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***L*-series of Artin stacks over finite fields**

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We develop the notion of stratifiability in the context of derived categories and the six operations for stacks. Then we reprove the Lefschetz trace formula for stacks, and give the meromorphic continuation of *L*-series (in particular, zeta functions) of \mathbb{F}_q -stacks. We also give an upper bound for the weights of the cohomology groups of stacks, and an “independence of ℓ ” result for a certain class of quotient stacks.

1. Introduction

In topology there is the famous Lefschetz–Hopf trace formula, which roughly says that if $f : X \rightarrow X$ is an endomorphism of a compact connected oriented space X with isolated fixed points, then the number of fixed points of f , counted with multiplicity, is equal to the alternating sum of the traces of f^* on the singular cohomology groups $H^i(X, \mathbb{Q})$.

There is also a trace formula in algebraic geometry, for schemes over finite fields, due to Grothendieck. It says that if X_0 is a scheme over \mathbb{F}_q , separated and of finite type, and F_q is the q -geometric Frobenius map, then

$$\#X_0(\mathbb{F}_q) = \sum_{i=0}^{2 \dim X_0} (-1)^i \operatorname{Tr}(F_q, H_c^i(X, \overline{\mathbb{Q}}_\ell)),$$

where $H_c^i(X, \overline{\mathbb{Q}}_\ell)$ is the ℓ -adic cohomology with compact support. In fact he proved the trace formula for an arbitrary constructible sheaf, see [Grothendieck 1965, Verdier 1967, Deligne 1977].

Behrend conjectured the trace formula for smooth algebraic stacks over \mathbb{F}_q in his thesis and [Behrend 1993], and proved it in [Behrend 2003]. However, he used ordinary cohomology and arithmetic Frobenius (rather than compact support cohomology and geometric Frobenius) to prove the “dual statement,” probably because at that time the theory of dualizing complexes of algebraic stacks, as well as compact support cohomology groups of stacks, were not developed. Later Laszlo and Olsson [2008a; 2008b] developed the theory of the six operations for

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algebraic stacks, which makes it possible to reprove the trace formula, and remove the smoothness assumption in Behrend's result. Also we will work with a fixed isomorphism of fields $\iota: \overline{\mathbb{Q}}_\ell \xrightarrow{\sim} \mathbb{C}$, namely we will work with ι -mixed complexes, rather than mixed ones, and this is a more general setting (see Remark 2.8.1).

Once we have the trace formula, we get a factorization of the zeta function into a possibly infinite product of L -factors, and from this one can deduce the meromorphic continuation of the zeta functions, generalizing a result of Behrend [1993, 3.2.4]. Furthermore, to locate the zeros and poles of the zeta functions, we give a result on the weights of cohomology groups of stacks.

We briefly mention the technical issues. As pointed out in [Behrend 2003], a big difference between schemes and stacks is the following. If

$$f: X_0 \rightarrow Y_0$$

is a morphism of \mathbb{F}_q -schemes of finite type, and $K_0 \in D_c^b(X_0, \overline{\mathbb{Q}}_\ell)$, then f_*K_0 and $f_!K_0$ are also bounded complexes. Since often we are mainly interested in the simplest case when K_0 is a sheaf concentrated in degree 0 (for instance, the constant sheaf $\overline{\mathbb{Q}}_\ell$), and D_c^b is stable under f_* and $f_!$, it is enough to consider D_c^b only. But for a morphism

$$f: \mathcal{X}_0 \rightarrow \mathcal{Y}_0$$

of \mathbb{F}_q -algebraic stacks of finite type, f_* and $f_!$ do not necessarily preserve boundedness. For instance, the cohomology ring $H^*(B\mathbb{G}_m, \overline{\mathbb{Q}}_\ell)$ is the polynomial ring $\overline{\mathbb{Q}}_\ell[T]$ with $\deg(T) = 2$. So for stacks we have to consider unbounded complexes, even if we are only interested in the constant sheaf $\overline{\mathbb{Q}}_\ell$. In order to define the trace of the Frobenius on cohomology groups, we need to consider the convergence of the complex series of the traces. This leads to the notion of an ι -convergent complex of sheaves (see Definition 4.1).

Another issue is the following. In the scheme case one considers bounded complexes, and for any bounded complex K_0 on a scheme X_0 , there exists a stratification of X_0 that "trivializes the complex K_0 " (that is, the restrictions of all cohomology sheaves $\mathcal{H}^i K_0$ to each stratum are lisse). But in the stack case we have to consider unbounded complexes, and in general there might be no stratification of the stack that trivializes every cohomology sheaf. This leads to the notion of a stratifiable complex of sheaves (see Definition 3.1). We need the stratifiability condition to control the dimensions of cohomology groups (see Lemma 3.10). All bounded complexes are stratifiable by Lemma 3.4 (v).

Also we will have to impose the condition of ι -mixedness, due to unboundedness. For bounded complexes on schemes, the trace formula can be proved without using this assumption, although the conjecture of Deligne [1980, 1.2.9] that all sheaves are ι -mixed is proved by Laurent Lafforgue, see Remark 2.8.1.

We briefly introduce the main results of this paper.

Fixed point formula.

Theorem 1.1. *Let \mathcal{X}_0 be an Artin stack of finite type over \mathbb{F}_q , and let $[\mathcal{X}_0(\mathbb{F}_q)]$ be the set of isomorphism classes of the groupoid of \mathbb{F}_q -points of \mathcal{X}_0 . Then the series*

$$\sum_{n \in \mathbb{Z}} (-1)^n \operatorname{Tr}(F_q, H_c^n(\mathcal{X}, \overline{\mathbb{Q}}_\ell)),$$

regarded as a complex series via ι , is absolutely convergent, and its limit is “the number of \mathbb{F}_q -points of \mathcal{X}_0 ”, namely

$$\#\mathcal{X}_0(\mathbb{F}_q) := \sum_{x \in [\mathcal{X}_0(\mathbb{F}_q)]} \frac{1}{\#\operatorname{Aut}_x \mathbb{F}_q}.$$

Here F_q denotes the q -geometric Frobenius. To generalize, one wants to impose some condition (P) on complexes $K_0 \in D_c^-(\mathcal{X}_0, \overline{\mathbb{Q}}_\ell)$ such that:

- (1) The condition (P) is preserved by $f_!$.
- (2) If a complex K_0 satisfies (P), then the “naive local terms” are well-defined.
- (3) Trace formula holds in this case.

The condition (P) on K_0 turns out to be a combination of three parts: ι -convergence (which implies (2) for K_0), ι -mixedness and stratifiability (which, together with the first part, implies (2) for $f_!K_0$). See Theorem 4.2 for the general statement. These conditions are due to Behrend [2003].

Meromorphic continuation. The rationality in Weil conjecture was first proved by Dwork, namely the zeta function $Z(X_0, t)$ of every variety X_0 over \mathbb{F}_q is a rational function in t . Later, this was reproved using the fixed point formula [Grothendieck 1965, Illusie 1977]. Following Behrend [1993, 3.2.3], we define the zeta functions of stacks as follows.

Definition 1.2. *For an \mathbb{F}_q -algebraic stack \mathcal{X}_0 of finite type, define the zeta function*

$$Z(\mathcal{X}_0, t) = \exp \left(\sum_{v \geq 1} \frac{t^v}{v} \sum_{x \in [\mathcal{X}_0(\mathbb{F}_{q^v})]} \frac{1}{\#\operatorname{Aut}_x \mathbb{F}_{q^v}} \right),$$

as a formal power series in the variable t .

Notice that in general, the zeta function is not rational (see Section 7). Behrend [1993, 3.2.4, 3.2.5] proved that if \mathcal{X}_0 is a smooth algebraic stack, and it is a quotient of an algebraic space by a linear algebraic group, then its zeta function $Z(\mathcal{X}_0, t)$ is a meromorphic function in the complex t -plane; if \mathcal{X}_0 is a smooth Deligne–Mumford stack, then $Z(\mathcal{X}_0, t)$ is a rational function. These results can be generalized as follows.

Theorem 1.3. *For every \mathbb{F}_q -algebraic stack \mathcal{X}_0 of finite type, its zeta function $Z(\mathcal{X}_0, t)$ defines a meromorphic function in the whole complex t -plane. If \mathcal{X}_0 is Deligne–Mumford, then $Z(\mathcal{X}_0, t)$ is a rational function.*

See Proposition 7.3.1 and Theorem 8.1 for the general statement.

A theorem of weights. One of the main results in [Deligne 1980] is that if X_0 is an \mathbb{F}_q -scheme, separated and of finite type, and \mathcal{F}_0 is an ι -mixed sheaf on X_0 of punctual ι -weights $\leq w \in \mathbb{R}$, then for every n , the punctual ι -weights of $H_c^n(X, \mathcal{F})$ are $\leq w + n$. The cohomology groups are zero unless $0 \leq n \leq 2 \dim X_0$. We will see in Remark 7.2.1 that the upper bound $w + n$ for the punctual ι -weights does not work in general for algebraic stacks. We will give an upper bound that applies to all algebraic stacks. Deligne’s upper bound of weights still applies to stacks for which all the automorphism groups are affine.

Theorem 1.4. *Let \mathcal{X}_0 be an \mathbb{F}_q -algebraic stack of finite type, and let \mathcal{F}_0 be an ι -mixed sheaf on \mathcal{X}_0 , with punctual ι -weights $\leq w$, for some $w \in \mathbb{R}$. Then the ι -weights of $H_c^n(\mathcal{X}, \mathcal{F})$ are $\leq \dim \mathcal{X}_0 + \frac{n}{2} + w$, and they are congruent mod \mathbb{Z} to weights that appear in \mathcal{F}_0 . If $n > 2 \dim \mathcal{X}_0$, then $H_c^n(\mathcal{X}, -) = 0$ on sheaves. If for all points $\bar{x} \in \mathcal{X}(\mathbb{F})$ in the support of \mathcal{F} , the automorphism group schemes $\text{Aut}_{\bar{x}}$ are affine, then the ι -weights of $H_c^n(\mathcal{X}, \mathcal{F})$ are $\leq n + w$.*

Organization. In Section 2 we review some preliminaries on derived categories of ℓ -adic sheaves on algebraic stacks over \mathbb{F}_q and ι -mixed complexes, and show that ι -mixedness is stable under the six operations.

In Section 3 we develop the notion of stratifiable complexes in the context of Laszlo and Olsson’s ℓ -adic derived categories, and prove its stability under the six operations.

In Section 4 we discuss convergent complexes, and show that they are preserved by $f_!$. In Section 5 we prove the trace formula for stacks. These two theorems are stated and proved in [Behrend 2003] in terms of ordinary cohomology and arithmetic Frobenius, and the proof we give here uses geometric Frobenius.

In Section 6 we discuss convergence of infinite products of formal power series, which will be used in the proof of the meromorphic continuation. In Section 7 we give some examples of zeta functions of stacks, and give the functional equation of the zeta functions and independence of ℓ of Frobenius eigenvalues for proper varieties with quotient singularities in Proposition 7.3.2.

In Section 8 and Section 9, we prove the meromorphic continuation and the weight theorem respectively. Finally in Section 10 we discuss “independence of ℓ ” for stacks, and prove Proposition 10.5 that for the quotient stack $[X_0/G_0]$, where X_0 is a proper smooth variety and G_0 is a linear algebraic group acting on X_0 , the Frobenius eigenvalues on its cohomology groups are independent of ℓ .

Notation and conventions.

1.5.1. We fix a prime power $q = p^a$ and an algebraic closure \mathbb{F} of the finite field \mathbb{F}_q with q elements. Let F or F_q be the q -geometric Frobenius, namely the q -th root automorphism on \mathbb{F} . Let ℓ be a prime number, $\ell \neq p$, and fix an isomorphism of fields $\overline{\mathbb{Q}}_\ell \xrightarrow{\iota} \mathbb{C}$. For simplicity, let $|\alpha|$ denote the complex absolute value $|\iota\alpha|$, for $\alpha \in \overline{\mathbb{Q}}_\ell$.

1.5.2. In this paper, by an Artin stack (or an algebraic stack) over a base scheme S , we mean an S -algebraic stack in the sense of M. Artin [Laumon and Moret-Bailly 2000, 4.1] of *finite type*. When we want the more general setting of Artin stacks *locally of finite type*, we will mention that explicitly.

1.5.3. Objects over \mathbb{F}_q will be denoted with an index $_0$. For instance, \mathcal{X}_0 will denote an \mathbb{F}_q -Artin stack; if \mathcal{F}_0 is a lisse-étale sheaf (or more generally, a Weil sheaf 2.4) on \mathcal{X}_0 , then \mathcal{F} denotes its inverse image on $\mathcal{X} := \mathcal{X}_0 \otimes_{\mathbb{F}_q} \mathbb{F}$.

1.5.4. For a field k , let $\text{Gal } k$ denote its absolute Galois group $\text{Gal}(k^{\text{sep}}/k)$. By a variety over k we mean a separated reduced k -scheme of finite type. Let $W(\mathbb{F}_q)$ be the *Weil group* $F_q^{\mathbb{Z}}$ of \mathbb{F}_q .

1.5.5. For a profinite group H , by $\overline{\mathbb{Q}}_\ell$ -representations of H we always mean finite-dimensional continuous representations (see [Deligne 1980], 1.1.6), and denote by $\text{Rep}_{\overline{\mathbb{Q}}_\ell} H$ the category of such representations.

1.5.6. For a scheme X , we denote by $|X|$ the set of its closed points. For a category \mathcal{C} we write $[\mathcal{C}]$ for the collection of isomorphism classes of objects in \mathcal{C} . For example, if $v \geq 1$ is an integer, then $[\mathcal{X}_0(\mathbb{F}_{q^v})]$ denotes the set of isomorphism classes of \mathbb{F}_{q^v} -points of the stack \mathcal{X}_0 . This is a finite set.

For $x \in \mathcal{X}_0(\mathbb{F}_{q^v})$ we will write $k(x)$ for the field \mathbb{F}_{q^v} . For an \mathbb{F}_q -scheme X_0 (always of finite type) and $x \in |X_0|$, we denote by $k(x)$ the residue field of x . In both cases, let $d(x)$ be the degree of the field extension $[k(x) : \mathbb{F}_q]$, and let $N(x) = q^{d(x)} = \#k(x)$. Also in both cases let $x : \text{Spec } \mathbb{F}_{q^v} \rightarrow \mathcal{X}_0$ (or X_0) be the natural map ($v = d(x)$), although in the second case the map is defined only up to an automorphism in $\text{Gal}(k(x)/\mathbb{F}_q)$. Given a $K_0 \in D_c(\mathcal{X}_0, \overline{\mathbb{Q}}_\ell)$ (see Section 2), the pullback $x^*K_0 \in D_c(\text{Spec } k(x), \overline{\mathbb{Q}}_\ell) = D_c(\text{Rep}_{\overline{\mathbb{Q}}_\ell} \text{Gal } k(x))$ gives a complex $K_{\bar{x}}$ of representations of $\text{Gal } k(x)$, and we let F_x be the geometric Frobenius generator $F_{q^{d(x)}}$ of this group, called “the local Frobenius”.

