq"'] & \widehat{M} \arrow[r] \arrow[d] & \widehat{C} \arrow[r] \arrow[d] & \widehat{N}[1] \arrow[d] \\ \widehat{N} \arrow[r, "u"] & M \arrow[r] & C \arrow[r] & N[1]. \end{tikzcd}$$

Since C is almost AR-null, we have $\widehat{C} = 0$ by Laszlo and Olsson [2008b, 2.2.2], and so $\widehat{u}$ is an isomorphism. $\square$

Now let $f : \mathscr{Y} \to \mathscr{X}$ be a morphism of S-algebraic stacks, and let $M \in \mathscr{D}_c(\mathscr{A}(\mathscr{X}))$. We claim that $f^*\widehat{M} \simeq \widehat{f^*M}$. Applying f^* to the exact triangle

$$\widehat{M} \longrightarrow M \longrightarrow C \longrightarrow$$

we get

$$f^*\widehat{M} \longrightarrow f^*M \longrightarrow f^*C \longrightarrow .$$

By Laszlo and Olsson [2008a, 4.3.2], $\widehat{M}_n = \mathrm{hocolim}_N \tau_{\le N} \widehat{M}_n$, and $- \otimes^L_{\Lambda_n} \Lambda_{n-1}$ and f^* preserve homotopy colimit because they preserve infinite direct sums. Now that $\tau_{\le N}\widehat{M}_n$ and Λ_{n-1} are bounded above complexes, we have

$$f^*(\tau_{\le N}\widehat{M}_n \otimes^L_{\Lambda_n} \Lambda_{n-1}) \simeq f^*\tau_{\le N}\widehat{M}_n \otimes^L_{\Lambda_n} \Lambda_{n-1}$$

(see the proof of [ibid., 4.5.3]). Hence applying f^* to the isomorphism

$$\widehat{M}_n \otimes^L_{\Lambda_n} \Lambda_{n-1} \longrightarrow \widehat{M}_{n-1}$$

we get an isomorphism

$$f^*\widehat{M}_n \otimes^L_{\Lambda_n} \Lambda_{n-1} \longrightarrow f^*\widehat{M}_{n-1},$$

and by [ibid., 3.0.10], $f^*\widehat{M}$ is normalized. Also it is clear that f^*C is AR-null. By Lemma 3.6.1 we have $f^*\widehat{M} \simeq \widehat{f^*M}$.

Therefore, the "if" part follows from Lemma 3.4 (iii) and Proposition 3.5, since $\widehat{K}|_V \simeq \widehat{(K|_V)}$.

(ii) This follows from Lemma 3.4 (vii), since $\widehat{K}(1) = \widehat{K(1)}$.

(iii) For P^*K, the "only if" part follows from Lemma 3.4 (ii), and the "if" part follows from Lemma 3.4 (iv) and Proposition 3.5, since $P^*\widehat{K} = \widehat{(P^*K)}$ [ibid., 2.2.1, 3.0.11].

Since P is smooth of relative dimension d, for some function $d : \pi_0(X) \to \mathbb{N}$, we have $P^!K \simeq P^*K(d)[2d]$, so by (ii), P^*K is stratifiable if and only if $P^!K$ is. $\square$

Before proving the main result of this section, we prove some special cases.

3.7. Let $f : X \to Y$ be a morphism of S-schemes. Then the Λ_n-dualizing complexes $K_{X,n}$ and $K_{Y,n}$ of X and Y respectively have finite quasi-injective dimensions, and are bounded by some integer independent of n. Together with the base change theorem for $f_!$, we see that there exists an integer $N > 0$ depending only on X, Y and f, such that for any integers a, b and n with $n \geq 0$ and any $M \in \mathscr{D}_c^{[a,b]}(X, \Lambda_n)$, we have $f_*M \in \mathscr{D}_c^{[a,b+N]}(Y, \Lambda_n)$. This implies that for each n, the functor (defined using K-injective resolutions, see [Spaltenstein 1988, 6.7])

$$f_* : \mathscr{D}(X, \Lambda_n) \to \mathscr{D}(Y, \Lambda_n)$$

restricts to

$$f_* : \mathscr{D}_c(X, \Lambda_n) \to \mathscr{D}_c(Y, \Lambda_n).$$

Moreover, for $M \in \mathscr{D}(\mathscr{A}(X))$ with constructible $\mathscr{H}^j(M_n)$'s (for all j and n) and for each $i \in \mathbb{Z}$, there exist integers $a < b$ such that

$$R^i f_*M \simeq R^i f_*\tau_{[a,b]}M.$$

In particular, if M is a λ-complex on X, then $R^i f_*M$ is AR-adic for each i, and hence $f_*M = (f_*M_n)_n$ is a λ-complex on Y.

This enables us to define

$$f_* : D_c(X, \Lambda) \to D_c(Y, \Lambda)$$

to be $K \mapsto Qf_*\widehat{K}$, where $Q : \mathscr{D}_c(\mathscr{A}(Y)) \to D_c(Y, \Lambda)$ is the localization functor. It agrees with the definition in [Laszlo and Olsson 2008b, 8] when restricted to

$D_c^+(X, \Lambda)$, and for each $i \in \mathbb{Z}$ and $K \in D_c(X, \Lambda)$, there exist integers $a < b$ such that $R^i f_* K \simeq R^i f_* \tau_{[a,b]} K$.

Lemma 3.8. (i) *If $f : X \to Y$ is a morphism of S-schemes, and $K \in D_c(X, \Lambda)$ is trivialized by $(\{X\}, \mathscr{L})$ for some $\mathscr{L}$, then $f_* K$ is stratifiable.*

(ii) *Let $\mathscr{X}$ be an S-algebraic stack that has a connected presentation* (*that is, there exists a presentation $P : X \to \mathscr{X}$ with X a connected S-scheme*). *Let $K_{\mathscr{X}}$ and $K'_{\mathscr{X}}$ be two Λ-dualizing complexes on $\mathscr{X}$, and let D and D' be the two associated dualizing functors, respectively. Let $K \in D_c(\mathscr{X}, \Lambda)$. If DK is trivialized by a pair $(\mathscr{S}, \mathscr{L})$, where all strata in $\mathscr{S}$ are connected, then $D'K$ is trivialized by $(\mathscr{S}, \mathscr{L}')$ for some other $\mathscr{L}'$. In particular, for stacks with connected presentation, the property of the Verdier dual of K being stratifiable is independent of the choice of the dualizing complex.*

(iii) *Let $\mathscr{X}$ be an S-algebraic stack that has a connected presentation, and assume that the constant sheaf Λ on $\mathscr{X}$ is a dualizing complex. If $K \in D_c(\mathscr{X}, \Lambda)$ is trivialized by a pair $(\{\mathscr{X}\}, \mathscr{L})$, then $D_{\mathscr{X}} K$ is trivialized by $(\{\mathscr{X}\}, \mathscr{L}')$ for some $\mathscr{L}'$.*

Proof. (i) Since $f_* K$ is the image of $f_* \widehat{K}$, it suffices to show that $f_* \widehat{K}$ is stratifiable. Since $f_* L$ is bounded for each $L \in \mathscr{L}$, there exists a pair $(\mathscr{S}_Y, \mathscr{L}_Y)$ on Y that trivializes $f_* L$, for all $L \in \mathscr{L}$. We claim that this pair trivializes $R^i f_* \widehat{K}_n$, for each i and n.

Since $R^i f_* \widehat{K}_n = R^i f_* \tau_{[a,b]} \widehat{K}_n$ for some $a < b$, and $\tau_{[a,b]} \widehat{K}_n$ is trivialized by $(\{X\}, \mathscr{L})$, we may assume $\widehat{K}_n$ is bounded. The claim then follows from the spectral sequence

$$R^p f_* \mathscr{H}^q((\widehat{K})_n) \Longrightarrow R^{p+q} f_*((\widehat{K})_n)$$

and Remark 3.4.1.

(ii) Recall that the dualizing complex $K_{\mathscr{X}}$ (respectively $K'_{\mathscr{X}}$) is defined to be the image of a normalized complex $K_{\mathscr{X},\bullet}$ (respectively $K'_{\mathscr{X},\bullet}$), where each $K_{\mathscr{X},n}$ (respectively $K'_{\mathscr{X},n}$) is a Λ_n-dualizing complex. See [ibid., 7.2.3, 7.2.4].

Let $P : X \to \mathscr{X}$ be a presentation where X is a connected scheme. Then we have

$$P^* R\mathscr{H}om(K_{\mathscr{X},n}, K'_{\mathscr{X},n}) = R\mathscr{H}om(P^* K_{\mathscr{X},n}, P^* K'_{\mathscr{X},n}) = R\mathscr{H}om(P^! K_{\mathscr{X},n}, P^! K'_{\mathscr{X},n}).$$

Since $P^! K_{\mathscr{X},n}$ and $P^! K'_{\mathscr{X},n}$ are Λ_n-dualizing complexes on X, by [Illusie 1977, Exp. I, 2], we see that $P^* R\mathscr{H}om(K_{\mathscr{X},n}, K'_{\mathscr{X},n})$ (and hence $R\mathscr{H}om(K_{\mathscr{X},n}, K'_{\mathscr{X},n})$) is cohomologically concentrated in one degree, therefore it is quasi-isomorphic to this nontrivial cohomology sheaf, once it has been appropriately shifted. So let $R\mathscr{H}om(K_{\mathscr{X},n}, K'_{\mathscr{X},n}) \simeq L_n[r_n]$ for some sheaf L_n and integer r_n. Since $P^* L_n$ is invertible and hence lcc (see [Illusie 1977, p. 19]), the sheaf L_n is lcc [Olsson 2007, 9.1].

For every stratum $\mathcal{U} \in \mathcal{S}$, let $\mathcal{L}_0(\mathcal{U})$ be the union of $\mathcal{L}(\mathcal{U})$ and the set of isomorphism classes of the Jordan–Hölder components of the lcc sheaf $L_0|_{\mathcal{U}}$. Since all strata in $\mathcal{S}$ are connected, there exists a tensor closed hull of $(\mathcal{S}, \mathcal{L}_0)$ of the form $(\mathcal{S}, \mathcal{L}')$, that is, they have the same stratification $\mathcal{S}$.

By Laszlo and Olsson [2008b, 4.0.8], the system

$$(L_n[r_n])_n = R\mathcal{H}om((K_{\mathcal{X},n})_n, (K'_{\mathcal{X},n})_n)$$

is normalized, so by Lemma 3.6.1, $\widehat{D'K_{\mathcal{X}}} = (L_n[r_n])_n$, and by Lemma 3.4 (vi), it is trivialized by $(\mathcal{S}, \mathcal{L}')$. Since DK is trivialized by $(\mathcal{S}, \mathcal{L}')$, so also is $D'K$, because $\widehat{D'K} \simeq \widehat{DK} \otimes^L \widehat{D'K_{\mathcal{X}}}$.

(iii) The assumption implies in particular that $\mathcal{X}$ is connected, so by (ii), the question is independent of the choice of the dualizing complex. By definition, $\widehat{K}$ is trivialized by $(\{\mathcal{X}\}, \mathcal{L})$, so are truncations of $\widehat{K}$. The essential image of $R\mathcal{H}om(\widehat{K}, \Lambda_\bullet)$ in $D_c(\mathcal{X}, \Lambda)$ is DK, so by 3.5.1 it suffices to show that $R\mathcal{H}om(\widehat{K}, \Lambda_\bullet) \in \mathcal{D}_{\{\mathcal{X}\},\mathcal{L}'}(\mathcal{A})$ for some $\mathcal{L}'$.

Since $\mathcal{X}$ is quasi-compact, each Λ_n-dualizing complex is of finite quasi-injective dimension, so for each integer i, there exist integers a and b such that

$$\mathcal{H}^i R\mathcal{H}om(\widehat{K}_n, \Lambda_n) = \mathcal{H}^i R\mathcal{H}om(\tau_{[a,b]}\widehat{K}_n, \Lambda_n).$$

Using truncation triangles, we may further replace $\tau_{[a,b]}\widehat{K}_n$ by the cohomology sheaves $\mathcal{H}^j \widehat{K}_n$, and hence by their Jordan–Hölder components. Therefore, it suffices to find an $\mathcal{L}'$ that trivializes $\mathcal{H}^i R\mathcal{H}om(L, \Lambda_0)$, for all $i \in \mathbb{Z}$ and $L \in \mathcal{L}$. Note that $R\mathcal{H}om(L, \Lambda_0) = \mathcal{H}om(L, \Lambda_0) = L^\vee$ is a simple Λ_0-sheaf, so one can take $\mathcal{L}' = \{L^\vee | L \in \mathcal{L}\}$. □

Remark 3.8.1. For any S-algebraic stack $\mathcal{X}$, the Verdier dual of a complex K in $D_c(\mathcal{X}, \Lambda)$ being stratifiable or not is independent of the choice of the dualizing complex. Let $K_{\mathcal{X}}$ and $K'_{\mathcal{X}}$ be two dualizing complexes on $\mathcal{X}$, defining dualizing functors D and D', respectively. Let $P : X \to \mathcal{X}$ be a presentation, let $K_X = P^! K_{\mathcal{X}}$ and let $K'_X = P^! K'_{\mathcal{X}}$, defining dualizing functors D_X and D'_X on X, respectively. Suppose DK is stratifiable. To show $D'K$ is also stratifiable, by Corollary 3.6 (iii) it suffices to show $P^! D'K = D'_X P^* K$ is stratifiable. Since $D_X P^* K = P^! DK$ is stratifiable by assumption, we may assume $\mathcal{X} = X$ is a scheme. Since X is noetherian, it has finitely many connected components, each of which is both open and closed. Then the result follows from Corollary 3.6 (i) and Lemma 3.8 (ii).

Next we prove the main result of this section.

Theorem 3.9. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of S-algebraic stacks. Then the operations f_*, $f_!$, f^*, $f^!$, $D_{\mathcal{X}}$, $- \otimes^L -$ and $R\mathcal{H}om(-, -)$ all preserve stratifiability,* *namely, they induce functors*

$$f_* : D_c^{+,\mathrm{stra}}(\mathcal{X}, \Lambda) \longrightarrow D_c^{+,\mathrm{stra}}(\mathcal{Y}, \Lambda), \qquad f_! : D_c^{-,\mathrm{stra}}(\mathcal{X}, \Lambda) \longrightarrow D_c^{-,\mathrm{stra}}(\mathcal{Y}, \Lambda),$$
$$f^* : D_c^{\mathrm{stra}}(\mathcal{Y}, \Lambda) \longrightarrow D_c^{\mathrm{stra}}(\mathcal{X}, \Lambda), \qquad f^! : D_c^{\mathrm{stra}}(\mathcal{Y}, \Lambda) \longrightarrow D_c^{\mathrm{stra}}(\mathcal{X}, \Lambda),$$
$$R\mathcal{H}om(-,-) : D_c^{-,\mathrm{stra}}(\mathcal{X}, \Lambda)^{\mathrm{op}} \times D_c^{+,\mathrm{stra}}(\mathcal{X}, \Lambda) \longrightarrow D_c^{+,\mathrm{stra}}(\mathcal{X}, \Lambda),$$
$$\otimes^L : D_c^{-,\mathrm{stra}}(\mathcal{X}, \Lambda) \times D_c^{-,\mathrm{stra}}(\mathcal{X}, \Lambda) \longrightarrow D_c^{-,\mathrm{stra}}(\mathcal{X}, \Lambda),$$
$$D : D_c^{\mathrm{stra}}(\mathcal{X}, \Lambda) \longrightarrow D_c^{\mathrm{stra}}(\mathcal{X}, \Lambda)^{\mathrm{op}}.$$

Proof. We may assume all stacks involved are reduced.

We consider the Verdier dual functor D first. Let $P : X \to \mathcal{X}$ be a presentation. Since $P^*D = DP^!$, by Corollary 3.6 (iii) we can assume $\mathcal{X} = X$ is a scheme. Let K be a complex on X trivialized by $(\mathcal{S}, \mathcal{L})$. Refining if necessary, we may assume all strata in $\mathcal{S}$ are connected and regular. Let $j : U \to X$ be the immersion of an open stratum in $\mathcal{S}$ with complement $i : Z \to X$. Shrinking U if necessary, we may assume there is a dimension function on U [Riou 2007, Définition 2.1], hence by a result of Gabber [ibid., Théorème 0.2], the constant sheaf Λ on U is a dualizing complex. Consider the exact triangle

$$i_*D_Z(K|_Z) \longrightarrow D_X K \longrightarrow j_*D_U(K|_U) \longrightarrow .$$

By Lemma 3.8 (iii) we see that $D_U(K|_U)$ is trivialized by $(\{U\}, \mathcal{L}')$ for some $\mathcal{L}'$, so $j_*D_U(K|_U)$ is stratifiable by Lemma 3.8 (i). By noetherian induction we may assume $D_Z(K|_Z)$ is stratifiable, and it is clear that i_* preserves stratifiability. Therefore by Lemma 3.4 (i), $D_X K$ is stratifiable.

The case of f^* (and hence $f^!$) is proved in Lemma 3.4 (ii).

Next we prove the case of $\otimes^L$. For $i = 1, 2$, let $K_i \in D_c^-(\mathcal{X}, \Lambda)$, trivialized by $(\mathcal{S}_i, \mathcal{L}_i)$. Let $(\mathcal{S}, \mathcal{L})$ be a common tensor closed refinement (by Lemma 3.3) of $(\mathcal{S}_i, \mathcal{L}_i)$, $i = 1, 2$. The total tensor product $K_1 \otimes^L K_2$ is defined to be the image in $D_c(\mathcal{X}, \Lambda)$ of $\widehat{K}_1 \otimes^L_{\Lambda_\bullet} \widehat{K}_2$, which by Laszlo and Olsson [2008b, 3.0.10] is normalized, so it suffices to show (by Lemma 3.4 (vi)) that

$$\widehat{K}_{1,0} \otimes^L_{\Lambda_0} \widehat{K}_{2,0} = \widehat{K}_{1,0} \otimes_{\Lambda_0} \widehat{K}_{2,0}$$

is trivialized by $(\mathcal{S}, \mathcal{L})$. This follows from

$$\mathcal{H}^r(\widehat{K}_{1,0} \otimes_{\Lambda_0} \widehat{K}_{2,0}) = \bigoplus_{i+j=r} \mathcal{H}^i(\widehat{K}_{1,0}) \otimes_{\Lambda_0} \mathcal{H}^j(\widehat{K}_{2,0})$$

and the assumption that $(\mathcal{S}, \mathcal{L})$ is tensor closed.

The case of $R\mathcal{H}om(K_1, K_2) = D(K_1 \otimes^L DK_2)$ follows.

Finally we prove the case of f_* and $f_!$. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of S-algebraic stacks, and let $K \in D^-_{\mathcal{S},\mathcal{L}}(\mathcal{X}, \Lambda)$ for some pair $(\mathcal{S}, \mathcal{L})$. We want to show

$f_!K$ is stratifiable. Let $j : \mathcal{U} \to \mathcal{X}$ be the immersion of an open stratum in $\mathcal{S}$, with complement $i : \mathcal{Z} \to \mathcal{X}$. From the exact triangle

$$(fj)_!j^*K \longrightarrow f_!K \longrightarrow (fi)_!i^*K \longrightarrow$$

we see that it suffices to prove the claim for fj and fi. By noetherian induction we can replace $\mathcal{X}$ by $\mathcal{U}$. By Corollary 3.6 (iii) and smooth base change [Laszlo and Olsson 2008b, 12.1], we can replace $\mathcal{Y}$ by a presentation Y, and by Corollary 3.6 (i) and [ibid., 12.3] we can shrink Y to an open subscheme. After these reductions, we assume that $\mathcal{Y} = Y$ is a regular irreducible affine S-scheme that has a dimension function on it, that K is trivialized by $(\{\mathcal{X}\}, \mathcal{L})$, and that the relative inertia stack $\mathcal{I}_f := \mathcal{X} \times_{\Delta, \mathcal{X} \times_Y \mathcal{X}, \Delta} \mathcal{X}$ is flat and has components over $\mathcal{X}$ [Behrend 2003, 5.1.14]. Therefore by [ibid., 5.1.13], f factors as

$$\mathcal{X} \xrightarrow{g} \mathcal{Z} \xrightarrow{h} Y,$$

where g is gerbe-like and h is representable (see [ibid., 5.1.3–5.1.6] for relevant notions). So we reduce to two cases: f is representable, or f is gerbe-like.

Case when f is representable. By shrinking the S-algebraic space $\mathcal{X}$ we can assume $\mathcal{X} = X$ is a regular connected scheme that has a dimension function, so that the constant sheaf Λ on X is a dualizing complex. By Lemma 3.8 (iii) we see that DK is trivialized by some $(\{X\}, \mathcal{L}')$, and by Lemma 3.8 (i), f_*DK is stratifiable. Therefore $f_!K = Df_*DK$ is also stratifiable.

Case when f is gerbe-like. In this case f is smooth [Behrend 2003, 5.1.5], hence étale locally on Y it has a section. Replacing Y by an étale cover, we may assume that f is a neutral gerbe, so $f : B(G/Y) \to Y$ is the structural map, for some flat group space G of finite type over Y [Laumon and Moret-Bailly 2000, 3.21]. By [ibid., 5.1.1] and Corollary 3.6 (i) we may assume G is a Y-group scheme. Next we reduce to the case when G is smooth over Y.

By assumption Y is integral. Let $k(Y)$ be the function field of Y and $\overline{k(Y)}$ an algebraic closure. Then $G_{\overline{k(Y)},\mathrm{red}}$ is smooth over $\overline{k(Y)}$, so there exists a finite extension L over $k(Y)$ such that $G_{L,\mathrm{red}}$ is smooth over L. Let Y' be the normalization of Y in L, which is a scheme of finite type over S, and the natural map $Y' \to Y$ is finite surjective. It factors through $Y' \to Z \to Y$, where Z is the normalization of Y in the separable closure of $k(Y)$ in $L = k(Y')$. So $Z \to Y$ is generically étale, and $Y' \to Z$ is purely inseparable, hence a universal homeomorphism, so Y' and Z have equivalent étale sites. Replacing Y' by Z and shrinking Y we can assume $Y' \to Y$ is finite étale. Replacing Y by Y' (by Corollary 3.6 (ii)) we assume G_{red} over Y has smooth generic fiber, and by shrinking Y we assume G_{red} is smooth over Y.

G_{red} is a subgroup scheme of G [Grothendieck and Demazure 1970, Exposé VI_{A}, 0.2]; we write $h : G_{\mathrm{red}} \hookrightarrow G$ for the associated closed immersion. Then

$Bh : B(G_{\mathrm{red}}/Y) \to B(G/Y)$ is faithful and hence representable. It is also radicial: consider the diagram where the square is 2-Cartesian

$$\begin{tikzcd} Y \arrow[r, hook, "i"] & G/G_{\mathrm{red}} \arrow[r, "g"] \arrow[d] & Y \arrow[d, "P"] \\ & B(G_{\mathrm{red}}/Y) \arrow[r, "Bh"'] & B(G/Y). \end{tikzcd}$$

The map i is a nilpotent closed embedding, so g is radicial. Since P is faithfully flat, Bh is also radicial. This shows that

$$(Bh)^* : D_c^-(B(G/Y), \Lambda) \to D_c^-(B(G_{\mathrm{red}}/Y), \Lambda)$$

is an equivalence of categories. Replacing G by G_{red} we assume G is smooth over Y, and hence $P : Y \to B(G/Y)$ is a connected presentation.

Let d be the relative dimension of G over Y. By assumption, the constant sheaf Λ on Y is a dualizing complex, and so $f^!\Lambda = \Lambda\langle -d\rangle$ (and hence the constant sheaf Λ on $\mathscr{X}$) is a dualizing complex on $\mathscr{X}$. By Lemma 3.8 (iii), we see that DK is trivialized by a pair of the form $(\{\mathscr{X}\}, \mathscr{L}')$. To show $f_!K$ is stratifiable is equivalent to showing that $Df_!K = f_*DK$ is stratifiable. So replacing K by DK, it suffices to show that f_*K is stratifiable, where $K \in D^+_{\{\mathscr{X}\},\mathscr{L}}(\mathscr{X}, \Lambda)$ for some $\mathscr{L}$.

Consider the strictly simplicial smooth hypercover associated to the presentation $P : Y \to B(G/Y)$, and let $f_i : \prod_i G \to Y$ be the structural map. As in the proof of Lemma 3.8 (i), it suffices to show the existence of a pair $(\mathscr{S}_Y, \mathscr{L}_Y)$ on Y that trivializes $R^n f_* L$, for all $L \in \mathscr{L}$ and $n \in \mathbb{Z}$. From the spectral sequence [Laszlo and Olsson 2008b, 10.0.9]

$$E_1^{ij} = R^j f_{i*} f_i^* P^* L \Longrightarrow R^{i+j} f_* L,$$

we see that it suffices for the pair $(\mathscr{S}_Y, \mathscr{L}_Y)$ to trivialize all the E_1^{ij}-terms. Assume $i \geq 1$. If we regard the map $f_i : \prod_i G \to Y$ as the product map

$$\prod_i f_1 : \prod_i G \to \prod_i Y,$$

where the products are fiber products over Y, then we can write $f_i^* P^* L$ as

$$f_1^* P^* L \boxtimes_{\Lambda_0} \Lambda_0 \boxtimes_{\Lambda_0} \cdots \boxtimes_{\Lambda_0} \Lambda_0.$$

Note that the scheme Y satisfies the condition (LO). By Künneth formula [Laszlo and Olsson 2008b, 11.0.14] we have

$$f_{i*} f_i^* P^* L = f_{1*} f_1^* P^* L \otimes_{\Lambda_0} f_{1*}\Lambda_0 \otimes_{\Lambda_0} \cdots \otimes_{\Lambda_0} f_{1*}\Lambda_0.$$

Since $f_{1*}f_1^*P^*L$ and $f_{1*}\Lambda_0$ are bounded complexes (by a theorem of Gabber [Illusie et al. 2008]), there exists a tensor closed pair $(\mathcal{S}_Y, \mathcal{L}_Y)$ that trivializes them, for all $L \in \mathcal{L}$. The proof is finished. □

Consequently, the theorem also holds for $\overline{\mathbb{Q}}_\ell$-coefficients.

Finally we give a lemma which will be used in the next section. This will play the same role as [Behrend 2003, 6.3.16].

Lemma 3.10. *Let X be a connected variety over an algebraically closed field k of characteristic not equal to ℓ, and let $\mathcal{L}$ be a finite set of isomorphism classes of simple lcc Λ_0-sheaves on X. Then there exists an integer d (depending only on $\mathcal{L}$) such that, for every lisse Λ-adic sheaf $\mathcal{F}$ on X trivialized by $\mathcal{L}$, and for every integer i, we have*

$$\dim_E H_c^i(X, \mathcal{F} \otimes_\Lambda E) \leq d \cdot \mathrm{rank}_E(\mathcal{F} \otimes_\Lambda E),$$

where E is the fraction field of Λ.

Proof. Since $\mathcal{L}$ is finite and $0 \leq i \leq 2\dim X$, there exists an integer $d > 0$ such that $\dim_{\Lambda_0} H_c^i(X, L) \leq d \cdot \mathrm{rank}_{\Lambda_0} L$, for every i and every $L \in \mathcal{L}$. For a short exact sequence of lcc Λ_0-sheaves

$$0 \longrightarrow \mathcal{G}' \longrightarrow \mathcal{G} \longrightarrow \mathcal{G}'' \longrightarrow 0$$

on X, the cohomological sequence

$$\cdots \longrightarrow H_c^i(X, \mathcal{G}') \longrightarrow H_c^i(X, \mathcal{G}) \longrightarrow H_c^i(X, \mathcal{G}'') \longrightarrow \cdots$$

implies that $\dim_{\Lambda_0} H_c^i(X, \mathcal{G}) \leq \dim_{\Lambda_0} H_c^i(X, \mathcal{G}') + \dim_{\Lambda_0} H_c^i(X, \mathcal{G}'')$. So it is clear that if $\mathcal{G}$ is trivialized by $\mathcal{L}$, then $\dim_{\Lambda_0} H_c^i(X, \mathcal{G}) \leq d \cdot \mathrm{rank}_{\Lambda_0} \mathcal{G}$, for every i.

Since we only consider $\mathcal{F} \otimes_\Lambda E$, we may assume $\mathcal{F} = (\mathcal{F}_n)_n$ is flat, of some constant rank over Λ (since X is connected), and this Λ-rank is equal to

$$\mathrm{rank}_{\Lambda_0} \mathcal{F}_0 = \mathrm{rank}_E(\mathcal{F} \otimes_\Lambda E).$$

Recall that $H_c^i(X, \mathcal{F})$ is a finitely generated Λ-module [Illusie 1977, VI, 2.2.2], so by Nakayama's lemma the minimal number of generators of the module is at most $\dim_{\Lambda_0}(\Lambda_0 \otimes_\Lambda H_c^i(X, \mathcal{F}))$. Similar to ordinary cohomology groups [Milne 2008, 19.2], we have an injection

$$\Lambda_0 \otimes_\Lambda H_c^i(X, \mathcal{F}) \hookrightarrow H_c^i(X, \mathcal{F}_0)$$

of Λ_0-vector spaces. Therefore, $\dim_E H_c^i(X, \mathcal{F} \otimes_\Lambda E)$ is less than or equal to the minimal number of generators of $H_c^i(X, \mathcal{F})$ over Λ, which is at most

$$\begin{aligned} \dim_{\Lambda_0}(\Lambda_0 \otimes_\Lambda H_c^i(X, \mathcal{F})) &\leq \dim_{\Lambda_0} H_c^i(X, \mathcal{F}_0) \\ &\leq d \cdot \mathrm{rank}_{\Lambda_0} \mathcal{F}_0 = d \cdot \mathrm{rank}_E(\mathcal{F} \otimes_\Lambda E). \end{aligned}$$

□

4. Convergent complexes and finiteness

We return to $\mathbb{F}_q$-algebraic stacks $\mathscr{X}_0, \mathscr{Y}_0, \dots$ of finite type. A complex K_0 in $W(\mathscr{X}_0, \overline{\mathbb{Q}}_\ell)$ is said to be *stratifiable* if K on $\mathscr{X}$ is stratifiable, and we denote by $W^{\mathrm{stra}}(\mathscr{X}_0, \overline{\mathbb{Q}}_\ell)$ the full subcategory of such complexes. Note that if K_0 is a lisse-étale complex, and it is stratifiable on $\mathscr{X}_0$, then it is geometrically stratifiable (that is, K on $\mathscr{X}$ is stratifiable). In turns out that in order for the trace formula to hold, it suffices to make this weaker assumption of geometric stratifiability. So we will only discuss stratifiable Weil complexes. Again, by a sheaf we mean a Weil $\overline{\mathbb{Q}}_\ell$-sheaf.

Definition 4.1. (i) *Let $K \in D_c(\overline{\mathbb{Q}}_\ell)$ and $\varphi : K \to K$ an endomorphism. The pair (K, φ) is said to be an ι-*convergent complex *(or just a convergent complex, since we fixed ι) if the complex series in two directions*

$$\sum_{n\in\mathbb{Z}} \sum_{H^n(K),H^n(\varphi)} |\alpha|^s$$

is convergent, for every real number $s > 0$. In this case let $\mathrm{Tr}(\varphi, K)$ *be the absolutely convergent complex series*

$$\sum_n (-1)^n \iota \,\mathrm{Tr}(H^n(\varphi), H^n(K))$$

or its limit.

(ii) *Let $K_0 \in W^-(\mathscr{X}_0, \overline{\mathbb{Q}}_\ell)$. We call K_0 an ι-*convergent complex of sheaves *(or just a convergent complex of sheaves), if for every integer $v \geq 1$ and every point $x \in \mathscr{X}_0(\mathbb{F}_{q^v})$, the pair $(K_{\overline{x}}, F_x)$ is a convergent complex. In particular, all bounded complexes are convergent.*

(iii) *Let $K_0 \in W^-(\mathscr{X}_0, \overline{\mathbb{Q}}_\ell)$ be a convergent complex of sheaves. Define*

$$c_v(\mathscr{X}_0, K_0) = \sum_{x\in[\mathscr{X}_0(\mathbb{F}_{q^v})]} \frac{1}{\#\,\mathrm{Aut}_x\,\mathbb{F}_{q^v}} \mathrm{Tr}(F_x, K_{\overline{x}}) \in \mathbb{C},$$

*and define the L-*series *of K_0 to be the formal power series*

$$L(\mathscr{X}_0, K_0, t) = \exp\left(\sum_{v\geq 1} c_v(\mathscr{X}_0, K_0)\frac{t^v}{v}\right) \in \mathbb{C}[[t]].$$

The zeta function $Z(\mathscr{X}_0, t)$ in Definition 1.2 is a special case of this definition as $Z(\mathscr{X}_0, t) = L(\mathscr{X}_0, \overline{\mathbb{Q}}_\ell, t)$. It has rational coefficients.

Notation 4.1.1. We sometimes write $c_v(K_0)$ for $c_v(\mathscr{X}_0, K_0)$, if it is clear that K_0 is on $\mathscr{X}_0$. We also write $c_v(\mathscr{X}_0)$ for $c_v(\mathscr{X}_0, \overline{\mathbb{Q}}_\ell)$.

Remark 4.1.2. (i) Behrend [2003, 6.2.3] defined convergent complexes with respect to arithmetic Frobenius elements, and our definition is for geometric Frobenius, and it is essentially the same as Behrend's definition, except we work with ι-mixed Weil complexes (which means *all* Weil complexes, by Remark 2.8.1) for an arbitrary isomorphism $\iota : \overline{\mathbb{Q}}_\ell \to \mathbb{C}$, while Behrend [2003] works with pure or mixed lisse-étale sheaves with integer weights. Also our definition is a little different from that in [Olsson 2008a]; the condition there is weaker.

(ii) Recall that Aut_x is defined to be the fiber over x of the inertia stack $\mathscr{I}_0 \to \mathscr{X}_0$. It is a group scheme of finite type [Laumon and Moret-Bailly 2000, 4.2] over $k(x)$, so $\mathrm{Aut}_x\, k(x)$ is a finite group.

(iii) If we have the following commutative diagram

$$\begin{tikzcd} \operatorname{Spec} \mathbb{F}_{q^{vd}} \arrow[r] \arrow[dr, "x'"'] & \operatorname{Spec} \mathbb{F}_{q^v} \arrow[d, "x"] \\ & \mathscr{X}_0, \end{tikzcd}$$

then $(K_{\overline{x}}, F_x)$ is convergent if and only if $(K_{\overline{x'}}, F_{x'})$ is convergent, because we have $F_{x'} = F_x^d$ and $s \mapsto sd : \mathbb{R}^{>0} \to \mathbb{R}^{>0}$ is a bijection. In particular, for a lisse-étale complex of sheaves, the property of being a convergent complex is independent of q and the structural morphism $\mathscr{X}_0 \to \operatorname{Spec} \mathbb{F}_q$. Also note that, for every integer $v \geq 1$, a complex K_0 on $\mathscr{X}_0$ is convergent if and only if $K_0 \otimes \mathbb{F}_{q^v}$ on $\mathscr{X}_0 \otimes \mathbb{F}_{q^v}$ is convergent.

We restate the main theorem in [Behrend 2003] using compactly supported cohomology as follows. It generalizes Theorem 1.1. We will prove it in this section and the next.

Theorem 4.2. *Let $f : \mathscr{X}_0 \to \mathscr{Y}_0$ be a morphism of $\mathbb{F}_q$-algebraic stacks, and let $K_0 \in W_m^{-,\mathrm{stra}}(\mathscr{X}_0, \overline{\mathbb{Q}}_\ell)$ be a convergent complex of sheaves. Then*:

(i) (Finiteness) *$f_! K_0$ is a convergent complex of sheaves on $\mathscr{Y}_0$.*

(ii) (Trace formula) $c_v(\mathscr{X}_0, K_0) = c_v(\mathscr{Y}_0, f_! K_0)$ *for every integer* $v \geq 1$.

First we give a few lemmas.

Lemma 4.3. *Let*

$$\begin{tikzcd} K' \arrow[r] \arrow[d, "\varphi'"'] & K \arrow[r] \arrow[d, "\varphi"'] & K'' \arrow[r] \arrow[d, "\varphi''"] & K'[1] \arrow[d, "\varphi'[1]"] \\ K' \arrow[r] & K \arrow[r] & K'' \arrow[r] & K'[1]. \end{tikzcd}$$

be an endomorphism of an exact triangle $K' \to K \to K'' \to K'[1]$ in $D_c^-(\overline{\mathbb{Q}}_\ell)$. If any two of the three pairs (K', φ'), (K'', φ'') and (K, φ) are convergent, then so is *the third, and*

$$\mathrm{Tr}(\varphi, K) = \mathrm{Tr}(\varphi', K') + \mathrm{Tr}(\varphi'', K'').$$

Proof. By the rotation axiom we can assume (K', φ') and (K'', φ'') are convergent. We have the exact sequence

$$\cdots \longrightarrow H^n(K') \longrightarrow H^n(K) \longrightarrow H^n(K'') \longrightarrow H^{n+1}(K') \longrightarrow \cdots .$$

Since $H^n(K)$ is an extension of a subobject of $H^n(K'')$ by a quotient object of $H^n(K')$, we have

$$\sum_{H^n(K),\varphi} |\alpha|^s \le \sum_{H^n(K'),\varphi'} |\alpha|^s + \sum_{H^n(K''),\varphi''} |\alpha|^s$$

for every real $s > 0$, so (K, φ) is convergent.

Since the series $\sum_{n\in\mathbb{Z}}(-1)^n \sum_{H^n(K),\varphi} \iota\alpha$ converges absolutely, we can change the order of summation, and the second assertion follows if we split the long exact sequence above into short exact sequences. $\square$

Corollary 4.4. *If $K_0' \to K_0 \to K_0'' \to K_0'[1]$ is an exact triangle in $W^-(\mathscr{X}_0, \overline{\mathbb{Q}}_\ell)$, and two of the three complexes K_0', K_0'' and K_0 are convergent complexes, then so is the third, and $c_v(K_0) = c_v(K_0') + c_v(K_0'')$.*

Proof. For every $x \in \mathscr{X}_0(\mathbb{F}_{q^v})$, we have an exact triangle

$$K'_{\overline{x}} \longrightarrow K_{\overline{x}} \longrightarrow K''_{\overline{x}} \longrightarrow$$

in $D_c^-(\overline{\mathbb{Q}}_\ell)$, equivariant under the action of F_x. Then apply Lemma 4.3. $\square$

Lemma 4.5. Theorem 4.2 *holds for* $f : \mathrm{Spec}\, \mathbb{F}_{q^d} \to \mathrm{Spec}\, \mathbb{F}_q$.

Proof. We have an equivalence of triangulated categories

$$W^-(\mathrm{Spec}\, \mathbb{F}_q, \overline{\mathbb{Q}}_\ell) \xrightarrow{\sim} D_c^-(\mathrm{Rep}_{\overline{\mathbb{Q}}_\ell} G),$$

where G is the Weil group $F^{\mathbb{Z}}$ of $\mathbb{F}_q$. Let H be the subgroup $F^{d\mathbb{Z}}$, the Weil group of $\mathbb{F}_{q^d}$. Since $f : \mathrm{Spec}\, \mathbb{F}_{q^d} \to \mathrm{Spec}\, \mathbb{F}_q$ is finite, we have $f_! = f_*$, and it is the induced-module functor

$$\mathit{Hom}_{\overline{\mathbb{Q}}_\ell[H]}\big(\overline{\mathbb{Q}}_\ell[G], -\big) : D_c^-(\mathrm{Rep}_{\overline{\mathbb{Q}}_\ell} H) \longrightarrow D_c^-(\mathrm{Rep}_{\overline{\mathbb{Q}}_\ell} G),$$

which is isomorphic to the coinduced-module functor $\overline{\mathbb{Q}}_\ell[G] \otimes_{\overline{\mathbb{Q}}_\ell[H]} -$. In particular, $f_!$ is exact on the level of sheaves.

Let A be a $\overline{\mathbb{Q}}_\ell$-representation of H, and $B = \overline{\mathbb{Q}}_\ell[G] \otimes_{\overline{\mathbb{Q}}_\ell[H]} A$. Let $x_1, \ldots, x_m$ be an ordered basis for A with respect to which F^d is an upper triangular matrix

$$\begin{bmatrix} \alpha_1 & * & * \\ & \ddots & * \\ & & \alpha_m \end{bmatrix}$$

with eigenvalues $\alpha_1, \dots, \alpha_m$. Then B has an ordered basis

$$\begin{array}{cccc} 1 \otimes x_1, & F \otimes x_1, & \cdots, & F^{d-1} \otimes x_1, \\ 1 \otimes x_2, & F \otimes x_2, & \cdots, & F^{d-1} \otimes x_2, \\ \vdots & \vdots & \ddots & \vdots \\ 1 \otimes x_m, & F \otimes x_m, & \cdots, & F^{d-1} \otimes x_m, \end{array}$$

with respect to which F is the matrix

$$\begin{bmatrix} M_1 & * & * \\ & \ddots & * \\ & & M_m \end{bmatrix}, \quad \text{where } M_i = \begin{bmatrix} 0 & \cdots & 0 & \alpha_i \\ 1 & \ddots & 0 & 0 \\ & \ddots & \ddots & \vdots \\ & & 1 & 0 \end{bmatrix}.$$

The characteristic polynomial of F on B is $\prod_{i=1}^m (t^d - \alpha_i)$.

Let K_0 be a complex of sheaves on $\operatorname{Spec} \mathbb{F}_{q^d}$. The eigenvalues of the Frobenius F on $\mathcal{H}^n(f_! K) = f_! \mathcal{H}^n(K)$ are all the d-th roots of the eigenvalues of F^d on $\mathcal{H}^n(K)$, so for every $s > 0$ we have

$$\sum_n \sum_{\mathcal{H}^n(f_! K), F} |\alpha|^s = d \sum_n \sum_{\mathcal{H}^n(K), F^d} |\alpha|^{s/d}.$$

This shows that $f_! K_0$ is a convergent complex on $\operatorname{Spec} \mathbb{F}_q$ if and only if K_0 is a convergent complex on $\operatorname{Spec} \mathbb{F}_{q^d}$.

Next we prove

$$c_v(\operatorname{Spec} \mathbb{F}_{q^d}, K_0) = c_v(\operatorname{Spec} \mathbb{F}_q, f_! K_0)$$

for every $v \geq 1$. Since $H^n(f_! K) = f_! H^n(K)$, and both sides are absolutely convergent series so that one can change the order of summation without changing the limit, it suffices to prove it when $K = A$ is a single representation concentrated in degree 0. Let us review this classical calculation. Use the notation above. For each i, fix a d-th root $\alpha_i^{1/d}$ of α_i, and let ζ_d be a primitive d-th root of unity. Then the eigenvalues of F on B are $\zeta_d^k \alpha_i^{1/d}$, for $i = 1, \dots, m$ and $k = 0, \dots, d-1$.

If $d \nmid v$, then $\operatorname{Hom}_{\mathbb{F}_q}(\mathbb{F}_{q^d}, \mathbb{F}_{q^v}) = \varnothing$, so $c_v(\operatorname{Spec} \mathbb{F}_{q^d}, A) = 0$. On the other hand,

$$c_v(\operatorname{Spec} \mathbb{F}_q, f_! A) = \operatorname{Tr}(F^v, B) = \sum_{i,k} \zeta_d^{vk} \alpha_i^{v/d} = \sum_i \alpha_i^{v/d} \sum_{k=0}^{d-1} \zeta_d^{vk} = 0.$$

If $d | v$, then $\operatorname{Hom}_{\mathbb{F}_q}(\mathbb{F}_{q^d}, \mathbb{F}_{q^v}) = \operatorname{Hom}_{\mathbb{F}_q}(\mathbb{F}_{q^d}, \mathbb{F}_{q^d}) = \mathbb{Z}/d\mathbb{Z}$. So

$$c_v(\mathbb{F}_{q^d}, A) = d \operatorname{Tr}(F^v, A) = d \sum_i \alpha_i^{v/d}.$$

On the other hand,

$$c_v(\mathbb{F}_q, B) = \mathrm{Tr}(F^v, B) = \sum_{i,k} \zeta_d^{vk} \alpha_i^{v/d} = \sum_{i,k} \alpha_i^{v/d} = d \sum_i \alpha_i^{v/d}. \qquad \square$$

Next, we consider BG_0, for a finite group scheme G_0 over $\mathbb{F}_q$.

Lemma 4.6. *Let G_0 be a finite $\mathbb{F}_q$-group scheme, and let $\mathscr{F}_0$ be a sheaf on BG_0. Then $H_c^r(BG, \mathscr{F}) = 0$ for all $r \neq 0$, and $H_c^0(BG, \mathscr{F}) \simeq H^0(BG, \mathscr{F})$ has dimension at most* $\mathrm{rank}(\mathscr{F})$. *Moreover, the set of ι-weights of $H_c^0(BG, \mathscr{F})$ is a subset of the ι-weights of $\mathscr{F}_0$.*

Proof. By [Olsson 2008a, 7.12–7.14] we can replace G_0 by its maximal reduced closed subscheme, and assume G_0 is reduced, hence étale. Then G_0 is the same as a finite group $G(\mathbb{F})$ with a continuous action of $\mathrm{Gal}(\mathbb{F}_q)$ [Milne 2012, XII, 2.11]. We will also write G for the group $G(\mathbb{F})$, if there is no confusion. Since $\mathrm{Spec}\,\mathbb{F} \to BG$ is surjective, we see that there is no nontrivial stratification on BG. In particular, all sheaves on BG are lisse, and they are just $\overline{\mathbb{Q}}_\ell$-representations of G.

BG is quasi-finite and proper over $\mathbb{F}$, with finite diagonal, so by [Olsson 2008a, 5.8], $H_c^r(BG, \mathscr{F}) = 0$ for all $r \neq 0$. From [ibid., 5.1], we see that if $\mathscr{F}$ is a sheaf on BG corresponding to the representation V of G, then $H_c^0(BG, \mathscr{F}) = V_G$ and $H^0(BG, \mathscr{F}) = V^G$, and there is a natural isomorphism

$$v \mapsto \sum_{g \in G} gv : V_G \longrightarrow V^G.$$

Therefore

$$h_c^0(BG, \mathscr{F}) = \dim V_G \leq \dim V = \mathrm{rank}(\mathscr{F}),$$

and the weights of V_G form a subset of the weights of V (counted with multiplicities). $\square$

4.7. (i) If k is a field, by a *k-algebraic group G* we mean a smooth k-group scheme of finite type. If G is connected, then it is geometrically connected [Grothendieck and Demazure 1970, Exposé $\mathrm{VI_A}$, 2.1.1].

(ii) For a connected k-algebraic group G, let $a : BG \to \mathrm{Spec}\,k$ be the structural map. Then

$$a^* : \Lambda\text{-Sh}(\mathrm{Spec}\ k) \longrightarrow \Lambda\text{-Sh}(BG)$$

is an equivalence of categories. This is because

• BG has no nontrivial stratifications (it is covered by $\mathrm{Spec}\,k$ which has no nontrivial stratifications), and therefore

• any constructible Λ-adic sheaf on BG is lisse, given by an adic system $(M_n)_n$ of sheaves on $\mathrm{Spec}\,k$ with G-actions, and these actions are trivial since G is connected, see [Behrend 2003, 5.2.9].