1.5.7. Let V be a finite dimensional $\overline{\mathbb{Q}}_\ell$ -vector space and φ an endomorphism of V . For a function $f : \overline{\mathbb{Q}}_\ell \rightarrow \mathbb{C}$, we denote by $\sum_{V, \varphi} f(\alpha)$ the sum of values of f in α , with α ranging over all the eigenvalues of φ on V with multiplicities. For instance, $\sum_{V, \varphi} \alpha = \text{Tr}(\varphi, V)$.

A 0×0 -matrix has trace 0 and determinant 1. For $K \in D_c^b(\overline{\mathbb{Q}}_\ell)$ and an endomorphism φ of K , we define, following Deligne [1977],

$$\begin{aligned} \mathrm{Tr}(\varphi, K) &:= \sum_{n \in \mathbb{Z}} (-1)^n \mathrm{Tr}(H^n(\varphi), H^n(K)), \quad \text{and} \\ \det(1 - \varphi t, K) &:= \prod_{n \in \mathbb{Z}} \det(1 - H^n(\varphi)t, H^n(K))^{(-1)^n}. \end{aligned}$$

For unbounded complexes K we use similar notations if the series (respectively the infinite product) converges (converges term by term; see Definition 6.2).

1.5.8. For a map $f : X \rightarrow Y$ and a sheaf \mathcal{F} on Y , we sometimes write $H^n(X, \mathcal{F})$ for $H^n(X, f^*\mathcal{F})$. We write $H^n(\mathcal{X})$ for $H^n(\mathcal{X}, \overline{\mathbb{Q}}_\ell)$, and $h^n(\mathcal{X}, \mathcal{F})$ for $\dim H^n(\mathcal{X}, \mathcal{F})$, and similarly for $H_c^n(\mathcal{X})$ and $h_c^n(\mathcal{X}, \mathcal{F})$.

1.5.9. For an \mathbb{F}_q -algebraic stack \mathcal{X}_0 and a Weil complex K_0 on \mathcal{X}_0 , by $R\Gamma(\mathcal{X}_0, K_0)$ (respectively $R\Gamma_c(\mathcal{X}_0, K_0)$) we mean Ra_*K_0 (respectively $Ra_!K_0$), where the morphism $a : \mathcal{X}_0 \rightarrow \mathrm{Spec} \mathbb{F}_q$ is the structural map.

The derived functors Rf_* , $Rf_!$, Lf^* and $Rf^!$ are usually abbreviated as f_* , $f_!$, f^* and $f^!$. But we reserve \otimes , $\mathcal{H}om$ and Hom for the ordinary sheaf tensor product, sheaf Hom and Hom group, respectively, and use \otimes^L , $R\mathcal{H}om$ and $RHom$ for their derived functors.

2. Derived category of ℓ -adic sheaves and mixedness

We briefly review the definition in [Laszlo and Olsson 2008a; 2008b] for derived category of ℓ -adic sheaves on stacks. Then we show that ι -mixedness is stable under the six operations. As a consequence of Lafforgue's result from Remark 2.8.1, this is automatic, but we still want to give a much more elementary argument. The proof works for *mixed* complexes as well, see Remark 2.12. One can also generalize the structure theorem of ι -mixed sheaves in [Deligne 1980] to algebraic stacks as in Remark 2.7.1.

2.1. Let Λ be a complete discrete valuation ring with maximal ideal \mathfrak{m} and residual characteristic ℓ . Let $\Lambda_n = \Lambda/\mathfrak{m}^{n+1}$, and let Λ_\bullet be the pro-ring $(\Lambda_n)_n$. We take the base scheme S to be a scheme that satisfies the following condition:

(LO): S is noetherian, affine, excellent, finite-dimensional, in which ℓ is invertible, and all S -schemes of finite type have finite ℓ -cohomological dimension.

We denote by $\mathcal{X}, \mathcal{Y}, \dots$ Artin stacks locally of finite type over S .

Consider the ringed topos $\mathcal{A} = \mathcal{A}(\mathcal{X}) := \mathrm{Mod}(\mathcal{X}_{\mathrm{lis-ét}}^{\mathbb{N}}, \Lambda_\bullet)$ of projective systems $(M_n)_n$ of $\mathrm{Ab}(\mathcal{X}_{\mathrm{lis-ét}})$ such that M_n is a Λ_n -module for each n , and the transition maps are Λ -linear. An object $M \in \mathcal{A}$ is said to be *AR-null*, if there exists an integer $r > 0$ such that for every integer n , the composed map $M_{n+r} \rightarrow M_n$ is the zero

map. A complex K in \mathcal{A} is called *AR-null*, if all cohomology systems $\mathcal{H}^i(K)$ are AR-null; it is called *almost AR-null*, if for every U in $\text{Lis-ét}(\mathcal{X})$ (assumed to be of finite type over S), the restriction of $\mathcal{H}^i(K)$ to $\dot{\text{Ét}}(U)$ is AR-null. Let $\mathcal{D}(\mathcal{A})$ be the ordinary derived category of \mathcal{A} . See [Laumon and Moret-Bailly 2000, 18.1.4] for the definition of constructible sheaves on $\mathcal{X}_{\text{lis-ét}}$.

Definition 2.2. *An object $M = (M_n)_n \in \mathcal{A}$ is adic if all the M_n 's are constructible, and for every n , the natural map*

$$\Lambda_n \otimes_{\Lambda_{n+1}} M_{n+1} \rightarrow M_n$$

is an isomorphism. It is called almost adic if all the M_n 's are constructible, and for every object U in $\text{Lis-ét}(\mathcal{X})$, the restriction $M|_U$ is AR-adic, that is, there exists an adic $N_U \in \text{Mod}(U_{\dot{\text{Ét}}}^{\mathbb{N}}, \Lambda_{\bullet})$ and a morphism $N_U \rightarrow M|_U$ with AR-null kernel and cokernel.

A complex K in \mathcal{A} is a λ -complex if $\mathcal{H}^i(K) \in \mathcal{A}$ are almost adic for all i . Let $\mathcal{D}_c(\mathcal{A})$ be the full triangulated subcategory of $\mathcal{D}(\mathcal{A})$ consisting of λ -complexes, and let $D_c(\mathcal{X}, \Lambda)$ be the quotient of $\mathcal{D}_c(\mathcal{A})$ by the thick full subcategory of those which are almost AR-null. This is called the derived category of Λ -adic sheaves on \mathcal{X} .

Remark 2.2.1. (i) $D_c(\mathcal{X}, \Lambda)$ is a triangulated category with a natural t -structure, and its heart is the quotient of the category of almost adic systems in \mathcal{A} by the thick full subcategory of almost AR-null systems. One can use this t -structure to define the subcategories $D_c^{\dagger}(\mathcal{X}, \Lambda)$ for $\dagger = \pm, b$.

If \mathcal{X}/S is of finite type (in particular, quasi-compact), it is clear that $K \in \mathcal{D}_{\text{cart}}(\mathcal{A})$ is AR-null if it is almost AR-null. Also if $M \in \mathcal{A}$ is almost adic, the adic system N_U and the map $N_U \rightarrow M|_U$ in the definition above are unique up to unique isomorphism, for each U , so by [Laumon and Moret-Bailly 2000, 12.2.1] they give an adic system N of Cartesian sheaves on $\mathcal{X}_{\text{lis-ét}}$, and an AR-isomorphism $N \rightarrow M$. This shows that an almost adic system is AR-adic, and it is clear [Illusie 1977, p. 234] that the natural functor

$$\Lambda\text{-Sh}(\mathcal{X}) \rightarrow \text{heart } D_c(\mathcal{X}, \Lambda)$$

is an equivalence of categories, where $\Lambda\text{-Sh}(\mathcal{X})$ denotes the category of Λ -adic (in particular, constructible) systems.

(ii) $D_c(\mathcal{X}, \Lambda)$ is different from the ordinary derived category of $\text{Mod}(\mathcal{X}_{\text{lis-ét}}, \Lambda)$ with constructible cohomology; the latter can be denoted by $\mathcal{D}_c(\mathcal{X}, \Lambda)$. Here $\text{Mod}(\mathcal{X}_{\text{lis-ét}}, \Lambda)$ denotes the abelian category of $\Lambda_{\mathcal{X}}$ -modules, not adic sheaves $\Lambda\text{-Sh}(\mathcal{X})$.

(iii) When $S = \text{Spec } k$ for k a finite field or an algebraically closed field, and $\mathcal{X} = X$ is a separated S -scheme, [Laszlo and Olsson 2008b, 3.1.6] gives a natural

equivalence of triangulated categories between $D_c^b(X, \Lambda)$ and Deligne's definition $\mathcal{D}_c^b(X, \Lambda)$ in [Deligne 1980, 1.1.2].

2.3. Let $\pi : \mathcal{X}_{\text{lis-ét}}^{\mathbb{N}} \rightarrow \mathcal{X}_{\text{lis-ét}}$ be the morphism of topoi where π^{-1} takes a sheaf F to the constant projective system $(F)_n$, and π_* takes a projective system to the inverse limit. This morphism induces a morphism of ringed topoi

$$(\pi^*, \pi_*) : (\mathcal{X}_{\text{lis-ét}}^{\mathbb{N}}, \Lambda_\bullet) \rightarrow (\mathcal{X}_{\text{lis-ét}}, \Lambda).$$

The functor $R\pi_* : \mathcal{D}_c(\mathcal{A}) \rightarrow \mathcal{D}(\mathcal{X}, \Lambda)$ vanishes on almost AR-null objects [Laszlo and Olsson 2008b, 2.2.2], hence factors through $D_c(\mathcal{X}, \Lambda)$. In [ibid., 3.0.8], the normalization functor is defined to be

$$K \mapsto \widehat{K} := L\pi^* R\pi_* K : D_c(\mathcal{X}, \Lambda) \rightarrow \mathcal{D}(\mathcal{A}).$$

This functor plays an important role in defining the six operations (ibid.). For instance:

- For $F \in D_c^-(\mathcal{X}, \Lambda)$ and $G \in D_c^+(\mathcal{X}, \Lambda)$, $R\mathcal{H}om(F, G)$ is defined to be the image of $R\mathcal{H}om_{\Lambda_\bullet}(\widehat{F}, \widehat{G})$ in $D_c(\mathcal{X}, \Lambda)$.
- For $F, G \in D_c^-(\mathcal{X}, \Lambda)$, the derived tensor product $F \otimes^L G$ is defined to be the image of $\widehat{F} \otimes_{\Lambda_\bullet}^L \widehat{G}$.
- For a morphism $f : \mathcal{X} \rightarrow \mathcal{Y}$ and $F \in D_c^+(\mathcal{X}, \Lambda)$, the derived direct image $f_* F$ is defined to be the image of $f_*^{\mathbb{N}} \widehat{F}$.

Let E_λ be a finite extension of \mathbb{Q}_ℓ with ring of integers \mathbb{O}_λ . Following Laszlo and Olsson [2008b] we define $D_c(\mathcal{X}, E_\lambda)$ to be the quotient of $D_c(\mathcal{X}, \mathbb{O}_\lambda)$ by the full subcategory consisting of complexes K such that, for every integer i , there exists an integer $n_i \geq 1$ such that $\mathcal{H}^i(K)$ is annihilated by λ^{n_i} . Then we define

$$D_c(\mathcal{X}, \overline{\mathbb{Q}}_\ell) = 2\text{-colim}_{E_\lambda} D_c(\mathcal{X}, E_\lambda),$$

where E_λ ranges over all finite subextensions of $\overline{\mathbb{Q}}_\ell/\mathbb{Q}_\ell$, and the transition functors are

$$E_{\lambda'} \otimes_{E_\lambda} - : D_c(\mathcal{X}, E_\lambda) \rightarrow D_c(\mathcal{X}, E_{\lambda'})$$

for $E_\lambda \subset E_{\lambda'}$.

2.4. From now on in this section, $S = \text{Spec } \mathbb{F}_q$. We recall some notions of weights and mixedness from [Deligne 1980], generalized to \mathbb{F}_q -algebraic stacks.

2.4.1. Frobenius endomorphism. For an \mathbb{F}_q -scheme X_0 , let $F_{X_0} : X_0 \rightarrow X_0$ be the morphism that is identity on the underlying topological space and q -th power on the structure sheaf \mathbb{O}_{X_0} ; this is an \mathbb{F}_q -morphism. Let $F_X : X \rightarrow X$ be the induced \mathbb{F} -morphism $F_{X_0} \times \text{id}_{\mathbb{F}}$ on $X = X_0 \otimes \mathbb{F}$.

By functoriality of the maps F_{X_0} , we can extend it to stacks. For an \mathbb{F}_q -algebraic stack \mathcal{X}_0 , define $F_{\mathcal{X}_0} : \mathcal{X}_0 \rightarrow \mathcal{X}_0$ to be such that for every \mathbb{F}_q -scheme X_0 , the map

$$F_{\mathcal{X}_0}(X_0) : \mathcal{X}_0(X_0) \rightarrow \mathcal{X}_0(X_0)$$

sends x to $x \circ F_{X_0}$. We also define $F_{\mathcal{X}} : \mathcal{X} \rightarrow \mathcal{X}$ to be $F_{\mathcal{X}_0} \times \text{id}_{\mathbb{F}}$. This morphism is a universal homeomorphism, hence $F_{\mathcal{X}}^*$ and $F_{\mathcal{X}*}$ are quasi-inverse to each other, and $F_{\mathcal{X}}^* \simeq F_{\mathcal{X}}^!$, $F_{\mathcal{X}*} \simeq F_{\mathcal{X}!}$.

2.4.2. Weil complexes. A *Weil complex* K_0 on \mathcal{X}_0 is a pair (K, φ) , where K is in $D_c(\mathcal{X}, \overline{\mathbb{Q}}_\ell)$ and $\varphi : F_{\mathcal{X}}^* K \rightarrow K$ is an isomorphism. A morphism of Weil complexes on \mathcal{X}_0 is a morphism of complexes on \mathcal{X} commuting with φ . We also call K_0 a *Weil sheaf* if K is a sheaf. Let $W(\mathcal{X}_0, \overline{\mathbb{Q}}_\ell)$ be the category of Weil complexes on \mathcal{X}_0 ; it is a triangulated category with the standard t -structure, and its core is the category of Weil sheaves. There is a natural fully faithful triangulated functor

$$D_c(\mathcal{X}_0, \overline{\mathbb{Q}}_\ell) \rightarrow W(\mathcal{X}_0, \overline{\mathbb{Q}}_\ell).$$

The usual six operations are well-defined on Weil complexes.

- Verdier duality. The Weil complex structure on $D_{\mathcal{X}} K$ is given by the inverse of the isomorphism

$$D_{\mathcal{X}} K \xrightarrow{D\varphi} D_{\mathcal{X}} F_{\mathcal{X}}^* K \xrightarrow{\sim} F_{\mathcal{X}}^* D_{\mathcal{X}} K.$$

- Tensor product. Let K_0 and L_0 be two Weil complexes such that $K \otimes^L L$ (which is $K \otimes L$ since they are of $\overline{\mathbb{Q}}_\ell$ -coefficients) is constructible. This is the case when they are both bounded above. The Weil complex structure on $K \otimes L$ is given by

$$F_{\mathcal{X}}^*(K \otimes L) \xrightarrow{\sim} F_{\mathcal{X}}^* K \otimes F_{\mathcal{X}}^* L \xrightarrow{\varphi_K \otimes \varphi_L} K \otimes L.$$

- Pullback. This is clear:

$$F_{\mathcal{X}}^* f^* K \xrightarrow{\sim} f^* F_{\mathcal{Y}}^* K \xrightarrow{f^* \varphi} f^* K.$$

Here $f : \mathcal{X}_0 \rightarrow \mathcal{Y}_0$ is an \mathbb{F}_q -morphism and (K, φ) is a Weil complex on \mathcal{Y}_0 .

- Pushforward. Let $f : \mathcal{X}_0 \rightarrow \mathcal{Y}_0$ and $K_0 \in W^+(\mathcal{X}_0, \overline{\mathbb{Q}}_\ell)$. The Weil complex structure on $f_* K$ is given by

$$F_{\mathcal{Y}}^* f_* K \longrightarrow f_* F_{\mathcal{X}}^* K \xrightarrow{f^* \varphi} f_* K,$$

where the first arrow is an isomorphism, because it is adjoint to

$$f_* K \rightarrow F_{\mathcal{Y}*} f_* F_{\mathcal{X}}^* K \simeq f_* F_{\mathcal{X}*} F_{\mathcal{X}}^* K$$

obtained by applying f_* to the adjunction morphism $K \rightarrow F_{\mathcal{X}*} F_{\mathcal{X}}^* K$, which is an isomorphism.

- The remaining cases $f^!$, $f_!$ and $R\mathcal{H}om$ follow from the previous cases.

In this article, when discussing stacks over \mathbb{F}_q , by a “sheaf” or “complex of sheaves,” we usually mean a “Weil sheaf” or “Weil complex,” whereas a “lisse-étale sheaf or complex” will be an ordinary constructible $\overline{\mathbb{Q}}_\ell$ -sheaf or complex on the lisse-étale site of \mathcal{X}_0 .

For $x \in \mathcal{X}_0(\mathbb{F}_{q^v})$, it is a fixed point of $F_{\mathcal{X}}^v$, hence there is a “local Frobenius automorphism” $F_x : K_{\bar{x}} \rightarrow K_{\bar{x}}$ for every Weil complex K_0 , defined to be

$$K_{\bar{x}} \simeq K_{F_{\mathcal{X}}(\bar{x})} = (F_{\mathcal{X}}^* K)_{\bar{x}} \xrightarrow{\varphi} K_{\bar{x}}.$$

2.4.3. ι -Weights and ι -mixedness. Recall that ι is a fixed isomorphism $\overline{\mathbb{Q}}_\ell \rightarrow \mathbb{C}$. For $\alpha \in \overline{\mathbb{Q}}_\ell^*$, let $w_q(\alpha) := 2 \log_q |\iota\alpha|$, called the ι -weight of α relative to q . For a real number β , a sheaf \mathcal{F}_0 on \mathcal{X}_0 is said to be *punctually ι -pure of weight β* , if for every integer $v \geq 1$ and every $x \in \mathcal{X}_0(\mathbb{F}_{q^v})$, and every eigenvalue α of F_x acting on $\mathcal{F}_{\bar{x}}$, we have $w_{N(x)}(\alpha) = \beta$. We say \mathcal{F}_0 is *ι -mixed* if it has a finite filtration with successive quotients punctually ι -pure, and the weights of these quotients are called the punctual ι -weights of \mathcal{F}_0 . A complex $K_0 \in W(\mathcal{X}_0, \overline{\mathbb{Q}}_\ell)$ is said to be *ι -mixed* if all the cohomology sheaves $\mathcal{H}^i K_0$ are ι -mixed. Let $W_m(\mathcal{X}_0, \overline{\mathbb{Q}}_\ell)$ (respectively $D_m(\mathcal{X}_0, \overline{\mathbb{Q}}_\ell)$) be the full subcategory of ι -mixed complexes in $W(\mathcal{X}_0, \overline{\mathbb{Q}}_\ell)$ (respectively $D_c(\mathcal{X}_0, \overline{\mathbb{Q}}_\ell)$).

One can also define *punctually pure sheaves*, *mixed sheaves* and *mixed complexes* for algebraic stacks.

2.4.4. Twists. For $b \in \overline{\mathbb{Q}}_\ell^*$, let $\overline{\mathbb{Q}}_\ell^{(b)}$ be the Weil sheaf on $\mathrm{Spec} \mathbb{F}_q$ of rank one, where F acts by multiplication by b . This is an étale sheaf if and only if b is an ℓ -adic unit [Deligne 1980, 1.2.7]. For an algebraic stack $\mathcal{X}_0/\mathbb{F}_q$, we also denote by $\overline{\mathbb{Q}}_\ell^{(b)}$ the inverse image on \mathcal{X}_0 of the above Weil sheaf via the structural map. If \mathcal{F}_0 is a sheaf on \mathcal{X}_0 , we denote by $\mathcal{F}_0^{(b)}$ the tensor product $\mathcal{F}_0 \otimes \overline{\mathbb{Q}}_\ell^{(b)}$, and say that $\mathcal{F}_0^{(b)}$ is deduced from \mathcal{F}_0 by a generalized Tate twist. Note that the operation $\mathcal{F}_0 \mapsto \mathcal{F}_0^{(b)}$ adds the weights by $w_q(b)$. For an integer d , the usual Tate twist $\overline{\mathbb{Q}}_\ell(d)$ is $\overline{\mathbb{Q}}_\ell^{(q^{-d})}$. We denote by $\langle d \rangle$ the operation $(d)[2d]$ on complexes of sheaves, where $[2d]$ means shifting $2d$ to the left. Note that ι -mixedness is stable under the operation $\langle d \rangle$.

Lemma 2.5. *Let \mathcal{X}_0 be an \mathbb{F}_q -algebraic stack.*

- (i) *If \mathcal{F}_0 is an ι -mixed sheaf on \mathcal{X}_0 , then so is every subquotient of \mathcal{F}_0 .*
- (ii) *If $0 \rightarrow \mathcal{F}'_0 \rightarrow \mathcal{F}_0 \rightarrow \mathcal{F}''_0 \rightarrow 0$ is an exact sequence of sheaves on \mathcal{X}_0 , and \mathcal{F}'_0 and \mathcal{F}''_0 are ι -mixed, then so is \mathcal{F}_0 .*
- (iii) *The full subcategory $W_m(\mathcal{X}_0, \overline{\mathbb{Q}}_\ell)$ (respectively $D_m(\mathcal{X}_0, \overline{\mathbb{Q}}_\ell)$) of $W(\mathcal{X}_0, \overline{\mathbb{Q}}_\ell)$ (respectively $D_c(\mathcal{X}_0, \overline{\mathbb{Q}}_\ell)$) is a triangulated subcategory with induced standard t -structure.*

(iv) If f is a morphism of \mathbb{F}_q -algebraic stacks, then f^* on complexes of sheaves preserves ι -mixedness.

(v) If $j : \mathcal{U}_0 \hookrightarrow \mathcal{X}_0$ is an open immersion and $i : \mathcal{X}_0 \hookrightarrow \mathcal{X}_0$ is its complement, then $K_0 \in W(\mathcal{X}_0, \overline{\mathbb{Q}}_\ell)$ is ι -mixed if and only if j^*K_0 and i^*K_0 are ι -mixed.

Proof. (i) If \mathcal{F}_0 is punctually ι -pure of weight β , then so is every subquotient of it. Now suppose \mathcal{F}_0 is ι -mixed and \mathcal{F}'_0 is a subsheaf of \mathcal{F}_0 . Let W be a finite filtration

$$0 \subset \dots \subset \mathcal{F}_0^{i-1} \subset \mathcal{F}_0^i \subset \dots \subset \mathcal{F}_0$$

of \mathcal{F}_0 such that $\mathrm{Gr}_i^W \mathcal{F}_0 := \mathcal{F}_0^i / \mathcal{F}_0^{i-1}$ is punctually ι -pure for every i . Let W' be the induced filtration $W \cap \mathcal{F}'_0$ of \mathcal{F}'_0 . Then $\mathrm{Gr}_i^{W'} \mathcal{F}'_0$ is the image of

$$\mathcal{F}_0^i \cap \mathcal{F}'_0 \subset \mathcal{F}_0^i \twoheadrightarrow \mathrm{Gr}_i^W \mathcal{F}_0,$$

so it is punctually ι -pure. Let $\mathcal{F}''_0 = \mathcal{F}_0 / \mathcal{F}'_0$ be a quotient of \mathcal{F}_0 , and let W'' be the induced filtration of \mathcal{F}''_0 , namely $(\mathcal{F}''_0)^i := \mathcal{F}_0^i / (\mathcal{F}'_0 \cap \mathcal{F}_0^i)$. Then

$$\mathrm{Gr}_i^{W''} \mathcal{F}''_0 = \mathcal{F}_0^i / (\mathcal{F}_0^{i-1} + \mathcal{F}'_0 \cap \mathcal{F}_0^i),$$

which is a quotient of $\mathcal{F}_0^i / \mathcal{F}_0^{i-1} = \mathrm{Gr}_i^W \mathcal{F}_0$, so it is punctually ι -pure. Hence every subquotient of \mathcal{F}_0 is ι -mixed.

(ii) Let W' and W'' be finite filtrations of \mathcal{F}'_0 and \mathcal{F}''_0 respectively, such that $\mathrm{Gr}_i^{W'} \mathcal{F}'_0$ and $\mathrm{Gr}_i^{W''} \mathcal{F}''_0$ are punctually ι -pure for every i . Then W' can be regarded as a finite filtration of \mathcal{F}_0 such that every member of the filtration is contained in \mathcal{F}'_0 , and W'' can be regarded as a finite filtration of \mathcal{F}_0 such that every member contains \mathcal{F}'_0 . Putting these two filtrations together, we get the desired filtration for \mathcal{F}_0 .

(iii) Being a triangulated subcategory means [Deligne 1977, p. 271] that if the sequence $K'_0 \rightarrow K_0 \rightarrow K''_0 \rightarrow K'_0[1]$ is an exact triangle in $W(\mathcal{X}_0, \overline{\mathbb{Q}}_\ell)$, and two of the three complexes are ι -mixed, then so is the third. By the rotation axiom of a triangulated category, we can assume K'_0 and K''_0 are ι -mixed. We have the exact sequence

$$\dots \longrightarrow \mathcal{H}^n K'_0 \longrightarrow \mathcal{H}^n K_0 \longrightarrow \mathcal{H}^n K''_0 \longrightarrow \dots,$$

and by (i) and (ii) we see that $\mathcal{H}^n K_0$ is ι -mixed.

$W_m(\mathcal{X}_0, \overline{\mathbb{Q}}_\ell)$ has the induced t -structure because if K_0 is ι -mixed, then its truncations $\tau_{\leq n} K_0$ and $\tau_{\geq n} K_0$ are ι -mixed.

(iv) On sheaves, f^* preserves stalks, so it is exact and preserves punctual ι -purity on sheaves. Let $f : \mathcal{X}_0 \rightarrow \mathcal{Y}_0$. Given an ι -mixed sheaf \mathcal{F}_0 on \mathcal{Y}_0 , let W be a finite filtration of \mathcal{F}_0 such that each $\mathrm{Gr}_i^W \mathcal{F}_0$ is punctually ι -pure. Then f^*W gives a finite filtration of $f^*\mathcal{F}_0$ and each $\mathrm{Gr}_i^{f^*W} f^*\mathcal{F}_0 = f^* \mathrm{Gr}_i^W \mathcal{F}_0$ is punctually ι -pure. So the sheaf $f^*\mathcal{F}_0$ is ι -mixed. For an ι -mixed complex K_0 on \mathcal{Y}_0 , note that $\mathcal{H}^n(f^*K_0) = f^*\mathcal{H}^n(K_0)$, hence f^*K_0 is ι -mixed.

(v) One direction follows from (iv). For the other direction, note that $j_!$ and i_* are exact and preserve punctual ι -purity on sheaves. If \mathcal{F}_0 is an ι -mixed sheaf on \mathcal{U}_0 , with a finite filtration W such that each $\mathrm{Gr}_i^W \mathcal{F}_0$ is punctually ι -pure, then for the induced filtration $j_!W$ of $j_!\mathcal{F}_0$, we see that $\mathrm{Gr}_i^{j_!W} j_!\mathcal{F}_0 = j_! \mathrm{Gr}_i^W \mathcal{F}_0$ is punctually ι -pure, so $j_!\mathcal{F}_0$ is ι -mixed. For an ι -mixed complex K_0 on \mathcal{U}_0 , we use that $\mathcal{H}^n(j_!K_0) = j_!\mathcal{H}^n(K_0)$. Similarly i_* also preserves ι -mixedness on complexes. Finally the result follows from (iii) and the exact triangle

$$j_!j^*K_0 \longrightarrow K_0 \longrightarrow i_*i^*K_0 \longrightarrow . \quad \square$$

To show that ι -mixedness is stable under the six operations, we need to show that ι -mixedness of complexes on stacks can be checked locally on their presentations. To descend a filtration on a presentation to the stack, we generalize the structure theorem of ι -mixed sheaves to algebraic spaces. Recall the following theorem of Deligne [1980, 3.4.1].

Theorem 2.6. *Let \mathcal{F}_0 be an ι -mixed sheaf on a scheme X_0 over \mathbb{F}_q .*

(i) *\mathcal{F}_0 has a unique decomposition $\mathcal{F}_0 = \bigoplus_{b \in \mathbb{R}/\mathbb{Z}} \mathcal{F}_0(b)$, called the decomposition according to the weights mod \mathbb{Z} , such that the punctual ι -weights of $\mathcal{F}_0(b)$ are all in the coset b . This decomposition, in which almost all the $\mathcal{F}_0(b)$ are zero, is functorial in \mathcal{F}_0 . Note that each $\mathcal{F}_0(b)$ is deduced by twist from an ι -mixed sheaf with integer punctual weights.*

(ii) *If the punctual weights of \mathcal{F}_0 are integers and \mathcal{F}_0 is lisse, \mathcal{F}_0 has a unique finite increasing filtration W by lisse subsheaves, called the filtration by punctual weights, such that $\mathrm{Gr}_i^W \mathcal{F}_0$ is punctually ι -pure of weight i . This filtration is functorial in \mathcal{F}_0 . More precisely, any morphism between ι -mixed lisse sheaves of integer punctual weights is strictly compatible with their filtrations by punctual weights.*

(iii) *If \mathcal{F}_0 is lisse and punctually ι -pure, and X_0 is normal, then the sheaf \mathcal{F} on X is semisimple.*

Remark 2.6.1. (i) If \mathcal{C} is an abelian category and \mathcal{D} is an abelian full subcategory of \mathcal{C} , and C is an object in \mathcal{D} , then every direct summand of C in \mathcal{C} lies in \mathcal{D} (or isomorphic to some object in \mathcal{D}). This is because the kernel of the composition

$$A \oplus B \xrightarrow{pr_A} A \xrightarrow{i_A} A \oplus B$$

is B . So direct summands of a lisse sheaf are lisse. If \mathcal{F}_0 in Theorem 2.6 (i) is lisse, then each $\mathcal{F}_0(b)$ is lisse.

(ii) If the $\overline{\mathbb{Q}}_\ell$ -sheaf \mathcal{F}_0 is defined over some finite subextension E_λ of $\overline{\mathbb{Q}}_\ell/\mathbb{Q}_\ell$, then its decomposition in Theorem 2.6 (i) and filtration in Theorem 2.6 (ii) are defined over E_λ . This is because the E_λ -action commutes with the Galois action.