(iii) Let G_0 be a connected $\mathbb{F}_q$-algebraic group. By a theorem of Lang [1956, Theorem 2], every G_0-torsor over $\operatorname{Spec}\mathbb{F}_q$ is trivial, with automorphism group G_0, therefore

$$c_v(BG_0) = \frac{1}{c_v(G_0)} = \frac{1}{\#G_0(\mathbb{F}_{q^v})}.$$

Recall the following theorem of Borel as in [Behrend 2003, 6.1.6].

Theorem 4.8. *Let k be a field and G a connected k-algebraic group. Consider the Leray spectral sequence given by the projection $f : \operatorname{Spec} k \to BG$,*

$$E_2^{rs} = H^r(BG_{\overline{k}}) \otimes H^s(G_{\overline{k}}) \Longrightarrow \overline{\mathbb{Q}}_\ell.$$

Let $N^s = E_{s+1}^{0,s} \subset H^s(G_{\overline{k}})$ be the transgressive subspaces, for $s \geq 1$, and let N be the graded $\overline{\mathbb{Q}}_\ell$-vector space $\bigoplus_{s\geq 1} N^s$. We have:

(a) *$N^s = 0$ if s is even.*

(b) *The canonical map $\bigwedge N \to H^*(G_{\overline{k}})$ is an isomorphism of graded $\overline{\mathbb{Q}}_\ell$-algebras.*

(c) *The spectral sequence above induces an epimorphism of graded $\overline{\mathbb{Q}}_\ell$-vector spaces $H^*(BG_{\overline{k}}) \twoheadrightarrow N[-1]$. Any section induces an isomorphism*

$$\operatorname{Sym}^*(N[-1]) \xrightarrow{\sim} H^*(BG_{\overline{k}}).$$

Remark 4.8.1. (i) The E_2^{rs}-term in Theorem 4.8 should be $H^r(BG_{\overline{k}}, R^s f_*\overline{\mathbb{Q}}_\ell)$, and $R^s f_*\overline{\mathbb{Q}}_\ell$ is a constructible sheaf on BG. By 4.7 (ii), the sheaf $R^s f_*\overline{\mathbb{Q}}_\ell$ is isomorphic to $a^*f^*R^s f_*\overline{\mathbb{Q}}_\ell = a^*H^s(G_{\overline{k}})$, where $a : BG \to \operatorname{Spec} k$ is the structural map and $H^s(G_{\overline{k}})$ is the $\operatorname{Gal}(k)$-module regarded as a sheaf on $\operatorname{Spec} k$. Therefore by projection formula, $E_2^{rs} = H^r(BG_{\overline{k}}) \otimes H^s(G_{\overline{k}})$.

(ii) Since the spectral sequence converges to $\overline{\mathbb{Q}}_\ell$ sitting in degree 0, all E_∞^{rs} are zero, except E_∞^{00}. For each $s \geq 1$, consider the differential map

$$d_{s+1}^{0,s} : E_{s+1}^{0,s} \longrightarrow E_{s+1}^{s+1,0}$$

on the $(s+1)$st page. This map must be injective (respectively surjective) because it is the last possibly nonzero map from $E_*^{0,s}$ (respectively into $E_*^{s+1,0}$). Therefore, it is an isomorphism. Here $N^s = E_{s+1}^{0,s}$ is a subspace of $E_2^{0,s} = H^s(G_{\overline{k}})$, and $E_{s+1}^{s+1,0}$ is a quotient of $E_2^{s+1,0} = H^{s+1}(BG_{\overline{k}})$. We get the surjection $H^{s+1}(BG_{\overline{k}}) \to N^s$ by using the isomorphism $d_{s+1}^{0,s}$.

4.8.2. Let G_0 be a connected $\mathbb{F}_q$-algebraic group of dimension d. We intend to apply Theorem 4.8 to investigate the compact support cohomology groups $H_c^*(BG)$.

We have graded Galois-invariant subspaces $N = \bigoplus_{r\geq 1} N^r \subset \bigoplus_{r\geq 0} H^r(G)$ concentrated in odd degrees, such that the induced map

$$\bigwedge N \longrightarrow H^*(G)$$

is an isomorphism, and such that $H^*(BG) \cong \mathrm{Sym}^* N[-1]$. Let $n_r = \dim N^r$, and let $v_{r1}, \dots, v_{rn_r}$ be a basis for N^r with respect to which the Frobenius acting on N^r is upper triangular

$$\begin{bmatrix} \alpha_{r1} & * & * \\ & \ddots & * \\ & & \alpha_{rn_r} \end{bmatrix}$$

with eigenvalues $\alpha_{r1}, \dots, \alpha_{rn_r}$. By Deligne [1980, 3.3.5], the weights of $H^r(G)$ are $\geq r$, so $|\alpha_{ri}| \geq q^{r/2} > 1$. We have

$$H^*(BG) = \mathrm{Sym}^* \overline{\mathbb{Q}}_\ell \langle v_{ij} \mid \text{for all } i, j \rangle = \overline{\mathbb{Q}}_\ell [v_{ij}],$$

with $\deg(v_{ij}) = i+1$. Note that all $i+1$ are even. In particular, $H^{2r-1}(BG) = 0$ and

$$\begin{aligned} H^{2r}(BG) &= \{\text{homogeneous polynomials of degree } 2r \text{ in } v_{ij}\} \\ &= \overline{\mathbb{Q}}_\ell \Big\langle \prod_{i,j} v_{ij}^{m_{ij}}\ ;\ \sum_{i,j} m_{ij}(i+1) = 2r \Big\rangle. \end{aligned}$$

With respect to an appropriate order of the basis, the matrix representing F acting on $H^{2r}(BG)$ is upper triangular, with eigenvalues

$$\prod_{i,j} \alpha_{ij}^{m_{ij}}, \qquad \text{for } \sum_{i,j} m_{ij}(i+1) = 2r.$$

By Poincaré duality, the eigenvalues of F acting on $H_c^{-2r-2d}(BG)$ are

$$q^{-d} \prod_{i,j} \alpha_{ij}^{-m_{ij}}, \quad \text{where } \sum_{i,j} m_{ij}(i+1) = 2r.$$

Here $(m_{ij})_{i,j}$ are tuples of nonnegative integers. Therefore the reciprocal characteristic polynomial of F on $H_c^{-2r-2d}(BG)$ is

$$P_{-2r-2d}(BG_0, t) \;=\; \prod_{\substack{m_{ij} \geq 0 \\ \sum_{i,j} m_{ij}(i+1) = 2r}} \Big(1 - q^{-d} \prod_{i,j} \alpha_{ij}^{-m_{ij}} \cdot t\Big).$$

In the following two lemmas we prove two key cases of Theorem 4.2 (i).

Lemma 4.9. *Let G_0 be an $\mathbb{F}_q$-group scheme of finite type. Then* Theorem 4.2 (i) *holds for the structural map $f : BG_0 \to \mathrm{Spec}\, \mathbb{F}_q$ and any convergent complex $K_0 \in W^-(BG_0, \overline{\mathbb{Q}}_\ell)$.*

Proof. By Olsson [2008a, 7.12–7.14] we may assume that G_0 is reduced (hence smooth), so that the natural projection $\mathrm{Spec}\, \mathbb{F}_q \to BG_0$ is a presentation. Note that then the assumptions of ι-mixedness and stratifiability on K_0 are verified automatically, by Proposition 2.8 and Corollary 3.6 (iii), even though we will not use them explicitly in the proof.

Let G_0^0 be the identity component of G_0 and consider the exact sequence of algebraic groups

$$1 \longrightarrow G_0^0 \longrightarrow G_0 \longrightarrow \pi_0(G_0) \longrightarrow 1.$$

The fibers of the induced map $BG_0 \to B\pi_0(G_0)$ are isomorphic to BG_0^0, so we reduce to prove two cases: G_0 is finite étale (or even a finite constant group scheme, by Remark 4.1.2 (iii)), or G_0 is connected and smooth.

Case of G_0 finite constant. Let $G_0/\mathbb{F}_q$ be the finite constant group scheme associated with a finite group G, and let $K_0 \in W^-(BG_0, \overline{\mathbb{Q}}_\ell)$. Again we denote by G both the group scheme $G_0 \otimes \mathbb{F}$ and the finite group $G_0(\mathbb{F})$, if no confusion arises. Let y be the unique point in $\operatorname{Spec} \mathbb{F}_q$,

$$\begin{array}{ccc} BG & \longrightarrow & BG_0 \\ {\scriptstyle f_{\overline{y}}}\downarrow & & \downarrow{\scriptstyle f} \\ \operatorname{Spec} \mathbb{F} & \xrightarrow{\overline{y}} & \operatorname{Spec} \mathbb{F}_q \end{array}$$

is identified with the coinvariance functor

$$(\ )_G : D_c^-(\operatorname{Rep}_{\overline{\mathbb{Q}}_\ell} G) \longrightarrow D_c^-(\overline{\mathbb{Q}}_\ell),$$

which is exact on the level of modules, since the category $\operatorname{Rep}_{\overline{\mathbb{Q}}_\ell} G$ is semisimple. So $(f_!K_0)_{\overline{y}} = (f_{\overline{y}})_!K = K_G$ and $\mathscr{H}^n(K_G) = \mathscr{H}^n(K)_G$. Therefore

$$\sum_{\mathscr{H}^n((f_{\overline{y}})_!K),F} |\alpha|^s \leq \sum_{\mathscr{H}^n(K),F} |\alpha|^s$$

for every $n \in \mathbb{Z}$ and $s > 0$. Therefore, if K_0 is a convergent complex, so is $f_!K_0$.

Case of G_0 smooth and connected. In this case

$$f^* : \overline{\mathbb{Q}}_\ell\text{-Sh}(\operatorname{Spec} \mathbb{F}_q) \longrightarrow \overline{\mathbb{Q}}_\ell\text{-Sh}(BG_0)$$

is an equivalence of categories by 4.7 (ii). Let $d = \dim G_0$, and let $\mathscr{F}_0$ be a sheaf on BG_0, corresponding to a representation V of the Weil group $W(\mathbb{F}_q)$, with $\beta_1, \dots, \beta_m$ as eigenvalues of F. By the projection formula [Laszlo and Olsson 2008b, 9.1.i] we have $H_c^n(BG, \mathscr{F}) \simeq H_c^n(BG) \otimes V$, and by 4.8.2 the eigenvalues of F on $H_c^{-2r-2d}(BG) \otimes V$ are (using the notation in 4.8.2)

$$q^{-d}\beta_k \prod_{i,j} \alpha_{ij}^{-m_{ij}},$$

for $k=1,\dots,m$ and tuples (m_{ij}) such that $\sum_{i,j} m_{ij}(i+1)=2r$. For every $s>0$,

$$\begin{aligned}\sum_{n\in\mathbb{Z}}\sum_{H_c^n(BG)\otimes V,F}|\alpha|^s &= \sum_{m_{ij},k} q^{-ds}\big|\beta_k\big|^s\prod_{i,j}\big|\alpha_{ij}^{-m_{ij}}\big|^s\\ &= \Big(\sum_{k=1}^m\big|\beta_k\big|^s\Big)\Big(\sum_{m_{ij}} q^{-ds}\prod_{i,j}\big|\alpha_{ij}\big|^{-m_{ij}s}\Big),\end{aligned}$$

which converges to

$$q^{-ds}\Big(\sum_{k=1}^m|\beta_k|^s\Big)\prod_{i,j}\frac{1}{1-|\alpha_{ij}|^{-s}},$$

since $|\alpha_{ij}|^{-s}<1$ and the product above is taken over finitely many indices.

Let K_0 be a convergent complex on BG_0, and for each $k\in\mathbb{Z}$, let V_k be a $W(\mathbb{F}_q)$-module corresponding to $\mathscr{H}^kK_0$. For every $x\in BG_0(\mathbb{F}_q)$ (for instance the trivial G_0-torsor), the pair $(\mathscr{H}^k(K)_{\overline{x}}, F_x)$ is isomorphic to (V_k, F). Consider the $W(\mathbb{F}_q)$-equivariant spectral sequence

$$H_c^r(BG,\mathscr{H}^k(K))\Longrightarrow H_c^{r+k}(BG,K).$$

We have

$$\begin{aligned}\sum_{n\in\mathbb{Z}}\sum_{H_c^n(BG,K),F}|\alpha|^s &\le \sum_{n\in\mathbb{Z}}\sum_{r+k=n}\sum_{H_c^r(BG,\mathscr{H}^kK),F}|\alpha|^s\\ &= \sum_{r,k\in\mathbb{Z}}\sum_{H_c^r(BG)\otimes V_k,F}|\alpha|^s\\ &= \sum_{k\in\mathbb{Z}}\sum_{r\in\mathbb{Z}}\sum_{H_c^r(BG)\otimes V_k,F}|\alpha|^s\\ &= \sum_{k\in\mathbb{Z}} q^{-ds}\Big(\sum_{V_k,F}|\alpha|^s\Big)\prod_{i,j}\frac{1}{1-|\alpha_{ij}|^{-s}}\\ &= \Big(\sum_{k\in\mathbb{Z}}\sum_{V_k,F}|\alpha|^s\Big)\Big(q^{-ds}\prod_{i,j}\frac{1}{1-|\alpha_{ij}|^{-s}}\Big),\end{aligned}$$

where the first factor is convergent by assumption, and the product in the second factor is taken over finitely many indices. This shows that $f_!K_0$ is a convergent complex. $\square$

Let E_λ be a finite subextension of $\overline{\mathbb{Q}}_\ell/\mathbb{Q}_\ell$ with ring of integers Λ and residue field Λ_0, and let $(\mathscr{S},\mathscr{L})$ be a pair on $\mathscr{X}$ defined by simple lcc Λ_0-sheaves on strata. A complex $K_0\in W(\mathscr{X}_0,\overline{\mathbb{Q}}_\ell)$ is said to be $(\mathscr{S},\mathscr{L})$-*stratifiable* (or *trivialized by* $(\mathscr{S},\mathscr{L})$), if K is defined over E_λ, with an integral model over Λ trivialized by $(\mathscr{S},\mathscr{L})$.

Lemma 4.10. *Let $X_0/\mathbb{F}_q$ be a geometrically connected variety, E_λ a finite subextension of $\overline{\mathbb{Q}}_\ell/\mathbb{Q}_\ell$ with ring of integers Λ, and let $\mathscr{L}$ be a finite set of simple lcc Λ_0-sheaves on X. Then* Theorem 4.2 (i) *holds for the structural morphism $f : X_0 \to \operatorname{Spec} \mathbb{F}_q$ and all lisse ι-mixed convergent complexes K_0 on X_0 that are trivialized by* $(\{X\}, \mathscr{L})$.

Proof. Let $N = \dim X_0$. From the spectral sequence

$$E_2^{rk} = H_c^r(X, \mathscr{H}^k K) \Longrightarrow H_c^{r+k}(X, K)$$

we see that

$$\sum_{n\in\mathbb{Z}} \sum_{H_c^n(X,K),F} |\alpha|^s \le \sum_{n\in\mathbb{Z}} \sum_{r+k=n} \sum_{H_c^r(X,\mathscr{H}^k K),F} |\alpha|^s = \sum_{\substack{0\le r\le 2N\\ k\in\mathbb{Z}}} \sum_{H_c^r(X,\mathscr{H}^k K),F} |\alpha|^s,$$

therefore it suffices to show that the series $\sum_{k\in\mathbb{Z}} \sum_{H_c^r(X,\mathscr{H}^k K),F} |\alpha|^s$ converges for each $0 \le r \le 2N$.

Let d be the number in Lemma 3.10 for $\mathscr{L}$. Each cohomology sheaf $\mathscr{H}^n K_0$ is ι-mixed and lisse on X_0, so by Theorem 2.6 (i) we have the decomposition

$$\mathscr{H}^n K_0 = \bigoplus_{b\in\mathbb{R}/\mathbb{Z}} (\mathscr{H}^n K_0)(b)$$

according to the weights mod $\mathbb{Z}$, defined over E_λ, see Remark 2.6.1 (ii). For each coset b, we choose a representative $b_0 \in b$, and take a $b_1 \in \overline{\mathbb{Q}}_\ell^*$ such that $w_q(b_1) = -b_0$. Then the sheaf $(\mathscr{H}^n K_0)(b)^{(b_1)}$ deduced by twist is lisse with integer punctual weights. Let W be the filtration by punctual weights Theorem 2.6 (ii) of $(\mathscr{H}^n K_0)(b)^{(b_1)}$. For every $v \ge 1$ and $x \in X_0(\mathbb{F}_{q^v})$, and every real $s > 0$, we have

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} \sum_{(\mathscr{H}^n K_0)_{\bar{x}},F_x} |\alpha|^{s/v} &= \sum_{\substack{n\in\mathbb{Z}\\ b\in\mathbb{R}/\mathbb{Z}}} \sum_{(\mathscr{H}^n K_0)(b)_{\bar{x}},F_x} |\alpha|^{s/v} \\
&= \sum_{\substack{n\in\mathbb{Z}\\ b\in\mathbb{R}/\mathbb{Z}}} \sum_{(\mathscr{H}^n K_0)(b)_{\bar{x}}^{(b_1)},F_x} |\alpha/b_1^v|^{s/v} \\
&= \sum_{\substack{n\in\mathbb{Z}\\ b\in\mathbb{R}/\mathbb{Z}}} q^{b_0 s/2} \sum_{(\mathscr{H}^n K_0)(b)_{\bar{x}}^{(b_1)},F_x} |\alpha|^{s/v} \\
&= \sum_{\substack{n\in\mathbb{Z}\\ b\in\mathbb{R}/\mathbb{Z}}} q^{b_0 s/2} \sum_{i\in\mathbb{Z}} \sum_{\mathrm{Gr}_i^W((\mathscr{H}^n K_0)(b)^{(b_1)})_{\bar{x}},F_x} |\alpha|^{s/v} \\
&= \sum_{\substack{n\in\mathbb{Z}\\ b\in\mathbb{R}/\mathbb{Z}}} q^{b_0 s/2} \sum_{i\in\mathbb{Z}} q^{is/2} \cdot \operatorname{rank} \mathrm{Gr}_i^W(\mathscr{H}^n K_0)(b)^{(b_1)}.
\end{aligned}$$

Since K_0 is a convergent complex, this series is convergent.

For each $n \in \mathbb{Z}$, every direct summand $(\mathscr{H}^n K_0)(b)$ of $\mathscr{H}^n K_0$ is trivialized by $(\{X\}, \mathscr{L})$. The filtration W of each $(\mathscr{H}^n K_0)(b)^{(b_1)}$ gives a filtration of $(\mathscr{H}^n K_0)(b)$ (also denoted W) by twisting back, and it is clear that this latter filtration is defined over E_λ. We have $\mathrm{Gr}_i^W((\mathscr{H}^n K_0)(b)^{(b_1)}) = (\mathrm{Gr}_i^W(\mathscr{H}^n K_0)(b))^{(b_1)}$, and each $\mathrm{Gr}_i^W(\mathscr{H}^n K_0)(b)$ is trivialized by $(\{X\}, \mathscr{L})$. By Lemma 3.10,

$$\begin{aligned} h_c^r\big(X, \mathrm{Gr}_i^W\big((\mathscr{H}^n K)(b)^{(b_1)}\big)\big) &= h_c^r(X, \mathrm{Gr}_i^W(\mathscr{H}^n K)(b)) \\ &\le d \cdot \operatorname{rank} \mathrm{Gr}_i^W(\mathscr{H}^n K)(b) \\ &= d \cdot \operatorname{rank} \mathrm{Gr}_i^W\big((\mathscr{H}^n K)(b)^{(b_1)}\big), \end{aligned}$$

where the first equality follows from [Laszlo and Olsson 2008b, 9.1.i]. Therefore

$$\begin{aligned} \sum_{n\in\mathbb{Z}} \sum_{H_c^r(X,\mathscr{H}^n K),F} |\alpha|^s &= \sum_{\substack{n\in\mathbb{Z}\\ b\in\mathbb{R}/\mathbb{Z}}} \sum_{H_c^r(X,(\mathscr{H}^n K)(b)),F} |\alpha|^s \\ &= \sum_{\substack{n\in\mathbb{Z}\\ b\in\mathbb{R}/\mathbb{Z}}} \sum_{H_c^r(X,(\mathscr{H}^n K)(b)^{(b_1)}),F} |b_1^{-1}\alpha|^s \\ &\le \sum_{\substack{n\in\mathbb{Z}\\ b\in\mathbb{R}/\mathbb{Z}}} q^{b_0 s/2} \sum_{i\in\mathbb{Z}} \sum_{H_c^r(X,\mathrm{Gr}_i^W((\mathscr{H}^n K)(b)^{(b_1)})),F} |\alpha|^s \\ &\le \sum_{\substack{n\in\mathbb{Z}\\ b\in\mathbb{R}/\mathbb{Z}}} q^{b_0 s/2} \sum_{i\in\mathbb{Z}} q^{(i+r)s/2} \cdot h_c^r\big(X, \mathrm{Gr}_i^W\big((\mathscr{H}^n K)(b)^{(b_1)}\big)\big) \\ &\le q^{rs/2} d \sum_{\substack{n\in\mathbb{Z}\\ b\in\mathbb{R}/\mathbb{Z}}} q^{b_0 s/2} \sum_{i\in\mathbb{Z}} q^{is/2} \cdot \operatorname{rank} \mathrm{Gr}_i^W\big((\mathscr{H}^n K)(b)^{(b_1)}\big), \end{aligned}$$

and it converges. □

Now we prove Theorem 4.2 (i) in general.

Proof. We may assume all stacks involved are reduced. From Theorem 2.11 and Theorem 3.9 we know that $f_! K_0 \in W_m^{-,\mathrm{stra}}(\mathscr{Y}_0, \overline{\mathbb{Q}}_\ell)$.

Let $y \in \mathscr{Y}_0(\mathbb{F}_{q^v})$, we want to show that $((f_! K_0)_{\bar{y}}, F_y)$ is a convergent complex. Since the property of being convergent depends only on the cohomology sheaves, by base change [Laszlo and Olsson 2008b, 12.5.3] we reduce to the case when $\mathscr{Y}_0 = \operatorname{Spec} \mathbb{F}_{q^v}$. Replacing q by q^v, we may assume $v = 1$. By Remark 4.1.2 (iii) we only need to show that $(R\Gamma_c(\mathscr{X}, K), F)$ is convergent.

If $j : \mathscr{U}_0 \hookrightarrow \mathscr{X}_0$ is an open substack with complement $i : \mathscr{Z}_0 \hookrightarrow \mathscr{X}_0$, then we have an exact triangle

$$j_! j^* K_0 \longrightarrow K_0 \longrightarrow i_* i^* K_0 \longrightarrow$$

in $W^-(\mathscr{X}_0, \overline{\mathbb{Q}}_\ell)$, which induces an exact triangle

$$R\Gamma_c(\mathscr{U}_0, j^*K_0) \longrightarrow R\Gamma_c(\mathscr{X}_0, K_0) \longrightarrow R\Gamma_c(\mathscr{Z}_0, i^*K_0) \longrightarrow$$

in $W^-(\operatorname{Spec} \mathbb{F}_q, \overline{\mathbb{Q}}_\ell)$. So by Corollary 4.4 and noetherian induction, it suffices to prove Theorem 4.2 (i) for a nonempty open substack. By [Behrend 2003, 5.1.14] we may assume that the inertia stack $\mathscr{I}_0$ is flat over $\mathscr{X}_0$. Then we can form the rigidification $\pi : \mathscr{X}_0 \to X_0$ with respect to $\mathscr{I}_0$ [Olsson 2008b, §1.5], where X_0 is an $\mathbb{F}_q$-algebraic space of quasi-compact diagonal. X_0 contains an open dense subscheme [Knutson 1971, II, 6.7]. Replacing $\mathscr{X}_0$ by the inverse image of this scheme, we can assume X_0 is a scheme.

If Theorem 4.2 (i) holds for two composable morphisms f and g, then it holds for their composition $g \circ f$. Since $R\Gamma_c(\mathscr{X}_0, -) = R\Gamma_c(X_0, -) \circ \pi_!$, we reduce to proving Theorem 4.2 (i) for these two morphisms. For every $x \in X_0(\mathbb{F}_{q^v})$, the fiber of π over x is a gerbe over $\operatorname{Spec} k(x)$. Extending the base $k(x)$ (see Remark 4.1.2 (iii)) one can assume it is a neutral gerbe (in fact all gerbes over a finite field are neutral; see [Behrend 2003, 6.4.2]). This means the following diagram is 2-Cartesian:

$$\begin{tikzcd} B\,\mathrm{Aut}_x \arrow[r] \arrow[d] & \mathscr{X}_0 \arrow[d, "\pi"] \\ \operatorname{Spec} \mathbb{F}_{q^v} \arrow[r, "x"] & X_0. \end{tikzcd}$$

So we reduce to two cases: $\mathscr{X}_0 = BG_0$ for an $\mathbb{F}_q$-algebraic group G_0, or $\mathscr{X}_0 = X_0$ is an $\mathbb{F}_q$-scheme. The first case is proved in Lemma 4.9.

For the second case, given a convergent complex $K_0 \in W_m^{-,\mathrm{stra}}(X_0, \overline{\mathbb{Q}}_\ell)$, defined over some E_λ with ring of integers Λ, and trivialized by a pair $(\mathscr{S}, \mathscr{L})$ ($\mathscr{L}$ being defined over Λ_0) on X, we can refine this pair so that every stratum is connected, and replace X_0 by models of the strata defined over some finite extension of $\mathbb{F}_q$ Remark 4.1.2 (iii). This case is proved in Lemma 4.10. ☐

5. Trace formula for stacks

We prove two special cases of Theorem 4.2 (ii) in the following two propositions.

Proposition 5.1. *Let G_0 be a finite étale group scheme over $\mathbb{F}_q$, and $\mathscr{F}_0$ a sheaf on BG_0. Then*

$$c_1(BG_0, \mathscr{F}_0) = c_1(\operatorname{Spec} \mathbb{F}_q, R\Gamma_c(BG_0, \mathscr{F}_0)).$$

Proof. This is a special case of [Olsson 2008a, 8.6] on correspondences given by group homomorphisms. ☐

Proposition 5.2. *Let G_0 be a connected $\mathbb{F}_q$-algebraic group, and let $\mathscr{F}_0$ be a sheaf on BG_0. Then*

$$c_1(BG_0, \mathscr{F}_0) = c_1(\operatorname{Spec} \mathbb{F}_q, R\Gamma_c(BG_0, \mathscr{F}_0)).$$

Proof. Let $f : BG_0 \to \operatorname{Spec} \mathbb{F}_q$ be the structural map and $d = \dim G_0$. By 4.7 (ii), the sheaf $\mathscr{F}_0$ on BG_0 takes the form f^*V, for some sheaf V on $\operatorname{Spec} \mathbb{F}_q$. By 4.7 (iii), we have

$$c_1(BG_0, \mathscr{F}_0) = \frac{1}{\#G_0(\mathbb{F}_q)} \operatorname{Tr}(F_x, \mathscr{F}_{\overline{x}}) = \frac{\operatorname{Tr}(F, V)}{\#G_0(\mathbb{F}_q)}.$$

By the projection formula we have $H^n_c(BG, \mathscr{F}) \simeq H^n_c(BG) \otimes V$, so

$$\operatorname{Tr}(F, H^n_c(BG, \mathscr{F})) = \operatorname{Tr}(F, H^n_c(BG)) \cdot \operatorname{Tr}(F, V).$$

Then

$$\begin{aligned} c_1(\operatorname{Spec} \mathbb{F}_q, R\Gamma_c(BG_0, \mathscr{F}_0)) &= \sum_n (-1)^n \operatorname{Tr}(F, H^n_c(BG, \mathscr{F})) \\ &= \operatorname{Tr}(F, V) \sum_n (-1)^n \operatorname{Tr}(F, H^n_c(BG)), \end{aligned}$$

so we can assume $\mathscr{F}_0 = \overline{\mathbb{Q}}_\ell$. Using the notations in 4.8.2 we have

$$\begin{aligned} \sum_n (-1)^n \operatorname{Tr}(F, H^n_c(BG)) &= \sum_{r \geq 0} \operatorname{Tr}(F, H_c^{-2r-2d}(BG)) \\ &= \sum_{r \geq 0} \sum_{\substack{\sum m_{ij}(i+1) = 2r \\ m_{ij} \geq 0}} q^{-d} \prod_{i,j} \alpha_{ij}^{-m_{ij}} = q^{-d} \sum_{m_{ij} \geq 0} \prod_{i,j} \alpha_{ij}^{-m_{ij}} \\ &= q^{-d} \prod_{i,j} \big(1 + \alpha_{ij}^{-1} + \alpha_{ij}^{-2} + \cdots\big) = q^{-d} \prod_{i,j} \frac{1}{1 - \alpha_{ij}^{-1}}. \end{aligned}$$

It remains to show that

$$\#G_0(\mathbb{F}_q) = q^d \prod_{i,j} \big(1 - \alpha_{ij}^{-1}\big).$$

In 4.8.2, we saw that if each N^i has an ordered basis $v_{i1}, \ldots, v_{in_i}$ with respect to which F is upper triangular, then since $H^*(G) = \bigwedge N$, $H^i(G)$ has a basis

$$v_{i_1 j_1} \wedge v_{i_2 j_2} \wedge \cdots \wedge v_{i_m j_m},$$

such that $\sum_{r=1}^m i_r = i$, $i_r \leq i_{r+1}$, and if $i_r = i_{r+1}$, then $j_r < j_{r+1}$. The eigenvalues of F on $H^i(G)$ are $\alpha_{i_1 j_1} \cdots \alpha_{i_m j_m}$ for such indices. By Poincaré duality, the eigenvalues of F on $H_c^{2d-i}(G)$ are $q^d(\alpha_{i_1 j_1} \cdots \alpha_{i_m j_m})^{-1}$. Note that all the i_r are odd, so

$$2d - i \equiv i = \sum_{r=1}^m i_r \equiv m \mod 2.$$

Applying the classical trace formula to G_0, we have

$$\#G_0(\mathbb{F}_q)=\sum(-1)^m q^d \alpha_{i_1j_1}^{-1}\cdots\alpha_{i_mj_m}^{-1}=q^d\prod_{i,j}\bigl(1-\alpha_{ij}^{-1}\bigr).$$

This finishes the proof. $\square$

5.2.1. Note that in Propositions 5.1 and 5.2 we did not make explicit use of the fact that $\mathscr{F}_0$ is ι-mixed.

Now we prove Theorem 4.2 (ii) in general.

Proof. Since $c_v(\mathscr{X}_0, K_0) = c_1(\mathscr{X}_0 \otimes \mathbb{F}_{q^v}, K_0 \otimes \mathbb{F}_{q^v})$, we can assume $v = 1$. We shall reduce to proving Theorem 4.2 (ii) for all fibers of f over $\mathbb{F}_q$-points of $\mathscr{Y}_0$, following the approach of Behrend [2003, 6.4.9].

Let $y \in \mathscr{Y}_0(\mathbb{F}_q)$ and $(\mathscr{X}_0)_y$ be the fiber over y. Then $(\mathscr{X}_0)_y(\mathbb{F}_q)$ is the groupoid of pairs (x, α), where $x \in \mathscr{X}_0(\mathbb{F}_q)$ and $\alpha : f(x) \to y$ is an isomorphism in $\mathscr{Y}_0(\mathbb{F}_q)$. Suppose $(\mathscr{X}_0)_y(\mathbb{F}_q) \neq \varnothing$, and fix an $x \in (\mathscr{X}_0)_y(\mathbb{F}_q)$. Then $\mathrm{Isom}(f(x), y)(\mathbb{F}_q)$ is a trivial left $\mathrm{Aut}_y(\mathbb{F}_q)$-torsor. There is also a natural right action of $\mathrm{Aut}_x(\mathbb{F}_q)$ on $\mathrm{Isom}(f(x), y)(\mathbb{F}_q)$, namely $\varphi \in \mathrm{Aut}_x(\mathbb{F}_q)$ takes α to $\alpha \circ f(\varphi)$. Under this action, for α and α' to be in the same orbit, there should be a $\varphi \in \mathrm{Aut}_x(\mathbb{F}_q)$ such that the diagram

$$\begin{tikzcd} f(x) \arrow[rr, "f(\varphi)"] \arrow[dr, "\alpha'"'] & & f(x) \arrow[dl, "\alpha"] \\ & y & \end{tikzcd}$$

commutes; by definition this means (x, α) is isomorphic to (x, α') in $(\mathscr{X}_0)_y(\mathbb{F}_q)$. So the set of orbits $\mathrm{Isom}(f(x), y)(\mathbb{F}_q)/\mathrm{Aut}_x(\mathbb{F}_q)$ is identified with the inverse image of the class of x along the map $[(\mathscr{X}_0)_y(\mathbb{F}_q)] \to [\mathscr{X}_0(\mathbb{F}_q)]$. The stabilizer group of $\alpha \in \mathrm{Isom}(f(x), y)(\mathbb{F}_q)$ is $\mathrm{Aut}_{(x,\alpha)}(\mathbb{F}_q)$, the automorphism group of (x, α) in $(\mathscr{X}_0)_y(\mathbb{F}_q)$.

In general, if a finite group G acts on a finite set S, then we have

$$\sum_{[x]\in S/G}\frac{\#G}{\#\mathrm{Stab}_G(x)}=\sum_{[x]\in S/G}\#\mathrm{Orb}_G(x)=\#S.$$

Now for $S = \mathrm{Isom}(f(x), y)(\mathbb{F}_q)$ and $G = \mathrm{Aut}_x(\mathbb{F}_q)$, we have

$$\sum_{\substack{(x,\alpha)\in[(\mathscr{X}_0)_y(\mathbb{F}_q)]\\(x,\alpha)\mapsto x}}\frac{\#\mathrm{Aut}_x(\mathbb{F}_q)}{\#\mathrm{Aut}_{(x,\alpha)}(\mathbb{F}_q)}=\#\mathrm{Isom}(f(x),y)(\mathbb{F}_q)=\#\mathrm{Aut}_y(\mathbb{F}_q);$$

the last equality follows from the fact that S is a trivial $\mathrm{Aut}_y(\mathbb{F}_q)$-torsor.

If we assume Theorem 4.2 (ii) holds for the fibers $f_y : (\mathscr{X}_0)_y \to \operatorname{Spec}\mathbb{F}_q$ of f, for all $y \in \mathscr{Y}_0(\mathbb{F}_q)$, then

$$\begin{aligned}
c_1(\mathscr{Y}_0, f_!K_0) &= \sum_{y\in[\mathscr{Y}_0(\mathbb{F}_q)]} \frac{\operatorname{Tr}(F_y, (f_!K)_{\bar{y}})}{\#\operatorname{Aut}_y(\mathbb{F}_q)} \\
&= \sum_{y\in[\mathscr{Y}_0(\mathbb{F}_q)]} \frac{\operatorname{Tr}(F_y, (f_{y!}K)_{\bar{y}})}{\#\operatorname{Aut}_y(\mathbb{F}_q)} \\
&= \sum_{y\in[\mathscr{Y}_0(\mathbb{F}_q)]} \frac{1}{\#\operatorname{Aut}_y(\mathbb{F}_q)} \sum_{(x,\alpha)\in[(\mathscr{X}_0)_y(\mathbb{F}_q)]} \frac{\operatorname{Tr}(F_x, K_{\bar{x}})}{\#\operatorname{Aut}_{(x,\alpha)}(\mathbb{F}_q)} \\
&= \sum_{y\in[\mathscr{Y}_0(\mathbb{F}_q)]} \frac{1}{\#\operatorname{Aut}_y(\mathbb{F}_q)} \sum_{\substack{x\in[\mathscr{X}_0(\mathbb{F}_q)]\\ x\mapsto y}} \Bigg(\sum_{\substack{(x,\alpha)\in[(\mathscr{X}_0)_y(\mathbb{F}_q)]\\ (x,\alpha)\mapsto x}} \frac{\operatorname{Tr}(F_x, K_{\bar{x}})}{\#\operatorname{Aut}_{(x,\alpha)}(\mathbb{F}_q)} \Bigg) \\
&= \sum_{y\in[\mathscr{Y}_0(\mathbb{F}_q)]} \frac{1}{\#\operatorname{Aut}_y(\mathbb{F}_q)} \sum_{\substack{x\in[\mathscr{X}_0(\mathbb{F}_q)]\\ x\mapsto y}} \frac{1}{\#\operatorname{Aut}_x(\mathbb{F}_q)} \\
&\qquad\qquad \Bigg(\sum_{\substack{(x,\alpha)\in[(\mathscr{X}_0)_y(\mathbb{F}_q)]\\ (x,\alpha)\mapsto x}} \frac{\#\operatorname{Aut}_x(\mathbb{F}_q)}{\#\operatorname{Aut}_{(x,\alpha)}(\mathbb{F}_q)} \Bigg) \operatorname{Tr}(F_x, K_{\bar{x}}) \\
&= \sum_{y\in[\mathscr{Y}_0(\mathbb{F}_q)]} \frac{1}{\#\operatorname{Aut}_y(\mathbb{F}_q)} \sum_{\substack{x\in[\mathscr{X}_0(\mathbb{F}_q)]\\ x\mapsto y}} \frac{\operatorname{Tr}(F_x, K_{\bar{x}})}{\#\operatorname{Aut}_x(\mathbb{F}_q)} \#\operatorname{Aut}_y(\mathbb{F}_q) \\
&= \sum_{x\in[\mathscr{X}_0(\mathbb{F}_q)]} \frac{\operatorname{Tr}(F_x, K_{\bar{x}})}{\#\operatorname{Aut}_x(\mathbb{F}_q)} =: c_1(\mathscr{X}_0, K_0).
\end{aligned}$$

Here the second equality follows from [Laszlo and Olsson 2008b, 12.5.3]. Thus we reduce to the case when $\mathscr{Y}_0 = \operatorname{Spec}\mathbb{F}_q$.

If $K_0' \to K_0 \to K_0'' \to K_0'[1]$ is an exact triangle of convergent complexes in $W_m^{-,\mathrm{stra}}(\mathscr{X}_0, \overline{\mathbb{Q}}_\ell)$, then by Corollary 4.4 and Theorem 4.2 (i) we have

$$\begin{aligned}
c_1(\mathscr{X}_0, K_0) &= c_1(\mathscr{X}_0, K_0') + c_1(\mathscr{X}_0, K_0'') \quad \text{and} \\
c_1(\mathscr{Y}_0, f_!K_0) &= c_1(\mathscr{Y}_0, f_!K_0') + c_1(\mathscr{Y}_0, f_!K_0'').
\end{aligned}$$

If $j : \mathscr{U}_0 \to \mathscr{X}_0$ is an open substack with complement $i : \mathscr{Z}_0 \to \mathscr{X}_0$, then

$$c_1(\mathscr{X}_0, j_!j^*K_0) = c_1(\mathscr{U}_0, j^*K_0) \quad \text{and} \quad c_1(\mathscr{X}_0, i_*i^*K_0) = c_1(\mathscr{Z}_0, i^*K_0).$$

By noetherian induction we can shrink $\mathscr{X}_0$ to a nonempty open substack. So we may assume the inertia stack $\mathscr{I}_0$ is flat over $\mathscr{X}_0$, with rigidification $\pi : \mathscr{X}_0 \to X_0$, where X_0 is a scheme. If Theorem 4.2 (ii) holds for two composable morphisms f and g, then it holds for $g \circ f$. So we reduce to two cases as before: $\mathscr{X}_0 = X_0$

is a scheme, or $\mathscr{X}_0 = BG_0$, where G_0 is either a connected algebraic group, or a finite étale algebraic group over $\mathbb{F}_q$. We may assume X_0 is separated, by further shrinking (for instance to an affine open subscheme).

For a complex of sheaves K_0 and an integer n, we have an exact triangle

$$\tau_{<n}K_0 \longrightarrow \tau_{<n+1}K_0 \longrightarrow \mathscr{H}^n(K_0)[-n] \longrightarrow,$$

so

$$\begin{aligned} c_1(\tau_{<n+1}K_0) &= c_1(\tau_{<n}K_0) + c_1(\mathscr{H}^n(K_0)[-n]) \\ &= c_1(\tau_{<n}K_0) + (-1)^n c_1(\mathscr{H}^n(K_0)). \end{aligned}$$

Since K_0 is bounded above, $\tau_{<N}K_0 \simeq K_0$ for $N \gg 0$. Since K_0 is convergent, $c_1(\tau_{<n}K_0) \to 0$ absolutely as $n \to -\infty$, so the series $\sum_{n\in\mathbb{Z}}(-1)^n c_1(\mathscr{H}^n(K_0))$ converges absolutely to $c_1(K_0)$.

Applying $R\Gamma_c$ we get an exact triangle

$$R\Gamma_c(\mathscr{X}_0, \tau_{<n}K_0) \longrightarrow R\Gamma_c(\mathscr{X}_0, \tau_{<n+1}K_0) \longrightarrow R\Gamma_c(\mathscr{X}_0, \mathscr{H}^n K_0)[-n] \longrightarrow$$

in $W^-(\operatorname{Spec}\mathbb{F}_q, \overline{\mathbb{Q}}_\ell)$. We claim that, for $\mathscr{X}_0 = X_0$ a scheme, or BG_0, we have

$$\lim_{n\to-\infty} c_1(\operatorname{Spec}\mathbb{F}_q, R\Gamma_c(\mathscr{X}_0, \tau_{<n}K_0)) = 0$$

absolutely. Recall that $c_1(R\Gamma_c(\tau_{<n}K_0)) = \sum_{i\in\mathbb{Z}}(-1)^i\iota\operatorname{Tr}(F, H^i_c(\mathscr{X}, \tau_{<n}K))$, so we need to show that

$$\sum_{i\in\mathbb{Z}} \sum_{H^i_c(\mathscr{X},\tau_{<n}K),F} |\alpha| \to 0 \quad \text{as } n \to -\infty.$$

From the spectral sequence

$$H^r_c(\mathscr{X}, \mathscr{H}^k\tau_{<n}K) \Longrightarrow H^{r+k}_c(\mathscr{X}, \tau_{<n}K)$$

we see that

$$\sum_{i\in\mathbb{Z}} \sum_{H^i_c(\mathscr{X},\tau_{<n}K),F} |\alpha| \le \sum_{i\in\mathbb{Z}} \sum_{r+k=i} \sum_{H^r_c(\mathscr{X},\mathscr{H}^k\tau_{<n}K),F} |\alpha| = \sum_{i\in\mathbb{Z}} \sum_{\substack{r+k=i \\ k<n}} \sum_{H^r_c(\mathscr{X},\mathscr{H}^k K),F} |\alpha|.$$

Let $d = \dim\mathscr{X}_0$ (see 9.1). In the cases where $\mathscr{X}_0$ is a scheme or BG_0, we have $H^r_c(\mathscr{X}, \mathscr{F}) = 0$ for every sheaf $\mathscr{F}$ unless $r \le 2d$ (see 4.8.2 and Lemma 4.6). Therefore

$$\sum_{i\in\mathbb{Z}} \sum_{\substack{r+k=i \\ k<n}} \sum_{H^r_c(\mathscr{X},\mathscr{H}^k K),F} |\alpha| \;\le\; \sum_{i<n+2d} \sum_{r+k=i} \sum_{H^r_c(\mathscr{X},\mathscr{H}^k K),F} |\alpha|,$$

and it suffices to show that the series

$$\sum_{i\in\mathbb{Z}} \sum_{r+k=i} \sum_{H^r_c(\mathscr{X},\mathscr{H}^k K),F} |\alpha|$$

converges. We already proved this for BG_0 in Lemma 4.9, and for schemes X_0 in Lemma 4.10 (we may shrink X_0 so that the assumption in Lemma 4.10 is satisfied).

Note that in the two cases $\mathscr{X}_0 = X_0$ or BG_0, Theorem 4.2 (ii) holds when K_0 is a sheaf concentrated in degree 0. For separated schemes X_0, this is a classical result of Grothendieck [1965] and Verdier [1967]; for BG_0, this is done in Propositions 5.1 and 5.2. Therefore, for a general convergent complex K_0, we have

$$\begin{aligned} c_1(R\Gamma_c(\tau_{<n+1}K_0)) &= c_1(R\Gamma_c(\tau_{<n}K_0)) + c_1(R\Gamma_c(\mathscr{H}^n K_0)[-n]) \\ &= c_1(R\Gamma_c(\tau_{<n}K_0)) + (-1)^n c_1(\mathscr{H}^n K_0), \end{aligned}$$

and so

$$c_1(R\Gamma_c(K_0)) = \sum_{n\in\mathbb{Z}} (-1)^n c_1(\mathscr{H}^n K_0) + \lim_{n\to-\infty} c_1(R\Gamma_c(\tau_{<n}K_0)) = c_1(K_0). \qquad \square$$

Corollary 5.3. *Let $f : \mathscr{X}_0 \to \mathscr{Y}_0$ be a morphism of $\mathbb{F}_q$-algebraic stacks, and let $K_0 \in W_m^{-,\mathrm{stra}}(\mathscr{X}_0, \overline{\mathbb{Q}}_\ell)$ be a convergent complex of sheaves. Then*

$$L(\mathscr{X}_0, K_0, t) = L(\mathscr{Y}_0, f_! K_0, t).$$

6. Infinite products

For a convergent complex K_0 on $\mathscr{X}_0$, the series $\sum_{v\geq 1} c_v(K_0)t^v/v$ (and hence the L-series $L(\mathscr{X}_0, K_0, t)$) usually has a finite radius of convergence. For instance, we have the following lemma.

Lemma 6.1. *Let $X_0/\mathbb{F}_q$ be a variety of dimension d. Then the radius of convergence of $\sum_{v\geq 1} c_v(X_0)t^v/v$ is $1/q^d$.*

Proof. Let $f_{X_0}(t) = \sum_{v\geq 1} c_v(X_0)t^v/v$. Let Y_0 be an irreducible component of X_0 with complement U_0. Then $c_v(X_0) = c_v(Y_0) + c_v(U_0)$, and since all the c_v-terms are nonnegative, we see that the radius of convergence of $f_{X_0}(t)$ is the minimum of that of $f_{Y_0}(t)$ and that of $f_{U_0}(t)$. Since $\max\{\dim(Y_0), \dim(U_0)\} = d$, and U_0 has fewer irreducible component than X_0, by induction we can assume X_0 is irreducible.

Then there exists an open dense subscheme $U_0 \subset X_0$ that is smooth over $\operatorname{Spec}\mathbb{F}_q$. Let $Z_0 = X_0 - U_0$, then $\dim(Z_0) < \dim(X_0) = d$. From the cohomology sequence

$$H_c^{2d-1}(Z) \longrightarrow H_c^{2d}(U) \longrightarrow H_c^{2d}(X) \longrightarrow H_c^{2d}(Z)$$

we see that $H_c^{2d}(X) = H_c^{2d}(U) = \overline{\mathbb{Q}}_\ell(-d)$. The Frobenius eigenvalues $\{\alpha_{ij}\}_j$ on $H_c^i(X)$ have ι-weights $\leq i$, for $0 \leq i < 2d$ [Deligne 1980, 3.3.4]. By the fixed point formula,

$$\frac{c_v(X_0)}{c_{v+1}(X_0)}=\frac{q^{vd}+\sum_{0\le i<2d}(-1)^i\sum_j\alpha_{ij}^v}{q^{(v+1)d}+\sum_{0\le i<2d}(-1)^i\sum_j\alpha_{ij}^{v+1}}$$
$$=\frac{\frac{1}{q^d}+\frac{1}{q^d}\sum_{0\le i<2d}(-1)^i\sum_j\left(\frac{\alpha_{ij}}{q^d}\right)^v}{1+\sum_{0\le i<2d}(-1)^i\sum_j\left(\frac{\alpha_{ij}}{q^d}\right)^{v+1}},$$

which converges to $1/q^d$ as $v\to\infty$, therefore the radius of convergence of $f_{X_0}(t)$ is

$$\lim_{v\to\infty}\frac{c_v(X_0)/v}{c_{v+1}(X_0)/(v+1)}=\frac{1}{q^d}.\qquad\square$$

In order to prove the meromorphic continuation of Theorem 8.1, we want to express the L-series as a possibly infinite product. For schemes, if we consider only bounded complexes, the L-series can be expressed as a finite alternating product of polynomials $P_n(X_0, K_0, t)$, so it is rational [Grothendieck 1965]. In the stack case, even for the sheaf $\overline{\mathbb{Q}}_\ell$, there might be infinitely many nonzero compact cohomology groups, and we need to consider the issue of convergence of the coefficients in an infinite products.