(iii) Deligne [1980] made the assumption that all schemes are separated, at least in order to use Nagata compactification to define f_i . After the work of Laszlo and Olsson [2008a, 2008b], one can remove this assumption, and many results in [Deligne 1980], for instance this one and (3.3.1), remain valid. For [Deligne 1980, 3.4.1] one can take a cover of a not necessarily separated scheme X_0 by open affines (which are separated), and use the functoriality to glue the decomposition or filtration on intersections.

Lemma 2.7. *Let X_0 be an \mathbb{F}_q -algebraic space, and \mathcal{F}_0 an ι -mixed sheaf on X_0 .*

(i) *\mathcal{F}_0 has a unique decomposition $\mathcal{F}_0 = \bigoplus_{b \in \mathbb{R}/\mathbb{Z}} \mathcal{F}_0(b)$, the decomposition according to the weights mod \mathbb{Z} , with the same property as in Theorem 2.6 (i). This decomposition is functorial in \mathcal{F}_0 .*

(ii) *If the punctual ι -weights of \mathcal{F}_0 are integers and \mathcal{F}_0 is lisse, \mathcal{F}_0 has a unique finite increasing filtration W by lisse subsheaves, called the filtration by punctual weights, with the same property as in Theorem 2.6 (ii). This filtration is functorial in \mathcal{F}_0 .*

Proof. Let $P : X'_0 \rightarrow X_0$ be an étale presentation, and let $\mathcal{F}'_0 = P^* \mathcal{F}_0$, which is also ι -mixed by Lemma 2.5 (iv). Let X''_0 be the fiber product

$$\begin{array}{ccc} X''_0 = X'_0 \times_{X_0} X'_0 & \xrightarrow{p_1} & X'_0 \\ p_2 \downarrow & & \downarrow P \\ X'_0 & \xrightarrow{P} & X_0. \end{array}$$

Then X''_0 is an \mathbb{F}_q -scheme of finite type.

(i) Applying Theorem 2.6 (i) to \mathcal{F}'_0 we get a decomposition $\mathcal{F}'_0 = \bigoplus_{b \in \mathbb{R}/\mathbb{Z}} \mathcal{F}'_0(b)$. For $j = 1, 2$, applying p_j^* we get a decomposition

$$p_j^* \mathcal{F}'_0 = \bigoplus_{b \in \mathbb{R}/\mathbb{Z}} p_j^* \mathcal{F}'_0(b).$$

Since p_j^* preserves weights, by the uniqueness in Theorem 2.6 (i), this decomposition is the decomposition of $p_j^* \mathcal{F}'_0$ according to the weights mod \mathbb{Z} . By the functoriality in Theorem 2.6 (i), the canonical isomorphism $\mu : p_1^* \mathcal{F}'_0 \rightarrow p_2^* \mathcal{F}'_0$ takes the form $\bigoplus_{b \in \mathbb{R}/\mathbb{Z}} \mu_b$, where $\mu_b : p_1^* \mathcal{F}'_0(b) \rightarrow p_2^* \mathcal{F}'_0(b)$ is an isomorphism satisfying cocycle condition as μ does. Therefore the decomposition $\mathcal{F}'_0 = \bigoplus_{b \in \mathbb{R}/\mathbb{Z}} \mathcal{F}'_0(b)$ descends to a decomposition $\mathcal{F}_0 = \bigoplus_{b \in \mathbb{R}/\mathbb{Z}} \mathcal{F}_0(b)$. We still need to show each direct summand $\mathcal{F}_0(b)$ is ι -mixed.

Fix a coset b and consider the summand $\mathcal{F}_0(b)$. Twisting it appropriately, we can assume that its inverse image $\mathcal{F}'_0(b)$ is ι -mixed with integer punctual ι -weights. By Lemma 2.5 (v) and noetherian induction, we can shrink X_0 to a nonempty

open subspace and assume $\mathcal{F}_0(b)$ is lisse. Then $\mathcal{F}'_0(b)$ is also lisse, and applying Theorem 2.6 (ii) we get a finite increasing filtration W' of $\mathcal{F}'_0(b)$ by lisse subsheaves $\mathcal{F}'_0(b)^i$, such that each $\mathrm{Gr}_i^{W'} \mathcal{F}'_0(b)$ is punctually ι -pure of weight i . Pulling back this filtration via p_j , we get a finite increasing filtration $p_j^* W'$ of $p_j^* \mathcal{F}'_0(b)$, and since

$$\mathrm{Gr}_i^{p_j^* W'} p_j^* \mathcal{F}'_0(b) = p_j^* \mathrm{Gr}_i^{W'} \mathcal{F}'_0(b)$$

is punctually ι -pure of weight i , it is the filtration by punctual weights given by Theorem 2.6 (ii), hence it is functorial. It follows that the canonical isomorphism $\mu_b : p_1^* \mathcal{F}'_0(b) \rightarrow p_2^* \mathcal{F}'_0(b)$ maps $p_1^* \mathcal{F}'_0(b)^i$ isomorphically onto $p_2^* \mathcal{F}'_0(b)^i$, satisfying cocycle condition. Therefore the filtration W' of $\mathcal{F}'_0(b)$ descends to a filtration W of $\mathcal{F}_0(b)$, and $P^* \mathrm{Gr}_i^W \mathcal{F}_0(b) = \mathrm{Gr}_i^{W'} \mathcal{F}'_0(b)$ is punctually ι -pure of weight i . Note that P is surjective, so every point $x \in X_0(\mathbb{F}_{q^v})$ can be lifted to a point $x' \in X'_0(\mathbb{F}_{q^{nv}})$ after some base extension $\mathbb{F}_{q^{nv}}$ of \mathbb{F}_{q^v} . This shows $\mathrm{Gr}_i^W \mathcal{F}_0(b)$ is punctually ι -pure of weight i , therefore $\mathcal{F}_0(b)$ is ι -mixed. This proves the existence of the decomposition in (i).

For uniqueness, let $\tilde{\mathcal{F}}_0 = \bigoplus \tilde{\mathcal{F}}_0(b)$ be another decomposition with the desired property. Then their restrictions to X'_0 are both equal to the decomposition of \mathcal{F}'_0 , which is unique Theorem 2.6 (i), so they are both obtained by descending this decomposition, and so they are isomorphic, that is, for every coset b there exists an isomorphism making the diagram commute:

$$\begin{array}{ccc} F_0(b) & \xrightarrow{\sim} & \tilde{\mathcal{F}}_0(b) \\ & \searrow & \swarrow \\ & \mathcal{F}_0 & \end{array}$$

For functoriality, let $\mathcal{G}_0 = \bigoplus \mathcal{G}_0(b)$ be another ι -mixed sheaf with decomposition on X_0 , and let $\varphi : \mathcal{F}_0 \rightarrow \mathcal{G}_0$ be a morphism of sheaves. Pulling φ back via P we get a morphism $\varphi' : \mathcal{F}'_0 \rightarrow \mathcal{G}'_0$ on X'_0 , and the diagram

$$\begin{array}{ccc} p_1^* \mathcal{F}'_0 & \xrightarrow{\mu_{\tilde{\mathcal{F}}_0}} & p_2^* \mathcal{F}'_0 \\ p_1^* \varphi' \downarrow & & \downarrow p_2^* \varphi' \\ p_1^* \mathcal{G}'_0 & \xrightarrow{\mu_{\mathcal{G}_0}} & p_2^* \mathcal{G}'_0 \end{array}$$

commutes. By Theorem 2.6 (i) $\varphi' = \bigoplus \varphi'(b)$ for morphisms $\varphi'(b) : \mathcal{F}'_0(b) \rightarrow \mathcal{G}'_0(b)$, and the diagram

$$\begin{array}{ccc}
p_1^* \mathcal{F}'_0(b) & \xrightarrow{\text{can}} & p_2^* \mathcal{F}'_0(b) \\
p_1^* \varphi' \downarrow & & \downarrow p_2^* \varphi' \\
p_1^* \mathcal{G}'_0(b) & \xrightarrow{\text{can}} & p_2^* \mathcal{G}'_0(b)
\end{array}$$

commutes for each b . It follows that the morphisms $\varphi'(b)$ descend to morphisms $\varphi(b) : \mathcal{F}_0(b) \rightarrow \mathcal{G}_0(b)$ such that $\varphi = \bigoplus \varphi(b)$.

(ii) The proof is similar to part (i). Applying Theorem 2.6 (ii) to \mathcal{F}'_0 on X'_0 we get a finite increasing filtration W' of \mathcal{F}'_0 by lisse subsheaves $\mathcal{F}'_0{}^i$ with desired property. Pulling back this filtration via $p_j : X''_0 \rightarrow X'_0$ we get the filtration by punctual weights of $p_j^* \mathcal{F}'_0$. By functoriality in Theorem 2.6 (ii), the canonical isomorphism $\mu : p_1^* \mathcal{F}'_0 \rightarrow p_2^* \mathcal{F}'_0$ maps $p_1^* \mathcal{F}'_0{}^i$ isomorphically onto $p_2^* \mathcal{F}'_0{}^i$ satisfying cocycle condition, therefore the filtration W' descends to a finite increasing filtration W of \mathcal{F}_0 by certain subsheaves $\mathcal{F}_0{}^i$. By Olsson [2007, 9.1] they are lisse subsheaves.

For uniqueness, if \tilde{W} is another filtration of \mathcal{F}_0 by certain subsheaves $\tilde{\mathcal{F}}_0{}^i$ with desired property, then their restrictions to X'_0 are both equal to the filtration W' by punctual weights, which is unique Theorem 2.6 (ii), so they are both obtained by descending this filtration W' , and therefore they are isomorphic.

For functoriality, let \mathcal{G}_0 be another lisse ι -mixed sheaf with integer punctual ι -weights, and let V be its filtration by punctual weights, and let $\varphi : \mathcal{F}_0 \rightarrow \mathcal{G}_0$ be a morphism. Pulling φ back via P we get a morphism $\varphi' : \mathcal{F}'_0 \rightarrow \mathcal{G}'_0$ on X'_0 , and the diagram

$$\begin{array}{ccc}
p_1^* \mathcal{F}'_0 & \xrightarrow{\mu_{\mathcal{F}'_0}} & p_2^* \mathcal{F}'_0 \\
p_1^* \varphi' \downarrow & & \downarrow p_2^* \varphi' \\
p_1^* \mathcal{G}'_0 & \xrightarrow{\mu_{\mathcal{G}'_0}} & p_2^* \mathcal{G}'_0
\end{array}$$

commutes. By Theorem 2.6 (ii) we have $\varphi'(\mathcal{F}'_0{}^i) \subset \mathcal{G}'_0{}^i$, and the diagram

$$\begin{array}{ccc}
p_1^* \mathcal{F}'_0{}^i & \xrightarrow{\mu_{\mathcal{F}'_0}} & p_2^* \mathcal{F}'_0{}^i \\
p_1^* \varphi' \downarrow & & \downarrow p_2^* \varphi' \\
p_1^* \mathcal{G}'_0{}^i & \xrightarrow{\mu_{\mathcal{G}'_0}} & p_2^* \mathcal{G}'_0{}^i
\end{array}$$

commutes for each i . Let $\varphi'^i : \mathcal{F}'_0{}^i \rightarrow \mathcal{G}'_0{}^i$ be the restriction of φ' . Then they descend to morphisms $\varphi^i : \mathcal{F}_0{}^i \rightarrow \mathcal{G}_0{}^i$, which are restrictions of φ . \square

Remark 2.7.1. One can prove a similar structure theorem of ι -mixed sheaves on algebraic stacks over \mathbb{F}_q : the proof of Lemma 2.7 carries over verbatim to the

case of algebraic stacks, except that for a presentation $X'_0 \rightarrow \mathcal{X}_0$, the fiber product $X''_0 = X'_0 \times_{\mathcal{X}_0} X'_0$ may not be a scheme, so we use the case for algebraic spaces and replace every “2.6” in the proof by “2.7”. It turns out that Theorem 2.6 (iii) also holds for algebraic stacks, as a consequence of the proof of Theorem 1.4. As we will not use these results in this paper, we do not give the proof in detail here, but refer to [Sun 2012, 2.1].

Proposition 2.8. *Let \mathcal{X}_0 be an \mathbb{F}_q -algebraic stack, and let $P : X_0 \rightarrow \mathcal{X}_0$ be a presentation (that is, a smooth surjection with X_0 a scheme). Then a complex $K_0 \in W(\mathcal{X}_0, \overline{\mathbb{Q}}_\ell)$ is ι -mixed if and only if P^*K_0 (respectively $P!K_0$) is ι -mixed.*

Proof. We consider P^*K_0 first. The “only if” part follows from Lemma 2.5 (iv). For the “if” part, since P^* is exact on sheaves and so $\mathcal{H}^i(P^*K_0) = P^*\mathcal{H}^i(K_0)$, we reduce to the case when $K_0 = \mathcal{F}_0$ is a sheaf. So we assume the sheaf $\mathcal{F}'_0 := P^*\mathcal{F}_0$ on X_0 is ι -mixed, and want to show \mathcal{F}_0 is also ι -mixed. The proof is similar to the argument in Lemma 2.7.

Let X''_0 be the fiber product

$$\begin{array}{ccc} X''_0 = X_0 \times_{\mathcal{X}_0} X_0 & \xrightarrow{p_1} & X_0 \\ p_2 \downarrow & & \downarrow P \\ X_0 & \xrightarrow{P} & \mathcal{X}_0. \end{array}$$

Then X''_0 is an algebraic space of finite type. Applying Theorem 2.6 (i) to \mathcal{F}'_0 we get a decomposition $\mathcal{F}'_0 = \bigoplus_{b \in \mathbb{R}/\mathbb{Z}} \mathcal{F}'_0(b)$. For $j = 1, 2$, applying p_j^* we get a decomposition

$$p_j^*\mathcal{F}'_0 = \bigoplus_{b \in \mathbb{R}/\mathbb{Z}} p_j^*\mathcal{F}'_0(b),$$

which is the decomposition of $p_j^*\mathcal{F}'_0$ according to the weights mod \mathbb{Z} . By the functoriality in Lemma 2.7 (i), the canonical isomorphism $\mu : p_1^*\mathcal{F}'_0 \rightarrow p_2^*\mathcal{F}'_0$ takes the form $\bigoplus_{b \in \mathbb{R}/\mathbb{Z}} \mu_b$, where $\mu_b : p_1^*\mathcal{F}'_0(b) \rightarrow p_2^*\mathcal{F}'_0(b)$ is an isomorphism satisfying cocycle condition as μ does. Therefore the decomposition of \mathcal{F}'_0 descends to a decomposition $\mathcal{F}_0 = \bigoplus_{b \in \mathbb{R}/\mathbb{Z}} \mathcal{F}_0(b)$. The ι -weights of the local Frobenius eigenvalues of $\mathcal{F}_0(b)$ at each point of \mathcal{X}_0 are in the coset b . Next we show that $\mathcal{F}_0(b)$'s are ι -mixed.

Replacing \mathcal{F}_0 by a direct summand $\mathcal{F}_0(b)$ and then twisting it appropriately, we may assume its inverse image \mathcal{F}'_0 is ι -mixed with integer punctual ι -weights. By Lemma 2.5 (v) we can shrink \mathcal{X}_0 to a nonempty open substack and assume \mathcal{F}_0 is lisse. Then \mathcal{F}'_0 is also lisse, and applying Theorem 2.6 (ii) we get a finite increasing filtration W' of \mathcal{F}'_0 by lisse subsheaves \mathcal{F}'_0^i , such that each $\mathrm{Gr}_i^{W'} \mathcal{F}'_0$ is punctually

ι -pure of weight i . Pulling back this filtration via p_j , we get a finite increasing filtration $p_j^* W'$ of $p_j^* \mathcal{F}'_0$, and since

$$\mathrm{Gr}_i^{p_j^* W'} p_j^* \mathcal{F}'_0 = p_j^* \mathrm{Gr}_i^{W'} \mathcal{F}'_0$$

is punctually ι -pure of weight i , it is the filtration by punctual weights given by Lemma 2.7 (ii). By functoriality, the canonical isomorphism $\mu : p_1^* \mathcal{F}'_0 \rightarrow p_2^* \mathcal{F}'_0$ maps $p_1^* \mathcal{F}'_0$ isomorphically onto $p_2^* \mathcal{F}'_0$, satisfying cocycle condition. Therefore the filtration W' of \mathcal{F}'_0 descends to a filtration W of \mathcal{F}_0 , and $P^* \mathrm{Gr}_i^{W'} \mathcal{F}'_0 = \mathrm{Gr}_i^W \mathcal{F}_0$ is punctually ι -pure of weight i . Since π is surjective, $\mathrm{Gr}_i^W \mathcal{F}_0$ is also punctually ι -pure of weight i , therefore \mathcal{F}_0 is ι -mixed.

Next we consider $P^! K_0$. We know that P is smooth of relative dimension d , for some function $d : \pi_0(X_0) \rightarrow \mathbb{N}$. Let X_0^0 be a connected component of X_0 . Since $\pi_0(X_0)$ is finite, X_0^0 is both open and closed in X_0 , so

$$f : X_0^0 \xrightarrow{j} X_0 \xrightarrow{P} \mathcal{X}_0$$

is smooth of relative dimension $d(X_0^0)$. Then $P^* K_0$ is ι -mixed if and only if $f^* K_0 = j^* P^* K_0$ is ι -mixed for the inclusion j of every connected component, if and only if $f^! K_0 = f^* K_0 \langle d(X_0^0) \rangle$ is ι -mixed, if and only if $P^! K_0$ is ι -mixed, since $f^! = j^! P^! = j^* P^!$. \square

Remark 2.8.1. As a consequence of Lafforgue's theorem on the Langlands correspondence for function fields and a Ramanujan–Petersson type of result, one deduces that all complexes on any \mathbb{F}_q -algebraic stack is ι -mixed, for any ι . To see this, by Proposition 2.8 and Lemma 2.5 (ii,v), we reduce to the case of an irreducible lisse sheaf on a smooth (in particular, normal) \mathbb{F}_q -scheme. By [Deligne 1980, 1.3.6] we reduce to the case where the determinant of the lisse sheaf has finite order, and Lafforgue's result [Laumon 2002, 1.3] applies. In the following, when we want to emphasize the assumption of ι -mixedness, we will still write $W_m(\mathcal{X}_0, \overline{\mathbb{Q}}_\ell)''$, although it equals the full category $W(\mathcal{X}_0, \overline{\mathbb{Q}}_\ell)$.

Next we show the stability of ι -mixedness, first for a few operations on complexes on algebraic spaces, and then for all the six operations on stacks. Denote by $D_{\mathcal{X}_0}$ or just D the dualizing functor $R\mathcal{H}om(-, K_{\mathcal{X}_0})$, where $K_{\mathcal{X}_0}$ is a dualizing complex on \mathcal{X}_0 [Laszlo and Olsson 2008b, §7].

2.9. Recall [Kiehl and Weissauer 2001, II 12.2] that, for \mathbb{F}_q -schemes and bounded complexes of sheaves on them, the operations f_* , $f_!$, f^* , $f^!$, D and $-\otimes^L -$ all preserve ι -mixedness. Since we are working with $\overline{\mathbb{Q}}_\ell$ -coefficients, $\otimes^L = \otimes$.

Lemma 2.10. *Let $f : X_0 \rightarrow Y_0$ be a morphism of \mathbb{F}_q -algebraic spaces. Then the operations $-\otimes -$, D_{X_0} , f_* and $f_!$ all preserve ι -mixedness, namely, they induce*

functors

$$\begin{aligned}
- \otimes - &: W_m^-(X_0, \overline{\mathbb{Q}}_\ell) \times W_m^-(X_0, \overline{\mathbb{Q}}_\ell) \longrightarrow W_m^-(X_0, \overline{\mathbb{Q}}_\ell), \\
D &: W_m(X_0, \overline{\mathbb{Q}}_\ell) \longrightarrow W_m(X_0, \overline{\mathbb{Q}}_\ell)^{\text{op}}, \\
f_* &: W_m^+(X_0, \overline{\mathbb{Q}}_\ell) \longrightarrow W_m^+(Y_0, \overline{\mathbb{Q}}_\ell), \\
f_! &: W_m^-(X_0, \overline{\mathbb{Q}}_\ell) \longrightarrow W_m^-(Y_0, \overline{\mathbb{Q}}_\ell).
\end{aligned}$$

Proof. We will reduce to the case of unbounded complexes on schemes, and then prove the scheme case. Let $P : X'_0 \rightarrow X_0$ be an étale presentation.

Reduction for \otimes . For objects K_0 and L_0 in $W_m^-(X_0, \overline{\mathbb{Q}}_\ell)$, we have that

$$P^*(K_0 \otimes L_0) = (P^*K_0) \otimes (P^*L_0),$$

and the reduction follows from Proposition 2.8.

Reduction for D . For $K_0 \in W_m(X_0, \overline{\mathbb{Q}}_\ell)$, we have $P^*DK_0 = DP^*K_0$, so the reduction follows from Proposition 2.8.

Reduction for f_* and $f_!$. By definition [Laszlo and Olsson 2008b, 9.1] we have $f_* = Df_!D$, so it suffices to prove the case for $f_!$. Let $K_0 \in W_m^-(X_0, \overline{\mathbb{Q}}_\ell)$, and let $P' : Y'_0 \rightarrow Y_0$ and $X'_0 \rightarrow X_0 \times_{Y_0} Y'_0$ be étale presentations:

$$\begin{array}{ccccc}
& & g & & \\
& \curvearrowright & & \curvearrowleft & \\
X'_0 & \longrightarrow & (X_0)_{Y'_0} & \xrightarrow{f'} & Y'_0 \\
& \searrow P & \downarrow h & & \downarrow P' \\
& & X_0 & \xrightarrow{f} & Y_0.
\end{array}$$

By smooth base change [Laszlo and Olsson 2008b, 12.1] we get $P'^*f_!K_0 = f'_!h^*K_0$. Replacing f by f' we can assume Y_0 is a scheme. Let $j : U_0 \rightarrow X_0$ be an open dense subscheme [Knutson 1971, II 6.7], with complement $i : Z_0 \rightarrow X_0$. Applying $f_!$ to the exact triangle

$$j_!j^*K_0 \longrightarrow K_0 \longrightarrow i_*i^*K_0 \longrightarrow$$

we get

$$(fj)_!j^*K_0 \longrightarrow f_!K_0 \longrightarrow (fi)_!i^*K_0 \longrightarrow$$

By Lemma 2.5 (iii) and noetherian induction, we can replace X_0 by U_0 , and reduce to the case where f is a morphism between schemes.

This finishes the reduction to the case of unbounded complexes on schemes, and now we prove this case.

For the Verdier dual D_{X_0} , since the dualizing complex K_{X_0} has finite quasi-injective dimension, for every $K_0 \in W_m(X_0, \overline{\mathbb{Q}}_\ell)$ and every integer i , there exist

integers a and b such that

$$\mathcal{H}^i(D_{X_0}K_0) \simeq \mathcal{H}^i(D_{X_0}\tau_{[a,b]}K_0),$$

and by 2.9, we see that $D_{X_0}K_0$ is ι -mixed.

Next we prove the case of \otimes . For K_0 and $L_0 \in W_m^-(X_0, \overline{\mathbb{Q}}_\ell)$, we have

$$\mathcal{H}^r(K_0 \otimes L_0) = \bigoplus_{i+j=r} \mathcal{H}^i(K_0) \otimes \mathcal{H}^j(L_0).$$

The result follows from 2.9.

Finally we prove the case of f_* and $f_!$. Let $K_0 \in W_m^+(X_0, \overline{\mathbb{Q}}_\ell)$. Then we have the spectral sequence

$$E_2^{ij} = R^i f_* (\mathcal{H}^j K_0) \implies R^{i+j} f_* K_0,$$

and the result follows from 2.9 and Lemma 2.5 (i,ii). The case for $f_! = Df_*D$ also follows. \square

Finally we prove the main result of this section. This generalizes [Behrend 2003, 6.3.7].

Theorem 2.11. *Let $f : \mathcal{X}_0 \rightarrow \mathcal{Y}_0$ be a morphism of \mathbb{F}_q -algebraic stacks. Then the operations f_* , $f_!$, f^* , $f^!$, $D_{\mathcal{X}_0}$, $-\otimes-$ and $R\mathcal{H}om(-, -)$ all preserve ι -mixedness, namely, they induce functors*

$$\begin{aligned} f_* : W_m^+(\mathcal{X}_0, \overline{\mathbb{Q}}_\ell) &\longrightarrow W_m^+(\mathcal{Y}_0, \overline{\mathbb{Q}}_\ell), & f_! : W_m^-(\mathcal{X}_0, \overline{\mathbb{Q}}_\ell) &\longrightarrow W_m^-(\mathcal{Y}_0, \overline{\mathbb{Q}}_\ell), \\ f^* : W_m(\mathcal{Y}_0, \overline{\mathbb{Q}}_\ell) &\longrightarrow W_m(\mathcal{X}_0, \overline{\mathbb{Q}}_\ell), & f^! : W_m(\mathcal{Y}_0, \overline{\mathbb{Q}}_\ell) &\longrightarrow W_m(\mathcal{X}_0, \overline{\mathbb{Q}}_\ell), \\ R\mathcal{H}om(-, -) : W_m^-(\mathcal{X}_0, \overline{\mathbb{Q}}_\ell)^{\text{op}} \times W_m^+(\mathcal{X}_0, \overline{\mathbb{Q}}_\ell) &\longrightarrow W_m^+(\mathcal{X}_0, \overline{\mathbb{Q}}_\ell), \\ \otimes : W_m^-(\mathcal{X}_0, \overline{\mathbb{Q}}_\ell) \times W_m^-(\mathcal{X}_0, \overline{\mathbb{Q}}_\ell) &\longrightarrow W_m^-(\mathcal{X}_0, \overline{\mathbb{Q}}_\ell), \\ D : W_m(\mathcal{X}_0, \overline{\mathbb{Q}}_\ell) &\longrightarrow W_m(\mathcal{X}_0, \overline{\mathbb{Q}}_\ell)^{\text{op}}. \end{aligned}$$

Proof. Recall from [Laszlo and Olsson 2008b, 9.1] that we have $f_! := Df_*D$ and $f^! := Df^*D$. By [ibid., 6.0.12, 7.3.1], for $K_0 \in W^-(\mathcal{X}_0, \overline{\mathbb{Q}}_\ell)$ and $L_0 \in W^+(\mathcal{X}_0, \overline{\mathbb{Q}}_\ell)$, we have

$$\begin{aligned} D(K_0 \otimes DL_0) &= R\mathcal{H}om(K_0 \otimes DL_0, K_{\mathcal{X}_0}) = R\mathcal{H}om(K_0, R\mathcal{H}om(DL_0, K_{\mathcal{X}_0})) \\ &= R\mathcal{H}om(K_0, DDL_0) = R\mathcal{H}om(K_0, L_0). \end{aligned}$$

Therefore it suffices to prove the result for f_* , f^* , D and $-\otimes-$. The case of f^* is proved in Lemma 2.5 (iv).

For D : Let $P : X_0 \rightarrow \mathcal{X}_0$ be a presentation. Since $P^*D = DP^!$, the result follows from Proposition 2.8 and Lemma 2.10.

For \otimes : Since we have $P^*(K_0 \otimes L_0) = P^*K_0 \otimes P^*L_0$, the result follows from Proposition 2.8 and Lemma 2.10.

For f_* and $f_!$: We will start with $f_!$, in order to use smooth base change to reduce to the case when \mathcal{Y}_0 is a scheme, and then turn to f_* in order to use cohomological descent.

Let $K_0 \in W_m^-(\mathcal{X}_0, \overline{\mathbb{Q}}_\ell)$, and let $P : Y_0 \rightarrow \mathcal{Y}_0$ be a presentation and the following diagram be 2-Cartesian:

$$\begin{array}{ccc} (\mathcal{X}_0)_{Y_0} & \xrightarrow{f'} & Y_0 \\ P' \downarrow & & \downarrow P \\ \mathcal{X}_0 & \xrightarrow{f} & \mathcal{Y}_0 \end{array}$$

We have [ibid., 12.1] that $P^* f_! K_0 = f'_! P'^* K_0$, so by 2.8 we can assume $\mathcal{Y}_0 = Y_0$ is a scheme.

Now we switch to f_* , where $f : \mathcal{X}_0 \rightarrow Y_0$, and $K_0 \in W_m^+(\mathcal{X}_0, \overline{\mathbb{Q}}_\ell)$. Let $X_0 \rightarrow \mathcal{X}_0$ be a presentation. Then it gives a strictly simplicial smooth hypercover $X_{0,\bullet}$ of \mathcal{X}_0 :

$$X_{0,n} := \underbrace{X_0 \times_{\mathcal{X}_0} \cdots \times_{\mathcal{X}_0} X_0}_{n+1 \text{ factors}}$$

where each $X_{0,n}$ is an \mathbb{F}_q -algebraic space of finite type. Let $f_n : X_{0,n} \rightarrow Y_0$ be the restriction of f to $X_{0,n}$. Then we have the spectral sequence [Laszlo and Olsson 2008b, 10.0.9]

$$E_1^{ij} = R^j f_{i*}(K_0|_{X_{0,i}}) \implies R^{i+j} f_* K_0.$$

Since f_i 's are morphisms of algebraic spaces, the result follows from Lemma 2.10 and Lemma 2.5 (i, ii). \square

Remark 2.12. In fact, we can take the dualizing complex $K_{\mathcal{X}_0}$ to be *mixed*, and results in this section hold (and can be proved verbatim) for *mixed* complexes. In particular, mixedness is preserved by the six operations and the Verdier dualizing functor for stacks (if we take a mixed dualizing complex).