Definition 6.2. *Let $f_n(t)=\sum_{k\ge 0}a_{nk}t^k\in\mathbb{C}[[t]]$ be a sequence of power series over $\mathbb{C}$. The sequence is said to be* convergent term by term, *if for each k, the sequence $(a_{nk})_n$ converges, and the series*

$$\lim_{n\to\infty}f_n(t):=\sum_{k\ge 0}t^k\lim_{n\to\infty}a_{nk}$$

is called the limit of the sequence $(f_n(t))_n$.

6.2.1. Strictly speaking, a series (respectively infinite product) is defined to be a sequence $(a_n)_n$, usually written as an "infinite sum" (respectively "infinite product") so that $(a_n)_n$ is the sequence of finite partial sums (respectively finite partial products) of it. So the definition above applies to series and infinite products.

Recall that $\log(1+g)=\sum_{m\ge 1}(-1)^{m+1}g^m/m$ for $g\in t\mathbb{C}[[t]]$.

Lemma 6.3. (i) *Let $f_n(t)=1+\sum_{k\ge 1}a_{nk}t^k\in\mathbb{C}[[t]]$ be a sequence of power series. Then $(f_n(t))_n$ is convergent term by term if and only if* $(\log f_n(t))_n$ *is convergent term by term, and*

$$\lim_{n\to\infty}\log f_n(t)=\log\lim_{n\to\infty}f_n(t).$$

(ii) *Let f and g be two power series with constant term* 1. *Then*

$$\log(fg)=\log(f)+\log(g).$$

(iii) *Let* $f_n(t) \in 1 + t\mathbb{C}[[t]]$ *be a sequence as in* (i). *Then the infinite product* $\prod_{n\geq 1} f_n(t)$ *converges term by term if and only if the series* $\sum_{n\geq 1} \log f_n(t)$ *converges term by term, and*

$$\sum_{n\geq 1} \log f_n(t) = \log \prod_{n\geq 1} f_n(t).$$

Proof. (i) We have

$$\begin{aligned}\log f_n(t) &= \sum_{m\geq 1} (-1)^{m+1} \Big(\sum_{k\geq 1} a_{nk} t^k\Big)^m / m \\ &= t \cdot a_{n1} + t^2\big(a_{n2} - \tfrac{1}{2}a_{n1}^2\big) + t^3\big(a_{n3} - a_{n1}a_{n2} + \tfrac{1}{3}a_{n1}^3\big) \\ &\qquad + t^4\big(a_{n4} - a_{n1}a_{n3} - \tfrac{1}{2}a_{n2}^2 + a_{n1}^2 a_{n2}\big) + \cdots =: \sum_{k\geq 1} A_{nk} t^k.\end{aligned}$$

In particular, for each k, the function $A_{nk} - a_{nk} = h(a_{n1}, \ldots, a_{n,k-1})$ is a polynomial in $a_{n1}, \ldots, a_{n,k-1}$ with rational coefficients. So if $(a_{nk})_n$ converges for each k, then $(A_{nk})_n$ also converges, and by induction the converse also holds. If we have $\lim_{n\to\infty} a_{nk} = a_k$, then $\lim_{n\to\infty} A_{nk} = a_k + h(a_1, \ldots, a_{k-1})$, and

$$\log \lim_{n\to\infty} f_n(t) = \log\Big(1 + \sum_{k\geq 1} a_k t^k\Big) = \sum_{k\geq 1} (a_k + h(a_1, \ldots, a_{k-1}))t^k = \lim_{n\to\infty} \log f_n(t).$$

(ii) log and exp are inverse to each other on power series, so it suffices to prove that for f and $g \in t\mathbb{C}[[t]]$, we have

$$\exp(f+g) = \exp(f)\exp(g).$$

This follows from the binomial formula:

$$\begin{aligned}\exp(f+g) &= \sum_{n\geq 0} (f+g)^n/n! = \sum_{n\geq 0} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} f^k g^{n-k} = \sum_{n\geq 0} \sum_{k=0}^{n} \frac{f^k}{k!} \cdot \frac{g^{n-k}}{(n-k)!} \\ &= \sum_{i,j\geq 0} \frac{f^i}{i!} \cdot \frac{g^j}{j!} = \Big(\sum_{i\geq 0} f^i/i!\Big)\Big(\sum_{j\geq 0} g^j/j!\Big) = \exp(f)\exp(g).\end{aligned}$$

(iii) Let $F_N(t) = \prod_{n=1}^{N} f_n(t)$. Applying (i) to the sequence $(F_N(t))_N$, we see that the infinite product $\prod_{n\geq 1} f_n(t)$ converges term by term if and only if (by definition) $(F_N(t))_N$ converges term by term, if and only if the sequence $(\log F_N(t))_N$ converges term by term, if and only if (by definition) the series $\sum_{n\geq 1} \log f_n(t)$ converges term by term, since by (ii)

$$\log \prod_{n=1}^{N} f_n(t) = \sum_{n=1}^{N} \log f_n(t)$$

Also

$$\log\prod_{n\geq 1}f_n(t)=\log\lim_{N\to\infty}F_N(t)=\lim_{N\to\infty}\log F_N(t)$$
$$=\lim_{N\to\infty}\sum_{n=1}^{N}\log f_n(t)=:\sum_{n\geq 1}\log f_n(t).\qquad\square$$

6.4. For a complex of sheaves K_0 on $\mathscr{X}_0$ and $n\in\mathbb{Z}$, define

$$P_n(\mathscr{X}_0,K_0,t):=\det(1-Ft,H^n_c(\mathscr{X},K)).$$

We regard $P_n(\mathscr{X}_0,K_0,t)^{\pm 1}$ as a complex power series with constant term 1 via ι.

Proposition 6.5. *For every convergent complex of sheaves $K_0\in W_m^{-,\mathrm{stra}}(\mathscr{X}_0,\overline{\mathbb{Q}}_\ell)$, the infinite product*

$$\prod_{n\in\mathbb{Z}}P_n(\mathscr{X}_0,K_0,t)^{(-1)^{n+1}}$$

is convergent term by term to the L-series $L(\mathscr{X}_0,K_0,t)$.

Proof. The complex $R\Gamma_c(\mathscr{X},K)$ is bounded above, so $P_n(\mathscr{X}_0,K_0,t)=1$ for $n\gg 0$, and the infinite product is a one-direction limit, namely $n\to-\infty$.

Let $\alpha_{n1},\dots,\alpha_{nm_n}$ be the eigenvalues (counted with multiplicity) of the morphism F on $H^n_c(\mathscr{X},K)$, regarded as complex numbers via ι, so that

$$P_n(t)=P_n(\mathscr{X}_0,K_0,t)=(1-\alpha_{n1}t)\cdots(1-\alpha_{nm_n}t).$$

By Lemma 6.3 (iii) it suffices to show that the series

$$\sum_{n\in\mathbb{Z}}(-1)^{n+1}\log P_n(t)$$

converges term by term to $\sum_{v\geq 1}c_v(K_0)t^v/v$.

We have

$$\sum_{n\in\mathbb{Z}}(-1)^{n+1}\log P_n(t)=\sum_{n\in\mathbb{Z}}(-1)^{n+1}\log\prod_i(1-\alpha_{ni}t)$$
$$=\sum_{n\in\mathbb{Z}}(-1)^n\sum_i\sum_{v\geq 1}\frac{\alpha_{ni}^v t^v}{v}=\sum_{v\geq 1}\frac{t^v}{v}\sum_{n\in\mathbb{Z}}(-1)^n\sum_i\alpha_{ni}^v=\sum_{v\geq 1}\frac{t^v}{v}c_v(R\Gamma_c(K_0)),$$

which converges term by term by Theorem 4.2 (i), and is equal to $\sum_{v\geq 1}c_v(K_0)t^v/v$ by Theorem 4.2 (ii). $\square$

Remark 6.5.1. In particular we have

$$Z(\mathscr{X}_0,t)=\prod_{n\in\mathbb{Z}}P_n(\mathscr{X}_0,t)^{(-1)^{n+1}},$$

where $P_n(\mathscr{X}_0, t) = P_n(\mathscr{X}_0, \overline{\mathbb{Q}}_\ell, t)$. This generalizes the classical result for schemes [Grothendieck 1965, 5.1]. When we want to emphasize the dependence on the prime ℓ, we will write $P_{n,\ell}(\mathscr{X}_0, t)$.

If G_0 is a connected $\mathbb{F}_q$-algebraic group, 4.8.2 shows that the zeta function of BG_0 is given by

$$Z(BG_0, t) = \prod_{r\geq 0} \prod_{\substack{m_{ij}\geq 0 \\ \sum_{i,j} m_{ij}(i+1)=2r}} \Big(1-q^{-d}\prod_{i,j}\alpha_{ij}^{-m_{ij}}\cdot t\Big)^{-1} = \prod_{m_{ij}\geq 0}\Big(1-q^{-d}\prod_{i,j}\alpha_{ij}^{-m_{ij}}\cdot t\Big)^{-1}.$$

7. Examples of zeta functions

In this section we compute the zeta functions of some stacks, and in each example we do it in two ways: by counting rational points and by computing cohomology groups. Also we investigate some analytic properties.

Example 7.1. $B\mathbb{G}_m$. By 4.7 (iii) we have $c_v(B\mathbb{G}_m) = 1/c_v(\mathbb{G}_m)$, so the zeta function is

$$Z(B\mathbb{G}_m, t) = \exp\Big(\sum_{v\geq 1} c_v(B\mathbb{G}_m)\frac{t^v}{v}\Big) = \exp\Big(\sum_{v\geq 1}\frac{1}{q^v-1}\frac{t^v}{v}\Big).$$

Using Borel's theorem 4.8 one can show (or see [Laumon and Moret-Bailly 2000, 19.3.2]) that the cohomology ring $H^*(B\mathbb{G}_m)$ is a polynomial ring $\overline{\mathbb{Q}}_\ell[T]$, generated by a variable T of degree 2, and that the Frobenius action on cohomology is given by $FT^n = q^nT^n$. So by Poincaré duality, we have

$$\begin{aligned}\mathrm{Tr}(F, H_c^{-2n-2}(B\mathbb{G}_m)) &= \mathrm{Tr}(F, H_c^{-2n-2}(B\mathbb{G}_m, \overline{\mathbb{Q}}_\ell(-1)))/q \\ &= \mathrm{Tr}(F^{-1}, H^{2n}(B\mathbb{G}_m))/q = q^{-n-1}.\end{aligned}$$

This gives

$$\prod_{n\in\mathbb{Z}} P_n(B\mathbb{G}_m, t)^{(-1)^{n+1}} = \prod_{n\geq 1}(1-q^{-n}t)^{-1}.$$

It is easy to verify the result in Remark 6.5.1 directly:

$$\begin{aligned}\exp\Big(\sum_{v\geq 1}\frac{1}{q^v-1}\frac{t^v}{v}\Big) &= \exp\Big(\sum_{v\geq 1}\frac{1/q^v}{1-1/q^v}\frac{t^v}{v}\Big) = \exp\Big(\sum_{v\geq 1}\frac{t^v}{v}\sum_{n\geq 1}\frac{1}{q^{nv}}\Big) \\ &= \prod_{n\geq 1}\exp\Big(\sum_{v\geq 1}\frac{(t/q^n)^v}{v}\Big) = \prod_{n\geq 1}(1-t/q^n)^{-1}.\end{aligned}$$

There is also a functional equation

$$Z(B\mathbb{G}_m, qt) = \frac{1}{1-t}Z(B\mathbb{G}_m, t),$$

which implies that $Z(B\mathbb{G}_m, t)$ has a meromorphic continuation to the whole complex plane, with simple poles at $t = q^n$, for $n \geq 1$.

$H_c^{-2n-2}(B\mathbb{G}_m)$ is pure of weight $-2n-2$. A natural question is if Deligne's theorem of weights [Deligne 1980, 3.3.4] still holds for algebraic stacks. Olsson told me that it does not hold in general, as the following example shows.

Example 7.2. BE, where E is an elliptic curve over $\mathbb{F}_q$. Again by 4.7 (iii) we have

$$c_v(BE) = 1/\#E(\mathbb{F}_{q^v}).$$

Let α and β be the roots of the reciprocal characteristic polynomial of the Frobenius on $H^1(E)$:

$$x^2 - (1 + q - c_1(E))x + q = 0. \tag{7.2.1}$$

Then for every $v \geq 1$, we have $c_v(E) = 1 - \alpha^v - \beta^v + q^v = (1-\alpha^v)(1-\beta^v)$. So

$$\begin{aligned} c_v(BE) = \frac{1}{(1-\alpha^v)(1-\beta^v)} &= \frac{\alpha^{-v}}{1-\alpha^{-v}} \cdot \frac{\beta^{-v}}{1-\beta^{-v}} \\ &= \Big(\sum_{n\geq 1} \alpha^{-nv}\Big)\Big(\sum_{m\geq 1} \beta^{-nv}\Big) = \sum_{n,m\geq 1} \Big(\frac{1}{\alpha^n\beta^m}\Big)^v, \end{aligned}$$

and the zeta function is

$$\begin{aligned} Z(BE, t) &= \exp\Big(\sum_{v\geq 1} c_v(BE)\frac{t^v}{v}\Big) \\ &= \exp\Big(\sum_{\substack{n,m\geq 1\\ v\geq 1}} \Big(\frac{t}{\alpha^n\beta^m}\Big)^v / v\Big) = \prod_{n,m\geq 1}\Big(1 - \frac{t}{\alpha^n\beta^m}\Big)^{-1}. \end{aligned}$$

To compute its cohomology, one can apply Borel's theorem 4.8 to E, and we have $N = N^1 = H^1(E)$, so $N[-1]$ is a 2-dimensional vector space sitting in degree 2, on which F has eigenvalues α and β. Then $H^*(BE)$ is a polynomial ring $\overline{\mathbb{Q}}_\ell[a, b]$ in two variables, both sitting in degree 2, and the basis a, b can be chosen so that the Frobenius action F on $H^2(BE)$ is upper triangular (or even diagonal)

$$\begin{bmatrix} \alpha & \gamma \\ & \beta \end{bmatrix}.$$

Then F acting on

$$H^{2n}(BE) = \mathrm{Sym}^n N[-1] = \overline{\mathbb{Q}}_\ell\langle a^n, a^{n-1}b, \ldots, b^n\rangle$$

can be represented by

$$\begin{bmatrix} \alpha^n & * & * & * \\ & \alpha^{n-1}\beta & * & * \\ & & \ddots & * \\ & & & \beta^n \end{bmatrix},$$

with eigenvalues $\alpha^n, \alpha^{n-1}\beta, \dots, \beta^n$. So the eigenvalues of F on $H_c^{-2-2n}(BE)$ are $q^{-1}\alpha^{-n}, q^{-1}\alpha^{1-n}\beta^{-1}, \dots, q^{-1}\beta^{-n}$ and

$$\prod_{n\in\mathbb{Z}} P_n(BE,t)^{(-1)^{n+1}} = 1\big/(1-q^{-1}t)\cdot\big[(1-q^{-1}\alpha^{-1}t)(1-q^{-1}\beta^{-1}t)\big] \cdot\big[(1-q^{-1}\alpha^{-2}t)(1-q^{-1}\alpha^{-1}\beta^{-1}t)(1-q^{-1}\beta^{-2}t)\big]\cdots.$$

Note that the right hand side is the same as $Z(BE,t)$ above (since $\alpha\beta=q$).

Let $Z_1(t) := Z(BE,qt)$. Its radius of convergence is 1, since by Lemma 6.1

$$\lim_{v\to\infty}\frac{c_v(BE)}{c_{v+1}(BE)} = \lim_{v\to\infty}\frac{c_{v+1}(E)}{c_v(E)} = q.$$

There is also a functional equation

$$Z_1(\alpha t) = \frac{1}{1-\alpha t} Z_1(t)\, Z_2(t),$$

where

$$Z_2(t) = \frac{1}{(1-\alpha\beta^{-1}t)(1-\alpha\beta^{-2}t)(1-\alpha\beta^{-3}t)\cdots}.$$

$Z_2(t)$ is holomorphic in the open unit disk and satisfies the functional equation

$$Z_2(\beta t) = \frac{1}{1-\alpha t} Z_2(t).$$

Therefore $Z_2(t)$, and hence $Z(BE,t)$, has a meromorphic continuation to the whole complex t-plane with the obvious poles.

Remark 7.2.1. $H_c^{-2-2n}(BE)$ is pure of weight $-2-n$, which is not $\leq -2-2n$ unless $n=0$. So the upper bound of weights for schemes fails for BE. This also leads to the failure of the decomposition theorem for BE; see [Sun 2012, §1], for the example of a pure complex on BE which is not geometrically semisimple.

Also note that, the Equation (7.2.1) is independent of ℓ, so the polynomials $P_{n,\ell}(BE,t)$ are independent of ℓ.

Example 7.3. BG_0, where G_0 is a finite étale $\mathbb{F}_q$-group scheme, corresponding to a finite group G and a Frobenius automorphism σ on it. Then $BG_0(\mathbb{F}_{q^v}) \simeq G/\rho^{(v)}$, where $\rho^{(v)}$ is the right action of G on the set G given by $h : g \mapsto \sigma^v(h^{-1})gh$. So

$$c_v(BG_0) = \sum_{[g]\in G/\rho^{(v)}} \frac{1}{\#\operatorname{Stab}_{\rho^{(v)}}(g)} = \frac{\#G}{\#G} = 1,$$

and the zeta function is

$$Z(BG_0,t) = \frac{1}{1-t}.$$

Its cohomology groups are given in Lemma 4.6: $H_c^0(BG) = \overline{\mathbb{Q}}_\ell$, and other $H_c^i = 0$. This verifies Remark 6.5.1.

Note that $Z(BG_0, t)$ is the same as the zeta function of its coarse moduli space $\operatorname{Spec}\mathbb{F}_q$. As a consequence, for every $\mathbb{F}_q$-algebraic stack $\mathscr{X}_0$, with finite inertia $\mathscr{I}_0 \to \mathscr{X}_0$ and coarse moduli space $\pi : \mathscr{X}_0 \to X_0$ [Conrad 2005, 1.1], we have $Z(\mathscr{X}_0, t) = Z(X_0, t)$, and hence it is a rational function. This is because for every $x \in X_0(\mathbb{F}_{q^v})$, the fiber $\pi^{-1}(x)$ is a neutral gerbe over $\operatorname{Spec} k(x)$, and from the above we see that $c_v(\pi^{-1}(x)) = 1$, and hence $c_v(\mathscr{X}_0) = c_v(X_0)$. The fact that $Z(X_0, t)$ is a rational function follows from [Knutson 1971, II, 6.7] and noetherian induction. More generally, we have the following.

Proposition 7.3.1. *Let $\mathscr{X}_0$ be an $\mathbb{F}_q$-algebraic stack. Suppose that $\mathscr{X}_0$ either has finite inertia, or is Deligne–Mumford (not necessarily separated). Then for every $K_0 \in W^b(\mathscr{X}_0, \overline{\mathbb{Q}}_\ell)$, the L-series $L(\mathscr{X}_0, K_0, t)$ is a rational function.*

Proof. It suffices to show that Theorem 4.2 holds for the structural morphism $\mathscr{X}_0 \to \operatorname{Spec}\mathbb{F}_q$ and $K_0 \in W^b(\mathscr{X}_0, \overline{\mathbb{Q}}_\ell)$ in these two cases. We will not make explicit use of the fact from Remark 2.8.1 that K_0 is ι-mixed.

Case when $\mathscr{X}_0$ has finite inertia. Let $\pi : \mathscr{X}_0 \to X_0$ be its coarse moduli space. For any sheaf $\mathscr{F}_0$ on $\mathscr{X}_0$, we have isomorphisms $H_c^r(X, R^0\pi_!\mathscr{F}) \simeq H_c^r(\mathscr{X}, \mathscr{F})$ by Lemma 4.6, so $R\Gamma_c(\mathscr{X}_0, \mathscr{F}_0)$ is a bounded complex, hence a convergent complex. To prove the trace formula for $\mathscr{X}_0 \to \operatorname{Spec}\mathbb{F}_q$ and the sheaf $\mathscr{F}_0$, it suffices to prove it for $\mathscr{X}_0 \to X_0$ and $X_0 \to \operatorname{Spec}\mathbb{F}_q$. The first case, when passing to fibers, is reduced to BG_0, and when passing to fibers again, it is reduced to the two subcases: when G_0 is finite, or when G_0 is connected. In both of these two cases as well as the case of an algebraic space $X_0 \to \operatorname{Spec}\mathbb{F}_q$, the trace formula can be proved without using ι-mixedness 5.2.1. Therefore, Theorem 4.2 holds for $\mathscr{X}_0 \to \operatorname{Spec}\mathbb{F}_q$ and any sheaf, hence any bounded complex, on $\mathscr{X}_0$.

The trace formula is equivalent to the equality of power series

$$L(\mathscr{X}_0, K_0, t) = \prod_{i \in \mathbb{Z}} P_i(\mathscr{X}_0, K_0, t)^{(-1)^{i+1}},$$

and the right-hand side is a finite product, because $R\Gamma_c(\mathscr{X}_0, K_0)$ is bounded. Therefore, $L(\mathscr{X}_0, K_0, t)$ is rational.

Case when $\mathscr{X}_0$ is Deligne–Mumford. For both (i) and (ii) of Theorem 4.2, we may replace $\mathscr{X}_0$ by a nonempty open substack, hence by [Laumon and Moret-Bailly 2000, 6.1.1] we may assume $\mathscr{X}_0$ is the quotient stack $[X_0'/G]$, where X_0' is an affine $\mathbb{F}_q$-scheme of finite type and G is a finite group acting on X_0'. This stack has finite diagonal, and hence finite inertia, so by the previous case we are done. Also, we know that $R\Gamma_c(\mathscr{X}_0, K_0)$ is bounded, therefore $L(\mathscr{X}_0, K_0, t)$ is rational. □

If one wants to use Poincaré duality to get a functional equation for the zeta function, [Olsson 2008a, 5.17] and [Laszlo and Olsson 2008b, 9.1.2] suggest that we should assume $\mathscr{X}_0$ to be proper smooth and of finite diagonal. Under these assumptions, one gets the expected functional equation for the zeta function, as well as the independence of ℓ for the coarse moduli space, which is proper but possibly singular. Examples of such stacks include the moduli stack of pointed stable curves $\overline{\mathcal{M}}_{g,n}$ over $\mathbb{F}_q$.

Proposition 7.3.2. *Let $\mathscr{X}_0$ be a proper smooth $\mathbb{F}_q$-algebraic stack of equidimension d, with finite diagonal, and let $\pi : \mathscr{X}_0 \to X_0$ be its coarse moduli space. Then $Z(X_0, t)$ satisfies the usual functional equation*

$$Z\Big(X_0, \frac{1}{q^d t}\Big) = \pm\, q^{d\chi/2} t^{\chi} Z(X_0, t),$$

where $\chi := \sum_{i=0}^{2d}(-1)^i \deg P_{i,\ell}(X_0, t)$. Moreover, $H^i(X)$ is pure of weight i, for every $0 \le i \le 2d$, and the reciprocal roots of each $P_{i,\ell}(X_0, t)$ are algebraic integers independent of ℓ.

Proof. First we show that the adjunction map $\overline{\mathbb{Q}}_\ell \to \pi_*\pi^*\overline{\mathbb{Q}}_\ell = \pi_*\overline{\mathbb{Q}}_\ell$ is an isomorphism. Since π is quasi-finite and proper [Conrad 2005, 1.1], we have $\pi_* = \pi_!$ [Olsson 2008a, 5.1] and $R^r\pi_!\overline{\mathbb{Q}}_\ell = 0$ for $r \neq 0$ [Olsson 2008a, 5.8]. The natural map $\overline{\mathbb{Q}}_\ell \to R^0\pi_*\overline{\mathbb{Q}}_\ell$ is an isomorphism, since the geometric fibers of π are connected.

Therefore $R\Gamma(\mathscr{X}_0, \overline{\mathbb{Q}}_\ell) = R\Gamma(X_0, \pi_*\overline{\mathbb{Q}}_\ell) = R\Gamma(X_0, \overline{\mathbb{Q}}_\ell)$, and hence

$$H^i(\mathscr{X}) \simeq H^i_c(\mathscr{X}) \simeq H^i(X) \simeq H^i_c(X)$$

for all i [Olsson 2008a, 5.17]. Let $P_i(t) = P_i(\mathscr{X}_0, t) = P_i(X_0, t)$. Since X_0 is an algebraic space of dimension d, $P_i(t) = 1$ if $i \notin [0, 2d]$. Since $\mathscr{X}_0$ is proper and smooth, Poincaré duality gives a perfect pairing

$$H^i(\mathscr{X}) \times H^{2d-i}(\mathscr{X}) \longrightarrow \overline{\mathbb{Q}}_\ell(-d).$$

Following the standard proof for proper smooth varieties, see [Milne 2008, 27.12], we get the expected functional equation for $Z(\mathscr{X}_0, t) = Z(X_0, t)$.

$H^i(X)$ is mixed of weights $\le i$ [Deligne 1980, 3.3.4], so by Poincaré duality, it is pure of weight i. Following the proof in [Deligne 1974a, p. 276)], this purity implies that the polynomials $P_{i,\ell}(X_0, t)$ have integer coefficients independent of ℓ. □

Remark 7.3.3. Weizhe Zheng suggested Proposition 7.3.1 to me. He also suggested that we give a functional equation relating $L(\mathscr{X}_0, DK_0, t)$ and $L(\mathscr{X}_0, K_0, t)$, for $K_0 \in W^b(\mathscr{X}_0, \overline{\mathbb{Q}}_\ell)$, where $\mathscr{X}_0$ is a proper $\mathbb{F}_q$-algebraic stack with finite diagonal, of equidimension d, but not necessarily smooth. Here is the functional equation:

$$L(\mathscr{X}_0, K_0, t^{-1}) = t^{\chi_c} \cdot Q \cdot L(\mathscr{X}_0, DK_0, t),$$

where $\chi_c = \sum_{i=0}^{2d}(-1)^i h_c^i(\mathscr{X}, K)$ and $Q = (t^{\chi_c} L(\mathscr{X}_0, K_0, t))|_{t=\infty}$. Note that the rational function $L(\mathscr{X}_0, K_0, t)$ has degree $-\chi_c$, hence Q is well-defined. The proof is similar to the above.

Example 7.4. BGL_N. We have

$$\# GL_N(\mathbb{F}_{q^v}) = \left(q^{vN}-1\right)\left(q^{vN}-q^v\right)\cdots\left(q^{vN}-q^{v(N-1)}\right),$$

so one can use $c_v(BGL_N) = 1/c_v(GL_N)$ to compute $Z(BGL_N, t)$. One can also compute the cohomology groups of BGL_N using Borel's theorem 4.8. We refer to [Behrend 1993, 2.3.2] for the result. Let us consider the case $N = 2$ only. The general case is similar.

We have

$$c_v(BGL_2) = \frac{1}{q^{4v}}\left(1+\frac{1}{q^v}+\frac{2}{q^{2v}}+\frac{2}{q^{3v}}+\frac{3}{q^{4v}}+\frac{3}{q^{5v}}+\cdots\right),$$

and therefore

$$\begin{aligned} Z(BGL_2, t) &= \exp\Big(\sum_v \frac{(t/q^4)^v}{v}\Big)\cdot\exp\Big(\sum_v \frac{(t/q^5)^v}{v}\Big)\cdot\exp\Big(\sum_v \frac{2(t/q^6)^v}{v}\Big)\cdots \\ &= \frac{1}{1-t/q^4}\cdot\frac{1}{1-t/q^5}\cdot\Big(\frac{1}{1-t/q^6}\Big)^2\cdot\Big(\frac{1}{1-t/q^7}\Big)^2\cdot\Big(\frac{1}{1-t/q^8}\Big)^3\cdots. \end{aligned}$$

So $Z(BGL_2, qt) = Z(BGL_2, t)\cdot Z_1(t)$, where

$$Z_1(t) = \frac{1}{(1-t/q^3)(1-t/q^5)(1-t/q^7)(1-t/q^9)\cdots}.$$

$Z_1(t)$ satisfies the functional equation

$$Z_1(q^2 t) = \frac{1}{1-t/q}\cdot Z_1(t),$$

So $Z_1(t)$, and hence $Z(BGL_2, t)$, has a meromorphic continuation with the obvious poles.

The nonzero compactly supported cohomology groups of BGL_2 are given as follows:

$$H_c^{-8-2n}(BGL_2) = \overline{\mathbb{Q}}_\ell(n+4)^{\oplus\left(\lfloor\frac{n}{2}\rfloor+1\right)}, \quad n \geq 0.$$

This gives

$$\prod_{n\in\mathbb{Z}} P_n(BGL_2, t)^{(-1)^{n+1}} = \frac{1}{(1-t/q^4)(1-t/q^5)(1-t/q^6)^2(1-t/q^7)^2\cdots},$$

and Remark 6.5.1 is verified. Note that the eigenvalues are $1/q^{n+4}$, which are independent of ℓ.

8. Analytic continuation

We state and prove a generalized version of Theorem 1.3.

Theorem 8.1. *Let $\mathscr{X}_0$ be an $\mathbb{F}_q$-algebraic stack, and let $K_0 \in W_m^{-,\mathrm{stra}}(\mathscr{X}_0, \overline{\mathbb{Q}}_\ell)$ be a convergent complex. Then $L(\mathscr{X}_0, K_0, t)$ has a meromorphic continuation to the whole complex t-plane, and its poles can only be zeros of the polynomials $P_{2n}(\mathscr{X}_0, K_0, t)$ for some integers n.*

We need a preliminary lemma. For an open subset $U \subset \mathbb{C}$, let $\mathbb{O}(U)$ be the set of analytic functions on U. There exists a sequence $\{K_n\}_{n\geq 1}$ of compact subsets of U such that $U = \bigcup_n K_n$ and $K_n \subset (K_{n+1})^\circ$. For f and g in $\mathbb{O}(U)$, define

$$\rho_n(f, g) = \sup\{|f(z) - g(z)|; z \in K_n\} \quad \text{and} \quad \rho(f, g) = \sum_{n=1}^{\infty} \left(\tfrac{1}{2}\right)^n \frac{\rho_n(f, g)}{1 + \rho_n(f, g)}.$$

Then ρ is a metric on $\mathbb{O}(U)$ and the topology it induces is independent of the subsets $\{K_n\}_n$ chosen (see [Conway 1973, VII, §1]).

The following lemma is from [Conway 1973, p. 167, 5.9].

Lemma 8.2. *Let $U \subset \mathbb{C}$ be connected and open and let $(f_n)_n$ be a sequence in $\mathbb{O}(U)$ such that no f_n is identically zero. If $\sum_n (f_n(z) - 1)$ converges absolutely and uniformly on compact subsets of U, then $\prod_{n\geq 1} f_n(z)$ converges in $\mathbb{O}(U)$ to an analytic function $f(z)$. If z_0 is a zero of f, then z_0 is a zero of only a finite number of the functions f_n, and the multiplicity of the zero of f at z_0 is the sum of the multiplicities of the zeros of the functions f_n at z_0.*

Now we prove Theorem 8.1.

Proof. Factorize $P_n(\mathscr{X}_0, K_0, t)$ as $\prod_{j=1}^{m_n}(1 - \alpha_{nj} t)$ in $\mathbb{C}$. Since $R\Gamma_c(\mathscr{X}_0, K_0)$ is a convergent complex by Theorem 4.2 (i), the series $\sum_{n,j} |\alpha_{nj}|$ converges.

By Proposition 6.5 we have

$$L(\mathscr{X}_0, K_0, t) = \prod_{n\in\mathbb{Z}} \Big(\prod_{j=1}^{m_n} (1 - \alpha_{nj} t) \Big)^{(-1)^{n+1}}$$

as formal power series. Take U to be the region $\mathbb{C} - \{\alpha_{nj}^{-1}; n \text{ even}\}$ with the intention of applying Lemma 8.2. Take the lexicographical order on the set of all factors

$$1 - \alpha_{nj} t, \ \text{for } n \text{ odd}; \qquad \frac{1}{1-\alpha_{nj} t}, \ \text{for } n \text{ even}.$$

Each factor is an analytic function on U. The sum $\sum_n (f_n(z) - 1)$ here is equal to

$$\sum_{n \text{ odd}, j} (-\alpha_{nj} t) \ + \sum_{n \text{ even}, j} \frac{\alpha_{nj} t}{1 - \alpha_{nj} t}.$$

Let

$$g_n(t)=\begin{cases}\displaystyle\sum_{j=1}^{m_n}|\alpha_{nj}t|, & n \text{ odd},\\ \displaystyle\sum_{j=1}^{m_n}\frac{|\alpha_{nj}t|}{|1-\alpha_{nj}t|}, & n \text{ even}.\end{cases}$$

We need to show that $\sum_n g_n(t)$ is pointwise convergent, uniformly on compact subsets of U. Precisely, we want to show that for any compact subset $B\subset U$, and for any $\varepsilon>0$, there exists a constant $N_B\in\mathbb{Z}$ such that

$$\sum_{n\le N} g_n(t)<\varepsilon$$

for all $N\le N_B$ and $t\in B$. Since $g_n(t)$ are nonnegative, it suffices to do this for $N=N_B$. There exists a constant M_B such that $|t|<M_B$ for all $t\in B$. Since $\sum_{n,j}|\alpha_{nj}|$ converges, $|\alpha_{nj}|\to 0$ as $n\to-\infty$, and there exists a constant $L_B\in\mathbb{Z}$ such that $|\alpha_{nj}|<1/(2M_B)$ for all $n<L_B$. So

$$g_n(t)\le 2\sum_{j=1}^{m_n}|\alpha_{nj}t|$$

for all $n<L_B$ and $t\in B$. There exists a constant $N_B<L_B$ such that

$$\sum_{n\le N_B}\sum_j|\alpha_{nj}|<\varepsilon/(2M_B)$$

and hence

$$\sum_{n\le N_B}g_n(t)\le 2\sum_{n\le N_B}\sum_j|\alpha_{nj}t|\le 2M_B\sum_{n\le N_B}\sum_j|\alpha_{nj}|<\varepsilon.$$

By Lemma 8.2, $L(\mathscr{X}_0,K_0,t)$ extends to an analytic function on U. By the second part of Lemma 8.2, the α_{nj}^{-1}'s, for n even, are at worst poles rather than essential singularities, therefore the L-series is meromorphic on $\mathbb{C}$. □

Now $L(\mathscr{X}_0,K_0,t)$ can be called an "L-function".

9. Weight theorem for algebraic stacks

9.1. We prove Theorem 1.4 in this section. For the reader's convenience, we briefly review the definition of the *dimension* of a locally noetherian S-algebraic stack $\mathscr{X}$ from [Laumon and Moret-Bailly 2000, Chapter 11].

If X is a locally noetherian S-algebraic space and x is a point of X, the dimension $\dim_x X$ of X at x is defined to be $\dim_{x'} X'$, for any pair (X',x') where X' is an S-scheme étale over X and $x'\in X'$ maps to x. This is independent of the choice of the pair. If $f : X \to Y$ is a morphism of S-algebraic spaces, locally of finite type, and x is a point of X with image y in Y, then the relative dimension $\dim_x f$ of f at x is defined to be $\dim_x X_y$.

Let $P : X \to \mathscr{X}$ be a presentation of an S-algebraic stack $\mathscr{X}$, and let x be a point of X. Then the relative dimension $\dim_x P$ of P at x is defined to be the relative dimension at (x, x) of the smooth morphism $\mathrm{pr}_1 : X \times_{\mathscr{X}} X \to X$ of S-algebraic spaces.

If $\mathscr{X}$ is a locally noetherian S-algebraic stack and if ξ is a point of $\mathscr{X}$, the dimension of $\mathscr{X}$ at ξ is defined to be $\dim_\xi \mathscr{X} = \dim_x X - \dim_x P$, where $P : X \to \mathscr{X}$ is an arbitrary presentation of $\mathscr{X}$ and x is an arbitrary point of X lying over ξ. This definition is independent of all the choices made. At last one defines the dimension of $\mathscr{X}$ by $\dim \mathscr{X} = \sup_\xi \dim_\xi \mathscr{X}$. For quotient stacks we have $\dim\,[X/G] = \dim X - \dim G$.

Now we prove Theorem 1.4.

Proof. If $j : \mathscr{U}_0 \to \mathscr{X}_0$ is an open substack with complement $i : \mathscr{Z}_0 \to \mathscr{X}_0$, then we have an exact sequence

$$\cdots \longrightarrow H^n_c(\mathscr{U}, j^*\mathscr{F}) \longrightarrow H^n_c(\mathscr{X}, \mathscr{F}) \longrightarrow H^n_c(\mathscr{Z}, i^*\mathscr{F}) \longrightarrow \cdots .$$

If both $H^n_c(\mathscr{U}, j^*\mathscr{F})$ and $H^n_c(\mathscr{Z}, i^*\mathscr{F})$ are zero (respectively have all ι-weights $\leq m$ for some number m), then so is $H^n_c(\mathscr{X}, \mathscr{F})$. Since the dimensions of $\mathscr{U}_0$ and $\mathscr{Z}_0$ are no more than that of $\mathscr{X}_0$, and the set of punctual ι-weights of $i^*\mathscr{F}_0$ and of $j^*\mathscr{F}_0$ is the same as that of $\mathscr{F}_0$, we may shrink $\mathscr{X}_0$ to a nonempty open substack. We can also make any finite base change on $\mathbb{F}_q$. To simplify notation, we may use twist (see 2.4) and projection formula to assume $w = 0$. As before, we reduce to the case when $\mathscr{X}_0$ is geometrically connected, and the inertia $f : \mathscr{I}_0 \to \mathscr{X}_0$ is flat, with rigidification $\pi : \mathscr{X}_0 \to X_0$, where X_0 is a scheme. The squares in the following diagram are 2-Cartesian:

$$\begin{tikzcd}
 & & \mathscr{I}_0 \arrow[d, "f"'] & \mathrm{Aut}_y \arrow[l] \arrow[d] \\
B\,\mathrm{Aut}_{\overline{x}} \arrow[r] \arrow[d, "\pi_{\overline{x}}"'] & B\,\mathrm{Aut}_x \arrow[r] \arrow[d, "\pi_x"'] & \mathscr{X}_0 \arrow[d, "\pi"] & \mathrm{Spec}\,\mathbb{F}_{q^v} \arrow[l, "y"'] \\
\mathrm{Spec}\,\mathbb{F} \arrow[r, "\overline{x}"'] & \mathrm{Spec}\,\mathbb{F}_{q^v} \arrow[r, "x"'] & X_0 &
\end{tikzcd}$$

We have $(R^k\pi_!\mathscr{F}_0)_{\overline{x}} = H^k_c(B\,\mathrm{Aut}_{\overline{x}}, \mathscr{F})$. Since f is representable and flat, and $\mathscr{X}_0$ is connected, all automorphism groups Aut_x have the same dimension, say d.

Assume Theorem 1.4 holds for all BG_0, where G_0 are $\mathbb{F}_q$-algebraic groups. Then $R^k\pi_!\mathscr{F}_0 = 0$ for $k > -2d$, and for $k \leq -2d$, the punctual ι-weights of

$R^k\pi_!\mathscr{F}_0$ are $\le \frac{k}{2}-d$, hence by [Deligne 1980, 3.3.4], the punctual ι-weights of $H^r_c(X, R^k\pi_!\mathscr{F})$ are $\le \frac{k}{2}-d+r$. Consider the Leray spectral sequence

$$E_2^{rk}=H^r_c(X, R^k\pi_!\mathscr{F})\Longrightarrow H^{r+k}_c(\mathscr{X},\mathscr{F}).$$

If we maximize $\frac{k}{2}-d+r$ under the constraints

$$r+k=n,\quad 0\le r\le 2\dim X_0\quad\text{and}\quad k\le -2d,$$

we find that $H^n_c(\mathscr{X},\mathscr{F})=0$ for $n>2\dim X_0-2d=2\dim\mathscr{X}_0$, and for $n\le 2\dim\mathscr{X}_0$, the punctual ι-weights of $H^n_c(\mathscr{X},\mathscr{F})$ are $\le \dim X_0+\frac{n}{2}-d=\dim\mathscr{X}_0+\frac{n}{2}$.

So we reduce to the case $\mathscr{X}_0=BG_0$. By Lemma 4.6 the Leray spectral sequence for $h: BG_0\to B\pi_0(G_0)$ degenerates to isomorphisms

$$H^0_c(B\pi_0(G), R^nh_!\mathscr{F})\simeq H^n_c(BG,\mathscr{F}).$$

The fibers of h are isomorphic to BG_0^0, so by base change and Lemma 4.6 we reduce to the case when G_0 is connected. Let $d=\dim G_0$ and $f: BG_0\to \operatorname{Spec}\mathbb{F}_q$ be the structural map. In this case, $\mathscr{F}_0\cong f^*V$ for some $\overline{\mathbb{Q}}_\ell$-representation V of $W(\mathbb{F}_q)$, and hence $\mathscr{F}_0$ and V have the same punctual ι-weights. Using the natural isomorphism $H^n_c(BG)\otimes V\simeq H^n_c(BG,\mathscr{F})$, we reduce to the case when $\mathscr{F}_0=\overline{\mathbb{Q}}_\ell$. In 4.8.2 we see that, if $\alpha_{i1},\dots,\alpha_{in_i}$ are the eigenvalues of F on N^i, $i\ge 1$ odd, then the eigenvalues of F on $H^{-2k-2d}_c(BG)$ are

$$q^{-d}\prod_{i,j}\alpha_{ij}^{-m_{ij}},\quad\text{where }\sum_{i,j}m_{ij}(i+1)=2k.$$

Since $i\ge 1$, we have $\sum im_{ij}\ge k$; together with $|\alpha_{ij}|\ge q^{i/2}$ one deduces

$$\Bigl|q^{-d}\prod_{i,j}\alpha_{ij}^{-m_{ij}}\Bigr|\le q^{(-k-2d)/2},$$

so the punctual ι-weights of $H^{-2k-2d}_c(BG)$ are $\le -k-2d$ for $k\ge 0$, and the other compactly supported cohomology groups are zero.

It is clear from the proof and [Deligne 1980, 3.3.10] that the weights of $H^n_c(\mathscr{X},\mathscr{F})$ differ from the weights of $\mathscr{F}_0$ by integers.

Recall that $H^{2k}(BG)$ is pure of weight $2k$, for a linear algebraic group G_0 over $\mathbb{F}_q$ [Deligne 1974b, 9.1.4]. Therefore, $H^{-2k-2d}_c(BG)$ is pure of weight $-2k-2d$, and following the same proof as above, we are done. □

Remark 9.2. When $\mathscr{X}_0=X_0$ is a scheme, and $n\le 2\dim X_0$, we have

$$\dim X_0+\frac{n}{2}+w\ge n+w,$$

so our bound for weights is worse than the bound in [Deligne 1980, 3.3.4]. For an $\mathbb{F}_q$-abelian variety A, our bound for the weights of $H_c^n(BA)$ is sharp: the weights are exactly $\dim(BA)+\frac{n}{2}$, whenever the cohomology group is nonzero.

We hope Theorem 1.4 has useful and interesting applications, for instance for generalizing the decomposition theorem of Beilinson–Bernstein–Deligne–Gabber (see [Sun 2012]) to stacks with affine stabilizers, and for studying the Hasse–Weil zeta functions of Artin stacks over number fields. For instance, it implies that the Hasse–Weil zeta function is analytic in some right half complex s-plane.

Using Theorem 1.4 we can show certain stacks have $\mathbb{F}_q$-points.

Example 9.3. Let $\mathscr{X}_0$ be a form of $B\mathbb{G}_m$, that is, $\mathscr{X}\cong B\mathbb{G}_{m,\mathbb{F}}$ over $\mathbb{F}$. Then all the automorphism group schemes in $\mathscr{X}_0$ are affine, and for any $n\geq 0$ we have $h_c^{-2-2n}(\mathscr{X})=h_c^{-2-2n}(B\mathbb{G}_m)=1$. Let α_{-2-2n} be the eigenvalue of F on $H_c^{-2-2n}(\mathscr{X})$. Then by Theorem 1.4 we have $|\alpha_{-2-2n}|\leq q^{-1-n}$. Smoothness is fppf local on the base, so $\mathscr{X}_0$ is smooth and connected, hence $H_c^{-2}(\mathscr{X})=\overline{\mathbb{Q}}_\ell(1)$ and $\alpha_{-2}=q^{-1}$. So

$$\begin{aligned}\#\mathscr{X}_0(\mathbb{F}_q)=\sum_{n\geq 0}\mathrm{Tr}(F,H_c^{-2-2n}(\mathscr{X}))&=q^{-1}+\alpha_{-4}+\alpha_{-6}+\cdots\\&\geq q^{-1}-q^{-2}-q^{-3}+\cdots=q^{-1}-\frac{q^{-1}}{q-1}>0\end{aligned}$$

when $q\neq 2$. In fact, since there exists an integer $r\geq 1$ such that $\mathscr{X}_0\otimes\mathbb{F}_{q^r}\cong B\mathbb{G}_{m,\mathbb{F}_{q^r}}$, we see that all cohomology groups $H_c^{-2-2n}(\mathscr{X})$ are pure, that is, $|\alpha_{-2-2n}|=q^{-1-n}$.

In fact, one can classify the forms of $B\mathbb{G}_{m,\mathbb{F}_q}$ as follows. If $\mathscr{X}_0$ is a form, then it is also a gerbe over $\operatorname{Spec}\mathbb{F}_q$, hence a neutral gerbe BG_0 for some algebraic group G_0 by Behrend [2003, 6.4.2]. By comparing the automorphism groups, we see that G_0 is a form of $\mathbb{G}_{m,\mathbb{F}_q}$. There is only one nontrivial form of $\mathbb{G}_{m,\mathbb{F}_q}$, because

$$H^1(\mathbb{F}_q,\mathrm{Aut}(\mathbb{G}_m))=H^1(\mathbb{F}_q,\mathbb{Z}/2\mathbb{Z})=\mathbb{Z}/2\mathbb{Z},$$

and this form is the kernel $R^1_{\mathbb{F}_{q^2}/\mathbb{F}_q}\mathbb{G}_{m,\mathbb{F}_{q^2}}$ of the norm map

$$R_{\mathbb{F}_{q^2}/\mathbb{F}_q}\mathbb{G}_{m,\mathbb{F}_{q^2}}\xrightarrow{\mathrm{Nm}}\mathbb{G}_{m,\mathbb{F}_q},$$

where $R_{\mathbb{F}_{q^2}/\mathbb{F}_q}$ is the operation of Weil's restriction of scalars. Therefore, the only nontrivial form of $B\mathbb{G}_{m,\mathbb{F}_q}$ is $B(R^1_{\mathbb{F}_{q^2}/\mathbb{F}_q}\mathbb{G}_{m,\mathbb{F}_{q^2}})$. In particular, they all have $\mathbb{F}_q$-points, even when $q=2$.

Example 9.4. Consider the projective line $\mathbb{P}^1$ with the following action of $\mathbb{G}_m$: it acts by multiplication on the open part $\mathbb{A}^1\subset\mathbb{P}^1$, and leaves the point ∞ fixed. So we get a quotient stack $[\mathbb{P}^1/\mathbb{G}_m]$ over $\mathbb{F}_q$. Let $\mathscr{X}_0$ be a form of $[\mathbb{P}^1/\mathbb{G}_m]$. We want to find an $\mathbb{F}_q$-point on $\mathscr{X}_0$, or even better, an $\mathbb{F}_q$-point on $\mathscr{X}_0$ which, when considered as a point in $\mathscr{X}(\mathbb{F})\cong[\mathbb{P}^1/\mathbb{G}_m](\mathbb{F})$, lies in the open dense orbit $[\mathbb{G}_m/\mathbb{G}_m](\mathbb{F})$.

9.4.1. Consider the following general situation. Let G_0 be a connected $\mathbb{F}_q$-algebraic group, and let X_0 be a proper smooth variety with a G_0-action over $\mathbb{F}_q$. Let

$$[X_0/G_0] \xrightarrow{f} BG_0 \xrightarrow{g} \operatorname{Spec} \mathbb{F}_q$$

be the natural maps, and let $\mathscr{X}_0$ be a form of $[X_0/G_0]$. Then f is representable and proper. For every k, $R^k f_*\overline{\mathbb{Q}}_\ell$ is a lisse sheaf, and takes the form g^*V_k for some sheaf V_k on $\operatorname{Spec}\mathbb{F}_q$. Consider the Leray spectral sequence

$$E_2^{rk} = R^r g_! R^k f_* \overline{\mathbb{Q}}_\ell \Longrightarrow R^{r+k}(gf)_!\overline{\mathbb{Q}}_\ell.$$

Since $R^r g_! R^k f_*\overline{\mathbb{Q}}_\ell = R^r g_!(g^*V_k) = (R^r g_!\overline{\mathbb{Q}}_\ell) \otimes V_k$, we have

$$h_c^n(\mathscr{X}) = h_c^n([X/G]) \le \sum_{r+k=n} h_c^r(BG)\cdot \dim V_k = \sum_{r+k=n} h_c^r(BG)\cdot h^k(X).$$

Now we return to $[\mathbb{P}^1/\mathbb{G}_m]$. Since $h^0(\mathbb{P}^1) = h^2(\mathbb{P}^1) = 1$ and $h_c^{-2i}(B\mathbb{G}_m) = 1$ for $i \ge 1$, we see that $h_c^n(\mathscr{X}) = 0$ for n odd and

$$h_c^{2n}(\mathscr{X}) \le h^0(\mathbb{P}^1)\, h_c^{2n}(B\mathbb{G}_m) + h^2(\mathbb{P}^1)\, h_c^{2n-2}(B\mathbb{G}_m) = \begin{cases} 0, & n \ge 1,\\ 1, & n = 0,\\ 2, & n < 0.\end{cases}$$

Since $\mathscr{X}_0$ is connected and smooth of dimension 0, we have $H_c^0(\mathscr{X}) = \overline{\mathbb{Q}}_\ell$. By Theorem 1.4, the ι-weights of $H_c^{2n}(\mathscr{X})$ are $\le 2n$. The trace formula gives

$$\begin{aligned}\#\mathscr{X}_0(\mathbb{F}_q) = \sum_{n\le 0}\operatorname{Tr}(F, H_c^{2n}(\mathscr{X})) &= 1 + \sum_{n<0}\operatorname{Tr}(F, H_c^{2n}(\mathscr{X}))\\ &\ge 1 - 2\sum_{n<0} q^n = 1 - \frac{2}{q-1} > 0\end{aligned}$$

when $q \ge 4$.

In order for the rational point to be in the open dense orbit, we need an upper bound for the number of $\mathbb{F}_q$-points on the closed orbits. When passing to $\mathbb{F}$, there are 2 closed orbits, both having stabilizer $\mathbb{G}_{m,\mathbb{F}}$. So in $[\mathscr{X}_0(\mathbb{F}_q)]$ there are at most 2 points whose automorphism groups are forms of the algebraic group $\mathbb{G}_{m,\mathbb{F}_q}$. From the cohomology sequence

$$1 \longrightarrow (R^1_{\mathbb{F}_{q^2}/\mathbb{F}_q}\mathbb{G}_{m,\mathbb{F}_{q^2}})(\mathbb{F}_q) \longrightarrow \mathbb{F}_{q^2}^* \xrightarrow{\mathrm{Nm}} \mathbb{F}_q^*$$

we see that

$$\#(R^1_{\mathbb{F}_{q^2}/\mathbb{F}_q}\mathbb{G}_{m,\mathbb{F}_{q^2}})(\mathbb{F}_q) = q+1.$$

Since $1/(q+1) \le 1/(q-1)$, the space that the closed orbits can take is at most $2/(q-1)$, and equality holds only when the two closed orbits are both defined over $\mathbb{F}_q$ with stabilizer $\mathbb{G}_m$. In order for there to exist an $\mathbb{F}_q$-point in the open dense orbit, we need

$$1-\frac{2}{q-1}>\frac{2}{q-1},$$

and this is so when $q \geq 7$.

10. About independence of ℓ

The coefficients of the expansion of the infinite product

$$Z(\mathscr{X}_0,t)=\prod_{i\in\mathbb{Z}} P_{i,\ell}(\mathscr{X}_0,t)^{(-1)^{i+1}}$$

are rational numbers and are independent of ℓ, because the $c_v(\mathscr{X}_0)$'s are rational numbers independent of ℓ. A famous conjecture is that this is also true for each $P_{i,\ell}(\mathscr{X}_0,t)$. First we show that the roots of $P_{i,\ell}(\mathscr{X}_0,t)$ are Weil q-numbers. Note that $P_{i,\ell}(\mathscr{X}_0,t)\in\mathbb{Q}_\ell[t]$.

Definition 10.1. *An algebraic number is called a* Weil q-number *if all of its conjugates have the same weight relative to q, and this weight is a rational integer. It is called a* Weil q-integer *if in addition it is an algebraic integer. A number in $\overline{\mathbb{Q}}_\ell$ is called a* Weil q-number *if it is a Weil q-number via ι.*

For $\alpha\in\overline{\mathbb{Q}}_\ell$, being a Weil q-number or not is independent of ι; in fact the images in $\mathbb{C}$ under various ι's are conjugate.

For an $\mathbb{F}_q$-variety X_0, not necessarily smooth or proper, [Deligne 1980, 3.3.4] implies all Frobenius eigenvalues of $H^i_c(X)$ are Weil q-integers. The following lemma generalizes this.

Lemma 10.2. *For every $\mathbb{F}_q$-algebraic stack $\mathscr{X}_0$, and a prime number $\ell\neq p$, the roots of each $P_{i,\ell}(\mathscr{X}_0,t)$ are Weil q-numbers. In particular, the coefficients of $P_{i,\ell}(\mathscr{X}_0,t)$ are algebraic numbers in $\mathbb{Q}_\ell$ (that is, algebraic over $\mathbb{Q}$).*

Proof. For an open immersion $j:\mathscr{U}_0\to\mathscr{X}_0$ with complement $i:\mathscr{Z}_0\to\mathscr{X}_0$, we have an exact sequence

$$\cdots\longrightarrow H^i_c(\mathscr{U})\longrightarrow H^i_c(\mathscr{X})\longrightarrow H^i_c(\mathscr{Z})\longrightarrow\cdots,$$

thus we may shrink to a nonempty open substack. In particular, Lemma 10.2 holds for algebraic spaces, by Knutson [1971, II 6.7] and Deligne [1980, 3.3.4].

We may assume $\mathscr{X}_0$ is smooth and connected. By Poincaré duality, it suffices to show that the Frobenius eigenvalues of $H^i(\mathscr{X})$ are Weil q-numbers, for all i. Take a presentation $X_0\to\mathscr{X}_0$ and consider the associated strictly simplicial smooth covering $X_0^\bullet\to\mathscr{X}_0$ by algebraic spaces. Then there is a spectral sequence [Laszlo and Olsson 2008b, 10.0.9]

$$E_1^{rk} = H^k(X^r) \Longrightarrow H^{r+k}(\mathscr{X}),$$

and the assertion for $\mathscr{X}_0$ follows from the assertion for algebraic spaces. □

Problem 10.3. Is each

$$P_{i,\ell}(\mathscr{X}_0, t) = \det(1 - Ft, H_c^i(\mathscr{X}, \mathbb{Q}_\ell))$$

a polynomial with coefficients in $\mathbb{Q}$, and the coefficients are independent of ℓ?

Remark 10.3.1. (i) Note that, unlike the case for varieties, we cannot expect the coefficients to be integers (for instance, for $B\mathbb{G}_m$, the coefficients are $1/q^i$).

(ii) Problem 10.3 is known to be true for smooth proper varieties [Deligne 1980, 3.3.9], and (coarse moduli spaces of) proper smooth algebraic stacks of finite diagonal (see Proposition 7.3.2). It remains open for general varieties. Even the Betti numbers are not known to be independent of ℓ for a general variety, see [Illusie 2006].

Let us give positive answer to Problem 10.3 in some special cases of algebraic stacks. In Section 7 we see that it holds for BE and BGL_N. We can generalize these two cases as follows.

Lemma 10.4. *Problem 10.3 has a positive answer for*:

(i) BA, *where A is an $\mathbb{F}_q$-abelian variety.*

(ii) BG_0, *where G_0 is a linear algebraic group over $\mathbb{F}_q$.*

Proof. (i) Let $g = \dim A$. Then $N = H^1(A)$ is a $2g$-dimensional vector space, with eigenvalues $\alpha_1, \dots, \alpha_{2g}$ for the Frobenius action F, and N is pure of weight 1. Let $a_1, \dots, a_{2g}$ be a basis for N so that F is given by the upper-triangular matrix

$$\begin{bmatrix} \alpha_1 & * & * \\ & \ddots & * \\ & & \alpha_{2g} \end{bmatrix}.$$

Then $H^*(BA) = \mathrm{Sym}^* N[-1] = \overline{\mathbb{Q}}_\ell[a_1, \dots, a_{2g}]$, where each a_i sits in degree 2. In degree $2n$, $H^{2n}(BA) = \overline{\mathbb{Q}}_\ell\langle a_{i_1} \cdots a_{i_n} \mid 1 \le i_1, \dots, i_n \le 2g\rangle$, and the eigenvalues are $\alpha_{i_1} \cdots \alpha_{i_n}$. By Poincaré duality

$$H_c^{-2n-2g}(BA) = H^{2n}(BA)^\vee \otimes \overline{\mathbb{Q}}_\ell(g)$$

we see that the eigenvalues of F on $H_c^{-2g-2n}(BA)$ are

$$q^{-g} \cdot \alpha_{i_1}^{-1} \cdots \alpha_{i_n}^{-1}.$$

Each factor

$$P_{-2g-2n}(q^g t) = \prod_{1\le i_1,\dots,i_n\le 2g} \big(1-(\alpha_{i_1}\cdots\alpha_{i_n})^{-1}t\big)$$

stays unchanged if we permute the α_i's arbitrarily, so the coefficients are symmetric polynomials in the α_i^{-1}'s with integer coefficients, hence are polynomials in the elementary symmetric functions, which are coefficients of $\prod_{i=1}^{2g}(t-\alpha_i^{-1})$. The polynomial

$$\prod_{i=1}^{2g}(1-\alpha_i t) = \det\big(1-Ft, H^1(A,\mathbb{Q}_\ell)\big)$$

also has roots α_i^{-1}, and this is a polynomial with integer coefficients, independent of ℓ, since A is smooth and proper. Let $m=\pm q^g$ be leading coefficient of it. Then

$$\prod_{i=1}^{2g}(t-\alpha_i^{-1}) = \frac{1}{m}\prod_{i=1}^{2g}(1-\alpha_i t).$$

This verifies Problem 10.3 for BA.

(ii) Let $d=\dim G_0$. For every $k\ge 0$, $H^{2k}(BG)$ is pure of weight $2k$ [Deligne 1974b, 9.1.4], hence $H_c^{-2d-2k}(BG)$ is pure of weight $-2d-2k$ by Poincaré duality. The entire function

$$\frac{1}{Z(BG_0,t)} = \prod_{k\ge 0} P_{-2d-2k}(BG_0,t) \in \mathbb{Q}[[t]]$$

is independent of ℓ, and invariant under the action of $\mathrm{Gal}(\mathbb{Q})$ on the coefficients of the Taylor expansion. Therefore the roots of $P_{-2d-2k}(BG_0,t)$ can be described as

$$\text{“zeros of } \frac{1}{Z(BG_0,t)} \text{ that have weight } 2d+2k \text{ relative to } q\text{,”}$$

which is a description independent of ℓ, and these roots (which are algebraic numbers) are permuted under $\mathrm{Gal}(\mathbb{Q})$. Hence $P_{-2d-2k}(BG_0,t)$ has rational coefficients. □

The following proposition generalizes Proposition 7.3.2 and Lemma 10.4 (ii).

Proposition 10.5. *Let X_0 be the coarse moduli space of a proper smooth $\mathbb{F}_q$-algebraic stack of finite diagonal, and let G_0 be a linear $\mathbb{F}_q$-algebraic group that acts on X_0, and let $\mathscr{X}_0$ be a form of the quotient stack $[X_0/G_0]$. Then Problem 10.3 is verified for $\mathscr{X}_0$.*

Proof. It suffices to show that $H_c^n(\mathscr{X})$ is pure of weight n, for every n. To show this, we can make a finite extension of the base field $\mathbb{F}_q$, so we may assume that $\mathscr{X}_0=[X_0/G_0]$. Let

$$\mathscr{X}_0 \xrightarrow{f} BG_0 \xrightarrow{h} B\pi_0(G_0)$$

be the natural maps.

Let $d = \dim G_0$. Consider the spectral sequence

$$H_c^{-2d-2r}(BG, R^k f_! \overline{\mathbb{Q}}_\ell) \Longrightarrow H_c^{-2d-2r+k}(\mathscr{X}).$$

The E_2-terms can be computed from the degenerate Leray spectral sequence for h:

$$H_c^{-2d-2r}(BG, R^k f_! \overline{\mathbb{Q}}_\ell) \simeq H_c^0(B\pi_0(G), R^{-2d-2r} h_! R^k f_! \overline{\mathbb{Q}}_\ell).$$

We remark that the restriction of $R^{-2d-2r} h_! R^k f_! \overline{\mathbb{Q}}_\ell$ along the natural projection $\operatorname{Spec} \mathbb{F}_q \to B\pi_0(G_0)$ is isomorphic to the Galois module $H_c^{-2d-2r}(BG^0, R^k f_! \overline{\mathbb{Q}}_\ell)$, and since G_0^0 is connected, $(R^k f_! \overline{\mathbb{Q}}_\ell)|_{BG_0^0}$ is the inverse image of some sheaf V_k via the structural map $BG_0^0 \to \operatorname{Spec} \mathbb{F}_q$. By base change, we see that the sheaf V_k, regarded as a $\operatorname{Gal}(\mathbb{F}_q)$-module, is $H^k(X)$. By projection formula we have

$$H_c^{-2d-2r}(BG^0, R^k f_! \overline{\mathbb{Q}}_\ell) \simeq H_c^{-2d-2r}(BG^0) \otimes H^k(X)$$

as representations of $\operatorname{Gal}(\mathbb{F}_q)$, and by Proposition 7.3.2, the right hand side is pure of weight $-2d-2r+k$. By Lemma 4.6, $H_c^{-2d-2r}(BG, R^k f_! \overline{\mathbb{Q}}_\ell)$ is also pure of weight $-2d-2r+k$, therefore $H_c^n(\mathscr{X})$ is pure of weight n, for every n. □

10.6. Finally, let us consider the following much weaker version of independence of ℓ. For $\mathscr{X}_0$ and $i \in \mathbb{Z}$, let $\Psi(\mathscr{X}_0, i)$ be the following property: the Frobenius eigenvalues of $H_c^i(\mathscr{X}, \overline{\mathbb{Q}}_\ell)$, counted with multiplicity, for all $\ell \neq p$, are contained in a finite set of algebraic numbers with multiplicities assigned, and this set together with the assignment of multiplicity, depends only on $\mathscr{X}_0$ and i. In particular it is independent of ℓ. In other words, there is a finite decomposition of the set of all prime numbers $\ell \neq p$ into disjoint union of some subsets, such that the Frobenius eigenvalues of $H_c^i(\mathscr{X}, \overline{\mathbb{Q}}_\ell)$ depends only on the subset that ℓ belongs to. If this property holds, we also denote such a finite set of algebraic numbers (which is not unique) by $\Psi(\mathscr{X}_0, i)$, if there is no confusion.

Proposition 10.6.1. *The property $\Psi(\mathscr{X}_0, i)$ holds for every $\mathscr{X}_0$ and i.*

Proof. If $\mathscr{U}_0$ is an open substack of $\mathscr{X}_0$ with complement $\mathscr{Z}_0$, and properties $\Psi(\mathscr{U}_0, i)$ and $\Psi(\mathscr{Z}_0, i)$ hold, then $\Psi(\mathscr{X}_0, i)$ also holds, and the finite set $\Psi(\mathscr{X}_0, i)$ a subset of $\Psi(\mathscr{U}_0, i) \cup \Psi(\mathscr{Z}_0, i)$.

Firstly we prove this for schemes X_0. By shrinking X_0 we can assume it is a connected smooth variety. By Poincaré duality it suffices to prove the similar statement $\Psi^*(X_0, i)$ for ordinary cohomology, that is, with H_c^i replaced by H^i, for all i. This follows from [de Jong 1996] and [Deligne 1980, 3.3.9]. Therefore it also holds for all algebraic spaces.

For a general algebraic stack $\mathscr{X}_0$, by shrinking it we can assume it is connected smooth. By Poincaré duality, it suffices to prove $\Psi^*(\mathscr{X}_0, i)$ for all i. This can be done by taking a hypercover by simplicial algebraic spaces, and considering the associated spectral sequence. □

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Multiplicative mimicry and improvements to the Pólya–Vinogradov inequality

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We study exponential sums whose coefficients are completely multiplicative and belong to the complex unit disc. Our main result shows that such a sum has substantial cancellation unless the coefficient function is essentially a Dirichlet character. As an application we improve current bounds on odd-order character sums. Furthermore, conditionally on the generalized Riemann hypothesis we obtain a bound for odd-order character sums which is best possible.

1. Introduction

Character sums, which encode information on the distribution of primes in arithmetic progressions, have played a central role in the history of analytic number theory. On the assumption of the generalized Riemann hypothesis (GRH), Montgomery and Vaughan [1977] determined an upper bound on character sums which was known to be best-possible for quadratic characters. More recently, under the assumption of the GRH, Granville and Soundararajan [2007] proved that the Montgomery–Vaughan bound is optimal for characters of every even order. In the same work, they also made breakthroughs in our understanding of odd-order character sums. In the present paper, we develop their ideas further and (again conditionally on the GRH) obtain a best-possible bound on character sums for characters of every odd order, thus completing the story.

Our results on character sums will follow from a more general result, which we discuss first. Let $\mathbb{U}$ denote the closed complex unit disc $\{|z| \le 1\}$, and set

$$\mathcal{F} = \{ f : \mathbb{Z} \to \mathbb{U} \mid f \text{ is completely multiplicative} \}, \tag{1-1}$$

that is, for all integers m and n, $f(mn) = f(m)f(n)$ and $|f(n)| \le 1$. Consider the exponential sum

$$\sum_{n \le x} \frac{f(n)}{n} e(n\alpha), \tag{1-2}$$

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where $f \in \mathcal{F}$, $\alpha \in \mathbb{R}$, and $e(X) = e^{2\pi i X}$. By the triangle inequality, this sum has magnitude $\ll \log x$; moreover, this trivial bound is attained in the case $f(n) \equiv 1$ and $\alpha = 0$.[1] However, the sum cannot in general be this large unless there is a correlation between the behavior of $f(n)$ and $e(n\alpha)$, an unlikely event given that f is completely multiplicative and $e(n\alpha)$ has an additive structure. Perhaps surprisingly, this unlikely scenario does occur nontrivially: taking $f = \chi_{-4}$ (the nontrivial Dirichlet character (mod 4)) and $\alpha = \frac{1}{4}$, we see that $f(n) = e\left(-\frac{1}{4}\right)e(n\alpha)$ for all odd integers n, from which one can deduce that the magnitude of the exponential sum (1-2) is $\gg \log x$. Our first result (Theorem 1) shows that this is essentially the only type of pathological example; precisely, we will show that if the sum has large magnitude, then $f(n)$ must closely mimic the behavior of a function of the form $\xi(n)n^{it}$, where ξ is a Dirichlet character of small conductor and t is a small real number. Moreover, the twist by n^{it} is almost certainly superfluous (see Conjecture 2.6).

Results of this type have been obtained before. Halász [1971] realized that the mean value of $f \in \mathcal{F}$ is small (in fact, zero) unless $f(n)$ mimics the behavior of a function of the form n^{it}. Much more recently, Granville and Soundararajan [2007] proved that a character sum $\sum \chi(n)$ has small magnitude unless χ mimics the behavior of a Dirichlet character ξ of small conductor and opposite parity. The first part of the present paper is devoted to creating a hybrid of these two methods. When combined with results of Montgomery and Vaughan, this leads to strong bounds on exponential sums of the shape (1-2).

Before we can state our main results, we must set up some notation. A common feature in Halász's and Granville and Soundararajan's work is a measure of how closely one function in $\mathcal{F}$ mimics another. We call this the *multiplicative mimicry* (MM) *metric*:

Definition (multiplicative mimicry metric). For any $f, g \in \mathcal{F}$ and any positive X, set

$$\mathbb{D}(f, g; X) := \left(\sum_{p \le X} \frac{1 - \operatorname{Re} f(p)\overline{g(p)}}{p} \right)^{1/2}. \tag{1-3}$$

Note that because f and g are completely multiplicative, their behavior is entirely determined by their values at prime arguments, so the above definition uses all the data on the behavior of f and g (up to X). Granville and Soundararajan [2007] observed that this is a pseudometric — in particular, it satisfies a triangle inequality: $\mathbb{D}(f_1, g_1; X) + \mathbb{D}(f_2, g_2; X) \ge \mathbb{D}(f_1 f_2, g_1 g_2; X)$ for any $f_i, g_i \in \mathcal{F}$. (The only way in which this measure fails to be an honest metric is the possibility that the distance from f to itself might be nonzero.) Further discussion of this

[1] Here and throughout we use Vinogradov's notation $f \ll g$ to mean $f = O(g)$.

pseudometric and some unexpected applications of the triangle inequality can be found in [Granville and Soundararajan 2008].

Halász [1971] proved that the mean value of a function $f \in \mathcal{F}$ is 0 unless $\mathbb{D}(f(n), n^{it}, \infty) \ll 1$ for some $t \in \mathbb{R}$; moreover, if such a t exists, it is unique. Montgomery [1978] and, subsequently, Tenenbaum [1995, §III.4.3] found that to further quantify Halász's result it is convenient to introduce a measure which is closely related to the MM metric:

$$\mathcal{M}(f; X, T) := \min_{|t|\le T} \mathbb{D}\big(f(n), n^{it}; X\big)^2. \tag{1-4}$$

Essentially, this is measuring how closely f can mimic a function of the form n^{it}. Our main theorem will likewise be stated in terms of this quantity.

For our intended applications, we will need to control the size of the prime factors of the argument. To this end, let $\mathcal{S}(y)$ denote the set of y-smooth numbers:

$$\mathcal{S}(y) := \{n \ge 1 : p \le y \text{ for every prime } p|n\}. \tag{1-5}$$

We can now state a version of our main theorem (for a stronger but more technical statement, see Theorem 2.1):

Theorem 1. *Let $\mathcal{F}$, $\mathcal{M}$, and $\mathcal{S}(y)$ be defined as in* (1-1), (1-4), *and* (1-5), *respectively. Suppose that $x \ge 2$, $y \ge 16$, $\alpha \in \mathbb{R}$, $f \in \mathcal{F}$, and that as ψ ranges over all primitive Dirichlet characters of conductor less than* $\log y$, $\mathcal{M}(f\bar{\psi}; y, \log^2 y)$ *is minimized when $\psi = \xi$. Then*

$$\sum_{\substack{n\le x\\ n\in\mathcal{S}(y)}} \frac{f(n)}{n} e(n\alpha) \ll (\log y)e^{-\mathcal{M}(f\bar{\xi}; y, \log^2 y)} + (\log y)^{2/3+o(1)},$$

where the implicit constant is absolute and $o(1) \to 0$ as $y \to \infty$.

Remarks. (i) Colloquially, the theorem asserts that there is lots of cancellation in the exponential sum unless $f(n) \approx \xi(n)n^{it}$ for many small n, where ξ is some Dirichlet character of small conductor and t is a small real number.

(ii) Formally, the bound is independent of x. However, note that for all $y \ge x$ the condition $n \in \mathcal{S}(y)$ becomes superfluous, so if this is the case we can replace all appearances of y by x on the right-hand side of the bound.

(iii) As stated, the theorem is uniform in α. See Theorem 2.1 for a quantitative version which is explicit in the dependence on α.

In the second half of this paper we apply this method to the study of character sums. Given a Dirichlet character $\chi \pmod q$, we wish to understand the behavior of the associated character sum function

$$S_\chi(t) := \sum_{n \le t} \chi(n).$$

The importance of this function is perhaps most easily seen in its intimate connection to the Dirichlet L-functions: partial summation on $L(s, \chi)$ leads to the following expression, valid whenever $\operatorname{Re} s > 0$:

$$L(s, \chi) = s \int_1^\infty \frac{1}{t^{s+1}} S_\chi(t)\, dt.$$

In the reverse direction, Perron's formula shows that for any $c > 1$ and any $t \notin \mathbb{Z}$,

$$S_\chi(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} L(s, \chi) x^s \frac{ds}{s}.$$

The behavior of the character sum function is not well understood, but some progress has been made in studying its magnitude. The first breakthrough occurred in 1918, when Pólya and Vinogradov independently proved that for all t,

$$|S_\chi(t)| \ll \sqrt{q} \log q. \tag{1-6}$$

This is superior to the trivial bound $|S_\chi(t)| \le t$ for all t larger than $q^{1/2+\epsilon}$, and is close to being sharp; for all primitive $\chi \pmod q$,

$$\max_{t \le q} |S_\chi(t)| \gg \sqrt{q}.$$

(A slick proof of this is to apply partial summation to the Gauss sum

$$\tau(\chi) := \sum_{n \le q} \chi(n) e\Big(\frac{n}{q}\Big) \tag{1-7}$$

and use the classical result that for primitive $\chi \pmod q$, $|\tau(\chi)| = \sqrt{q}$.)

The Pólya–Vinogradov inequality naturally suggests two distinct research goals: to obtain nontrivial bounds for short character sums, and to improve (1-6) for long sums. Great progress has been made in the former by Burgess, although the current state of knowledge still falls far short of the bound $|S_\chi(t)| \ll_\epsilon q^\epsilon \sqrt{t}$ implied by the GRH. The other path, that of sharpening the Pólya–Vinogradov inequality for long sums, saw little progress until [Montgomery and Vaughan 1977], which proved on the assumption of the GRH that

$$|S_\chi(t)| \ll \sqrt{q} \log \log q. \tag{1-8}$$

Given the strength of the hypothesis this improvement may seem a bit precious, but in fact it is a best-possible result: Paley [1932] constructed an infinite class of quadratic characters $\{\chi_n \pmod{q_n}\}$ for which

$$\max_{t\le q}|S_{\chi_n}(t)| \gg \sqrt{q_n}\log\log q_n.$$

Unconditionally, however, there were no asymptotic improvements[2] of the Pólya–Vinogradov inequality for long sums until the recent breakthroughs of Granville and Soundararajan [2007]. Among other results, these authors showed that for primitive characters $\chi \pmod q$ of odd order one can unconditionally improve the Pólya–Vinogradov bound by a power of $\log q$ and, conditionally on the GRH, the Montgomery–Vaughan estimate by a power of $\log\log q$. The following theorem, which will be an immediate consequence of Theorems 2.9 and 2.10, improves Granville and Soundararajan's conditional and unconditional bounds alike; see the remarks following the theorem.

Theorem 2. *For every primitive Dirichlet character $\chi \pmod q$ of odd order g,*

$$|S_\chi(t)| \ll_g \sqrt{q}(\log Q)^{1-\delta_g+o(1)},$$

where $\delta_g := 1 - \frac{g}{\pi}\sin\frac{\pi}{g}$ and

$$Q = \begin{cases} q & \textit{unconditionally},\\ \log q & \textit{conditionally on the GRH.}\end{cases}$$

The implicit constant depends only on g, and $o(1) \to 0$ as $q \to \infty$.

Remarks. (i) Our conditional estimate was conjectured in [Granville and Soundararajan 2007].

(ii) $\delta_3 \approx 0.173$, so $1-\delta_3$ is slightly smaller than $5/6$.

(iii) Theorem 2 saves a factor of $(\log Q)^{\delta_g/2}$ over the Granville–Soundararajan bounds (see [Granville and Soundararajan 2007, Theorems 1 and 4]).

(iv) The only step in our argument requiring the GRH is Proposition 2.8.

Finally, we show that the conditional estimate in Theorem 2 is best-possible:

Theorem 3. *Assume the GRH. Then for any odd integer $g \ge 3$, there exists an infinite family of characters $\chi \pmod q$ of order g such that*

$$\max_{t\le q}|S_\chi(t)| \gg_{\epsilon,g} \sqrt{q}(\log\log q)^{1-\delta_g-\epsilon}.$$

In the following section, we state precise versions of our results and outline the arguments which go into proving them.

[2] There were several improvements of the implicit constant, however. Of particular note is Hildebrand [1988], which puts forward the idea that $S_\chi(t)$ can only have large magnitude if χ mimics closely the behavior of a character of very small conductor. It was the development of this idea which led to the work of Granville and Soundararajan, and subsequently to the present paper.

2. Precise statements of results and sketches of their proofs

It has long been understood that cancellation in exponential sums with arithmetic coefficients is closely related to the diophantine properties of α. To state this more precisely, recall Dirichlet's theorem on diophantine approximation: given any $M \geq 2$ there exists a rational number b/r such that

$$1 \leq r \leq M, \quad (b,r)=1, \quad \text{and} \quad \left|\alpha - \frac{b}{r}\right| \leq \frac{1}{rM}. \tag{2-1}$$

Montgomery and Vaughan [1977] showed that there is cancellation in the exponential sum (1-2) for α belonging to a "minor arc", that is, for those α admitting a diophantine approximation by a rational number with a large denominator. Our main result complements this by showing that there is substantial cancellation in the sum (1-2) even for those α not admitting such a rational approximation, unless both $f(n)$ and α are rather special: $f(n)$ must mimic a function of the form $\xi(n)n^{it}$ for some primitive Dirichlet character $\xi \pmod m$, and the denominator r of the diophantine approximation for α given by (2-1) must be a multiple of the "exceptional modulus" m. Formally:

Theorem 2.1. *Let $\mathcal{F}$, $\mathcal{M}$, and $\mathcal{S}(y)$ be defined as in* (1-1), (1-4), *and* (1-5), *respectively. Suppose that $x \geq 2$, $y \geq 16$, $\alpha \in \mathbb{R}$, $f \in \mathcal{F}$, and that as ψ ranges over all primitive Dirichlet characters of conductor less than* $\log y$, *$\mathcal{M}(f\overline{\psi}; y, \log^2 y)$ is minimized when $\psi = \xi \pmod m$. Set*

$$M = \exp\left(\exp\frac{\log\log y}{\log\log\log y}\right).$$

(I) *If there exists b/r satisfying* (2-1) *with $r > \log y$, then*

$$\sum_{\substack{n\leq x\\ n\in\mathcal{S}(y)}} \frac{f(n)}{n} e(n\alpha) \ll (\log y)^{\frac{1}{2}+o(1)}.$$

(II) *If there exists a rational number of the form b/r such that* (2-1) *holds with $r \leq \log y$ and $m \nmid r$, then*

$$\sum_{\substack{n\leq x\\ n\in\mathcal{S}(y)}} \frac{f(n)}{n} e(n\alpha) \ll (\log y)^{\frac{2}{3}+o(1)}.$$

(III) *If no rational numbers satisfy the hypotheses of* (I) *or* (II), *then*

$$\sum_{\substack{n\leq x\\ n\in\mathcal{S}(y)}} \frac{f(n)}{n} e(n\alpha) \ll \frac{\sqrt{m}}{\varphi(m)}(\log y)\, e^{-\mathcal{M}(f\bar{\xi};\, y, \log^2 y)} + \frac{1}{\sqrt{r}}(\log y)^{\frac{2}{3}+o(1)} + (\log y)^{\frac{1}{2}+o(1)}.$$

All implicit constants are absolute, and $o(1) \to 0$ as $y \to \infty$.

Remarks. (i) We expect that the twist by n^{it} is superfluous. In other words, taking $\xi \pmod m$ to be the nearest primitive Dirichlet character to $f(n)$ with respect to the MM metric, the above theorem should hold with $\mathcal{M}(f\overline{\psi};\, y, \log^2 y)$ replaced throughout by $\mathbb{D}(f, \psi; y)^2$. See Conjecture 2.6 and the discussion preceding it for a justification of this belief.

(ii) The methods used to prove Theorem 2.1 can be applied to obtain an analogous theorem for sums of the form $\sum f(n)e(n\alpha)$ with $f \in \mathcal{F}$. In this case, in contrast with the previous remark, the twist by n^{it} will be necessary. See the discussion preceding Conjecture 2.6.

(iii) With more work, it should be possible to adapt the argument to prove a similar result under the weaker hypothesis that $f(n)$ is multiplicative (as opposed to completely multiplicative). The hypothesis that $|f(n)| \le 1$ for all n is much more delicate, however. Proving an analogous result for $f(n)$ whose magnitude grows (however slowly) to infinity would find wide application, but the methods described here seem insufficient to attack this problem.

(iv) Theorem 2.1 immediately implies Theorem 1.

We split the proof into several steps.

Step 1: Handling the minor arcs. Montgomery and Vaughan [1977] made an important breakthrough in the study of character sums by proving the upper bound (1-8) on the assumption of the generalized Riemann hypothesis (GRH). Most of their paper is devoted to (unconditionally) obtaining cancellation in sums of the form $\sum f(n)e(n\alpha)$ with f multiplicative and α admitting a rational diophantine approximation with a large denominator. To accomplish this, they first reduce the problem to studying certain bilinear forms, then develop an intricate iterated version of Dirichlet's hyperbola method to estimate this form. For our purposes, we require a variant of their bound: first, we are interested in sums of the form $\sum (f(n)/n)e(n\alpha)$, and second, we will need to control the smoothness of the argument. In Section 3 we deduce the following from Montgomery and Vaughan's theorem:

Corollary 2.2. *Given $f \in \mathcal{F}$, $\alpha \in \mathbb{R}$, and a reduced fraction b/r such that $r \ge 2$ and $|\alpha - b/r| \le 1/r^2$, we have, for $x \ge 2$ and $y \ge 16$,*

$$\sum_{\substack{n \le x \\ n \in \mathcal{S}(y)}} \frac{f(n)}{n} e(n\alpha) \ll \log r + \frac{(\log r)^{5/2}}{\sqrt{r}} \log y + \log\log y,$$

where the implicit constant is absolute.

It is evident that this bound is particularly effective for those α which have a rational diophantine approximation with a large denominator. In the language of the circle method, such α constitute the *minor arcs*; all other α (that is, all of whose rational diophantine approximations have small denominators) comprise the *major arcs*. Thus, Corollary 2.2 handles the minor arcs, and it remains to tackle those α belonging to major arcs. A method to do this in the case that f is a character was developed in [Granville and Soundararajan 2007]. In addition to generalizing and streamlining Granville and Soundararajan's argument somewhat, we introduce a new ingredient: the work of Halász, Montgomery, and Tenenbaum on mean values of multiplicative functions. We describe how this is done in the next three steps of our outline.

Step 2: The Granville–Soundararajan identity. In Section 4 we prove Lemma 4.1, which will allow us to replace α by a rational diophantine approximation in the exponential sum at the cost of possibly shortening the range of summation slightly and adding a negligible error. More precisely, under a weak technical hypothesis (easily satisfied in our situation), it will assert the existence of an $N \leq x$ such that

$$\sum_{\substack{n\leq x\\ n\in\mathcal{S}(y)}} \frac{f(n)}{n} e(n\alpha) = \sum_{\substack{n\leq N\\ n\in\mathcal{S}(y)}} \frac{f(n)}{n} e\Big(\frac{b}{r}n\Big) + O(\log\log y).$$

It is worth noting that while our choice of N will be dependent on α, the implicit constant in the error term will be absolute.

This step allows us to focus on the case of rational α. An identity that is implicit in [Granville and Soundararajan 2007, Section 6.2], and whose proof can be found in Section 4, gets right to the heart of the matter:

Proposition 2.3 (Granville–Soundararajan identity[3]). *Given integers b and r such that $(b, r) = 1$ with $b \neq 0$ and $r \geq 1$, we have, for all $f \in \mathcal{F}$, $N \geq 2$ and $y \geq 2$,*

$$\sum_{\substack{n\leq N\\ n\in\mathcal{S}(y)}} \frac{f(n)}{n} e\Big(\frac{b}{r}n\Big) = \sum_{\substack{d\mid r\\ d\in\mathcal{S}(y)}} \frac{f(d)}{d}\cdot\frac{1}{\varphi(r/d)} \sum_{\psi\ (\mathrm{mod}\ r/d)} \tau(\psi)\overline{\psi}(b) \sum_{\substack{n\leq N/d\\ n\in\mathcal{S}(y)}} \frac{f(n)\overline{\psi}(n)}{n}.$$

In the case that α belongs to a major arc, r will be small, so the only factor on the right-hand side which can make a significant contribution is the innermost sum. We thus must turn our attention to sums of the form

$$\sum_{\substack{n\leq x\\ n\in\mathcal{S}(y)}} \frac{g(n)}{n}$$

for $g \in \mathcal{F}$; it is here that we introduce significant refinements to Granville and Soundararajan's ideas.

[3]Similar identities appear in [Montgomery and Vaughan 1977; Hildebrand 1988].

Step 3: A Halász-like result. As mentioned in the introduction, Halász [1971] realized that the mean value of $f \in \mathcal{F}$ can be large only if $f(n)$ mimics a function of the form n^{it}, where this mimicry is measured by the MM metric. Montgomery [1978] reworked Halász's method to bound the magnitude of $\sum_{n\le x} f(n)$ in terms of the behavior of the generating function of f,

$$F(s) := \sum_{n=1}^{\infty} \frac{f(n)}{n^s}, \tag{2-2}$$

in a vertical strip of the complex plane. In §III.4.3 of his excellent book, Tenenbaum [1995] outlines a method of bounding $F(s)$ in terms of the quantity

$$\mathcal{M}(f; X, T) := \min_{|t|\le T} \mathbb{D}\big(f(n), n^{it}; X\big)^2.$$

In combination with Montgomery's work, this leads to an elegant quantitative version of Halász's result.

Inspired by Montgomery's reworking of Halász's method, Montgomery and Vaughan [2001] bounded

$$\sum_{n\le x} \frac{f(n)}{n}$$

in terms of $F(s)$, the generating series of f defined in (2-2). In Section 5 we apply Tenenbaum's method to the Montgomery–Vaughan bound to prove the following:

Theorem 2.4. *For $f \in \mathcal{F}$, $x \ge 2$, and $T \ge 1$,*

$$\sum_{n\le x} \frac{f(n)}{n} \ll (\log x)\, e^{-\mathcal{M}(f;\, x, T)} + \frac{1}{\sqrt{T}},$$

where $\mathcal{M}$ is defined by (1-4).

From this it is not hard to deduce the following useful corollary.

Corollary 2.5. *For $f \in \mathcal{F}$, $x \ge 2$, $y \ge 2$, and $T \ge 1$,*

$$\sum_{\substack{n\le x \\ n\in \mathcal{S}(y)}} \frac{f(n)}{n} \ll (\log y)\, e^{-\mathcal{M}(f;\, y, T)} + \frac{1}{\sqrt{T}}.$$

Remark. Taking $y = x$ in the corollary immediately yields Theorem 2.4, so the two statements are in fact equivalent.

The above simultaneously refines and generalizes [Granville and Soundararajan 2007, Lemma 4.3], and is sufficiently strong for our intended application of an optimal bound on odd-order character sums. However, we suspect that more can be said. Colloquially, our bound indicates that $\sum f(n)/n$ can be large only if $f(n)$

mimics a function of the form n^{it}. This is an artifact from the proof of the Halász–Montgomery–Tenenbaum theorem, which drew the same conclusion for the sum $\sum f(n)$. In that case, n^{it} is an actual enemy since $\sum n^{it}$ is not $o(x)$. Our situation is quite different: if $f(n)$ closely mimics n^{it}, then

$$\sum_{n\le x} \frac{f(n)}{n} \approx \zeta(1-it),$$

which is bounded so long as t is neither too small nor too large. Therefore, for sums of the form considered in Theorem 2.4, n^{it} is no longer an enemy — the only real enemy is the constant function 1. This leads us to make the following conjecture:

Conjecture 2.6. *For $f \in \mathcal{F}$ and $2 \le y \le x$,*

$$\sum_{\substack{n\le x\\ n\in\mathcal{S}(y)}} \frac{f(n)}{n} \ll 1 + (\log y)\, e^{-\mathbb{D}(f,1;y)^2}.$$

Note that the restriction $y \le x$ is necessary, as shown by the example directly following [Granville and Soundararajan 2007, Lemma 4.3].

If some form of this conjecture holds, it would improve our main results (Theorems 1, 2, and 2.1) by removing the possible twist by n^{it}, and would allow us to state all the results purely in terms of the distance from $f(n)$ to the nearest primitive character.

Step 4: Handling the major arcs. One important discovery of Granville and Soundararajan in their study of the MM metric was a repulsion principle similar to the Deuring–Heilbronn phenomenon: f cannot mimic two different characters too closely. Thus, if we identify the "exceptional character" $\xi \pmod m$ which f most nearly mimics (in the sense made precise in the statement of Theorem 2.1), then f must be quite far from mimicking any other primitive character. In their study of mean values of multiplicative functions in arithmetic progressions, Balog et al. [2007] derived explicit lower bounds on $\mathcal{M}(f\overline{\psi}; y, \log^2 y)$ for all primitive $\psi \ne \xi$.

With this in mind, we turn to major arcs. Suppose that $\alpha \approx b/r$ with r small, so that the Montgomery–Vaughan result (Corollary 2.2) is not useful. Plugging the estimate of Corollary 2.5 into the right side of the Granville–Soundararajan identity (Proposition 2.3), we quickly find an upper bound on the magnitude of the left side in terms of the quantities $\mathcal{M}(f\overline{\psi}; N/d, T)$, where T is a parameter we can specify as we wish and ψ runs over all characters of modulus dividing r. If r is not a multiple of the exceptional modulus m, then none of the characters ψ are induced by the exceptional character ξ; the repulsion principle then implies that $\mathcal{M}(f\overline{\psi}; y, \log^2 y)$ is bounded from below for all ψ in the sum, meaning that the contribution from each character to the sum is not too large.

If on the other hand m divides r, then some of the characters we are summing over might be induced by the exceptional character ξ. In this case, once again using the repulsion principle, we can bound $\mathcal{M}(f\overline{\psi};\, y, \log^2 y)$ from below for all ψ which are not induced by ξ; however, there will now be a main term coming from the characters induced by the exceptional character. In Section 6 we make these arguments precise and deduce the following:

Theorem 2.7. *Assume $N \geq 2$, $y \geq 16$, $f \in \mathcal{F}$, and b/r is a reduced fraction*[4] *with $1 \leq r \leq \log y$. Suppose that as ψ ranges over all primitive characters of conductor less than r, the minimum of $\mathcal{M}(f\overline{\psi};\, y, \log^2 y)$ occurs when $\psi = \xi \pmod m$. Then*

$$\sum_{\substack{n\leq N\\ n\in\mathcal{S}(y)}} \frac{f(n)}{n} e\Big(\frac{b}{r}n\Big) \ll \frac{1}{\sqrt{r}}(\log y)^{\frac{2}{3}+o(1)} + \sqrt{r}\, e^{C\sqrt{\log\log y}} + \begin{cases} \dfrac{\sqrt{m}}{\varphi(m)}(\log y)\, e^{-\mathcal{M}(f\bar{\xi};\, y, \log^2 y)} & \text{if } m \mid r, \\ 0 & \text{otherwise,} \end{cases}$$

where both C and the implicit constant are absolute, and $o(1) \to 0$ as $y \to \infty$.

This result is complementary to Corollary 2.2, which bounded the same quantity effectively for large r; combining the two yields Theorem 2.1, as will be shown in Section 7.

Having sketched the proof of Theorem 2.1, we move on to sketching the proof of Theorem 2.

Application to character sums. In their proofs of the inequality (1-6), both Pólya and Vinogradov expanded the character sum function $S_\chi(t)$ as a Fourier series (Vinogradov had earlier proved the inequality via other means). Pólya's version of the Fourier expansion is as follows: for any N,

$$S_\chi(t) = \frac{\tau(\chi)}{2\pi i} \sum_{1\leq |n|\leq N} \frac{\overline{\chi}(n)}{n}\Big(1 - e\Big(-\frac{nt}{q}\Big)\Big) + O\Big(1 + \frac{q\log q}{N}\Big), \tag{2-3}$$

where $\tau(\chi)$ denotes the Gauss sum, defined in (1-7). For any primitive Dirichlet character $\chi \pmod q$, $|\tau(\chi)| = \sqrt{q}$, so we are left to study sums of the form

$$\sum_{1\leq |n|\leq N} \frac{\overline{\chi}(n)}{n} e(n\alpha). \tag{2-4}$$

Needless to say, this looks very similar to the sums seen in Theorems 1 and 2.1, aside from n running over both positive and negative values. Actually, we will be able to use this symmetry to our advantage. As a simple illustration of this, we note that if χ has odd order and $\alpha = 0$, the sum (2-4) vanishes.

[4]We adopt the convention that the reduced form of 0 is $\frac{0}{1}$.

One important consequence of the GRH is that, for some of the most fundamental sums which occur in multiplicative number theory, the bulk of the contribution comes from the so-called *smooth* arguments, that is, those with no large prime factors — see (1-5) for the precise definition.[5] The following proposition is due to Granville and Soundararajan, and is the only step in our argument which depends on the GRH.