3. Stratifiable complexes

In this section, we use the same notations and hypotheses in 2.1. For the purpose of this article, it suffices to take S to be $\text{Spec } k$ for an algebraically closed field k of characteristic not equal to ℓ , but we want to work in the general setting (namely, that of any scheme that satisfies (LO)) for future applications; for instance, when proving the generic base change. Let $\mathcal{X}, \mathcal{Y}, \dots$ be S -algebraic stacks of finite type. By “sheaves” we mean “lisse-étale sheaves”. “Locally constant constructible” is abbreviated as “lcc”. A *stratification* \mathcal{S} of an S -algebraic stack \mathcal{X} is a finite set of disjoint locally closed substacks that cover \mathcal{X} . If \mathcal{F} is a lcc $(\Lambda_n)_{\mathcal{X}}$ -module, a *decomposition series* of \mathcal{F} is a filtration by lcc $\Lambda_{\mathcal{X}}$ -subsheaves, such that the successive

quotients are simple $\Lambda_{\mathcal{X}}$ -modules. Note that the filtration is always finite, and the simple successive quotients, which are $(\Lambda_0)_{\mathcal{X}}$ -modules, are independent (up to order) of the decomposition series chosen. They are called the *Jordan–Hölder components* of \mathcal{F} .

Definition 3.1. (i) A complex $K = (K_n)_n \in \mathcal{D}_c(\mathcal{A})$ is said to be stratifiable if there exists a pair $(\mathcal{S}, \mathcal{L})$, where \mathcal{S} is a stratification of \mathcal{X} and \mathcal{L} is a function that assigns to every stratum $\mathcal{U} \in \mathcal{S}$ a finite set $\mathcal{L}(\mathcal{U})$ of isomorphism classes of simple (that is, irreducible) lcc Λ_0 -modules on $\mathcal{U}_{\text{lis-ét}}$, such that for each pair (i, n) of integers the restriction of the sheaf $\mathcal{H}^i(K_n) \in \text{Mod}_c(\mathcal{X}_{\text{lis-ét}}, \Lambda_n)$ to each stratum $\mathcal{U} \in \mathcal{S}$ is lcc, with Jordan–Hölder components (as a $\Lambda_{\mathcal{U}}$ -module) contained in $\mathcal{L}(\mathcal{U})$. We say that the pair $(\mathcal{S}, \mathcal{L})$ trivializes K (or K is $(\mathcal{S}, \mathcal{L})$ -stratifiable), and denote the full subcategory of $(\mathcal{S}, \mathcal{L})$ -stratifiable complexes by $\mathcal{D}_{\mathcal{S}, \mathcal{L}}(\mathcal{A})$. The full subcategory of stratifiable complexes in $\mathcal{D}_c(\mathcal{A})$ is denoted by $\mathcal{D}_c^{\text{stra}}(\mathcal{A})$.

(ii) Let $D_c^{\text{stra}}(\mathcal{X}, \Lambda)$ be the essential image of $\mathcal{D}_c^{\text{stra}}(\mathcal{A})$ in $D_c(\mathcal{X}, \Lambda)$, and we call the objects of $D_c^{\text{stra}}(\mathcal{X}, \Lambda)$ stratifiable complexes of sheaves.

(iii) Let E_λ be a finite extension of \mathbb{Q}_ℓ with ring of integers \mathbb{O}_λ . Then the definition above applies to $\Lambda = \mathbb{O}_\lambda$. Let $D_c^{\text{stra}}(\mathcal{X}, E_\lambda)$ be the essential image of $D_c^{\text{stra}}(\mathcal{X}, \mathbb{O}_\lambda)$ in $D_c(\mathcal{X}, E_\lambda)$. Finally we define

$$D_c^{\text{stra}}(\mathcal{X}, \overline{\mathbb{Q}}_\ell) = 2\text{-colim}_{E_\lambda} D_c^{\text{stra}}(\mathcal{X}, E_\lambda).$$

Remark 3.1.1. (i) This notion is due to Beilinson, Bernstein and Deligne [1982], and Behrend [2003] used it to define his derived category for stacks. Many results in this section are borrowed from [Behrend 2003], but reformulated and reproved in terms of the derived categories defined in [Laszlo and Olsson 2008b].

(ii) Let \mathcal{F} be a Λ_n -sheaf trivialized by a pair $(\mathcal{S}, \mathcal{L})$, and let \mathcal{G} be a subquotient sheaf of \mathcal{F} . Then \mathcal{G} is not necessarily trivialized by $(\mathcal{S}, \mathcal{L})$. But if \mathcal{G} is lcc on each stratum in \mathcal{S} , then it is necessarily trivialized by $(\mathcal{S}, \mathcal{L})$.

3.2. We say that the pair $(\mathcal{S}', \mathcal{L}')$ refines the pair $(\mathcal{S}, \mathcal{L})$, if \mathcal{S}' refines \mathcal{S} , and for every $\mathcal{V} \in \mathcal{S}'$, $\mathcal{U} \in \mathcal{S}$ and $L \in \mathcal{L}(\mathcal{U})$, such that $\mathcal{V} \subset \mathcal{U}$, the restriction $L|_{\mathcal{V}}$ is trivialized by $\mathcal{L}'(\mathcal{V})$. Given a pair $(\mathcal{S}, \mathcal{L})$ and a refined stratification \mathcal{S}' of \mathcal{S} , there is a canonical way to define \mathcal{L}' such that $(\mathcal{S}', \mathcal{L}')$ refines $(\mathcal{S}, \mathcal{L})$: for every $\mathcal{V} \in \mathcal{S}'$, we take $\mathcal{L}'(\mathcal{V})$ to be the set of isomorphism classes of Jordan–Hölder components of the lcc sheaves $L|_{\mathcal{V}}$ for $L \in \mathcal{L}(\mathcal{U})$, where \mathcal{U} ranges over all strata in \mathcal{S} that contains \mathcal{V} . It is clear that the set of all pairs $(\mathcal{S}, \mathcal{L})$ form a filtered direct system.

A pair $(\mathcal{S}, \mathcal{L})$ is said to be *tensor closed* if for every $\mathcal{U} \in \mathcal{S}$ and $L, M \in \mathcal{L}(\mathcal{U})$, the sheaf tensor product $L \otimes_{\Lambda_0} M$ has Jordan–Hölder components in $\mathcal{L}(\mathcal{U})$.

For a pair $(\mathcal{S}, \mathcal{L})$, a *tensor closed hull* of this pair is a tensor closed refinement.

Lemma 3.3. *Every pair $(\mathcal{S}, \mathcal{L})$ can be refined to a tensor closed pair $(\mathcal{S}', \mathcal{L}')$.*

Proof. First we show that for a lcc sheaf of sets \mathcal{F} on $\mathcal{X}_{\text{lis-ét}}$, there exists a finite étale morphism $f : \mathcal{Y} \rightarrow \mathcal{X}$ of algebraic S -stacks such that $f^{-1}\mathcal{F}$ is constant. Consider the total space $[\mathcal{F}]$ of the sheaf \mathcal{F} . Precisely, this is the category fibered in groupoids over (Aff/S) with the underlying category described as follows. Its objects are triples $(U \in \text{obj}(\text{Aff}/S), u \in \text{obj } \mathcal{X}(U), s \in (u^{-1}\mathcal{F})(U))$, and morphisms from (U, u, s) to (V, v, t) are pairs $(f : U \rightarrow V, \alpha : vf \Rightarrow u)$ such that t is mapped to s under the identification $\alpha : f^{-1}v^{-1}\mathcal{F} \cong u^{-1}\mathcal{F}$. The map $(U, u, s) \mapsto (U, u)$ gives a map $g : [\mathcal{F}] \rightarrow \mathcal{X}$, which is representable finite étale (because it is so locally). The pullback sheaf $g^{-1}\mathcal{F}$ on $[\mathcal{F}]$ has a global section, so the total space breaks up into two parts, one part being mapped isomorphically onto the base $[\mathcal{F}]$. By induction on the degree of g we are done.

Next we show that for a fixed representable finite étale morphism $\mathcal{Y} \rightarrow \mathcal{X}$, there are only finitely many isomorphism classes of simple lcc Λ_0 -sheaves on \mathcal{X} that become constant when pulled back to \mathcal{Y} . We can assume that both \mathcal{X} and \mathcal{Y} are connected. By the following Lemma 3.3.1, we reduce to the case where $\mathcal{Y} \rightarrow \mathcal{X}$ is Galois with group G , for some finite group G . Then simple lcc Λ_0 -sheaves on \mathcal{X} that become constant on \mathcal{Y} correspond to simple left $\Lambda_0[G]$ -modules, which are cyclic and hence isomorphic to $\Lambda_0[G]/I$ for left maximal ideals I of $\Lambda_0[G]$. There are only finitely many such ideals since $\Lambda_0[G]$ is a finite set.

Also note that a lcc subsheaf of a constant constructible sheaf on a connected stack is also constant. Let L be a lcc subsheaf on \mathcal{X} of the constant sheaf associated to a finite set M . Consider their total spaces. We have an inclusion of substacks $i : [L] \hookrightarrow \coprod_{m \in M} \mathcal{X}_m$, where each part \mathcal{X}_m is identified with \mathcal{X} . Then $i^{-1}(\mathcal{X}_m) \rightarrow \mathcal{X}_m$ is finite étale, and is the inclusion of a substack, hence is either an equivalence or the inclusion of the empty substack, since \mathcal{X} is connected. It is clear that L is also constant, associated to the subset of those $m \in M$ for which $i^{-1}(\mathcal{X}_m) \neq \emptyset$.

Finally we prove the lemma. Refining \mathcal{S} if necessary, we assume all strata are connected stacks. For each stratum $\mathcal{U} \in \mathcal{S}$, let $\mathcal{Y} \rightarrow \mathcal{U}$ be a representable finite étale morphism, such that all sheaves in $\mathcal{L}(\mathcal{U})$ become constant on \mathcal{Y} . Then define $\mathcal{L}'(\mathcal{U})$ to be the set of isomorphism classes of simple lcc Λ_0 -sheaves on $\mathcal{U}_{\text{lis-ét}}$ which become constant on \mathcal{Y} . For any L and $M \in \mathcal{L}'(\mathcal{U})$, since all lcc subsheaves of $L \otimes_{\Lambda_0} M$ are constant on \mathcal{Y} , we see that $L \otimes_{\Lambda_0} M$ has Jordan–Hölder components in $\mathcal{L}'(\mathcal{U})$ and hence $(\mathcal{S}, \mathcal{L}')$ is a tensor closed refinement of $(\mathcal{S}, \mathcal{L})$. \square

Lemma 3.3.1. *Let $\mathcal{Y} \rightarrow \mathcal{X}$ be a representable finite étale morphism between connected S -algebraic stacks. Then there exists a morphism $\mathcal{Z} \rightarrow \mathcal{Y}$, such that \mathcal{Z} is Galois over \mathcal{X} , that is, it is a G -torsor for some finite group G .*

Proof. Assume \mathcal{X} is nonempty, and take a geometric point $\bar{x} \rightarrow \mathcal{X}$. Let \mathcal{C} be the category $\text{FÉt}(\mathcal{X})$ of representable finite étale morphisms to \mathcal{X} , and let

$$F : \mathcal{C} \rightarrow \text{FSet}$$

be the fiber functor to the category of finite sets, namely $F(\mathcal{Y}) = \text{Hom}_{\mathcal{X}}(\bar{x}, \mathcal{Y})$. Note that this Hom, which is *a priori* a category, is a finite set, since $\mathcal{Y} \rightarrow \mathcal{X}$ is representable and finite. Then one can verify that $(\mathcal{C}, F : \mathcal{C} \rightarrow \text{FSet})$ satisfies the axioms of Galois formalism in [Grothendieck and Raynaud 1971, Exp. V, 4], and use the consequence g) on p. 121 in (loc. cit.) For the reader's convenience, we follow Olsson's suggestion and explain the proof briefly. Basically, we will verify certain axioms of (G1)–(G6), and deduce the conclusion as in (loc. cit.).

First note that \mathcal{C} , which is *a priori* a 2-category, is a 1-category. This is because for any 2-commutative diagram

$$\begin{array}{ccc} \mathcal{Y} & \xrightarrow{f} & \mathcal{Z} \\ & \searrow & \swarrow \\ & \mathcal{X} & \end{array}$$

where \mathcal{Y} and \mathcal{Z} are in \mathcal{C} , the morphism f is also representable (and finite étale), so $\text{Hom}_{\mathcal{X}}(\mathcal{Y}, \mathcal{Z})$ is discrete. By definition, the functor F preserves fiber-products, and $F(\mathcal{X})$ is a one-point set.

Let $f : \mathcal{Y} \rightarrow \mathcal{Z}$ be a morphism in \mathcal{C} , then it is finite étale. So if the degree of f is 1, then f is an isomorphism. This implies that the functor F is *conservative*, that is, f is an isomorphism if $F(f)$ is. In particular, f is a monomorphism if and only if $F(f)$ is. This is because f is a monomorphism if and only if $p_1 : \mathcal{Y} \times_{\mathcal{Z}} \mathcal{Y} \rightarrow \mathcal{Y}$ is an isomorphism, and F preserves fiber-products.

Since $f : \mathcal{Y} \rightarrow \mathcal{Z}$ is finite étale, its image stack $\mathcal{Y}' \subset \mathcal{Z}$ is both open and closed, hence $\mathcal{Y}' \rightarrow \mathcal{Z}$ is a monomorphism that is an isomorphism onto a direct summand of \mathcal{Z} (that is, $\mathcal{Z} = \mathcal{Y}' \amalg \mathcal{Y}''$ for some other open and closed substack $\mathcal{Y}'' \subset \mathcal{Z}$). Also, since $\mathcal{Y} \rightarrow \mathcal{Y}'$ is epic and finite étale, it is *strictly epic*, that is, for every $\mathcal{Z} \in \mathcal{C}$, the diagram

$$\text{Hom}(\mathcal{Y}', \mathcal{Z}) \rightarrow \text{Hom}(\mathcal{Y}, \mathcal{Z}) \rightrightarrows \text{Hom}(\mathcal{Y} \times_{\mathcal{Y}'} \mathcal{Y}, \mathcal{Z})$$

is an equalizer.

Every object \mathcal{Y} in \mathcal{C} is artinian: for a chain of monomorphisms

$$\dots \rightarrow \mathcal{Y}_n \rightarrow \dots \rightarrow \mathcal{Y}_2 \rightarrow \mathcal{Y}_1 \rightarrow \mathcal{Y},$$

we get a chain of injections

$$\dots \rightarrow F(\mathcal{Y}_n) \rightarrow \dots \rightarrow F(\mathcal{Y}_1) \rightarrow F(\mathcal{Y}),$$

which is stable since $F(\mathcal{Y})$ is a finite set, and so the first chain is also stable since F is conservative.

Since F is left exact and every object in \mathcal{C} is artinian, by [Grothendieck 1960, 3.1] the functor F is *strictly pro-representable*, that is, there exists a projective system $P = \{P_i; i \in I\}$ of objects in \mathcal{C} indexed by a filtered partially ordered set I ,

with epic transition morphisms $\varphi_{ij} : P_j \rightarrow P_i$ ($i \leq j$), such that there is a natural isomorphism of functors

$$F \xrightarrow{\sim} \text{Hom}(P, -) := \text{colim}_I \text{Hom}(P_i, -).$$

Let $\psi_i : P \rightarrow P_i$ be the canonical projection in the category $\text{Pro}(\mathcal{C})$ of pro-objects of \mathcal{C} . We may assume that every epimorphism $P_j \rightarrow \mathcal{X}$ in \mathcal{C} is isomorphic to $P_j \xrightarrow{\varphi_{ij}} P_i$ for some $i \leq j$. This is because one can add $P_j \rightarrow \mathcal{X}$ into the projective system P without changing the functor it represents. Also one can show that the P_i 's are connected [Grothendieck 1960], and morphisms in \mathcal{C} between connected stacks are strictly epic.