Proposition 2.8. *Assume the GRH. Then for all primitive Dirichlet characters* $\chi \pmod q$ *we have*

$$\sum_{n\le x} \frac{\overline{\chi}(n)}{n} e(n\alpha) = \sum_{\substack{n\le x\\ n\in\mathcal{S}(y)}} \frac{\overline{\chi}(n)}{n} e(n\alpha) + O\Big(\frac{(\log q)(\log ex)}{y^{1/6}}\Big)$$

uniformly for $1 \le x \le q^{3/2}$, $y \ge 1$, *and all* α.

Proof. This follows immediately from [Granville and Soundararajan 2007, Lemma 5.2] by partial summation. □

A precursor of this result, with $\alpha = 0$, was proved in [Montgomery and Vaughan 1977, Lemma 2].

Very slightly modifying the method used to prove Theorem 2.1, we will show (in Section 7) that

$$\sum_{1\le |n|\le q} \frac{\overline{\chi}(n)}{n} e(n\alpha) \ll \big(1-\chi(-1)\xi(-1)\big)\frac{\sqrt{m}}{\varphi(m)}(\log Q)e^{-\mathcal{M}(\chi\bar{\xi};\, Q, \log^2 Q)} + (\log Q)^{2/3+o(1)},$$

where the implicit constant is absolute and $o(1) \to 0$ as $q \to \infty$. Colloquially, this indicates that there is a lot of cancellation in the sum on the left-hand side unless $\chi(n)$ mimics $\xi(n)n^{it}$ for some primitive Dirichlet character ξ of opposite parity and small conductor, and some small real number t.

Combining this bound with Pólya's Fourier expansion (2-3) we immediately deduce the following:

Theorem 2.9. *Given a primitive Dirichlet character* $\chi \pmod q$, *set*

$$Q = \begin{cases} q & \textit{unconditionally}, \\ (\log q)^{12} & \textit{conditionally on the GRH.} \end{cases}$$

[5] Recall, for example, Littlewood's celebrated result that, on the GRH, $L(1,\chi)$ is well approximated by a short Euler product for any primitive Dirichlet character $\chi \pmod q$. Expanding the product, his result can be roughly written down in the following form: assuming the GRH, $L(1,\chi) \approx \sum_{n\in\mathcal{S}((\log q)^2)} \chi(n)/n$. See [Littlewood 1928] for the original argument, or [Granville and Soundararajan 2003, Section 2] for some unconditional versions.

Suppose that as ψ ranges over all primitive characters of conductor less than $\log Q$, $\mathcal{M}(\chi\overline{\psi};\, Q, \log^2 Q)$ *is minimized when* $\psi = \xi \pmod m$. *Then*

$$\max_{t\le q}|S_\chi(t)| \\ \ll \big(1-\chi(-1)\xi(-1)\big)\frac{\sqrt{m}}{\varphi(m)}\sqrt{q}(\log Q)\,e^{-\mathcal{M}(\chi\bar{\xi};\, Q,\log^2 Q)} + \sqrt{q}(\log Q)^{2/3+o(1)},$$

where the implicit constant is absolute and $o(1)\to 0$ *as* $q\to\infty$.

Remark. This refines the main term and sharpens the error term of [Granville and Soundararajan 2007, Theorems 2.1 and 2.4].

To conclude the proof of Theorem 2, it remains only to show that given any primitive Dirichlet character $\chi \pmod q$ of odd order, and any primitive character ξ of small conductor and opposite parity, $\chi(n)$ cannot mimic too closely the behavior of $\xi(n)n^{it}$ for small t. This is reminiscent of [Granville and Soundararajan 2007, Lemma 3.2], wherein Granville and Soundararajan proved the same statement in the special case that $t=0$. Unfortunately, their argument does not generalize easily, and we are forced to introduce several new ingredients. These are discussed at the beginning of Section 8, in which we will prove the following:

Theorem 2.10. *Let* $y\ge 3$, *a primitive character* $\chi \pmod q$ *of odd order* g, *and an odd character* $\xi \pmod m$ *with* $m<(\log y)^A$ *be given. Then*

$$\mathcal{M}(\chi\bar{\xi};\, y, \log^2 y) \ge \big(\delta_g + o(1)\big)\log\log y,$$

where $o(1)\to 0$ *as* $y\to\infty$ *for any fixed values of* g *and* A.

Using the bound from Theorem 2.10 in the one from Theorem 2.9, we deduce Theorem 2.

We conclude the paper with a proof of Theorem 3, which shows that, conditionally on the GRH, our bound on odd-order character sums is best possible.

This concludes our outline. We summarize it, more briefly, before carrying out the arguments. Section 3 builds on the work of Montgomery and Vaughan estimating the minor arc contributions to the exponential sum $\sum(f(n)/n)e(n\alpha)$, culminating in Corollary 2.2. In Section 4 we prove two elementary results which inform the rest of our arguments: Lemma 4.1 shows that it suffices to consider the case of rational α, and an identity of Granville and Soundararajan further reduces the problem to considering a sum of a type previously investigated by Montgomery and Vaughan. In Section 5 we apply Tenenbaum's method to Montgomery and Vaughan's bound to obtain Corollary 2.5, a variation on the Halász–Montgomery–Tenenbaum bound for mean values of multiplicative functions. This puts us in the position to treat the major arcs and prove Theorem 2.7, which we do in Section 6. In Section 7 we combine the major arc and minor arc estimates to obtain Theorem 2.1, and subsequently deduce the bound on character sums given by Theorem 2.9. In

Section 8, we show that a primitive character of odd order cannot mimic too closely any function of the form $\xi(n)n^{it}$, where ξ is a character of even order and small conductor; this is Theorem 2.10. Finally, in Section 9, we prove Theorem 3.

3. The minor arc case: Proof of Corollary 2.2

We begin by recalling a result of Montgomery and Vaughan:

Theorem 3.1 (Montgomery–Vaughan). *Suppose $f \in \mathcal{F}$ and $|\alpha - b/r| \le 1/r^2$ with $(b,r) = 1$. Then for every $R \in [2, r]$ and any $N \ge Rr$ we have*

$$\sum_{Rr \le n \le N} \frac{f(n)}{n} e(n\alpha) \ll \log\log N + \frac{(\log R)^{3/2}}{\sqrt{R}} \log N,$$

where the implicit constant is absolute.

Proof. This follows immediately from [Montgomery and Vaughan 1977, Corollary 1] by partial summation. Our formulation of this theorem is lifted from [Granville and Soundararajan 2007, Lemma 4.2]. □

Montgomery and Vaughan's proof of the above theorem required both ingenuity and hard analysis, as might be expected in a minor arc estimate. With their result in hand, we can deduce the following corollary (which is modeled on [Granville and Soundararajan 2007, Lemma 6.1]) without much exertion.

Corollary 2.2. *Given $f \in \mathcal{F}$, $\alpha \in \mathbb{R}$, and a reduced fraction b/r such that $r \ge 2$ and $|\alpha - b/r| \le 1/r^2$, we have, for $x \ge 2$ and $y \ge 16$,*

$$\sum_{\substack{n \le x \\ n \in \mathcal{S}(y)}} \frac{f(n)}{n} e(n\alpha) \ll \log r + \frac{(\log r)^{5/2}}{\sqrt{r}} \log y + \log\log y,$$

where the implicit constant is absolute.

Prior to proving this, we introduce one more piece of notation. Given $f : \mathbb{Z} \to \mathbb{C}$ and any positive number y, we define the y-smoothed function f_y:

$$f_y(n) = \begin{cases} f(n) & \text{if } n \in \mathcal{S}(y), \\ 0 & \text{otherwise.} \end{cases} \tag{3-1}$$

Note that if $f \in \mathcal{F}$, then $f_y \in \mathcal{F}$ as well.

Proof. The bound is trivially true for $x \le r^2$, so we assume $x > r^2$.

First, note that for $x \le y^{\log r}$ the claim follows from Theorem 3.1 applied to f_y:

$$\sum_{\substack{n\le x\\ n\in\mathcal{S}(y)}} \frac{f(n)}{n}e(n\alpha) = \sum_{n\le x} \frac{f_y(n)}{n}e(n\alpha) = \sum_{n<r^2} \frac{f_y(n)}{n}e(n\alpha) + \sum_{r^2\le n\le x} \frac{f_y(n)}{n}e(n\alpha)$$
$$\ll \log r + \frac{(\log r)^{3/2}}{\sqrt{r}}\log x + \log\log x$$
$$\ll \log r + \frac{(\log r)^{5/2}}{\sqrt{r}}\log y + \log\log y.$$

It therefore suffices to bound

$$\sum_{\substack{y^{\log r}<n\le x\\ n\in\mathcal{S}(y)}} \frac{f(n)}{n}e(n\alpha).$$

Since $n > y^{\log r}$ if and only if $n > r\cdot n^{1-1/(\log y)}$,

$$\sum_{\substack{y^{\log r}<n\le x\\ n\in\mathcal{S}(y)}} \frac{f(n)}{n}e(n\alpha) \ll \frac{1}{r}\sum_{\substack{y^{\log r}<n\le x\\ n\in\mathcal{S}(y)}} \frac{1}{n^{1-1/(\log y)}} \le \frac{1}{r}\prod_{p\le y}\left(1-\frac{1}{p^{1-1/(\log y)}}\right)^{-1}.$$

By the prime number theorem,

$$\log\prod_{p\le y}\left(1-\frac{1}{p^{1-1/(\log y)}}\right)^{-1} = \sum_{p\le y}\frac{1}{p^{1-1/(\log y)}} + O(1) = \log\log y + O(1).$$

It follows that

$$\sum_{\substack{y^{\log r}<n\le x\\ n\in\mathcal{S}(y)}} \frac{f(n)}{n}e(n\alpha) \ll \frac{1}{r}\log y,$$

and the corollary is proved. □

4. Reduction to rational α and the Granville–Soundararajan identity

We now begin our approach towards the major arcs. We begin by reducing the problem to the case of rational α. The following bound is inspired by [Granville and Soundararajan 2007, Lemma 6.2]:

Lemma 4.1. *Assume $f \in \mathcal{F}$, $\alpha \in \mathbb{R}$, $x \ge 16$, $y \ge 16$, and $M \ge 2$. Suppose the reduced fraction b/r with $r \le M$ is a rational diophantine approximation to α, that is,*

$$\left|\alpha - \frac{b}{r}\right| \le \frac{1}{rM}.$$

Set $N = \min\left\{x, \dfrac{1}{|r\alpha - b|}\right\}$. *Then for all* $R \in \left[2, \dfrac{N}{2}\right]$,

$$\sum_{\substack{n\le x\\ n\in\mathcal{S}(y)}} \frac{f(n)}{n} e(n\alpha) = \sum_{\substack{n\le N\\ n\in\mathcal{S}(y)}} \frac{f(n)}{n} e\Big(\frac{b}{r}n\Big) + O\left(\log R + \frac{(\log R)^{3/2}}{\sqrt{R}}(\log y)^2 + \log\log y\right),$$

where the implied constant in the error term is absolute. Moreover, the error term above can be replaced by $O(\log\log y)$ *if* $M \ge 2(\log y)^4 \log\log y$.

Remarks. (i) For our intended applications, we will be able to choose an M much larger than $2(\log y)^4 \log\log y$.

(ii) The actual value of N is unimportant; what is important is that $M \le N \le x$.

Proof. If $N = x$ then $\left|\alpha - \frac{b}{r}\right| \le \frac{1}{rx}$ whence

$$\sum_{\substack{n\le x\\ n\in\mathcal{S}(y)}} \frac{f(n)}{n}\left(e(n\alpha) - e\Big(\frac{b}{r}n\Big)\right) \ll \sum_{\substack{n\le x\\ n\in\mathcal{S}(y)}} \frac{1}{n}\cdot n\left|\alpha - \frac{b}{r}\right| \ll 1.$$

We therefore assume that $N = \dfrac{1}{|r\alpha - b|} < x$. Note that this immediately implies that $N \ge M$ and that

$$\left|\alpha - \frac{b}{r}\right| = \frac{1}{rN}.$$

By Dirichlet's theorem, there is a reduced fraction $\dfrac{b_1}{r_1}$ with $r_1 \le 2N$ such that

$$\left|\alpha - \frac{b_1}{r_1}\right| \le \frac{1}{2r_1N}.$$

Note that $\dfrac{b}{r} \ne \dfrac{b_1}{r_1}$, since $\left|\alpha - \dfrac{b_1}{r_1}\right| < \dfrac{1}{r_1N}$. Thus,

$$\frac{1}{rr_1} \le \left|\frac{b}{r} - \frac{b_1}{r_1}\right| \le \frac{1}{2r_1N} + \frac{1}{rN},$$

whence $r_1 \ge N - \dfrac{r}{2}$. Since $r \le M \le N$, we see that

$$\frac{N}{2} \le r_1 \le 2N,$$

so we can trivially bound the (possibly empty) sum

$$\sum_{\substack{N<n\le Rr_1\\ n\in\mathcal{S}(y)}} \frac{f(n)}{n} e(n\alpha) \ll \log\frac{Rr_1}{N} = \log R + O(1).$$

Once again applying Montgomery–Vaughan's Theorem 3.1 to f_y (which we can do since $R \le N/2 \le r_1$) we see that

$$\sum_{\substack{Rr_1 < n \le e^{(\log y)^2} \\ n \in \mathcal{S}(y)}} \frac{f(n)}{n} e(n\alpha) = \sum_{Rr_1 < n \le e^{(\log y)^2}} \frac{f_y(n)}{n} e(n\alpha) \ll \log\log y + \frac{(\log R)^{3/2}}{\sqrt{R}} (\log y)^2.$$

Finally, using the same device as in the proof of Corollary 2.2, we see that

$$\sum_{\substack{e^{(\log y)^2} < n \le x \\ n \in \mathcal{S}(y)}} \frac{f(n)}{n} e(n\alpha) \ll \sum_{\substack{e^{(\log y)^2} < n \le x \\ n \in \mathcal{S}(y)}} \frac{1}{n} \ll \frac{1}{y} \sum_{n \in \mathcal{S}(y)} \frac{1}{n^{1-1/\log y}} \ll 1.$$

Combining these three bounds, we deduce

$$\sum_{\substack{n \le x \\ n \in \mathcal{S}(y)}} \frac{f(n)}{n} e(n\alpha) = \sum_{\substack{n \le N \\ n \in \mathcal{S}(y)}} \frac{f(n)}{n} e(n\alpha) + O\Big(1 + \log R + \frac{(\log R)^{3/2}}{\sqrt{R}} (\log y)^2 + \log\log y\Big).$$

Just as at the start of the proof, we have

$$\sum_{\substack{n \le N \\ n \in \mathcal{S}(y)}} \frac{f(n)}{n} e(n\alpha) = \sum_{\substack{n \le N \\ n \in \mathcal{S}(y)}} \frac{f(n)}{n} e\Big(\frac{b}{r} n\Big) + O(1)$$

and we conclude the proof of the first part of the theorem.

For the second claim, if $M \ge 2(\log y)^4 \log\log y$, then

$$r_1 \ge N - \frac{r}{2} \ge M - \frac{M}{2} \ge (\log y)^4 \log\log y.$$

Taking $R = (\log y)^4 \log\log y$ renders the error $O(\log\log y)$. □

We now suppose that α is rational. The following identity, essentially due to Granville and Soundararajan, highlights the key contributors to the major arcs.

Proposition 2.3 (Granville–Soundararajan identity). *Given integers b and r such that $(b, r) = 1$ with $b \ne 0$ and $r \ge 1$, we have, for all $f \in \mathcal{F}$, $N \ge 2$ and $y \ge 2$,*

$$\sum_{\substack{n \le N \\ n \in \mathcal{S}(y)}} \frac{f(n)}{n} e\Big(\frac{b}{r} n\Big) = \sum_{\substack{d \mid r \\ d \in \mathcal{S}(y)}} \frac{f(d)}{d} \cdot \frac{1}{\varphi(r/d)} \sum_{\psi \pmod{r/d}} \tau(\psi)\overline{\psi}(b) \sum_{\substack{n \le N/d \\ n \in \mathcal{S}(y)}} \frac{f(n)\overline{\psi}(n)}{n}.$$

Thus for small r, the left-hand side can be large only if

$$\sum_{\substack{n \le N/d \\ n \in \mathcal{S}(y)}} \frac{f(n)\overline{\psi}(n)}{n}$$

is large for some Dirichlet character ψ of conductor dividing r.

Proof. We examine the left-hand side. Summing over all possible greatest common divisors d of n and r, and setting $a = n/d$ we find

$$\sum_{\substack{n \le N \\ n \in \mathcal{S}(y)}} \frac{f(n)}{n} e\Big(\frac{b}{r} n\Big) = \sum_{\substack{d \mid r \\ d \in \mathcal{S}(y)}} \frac{f(d)}{d} \sum_{\substack{a \le N/d \\ (a, r/d) = 1 \\ a \in \mathcal{S}(y)}} \frac{f(a)}{a} e\Big(\frac{ab}{r/d}\Big). \tag{4-1}$$

Now,

$$e\Big(\frac{ab}{r/d}\Big) = \sum_{k \,(\mathrm{mod}\ r/d)} e\Big(\frac{k}{r/d}\Big) \delta_{ab}(k),$$

where δ_x is the indicator function of x. By orthogonality of characters, we can express the indicator function in terms of characters:

$$\delta_{ab}(k) = \frac{1}{\varphi(r/d)} \sum_{\psi \,(\mathrm{mod}\ r/d)} \overline{\psi}(ab) \psi(k),$$

whence, switching the order of summation,

$$e\Big(\frac{ab}{r/d}\Big) = \frac{1}{\varphi(r/d)} \sum_{\psi \,(\mathrm{mod}\ r/d)} \tau(\psi) \overline{\psi}(ab).$$

Plugging this back into (4-1) and once again switching order of summation yields the identity. □

5. A Halász-like result: Proof of Theorem 2.4

Given $f \in \mathcal{F}$, set

$$F(s) := \sum_{n=1}^{\infty} \frac{f(n)}{n^s}.$$

Note that this generating series converges in the half-plane $\mathrm{Re}\, s > 1$.

Theorem 5.1 [Montgomery and Vaughan 2001]. *For any $f \in \mathcal{F}$ and $x \ge 3$, we have*

$$\sum_{n \le x} \frac{f(n)}{n} \ll \frac{1}{\log x} \int_{1/\log x}^{1} \frac{1}{\alpha} H(\alpha)\, d\alpha,$$

where

$$H(\alpha) := \bigg(\sum_{k \in \mathbb{Z}} \max_{s \in \mathcal{B}_k(\alpha)} \left| \frac{F(s)}{s-1} \right|^2 \bigg)^{1/2}$$

and $\mathcal{B}_k(\alpha)$ is the region in the complex plane defined by

$$\mathcal{B}_k(\alpha) := \big\{ s \in \mathbb{C} : 1 + \alpha \le \sigma \le 2 \textit{ and } |t - k| \le \tfrac{1}{2} \big\}.$$

In order to deduce Theorem 2.4 from this, we use the following:

Theorem 5.2 [Tenenbaum 1995]. *Assume f and F are as above and $x \geq 3$. Then we have*

$$F(1+\alpha+it) \ll \begin{cases} (\log x)\, e^{-\mathcal{M}(f;\, x, T)} & \text{for } |t| \leq T, \\ \dfrac{1}{\alpha} & \text{for } |t| > T, \end{cases}$$

uniformly for $\alpha \in \left[\dfrac{1}{\log x}, 1\right]$.

Proof of Theorem 2.4. Applying the bound of Theorem 5.2, we estimate $H(\alpha)$ from Montgomery and Vaughan's Theorem 5.1 as follows:

$$\begin{aligned} H(\alpha) &= \Bigg(\sum_{k\in\mathbb{Z}} \max_{s\in\mathcal{B}_k(\alpha)} \left|\frac{F(s)}{s-1}\right|^2\Bigg)^{1/2} \leq \Bigg(\sum_{k\in\mathbb{Z}} \frac{1}{k^2+\alpha^2} \max_{s\in\mathcal{B}_k(\alpha)} |F(s)|^2\Bigg)^{1/2} \\ &\ll (\log x)\, e^{-\mathcal{M}(f;\, x,T)} \Bigg(\sum_{|k|\leq T-\frac{1}{2}} \frac{1}{k^2+\alpha^2}\Bigg)^{1/2} + \frac{1}{\alpha}\Bigg(\sum_{|k|>T-\frac{1}{2}} \frac{1}{k^2+\alpha^2}\Bigg)^{1/2} \\ &\ll \frac{1}{\alpha}(\log x)\, e^{-\mathcal{M}(f;\, x,T)} + (\log x)\, e^{-\mathcal{M}(f;\, x,T)} \Bigg(\sum_{k\leq T} \frac{1}{k^2}\Bigg)^{1/2} + \frac{1}{\alpha}\Bigg(\sum_{k>T-\frac{1}{2}} \frac{1}{k^2}\Bigg)^{1/2} \\ &\ll \frac{1}{\alpha}(\log x)\, e^{-\mathcal{M}(f;\, x,T)} + \frac{1}{\alpha\sqrt{T}}. \end{aligned}$$

Using this bound in Theorem 5.1 immediately yields the result. □

Proof of Corollary 2.5. Recall from Section 3 the convenient notation

$$f_y(n) := \begin{cases} f(n) & \text{if } n \in \mathcal{S}(y), \\ 0 & \text{otherwise.} \end{cases}$$

As was noted there, $f \in \mathcal{F}$ implies that $f_y \in \mathcal{F}$. Therefore, by Theorem 2.4 we have

$$\sum_{\substack{n\leq x\\ n\in\mathcal{S}(y)}} \frac{f(n)}{n} = \sum_{n\leq x} \frac{f_y(n)}{n} \ll (\log x)\, e^{-\mathcal{M}(f_y;\, x,T)} + \frac{1}{\sqrt{T}}.$$

The following calculation completes the proof:

$$\begin{aligned} \mathcal{M}(f_y;\ x, T) &= \min_{|t|\leq T} \mathbb{D}\big(f_y(n), n^{it};\ x\big)^2 = \min_{|t|\leq T} \sum_{p\leq x} \frac{1-\operatorname{Re} f_y(p)p^{-it}}{p} \\ &= \min_{|t|\leq T} \Bigg(\sum_{p\leq y} \frac{1-\operatorname{Re} f(p)p^{-it}}{p} + \sum_{y<p\leq x} \frac{1}{p}\Bigg) \\ &= \mathcal{M}(f;\ y, T) + \log\frac{\log x}{\log y} + O(1). \end{aligned}$$

□

6. The major arc case: Proof of Theorem 2.7

We first derive the claimed bound for $b \neq 0$. In this case, we can apply the Granville–Soundararajan identity (Proposition 2.3), which we rewrite in the form

$$\sum_{\substack{n \le N \\ n \in \mathcal{S}(y)}} \frac{f(n)}{n} e\Big(\frac{b}{r} n\Big) = \sum_{\substack{d \mid r \\ d \in \mathcal{S}(y)}} \frac{f(d)}{d} a(d), \tag{6-1}$$

where

$$a(d) = \frac{1}{\varphi(r/d)} \sum_{\psi \,(\mathrm{mod}\ r/d)} \tau(\psi)\overline{\psi}(b)\Big(\sum_{\substack{n \le N/d \\ n \in \mathcal{S}(y)}} \frac{f(n)\overline{\psi}(n)}{n}\Big).$$

Because we are assuming $r < \log y$, the restriction $d \in \mathcal{S}(y)$ above is superfluous.

Our first goal is to identify the *exceptional character*, the one primitive character which is the primary contributor to our exponential sum. To this end, consider the set of all primitive characters with conductor not exceeding r, where we include the constant function $\mathbf{1}$ as the primitive character (mod 1) which induces all the principal characters to larger moduli. Enumerate all of these primitive characters as $\psi_k \pmod{m_k}$ in such a way that

$$\mathcal{M}(f\overline{\psi}_1; y, \log^2 y) \le \mathcal{M}(f\overline{\psi}_2; y, \log^2 y) \le \cdots .$$

It will be seen that $\psi_1 \pmod{m_1}$ is the exceptional character for f; this is the character we called $\xi \pmod{m}$ in the statement of the theorem, and its contribution to the sum is difficult to control. We will return to this point later in the proof.

The behavior of the characters (mod r/d) is determined by the set of primitive characters inducing them, so for ease of reference we define for each $d \mid r$ the set

$$\mathcal{K}_d = \Big\{k : m_k \;\Big|\; \frac{r}{d}\Big\}.$$

Note that $|\mathcal{K}_d| = \varphi(r/d)$. We can rewrite $a(d)$ in terms of the underlying primitive characters $\{\psi_k \pmod{m_k}\}_{k \in \mathcal{K}_d}$:

$$a(d) = \frac{1}{\varphi(r/d)} \sum_{k \in \mathcal{K}_d} \tau(\psi_k\chi_0)\overline{\psi}_k(b)\chi_0(b)\Big(\sum_{\substack{n \le N/d \\ n \in \mathcal{S}(y)}} \frac{f(n)\overline{\psi}_k(n)\chi_0(n)}{n}\Big),$$

where χ_0 is the principal character (mod r/d). A straightforward calculation shows that if a character $\psi \pmod{m}$ is induced by the primitive character $\psi^* \pmod{m^*}$, then

$$\tau(\psi) = \mu\Big(\frac{m}{m^*}\Big)\psi^*\Big(\frac{m}{m^*}\Big)\tau(\psi^*).$$

Therefore,

$$a(d) = \frac{\chi_0(b)}{\varphi(r/d)} \sum_{k \in \mathcal{K}_d} \mu\Big(\frac{r}{dm_k}\Big)\psi_k\Big(\frac{r}{dm_k}\Big)\tau(\psi_k)\overline{\psi}_k(b) \sum_{\substack{n \le N/d \\ n \in \mathcal{S}(y) \\ (n, r/d)=1}} \frac{f(n)\overline{\psi}_k(n)}{n}.$$

We make one final cosmetic adjustment prior to estimating this quantity. Lemma 5 of [Hildebrand 1988] asserts that, for any $g \in \mathcal{F}$ and $x \ge 1$,

$$\sum_{\substack{n \le x \\ (n,k)=1}} \frac{g(n)}{n} = \prod_{p \mid k}\Big(1 - \frac{g(p)}{p}\Big) \sum_{n \le x} \frac{g(n)}{n} + O\Big(\big(\log\log(k+2)\big)^3\Big),$$

where the implicit constant is absolute.[6] Set $g = f\overline{\psi}$ for any Dirichlet character ψ, and let g_y be the y-smoothed version of g (defined in (3-1)). Applying Hildebrand's lemma to g_y and using the inequalities $d \le r \le y$, we see that

$$\begin{aligned}
\sum_{\substack{n \le N/d \\ n \in \mathcal{S}(y) \\ (n, r/d)=1}} \frac{f(n)\overline{\psi}(n)}{n} = \sum_{\substack{n \le N/d \\ (n, r/d)=1}} \frac{g_y(n)}{n} &= \sum_{\substack{n \le N \\ (n, r/d)=1}} \frac{g_y(n)}{n} + O(\log d) \\
&= \prod_{p \mid r/d}\Big(1 - \frac{g_y(p)}{p}\Big) \sum_{n \le N} \frac{g_y(n)}{n} + O(\log r) \\
&= \prod_{p \mid r/d}\Big(1 - \frac{f(p)\overline{\psi}(p)}{p}\Big) \sum_{\substack{n \le N \\ n \in \mathcal{S}(y)}} \frac{f(n)\overline{\psi}(n)}{n} + O(\log r).
\end{aligned}$$

Therefore, continuing our calculation from above,

$$\begin{aligned}
a(d) = \frac{\chi_0(b)}{\varphi(r/d)} &\sum_{k \in \mathcal{K}_d} \mu\Big(\frac{r}{dm_k}\Big)\psi_k\Big(\frac{r}{dm_k}\Big)\tau(\psi_k)\overline{\psi}_k(b) \\
&\times \prod_{p \mid r/d}\Big(1 - \frac{f(p)\overline{\psi}_k(p)}{p}\Big) \sum_{\substack{n \le N \\ n \in \mathcal{S}(y)}} \frac{f(n)\overline{\psi}_k(n)}{n}
\end{aligned}$$

up to an error of size

$$\ll \frac{1}{\varphi(r/d)} \sum_{k \in \mathcal{K}_d} \sqrt{m_k}\log r \ll \sqrt{\frac{r}{d}}\log r, \tag{6-2}$$

[6] See [Granville and Soundararajan 2007, Lemma 4.4] for a substantially similar result.

since $m_k | r/d$ and $|\mathcal{K}_d| = \varphi(r/d)$. Before further refining our estimate for $a(d)$, we bound the accumulation of the error (6-2) in the sum

$$\sum_{\substack{n \le N \\ n \in \mathcal{S}(y)}} \frac{f(n)}{n} e\Big(\frac{b}{r} n\Big) = \sum_{d|r} \frac{f(d)}{d} a(d).$$

Since $r < \log y$, we find that the total possible contribution from the error terms is

$$\ll \sum_{d|r} \frac{1}{d} \sqrt{\frac{r}{d}} \log r \ll \sqrt{r} \log r \ll \sqrt{r} \log\log y. \tag{6-3}$$

In view of the bound claimed in Theorem 2.7, this is negligible.

We will now show that the contribution from all the nonexceptional characters $\{\psi_k\}_{k \ge 2}$ to $a(d)$ is not terribly large. From Corollary 2.5 we deduce that

$$\begin{aligned} &\frac{\chi_0(b)}{\varphi(r/d)} \sum_{\substack{k \in \mathcal{K}_d \\ k \ge 2}} \mu\Big(\frac{r}{dm_k}\Big) \psi_k\Big(\frac{r}{dm_k}\Big) \tau(\psi_k) \overline{\psi}_k(b) \\ &\qquad \times \prod_{p | r/d} \Big(1 - \frac{f(p)\overline{\psi}_k(p)}{p}\Big) \sum_{\substack{n \le N \\ n \in \mathcal{S}(y)}} \frac{f(n)\overline{\psi}_k(n)}{n} \\ &\ll \frac{1}{\varphi(r/d)} \sum_{\substack{k \in \mathcal{K}_d \\ k \ge 2}} \sqrt{m_k} \Big(\prod_{p | r/d} \Big(1 + \frac{1}{p}\Big)\Big) \Big((\log y) e^{-\mathcal{M}(f\overline{\psi}_k;\, y, \log^2 y)} + \frac{1}{\log y}\Big). \end{aligned}$$

Note that for any $g \in \mathcal{F}$ and any $T \ge 0$ we have $0 \le \mathcal{M}(g;\, y, T) \le 2 \log\log y + O(1)$, whence

$$(\log y) e^{-\mathcal{M}(f\overline{\psi}_k;\, y, \log^2 y)} \gg \frac{1}{\log y}.$$

Also, $m_k \le r/d$ for all $k \in \mathcal{K}_d$, and

$$\prod_{p | r/d} \Big(1 + \frac{1}{p}\Big) \ll \log\log\Big(\frac{r}{d} + 2\Big).$$

Therefore, the contribution from all the $k \ge 2$ to $a(d)$ is

$$\ll \frac{1}{\varphi(r/d)} \sqrt{\frac{r}{d}} \Big(\log\log\Big(\frac{r}{d} + 2\Big)\Big) (\log y) \sum_{\substack{k \in \mathcal{K}_d \\ k \ge 2}} e^{-\mathcal{M}(f\overline{\psi}_k;\, y, \log^2 y)}.$$

To make further progress, we need lower bounds on $\mathcal{M}(f\overline{\psi}_k;\, y, \log^2 y)$ for $k \ge 2$; in other words, we wish to show that $f(n)$ cannot mimic too closely a function of the form $\psi(n) n^{it}$ so long as ψ is not induced by the exceptional character ψ_1. Fortuitously, such bounds were determined by Balog et al. [2007] in their recent study of mean values of multiplicative functions over arithmetic progressions. Lemma 3.3 of [Balog et al. 2007] asserts that, for all $k \geq 2$,

$$\mathcal{M}(f\overline{\psi}_k;\, y, \log^2 y) \geq \left(\tfrac{1}{3} + o(1)\right) \log\log y. \tag{6-4}$$

For larger values of k we can do even better: from Lemma 3.1 of the same reference we deduce that for all $k > \sqrt{\log\log y}$,

$$\mathcal{M}(f\overline{\psi}_k;\, y, \log^2 y) \geq \log\log y + O(\sqrt{\log\log y}).$$

Using these bounds in our calculations above (and keeping in mind that $|\mathcal{K}_d| = \varphi(r/d)$) we find that the contribution to $a(d)$ from all those $k \geq 2$ which are in $\mathcal{K}_d$ is

$$\ll \frac{1}{\varphi(r/d)}\sqrt{\frac{r}{d}}\left(\log\log\left(\frac{r}{d}+2\right)\right)(\log y)^{2/3+o(1)} + \sqrt{\frac{r}{d}}\left(\log\log\left(\frac{r}{d}+2\right)\right)e^{O(\sqrt{\log\log y})}.$$

Going back to (6-1), we see that the total contribution of all such terms to the sum

$$\sum_{\substack{n \leq N \\ n \in \mathcal{S}(y)}} \frac{f(n)}{n} e\left(\frac{b}{r}n\right) = \sum_{d \mid r} \frac{f(d)}{d} a(d)$$

is

$$\begin{aligned} &\ll \sum_{d \mid r} \frac{1}{d}\left(\frac{1}{\varphi(r/d)}\sqrt{\frac{r}{d}}\left(\log\log\left(\frac{r}{d}+2\right)\right)(\log y)^{2/3+o(1)} \right. \\ &\qquad\qquad\qquad\qquad \left. + \sqrt{\frac{r}{d}}\left(\log\log\left(\frac{r}{d}+2\right)\right)e^{O(\sqrt{\log\log y})}\right) \\ &\ll \sqrt{r}\left(\log\log(r+2)\right) \sum_{d \mid r}\left(\frac{1}{d}\right)^{3/2}\left(\frac{1}{\varphi(r/d)}(\log y)^{2/3+o(1)} + e^{O(\sqrt{\log\log y})}\right) \\ &\ll \frac{1}{r}\left(\log\log(r+2)\right)(\log y)^{2/3+o(1)} \sum_{d \mid r} \frac{d^{3/2}}{\varphi(d)} + \sqrt{r}\left(\log\log(r+2)\right)e^{O(\sqrt{\log\log y})}, \end{aligned}$$

where we have used the change of variables $d \leftrightarrow \dfrac{r}{d}$ in the sum. Finally, recall that

$$\frac{n}{\varphi(n)} \ll \log\log n, \qquad \log d(n) \ll \frac{\log n}{\log\log n},$$

where $d(n)$ denotes the number of divisors of n; in particular, we deduce that $d(r) \ll (\log y)^{o(1)}$ where $o(1) \to 0$ as $y \to \infty$. Using these bounds in conjunction with our above results, we deduce that the total contribution of all the primitive characters ψ_k with $k \geq 2$ is

$$\ll \frac{1}{\sqrt{r}}(\log y)^{\frac{2}{3}+o(1)} + \sqrt{r}\, e^{C\sqrt{\log\log y}},$$

where both C and the implicit constant are absolute, and $o(1) \to 0$ as $y \to \infty$.

If $m_1 \nmid r$ then $1 \notin \mathcal{K}_d$ for all $d \mid r$, which means that the exceptional character $\psi_1 \pmod{m_1}$ does not contribute anything to our exponential sum. In this case, our above estimates tell the whole story, and we conclude the proof of the theorem.

Now suppose instead that $m_1 \mid r$; in this case, we must estimate the contribution from the exceptional character $\psi_1 \pmod{m_1}$ to each $a(d)$. This character appears in our sum precisely whenever $1 \in \mathcal{K}_d$ (that is, whenever ψ_1 induces a character $(\mathrm{mod}\ r/d)$), so the total contribution of this exceptional character is

$$\sum_{d \mid r/m_1} \frac{f(d)}{d} \cdot \frac{1}{\varphi(r/d)} \mu\Big(\frac{r}{dm_1}\Big) \psi_1\Big(\frac{r}{dm_1}\Big) \tau(\psi_1) \overline{\psi}_1(b) \\ \times \Bigg(\prod_{p \mid r/(dm_1)} \Big(1 - \frac{f\overline{\psi}_1(p)}{p}\Big)\Bigg) \sum_{\substack{n \le N \\ n \in \mathcal{S}(y)}} \frac{f\overline{\psi}_1(n)}{n}.$$

Note that the product now runs over only those p dividing $r/(dm_1)$, not just those dividing r/d (it is easily seen that this extra restriction does not change the value of the product). Making the change of variables $d \leftrightarrow r/(dm_1)$, we find that ψ_1's contribution can be rewritten in the form

$$\frac{m_1}{r} \tau(\psi_1) \overline{\psi}_1(b) \Bigg(\sum_{\substack{n \le N \\ n \in \mathcal{S}(y)}} \frac{f\overline{\psi}_1(n)}{n} \Bigg) \sum_{d \mid r/m_1} f\Big(\frac{r}{dm_1}\Big) A(d), \tag{6-5}$$

where

$$A(d) = \frac{d}{\varphi(dm_1)} \mu(d) \psi_1(d) \prod_{p \mid d} \Big(1 - \frac{f\overline{\psi}_1(p)}{p}\Big).$$

Note that $A(d) = 0$ whenever $(d, m_1) \ne 1$, so only those d which are coprime to m_1 contribute to the sum in (6-5). Moreover, the same reasoning shows that we need only consider squarefree d. Therefore,

$$A(d) = \frac{1}{\varphi(m_1)} \cdot \frac{d\,\mu(d)\psi_1(d)}{\varphi(d)} \prod_{p \mid d} \Big(1 - \frac{f\overline{\psi}_1(p)}{p}\Big) = \frac{1}{\varphi(m_1)} \prod_{p \mid d} \Big(\frac{f(p) - \psi_1(p) \cdot p}{\varphi(p)} \Big) \\ \ll \frac{1}{\varphi(m_1)} \prod_{p \mid d} \Big(\frac{p+1}{p-1}\Big) \ll \frac{1}{\varphi(m_1)} \big(\log\log(d+2)\big)^2.$$

Combining this with Corollary 2.5 and (6-5) and making elementary estimates as above, we conclude that the total contribution from ψ_1 is

$$\ll \frac{\sqrt{m_1}}{\varphi(m_1)} (\log y)\, e^{-\mathcal{M}(f\overline{\psi}_1;\, y, \log^2 y)};$$

this completes the proof of Theorem 2.7 in the case $b \ne 0$.

To show that the same bound holds for the case $b = 0$, we consider two separate cases: either ψ_1 is the trivial character $\mathbf{1}$, or it isn't. In the former scenario, $m_1 = 1$, so from Corollary 2.5 we deduce that

$$\sum_{\substack{n\le N\\ n\in\mathcal{S}(y)}} \frac{f(n)}{n} \ll \frac{\sqrt{m_1}}{\varphi(m_1)} e^{-\mathcal{M}(f\overline{\psi}_1;\, y, \log^2 y)}. \tag{6-6}$$

If, on the other hand, ψ_1 is not the trivial character, then by Corollary 2.5 together with the lower bound (6-4) we find

$$\sum_{\substack{n\le N\\ n\in\mathcal{S}(y)}} \frac{f(n)}{n} \ll \frac{1}{\sqrt{r}}(\log y)^{2/3+o(1)} \tag{6-7}$$

(recall our convention that the reduced form of 0 is $\frac{0}{1}$, so $r = 1$). In either case, these bounds are subsumed by those claimed. This concludes the proof.

7. Exponential sums with multiplicative coefficients and character sums: Proofs of Theorems 2.1 and 2.9

Having dealt with both the major and minor arcs, we can now prove Theorem 2.1 without too much difficulty.

Proof of Theorem 2.1. As in the statement of the theorem, set

$$M = \exp\left(\exp \frac{\log\log y}{\log\log\log y}\right).$$

By Dirichlet's theorem on diophantine approximation, there exists a reduced fraction b/r with $1 \le r \le M$, such that

$$\left|\alpha - \frac{b}{r}\right| \le \frac{1}{rM}. \tag{7-1}$$

If the hypotheses of (I) hold (that is, if α belongs to a minor arc), Corollary 2.2 immediately implies the result claimed.

Suppose instead that the hypotheses of (I) fail to hold (that is, α belongs to a major arc). By Lemma 4.1, since $M \ge 2(\log y)^4 \log\log y$ there exists an $N \in [M, x]$ such that

$$\sum_{\substack{n\le x\\ n\in\mathcal{S}(y)}} \frac{f(n)}{n} e(n\alpha) = \sum_{\substack{n\le N\\ n\in\mathcal{S}(y)}} \frac{f(n)}{n} e\left(\frac{b}{r}n\right) + O(\log\log y).$$

Applying Theorem 2.7 immediately yields the claim for scenarios (II) and (III). □

Theorem 2.9 is not much harder:

Proof of Theorem 2.9. Taking $N = q$ in Pólya's Fourier expansion (2-3) we see that we must bound the sum

$$\sum_{1\le|n|\le q} \frac{\overline{\chi}(n)}{n} e(n\alpha)$$

for $\alpha = 0$ or $-nt/q$. As in the proof of Theorem 2.7, we treat the cases $\alpha = 0$ and $\alpha \neq 0$ separately, starting with the latter.

Recall from the statement of the theorem that we set

$$Q = \begin{cases} q & \text{unconditionally,} \\ (\log q)^{12} & \text{conditionally on the GRH.} \end{cases}$$

We use Proposition 2.8 to restrict attention to smooth arguments, in the case that the GRH is assumed:

$$\sum_{1\le|n|\le q} \frac{\overline{\chi}(n)}{n} e(n\alpha) = \sum_{\substack{1\le|n|\le q \\ n\in\mathcal{S}(Q)}} \frac{\overline{\chi}(n)}{n} e(n\alpha) + O(1). \tag{7-2}$$

Note that this holds unconditionally as well, albeit with a superfluous error term. We next find a diophantine rational approximation to α, that is, a reduced fraction b/r with $1 \le r \le M$ such that

$$\left|\alpha - \frac{b}{r}\right| \le \frac{1}{rM}.$$

Lemma 4.1 asserts that for $M \ge 2(\log Q)^4 \log\log Q$ there exists $N \in [M, q]$ such that

$$\sum_{\substack{1\le|n|\le q \\ n\in\mathcal{S}(Q)}} \frac{\overline{\chi}(n)}{n} e(n\alpha) = \sum_{\substack{1\le|n|\le N \\ n\in\mathcal{S}(Q)}} \frac{\overline{\chi}(n)}{n} e\Big(\frac{b}{r} n\Big) + O(\log\log Q).$$

Finally, note that

$$\sum_{\substack{1\le|n|\le N \\ n\in\mathcal{S}(Q)}} \frac{\overline{\chi}(n)}{n} e\Big(\frac{b}{r} n\Big) = \sum_{\substack{n\le N \\ n\in\mathcal{S}(Q)}} \frac{\overline{\chi}(n)}{n} e\Big(\frac{b}{r} n\Big) - \chi(-1) \sum_{\substack{n\le N \\ n\in\mathcal{S}(Q)}} \frac{\overline{\chi}(n)}{n} e\Big(\frac{-b}{r} n\Big).$$

Since $M \to \infty$ with q while $\alpha \neq 0$ remains fixed, we must have $b \neq 0$. It follows that we can apply the Granville–Soundararajan identity (Proposition 2.3) to both of the expressions on the right-hand side of the above equation, deducing the relation

$$\sum_{\substack{1\le|n|\le N\\ n\in\mathcal{S}(Q)}} \frac{\overline{\chi}(n)}{n} e\Big(\frac{b}{r}n\Big)$$
$$= \sum_{\substack{d\mid r\\ d\in\mathcal{S}(Q)}} \frac{\overline{\chi}(d)}{d}\cdot\frac{1}{\varphi(r/d)} \sum_{\psi \,(\mathrm{mod}\ r/d)} \big(1-\chi(-1)\psi(-1)\big)\tau(\overline{\psi})\psi(b)\Bigg(\sum_{\substack{n\le N/d\\ n\in\mathcal{S}(Q)}} \frac{\overline{\chi}(n)\psi(n)}{n}\Bigg).$$

The arguments from the proofs of Theorems 2.1 and 2.7 carry over virtually verbatim, and we conclude that for $\alpha \neq 0$,

$$\sum_{1\le|n|\le q} \frac{\overline{\chi}(n)}{n} e(n\alpha)$$
$$\ll \big(1-\chi(-1)\xi(-1)\big)\frac{\sqrt{m}}{\varphi(m)}(\log Q)\, e^{-\mathcal{M}(\chi\overline{\xi};\, Q,\log^2 Q)} + (\log Q)^{2/3+o(1)},$$

where the implicit constant is absolute, and $o(1)\to 0$ as $q\to\infty$.

We now treat the case $\alpha = 0$; again, the arguments will be familiar. We begin as before, by using (7-2) to (potentially) restrict the sum

$$\sum_{1\le|n|\le q} \frac{\overline{\chi}(n)}{n} = \big(1-\chi(-1)\big)\sum_{n\le q}\frac{\overline{\chi}(n)}{n}$$

to Q-smooth arguments. We consider separately the two cases $\xi = \mathbf{1}$ and $\xi \neq \mathbf{1}$. In the former, $\xi(-1)=1$, whence

$$\big(1-\chi(-1)\big)\sum_{\substack{n\le q\\ n\in\mathcal{S}(Q)}} \frac{\overline{\chi}(n)}{n} = \big(1-\chi(-1)\xi(-1)\big)\sum_{\substack{n\le q\\ n\in\mathcal{S}(Q)}} \frac{\xi(n)\overline{\chi}(n)}{n}$$
$$\ll \big(1-\chi(-1)\xi(-1)\big)\frac{\sqrt{m}}{\varphi(m)} e^{-\mathcal{M}(\chi\overline{\xi};\, Q,\log^2 Q)},$$

by Corollary 2.5 (as in (6-6)). If $\xi\neq\mathbf{1}$, then from (6-7) we know that

$$\big(1-\chi(-1)\xi(-1)\big)\sum_{\substack{n\le q\\ n\in\mathcal{S}(Q)}} \frac{\overline{\chi}(n)}{n} \ll (\log Q)^{2/3+o(1)},$$

where the constant is absolute and $o(1)\to 0$ as $q\to\infty$.

Putting this all together with Pólya's Fourier expansion, we deduce the claimed bound on $S_\chi(t)$. □

8. Multiplicative nonmimicry: Proof of Theorem 2.10

Granville and Soundararajan [2007, Lemma 3.2] proved that for any primitive character χ (mod q) of odd order g, and any primitive character ξ of opposite parity and conductor smaller than a power of $\log y$,

$$\mathbb{D}(\chi, \xi; y)^2 \geq \big(\delta_g + o(1)\big) \log\log y. \tag{8-1}$$

Our goal in this section is to prove Theorem 2.10, which asserts that the same lower bound continues to hold for small perturbations of ξ. To be precise, we will show that under the same hypotheses on χ and ξ as above,

$$\mathbb{D}\big(\chi(n), \xi(n)n^{i\beta}; y\big)^2 \geq \big(\delta_g + o(1)\big) \log\log y \tag{8-2}$$

for all β of magnitude smaller than $\log^2 y$. For $\beta = o(\log\log y/\log y)$ this is straightforward:

$$\begin{aligned}
\mathbb{D}\big(\chi(n), \xi(n)n^{i\beta}; y\big)^2 &= \sum_{p\leq y} \frac{1}{p}\big(1 - \operatorname{Re} \chi\bar{\xi}(p) e^{-i\beta \log p}\big) \\
&= \sum_{p\leq y} \frac{1}{p}\Big(1 - \operatorname{Re} \chi\bar{\xi}(p)\big(1 + O(|\beta| \log p)\big)\Big) \\
&= \mathbb{D}(\chi, \xi; y)^2 + O\Big(|\beta| \sum_{p\leq y} \frac{\log p}{p}\Big) \\
&= \mathbb{D}(\chi, \xi; y)^2 + o(\log\log y),
\end{aligned}$$

and thus for such β, (8-2) follows from (8-1). For larger perturbations, however, the problem is more delicate.

Our plan of attack is as follows. Fix a primitive Dirichlet character $\chi \pmod q$ of odd order g, and a primitive $\xi \pmod m$ of opposite parity to χ. Since χ has odd order, $\chi(-1) = 1$, whence $\xi(-1) = -1$ and therefore ξ has even order k, say. We partition the interval $[2, y]$ into many small intervals of the form $(x, (1+\delta)x]$, where δ is small. For each prime p in such an interval, we approximate $p^{-i\beta}$ by $x^{-i\beta}$. This reduces our problem to estimating sums of the form

$$\sum_{\ell \ (\mathrm{mod}\ k)} \sum_{\substack{x<p\leq(1+\delta)x \\ \xi(p)=e(\ell/k)}} \frac{1}{p}\Big(1 - \operatorname{Re} \chi(p) e\Big(-\frac{\ell}{k}\Big) x^{-i\beta}\Big).$$

Following Granville and Soundararajan's proof of (8-1), we ignore the arithmetic properties of χ and view it as an arbitrary function from $\mathbb{Z}$ to $\mu_g \cup \{0\}$; here μ_g denotes the set of g-th roots of unity. This leads us to consider

$$\sum_{\ell \ (\mathrm{mod}\ k)} \sum_{\substack{x<p\leq(1+\delta)x \\ \xi(p)=e(\ell/k)}} \frac{1}{p} \min_{z\in\mu_g\cup\{0\}} \Big(1 - \operatorname{Re} z\, e\Big(-\frac{\ell}{k}\Big) x^{-i\beta}\Big),$$

and since the only factor dependent on p is the $1/p$ out front, we look at

$$\sum_{\substack{x<p\le(1+\delta)x \\ \xi(p)=e(\ell/k)}} \frac{1}{p}.$$

We expect $\xi(p)=e(\ell/k)$ for $1/k$ of the primes, so the natural guess is

$$\sum_{\substack{x<p\le(1+\delta)x \\ \xi(p)=e(\ell/k)}} \frac{1}{p} \approx \frac{1}{k} \sum_{x<p\le(1+\delta)x} \frac{1}{p} \approx \frac{\delta}{k\log x}.$$

A straightforward application of Siegel–Walfisz will make this estimate rigorous (see Lemma 8.1), and the remaining sum,

$$\sum_{\ell \ (\mathrm{mod}\ k)} \min_{z\in\mu_g\cup\{0\}} \Bigl(1-\operatorname{Re} z\, e\Bigl(-\frac{\ell}{k}\Bigr)x^{-i\beta}\Bigr),$$

can then be evaluated by arguments inspired by those of [Granville and Soundararajan 2007]. Summing over all the small intervals will yield the desired lower bound (8-2).

The contribution from short intervals. Our first goal is to obtain a lower bound on the sum over a short interval

$$\sum_{x<p\le(1+\delta)x} \frac{1}{p}\bigl(1-\operatorname{Re}\chi\bar{\xi}(p)p^{-i\beta}\bigr), \tag{8-3}$$

where

$$\delta \asymp \frac{1}{\log^3 y}.$$

Note that for any prime $p\in(x,(1+\delta)x]$, we may approximate $p^{i\beta}$ by $x^{i\beta}$: we have $0\le \log p-\log x\le\delta$, whence

$$|p^{-i\beta}-x^{-i\beta}| = |1-e^{i\beta(\log p-\log x)}| \le |\beta(\log p-\log x)| \le \delta|\beta|.$$

Therefore,

$$\begin{aligned}
&\sum_{x<p\le(1+\delta)x} \frac{1}{p}\bigl(1-\operatorname{Re}\chi\bar{\xi}(p)p^{-i\beta}\bigr) \\
&\qquad = \sum_{x<p\le(1+\delta)x} \frac{1}{p}\bigl(1-\operatorname{Re}\chi\bar{\xi}(p)x^{-i\beta}\bigr) + O\Bigl(\delta|\beta| \sum_{x<p\le(1+\delta)x} \frac{1}{p}\Bigr) \\
&\qquad = \sum_{x<p\le(1+\delta)x} \frac{1}{p}\bigl(1-\operatorname{Re}\chi\bar{\xi}(p)e(\theta_x)\bigr) + O\Bigl(\frac{\delta^2\log^2 y}{\log x}\Bigr),
\end{aligned} \tag{8-4}$$

where $\theta_x = -(\beta/(2\pi))\log x$. We bound the sum from below in terms of the orders of χ and ξ:

$$\sum_{x<p\le(1+\delta)x} \frac{1}{p}\bigl(1-\operatorname{Re}\chi\bar{\xi}(p)e(\theta_x)\bigr)$$
$$= \sum_{\ell \ (\mathrm{mod}\ k)} \sum_{\substack{x<p\le(1+\delta)x \\ \xi(p)=e(\ell/k)}} \frac{1}{p}\biggl(1-\operatorname{Re}\chi(p)e\Bigl(-\frac{\ell}{k}\Bigr)e(\theta_x)\biggr)$$
$$\ge \sum_{\ell \ (\mathrm{mod}\ k)} \sum_{\substack{x<p\le(1+\delta)x \\ \xi(p)=e(\ell/k)}} \frac{1}{p}\min_{z\in\mu_g\cup\{0\}}\biggl(1-\operatorname{Re} z\cdot e\Bigl(\theta_x-\frac{\ell}{k}\Bigr)\biggr).$$

We first estimate the interior sum over primes:

Lemma 8.1. *Suppose $\epsilon > 0$, $\xi \ (\mathrm{mod}\ m)$ is a nonprincipal character of order k, and $y \ge \exp(m^\epsilon)$. Then for $\delta \asymp (\log y)^{-3}$ and $x \ge \exp((\log y)^\epsilon)$,*

$$\sum_{\substack{x<p\le(1+\delta)x \\ \xi(p)=e(\ell/k)}} \frac{1}{p} = \frac{\delta}{k\log x}\bigl(1+o(1)\bigr),$$

where $o(1)\to 0$ as $y\to\infty$ and depends only on y and ϵ.

Note that this estimate is independent of ℓ. Thus the following general result, combined with Lemma 8.1, will furnish a lower bound on the sum (8-3):

Lemma 8.2. *Given $g \ge 3$ odd, $k \ge 2$ even, and $\theta \in \bigl(-\frac{1}{2}, \frac{1}{2}\bigr]$. Set $k^* = k/(g,k)$. Then*

$$\frac{1}{k}\sum_{\ell \ (\mathrm{mod}\ k)} \min_{z\in\mu_g\cup\{0\}}\biggl(1-\operatorname{Re} z\cdot e\Bigl(\theta-\frac{\ell}{k}\Bigr)\biggr) = 1-\frac{\sin(\pi/g)}{k^*\tan(\pi/(gk^*))}F_{gk^*}(-gk^*\theta), \tag{8-5}$$

where

$$F_N(\omega) = \cos\frac{2\pi\{\omega\}}{N} + \Bigl(\tan\frac{\pi}{N}\Bigr)\sin\frac{2\pi\{\omega\}}{N}.$$

To make sense of this lemma, we examine some properties of $F_N(\omega)$. First, since $F_N(\omega) = F_N(\{\omega\})$ we may assume that $\omega \in [0, 1)$. Second, since k^* must be even, $gk^* \ge 6$, and we can therefore assume that $N \ge 6$. Under these assumptions, one easily checks that

(i) $F_N(0) = 1$ and $F_N(0.5) = \dfrac{1}{\cos(\pi/N)}$,

(ii) $F_N(\omega)$ is concave down everywhere on $[0, 1)$,

(iii) On the unit interval, F_N is symmetric about $\omega = \frac{1}{2}$, and

(iv) The average value of F_N over the unit interval is $\dfrac{N}{\pi}\tan\dfrac{\pi}{N}$.

Thus, for the typical θ we expect the right side of (8-5) to be δ_g. It is appreciably larger than δ_g when $gk^*\theta$ is close to an integer, and somewhat smaller than δ_g when $gk^*\theta$ is close to a half-integer. In the context of [Granville and Soundararajan 2007], $\theta = 0$, which allowed Granville and Soundararajan to bound (8-5) from below by δ_g quite easily. Although our arguments are also not difficult, the computations are naturally somewhat more involved; we will isolate the proof in a separate subsection.

Before proving the two lemmata, we deduce from them a lower bound on (8-3). The main term of (8-4) can be bounded from below, for all $x \geq \exp((\log y)^\epsilon)$:

$$\begin{aligned}\sum_{x<p\leq(1+\delta)x} \frac{1}{p}\bigl(1-\operatorname{Re}\chi\bar{\xi}(p)e(\theta_x)\bigr) &\geq \sum_{\ell \ (\mathrm{mod}\, k)} \Biggl(\sum_{\substack{x<p\leq(1+\delta)x\\ \xi(p)=e(\ell/k)}} \frac{1}{p}\Biggr) \min_{z\in\mu_g\cup\{0\}} \biggl(1-\operatorname{Re} z\cdot e\Bigl(\theta_x-\frac{\ell}{k}\Bigr)\biggr)\\ &= \frac{\delta\bigl(1+o(1)\bigr)}{\log x}\biggl(1-\frac{\sin(\pi/g)}{k^*\tan(\pi/(gk^*))}F_{gk^*}(-gk^*\theta_x)\biggr).\end{aligned}$$

Let

$$G(t) = 1-\frac{\sin(\pi/g)}{k^*\tan(\pi/(gk^*))}F_{gk^*}\Bigl(\frac{\beta gk^*}{2\pi}t\Bigr).$$

Note that G is minimized at values of t for which F_{gk^*} is maximized, whence

$$G(t) \geq 1-\frac{\sin(\pi/g)}{k^*\sin(\pi/(gk^*))}.$$

It follows that as a function of t, $G(t)$ is bounded away from 0. This combined with our choice of δ of size $(\log y)^{-3}$ shows that we can bound (8-4) as follows:

$$\begin{aligned}\sum_{x<p\leq(1+\delta)x} \frac{1}{p}\bigl(1-\operatorname{Re}\chi\bar{\xi}(p)p^{-i\beta}\bigr) &\geq \frac{\bigl(1+o(1)\bigr)\delta}{\log x}G(\log x)+O\Bigl(\frac{\delta^2\log^2 y}{\log x}\Bigr)\\ &= \frac{\bigl(1+o(1)\bigr)\delta}{\log x}G(\log x), \qquad (8\text{-}6)\end{aligned}$$

where the $o(1)$ term in (8-6) tends to 0 as $y \to \infty$ and depends only on y, ϵ, g, and k.

We now go back and prove the two lemmata.

Proof of Lemma 8.1. A consequence of the Siegel–Walfisz theorem says that for any fixed $\epsilon > 0$ and $A > 0$, for all $X \geq \exp(m^\epsilon)$,

$$\theta(X;m,a) := \sum_{\substack{p\leq X\\ p\equiv a \ (\mathrm{mod}\, m)}} \log p = \frac{X}{\varphi(m)}\biggl(1+O\Bigl(\frac{1}{(\log X)^A}\Bigr)\biggr),$$

where the constant implicit in the O-term depends only upon A and ϵ. In particular, for all $X \geq \exp((\log y)^\epsilon)$,

$$\theta(X; m, a) = \frac{X}{\varphi(m)}\left(1 + O_\epsilon\left(\frac{1}{(\log X)^{4/\epsilon}}\right)\right), \tag{8-7}$$

where the implicit constant only depends on ϵ.

To apply Siegel–Walfisz, we must first express the sum in question as a sum over primes in arithmetic progressions:

$$\sum_{\substack{x<p\leq(1+\delta)x \\ \xi(p)=e(\ell/k)}} \frac{1}{p} = \sum_{\substack{a \ (\mathrm{mod}\ m) \\ \xi(a)=e(\ell/k)}} \sum_{\substack{x<p\leq(1+\delta)x \\ p\equiv a \ (\mathrm{mod}\ m)}} \frac{1}{p}.$$

Note that $x < p \leq (1+\delta)x$ is equivalent to $(1/(1+\delta))p \leq x < p$, whence

$$\frac{x \log x}{p \log p} = \frac{x}{p} \cdot \frac{\log x}{\log p} = \bigl(1 + O(\delta)\bigr) \cdot \left(1 + O\left(\frac{\delta}{\log p}\right)\right) = 1 + O(\delta).$$

Combining this with (8-7) and the hypotheses on the sizes of x and δ yields

$$\sum_{\substack{x<p\leq(1+\delta)x \\ p\equiv a \ (\mathrm{mod}\ m)}} \frac{1}{p} = \frac{1+O(\delta)}{x \log x} \sum_{\substack{x<p\leq(1+\delta)x \\ p\equiv a \ (\mathrm{mod}\ m)}} \log p = \frac{\delta}{\varphi(m) \log x}\left(1 + O_\epsilon\left(\frac{1}{\log y}\right)\right). \tag{8-8}$$

Since this estimate is independent of a, to prove the lemma it remains only to show

$$\sum_{\substack{a \ (\mathrm{mod}\ m) \\ \xi(a)=e(\ell/k)}} 1 = \frac{\varphi(m)}{k}. \tag{8-9}$$

For brevity, denote $(\mathbb{Z}/m\mathbb{Z})^*$ by G. Since ξ has order k, there is some $b \in G$ such that $1, \xi(b), \xi(b)^2, \ldots, \xi(b)^{k-1}$ are all distinct; on the other hand, all these must be k-th roots of unity. In particular, there exists some $g \in G$ such that $\xi(g) = e(1/k)$.

Let H be the kernel of ξ, that is, $H = \{a \in G : \xi(a) = 1\}$. This is a normal subgroup of G, and $g^\ell H = \{a \in G : \xi(a) = e(\ell/k)\}$. G can therefore be decomposed as a disjoint union of the k cosets $g^\ell H$ with $0 \leq \ell \leq k-1$. Since $|g^\ell H| = |H|$, (8-9) must hold. Combining this with (8-8) yields the lemma. □

Proof of Lemma 8.2. Recall that $g \geq 3$ is odd, $k \geq 2$ is even, and $\theta \in \left(-\frac{1}{2}, \frac{1}{2}\right]$. Let $d = (g, k)$ and set $k^* = k/d$ and $g^* = g/d$.

To prove (8-5), it suffices to show

$$\sum_{\ell \ (\mathrm{mod}\ k)} \max_{z \in \mu_g \cup \{0\}} \operatorname{Re} z \cdot e\left(\theta - \frac{\ell}{k}\right) = d \cdot \frac{\sin(\pi/g)}{\tan(\pi/(gk^*))} \cdot F_{gk^*}(-gk^*\theta). \tag{8-10}$$

Let $\mathcal{A}_0 = \{e(\beta) : -1/(2g) < \beta \leq 1/(2g)\}$ and set $\mathcal{A}_n = e(n/g)\mathcal{A}_0$; note that the disjoint union of $\mathcal{A}_n$ as n runs over any complete set of residues of $\mathbb{Z}/g\mathbb{Z}$ is the complex unit circle. In particular, for any $\ell \in \mathbb{Z}$ there is a unique $n_\ell \in (-g/2, g/2]$ such that $e(\theta-\ell/k) \in \mathcal{A}_{n_\ell}$. By definition, this means that $e(-n_\ell/g)e(\theta-\ell/k) \in \mathcal{A}_0$. Since $e(-n/g)e(\theta-\ell/k) \notin \mathcal{A}_0$ for all other $n \in (-g/2, g/2]$, we deduce that

$$\max_{z\in\mu_g\cup\{0\}} \operatorname{Re} z\cdot e\Big(\theta-\frac{\ell}{k}\Big) = \operatorname{Re} e\Big(-\frac{n_\ell}{g}\Big)e\Big(\theta-\frac{\ell}{k}\Big) = \operatorname{Re} e(\theta)\, e\Big(\frac{f(\ell)}{gk}\Big)$$

where $f:\mathbb{Z}\to\mathbb{Z}$ is defined by $f(\ell) = -(g\ell+kn_\ell)$. This allows us to rewrite the left-hand side of the inequality (8-10):

$$\sum_{\ell\ (\mathrm{mod}\ k)} \max_{z\in\mu_g\cup\{0\}} \operatorname{Re} z\cdot e\Big(\theta-\frac{\ell}{k}\Big) = \operatorname{Re} e(\theta) \sum_{\ell\ (\mathrm{mod}\ k)} e\Big(\frac{f(\ell)}{gk}\Big). \tag{8-11}$$

Our aim is rewrite the sum on the right side of (8-11) in terms of geometric series.

It is not hard to see that if $\ell_1 \equiv \ell_2 \pmod k$ then $f(\ell_1) \equiv f(\ell_2) \pmod{gk}$. However, more is true:

Lemma 8.3. $\ell_1 \equiv \ell_2 \pmod{k^*} \Longrightarrow f(\ell_1) \equiv f(\ell_2) \pmod{gk}$.

Proof. Assume $\ell_1 \equiv \ell_2 \pmod{k^*}$. Then k divides $g(\ell_2-\ell_1)$, since $g(\ell_2-\ell_1) = g^*k(\ell_2-\ell_1)/k^*$. Equivalently, there exists $m \in \mathbb{Z}$ such that $-\ell_1/k = -\ell_2/k + m/g$. Therefore, by the definition of n_ℓ, we find that both $e((m-n_{\ell_1})/g)$ and $e(-n_{\ell_2}/g)$ belong to the set $e(\ell_2/k-\theta)\mathcal{A}_0$. But this implies that $n_{\ell_1} \equiv m + n_{\ell_2} \pmod g$, whence, as needed,

$$e\Big(\frac{f(\ell_1)}{gk}\Big) = e\Big(\frac{f(\ell_2)}{gk}\Big). \qquad \square$$

Thus, we can restrict the sum on the right side of (8-11) to $\mathbb{Z}/k^*\mathbb{Z}$:

$$\sum_{\ell\ (\mathrm{mod}\ k)} e\Big(\frac{f(\ell)}{gk}\Big) = d\cdot \sum_{\ell^*\ (\mathrm{mod}\ k^*)} e\Big(\frac{f(\ell^*)}{gk}\Big). \tag{8-12}$$

We now prove a weaker form of Lemma 8.3, which has the advantage of a converse.

Lemma 8.4. $\ell_1 \equiv \ell_2 \pmod{k^*} \Longleftrightarrow f(\ell_1) \equiv f(\ell_2) \pmod k$.

Proof. We have

$$f(\ell_1) \equiv f(\ell_2) \pmod k \Longrightarrow k \mid g(\ell_2-\ell_1) \Longrightarrow k^* \mid g^*(\ell_2-\ell_1) \Longrightarrow \ell_1 \equiv \ell_2 \pmod{k^*},$$

since $(g^*, k^*) = 1$. On the other hand,

$$\ell_1 \equiv \ell_2 \pmod{k^*} \Longrightarrow k \mid d(\ell_2-\ell_1),$$

whence

$$\begin{aligned} f(\ell_1)-f(\ell_2) &= g(\ell_2-\ell_1) + k(n_{\ell_1}-n_{\ell_2}) \\ &= g^*d(\ell_2-\ell_1) + k(n_{\ell_1}-n_{\ell_2}) \equiv 0 \pmod k. \end{aligned} \qquad \square$$

Proposition 8.5. *The map f restricted to $[-k^*/2 + k^*\theta, k^*/2 + k^*\theta) \cap \mathbb{Z}$ is an injection into*

$$\Bigl(-\frac{k}{2} - gk\theta, \frac{k}{2} - gk\theta\Bigr] \cap \mathbb{Z}.$$

Proof. Injectivity follows immediately from Lemma 8.4, so it suffices to show that the image of $[-k^*/2+k^*\theta, k^*/2+k^*\theta) \cap \mathbb{Z}$ under f lands in the claimed target. In fact, we will show a slightly stronger statement. Observe that because $|\theta| \le \frac{1}{2}$,

$$\Bigl[-\frac{k^*}{2} + k^*\theta, \frac{k^*}{2} + k^*\theta\Bigr) \subseteq \Bigl[-\frac{k}{2} + k\theta, \frac{k}{2} + k\theta\Bigr);$$

we claim that the image under f of the larger set lands inside the claimed target.

Fix any $\ell \in [-k/2 + k\theta, k/2 + k\theta)$; this is equivalent to requiring $\theta - \ell/k \in \bigl(-\frac{1}{2}, \frac{1}{2}\bigr]$. By definition of n_ℓ we have $e(\theta - \ell/k) \in \mathscr{A}_{n_\ell}$, from which we deduce that for some integer N,

$$\theta - \frac{\ell}{k} \in \biggl(N + \frac{2n_\ell - 1}{2g}, N + \frac{2n_\ell + 1}{2g}\biggr].$$

By our restriction on ℓ, N must equal 0 (recall that $-(g-1)/2 \le n_\ell \le (g-1)/2$). It follows that $f(\ell) \in (-k/2 - gk\theta, k/2 - gk\theta]$. $\square$

Note that $d|f(\ell)$ for all ℓ. Combining this fact with Proposition 8.5 we conclude that

$$\Bigl\{f(\ell^*) : -\frac{k^*}{2} + k^*\theta \le \ell^* < \frac{k^*}{2} + k^*\theta\Bigr\}$$

is a set of k^* distinct multiples of d, all contained in $(-k/2 - gk\theta, k/2 - gk\theta]$. But by inspection, this interval contains *precisely* k^* multiples of d. Therefore:

$$\begin{aligned}\sum_{\ell^* \pmod{k^*}} e\Bigl(\frac{f(\ell^*)}{gk}\Bigr) &= \sum_{-\frac{k^*}{2}+k^*\theta \le \ell^* < \frac{k^*}{2}+k^*\theta} e\Bigl(\frac{f(\ell^*)}{gk}\Bigr)\\ &= \sum_{\frac{1}{d}(-\frac{k}{2}-gk\theta) < m \le \frac{1}{d}(\frac{k}{2}-gk\theta)} e\Bigl(\frac{md}{gk}\Bigr)\\ &= \sum_{-\frac{k^*}{2}-gk^*\theta < m \le \frac{k^*}{2}-gk^*\theta} e\Bigl(\frac{m}{gk^*}\Bigr). \end{aligned} \tag{8-13}$$

This is a k^*-term geometric series with first term $e\bigl((1/(gk^*))[k^*/2 - gk^*\theta]\bigr)$ and ratio $e(-1/(gk^*))$. Summing the series and performing standard algebraic manipulations, one finds

$$\sum_{-\frac{k^*}{2}-gk^*\theta < m \le \frac{k^*}{2}-gk^*\theta} e\Bigl(\frac{m}{gk^*}\Bigr) = e\Bigl(-\theta + \frac{1-2c}{2gk^*}\Bigr)\frac{\sin(\pi/g)}{\sin(\pi/(gk^*))},$$

where $c = \{-gk^*\theta\} \in [0, 1)$. Tracing back through (8-11)–(8-13) and simplifying, we see that

$$\sum_{\ell \ (\mathrm{mod}\ k)} \max_{z \in \mu_g \cup \{0\}} \operatorname{Re} z \cdot e\Big(\theta - \frac{\ell}{k}\Big) = d \cdot \frac{\sin(\pi/g)}{\sin(\pi/(gk^*))} \cdot \cos\Big(\frac{\pi}{gk^*}(1-2c)\Big)$$
$$= d \cdot \frac{\sin(\pi/g)}{\tan(\pi/(gk^*))} \cdot F_{gk^*}(-gk^*\theta),$$

proving (8-10), and thus the lemma. □

Completion of the proof of Theorem 2.10. Let $x_0 = \exp((\log y)^\epsilon)$ and set $x_r = x_0(1+\delta)^r$. Then from (8-6) we deduce

$$\begin{aligned}\mathbb{D}\big(\chi(n), \xi(n)n^{i\beta}; y\big)^2 &= \sum_{p \le y} \frac{1}{p}\big(1 - \operatorname{Re} \chi\bar{\xi}(p)p^{-i\beta}\big) \ge \sum_{x_0 < p \le y} \frac{1}{p}\big(1 - \operatorname{Re} \chi\bar{\xi}(p)p^{-i\beta}\big) \\ &\ge \sum_{\substack{r \ge 0 \\ x_{r+1} \le y}} \sum_{x_r < p \le x_{r+1}} \frac{1}{p}\big(1 - \operatorname{Re} \chi\bar{\xi}(p)p^{-i\beta}\big) \\ &\ge \sum_{\substack{r \ge 0 \\ x_{r+1} \le y}} \frac{(1+o(1))\delta}{\log x_r} G(\log x_r) \\ &\ge \big(1+o(1)\big) \log(1+\delta) \sum_{\substack{r \ge 0 \\ x_{r+1} \le y}} \frac{G(\log x_r)}{\log x_r}. \end{aligned} \tag{8-14}$$

We recognize the sum above as the left Riemann sum — with subintervals of length $\log(1+\delta)$ — for the integral

$$\int_{\log x_0}^{\log x_m} \frac{G(t)}{t}\, dt,$$

where m is the integer such that $x_m \le y < x_{m+1}$. Since

$$\left|\frac{d}{dt}\Big(\frac{G(t)}{t}\Big)\right| \le \left|\frac{G'(t)}{t}\right| + \left|\frac{G(t)}{t^2}\right| \le \frac{\dfrac{\sin(\pi g)}{k^* \tan(\pi/(gk^*))} F'_{gk^*}(0)}{\log x_0} + \frac{2}{(\log x_0)^2} \ll 1,$$

for all $t \ge \log x_0$, we have

$$\left|\log(1+\delta) \sum_{\substack{r \ge 0 \\ x_{r+1} \le y}} \frac{G(\log x_r)}{\log x_r} - \int_{\log x_0}^{\log y} \frac{G(t)}{t}\, dt\right|$$
$$\ll (\log y) \cdot \log(1+\delta) + \left|\int_{\log x_m}^{\log y} \frac{G(t)}{t}\, dt\right| \ll \frac{1}{\log^2 y}.$$

Therefore, continuing our calculation from where we left it in (8-14),

$$\mathbb{D}\big(\chi(n), \xi(n)n^{i\beta}; y\big)^2 \geq \big(1+o(1)\big) \int_{\log x_0}^{\log y} \frac{G(t)}{t}\, dt + O(1). \tag{8-15}$$

To prove Theorem 2.10 it remains only to bound the integral on the right side of (8-15) from below by $(\delta_g + o(1)) \log\log y$. Recall that

$$G(t) = 1 - \frac{\sin(\pi/g)}{k^* \tan(\pi/(gk^*))} F_{gk^*}\Big(\frac{\beta g k^*}{2\pi} t\Big),$$

where

$$F_N(\omega) = \cos \frac{2\pi\{\omega\}}{N} + \Big(\tan \frac{\pi}{N}\Big) \sin \frac{2\pi\{\omega\}}{N}$$

is concave down everywhere on the unit interval and symmetric about $t = \frac{1}{2}$, with minima at the endpoints of the interval. Furthermore, $\overline{F}_N$, the mean value of F_N on the unit interval, is $(N/\pi)\tan(\pi/N)$. Rewriting (8-15), we see that it suffices to prove that

$$\int_{a(y)}^{b(y)} \frac{1}{t} F_N(t)\, dt \leq \big(\overline{F}_N + o(1)\big) \log\log y,$$

where

$$a(y) = \frac{N|\beta|}{2\pi} (\log y)^{\epsilon} \quad \text{and} \quad b(y) = \frac{N|\beta|}{2\pi} \log y.$$

(Note that $a(y)$ and $b(y)$ are expressed in terms of the magnitude of β, a change of variables we can make because F_N is an even function.) Given any $x \geq 1$ we find

$$\int_1^x \frac{1}{t} F_N(t)\, dt = \overline{F}_N \cdot \log x + O(1),$$

by splitting the integral into unit intervals (with at most one exception) and on each interval bounding $1/t$ from above and below trivially. Thus if $a(y) \geq 1$, we immediately find

$$\int_{a(y)}^{b(y)} \frac{1}{t} F_N(t)\, dt = \overline{F}_N \cdot \log \frac{b(y)}{a(y)} + O(1) \leq \big(\overline{F}_N + o(1)\big) \log\log y.$$

Now we consider the case when $a(y) < 1$. Note that we may take $b(y) \geq 1$: from the discussion directly following (8-2) we see that we can assume

$$|\beta| \geq \frac{C_0 (\log\log y)^{1/2}}{\log y}$$

for any positive constant C_0, and since $y \geq 3$ and $N = gk^* \geq 6$, choosing

$$C_0 = \frac{2\pi}{6} (\log\log 3)^{-1/2}$$

makes $b(y) \geq 1$. Therefore,

$$\begin{aligned}\int_{a(y)}^{b(y)} \frac{1}{t} F_N(t)\,dt &= \int_{a(y)}^{1} \frac{1}{t} F_N(t)\,dt + \int_{1}^{b(y)} \frac{1}{t} F_N(t)\,dt \\ &= \int_{1}^{1/a(y)} \frac{1}{t} F_N\Big(\frac{1}{t}\Big)\,dt + \overline{F}_N \cdot \log b(y) + O(1).\end{aligned}$$

It remains only to show that

$$\int_{1}^{x} \frac{1}{t} F_N\Big(\frac{1}{t}\Big)\,dt \leq \overline{F}_N \cdot \log x + O(1). \tag{8-16}$$

Because F_N is concave down on $[0, 1)$, we see that for all sufficiently large x, $F_N(1/x) \leq \overline{F}_N$. Therefore,

$$\frac{d}{dx}\bigg(\int_{1}^{x} \frac{1}{t} F_N\Big(\frac{1}{t}\Big)\,dt\bigg) \leq \frac{d}{dx}(\overline{F}_N \cdot \log x)$$

for all large x. This implies (8-16), and Theorem 2.10 is proved. □

9. Proof of Theorem 3

All results stated and proved in this section are conditional on the generalized Riemann hypothesis.

In Theorem 2 we proved that

$$|S_\chi(t)| \ll_g \sqrt{q}(\log\log q)^{1-\delta_g+o(1)}$$

for any primitive character $\chi \pmod q$ of odd order $g \geq 3$. The goal of this section is to construct an infinite family of characters $\chi \pmod q$ of order g such that

$$\max_{t \leq q} |S_\chi(t)| \gg_{\epsilon,g} \sqrt{q}(\log\log q)^{1-\delta_g-\epsilon},$$

thus showing that the constant $1 - \delta_g$ in our upper bound cannot be improved. We note that when g is squarefree, the dependence of the implicit constant on g can be made explicit from our construction. We first recall the following:

Theorem 9.1 [Granville and Soundararajan 2007, Theorem 2.5]. *Assume the GRH. Given a primitive character $\chi \pmod q$, let $\xi \pmod m$ be a primitive character of opposite parity to χ. Then*

$$\max_{t \leq q} |S_\chi(t)| + \frac{\sqrt{m}}{\varphi(m)} \sqrt{q} \log\log\log q \gg \frac{\sqrt{m}}{\varphi(m)} \sqrt{q}(\log\log q)\, e^{-\mathbb{D}(\chi,\xi;\log q)^2}.$$

To prove Theorem 3 it therefore suffices to show that there is an odd character $\xi \pmod m$ and an infinite family of characters $\chi \pmod q$ of odd order g such that

$$\mathbb{D}(\chi, \xi; \log q)^2 \leq (\delta_g + \epsilon) \log\log\log q$$

or, equivalently, that

$$\sum_{p\le \log q} \frac{1}{p} \operatorname{Re} \chi(p)\overline{\xi(p)} \ge (1-\delta_g-\epsilon)\log\log\log q. \tag{9-1}$$

We will accomplish this in two steps. First, using ideas similar to those of the previous section, we will prove:

Proposition 9.2. *For any $\epsilon > 0$, there exists an odd character $\xi \pmod m$ such that for $y \ge \exp(m^{\epsilon})$,*

$$\sum_{p\le y} \frac{1}{p} \max_{z\in\mu_g\cup\{0\}} \operatorname{Re} z\overline{\xi(p)} \ge \big(1-\epsilon+o(1)\big)(1-\delta_g)\log\log y; \tag{9-2}$$

$o(1) \to 0$ as $y \to \infty$.

Given such a ξ, to deduce (9-1) it suffices to find a $\chi \pmod q$ whose values at primes up to $\log q$ coincide with the z that maximize each term of (9-2). Using a generalization of Eisenstein's reciprocity law and the Chinese remainder theorem, we will prove:

Proposition 9.3. *Fix an odd integer $g \ge 3$, and let $\psi : \mathbb{Z} \longrightarrow \mu_g \cup \{0\}$ be a completely multiplicative function. Then there exists a constant $C = C(g) > 0$ and infinitely many Dirichlet characters $\chi \pmod q$ of order g such that $\chi(n) = \psi(n)$ for all $n \le C\log q$ which are coprime to g.*

With these results in hand, Theorem 3 follows easily:

Proof of Theorem 3. Proposition 9.2 furnishes a character ξ such that (9-2) holds for all $y \ge \exp(m^{\epsilon})$. For any such y, choose $z_p \in \mu_g \cup \{0\}$ so that

$$\sum_{p\le y} \frac{1}{p} \max_{z\in\mu_g\cup\{0\}} \operatorname{Re} z\overline{\xi(p)} = \sum_{p\le y} \frac{1}{p} \operatorname{Re} z_p\overline{\xi(p)}.$$

By Proposition 9.3 we can find infinitely many characters $\chi \pmod q$ such that $\chi(p) = z_p$ for all $p \le C\log q$ which are coprime to g. For any such χ, we therefore have

$$\sum_{p\le C\log q} \frac{1}{p} \operatorname{Re} \chi(p)\overline{\xi(p)} = \sum_{p\le C\log q} \frac{1}{p} \operatorname{Re} z_p\overline{\xi(p)} + O\Big(\sum_{p\mid g} \frac{1}{p}\Big).$$

Since g is fixed, (9-2) implies (9-1); applying Theorem 9.1 yields Theorem 3. $\square$

It remains only to prove the two propositions.

Proof of Proposition 9.2. Let $\xi \pmod m$ be an odd character. Then its order k must be even, and (exactly as in the previous section) we have

$$\sum_{p\le y}\frac{1}{p}\max_{z\in\mu_g\cup\{0\}}\operatorname{Re} z\overline{\xi(p)} = \sum_{\ell \pmod k}\max_{z\in\mu_g\cup\{0\}}\operatorname{Re} z\, e\Big(-\frac{\ell}{k}\Big)\sum_{\substack{p\le y\\ \xi(p)=e(\ell/k)}}\frac{1}{p}.$$

Siegel–Walfisz implies that

$$\sum_{\substack{p\le y\\ \xi(p)=e(\ell/k)}}\frac{1}{p} = \frac{1+o(1)}{k}\log\log y$$

and relation (8-10) (with $\theta = 0$) gives

$$\sum_{\ell \pmod k}\max_{z\in\mu_g\cup\{0\}}\operatorname{Re} z\, e\Big(-\frac{\ell}{k}\Big) = (g,k)\frac{\sin(\pi/g)}{\tan(\pi/(gk^*))}.$$

Putting these estimates together yields

$$\sum_{p\le y}\frac{1}{p}\max_{z\in\mu_g\cup\{0\}}\operatorname{Re} z\overline{\xi(p)} = \big(1-\delta_g+o(1)\big)\frac{\pi/(gk^*)}{\tan(\pi/(gk^*))}\log\log y.$$

The function $x/\tan x$ tends to 1 from below as $x\to 0$, so to prove the proposition it suffices to find a sequence of k^* tending to infinity. Since g is fixed and $k^* = k/(g,k)$, this is easily achieved by choosing ξ of order k relatively prime to g. $\square$

Proof of Proposition 9.3. Let y be large (this is an auxiliary parameter which will tend to infinity). Given a prime $p\nmid g$, there exists an integer Q_p such that

$$\left(\frac{Q_p}{p}\right)_g = \psi(p),$$

where $\left(\frac{\cdot}{\cdot}\right)_g$ is the g-th order residue symbol. By the Chinese remainder theorem, there exists a $Q = Q(y)$ satisfying

(1) $Q \equiv Q_p \pmod p$ for all primes $p \le y$ such that $p\nmid g$;

(2) $Q \equiv 1 \pmod g$; and

(3) $g\prod_{\substack{p\le y\\ p\nmid g}} p < Q \le 2g\prod_{\substack{p\le y\\ p\nmid g}} p.$

It follows that

$$\left(\frac{Q}{p}\right)_g = \psi(p) \tag{9-3}$$

for all $p \le y$ coprime to g.

We now wish to use reciprocity for the g-th order residue symbol to obtain a g-th order character of modulus Q. For g an odd prime, this is given by the Eisenstein reciprocity law. Recently, Vostokov and Orlova [2008] gave a generalization of the reciprocity law to all odd g. In our situation, their result implies that

$$\left(\frac{Q}{p}\right)_g = \left(\frac{p}{Q}\right)_g$$

for all $p \nmid g$.

By the prime number theorem and our restriction on the size of Q, we see that

$$\log Q \asymp y + \log \frac{g}{\operatorname{rad} g},$$

where $\operatorname{rad} g$ denotes the radical of g. It follows that there exists a constant $C = C(g)$ such that $y \geq C \log Q$. Combining this with (9-3) and the Vostokov–Orlova reciprocity, we deduce that

$$\left(\frac{p}{Q}\right)_g = \psi(p)$$

for all $p \leq C \log Q$ relatively prime to g. By complete multiplicativity,

$$\left(\frac{n}{Q}\right)_g = \psi(n) \tag{9-4}$$

for all $n \leq C \log Q$ coprime to g. Letting y tend to infinity, we see that Q must also tend to infinity, whence we find infinitely many Q satisfying (9-4). This concludes the proof. □

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Quiver Grassmannians and degenerate flag varieties

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Quiver Grassmannians are varieties parametrizing subrepresentations of a quiver representation. It is observed that certain quiver Grassmannians for type A quivers are isomorphic to the degenerate flag varieties investigated earlier by Feigin. This leads to the consideration of a class of Grassmannians of subrepresentations of the direct sum of a projective and an injective representation of a Dynkin quiver. It is proved that these are (typically singular) irreducible normal local complete intersection varieties, which admit a group action with finitely many orbits and a cellular decomposition. For type A quivers, explicit formulas for the Euler characteristic (the median Genocchi numbers) and the Poincaré polynomials are derived.

1. Introduction

Motivation. Quiver Grassmannians, which are varieties parametrizing subrepresentations of a quiver representation, first appeared in [Crawley-Boevey 1989; Schofield 1992] in relation to questions on generic properties of quiver representations. It was observed in [Caldero and Chapoton 2006] that these varieties play an important role in cluster algebra theory [Fomin and Zelevinsky 2002]; namely, the cluster variables can be described in terms of the Euler characteristic of quiver Grassmannians. Subsequently, specific classes of quiver Grassmannians (for example, varieties of subrepresentations of exceptional quiver representations) were studied by several authors, with the principal aim of computing their Euler characteristic explicitly; see for example [Caldero and Reineke 2008; Cerulli Irelli 2011; Cerulli Irelli and Esposito 2010; Cerulli Irelli et al. 2010]. In recent papers, authors noticed that also the Poincaré polynomials of quiver Grassmannians play an important role in the study of quantum cluster algebras [Qin 2010; Berenstein and Zelevinsky 2005].

This paper originated with the observation that certain quiver Grassmannians can be identified with the $\mathfrak{sl}_n$-degenerate flag variety of [Feigin 2010; 2011; Feigin

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and Finkelberg 2011]. This led to the consideration of a class of Grassmannians of subrepresentations of the direct sum of a projective and an injective representation of a Dynkin quiver. It turns out that this class of varieties enjoys many of the favorable properties of quiver Grassmannians for exceptional representations. More precisely, they turn out to be (typically singular) irreducible normal local complete intersection varieties which admit a group action with finitely many orbits and a cellular decomposition. The proofs of the basic geometric properties are based on generalizations of the techniques of [Reineke 2008], where the case of Grassmannians of subrepresentations of injective quiver representations is treated.

Main results. Let Q be a quiver with set of vertices Q_0 of cardinality n and finite set of arrows Q_1. For a representation M of Q, we denote by M_i the space in M attached to the i-th vertex, and by $M_\alpha : M_i \to M_j$ the linear map attached to an arrow $\alpha : i \to j$. We also denote by $\langle \_\,, \_ \rangle$ the Euler form on $\mathbb{Z}Q_0$. Given a dimension vector $\boldsymbol{e} = (e_1, \dots, e_n) \in \mathbb{Z}_{\geq 0}Q_0$ and a representation M of Q, the quiver Grassmannian $\mathrm{Gr}_{\boldsymbol{e}}(M) \subset \prod_{i=1}^n \mathrm{Gr}_{e_i}(M_i)$ is the subvariety of collections of subspaces $V_i \subset M_i$ subject to the conditions $M_\alpha V_i \subset V_j$ for all $\alpha : i \to j \in Q_1$. In this paper we study a certain class of quiver Grassmannians for Dynkin quivers Q. Before describing this class, we first consider the following example.

Let Q be an equioriented quiver of type A_n with vertices $i = 1, \dots, n$ and arrows $i \to i+1$, and let $\mathbb{C}Q$ be the path algebra of Q. Then the quiver Grassmannian $\mathrm{Gr}_{\mathbf{dim}\,\mathbb{C}Q}(\mathbb{C}Q \oplus \mathbb{C}Q^*)$ is isomorphic to the complete degenerate flag variety $\mathcal{F}^a_{n+1}$ for $G = \mathrm{SL}_{n+1}$. We recall the definition from [Feigin 2010; 2011]. Let W be an $(n+1)$-dimensional vector space with basis $w_1, \dots, w_{n+1}$. Let $\mathrm{pr}_k : W \to W$ for $k = 1, \dots, n+1$ be the projection operators $\mathrm{pr}_k(\sum_{i=1}^{n+1} c_i w_i) = a \sum_{i \neq k} c_i w_i$. Then the degenerate flag variety consists of collections $(V_1, \dots, V_n)$ with $V_i \subset W$ and $\dim V_i = i$, subject to the conditions $\mathrm{pr}_{k+1} V_k \subset V_{k+1}$ for $k = 1, \dots, n-1$. These varieties are irreducible singular algebraic varieties enjoying many nice properties. In particular, they are flat degenerations of classical flag varieties SL_{n+1}/B. Now consider the representation M of Q such that $M_i = W$ and the maps $M_i \to M_{i+1}$ are given by pr_{i+1}. For example, for $n = 3$, M has the following coefficient quiver:

$$\begin{array}{ccccc}
\bullet & \to & \bullet & \to & \bullet \\
\bullet & \to & \bullet & & \bullet \\
\bullet & & \bullet & \to & \bullet \\
\bullet & \to & \bullet & \to & \bullet
\end{array}$$

where each dot represents basis vectors w_1, w_2, w_3, w_4 from bottom to top and arrows represent maps. Note that M is isomorphic to $\mathbb{C}Q \oplus \mathbb{C}Q^*$ as a representation of Q and moreover, we have

$$\mathcal{F}^a_{n+1} \simeq \mathrm{Gr}_{\mathbf{dim}\,\mathbb{C}Q}(\mathbb{C}Q \oplus \mathbb{C}Q^*). \tag{1-1}$$

Now let Q be a Dynkin quiver. Recall that the path algebra $\mathbb{C}Q$ (resp. its linear dual $\mathbb{C}Q^*$) is isomorphic as a representation of Q to the direct sum of all indecomposable projective (resp. injective) representations. Motivated by the isomorphism (1-1), we consider the quiver Grassmannians $\mathrm{Gr}_{\mathbf{dim}\,P}(P \oplus I)$, where P and I are projective resp. injective representations of Q. (We note that some of our results are valid for more general Grassmannians and we discuss it in the main body of the paper. However, in the introduction we restrict ourselves to the above mentioned class of varieties). We use the isomorphism (1-1) in two different ways. On the one hand, we generalize and expand the results about $\mathcal{F}^a_{n+1}$ to the case of the above quiver Grassmannians. On the other hand, we use general results and constructions from the theory of quiver representations to understand better the structure of $\mathcal{F}^a_{n+1}$.

Our first theorem is as follows:

Theorem 1.1. *The variety* $\mathrm{Gr}_{\mathbf{dim}\,P}(P \oplus I)$ *is of dimension* $\langle \mathbf{dim}\,P, \mathbf{dim}\,I \rangle$ *and irreducible. It is a normal local complete intersection variety.*

Our next goal is to construct cellular decompositions of the quiver Grassmannians and to compute their Poincaré polynomials. Let us consider the following stratification of $\mathrm{Gr}_{\mathbf{dim}\,P}(P \oplus I)$. For a point N we set $N_I = N \cap I$, $N_P = \pi N$, where $\pi : P \oplus I \to P$ is the projection. Then for a dimension vector $\boldsymbol{f} \in \mathbb{Z}_{\geq 0} Q_0$ we set

$$\mathcal{S}_{\boldsymbol{f}} = \{N \in \mathrm{Gr}_{\mathbf{dim}\,P}(P \oplus I) : \mathbf{dim}\,N_I = \boldsymbol{f}, \mathbf{dim}\,N_P = \mathbf{dim}\,P - \boldsymbol{f}\}.$$

We have natural surjective maps $\zeta_{\boldsymbol{f}} : \mathcal{S}_{\boldsymbol{f}} \to \mathrm{Gr}_{\boldsymbol{f}}(I) \times \mathrm{Gr}_{\mathbf{dim}\,P-\boldsymbol{f}}(P)$.

Theorem 1.2. *The map* $\zeta_{\boldsymbol{f}}$ *is a vector bundle. The fiber over a point* (N_P, N_I) *is isomorphic to* $Hom_Q(N_P, I/N_I)$ *which has dimension* $\langle \mathbf{dim}\,P - \boldsymbol{f}, \mathbf{dim}\,I - \boldsymbol{f} \rangle$.

Using this theorem, we construct a cellular decomposition for each stratum $X_{\boldsymbol{f}}$ and thus for the whole variety X as well. Moreover, since the Poincaré polynomials of $\mathrm{Gr}_{\boldsymbol{f}}(I)$ and of $\mathrm{Gr}_{\mathbf{dim}\,P-\boldsymbol{f}}(P)$ can be easily computed, we arrive at a formula for the Poincaré polynomial (and thus for the Euler characteristic) of X. Recall (see [Feigin 2010]) that the Euler characteristic of the variety $\mathcal{F}^a_{n+1}$ is given by the normalized median Genocchi number h_{n+1} (see [Dellac 1900; Dumont 1974; Dumont and Randrianarivony 1994; Dumont and Zeng 1994; Viennot 1982]). Using Theorem 1.2 we obtain an explicit formula for h_{n+1} in terms of binomial coefficients. Moreover, we give a formula for the Poincaré polynomial of $\mathcal{F}^a_{n+1}$, providing a natural q-version of h_{n+1}.