Given $\mathcal{Y} \in \mathcal{C}$, now we show that there exists an object $\mathcal{X} \rightarrow \mathcal{X}$ that is Galois and factors through \mathcal{Y} . Since $F(\mathcal{Y})$ is a finite set, there exists an index $j \in I$ such that all maps $P \rightarrow \mathcal{Y}$ factors through $P \xrightarrow{\psi_j} P_j$. This means that the canonical map

$$P \rightarrow \mathcal{Y}^J := \underbrace{\mathcal{Y} \times_{\mathcal{X}} \cdots \times_{\mathcal{X}} \mathcal{Y}}_{\#J \text{ factors}}, \quad \text{where } J := F(\mathcal{Y}) = \text{Hom}_{\text{Pro}(\mathcal{C})}(P, \mathcal{Y})$$

factors as

$$P \xrightarrow{\psi_j} P_j \xrightarrow{A} \mathcal{Y}^J.$$

Let $P_j \rightarrow P_i \xrightarrow{B} \mathcal{Y}^J$ be the factorization of A into a composition of an epimorphism and a monomorphism B . We claim that P_i is Galois over \mathcal{X} .

Since $F(P_i)$ is a finite set, there exists an index $k \in I$ such that all maps $P \rightarrow P_i$ factors through $P \xrightarrow{\psi_k} P_k$. Fix any $v : P_k \rightarrow P_i$. To show P_i is Galois, it suffices to show that $\text{Aut}(P_i)$ acts on $F(P_i) = \text{Hom}(P_k, P_i)$ transitively, that is, there exists a $\sigma \in \text{Aut}(P_i)$ making the triangle commute:

$$\begin{array}{ccc} P_k & \xrightarrow{v} & P_i \\ & \searrow \varphi_{ik} & \downarrow \sigma \\ & & P_i \end{array}$$

For every $u \in J = \text{Hom}(P_i, \mathcal{Y})$, we have $u \circ v \in \text{Hom}(P_k, \mathcal{Y})$, so there exists a $u' \in \text{Hom}(P_i, \mathcal{Y})$ making the diagram commute:

$$\begin{array}{ccc} P_k & \xrightarrow{v} & P_i \\ \varphi_{ik} \downarrow & & \downarrow u \\ P_i & \xrightarrow{u'} & \mathcal{Y}. \end{array}$$

Since v is epic, the function $u \mapsto u' : J \rightarrow J$ is injective, hence a bijection. Let $\alpha : \mathcal{Y}^J \rightarrow \mathcal{Y}^J$ be the isomorphism induced by the map $u \mapsto u'$. Then the diagram

$$\begin{array}{ccccc}
P_k & \xrightarrow{v} & P_i & \xrightarrow{B} & \mathcal{O}_Y^J \\
& \searrow \varphi_{ik} & & & \downarrow \alpha \\
& & P_i & \xrightarrow{B} & \mathcal{O}_Y^J
\end{array}$$

commutes. By the uniqueness of the factorization of the map $P_k \rightarrow \mathcal{O}_Y^J$ into the composition of an epimorphism and a monomorphism, there exists a $\sigma \in \text{Aut}(P_i)$ such that $\sigma \circ v = \varphi_{ik}$. This finishes the proof. \square

We give some basic properties of stratifiable complexes.

Lemma 3.4. (i) $\mathcal{D}_c^{\text{stra}}(\mathcal{A})$ (respectively $D_c^{\text{stra}}(\mathcal{X}, \Lambda)$) is a triangulated subcategory of $\mathcal{D}_c(\mathcal{A})$ (respectively $D_c(\mathcal{X}, \Lambda)$) with the induced standard t -structure.

(ii) If $f : \mathcal{X} \rightarrow \mathcal{Y}$ is an S -morphism, then $f^* : \mathcal{D}_c(\mathcal{A}(\mathcal{Y})) \rightarrow \mathcal{D}_c(\mathcal{A}(\mathcal{X}))$ (respectively $f^* : D_c(\mathcal{Y}, \Lambda) \rightarrow D_c(\mathcal{X}, \Lambda)$) preserves stratifiability.

(iii) If \mathcal{S} is a stratification of \mathcal{X} , then $K \in \mathcal{D}_c(\mathcal{A}(\mathcal{X}))$ is stratifiable if and only if $K|_V$ is stratifiable for every $V \in \mathcal{S}$.

(iv) Let $P : X \rightarrow \mathcal{X}$ be a presentation, and let $K = (K_n)_n \in \mathcal{D}_c(\mathcal{A}(\mathcal{X}))$. Then K is stratifiable if and only if P^*K is stratifiable.

(v) $D_c^{\text{stra}}(\mathcal{X}, \Lambda)$ contains $D_c^b(\mathcal{X}, \Lambda)$, and the heart of $D_c^{\text{stra}}(\mathcal{X}, \Lambda)$ is the same as that of $D_c(\mathcal{X}, \Lambda)$ in Remark 2.2.1 (i).

(vi) Let $K \in \mathcal{D}_c(\mathcal{A})$ be a normalized complex [Laszlo and Olsson 2008b, 3.0.8]. Then K is trivialized by a pair $(\mathcal{S}, \mathcal{L})$ if and only if K_0 is trivialized by this pair.

(vii) Let $K \in \mathcal{D}_c^{\text{stra}}(\mathcal{A})$. Then its Tate twist $K(1)$ is also stratifiable.

Proof. (i) To show $\mathcal{D}_c^{\text{stra}}(\mathcal{A})$ is a triangulated subcategory, it suffices to show [Deligne 1977, p. 271] that for every exact triangle $K' \rightarrow K \rightarrow K'' \rightarrow K'[1]$ in $\mathcal{D}_c(\mathcal{A})$, if K' and K'' are stratifiable, so also is K .

Using refinement we may assume that K' and K'' are trivialized by the same pair $(\mathcal{S}, \mathcal{L})$. Consider the cohomology sequence of this exact triangle at level n , restricted to a stratum $\mathcal{U} \in \mathcal{S}$. By Olsson [2007, 9.1], to show that a sheaf is lcc on \mathcal{U} , one can pass to a presentation U of the stack \mathcal{U} . Then by Milne [2008, 20.3] and the five-lemma, we see that the $\mathcal{H}^i(K_n)$'s are lcc on \mathcal{U} , with Jordan–Hölder components contained in $\mathcal{L}(\mathcal{U})$. Therefore $\mathcal{D}_c^{\text{stra}}(\mathcal{A})$ (and hence $D_c^{\text{stra}}(\mathcal{X}, \Lambda)$) is a triangulated subcategory.

The t -structure is inherited by $\mathcal{D}_c^{\text{stra}}(\mathcal{A})$ (and hence by $D_c^{\text{stra}}(\mathcal{X}, \Lambda)$) because, if $K \in \mathcal{D}_c(\mathcal{A})$ is stratifiable, so also are its truncations $\tau_{\leq r}K$ and $\tau_{\geq r}K$.

(ii) The functor f^* is exact on the level of sheaves, and takes a lcc sheaf to a lcc sheaf. If $(K_n)_n \in \mathcal{D}_c(\mathcal{A}(\mathcal{Y}))$ is trivialized by $(\mathcal{S}, \mathcal{L})$, then $(f^*K_n)_n$ is trivialized by $(f^*\mathcal{S}, f^*\mathcal{L})$, where $f^*\mathcal{S} = \{f^{-1}(V) | V \in \mathcal{S}\}$ and $(f^*\mathcal{L})(f^{-1}(V))$ is the set of

isomorphism classes of Jordan–Hölder components of f^*L , $L \in \mathcal{L}(V)$. The case of $D_c(-, \Lambda)$ follows easily.

(iii) The “only if” part follows from (ii). The “if” part is clear: if $(\mathcal{S}_V, \mathcal{L}_V)$ is a pair on V that trivializes $(K_n|_V)_n$, then the pair $(\mathcal{S}_{\mathcal{X}}, \mathcal{L})$ on \mathcal{X} , where $\mathcal{S}_{\mathcal{X}} = \cup \mathcal{S}_V$ and $\mathcal{L} = \{\mathcal{L}_V\}_{V \in \mathcal{S}}$, trivializes $(K_n)_n$.

(iv) The “only if” part follows from (ii). For the “if” part, assume P^*K is trivialized by a pair $(\mathcal{S}_X, \mathcal{L}_X)$ on X . Let $U \in \mathcal{S}_X$ be an open stratum, and let $V \subset \mathcal{X}$ be the image of U [Laumon and Moret-Bailly 2000, 3.7]. Recall that for every T in Aff/S , $V(T)$ is the full subcategory of $\mathcal{X}(T)$ consisting of objects x that are locally in the essential image of $U(T)$, that is, such that there exists an étale surjection $T' \rightarrow T$ in Aff/S and $u' \in U(T')$, such that the image of u' in $\mathcal{X}(T')$ and $x|_{T'}$ are isomorphic. Then V is an open substack of \mathcal{X} (hence also an algebraic stack) and $P|_U : U \rightarrow V$ is a presentation. Replacing $P : X \rightarrow \mathcal{X}$ by $P|_U : U \rightarrow V$ and using noetherian induction and (iii), we may assume $\mathcal{S}_X = \{X\}$.

It follows from a theorem of Gabber [Illusie et al. 2008] that P_* takes a bounded complex to a bounded complex. In fact, using base change by P , we may assume that $P : Y \rightarrow X$ is a morphism from an S -algebraic space Y to an S -scheme X . Let $j : U \rightarrow Y$ be an open dense subscheme of Y with complement $i : Z \rightarrow Y$. For a bounded complex L of Λ_n -sheaves on Y , we have the exact triangle

$$(Pi)_*i^!L \longrightarrow P_*L \longrightarrow (Pj)_*j^*L \longrightarrow .$$

Gabber’s theorem implies that $(Pj)_*j^*L$ is bounded, since $Pj : U \rightarrow X$ is a morphism between schemes. Note that the dualizing functor preserves boundedness, so does $i^! = D_Z i^* D_Y$, and therefore we may assume that $(Pi)_*i^!L$ is bounded by noetherian induction. It follows that P_*L is bounded.

Now take a pair $(\mathcal{S}, \mathcal{L})$ on \mathcal{X} that trivializes all P_*L ’s, for $L \in \mathcal{L}_X$; this is possible since each P_*L is bounded and \mathcal{L}_X is a finite set. We claim that K is trivialized by $(\mathcal{S}, \mathcal{L})$.

For each sheaf \mathcal{F} on \mathcal{X} , the natural map $\mathcal{F} \rightarrow R^0 P_* P^* \mathcal{F}$ is injective. This follows from the sheaf axioms for the lisse-lisse topology, and the fact that the lisse-étale topos and the lisse-lisse topos are the same. Explicitly, to verify the injectivity on $X_U \rightarrow U$, for any $u \in \mathcal{X}(U)$, since the question is étale local on U , one can assume $P : X_U \rightarrow U$ has a section $s : U \rightarrow X_U$. Then the composition $\mathcal{F}_U \rightarrow R^0 P_* P^* \mathcal{F}_U \rightarrow R^0 P_* R^0 s_* s^* P^* \mathcal{F}_U = \mathcal{F}_U$ of the two adjunctions is the adjunction for $P \circ s = \text{id}$, so the composite is an isomorphism, and the first map is injective.

We take $\overline{\mathcal{F}}$ to be the cohomology sheaves $\mathcal{H}^i(K_n)$. Since $P^* \mathcal{H}^i(K_n)$ is an iterated extension of sheaves in \mathcal{L}_X , we see that $P_* P^* \mathcal{H}^i(K_n)$, and in particular $R^0 P_* P^* \mathcal{H}^i(K_n)$, are trivialized by $(\mathcal{S}, \mathcal{L})$ by (i). Since $\mathcal{H}^i(K_n)$ is lcc [Olsson

2007, 9.1], by Remark 3.1.1 (ii) we see that $\mathcal{H}^i(K_n)$ (hence K) is trivialized by $(\mathcal{S}, \mathcal{L})$.

(v) By part (i) and Remark 2.2.1 (i) it is enough to show that all adic systems $M = (M_n)_n \in \mathcal{A}$ are stratifiable. By (iv) we may assume $\mathcal{X} = X$ is an S -scheme. Since X is noetherian, there exists a stratification [Illusie 1977, VI, 1.2.6] of X such that M is lisse on each stratum. By (iii) we may assume M is lisse on X .

Let \mathcal{L} be the set of isomorphism classes of Jordan–Hölder components of the Λ_0 -sheaf M_0 . We claim that \mathcal{L} trivializes M_n for all n . Suppose it trivializes M_{n-1} for some $n \geq 1$. Consider the sub- Λ_n -modules $\lambda M_n \subset M_n[\lambda^n] \subset M_n$, where $M_n[\lambda^n]$ is the kernel of the map $\lambda^n : M_n \rightarrow M_n$. Since M is adic, we have exact sequences of Λ_X -modules

$$\begin{aligned} 0 &\longrightarrow \lambda M_n \longrightarrow M_n \longrightarrow M_0 \longrightarrow 0, \\ 0 &\longrightarrow M_n[\lambda^n] \longrightarrow M_n \longrightarrow \lambda^n M_n \longrightarrow 0, \\ 0 &\longrightarrow \lambda^n M_n \longrightarrow M_n \longrightarrow M_{n-1} \longrightarrow 0. \end{aligned}$$

The natural surjection $M_n/\lambda M_n \rightarrow M_n/M_n[\lambda^n]$ implies that \mathcal{L} trivializes $\lambda^n M_n$, and therefore it also trivializes M_n . By induction on n we are done.

Since $D_c^b \subset D_c^{\text{stra}} \subset D_c$, and D_c^b and D_c have the same heart, it is clear that D_c^{stra} has the same heart as them.

(vi) Applying $-\otimes_{\Lambda_n}^L K_n$ to the following exact sequence, viewed as an exact triangle in $\mathcal{D}(\mathcal{X}, \Lambda_n)$

$$0 \longrightarrow \Lambda_{n-1} \xrightarrow{1 \rightarrow \lambda} \Lambda_n \longrightarrow \Lambda_0 \longrightarrow 0,$$

we get an exact triangle by Laszlo and Olsson [2008b, 3.0.10]

$$K_{n-1} \longrightarrow K_n \longrightarrow K_0 \longrightarrow .$$

By induction on n and Remark 3.4.1 below, we see that K is trivialized by $(\mathcal{S}, \mathcal{L})$ if K_0 is.

(vii) Let $K = (K_n)_n$. Recall that the Tate twist $K(1)$ is defined to be the system $(K_n(1))_n$, where $K_n(1) = K_n \otimes_{\Lambda_n}^L \Lambda_n(1)$. Note that the sheaf $\Lambda_n(1)$ is a flat Λ_n -module: to show that $-\otimes_{\Lambda_n} \Lambda_n(1)$ preserves injections, one can pass to stalks at geometric points, over which we have a trivialization $\Lambda_n \simeq \Lambda_n(1)$.