Finally we study the action of the group of automorphisms $Aut(P \oplus I)$ on the quiver Grassmannians. Let $G \subset \mathrm{Aut}(P \oplus I)$ be the group

$$G = \begin{bmatrix} \mathrm{Aut}_Q(P) & 0 \\ \mathrm{Hom}_Q(P, I) & \mathrm{Aut}_Q(I) \end{bmatrix}.$$

(We note that G coincides with the whole group of automorphisms unless Q is of type A_n.) We prove the following theorem:

Theorem 1.3. *The group G acts on $\mathrm{Gr}_{\mathbf{dim}\, P}(P \oplus I)$ with finitely many orbits, parametrized by pairs of isomorphism classes $([Q_P], [N_I])$ such that Q_P is a quotient of P, N_I is a subrepresentation of I, and $\mathbf{dim}\, Q_P = \mathbf{dim}\, N_I$. Moreover, if Q is equioriented of type A_n, then the orbits are cells parametrized by torus fixed points.*

Outline of the paper. In Section 2, we recall general facts about quiver Grassmannians and degenerate flag varieties. In Section 3, we prove that the quiver Grassmannians $\mathrm{Gr}_{\mathbf{dim}}(P \oplus I)$ are locally complete intersections and that they are flat degenerations of the Grassmannians in exceptional representations. In Section 4, we study the action of the automorphism group on $\mathrm{Gr}_{\mathbf{dim}}(P \oplus I)$, describe the orbits and prove the normality of $\mathrm{Gr}_{\mathbf{dim}}(P \oplus I)$. In Section 5, we construct a one-dimensional torus action on our quiver Grassmannians such that the attracting sets form a cellular decomposition. Sections 6 and 7 are devoted to the case of the equioriented quiver of type A. In Section 6, we compute the Poincaré polynomials of $\mathrm{Gr}_{\mathbf{dim}}(P \oplus I)$ and derive several new formulas for the Euler characteristics — the normalized median Genocchi numbers. In Section 7 we prove that the orbits studied in Section 4, are cells coinciding with the attracting cells constructed in Section 5. We also describe the connection with the degenerate group SL^a_{n+1}.

2. General facts on quiver Grassmannians and degenerate flag varieties

General facts on quivers. Let Q be a finite quiver with a finite set of vertices Q_0 and finite set of arrows Q_1; arrows will be written as $(\alpha : i \to j) \in Q_1$ for $i, j \in Q_0$. We assume Q to be without oriented cycles. Denote by $\mathbb{Z}Q_0$ the free abelian group generated by Q_0, and by $\mathbb{N}Q_0$ the subsemigroup of dimension vectors $\boldsymbol{d} = (d_i)_{i \in Q_0}$ for Q. Let $\langle \_\,, \_ \rangle$ be the Euler form on $\mathbb{Z}Q_0$ defined by

$$\langle \boldsymbol{d}, \boldsymbol{e} \rangle = \sum_{i \in Q_0} d_i e_i - \sum_{(\alpha: i \to j) \in Q_1} d_i e_j .$$

We consider finite dimensional representations M of Q over the complex numbers, viewed either as finite dimensional left modules over the path algebra $\mathbb{C}Q$ of Q, or as tuples $M = ((M_i)_{i \in Q_0}, (M_\alpha : M_i \to M_j)_{(\alpha: i \to j) \in Q_1})$ consisting of finite dimensional complex vector spaces M_i and linear maps M_α. The category $\mathrm{rep}(Q)$ of all such representations is hereditary (that is, $\mathrm{Ext}_Q^{\geq 2}(\_,\_) = 0$). Its Grothendieck group $K(\mathrm{rep}(Q))$ is isomorphic to $\mathbb{Z}Q_0$ through the the identification of the class of a representation M with its dimension vector $\mathbf{dim}\, M = (\dim M_i)_{i\in Q_0} \in \mathbb{Z}Q_0$. The Euler form defined above then identifies with the homological Euler form, that is,

$$\dim \mathrm{Hom}_Q(M, N) - \dim \mathrm{Ext}^1_Q(M, N) = \langle \mathbf{dim}\, M, \mathbf{dim}\, N\rangle$$

for all representations M and N.

Associated to a vertex $i \in Q_0$, we have the simple representation S_i of Q with $(\mathbf{dim}\, S_i)_j = \delta_{i,j}$ (the Kronecker delta), the projective indecomposable P_i, and the indecomposable injective I_i. The latter are determined as the projective cover (resp. injective envelope) of S_i; more explicitly, $(P_i)_j$ is the space generated by all paths from i to j, and the linear dual of $(I_i)_j$ is the space generated by all paths from j to i.

Given a dimension vector $\boldsymbol{d} \in \mathbb{N}Q_0$, we fix complex vector spaces M_i of dimension d_i for all $i \in Q_0$. We consider the affine space

$$R_{\boldsymbol{d}}(Q) = \bigoplus_{(\alpha:i\to j)} \mathrm{Hom}_{\mathbb{C}}(M_i, M_j);$$

its points canonically parametrize representations of Q of dimension vector $\boldsymbol{d}$. The reductive algebraic group $G_{\boldsymbol{d}} = \prod_{i\in Q_0} \mathrm{GL}(M_i)$ acts naturally on $R_{\boldsymbol{d}}(Q)$ via base change

$$(g_i)_i \cdot (M_\alpha)_\alpha = (g_j M_\alpha g_i^{-1})_{(\alpha:i\to j)},$$

such that the orbits $\mathbb{O}_M$ for this action naturally correspond to the isomorphism classes $[M]$ of representations of Q of dimension vector $\boldsymbol{d}$. Note that $\dim G_{\boldsymbol{d}} - \dim R_{\boldsymbol{d}}(Q) = \langle \boldsymbol{d}, \boldsymbol{d}\rangle$. The stabilizer under $G_{\boldsymbol{d}}$ of a point $M \in R_{\boldsymbol{d}}(Q)$ is isomorphic to the automorphism group $\mathrm{Aut}_Q(M)$ of the corresponding representation, which (being open in the endomorphism space $\mathrm{End}_Q(M)$) is a connected algebraic group of dimension $\dim \mathrm{End}_Q(M)$. In particular, we get the following formulas:

$$\dim \mathbb{O}_M = \dim G_d - \dim \mathrm{End}_Q(M), \quad \mathrm{codim}_{R_{\boldsymbol{d}}} \mathbb{O}_M = \dim \mathrm{Ext}^1_Q(M, M). \tag{2-1}$$

Basic facts on quiver Grassmannians. The constructions and results in this section follow [Caldero and Reineke 2008; Schofield 1992]. Additionally to fix another dimension vector $\boldsymbol{e}$ such that $\boldsymbol{e} \leq \boldsymbol{d}$ componentwise, and define the Q_0-graded Grassmannian $\mathrm{Gr}_{\boldsymbol{e}}(\boldsymbol{d}) = \prod_{i\in Q_0} \mathrm{Gr}_{e_i}(M_i)$ which is a projective homogeneous space for $G_{\boldsymbol{d}}$ of dimension $\sum_{i\in Q_0} e_i(d_i - e_i)$, namely $\mathrm{Gr}_{\boldsymbol{e}}(\boldsymbol{d}) \simeq G_{\boldsymbol{d}}/P_{\boldsymbol{e}}$ for a maximal parabolic $P_{\boldsymbol{e}} \subset G_{\boldsymbol{d}}$. We define $\mathrm{Gr}^Q_{\boldsymbol{e}}(\boldsymbol{d})$, the universal Grassmannian of $\boldsymbol{e}$-dimensional subrepresentations of $\boldsymbol{d}$-dimensional representations of Q, as the closed subvariety of $\mathrm{Gr}_{\boldsymbol{e}}(\boldsymbol{d}) \times R_{\boldsymbol{d}}(Q)$ consisting of tuples

$$((U_i \subset M_i)_{i\in Q_0}, (M_\alpha)_{\alpha\in Q_1})$$

such that $M_\alpha(U_i) \subset U_j$ for all arrows $(\alpha : i \to j) \in Q_1$. The group $G_{\boldsymbol{d}}$ acts on $\mathrm{Gr}^Q_{\boldsymbol{e}}(\boldsymbol{d})$ diagonally so that the projections $p_1 : \mathrm{Gr}^Q_{\boldsymbol{e}}(\boldsymbol{d}) \to \mathrm{Gr}_{\boldsymbol{e}}(\boldsymbol{d})$ and $p_2 : \mathrm{Gr}^Q_{\boldsymbol{e}}(\boldsymbol{d}) \to R_{\boldsymbol{d}}(Q)$ are $G_{\boldsymbol{d}}$-equivariant. In fact, the projection p_1 identifies $\mathrm{Gr}^Q_{\boldsymbol{e}}(\boldsymbol{d})$ as the total space of a homogeneous vector bundle over $\mathrm{Gr}_{\boldsymbol{e}}(\boldsymbol{d})$ of rank

$$\sum_{(\alpha:i\to j)\in Q_1} (d_i d_j + e_i e_j - e_i d_j).$$

Indeed, for a point $(U_i)_{i=1}^{\#Q_0}$ in $\mathrm{Gr}_{\boldsymbol{e}}(\boldsymbol{d})$, we can choose complements $M_i = U_i \oplus V_i$ and identify the fiber of p_1 over $(U_i)_{i=1}^{\#Q_0}$ with

$$\left(\begin{bmatrix} \mathrm{Hom}_Q(U_i, U_j) & \mathrm{Hom}_Q(V_i, U_j) \\ 0 & \mathrm{Hom}_Q(V_i, V_j) \end{bmatrix} \subset \mathrm{Hom}_Q(M_i, M_j) \right)_{(\alpha:i\to j)} \subset R_{\boldsymbol{d}}(Q).$$

In particular, the universal Grassmannian $\mathrm{Gr}^Q_{\boldsymbol{e}}(\boldsymbol{d})$ is smooth and irreducible of dimension

$$\dim \mathrm{Gr}^Q_{\boldsymbol{e}}(\boldsymbol{d}) = \langle \boldsymbol{e}, \boldsymbol{d} - \boldsymbol{e} \rangle + \dim R_{\boldsymbol{d}}(Q).$$

The projection p_2 is proper, thus its image is a closed $G_{\boldsymbol{d}}$-stable subvariety of $R_{\boldsymbol{d}}$, consisting of representations admitting a subrepresentation of dimension vector $\boldsymbol{e}$.

We define the quiver Grassmannian $\mathrm{Gr}_{\boldsymbol{e}}(M) = p_2^{-1}(M)$ as the fiber of p_2 over a point $M \in R_{\boldsymbol{d}}(Q)$; by definition, it parametrizes (more precisely, its closed points parametrize) $\boldsymbol{e}$-dimensional subrepresentations of the representation M.

Remark 2.1. Note that we have to view $\mathrm{Gr}_{\boldsymbol{e}}(M)$ as a scheme; in particular, it might be nonreduced. For example, if Q is the Kronecker quiver, $\boldsymbol{e}$ the isotropic root, and M is a regular indecomposable representation of dimension vector $2\boldsymbol{e}$, the quiver Grassmannian is Spec of the ring of dual numbers.

Recall that a representation M is called exceptional if $\mathrm{Ext}^1_Q(M, M) = 0$; thus, in view of (2-1), its orbit in $R_{\boldsymbol{d}}(Q)$ is open and dense.

Proposition 2.2. *Let M be an exceptional $\boldsymbol{d}$-dimensional representation of Q. Then $\mathrm{Gr}_{\boldsymbol{e}}(M)$ is nonempty if $\mathrm{Ext}^1_Q(N, L)$ vanishes for generic N of dimension vector $\boldsymbol{e}$ and generic L of dimension vector $\boldsymbol{d} - \boldsymbol{e}$. In this case, $\mathrm{Gr}_{\boldsymbol{e}}(M)$ is smooth of dimension $\langle \boldsymbol{e}, \boldsymbol{d} - \boldsymbol{e} \rangle$, and for all $\boldsymbol{d}$-dimensional representations N, every irreducible component of $\mathrm{Gr}_{\boldsymbol{e}}(N)$ has at least dimension $\langle \boldsymbol{e}, \boldsymbol{d} - \boldsymbol{e} \rangle$.*

Proof. The criterion for nonemptiness follows from [Schofield 1992, Theorem 3.3]. If $\mathrm{Gr}_{\boldsymbol{e}}(M)$ is nonempty, p_2 is surjective with $\mathrm{Gr}_{\boldsymbol{e}}(M)$ as its generic fiber. In particular, $\mathrm{Gr}_{\boldsymbol{e}}(M)$ is smooth of dimension $\langle \boldsymbol{e}, \boldsymbol{d} - \boldsymbol{e} \rangle$. For all other fibers, we obtain at least the desired estimate on dimensions of their irreducible components [Hartshorne 1977, Chapter II, Exercise 3.22 (b)]. □

We conclude this section by pointing out a useful isomorphism: let U be a point of $\mathrm{Gr}_{\boldsymbol{e}}(M)$ and let $T_U(\mathrm{Gr}_{\boldsymbol{e}}(M))$ denote the Zariski tangent space of $\mathrm{Gr}_{\boldsymbol{e}}(M)$ at U. As shown in [Schofield 1992; Caldero and Reineke 2008] we have the following scheme-theoretic description of the tangent space:

Lemma 2.3. *For $U \in \mathrm{Gr}_{\boldsymbol{e}}(M)$, we have $T_U(\mathrm{Gr}_{\boldsymbol{e}}(M)) \simeq \mathrm{Hom}_Q(U, M/U)$.*

Quotient construction of (universal) quiver Grassmannians and a stratification. We follow [Reineke 2008, Section 3.2]. Additionally to the choices before, fix vector spaces N_i of dimension e_i for $i \in Q_0$. We consider the universal variety $\mathrm{Hom}_Q(\boldsymbol{e}, \boldsymbol{d})$ of homomorphisms from an $\boldsymbol{e}$-dimensional to a $\boldsymbol{d}$-dimensional representation; explicitly, $\mathrm{Hom}_Q(\boldsymbol{e}, \boldsymbol{d})$ is the set of triples

$$((N_\alpha)_{\alpha\in Q_1}, (f_i : N_i \to M_i)_{i\in Q_0}, (M_\alpha)_{\alpha\in Q_1}) \in R_{\boldsymbol{e}} \times \prod_{i\in Q_0} \mathrm{Hom}(N_i, M_i) \times R_{\boldsymbol{d}}(Q)$$

such that $f_j N_\alpha = M_\alpha f_i$ for all $(\alpha : i \to j) \in Q_1$. This is an affine variety defined by quadratic relations, namely by the vanishing of the individual entries of the matrices $f_j N_\alpha - M_\alpha f_i$, on which $G_{\boldsymbol{e}} \times G_{\boldsymbol{d}}$ acts naturally. On the open subset $\mathrm{Hom}^0_Q(\boldsymbol{e}, \boldsymbol{d})$ where all $f_i : N_i \to M_i$ are injective maps, the action of $G_{\boldsymbol{e}}$ is free. By construction, we have an isomorphism

$$\mathrm{Hom}^0_Q(\boldsymbol{e}, \boldsymbol{d})/G_e \simeq \mathrm{Gr}^Q_{\boldsymbol{e}}(\boldsymbol{d})$$

which associates the orbit of a triple $((N_\alpha), (f_i), (M_\alpha))$ with the pair given by $(f_i(N_i) \subset M_i)_{i\in Q_0}, (M_\alpha)_{\alpha\in Q_1})$. Indeed, the maps N_α are uniquely determined in this situation, and they can be reconstructed algebraically from (f_i) and (M_α) (see [Reineke 2008, Lemma 3.5]).

Similarly to $\mathrm{Gr}^Q_{\boldsymbol{e}}(\boldsymbol{d})$, we have a projection $\tilde{p}_2 : \mathrm{Hom}^0_Q(\boldsymbol{e}, \boldsymbol{d}) \to R_{\boldsymbol{d}}(Q)$ with fibers $\tilde{p}_2^{-1}(M) = \mathrm{Hom}^0_Q(\boldsymbol{e}, M)$, and we have a local version of the previous isomorphism:

$$\mathrm{Hom}^0_Q(\boldsymbol{e}, M)/G_{\boldsymbol{e}} = \tilde{p}_2^{-1}(M)/G_{\boldsymbol{e}} \simeq \mathrm{Gr}_{\boldsymbol{e}}(M).$$

Note that the quotient map $\mathrm{Hom}^0_Q(\boldsymbol{e}, M) \to \mathrm{Gr}_{\boldsymbol{e}}(M)$ is locally trivial since it is induced by the quotient map

$$\mathrm{Hom}^0_Q(\boldsymbol{e}, \boldsymbol{d}) = \prod_{i\in Q_0} \mathrm{Hom}^0(N_i, M_i) \to \mathrm{Gr}_{\boldsymbol{e}}(\boldsymbol{d}),$$

which can be trivialized over the standard open affine coverings of Grassmannians.

Let p be the projection from $\mathrm{Hom}^0_Q(\boldsymbol{e}, M)$ to $R_{\boldsymbol{e}}(Q)$; its fiber over N is the space $\mathrm{Hom}^0_Q(N, M)$ of injective maps. For each isomorphism class $[N]$ of representations of dimension vector $\boldsymbol{e}$, we can consider the subset $\mathcal{S}_{[N]}$ of $\mathrm{Gr}_{\boldsymbol{e}}(M)$ corresponding under the previous isomorphism to $(p^{-1}(\mathbb{O}_N))/G_{\boldsymbol{e}}$. Therefore it consists of all subrepresentations $U \in \mathrm{Gr}_{\boldsymbol{e}}(M)$ which are isomorphic to N.

Lemma 2.4. *Each $\mathcal{S}_{[N]}$ is an irreducible locally closed subset of* $\mathrm{Gr}_{\boldsymbol{e}}(M)$ *of dimension* $\dim \mathrm{Hom}_Q(N, M) - \dim \mathrm{End}_Q(N)$.

Proof. Irreducibility of $\mathcal{S}_{[N]}$ follows from irreducibility of $\mathbb{O}_N$ by G_e-equivariance of p. Using the fact that the geometric quotient is closed and separating on G_e-stable subsets, induction over $\dim \mathbb{O}_N$ proves that all $\mathcal{S}_{[N]}$ are locally closed. The dimension is calculated as

$$\dim \mathcal{S}_{[N]} = \dim \mathbb{O}_N + \dim \mathrm{Hom}^0_Q(N, M) - \dim G_e. \qquad \square$$

Degenerate flag varieties. In this subsection we recall the definition of the degenerate flag varieties following [Feigin 2010; 2011; Feigin and Finkelberg 2011]. Let W be an n-dimensional vector space with a basis $w_1, \dots, w_n$. We denote by $\mathrm{pr}_k : W \to W$ the projections along w_k to the linear span of the remaining basis vectors, that is, $\mathrm{pr}_k \sum_{i=1}^n c_i w_i = \sum_{i \neq k} c_i w_i$.

Definition 2.5. The variety $\mathcal{F}^a_n$ is the set of collections of subspaces

$$(V_i \in \mathrm{Gr}_i(W))_{i=1}^{n-1}$$

subject to the conditions $\mathrm{pr}_{i+1} V_i \subset V_{i+1}$ for all $i = 1, \dots, n-2$.

The variety $\mathcal{F}^a_n$ is called the complete degenerate flag variety. It enjoys the following properties:

- $\mathcal{F}^a_n$ is a singular irreducible projective algebraic variety of dimension $\binom{n}{2}$.
- $\mathcal{F}^a_n$ is a flat degeneration of the classical complete flag variety SL_n/B.
- $\mathcal{F}^a_n$ is a normal local complete intersection variety.
- $\mathcal{F}^a_n$ can be decomposed into a disjoint union of complex cells.

We add some comments on the last property. The number of cells (which is equal to the Euler characteristic of $\mathcal{F}^a_n$) is given by the n-th normalized median Genocchi number h_n; see for example [Feigin 2011, Section 3]. These numbers have several definitions. Here, we will use the following one: h_n is the number of collections $(S_1, \dots, S_{n-1})$, where $S_i \subset \{1, \dots, n\}$ subject to the conditions

$$\# S_i = i, \ 1 \le i \le n-1; \quad S_i \subset S_{i+1} \cup \{i+1\}, \ 1 \le i \le n-2.$$

For $n = 1, 2, 3, 4, 5$ the numbers h_n are equal to $1, 2, 7, 38, 295$, respectively.

There exists a degeneration SL^a_n of the group SL_n acting on $\mathcal{F}^a_n$. Namely, the degenerate group SL^a_n is the semidirect product of the Borel subgroup B of SL_n and a normal abelian subgroup $\mathbb{G}_a^{n(n-1)/2}$, where $\mathbb{G}_a$ is the additive group of the field. The simplest way to describe the structure of the semidirect product is via the Lie algebra $\mathfrak{sl}^a_n$ of SL^a_n. Namely, let $\mathfrak{b} \in \mathfrak{sl}_n$ be the Borel subalgebra of upper-triangular matrices and $\mathfrak{n}^-$ be the nilpotent subalgebra of strictly lower-triangular matrices.

Let $(\mathfrak{n}^-)^a$ be the abelian Lie algebra with underlying vector space $\mathfrak{n}^-$. Then $\mathfrak{n}^-$ carries a natural structure of $\mathfrak{b}$-module induced by the adjoint action on the quotient $(\mathfrak{n}^-)^a \simeq \mathfrak{sl}_n/\mathfrak{b}$. Then $\mathfrak{sl}_n^a = \mathfrak{b} \oplus (\mathfrak{n}^-)^a$, where $(\mathfrak{n}^-)^a$ is abelian ideal and $\mathfrak{b}$ acts on $(\mathfrak{n}^-)^a$ as described above. The group SL_n^a (the Lie group of $\mathfrak{sl}_n^a$) acts on the variety $\mathcal{F}_n^a$ with an open $\mathbb{G}_a^{n(n-1)/2}$-orbit. We note that in contrast with the classical situation, the group SL_n^a acts on $\mathcal{F}_n^a$ with an infinite number of orbits.

For partial (parabolic) flag varieties of SL_n there exists a natural generalization of $\mathcal{F}_n^a$. Namely, consider an increasing collection $1 \le d_1 < \cdots < d_s < n$. In what follows we denote such a collection by $\boldsymbol{d}$. Let $\mathcal{F}_{\boldsymbol{d}}$ be the classical partial flag variety consisting of the collections $(V_i)_{i=1}^s$, $V_i \in \mathrm{Gr}_{d_i}(W)$ such that $V_i \subset V_{i+1}$.

Definition 2.6. The degenerate partial variety $\mathcal{F}_{\boldsymbol{d}}^a$ is the set of collections of subspaces $V_i \in \mathrm{Gr}_{d_i}(W)$ subject to the conditions $\mathrm{pr}_{d_i+1} \dots \mathrm{pr}_{d_{i+1}} V_i \subset V_{i+1}$ for all $i = 1, \dots, s-1$.

We still have the following properties:

- $\mathcal{F}_{\boldsymbol{d}}^a$ is a singular irreducible projective algebraic variety.
- $\mathcal{F}_{\boldsymbol{d}}^a$ is a flat degeneration of $\mathcal{F}_{\boldsymbol{d}}$.
- $\mathcal{F}_{\boldsymbol{d}}^a$ is a normal local complete intersection variety.
- $\mathcal{F}_{\boldsymbol{d}}^a$ is acted upon by the group SL_n^a with an open $\mathbb{G}_a^{n(n-1)/2}$-orbit.

Comparison between quiver Grassmannians and degenerate flag varieties. Let Q be an equioriented quiver of type A_n. We order the vertices of Q from 1 to n in such a way that the arrows of Q are of the form $i \to i+1$. Let P_i and I_i, be the projective and injective representations, respectively, attached to the i-th vertex for $i = 1, \dots, n$. In particular, $\mathbf{dim}\, P_i = (0, \dots, 0, 1, \dots, 1)$ with $i-1$ zeros and $\mathbf{dim}\, I_i = (1, \dots, 1, 0, \dots, 0)$ with $n-i$ zeros.

We will use the following basis of P_i and I_i: for each $j = i, \dots, n$ we fix nonzero elements $w_{i,j} \in (P_j)_i$ in such a way that $w_{i,j} \mapsto w_{i+1,j}$. Also, for $j = 1, \dots, i$, we fix nonzero elements $w_{i,j+1} \in (I_j)_i$ in such a way that $w_{i,j} \mapsto w_{i+1,j}$ unless $j = i+1$ and $w_{i,i+1} \mapsto 0$.

Let A be the path algebra $\mathbb{C}Q$. Viewed as a representation of Q, A is isomorphic to the direct sum $\bigoplus_{i=1}^n P_i$. In particular, $\mathbf{dim}\, A = (1, 2, \dots, n)$. The linear dual A^* is isomorphic to the direct sum of injective representations $\bigoplus_{i=1}^n I_i$.

Proposition 2.7. *The quiver Grassmannian* $\mathrm{Gr}_{\mathbf{dim}\, A}(A \oplus A^*)$ *is isomorphic to the degenerate flag variety* $\mathcal{F}_{n+1}^a$ *of* $\mathfrak{sl}_{n+1}$.

Proof. Consider $A \oplus A^* = \bigoplus_{i=1}^n (P_i \oplus I_i)$ as a representation of Q. Let W_j be the space attached to the j-th vertex, that is, $A \oplus A^* = (W_1, \dots, W_n)$. First, we note that $\dim W_j = n+1$ for all j. Second, we fix an $(n+1)$-dimensional vector space W with a basis $w_1, \dots, w_{n+1}$. We identify all W_j with W by sending $w_{i,j}$ to w_j.

Then the maps $W_j \to W_{j+1}$ coincide with pr_{j+1}. Now our proposition follows from the equality $\mathbf{dim}\, A = (1, 2, \dots, n)$. $\square$

The coefficient quiver of the representation $A \oplus A^*$ is given by ($n = 4$):

$$\begin{array}{ccccccc} w_{1,5} & \to & w_{2,5} & \to & w_{3,5} & \to & w_{4,5} \\ w_{1,4} & \to & w_{2,4} & \to & w_{3,4} & & w_{4,4} \\ w_{1,3} & \to & w_{2,3} & & w_{3,3} & \to & w_{4,3} \\ w_{1,2} & & w_{2,2} & \to & w_{3,2} & \to & w_{4,2} \\ w_{1,1} & \to & w_{2,1} & \to & w_{3,1} & \to & w_{4,1} \end{array} \tag{2-2}$$

Remark 2.8. We note that the classical SL_{n+1} flag variety has a similar realization. Namely, let $\tilde{M}$ be the representation of Q isomorphic to the direct sum of $n+1$ copies of P_1, so $\mathbf{dim}\, \tilde{M} = \mathbf{dim}\, (A \oplus A^*)$. Then the classical flag variety SL_{n+1}/B is isomorphic to the quiver Grassmannian $\mathrm{Gr}_{\mathbf{dim} A} \tilde{M}$. The Q-representation $\tilde{M}$ can be visualized as ($n = 4$)

$$\begin{array}{ccccccc} \bullet & \to & \bullet & \to & \bullet & \to & \bullet \\ \bullet & \to & \bullet & \to & \bullet & \to & \bullet \\ \bullet & \to & \bullet & \to & \bullet & \to & \bullet \\ \bullet & \to & \bullet & \to & \bullet & \to & \bullet \\ \bullet & \to & \bullet & \to & \bullet & \to & \bullet \end{array} \tag{2-3}$$

We can easily generalize Proposition 2.7 to degenerate partial flag varieties: Suppose we are given a sequence $\boldsymbol{d} = (0 = d_0 < d_1 < d_2 < \dots < d_s < d_{s+1} = n+1)$. Then we define

$$P = \bigoplus_{i=1}^{s} P_i^{d_i - d_{i-1}} \quad \text{and} \quad I = \bigoplus_{i=1}^{s} I_i^{d_{i+1} - d_i}$$

as representations of an equioriented quiver of type A_s.

Proposition 2.9. *The quiver Grassmannian* $\mathrm{Gr}_{\mathbf{dim}\, P}(P \oplus I)$ *is isomorphic to the degenerate partial flag variety* $\mathcal{F}^a_{\boldsymbol{d}}$ *of* $\mathfrak{sl}_{n+1}$.

Proof. The dimension vector of $P \oplus I$ is given by $(n+1, \dots, n+1)$ and the dimension vector of P equals $(d_1, \dots, d_s)$. Now let us identify the spaces $(P \oplus I)_j$ with W as in the proof of Proposition 2.7. Then the map

$$(P \oplus I)_j \to (P \oplus I)_{j+1}$$

corresponding to the arrow $j \to j+1$ coincides with $\mathrm{pr}_{d_j+1}, \dots, \mathrm{pr}_{d_{j+1}}$, which proves the proposition. $\square$

3. A class of well-behaved quiver Grassmannians

Geometric properties. From now on, let Q be a Dynkin quiver. Then $G_{\boldsymbol{d}}$ acts with finitely many orbits on $R_{\boldsymbol{d}}(Q)$ for every $\boldsymbol{d}$; in particular, for every $\boldsymbol{d} \in \mathbb{N}Q_0$, there exists a unique (up to isomorphism) exceptional representation of this dimension vector.

The subsets $\mathcal{S}_{[N]}$ defined just before Lemma 2.4 then define a finite stratification of each quiver Grassmannian $\mathrm{Gr}_{\boldsymbol{e}}(M)$ according to isomorphism type of the subrepresentation $N \subset M$.

Proposition 3.1. *Assume that X and Y are exceptional representations of Q such that* $\mathrm{Ext}^1_Q(X, Y) = 0$. *Define $M = X \oplus Y$ and $\boldsymbol{e} = \mathbf{dim}\, X$, $\boldsymbol{d} = \mathbf{dim}\,(X \oplus Y)$. Then*:

(i) $\dim \mathrm{Gr}_{\boldsymbol{e}}(M) = \langle \boldsymbol{e}, \boldsymbol{d} - \boldsymbol{e}\rangle$.

(ii) *The variety* $\mathrm{Gr}_{\boldsymbol{e}}(M)$ *is reduced, irreducible and rational.*

(iii) $\mathrm{Gr}_{\boldsymbol{e}}(M)$ *is a locally complete intersection scheme.*

Proof. The representation X obviously embeds into M; thus

$$\begin{aligned}\dim \mathrm{Gr}_{\boldsymbol{e}}(M) \geq \dim \mathcal{S}_{[X]} &= \dim \mathrm{Hom}_Q(X, M) - \dim \mathrm{End}_Q(X)\\ &= \dim \mathrm{Hom}_Q(X, Y).\end{aligned}$$

The tangent space to any point $U \in \mathcal{S}_{[X]}$ has dimension $\dim \mathrm{Hom}_Q(X, Y)$ too, thus $\overline{\mathcal{S}_{[X]}}$ is reduced. Moreover, a generic embedding of X into $X \oplus Y$ is of the form $[\mathrm{id}_X, f]$ for a map $f \in \mathrm{Hom}_Q(X, Y)$, and this identifies an open subset isomorphic to $\mathrm{Hom}_Q(X, Y)$ of $\mathcal{S}_{[X]}$, proving rationality of $\overline{\mathcal{S}_{[X]}}$. Now suppose N embeds into $M = X \oplus Y$ and $\mathbf{dim}\, N = \boldsymbol{e}$. Then $\mathrm{Ext}^1_Q(N, Y) = 0$ since $\mathrm{Ext}^1_Q(X \oplus Y, Y) = 0$ by assumption, and thus $\dim \mathrm{Hom}_Q(N, Y) = \langle \boldsymbol{e}, \boldsymbol{d} - \boldsymbol{e}\rangle = \dim \mathrm{Hom}_Q(X, Y)$. Therefore,

$$\begin{aligned}\dim \mathcal{S}_{[N]} &= \dim \mathrm{Hom}_Q(N, X) - \dim \mathrm{Hom}_Q(N, N) + \dim \mathrm{Hom}_Q(X, Y)\\ &\leq \dim \mathrm{Hom}_Q(X, Y),\end{aligned}$$

which proves that $\dim \mathrm{Gr}_{\boldsymbol{e}}(M) = \dim \mathrm{Hom}_Q(X, Y) = \langle \boldsymbol{e}, \boldsymbol{d} - \boldsymbol{e}\rangle$, and that the closure of $\mathcal{S}_{[X]}$ is an irreducible component of $\mathrm{Gr}_{\boldsymbol{e}}(M)$. Conversely, suppose that an irreducible component C of $\mathrm{Gr}_{\boldsymbol{e}}(M)$ is given. Then C is necessarily the closure of some stratum $\mathcal{S}_{[N]}$, and the dimension of C equals $\langle \boldsymbol{e}, \boldsymbol{d} - \boldsymbol{e}\rangle = \dim \mathrm{Hom}_Q(X, Y)$ by Proposition 2.2. By the dimension estimate above, we conclude that

$$\dim \mathrm{Hom}_Q(N, X) = \dim \mathrm{Hom}_Q(N, N).$$

By [Bongartz 1996, Theorem 2.4], this yields an embedding $N \subset X$, and thus $N = X$ by equality of dimensions. Therefore, $\mathrm{Gr}_{\boldsymbol{e}}(M)$ equals the closure of the stratum $\mathcal{S}_{[X]}$, thus it is irreducible, reduced and rational. The dimension of $\mathrm{Hom}^0_Q(\boldsymbol{e}, M)$

equals $\langle \boldsymbol{e}, \boldsymbol{d}-\boldsymbol{e}\rangle + \dim G_{\boldsymbol{e}}$, thus its codimension in $R_{\boldsymbol{e}}(Q) \times \mathrm{Hom}^0_Q(\boldsymbol{e}, \boldsymbol{d})$ equals

$$\dim R_{\boldsymbol{e}}(Q) + \sum_i e_i d_i - \langle \boldsymbol{e}, \boldsymbol{d}-\boldsymbol{e}\rangle - \dim G_{\boldsymbol{e}} = \sum_{(\alpha: i\to j)\in Q_1} e_i d_j.$$

But this value is exactly the number of equations defining $\mathrm{Hom}_Q(\boldsymbol{e}, M)$. Thus $\mathrm{Hom}^0_Q(\boldsymbol{e}, M)$ is locally a complete intersection. The map $\mathrm{Hom}^0_Q(\boldsymbol{e}, M) \to \mathrm{Gr}_{\boldsymbol{e}}(M)$ is locally trivial with smooth fiber $G_{\boldsymbol{e}}$, hence the last statement follows. □

On a quiver Grassmannian $\mathrm{Gr}_{\boldsymbol{e}}(M)$, the automorphism group $\mathrm{Aut}_Q(M)$ acts algebraically. In the present situation, this implies that the group

$$G = \begin{bmatrix} \mathrm{Aut}_Q(X) & 0 \\ \mathrm{Hom}_Q(X,Y) & \mathrm{Aut}_Q(Y) \end{bmatrix}$$

acts on $\mathrm{Gr}_{\boldsymbol{e}}(X \oplus Y)$.

Flat degeneration. Now let $\tilde{M}$ be the unique (up to isomorphism) exceptional representation of the same dimension vector as M. By Proposition 2.2, we also have $\dim \mathrm{Gr}_{\boldsymbol{e}}(\tilde{M}) = \langle \boldsymbol{e}, \boldsymbol{d}-\boldsymbol{e}\rangle$. It is thus reasonable to ask for good properties of the degeneration from $\mathrm{Gr}_{\boldsymbol{e}}(\widetilde{M})$ to $\mathrm{Gr}_{\boldsymbol{e}}(M)$.

Theorem 3.2. *Under the previous hypotheses, the quiver Grassmannian* $\mathrm{Gr}_{\boldsymbol{e}}(M)$ *is a flat degeneration of* $\mathrm{Gr}_{\boldsymbol{e}}(\widetilde{M})$.

Proof. Let Y be the open subset of $R_{\boldsymbol{d}}(Q)$ consisting of all representations Z whose orbit closure $\overline{\mathcal{O}_Z}$ contains the orbit $\mathcal{O}_M$; in particular, Y contains $\mathcal{O}_{\widetilde{M}}$. We consider the diagram

$$\mathrm{Gr}_{\boldsymbol{e}}(\boldsymbol{d}) \xleftarrow{p_1} \mathrm{Gr}^Q_{\boldsymbol{e}}(\boldsymbol{d}) \xrightarrow{p_2} R_{\boldsymbol{d}}(Q)$$

of the previous section. In particular, we consider the restriction $q : \widetilde{Y} \to Y$ of p_2 to $\widetilde{Y} = p_2^{-1}(Y)$. This is a proper morphism (since p_2 is so) between two smooth and irreducible varieties (since they are open subsets of the smooth varieties $R_{\boldsymbol{d}}(Q)$ and $\mathrm{Gr}^Q_{\boldsymbol{e}}(\boldsymbol{d})$, respectively). The general fiber of q is $\mathrm{Gr}_{\boldsymbol{e}}(\widetilde{M})$, since the orbit of $\widetilde{M}$, being exceptional, is open in Y, and the special fiber of q is $\mathrm{Gr}_{\boldsymbol{e}}(M)$, since the orbit of M is closed in Y by definition. By semicontinuity, all fibers of q have the same dimension $\langle \boldsymbol{e}, \boldsymbol{d}-\boldsymbol{e}\rangle$. By [Matsumura 1989, Corollary to Theorem 23.1], a proper morphism between smooth and irreducible varieties with constant fiber dimension is already flat. □

Remark 3.3. Theorem 3.2 generalizes Proposition 3.15 of [Feigin 2010], where the flatness of the degeneration $\mathcal{F}_n \to \mathcal{F}^a_n$ was proved using complicated combinatorial tools. (See also the sections on degenerate flag varieties in the present paper, pages 172–174.)

Note that the degeneration from $\widetilde{M}$ to M in $R_{\boldsymbol{d}}(Q)$ can be realized along a one-parameter subgroup of $G_{\boldsymbol{d}}$ in the following way:

Lemma 3.4. *Under the previous hypotheses, there exists a short exact sequence*

$$0 \to X \to \widetilde{M} \to Y \to 0.$$

Proof. By [Schofield 1992, Theorem 3.3], a generic representation Z of dimension vector $\boldsymbol{d}$ admits a subrepresentation of dimension vector $\boldsymbol{e}$ if $\mathrm{Ext}^1_Q(N, L)$ vanishes for generic N of dimension vector $\boldsymbol{e}$ and generic L of dimension vector $\boldsymbol{d}-\boldsymbol{e}$. In the present case, these generic representations are $Z = \widetilde{M}$, $N = X$ and $L = Y$, and the lemma follows. □

This lemma implies that $\widetilde{M}$ can be written, up to isomorphism, in the following form

$$\widetilde{M}_\alpha = \begin{bmatrix} X_\alpha & \zeta_\alpha \\ 0 & Y_\alpha \end{bmatrix}$$

for all $\alpha \in Q_1$. Conjugating with the one-parameter subgroup

$$\left(\begin{bmatrix} t \cdot \mathrm{id}X_i & 0 \\ 0 & \mathrm{id}Y_i \end{bmatrix}\right)_{i \in Q_0}$$

of $G_{\boldsymbol{d}}$ and passing to the limit $t = 0$, we arrive at the desired degeneration.

Since Q is a Dynkin quiver, the isomorphism classes of indecomposable representations of Q are parametrized by the positive roots Φ^+ of the corresponding root system. We view Φ^+ as a subset of $\mathbb{N}Q_0$ by identifying the simple root α_i with the vector having 1 at the i-th place and zeros everywhere else. Denote by V_α the indecomposable representation corresponding to $\alpha \in \Phi^+$; more precisely, $\mathbf{dim}\, V_\alpha = \alpha$. Using this parametrization of the indecomposables and the Auslander–Reiten quiver of Q, we can actually construct $\widetilde{M}$ explicitly from X and Y (or, more precisely, from their decompositions into indecomposables), using the algorithm of [Reineke 2001, Section 3].

4. The group action and normality

In this section we put $X = P$ and $Y = I$, where P and I are projective and injective representations of a Dynkin quiver Q. We consider the group

$$G = \begin{bmatrix} \mathrm{Aut}_Q(P) & 0 \\ \mathrm{Hom}_Q(P, I) & \mathrm{Aut}_Q(I) \end{bmatrix}.$$

Theorem 4.1. *The group G acts on $\mathrm{Gr}_{\mathbf{dim}\, P}(P \oplus I)$ with finitely many orbits, parametrized by pairs $([Q_P], [N_I])$ of isomorphism classes such that Q_P is a quotient of P, N_I is a subrepresentation of I, and Q_P and N_I have the same dimension vector.*

Proof. Suppose N is a subrepresentation of $P \oplus I$ of dimension vector $\mathbf{dim}\, N = \mathbf{dim}\, P$, and denote by $\iota : N \to P \oplus I$ the embedding. Define $N_I = N \cap I$ and $N_P = N/(N \cap I)$. Then $N_P \simeq (N+I)/I$ embeds into $(P \oplus I)/I \simeq P$, thus N_P is projective since $\mathrm{rep}(Q)$ is hereditary. Therefore, the short exact sequence

$$0 \to N_I \to N \to N_P \to 0$$

splits. We thus have a retraction $r : N_P \to N$ such that N is the direct sum of N_I and $r(N_P)$ and N_I embeds into the component I of $P \oplus I$ under ι. Without loss of generality, we can thus write the embedding of N into $P \oplus I$ as

$$\iota = \begin{bmatrix} \iota_P & 0 \\ f & \iota_I \end{bmatrix} : N_P \oplus N_I \to P \oplus I$$

for ι_P (resp. ι_I) an embedding of N_P (resp. N_I) into P (resp. I), and $f : N_P \to I$. Since I is injective, the map f factors through ι_P, yielding a map $x : P \to I$ such that $x\iota_P = f$. We can then conjugate ι with

$$\begin{bmatrix} 1 & 0 \\ -x & 1 \end{bmatrix} \in G.$$

We have thus proved that each G-orbit in $\mathrm{Gr}_{\mathbf{dim}_P}(P \oplus I)$ contains an embedding of the form

$$\begin{bmatrix} \iota_P & 0 \\ 0 & \iota_I \end{bmatrix} : N_P \oplus N_I \to P \oplus I,$$

such that N_I is a subrepresentation of I, the representation $Q_P = P/N_P$ is a quotient of P, and their dimension vectors obviously add up to $\mathbf{dim}\, P$. We now have to show that the isomorphism classes of such Q_P and N_I already characterize the corresponding G-orbit in $\mathrm{Gr}_{\mathbf{dim}\, P}(P \oplus I)$. To do this, suppose we are given two such embeddings,

$$\begin{bmatrix} \iota_P & 0 \\ 0 & \iota_I \end{bmatrix} : N_P \oplus N_I \to P \oplus I \quad \text{and} \quad \begin{bmatrix} \iota'_P & 0 \\ 0 & \iota'_I \end{bmatrix} : N'_P \oplus N'_I \to P \oplus I,$$

such that the cokernels Q_P and Q'_P of ι_P and ι'_P, respectively, are isomorphic, and such that N_I and N'_I are isomorphic. By [Reineke 2008, Lemma 6.3], an arbitrary isomorphism $\psi_I : N_I \to N'_I$ lifts to an automorphism φ_I of I, such that $\varphi_I \iota_I = \iota'_I \psi_I$. By the obvious dual version of the same lemma, an arbitrary isomorphism $\xi_P : Q_P \to Q'_P$ lifts to an automorphism φ_P of P, which in turn induces an isomorphism $\psi_P : N_P \to N'_P$ such that $\varphi_P \iota_P = \iota'_P \psi_P$. This proves that the two embeddings above are conjugate under G. Finally, given representations Q_P and N_I as above, we can define N_P as the kernel of the quotient map and get an embedding as above. $\square$

Remark 4.2. We can obtain an explicit parametrization of the orbits by writing

$$P = \bigoplus_{i \in Q_0} P_i^{a_i} \quad \text{and} \quad I = \bigoplus_{i \in Q_0} I_i^{b_i}.$$

By [Reineke 2008, Lemma 4.1] and its obvious dual version, we have:

- A representation N_I embeds into I if and only if $\dim \operatorname{Hom}_Q(S_i, N_I) \le b_i$ for all $i \in Q_0$.
- A representation Q_P is a quotient of P if and only if $\dim \operatorname{Hom}_Q(Q_P, S_i) \le a_i$ for all $i \in Q_0$.

The previous result establishes a finite decomposition of the quiver Grassmannians into orbits. In particular the tangent space is equidimensional along every such orbit. The following example shows that in general such orbits are not cells.

Example 4.3. Let

$$Q := \begin{tikzcd} & & 3 \\ 1 \arrow[r] & 2 \arrow[ur] \arrow[rr] & & 4 \end{tikzcd}$$

be a Dynkin quiver of type D_4. The quiver Grassmannian $\mathrm{Gr}_{(1211)}(I_3 \oplus I_4)$ is isomorphic to $\mathbb{P}^1$, with the points 0 and ∞ corresponding to two decomposable representations, whereas all points in $\mathbb{P}^1 \setminus \{0, \infty\}$, which is obviously not a cell, correspond to subrepresentations which are isomorphic to the indecomposable representation of dimension vector (1211).

We note the following generalization of the tautological bundles

$$\mathcal{U}_i = \{(U, x) \in \mathrm{Gr}_{\boldsymbol{e}}(X) \times X_i \ : \ x \in U_i\} \quad \text{on } \mathrm{Gr}_{\boldsymbol{e}}(X) :$$

Given a projective representation P, the trivial vector bundle $\operatorname{Hom}_Q(P, X)$ on $\mathrm{Gr}_{\boldsymbol{e}}(X)$ admits the subbundle

$$\mathcal{V}_P = \{(U, \alpha) \in \mathrm{Gr}_{\boldsymbol{e}}(X) \times \operatorname{Hom}_Q(P, X) \ : \ \alpha(P) \subset U\}.$$

We then have $\mathcal{V}_P \simeq \bigoplus_{i \in Q_0} \mathcal{V}_i^{m_i}$ if $P \simeq \bigoplus_{i \in Q_0} P_i^{m_i}$. Dually, given an injective representation I, the trivial vector bundle $\operatorname{Hom}_Q(X, I)$ admits the subbundle

$$\mathcal{V}_I = \{(U, \beta) \in \mathrm{Gr}_e(X) \times \operatorname{Hom}_Q(X, I) \ : \ \beta(U) = 0\}.$$

We then have $\mathcal{V}_I \simeq \bigoplus_{i \in Q_0} (\mathcal{V}_i^*)^{m_i}$ if $I \simeq \bigoplus_{i \in Q_0} I_i^{m_i}$.

Given a decomposition of the dimension vector $\mathbf{dim}\, P = \boldsymbol{e} = \boldsymbol{f} + \boldsymbol{g}$, recall the subvariety $\mathcal{S}_{\boldsymbol{f}}(P \oplus I) \subset \mathrm{Gr}_{\boldsymbol{e}}(P \oplus I)$ consisting of all representations N such that $\mathbf{dim}\, N \cap I = \boldsymbol{f}$ and $\mathbf{dim}\, \pi(N) = \boldsymbol{g}$, where $\pi : P \oplus I \to P$ is the natural projection. We have a natural surjective map $\zeta : \mathrm{Gr}_{\boldsymbol{f},\boldsymbol{g}}(P \oplus I) \to \mathrm{Gr}_{\boldsymbol{g}}(P) \times \mathrm{Gr}_{\boldsymbol{f}}(I)$. We note that since P is projective, all the points of $\mathrm{Gr}_{\boldsymbol{g}}(P)$ are isomorphic as representations of Q.

Also, since I is injective, for any two points $M_1, M_2 \in \mathrm{Gr}_{\boldsymbol{f}}(I)$ the representations I/M_1 and I/M_2 of Q are isomorphic. Therefore, the dimension of the vector space $\mathrm{Hom}_Q(N_P, I/N_I)$ is independent of the points $N_P \in \mathrm{Gr}_{\boldsymbol{g}}(P)$ and $N_I \in \mathrm{Gr}_{\boldsymbol{f}}(I)$. We denote this dimension by D.

Proposition 4.4. *The map ζ is a D-dimensional vector bundle (in the Zariski topology).*

Proof. Associated to N_P and N_I, we have exact sequences

$$0 \to N_P \to P \to Q_P \to 0 \quad \text{and} \quad 0 \to N_I \to I \to Q_I \to 0.$$

These induce the following commutative diagram with exact rows and columns, where the final zeroes arise from projectivity of N_P and injectivity of Q_I and we abbreviate $\mathrm{Hom}_Q(\_,\_)$ by $(\_,\_)$:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & (Q_P, N_I) & \to & (Q_P, I) & \to & (Q_P, Q_I) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & (P, N_I) & \to & (P, I) & \to & (P, Q_I) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & (N_P, N_I) & \to & (N_P, I) & \to & (N_P, Q_I) & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

This diagram yields an isomorphism

$$\mathrm{Hom}_Q(N_P, Q_I) \simeq \mathrm{Hom}_Q(P, I)/(\mathrm{Hom}_Q(P, N_I) + \mathrm{Hom}_Q(Q_P, I)).$$

Pulling back the tautological bundles constructed above via the projections

$$\mathrm{Gr}_{\boldsymbol{g}}(P) \overset{\mathrm{pr}_1}{\leftarrow} \mathrm{Gr}_{\boldsymbol{g}}(P) \times \mathrm{Gr}_{\boldsymbol{f}}(I) \overset{\mathrm{pr}_2}{\to} \mathrm{Gr}_{\boldsymbol{f}}(I),$$

we get subbundles $\mathrm{pr}_2^* \mathcal{V}_P$ and $\mathrm{pr}_1^* \mathcal{V}_I$ of the trivial bundle $\mathrm{Hom}_Q(P, I)$ on $\mathrm{Gr}_{\boldsymbol{g}}(P) \times \mathrm{Gr}_{\boldsymbol{f}}(I)$. By the above isomorphism, the quotient bundle

$$\mathrm{Hom}_Q(P, I)/(\mathrm{pr}_1^* \mathcal{V}_P + \mathrm{pr}_2^* \mathcal{V}_I)$$

identifies with the fibration $\zeta : \mathcal{S}_{\boldsymbol{f}}(P \oplus I) \to \mathrm{Gr}_{\boldsymbol{g}}(P) \times \mathrm{Gr}_{\boldsymbol{f}}(I)$, proving the Zariski local triviality of the latter. □

The methods established in the two previous proofs now allow us to prove normality of the quiver Grassmannians.

Theorem 4.5. *The quiver Grassmannian $\mathrm{Gr}_{\boldsymbol{e}}(P \oplus I)$ is a normal variety.*

Proof. We already know that $\mathrm{Gr}_{\boldsymbol{e}}(P \oplus I)$ is locally a complete intersection, thus normality is proved once we know that $\mathrm{Gr}_{\boldsymbol{e}}(P \oplus I)$ is regular in codimension 1. By the proof of Theorem 4.1, we know that a subrepresentation N of $P \oplus I$ of dimension vector $\mathbf{dim}\, P$ is of the form $N = N_P \oplus N_I$, with exact sequences

$$0 \to N_P \to P \to Q_P \to 0 \quad \text{and} \quad 0 \to N_I \to I \to Q_I \to 0$$

such that N_I and Q_P are of the same dimension vector $\boldsymbol{f}$. By the tangent space formula, N defines a singular point of $\mathrm{Gr}_{\boldsymbol{e}}(P \oplus I)$ if and only if

$$\mathrm{Ext}^1_Q(N_P \oplus N_I, Q_P \oplus Q_I) = \mathrm{Ext}^1_Q(N_I, Q_P)$$

is nonzero. In particular, singularity of the point N only depends on the isomorphism types of $N_I = N \cap I$ and $Q_P = (P \oplus I)/(N + I)$. Consider the locally closed subset Z of $\mathrm{Gr}_{\boldsymbol{e}}(P \oplus I)$ consisting of subrepresentations N' such that $N' \cap I \simeq N_I$ and $(P \oplus I)/(N' + I) \simeq Q_P$; thus $Z \subset \mathcal{S}_{\boldsymbol{f}}$. The vector bundle $\zeta : \mathcal{S}_{\boldsymbol{f}} \to \mathrm{Gr}_{\boldsymbol{f}}(I) \times \mathrm{Gr}_{\boldsymbol{e}-\boldsymbol{f}}(P)$ of the previous proposition restricts to a vector bundle

$$\zeta : Z \to Z_I \times Z_P,$$

where $Z_I = \mathcal{S}_{[N_I]} \subset \mathrm{Gr}_{\boldsymbol{f}}(I)$ consists of subrepresentations isomorphic to N_I, and $Z_P \subset \mathrm{Gr}_{\boldsymbol{e}-\boldsymbol{f}}(P)$ consists of subrepresentations with quotient isomorphic to Q_P. By the dimension formula for the strata $\mathcal{S}_{[N_I]}$, the codimension of Z_I in $\mathrm{Gr}_{\boldsymbol{f}}(I)$ equals $\dim \mathrm{Ext}^1_Q(N_I, N_I)$; dually, the codimension of Z_P in $\mathrm{Gr}_{\boldsymbol{e}-\boldsymbol{f}}(P)$ equals $\dim \mathrm{Ext}^1_Q(Q_P, Q_P)$. Since the rank of the bundle ζ is $\dim \mathrm{Hom}_Q(N_P, Q_I)$, we have

$$\begin{aligned}
&\dim \mathrm{Gr}_{\boldsymbol{e}}(P \oplus I) - \dim \zeta^{-1}(Z_I \times Z_P)\\
&\quad = \dim \mathrm{Gr}_{\boldsymbol{e}}(P \oplus I) - \dim \mathrm{Hom}_Q(N_P, Q_I) - (\dim \mathrm{Gr}_{\boldsymbol{f}}(I) - \dim \mathrm{Ext}^1_Q(N_I, N_I))\\
&\qquad\qquad - (\dim \mathrm{Gr}_{\boldsymbol{e}-\boldsymbol{f}}(P) - \dim \mathrm{Ext}^1_Q(Q_P, Q_P))\\
&\quad = \langle \boldsymbol{e}, \boldsymbol{d} \rangle - \langle \boldsymbol{e} - \boldsymbol{f}, \boldsymbol{d} - \boldsymbol{f} \rangle - \langle \boldsymbol{f}, \boldsymbol{d} - \boldsymbol{f} \rangle - \langle \boldsymbol{e} - \boldsymbol{f}, \boldsymbol{f} \rangle\\
&\qquad\qquad + \dim \mathrm{Ext}^1_Q(N_I, N_I) + \dim \mathrm{Ext}^1_Q(Q_P, Q_P)\\
&\quad = \langle \boldsymbol{f}, \boldsymbol{f} \rangle + \dim \mathrm{Ext}^1_Q(N_I, N_I) + \dim \mathrm{Ext}^1_Q(Q_P, Q_P)
\end{aligned}$$

for the codimension of Z in $\mathrm{Gr}_{\boldsymbol{e}}(P \oplus I)$. Assume that this codimension equals 1. Since the Euler form (Q being Dynkin) is positive definite, the summand $\langle \boldsymbol{f}, \boldsymbol{f} \rangle$ is nonnegative. If it equals 0, then $\boldsymbol{f}$ equals 0, and N_I and Q_P are just the zero representations, a contradiction to the assumption $\mathrm{Ext}^1_Q(N_I, Q_P) \neq 0$. Thus $\langle \boldsymbol{f}, \boldsymbol{f} \rangle = 1$ and both other summands are zero, thus N_I and Q_P are both isomorphic to the exceptional representation of dimension vector $\boldsymbol{f}$. But this implies vanishing of $\mathrm{Ext}^1_Q(N_I, Q_P)$ and thus nonsingularity of N. $\square$

5. Cell decomposition

Let Q be a Dynkin quiver, P and I respectively a projective and an injective representation of Q. Let $M := P \oplus I$ and let $\mathrm{Gr} = \mathrm{Gr}_{\boldsymbol{e}}(M)$ where $\boldsymbol{e} = \mathbf{dim}\, P$. In this section we construct a cellular decomposition of Gr.

The indecomposable direct summands of M are either injective or projective. In particular they are thin, that is, the vector space at every vertex is at most one–dimensional. The set of generators of these one–dimensional spaces form a linear basis of M which we denote by $\boldsymbol{B}$. To each indecomposable summand L of M we assign an integer $d(L)$, the *degree* of L, so that if $\mathrm{Hom}_Q(L, L') \neq 0$ then $d(L) < d(L')$ and so that all the degrees are different. In particular the degrees of the homogeneous vectors of I are strictly bigger than the ones of P (in case there is a projective–injective summand in both P and I we choose the degree of the copy in I to be bigger than the degree of the copy in P). To every vector of L we assign degree $d(L)$. In particular every element v of $\boldsymbol{B}$ has an assigned degree $d(v)$. In view of [Cerulli Irelli 2011] the one–dimensional torus $T = \mathbb{C}^*$ acts on Gr as follows: for every $v \in \boldsymbol{B}$ and every $\lambda \in T$ we define

$$\lambda \cdot v := \lambda^{d(v)} v. \tag{5-1}$$

This action extends uniquely to an action on M and induces an action on Gr. The T-fixed points are precisely the points of Gr generated by a part of $\boldsymbol{B}$, that is, the coordinate subrepresentations of $P \oplus I$ of dimension vector $\mathbf{dim}\, P$.

We denote the (finite) set of T-fixed points of Gr by Gr^T.

For every $L \in \mathrm{Gr}^T$, the torus acts on the tangent space $T_L(\mathrm{Gr}) \simeq \mathrm{Hom}_Q(L, M/L)$. More explicitly, the vector space $\mathrm{Hom}_Q(L, M/L)$ has a basis given by elements which associate to a basis vector $v \in L \cap \boldsymbol{B}$ a nonzero element $v' \in M/L \cap \boldsymbol{B}$ and such element is homogeneous of degree $d(v') - d(v)$ [Crawley-Boevey 1989]. We denote by $\mathrm{Hom}_Q(L, M/L)^+$ the vector subspace of $\mathrm{Hom}_Q(L, M/L)$ generated by the basis elements of *positive* degree.

Since Gr is projective, for every $N \in \mathrm{Gr}$ the limit $\lim_{\lambda\to 0} \lambda.N$ exists and moreover it is T-fixed; see for example [Chriss and Ginzburg 1997, Lemma 2.4.3]. For every $L \in \mathrm{Gr}^T$ we consider its *attracting set*

$$\mathscr{C}(L) = \{N \in \mathrm{Gr} \mid \lim_{\lambda\to 0} \lambda \cdot N = L\}.$$

The action (5-1) on Gr induces an action on $\mathrm{Gr}_{\boldsymbol{f}}(I)$ and $\mathrm{Gr}_{\boldsymbol{e}-\boldsymbol{f}}(P)$ so that the map

$$\zeta : \mathscr{S}_{\boldsymbol{f}} \to \mathrm{Gr}_{\boldsymbol{f}}(I) \times \mathrm{Gr}_{\boldsymbol{e}-\boldsymbol{f}}(P) \tag{5-2}$$

is T-equivariant. Since both $\mathrm{Gr}_{\boldsymbol{f}}(I)$ and $\mathrm{Gr}_{\boldsymbol{e}-\boldsymbol{f}}(P)$ are smooth (P and I being rigid), we apply [Białynicki-Birula 1973] and we get cellular decompositions into attracting sets

$$\mathrm{Gr}_{\boldsymbol{f}}(I)=\coprod_{L_I\in\mathrm{Gr}_{\boldsymbol{f}}(I)^T}\mathscr{C}(L_I)\ \text{ and }\ \mathrm{Gr}_{\boldsymbol{g}}(P)=\coprod_{L_P\in\mathrm{Gr}_{\boldsymbol{g}}(P)^T}\mathscr{C}(L_P),$$

and moreover $\mathscr{C}(L_I)\simeq\mathrm{Hom}_Q(L_I,I/L_I)^+$ and $\mathscr{C}(L_P)\simeq\mathrm{Hom}_Q(L_P,P/L_P)^+$.

Theorem 5.1. *The attracting set of each $L\in\mathrm{Gr}^T$ is an affine space isomorphic to $\mathrm{Hom}_Q(L,M/L)^+$. In particular, we get a cellular decomposition*

$$\mathrm{Gr}=\coprod_{L\in\mathrm{Gr}^T}\mathscr{C}(L).$$

Moreover,

$$\mathscr{C}(L)=\zeta^{-1}(\mathscr{C}(L_I)\times\mathscr{C}(L_P))\simeq\mathscr{C}(L_I)\times\mathscr{C}(L_P)\times\mathrm{Hom}_Q(L_P,I/L_I). \tag{5-3}$$

Proof. The subvariety $\mathscr{S}_{\boldsymbol{f}}:=\zeta^{-1}(\mathrm{Gr}_{\boldsymbol{f}}(I)\times\mathrm{Gr}_{\mathbf{dim}\,P-\boldsymbol{f}}(P))$ is smooth but not projective. Nevertheless it enjoys the following property:

$$\lim_{\lambda\to 0}\lambda\cdot N\in\mathscr{S}_{\boldsymbol{f}}\quad\text{for all }\ N\in\mathscr{S}_{\boldsymbol{f}}. \tag{5-4}$$

Indeed let N be a point of $\mathscr{S}_{\boldsymbol{f}}$, and let $w_1,\dots,w_{|\boldsymbol{e}|}$ be a basis of it (here $|\boldsymbol{e}|=\sum_{i\in Q_0}e_i$). We write every w_i in the basis $\boldsymbol{B}$, and we find a vector $v_i\in\boldsymbol{B}$ which has minimal degree in this linear combination and whose coefficient can be assumed to be 1. We call v_i the leading term of w_i. The subrepresentation $N_I=N\cap I$ is generated by those w_i which belong to I while $N_P=\pi(N)\simeq N/N_I$ is generated by the remaining ones. The torus action is chosen in such a way that the leading term of every $w_j\in N_P$ belongs to P. The limit point $L:=\lim_{\lambda\to 0}\lambda\cdot N$ has $v_1,\dots,v_{|\boldsymbol{e}|}$ as its basis. The subrepresentation $L_I=L\cap I$ is generated precisely by the v_i which are the leading terms of $w_i\in N_I$. In particular $\mathbf{dim}\,L_I=\mathbf{dim}\,N_I=\boldsymbol{f}$ and hence $L\in\mathrm{Gr}_{\boldsymbol{f}}$.

Since the map ζ is T-equivariant, (5-3) follows from (5-4).

It remains to prove that $\mathscr{C}(L)\simeq\mathrm{Hom}_Q(L,M/L)^+$. This is a consequence of

$$\begin{aligned}
&\mathscr{C}(L_I)\simeq\mathrm{Hom}_Q(L_I,I/L_I)^+, && \mathrm{Hom}_Q(L_I,P/L_I)^+=0,\\
&\mathscr{C}(L_P)\simeq\mathrm{Hom}_Q(L_P,P/L_P)^+, && \mathrm{Hom}_Q(L_P,I/L_I)^+=\mathrm{Hom}_Q(L_P,I/L_I),
\end{aligned}$$

together with the isomorphism (5-3). $\square$

The following example shows that for $L\in\mathrm{Gr}^T$ and $N\in\mathscr{C}(L)$, it is not true that the tangent spaces at N and L have the same dimension.

Example 5.2. Let

$$Q:=\quad\begin{tikzcd} & & 3 \\ 1 \arrow[r] & 2 \arrow[ur] \arrow[rr] & & 4 \end{tikzcd}$$

be a Dynkin quiver of type D_4. For every vertex $k \in Q_0$, let P_k and I_k be, respectively, the indecomposable projective and injective Q-representations at vertex k. Let $P := P_1 \oplus P_2 \oplus P_3 \oplus P_4$ and $I := I_1 \oplus I_2 \oplus I_3 \oplus I_4$. We consider the variety $\mathrm{Gr}_{(1233)}(I \oplus P)$. We assign degree $\deg(P_k) := 4 - k$ and $\deg(I_k) := 4 + k$ for $k = 1, 2, 3, 4$. We notice that $I_4 \oplus I_3 \oplus I_2$ has an indecomposable subrepresentation N_I of dimension vector (1211) such that $\lim_{\lambda \to 0} \lambda \cdot N_I = I_4 \oplus (0110) := L$, where (0110) denotes the indecomposable subrepresentation of I_3 of dimension vector (0110). We have $I/L_I \simeq I/N_I \simeq I_1 \oplus I_1 \oplus I_2$ and $\dim \mathrm{Hom}_Q(N_I, I/N_I) = \dim \mathrm{Hom}_Q(L_I, I/L_I) = 3$. Let us choose L_P inside P of dimension vector (0022) so that $L_I \oplus L_P \in \mathrm{Gr}$. We choose $L_P \simeq P_3^2 \oplus P_4^2$, where P_3^2 is a subrepresentation of $P_1 \oplus P_3$ and P_4^2 is in $P_1 \oplus P_2$. The quotient $P/L_P \simeq I_2 \oplus (0110) \oplus P_4$. Now $\dim \mathrm{Ext}^1_Q(N_I, P/L_P) = \dim \mathrm{Ext}^1_Q(N_I, P_4) = 1$, but $\dim \mathrm{Ext}^1_Q(L_I, P/L_P) = \dim \mathrm{Ext}^1_Q(I_4, (0110)) + \dim \mathrm{Ext}^1_Q((0110), P_4) = 2$.

6. Poincaré polynomials in type A and Genocchi numbers

In this section we compute the Poincaré polynomials of $\mathrm{Gr}_{\mathbf{dim}\, P}(P \oplus I)$ for equioriented quiver of type A and derive some combinatorial consequences.

Equioriented quiver of type A. For two nonnegative integers k and l, the q-binomial coefficient $\binom{k}{l}_q$ is defined by the formula

$$\binom{k}{l}_q = \frac{k_q!}{l_q!(k-l)_q!}, \quad \text{where } k_q! = (1-q)(1-q^2)\dots(1-q^k).$$

We also set $\binom{k}{l}_q = 0$ if $k < l$, $k < 0$, or $l < 0$.

Recall (see Proposition 2.7) that $\mathcal{F}^a_{n+1}$ is isomorphic to $\mathrm{Gr}_{\mathbf{dim}\, P}(P \oplus I)$, where P and I are the direct sums of, respectively, all projective and all injective indecomposable representations of Q. According to Proposition 4.4, in order to compute the Poincaré polynomial of $\mathrm{Gr}_{\boldsymbol{e}}(P \oplus I)$, we only need to compute the Poincaré polynomials of $\mathrm{Gr}_{\boldsymbol{g}}(P)$ and $\mathrm{Gr}_{\boldsymbol{f}}(I)$ for arbitrary dimension vectors $\boldsymbol{g} = (g_1, \dots, g_n)$ and $\boldsymbol{f} = (f_1, \dots, f_n)$. Let us compute these polynomials in a slightly more general setting. Namely, fix two collections of nonnegative integers $a_1, \dots, a_n$ and $b_1, \dots, b_n$, and set $P = \bigoplus_{i=1}^n P_i^{a_i}$ and $I = \bigoplus_{i=1}^n I_i^{b_i}$.

Lemma 6.1. *We have*

$$P_{Gr_{\boldsymbol{g}}(P)}(q) = \prod_{k=1}^{n} \binom{a_1 + \dots + a_k - g_{k-1}}{g_k - g_{k-1}}_q, \tag{6-1}$$

$$P_{Gr_{\boldsymbol{f}}(I)}(q) = \prod_{k=1}^{n} \binom{b_{n+1-k} + f_{n+2-k}}{f_{n+1-k}}_q \tag{6-2}$$

with the convention $g_0 = 0$, $f_{n+1} = 0$.

Proof. We first prove the first formula by induction on n. For $n = 1$, the formula reduces to the well-known formula for the Poincaré polynomials of the classical Grassmannians. Let $n > 1$. Consider the map $\mathrm{Gr}_{\boldsymbol{g}}(P) \to \mathrm{Gr}_{g_1}((P)_1)$. We claim that this map is a fibration with the base $\mathrm{Gr}_{g_1}(\mathbb{C}^{a_1})$ and a fiber isomorphic to $\mathrm{Gr}_{(g_2-g_1, g_3-g_1, \dots, g_n-g_1)}\big(P_1^{a_1-g_1} \bigoplus_{i=2}^n P_i^{a_i}\big)$. In fact, an element of $\mathrm{Gr}_{\boldsymbol{g}}(P)$ is a collection of spaces $(V_1, \dots, V_n)$ such that $V_i \subset (P)_i$. We note that all the maps in P corresponding to the arrows $i \to i+1$ are embeddings. Therefore, if one fixes a g_1-dimensional subspace $V_1 \subset P_1$, this automatically determines the g_1-dimensional subspaces to be contained in $V_2, \dots, V_n$. This proves the claim. Now formula (6-1) follows by induction.

In order to prove (6-2), we consider the map

$$\mathrm{Gr}_{\boldsymbol{f}}(I) \to \mathrm{Gr}_{\boldsymbol{f}^*}(I^*) : N \mapsto \{\varphi \in I^* \mid \varphi(N) = 0\},$$

where $I^* = \mathrm{Hom}_{\mathbb{C}}(I, \mathbb{C})$ and $\boldsymbol{f}^* = (f_1^*, \cdots, f_{n-1}^*) = \mathbf{dim}\, I - \boldsymbol{f}$ is defined by

$$f_k^* = b_k + b_{k+1} + \cdots + b_n - f_k.$$

Now I^* can be identified with $\bigoplus_{i=1}^n P_{n+1-i}^{b_i}$ by acting on the vertices of Q with the permutation $\omega : i \mapsto n - i$ for every $i = 1, 2, \cdots, n-1$. We hence have an isomorphism

$$\mathrm{Gr}_{\boldsymbol{f}}(I) \simeq \mathrm{Gr}_{\omega \boldsymbol{f}^*}\Big(\bigoplus_{i=1}^n P_i^{b_i}\Big).$$

Substituting into (6-1), we obtain (6-2). ☐

Theorem 6.2. *Let* $\mathrm{Gr} = \mathrm{Gr}_{\boldsymbol{e}}(I \oplus P)$ *with* I *and* P *as above. Then the Poincaré polynomial* $P_{\mathrm{Gr}}(q)$ *of* Gr *is given by*

$$\sum_{\boldsymbol{f}+\boldsymbol{g}=\boldsymbol{e}} q^{\sum_{i=1}^n g_i(a_i - f_i + f_{i+1})} \prod_{k=1}^n \binom{a_1 + \cdots + a_k - g_{k-1}}{g_k - g_{k-1}}_q \prod_{k=1}^n \binom{b_{n+1-k} + f_{n+2-k}}{f_{n+1-k}}_q.$$

Proof. Recall the decomposition $\mathrm{Gr}_{\boldsymbol{e}}(P \oplus I) = \sqcup_{\boldsymbol{f}} \mathcal{S}_{\boldsymbol{f}}$. Each stratum $\mathcal{S}_{\boldsymbol{f}}$ is a total space of a vector bundle over $\mathrm{Gr}_{\boldsymbol{g}}(P) \times \mathrm{Gr}_{\boldsymbol{f}}(I)$ with fiber over a point $(N_P, N_I) \in \mathrm{Gr}_{\boldsymbol{g}}(P) \times \mathrm{Gr}_{\boldsymbol{f}}(I)$ isomorphic to $\mathrm{Hom}_Q(N_P, I/N_I)$. Since $\mathrm{Ext}_Q^1(N_P, I/N_I) = 0$, we obtain $\dim \mathrm{Hom}_Q(N_P, I/N_I) = \langle \boldsymbol{g}, \mathbf{dim}\, I - \boldsymbol{f}\, \rangle$. Since Q is the equioriented quiver of type A_n, we obtain

$$\langle \boldsymbol{g}, \mathbf{dim}\, I - \boldsymbol{f} \rangle = \sum_{i=1}^n g_i(a_i - f_i + f_{i+1}). \tag{6-3}$$

Now our theorem follows from formulas (6-1) and (6-2). ☐

Now let $a_i = b_i = 1$, $i = 1, \dots, n$. Then the quiver Grassmannian $\mathrm{Gr}_{\mathbf{dim}\, P}(P \oplus I)$ is isomorphic to $\mathcal{F}_{n+1}^a$.

Corollary 6.3. *The Poincaré polynomial of the complete degenerate flag variety $\mathcal{F}^a_{n+1}$ is equal to*

$$\sum_{f_1,\dots,f_n\geq 0} q^{\sum_{k=1}^n (k-f_k)(1-f_k+f_{k+1})} \prod_{k=1}^n \binom{1+f_{k-1}}{f_k}_q \prod_{k=1}^n \binom{1+f_{k+1}}{f_k}_q \tag{6-4}$$

where $f_0 = f_{n+1} = 0$.

Now fix a collection $\boldsymbol{d} = (d_1, \dots, d_s)$ with $0 = d_0 < d_1 < \dots < d_s < d_{s+1} = n+1$.

Corollary 6.4. *Define $a_i = d_i - d_{i-1}$ and $b_i = d_{i+1} - d_i$. Then Theorem 6.2 gives the Poincaré polynomial of the partial degenerate flag variety $\mathcal{F}^a_{\boldsymbol{d}}$.*

Proof. This follows from Proposition 2.9. □

The normalized median Genocchi numbers. Recall that the Euler characteristic of $\mathcal{F}^a_{n+1}$ is equal to the $(n+1)$-st normalized median Genocchi number h_{n+1}; see [Feigin 2011, Proposition 3.1 and Corollary 3.7]. In particular, the Poincaré polynomial (6-4) provides natural q-deformation $h_{n+1}(q)$. We also arrive at the following formula.

Corollary 6.5. *With the convention $f_0 = f_{n+1} = 0$, we have*

$$h_{n+1} = \sum_{f_1,\dots,f_n\geq 0} \prod_{k=1}^n \binom{1+f_{k-1}}{f_k} \prod_{k=1}^n \binom{1+f_{k+1}}{f_k} \tag{6-5}$$

We note that formula (6-5) can be seen as a sum over the set M_{n+1} of Motzkin paths starting at $(0,0)$ and ending at $(n+1,0)$. Namely, we note that a term in (6-5) is zero unless, for all $i = 1, \dots, n$, we have $f_{i+1} = f_i$, $f_{i+1} = f_i + 1$, or $f_{i+1} = f_i - 1$ (recall that $f_i \geq 0$ and $f_0 = f_{n+1} = 0$). Therefore the terms in (6-5) are labeled by Motzkin paths; see for example [Donaghey and Shapiro 1977]. We can simplify the expression for h_{n+1}. Namely, for a Motzkin path $\boldsymbol{f} \in M_{n+1}$ let $l(\boldsymbol{f})$ be the number of rises ($f_{i+1} = f_i + 1$) plus the number of falls ($f_{i+1} = f_i - 1$). Then we obtain:

Corollary 6.6.
$$h_{n+1} = \sum_{\boldsymbol{f}\in M_{n+1}} \frac{\prod_{k=1}^n (1+f_k)^2}{2^{l(\boldsymbol{f})}}.$$

We note also that Remark 4.2 produces one more combinatorial definition of the numbers h_{n+1}. Namely, for $1 \leq i \leq j \leq n$, we denote by $S_{i,j}$ the indecomposable representation of Q such that

$$\mathbf{dim}\, S_{i,j} = (\underbrace{0, \dots, 0}_{i-1}, \underbrace{1, \dots, 1}_{j-i+1}, 0, \dots, 0).$$

In particular, the simple indecomposable representation S_i coincides with $S_{i,i}$. Then we have

$$\dim \mathrm{Hom}_Q(S_k, S_{i,j}) = \begin{cases} 1 & \text{if } k = j, \\ 0 & \text{otherwise;} \end{cases} \qquad \dim \mathrm{Hom}_Q(S_{i,j}, S_k) = \begin{cases} 1 & \text{if } i = k, \\ 0 & \text{otherwise.} \end{cases}$$

Recall (see Theorem 4.1) that the Euler characteristic of $\mathcal{F}^a_{n+1}$ is equal to the number of isomorphism classes of pairs $([Q_P], [N_I])$ such that N_I is embedded into $I = \bigoplus_{k=1}^n I_k$, Q_P is a quotient of $P = \bigoplus_{k=1}^n P_k$, and $\mathbf{dim}\, N_I = \mathbf{dim}\, Q_P$. Let

$$N_I = \bigoplus_{1 \le i \le j \le n} S_{i,j}^{r_{i,j}} \quad \text{and} \quad Q_P = \bigoplus_{1 \le i \le j \le n} S_{i,j}^{m_{i,j}}.$$

Then from Remark 4.2 we obtain the following proposition.

Proposition 6.7. *The normalized median Genocchi number h_{n+1} is equal to the number of pairs of collections of nonnegative integers $(r_{i,j})$, $(m_{i,j})$, $1 \le i \le j \le n$, subject to the following conditions for all $k = 1, \ldots, n$:*

$$\sum_{k=i}^{n} r_{i,k} \le 1, \quad \sum_{k=1}^{j} m_{k,j} \le 1, \quad \sum_{i \le k \le j} r_{i,j} = \sum_{i \le k \le j} m_{i,j}.$$

7. Cells and the group action in type A

In this section we fix Q to be the equioriented quiver of type A_n.

Description of the group. Let $P = \bigoplus_{i=1}^{n} P_i$ and $I = \bigoplus_{i=1}^{n} I_i$. As in the general case, we consider the group

$$G = \begin{bmatrix} \mathrm{Aut}_Q(P) & 0 \\ \mathrm{Hom}_Q(P, I) & \mathrm{Aut}_Q(I) \end{bmatrix},$$

which is a subgroup of $\mathrm{Aut}_Q(P \oplus I)$.

Remark 7.1. The whole group of automorphisms $\mathrm{Aut}_Q(P \oplus I)$ is generated by G and $\exp(\mathrm{Hom}_Q(I, P))$. We note that $\mathrm{Hom}_Q(I, P)$ is a one-dimensional space. In fact, $\mathrm{Hom}_Q(I_k, P_l) = 0$ unless $k = n$, $i = 1$, and $I_n \simeq P_1$. Thus G "almost" coincides with $\mathrm{Aut}(P \oplus I)$.

We now describe G explicitly.

Lemma 7.2. *The groups $\mathrm{Aut}_Q(P)$ and $\mathrm{Aut}_Q(I)$ are isomorphic to the Borel subgroup B_n of the Lie group GL_n, that is, to the group of nondegenerate upper-triangular matrices.*

Proof. For $g \in \mathrm{Aut}(P \oplus I)$, let g_i be the component acting on $(P \oplus I)_i$ (the vector space corresponding to the i-th vertex). Then the map $g \mapsto g_n$ gives a group isomorphism $\mathrm{Aut}_Q(P) \simeq B_n$. In fact, if $k > l$, $\mathrm{Hom}_Q(P_k, P_l) = 0$; otherwise,

$\mathrm{Hom}_Q(P_k, P_l)$ is one-dimensional and is completely determined by the n-th component. Similarly, the map $g \mapsto g_1$ gives a group isomorphism $\mathrm{Aut}_Q(I) \simeq B_n$. $\square$

In what follows, we denote $\mathrm{Aut}_Q(P)$ by B_P and $\mathrm{Aut}_Q(I)$ by B_I.

Proposition 7.3. *The group G is isomorphic to the semidirect product*

$$\mathbb{G}_a^{n(n+1)/2} \rtimes (B_P \times B_I).$$

Proof. First, the groups B_P and B_I commute inside G. Second, the group G is generated by B_P, B_I, and $\exp(\mathrm{Hom}_Q(P, I))$. The group $\exp(\mathrm{Hom}_Q(P, I))$ is abelian and isomorphic to $\mathbb{G}_a^{n(n+1)/2}$ (the abelian version of the unipotent subgroup of the lower-triangular matrices in SL_{n+1}). In fact, if $i > j$, $\mathrm{Hom}_Q(P_i, I_j)$ is trivial; otherwise, it is one-dimensional. Also, $\exp(\mathrm{Hom}_Q(P, I))$ is normal in G. $\square$

We now describe explicitly the structure of the semidirect product. For this we pass to the level of the Lie algebras. So let $\mathfrak{b}_P$ and $\mathfrak{b}_I$ be the Lie algebras of B_P and B_I, respectively ($\mathfrak{b}_P$ and $\mathfrak{b}_I$ are isomorphic to the Borel subalgebra of $\mathfrak{sl}_n$). Let $(\mathfrak{n}^-)^a$ be the abelian $n(n+1)/2$-dimensional Lie algebra, that is, the Lie algebra of the group $\mathbb{G}_a^{n(n+1)/2}$. Also, let $\mathfrak{b}$ be the Borel subalgebra of $\mathfrak{sl}_{n+1}$. Recall that the degenerate Lie algebra $\mathfrak{sl}_{n+1}^a$ is defined as $(\mathfrak{n}^-)^a \oplus \mathfrak{b}$, where $(\mathfrak{n}^-)^a$ is an abelian ideal and the action of $\mathfrak{b}$ on $(\mathfrak{n}^-)^a$ is induced by the adjoint action of $\mathfrak{b}$ on the quotient $(\mathfrak{n}^-)^a \simeq \mathfrak{sl}_{n+1}/\mathfrak{b}$. Consider the embedding $\imath_P : \mathfrak{b}_P \to \mathfrak{b}$, $E_{i,j} \mapsto E_{i,j}$, and the embedding $\imath_I : \mathfrak{b}_I \to \mathfrak{b}$, $E_{i,j} \mapsto E_{i+1,j+1}$. These embeddings define the structures of $\mathfrak{b}_P$- and $\mathfrak{b}_I$-modules on $(\mathfrak{n}^-)^a$.

Proposition 7.4. *The Lie algebra of the group G is isomorphic to* $(\mathfrak{n}^-)^a \oplus \mathfrak{b}_P \oplus \mathfrak{b}_I$, *where* $(\mathfrak{n}^-)^a$ *is an abelian ideal and the structure of* $\mathfrak{b}_P \oplus \mathfrak{b}_I$*-module on* $(\mathfrak{n}^-)^a$ *is defined by the embeddings* $\imath_P$ *and* $\imath_I$.

Proof. The Lie algebra of G is isomorphic to the direct sum $\mathrm{End}_Q(P) \oplus \mathrm{End}_Q(I) \oplus \mathrm{Hom}_Q(P, I)$. Recall that the identification $\mathrm{Hom}_Q(P, P) \simeq \mathfrak{b}_P$ is given by $a \mapsto a_n$ and the identification $\mathrm{Hom}_Q(I, I) \simeq \mathfrak{b}_I$ is given by $a \mapsto a_1$, where a_i denotes the i-th component of $a \in \mathrm{Hom}_Q(P \oplus I, P \oplus I)$. Recall from page 173 that $(P \oplus I)_1$ is spanned by the vectors $w_{1,j}$, $j = 1, \dots, n+1$, and that $w_{1,1} \in (P_1)_1$ and $w_{1,j} \in (I_{j-1})_1$ for $j > 1$. Therefore, we have a natural embedding $\mathfrak{b}_I \subset \mathfrak{b}$ mapping the matrix unit $E_{i,j}$ to $E_{i+1,j+1}$. Similarly, $(P \oplus I)_n$ is spanned by the vectors $w_{n,j}$, $j = 1, \dots, n+1$, together with $w_{n,n+1} \in (I_n)_n$ and $w_{n,j} \in (P_j)_n$ for $j < n+1$, giving the natural embedding $\mathfrak{b}_I \subset \mathfrak{b}$, $E_{i,j} \mapsto E_{i,j}$. With such a description, it is easy to compute the commutator of an element of $\mathfrak{b}_P \oplus \mathfrak{b}_I$ with an element of $\mathrm{Hom}_Q(P, I) \simeq (\mathfrak{n}^-)^a$. $\square$

We now compare G with SL_{n+1}^a. We note that the Lie algebra $\mathfrak{sl}_{n+1}^a$ and the Lie group SL_{n+1}^a have one-dimensional centers. Namely, let θ be the highest root of $\mathfrak{sl}_{n+1}$ and let $e_\theta = E_{1,n} \in \mathfrak{b} \subset \mathfrak{sl}_{n+1}$ be the corresponding element. Then e_θ

commutes with everything in $\mathfrak{sl}_{n+1}^a$ and thus the exponents $\exp(te_\theta) \in \mathrm{SL}_{n+1}^a$ form the center Z. From Proposition 7.4 we obtain the following corollary.

Corollary 7.5. *The group SL_{n+1}^a/Z is embedded into G.*

Bruhat-type decomposition. The goal of this subsection is to study the G-orbits on the degenerate flag varieties. So let $\boldsymbol{d} = (d_1, \ldots, d_s)$ for $0 = d_0 < d_1 < \cdots < d_s < d_{s+1} = n+1$.

Lemma 7.6. *The group G acts naturally on all degenerate flag varieties $\mathcal{F}_{\boldsymbol{d}}^a$.*

Proof. By definition, G acts on the degenerate flag variety $\mathcal{F}_{n+1}^a$. We note that there exists a map $\mathcal{F}_{n+1}^a \to \mathcal{F}_{\boldsymbol{d}}^a$ defined by $(V_1, \ldots, V_n) \mapsto (V_{d_1}, \ldots, V_{d_s})$. Since G acts fiberwise with respect to this projection, the G-action on $\mathcal{F}_{n+1}^a$ induces a G-action on $\mathcal{F}_{\boldsymbol{d}}^a$. □

We first work out the case $s = 1$, that is, the G-action on the classical Grassmannian $\mathrm{Gr}_d(n+1)$. We first recall the cellular decomposition from [Feigin 2011]. The cells are labeled by torus fixed points, that is, by collections $L = (l_1, \ldots, l_d)$ with $1 \le l_1 < \cdots < l_d \le n+1$. The corresponding cell is denoted by $\mathscr{C}_L$. Explicitly, the elements of $\mathscr{C}_L$ can be described as follows. Let k be an integer such that $l_k \le d < l_{k+1}$. Recall the basis $w_1, \ldots, w_{n+1}$ of $W = \mathbb{C}^{n+1}$. We denote by $x_L \in \mathrm{Gr}_{d+1}(n+1)$ the linear span of $w_{l_1}, \ldots, w_{l_d}$. Then a d-dimensional subspace V belongs to $\mathscr{C}_L$ if and only if it has a basis $e_1, \ldots, e_d$ such that for some constants c_p, we have

$$e_j = w_{l_j} + \sum_{p=1}^{l_j-1} c_p w_p + \sum_{p=d+1}^{n+1} c_p w_p \qquad \text{for } j = 1, \ldots, k; \tag{7-1}$$

$$e_j = w_{l_j} + \sum_{p=d+1}^{l_j-1} c_p w_p \qquad \text{for } j = k+1, \ldots, d. \tag{7-2}$$

For example, $x_L \in \mathscr{C}_L$.

Lemma 7.7. *Each G-orbit on the Grassmannian $\mathcal{F}_{(d)}^a$ contains exactly one torus fixed point x_L. The orbit $G \cdot x_L$ coincides with $\mathscr{C}_L$.*

Proof. Follows from the definition of G. □

We prove now that the G-orbits in $\mathrm{Gr}_{\mathbf{dim}\,P}(P \oplus I)$ described in Theorem 4.1 are cells. Moreover, we prove that this cellular decomposition coincides with the one of [Feigin 2011]. Let

$$P = \bigoplus_{i=1}^{s} P_i^{d_i - d_{i-1}} \quad \text{and} \quad I = \bigoplus_{i=1}^{s} I_i^{d_{i+1}-d_i}.$$

We start with the following lemma.

Lemma 7.8. *Let $N_I \subset I$ be a subrepresentation of I. Then there exists a unique torus fixed point $N_I^\circ \in \mathrm{Gr}_{\mathbf{dim} N_I}(I)$ such that $N_I \simeq N_I^\circ$. Similarly, for $N_P \subset P$ there exists a unique torus fixed point $N_P^\circ \in \mathrm{Gr}_{\mathbf{dim} N_P}(P)$ such that $P/N_P \simeq P/N_P^\circ$.*

Proof. We prove the first part; the second part can be proved similarly. Recall the vectors $w_{i,j} \in (I_{j-1})_i$, $i = 1, \dots, n$, $j = i+1, \dots, n+1$ such that $w_{i,j} \mapsto w_{i+1,j}$ if $j \neq i+1$ and $w_{i,j} \mapsto 0$ if $j = i+1$. For each indecomposable summand $S_{k,l}$ of N_I we construct the corresponding indecomposable summand of N_I°. Namely, we take the subrepresentation in I_l of dimension vector

$$(\underbrace{0, \dots, 0}_{k-1}, \underbrace{1, \dots, 1}_{l-k+1}, 0, \dots, 0).$$

Since each I_l is torus-fixed, our lemma is proved. □

Remark 7.9. This lemma is not true for injective modules over Dynkin quivers in general. Namely, consider the quiver from Example 4.3 and let $N_I \subset I_3 \oplus I_4$ be an indecomposable Q-module of dimension $(1, 2, 1, 1)$. Then, for such N_I, Lemma 7.8 does not hold.

Corollary 7.10. *Each G-orbit in $\mathrm{Gr}_{\mathbf{dim} P}(P \oplus I)$ contains exactly one torus fixed point, and each such point is contained in some orbit.*

Proof. Follows from Theorem 4.1 and Lemma 7.8. □

We note that any torus fixed point in $\mathcal{F}_{\boldsymbol{d}}^a$ is the product of fixed points in the Grassmannians $\mathcal{F}_{(d_i)}^a$, $i = 1, \dots, s$. Therefore, any such point is of the form $\prod_{i=1}^s x_{L^i}$. We denote this point by $x_{L^1,\dots,L^s}$.

Theorem 7.11. *The orbit $G \cdot x_{L^1,\dots,L^s}$ is the intersection of the quiver Grassmannian $\mathrm{Gr}_{\mathbf{dim} P}(P \oplus I)$ with the product of cells $\mathscr{C}_{L^i}$.*

Proof. First, obviously $G \cdot x_{L^1,\dots,L^s} \subset \mathcal{F}_{\boldsymbol{d}}^a \cap \prod_{i=1}^s \mathscr{C}_{L^i}$. Second, since each orbit contains exactly one torus fixed point and the intersection on the right hand side does not contain fixed points other than $x_{L^1,\dots,L^s}$, the theorem is proved. □

Corollary 7.12. *The G-orbits on $\mathcal{F}_{\boldsymbol{d}}^a$ produce the same cellular decomposition as the one constructed in* [Feigin 2011].

Proof. The cells from [Feigin 2011] are labeled by collections $L^1, \dots, L^s$ (whenever $x_{L^1,\dots,L^s} \in \mathcal{F}_{\boldsymbol{d}}^a$) and the corresponding cell $\mathscr{C}_{L^1,\dots,L^s}$ is given by

$$\mathscr{C}_{L^1,\dots,L^s} = \mathcal{F}_{\boldsymbol{d}}^a \cap \prod_{i=1}^s \mathscr{C}_{L^i}.$$ □

Cells and one-dimensional torus. In this subsection we show that the cellular decomposition described above coincides with the one constructed in Section 5. We describe the case of the complete flag varieties (in the parabolic case everything works in the same manner). Recall that the action of our torus is given by the formulas

$$\lambda \cdot w_{i,j} = \begin{cases} \lambda^{2n-j+1} w_{i,j} & \text{if } j > i, \\ \lambda^{n-j} w_{i,j} & \text{if } j \le i. \end{cases} \tag{7-3}$$

For $n = 4$ we have the following picture (compare with (2-2)):

$$\begin{array}{lrcccccc}
1 & & & & & & & w_{4,4} \\
\lambda & & & & & w_{3,3} & \to & w_{4,3} \\
\lambda^2 & & & w_{2,2} & \to & w_{3,2} & \to & w_{4,2} \\
\lambda^3 & w_{1,1} & \to & w_{2,1} & \to & w_{3,1} & \to & w_{4,1} \\
\lambda^4 & w_{1,5} & \to & w_{2,5} & \to & w_{3,5} & \to & w_{4,5} \\
\lambda^5 & w_{1,4} & \to & w_{2,4} & \to & w_{3,4} & & \\
\lambda^6 & w_{1,3} & \to & w_{2,3} & & & & \\
\lambda^7 & w_{1,2} & & & & & &
\end{array} \tag{7-4}$$

Proposition 7.13. *Given a fixed point x of the one-dimensional torus* (7-3), *the attracting* $(\lambda \to 0)$*-cell of x coincides with the G-orbit $G \cdot x$.*

Proof. First, consider the action of our torus on each Grassmannian $\mathrm{Gr}_d((P \oplus I)_d)$. Then formulas (7-1) and (7-2) imply that the attracting cells $(\lambda \to 0)$ coincide with the cells $\mathscr{C}_L$. Now Theorem 7.11 implies our proposition. □

We note that the one-dimensional torus (7-3) does not belong to SL^a_{n+1} (more precisely, to the image of SL^a_{n+1} in the group of automorphisms of the degenerate flag variety). However, it does belong to a one-dimensional extension

$$\mathrm{SL}^a_{n+1} \rtimes \mathbb{C}^*_{PBW}$$

of the degenerate group; see [Feigin 2011, Remark 1.1]. Recall that the extended group is the Lie group of the extended Lie algebra $\mathfrak{sl}^a_{n+1} \oplus \mathbb{C} d_{PBW}$, where d_{PBW} commutes with the generators $E_{i,j} \in \mathfrak{sl}_{n+1}$ as follows:

$$[d_{PBW}, E_{i,j}] = \begin{cases} 0 & \text{if } i < j, \\ E_{i,j} & \text{if } i > j. \end{cases}$$

In particular, the action of the torus $\mathbb{C}^*_{PBW} = \{\exp(\lambda d_{PBW}), \lambda \in \mathbb{C}\}$ on $w_{i,j}$ is given by the formulas: $\lambda \cdot w_{i,j} = w_{i,j}$ if $i \ge j$ and $\lambda \cdot w_{i,j} = \lambda w_{i,j}$ if $i < j$. For example, for $n = 4$ one has the following picture (vectors come equipped with the weights):

$$\begin{array}{ccccccc}
\lambda\bullet & \to & \lambda\bullet & \to & \lambda\bullet & \to & \lambda\bullet \\
\lambda\bullet & \to & \lambda\bullet & \to & \lambda\bullet & & 1\bullet \\
\lambda\bullet & \to & \lambda\bullet & & 1\bullet & \to & 1\bullet \\
\lambda\bullet & & 1\bullet & \to & 1\bullet & \to & 1\bullet \\
1\bullet & \to & 1\bullet & \to & 1\bullet & \to & 1\bullet
\end{array}$$

Proposition 7.14. *The one-dimensional torus* (7-3) *sits inside the extended group* $\mathrm{SL}^a_{n+1} \rtimes \mathbb{C}^*_{PBW}$.

Proof. For any collection of integers $k_1, \ldots, k_{n+1}$ there exists a one-dimensional torus $\mathbb{C}^*_{(k_1,\ldots,k_{n+1})}$ inside the Cartan subgroup of SL^a_{n+1} which acts on $w_{i,j}$ by the formula $w_{i,j} \mapsto \lambda^{k_j} w_{i,j}$. A direct check shows that the torus (7-3) acts as

$$\mathbb{C}^*_{(n,n+1,\ldots,2n)} \times (\mathbb{C}^*_{PBW})^{-n-1}. \qquad \square$$

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