Suppose K is $(\mathcal{S}, \mathcal{L})$ -stratifiable. Using the isomorphism

$$\mathcal{H}^i(K_n) \otimes_{\Lambda_n} \Lambda_n(1) = \mathcal{H}^i(K_n \otimes_{\Lambda_n}^L \Lambda_n(1)),$$

it suffices to show the existence of a pair $(\mathcal{S}, \mathcal{L}')$ such that for each $\mathcal{U} \in \mathcal{S}$, the Jordan–Hölder components of the lcc sheaves $L \otimes_{\Lambda_n} \Lambda_n(1)$ lie in $\mathcal{L}'(\mathcal{U})$, for all

$L \in \mathcal{L}(\mathcal{U})$. Since L is a Λ_0 -module, we have

$$\begin{aligned} L \otimes_{\Lambda_n} \Lambda_n(1) &= (L \otimes_{\Lambda_n} \Lambda_0) \otimes_{\Lambda_n} \Lambda_n(1) = L \otimes_{\Lambda_n} (\Lambda_0 \otimes_{\Lambda_n} \Lambda_n(1)) \\ &= L \otimes_{\Lambda_n} \Lambda_0(1) = L \otimes_{\Lambda_0} \Lambda_0(1), \end{aligned}$$

and we can take $\mathcal{L}'(\mathcal{U})$ to be a tensor closed hull of $\{\Lambda_0(1), L \in \mathcal{L}(\mathcal{U})\}$. \square

Remark 3.4.1. In fact the proof of Lemma 3.4 (i) shows that $\mathcal{D}_{\mathcal{G}, \mathcal{L}}(\mathcal{A})$ is a triangulated subcategory with induced standard t -structure, for each fixed pair $(\mathcal{S}, \mathcal{L})$. Let $D_{\mathcal{G}, \mathcal{L}}(\mathcal{X}, \Lambda)$ be the essential image of $\mathcal{D}_{\mathcal{G}, \mathcal{L}}(\mathcal{A})$ in $D_c(\mathcal{X}, \Lambda)$, and this is also a triangulated subcategory with induced standard t -structure.

Also if $E_r^{ij} \implies E^n$ is a spectral sequence in the category $\mathcal{A}(\mathcal{X})$, and the E_r^{ij} 's are trivialized by $(\mathcal{S}, \mathcal{L})$ for all i, j , then all the E^n 's are trivialized by $(\mathcal{S}, \mathcal{L})$.

We denote by $D_c^{\dagger, \text{stra}}(\mathcal{X}, \Lambda)$, for $\dagger = \pm, b$, the full subcategory of \dagger -bounded stratifiable complexes, using the induced t -structure.

The following is a key result for showing the stability of stratifiability under the six operations later. Recall that $M \mapsto \widehat{M} = L\pi^* R\pi_* M$ is the normalization functor, where $\pi : \mathcal{X}^{\mathbb{N}} \rightarrow \mathcal{X}$ is the morphism of topoi in [Laszlo and Olsson 2008b, 2.1], mentioned in 2.3.

Proposition 3.5. *For a pair $(\mathcal{S}, \mathcal{L})$ on \mathcal{X} , if $M \in \mathcal{D}_{\mathcal{G}, \mathcal{L}}(\mathcal{A})$, then $\widehat{M} \in \mathcal{D}_{\mathcal{G}, \mathcal{L}}(\mathcal{A})$, too. In particular, if $K \in D_c(\mathcal{X}, \Lambda)$, then K is stratifiable if and only if its normalization $\widehat{K} \in \mathcal{D}_c(\mathcal{A})$ is stratifiable.*

Proof. First, we will reduce to the case where M is essentially bounded (that is, $\mathcal{H}^i M$ is AR-null for $|i| \gg 0$). Let $P : X \rightarrow \mathcal{X}$ be a presentation. The ℓ -cohomological dimension of $X_{\text{ét}}$ is finite, by the assumption (LO) on S . Therefore, by Laszlo and Olsson [2008b, 2.1.i], the normalization functor for X has finite cohomological dimension, and the same is true for \mathcal{X} since $P^* \widehat{M} = \widehat{P^* M}$, by [ibid., 2.2.1, 3.0.11]. This implies that for each integer i , there exist integers a and b with $a \leq b$, such that $\mathcal{H}^i(\widehat{M}) = \mathcal{H}^i(\widehat{\tau_{[a,b]}} M)$. Since $\tau_{[a,b]} M$ is also trivialized by $(\mathcal{S}, \mathcal{L})$, we may assume $M \in \mathcal{D}_{\mathcal{G}, \mathcal{L}}^b(\mathcal{A}(\mathcal{X}))$.

Since \widehat{M} is normalized, by Lemma 3.4 (vi), it suffices to show that $(\widehat{M})_0$ is trivialized by $(\mathcal{S}, \mathcal{L})$. Using projection formula and the flat resolution of Λ_0

$$0 \longrightarrow \Lambda \xrightarrow{\lambda} \Lambda \xrightarrow{\varepsilon} \Lambda_0 \longrightarrow 0,$$

we have [ibid., p.176]

$$(\widehat{M})_0 = \Lambda_0 \otimes_{\Lambda}^L R\pi_* M = R\pi_*(\pi^* \Lambda_0 \otimes_{\Lambda}^L M),$$

where $\pi^* \Lambda_0$ is the constant projective system defined by Λ_0 . Let $C \in \mathcal{D}(\mathcal{A})$ be the complex of projective systems $\pi^* \Lambda_0 \otimes_{\Lambda}^L M$; it is a λ -complex, and we have $C_n = \Lambda_0 \otimes_{\Lambda_n}^L M_n \in \mathcal{D}_c(\mathcal{X}, \Lambda_0)$.

Recall [Illusie 1977, V, 3.2.3] that a projective system $(K_n)_n$ ringed by Λ_\bullet in an abelian category is AR-adic if and only if

- it satisfies the condition (MLAR) [Illusie 1977, V, 2.1.1], hence (ML), and denote by $(N_n)_n$ the projective system of the universal images of $(K_n)_n$;
- there exists an integer $k \geq 0$ such that the projective system $(L_n)_n := (N_{n+k} \otimes \Lambda_n)_n$ is adic.

Moreover, $(K_n)_n$ is AR-isomorphic to $(L_n)_n$. Now for each i , the projective system $\mathcal{H}^i(C)$ is AR-adic Remark 2.2.1 (i). Let $N^i = (N_n^i)_n$ be the projective system of the universal images of $\mathcal{H}^i(C)$, and choose an integer $k \geq 0$ such that the system $L^i = (L_n^i)_n = (N_{n+k}^i \otimes \Lambda_n)_n$ is adic. Since $N_{n+k}^i \subset \mathcal{H}^i(C_{n+k})$ is annihilated by λ , we have $L_n^i = N_{n+k}^i$, and the transition morphism gives an isomorphism

$$L_n^i \simeq L_n^i \otimes_{\Lambda_n} \Lambda_{n-1} \xrightarrow{\sim} L_{n-1}^i.$$

This means the projective system L^i is the constant system $\pi^* L_0^i$. By Laszlo and Olsson [2008b, 2.2.2] we have $R\pi_* \mathcal{H}^i(C) \simeq R\pi_* L^i$, which is just L_0^i by [ibid., 2.2.3].

The spectral sequence

$$R^j \pi_* \mathcal{H}^i(C) \implies \mathcal{H}^{i+j}((\widehat{M})_0)$$

degenerates to isomorphisms $L_0^i \simeq \mathcal{H}^i((\widehat{M})_0)$, so we only need to show that L_0^i is trivialized by $(\mathcal{S}, \mathcal{L})$. Using the periodic Λ_n -flat resolution of Λ_0

$$\dots \longrightarrow \Lambda_n \xrightarrow{\lambda} \Lambda_n \xrightarrow{\lambda^n} \Lambda_n \xrightarrow{\lambda} \Lambda_n \xrightarrow{\varepsilon} \Lambda_0 \longrightarrow 0,$$

we see that $\Lambda_0 \otimes_{\Lambda_n}^L \mathcal{H}^j(M_n)$ is represented by the complex

$$\dots \longrightarrow \mathcal{H}^j(M_n) \xrightarrow{\lambda^n} \mathcal{H}^j(M_n) \xrightarrow{\lambda} \mathcal{H}^j(M_n) \longrightarrow 0,$$

so $\mathcal{H}^i(\Lambda_0 \otimes_{\Lambda_n}^L \mathcal{H}^j(M_n))$ are trivialized by $(\mathcal{S}, \mathcal{L})$, for all i, j . Since M is essentially bounded, we have the spectral sequence

$$\mathcal{H}^i(\Lambda_0 \otimes_{\Lambda_n}^L \mathcal{H}^j(M_n)) \implies \mathcal{H}^{i+j}(C_n),$$

from which we deduce (by Remark 3.4.1) that the $\mathcal{H}^i(C_n)$'s are trivialized by $(\mathcal{S}, \mathcal{L})$. The universal image N_n^i is the image of $\mathcal{H}^i(C_{n+r}) \rightarrow \mathcal{H}^i(C_n)$ for some $r \gg 0$, therefore the N_n^i 's (and hence the L_n^i 's) are trivialized by $(\mathcal{S}, \mathcal{L})$.

For the second claim, let $K \in D_c(\mathcal{X}, \Lambda)$. Since K is isomorphic to the image of \widehat{K} under the localization $\mathcal{D}_c(\mathcal{A}) \rightarrow D_c(\mathcal{X}, \Lambda)$ [Laszlo and Olsson 2008b, 3.0.14], we see that K is stratifiable if \widehat{K} is. Conversely, if K is stratifiable, which means that it is isomorphic to the image of some $M \in \mathcal{D}_c^{\text{stra}}(\mathcal{A})$, then $\widehat{K} = \widehat{M}$ is also stratifiable. \square

3.5.1. For $K \in D_c(\mathcal{X}, \Lambda)$, we say that K is $(\mathcal{S}, \mathcal{L})$ -stratifiable if \widehat{K} is, and then Proposition 3.5 implies that $K \in D_{\mathcal{S}, \mathcal{L}}(\mathcal{X}, \Lambda)$ (see Remark 3.4.1) if and only if K is $(\mathcal{S}, \mathcal{L})$ -stratifiable.

Corollary 3.6. (i) If \mathcal{S} is a stratification of \mathcal{X} , then $K \in D_c(\mathcal{X}, \Lambda)$ is stratifiable if and only if $K|_V$ is stratifiable for every $V \in \mathcal{S}$.

(ii) Let $K \in D_c(\mathcal{X}, \Lambda)$. Then K is stratifiable if and only if its Tate twist $K(1)$ is.

(iii) Let $P : X \rightarrow \mathcal{X}$ be a presentation, and let $K \in D_c(\mathcal{X}, \Lambda)$. Then K is stratifiable if and only if P^*K (respectively $P^!K$) is stratifiable.

Proof. (i) The “only if” part follows from Lemma 3.4 (ii). For the “if” part, we first prove the following.

Lemma 3.6.1. For an S -algebraic stack \mathcal{X} locally of finite type, let

$$N \xrightarrow{u} M \longrightarrow C \longrightarrow N[1]$$

be an exact triangle in $\mathcal{D}_c(\mathcal{A})$, where N is a normalized complex and C is almost AR-null. Then the morphism u is isomorphic to the natural map $\widehat{M} \rightarrow M$.

Proof. Consider the following diagram

$$\begin{array}{ccccccc} \widehat{N} & \xrightarrow{\widehat{u}} & \widehat{M} & \longrightarrow & \widehat{C} & \longrightarrow & \widehat{N}[1] \\ \cong \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \widehat{N} & \xrightarrow{u} & M & \longrightarrow & C & \longrightarrow & N[1]. \end{array}$$

Since C is almost AR-null, we have $\widehat{C} = 0$ by Laszlo and Olsson [2008b, 2.2.2], and so \widehat{u} is an isomorphism. \square

Now let $f : \mathcal{V} \rightarrow \mathcal{X}$ be a morphism of S -algebraic stacks, and let $M \in \mathcal{D}_c(\mathcal{A}(\mathcal{X}))$. We claim that $f^*\widehat{M} \simeq \widehat{f^*M}$. Applying f^* to the exact triangle

$$\widehat{M} \longrightarrow M \longrightarrow C \longrightarrow$$

we get

$$f^*\widehat{M} \longrightarrow f^*M \longrightarrow f^*C \longrightarrow .$$

By Laszlo and Olsson [2008a, 4.3.2], $\widehat{M}_n = \text{hocolim}_N \tau_{\leq N} \widehat{M}_n$, and $- \otimes_{\Lambda_n}^L \Lambda_{n-1}$ and f^* preserve homotopy colimit because they preserve infinite direct sums. Now that $\tau_{\leq N} \widehat{M}_n$ and Λ_{n-1} are bounded above complexes, we have

$$f^*(\tau_{\leq N} \widehat{M}_n \otimes_{\Lambda_n}^L \Lambda_{n-1}) \simeq f^* \tau_{\leq N} \widehat{M}_n \otimes_{\Lambda_n}^L \Lambda_{n-1}$$

(see the proof of [ibid., 4.5.3]). Hence applying f^* to the isomorphism

$$\widehat{M}_n \otimes_{\Lambda_n}^L \Lambda_{n-1} \longrightarrow \widehat{M}_{n-1}$$

we get an isomorphism

$$f^* \widehat{M}_n \otimes_{\Lambda_n}^L \Lambda_{n-1} \longrightarrow f^* \widehat{M}_{n-1},$$

and by [ibid., 3.0.10], $f^* \widehat{M}$ is normalized. Also it is clear that $f^* C$ is AR-null. By Lemma 3.6.1 we have $f^* \widehat{M} \simeq \widehat{f^* M}$.

Therefore, the “if” part follows from Lemma 3.4 (iii) and Proposition 3.5, since $\widehat{K}|_V \simeq \widehat{(K|_V)}$.

(ii) This follows from Lemma 3.4 (vii), since $\widehat{K}(1) = \widehat{K}(1)$.

(iii) For $P^* K$, the “only if” part follows from Lemma 3.4 (ii), and the “if” part follows from Lemma 3.4 (iv) and Proposition 3.5, since $P^* \widehat{K} = \widehat{(P^* K)}$ [ibid., 2.2.1, 3.0.11].

Since P is smooth of relative dimension d , for some function $d : \pi_0(X) \rightarrow \mathbb{N}$, we have $P^! K \simeq P^* K(d)[2d]$, so by (ii), $P^* K$ is stratifiable if and only if $P^! K$ is. \square

Before proving the main result of this section, we prove some special cases.

3.7. Let $f : X \rightarrow Y$ be a morphism of S -schemes. Then the Λ_n -dualizing complexes $K_{X,n}$ and $K_{Y,n}$ of X and Y respectively have finite quasi-injective dimensions, and are bounded by some integer independent of n . Together with the base change theorem for $f_!$, we see that there exists an integer $N > 0$ depending only on X, Y and f , such that for any integers a, b and n with $n \geq 0$ and any $M \in \mathcal{D}_c^{[a,b]}(X, \Lambda_n)$, we have $f_* M \in \mathcal{D}_c^{[a,b+N]}(Y, \Lambda_n)$. This implies that for each n , the functor (defined using K -injective resolutions, see [Spaltenstein 1988, 6.7])

$$f_* : \mathcal{D}(X, \Lambda_n) \rightarrow \mathcal{D}(Y, \Lambda_n)$$

restricts to

$$f_* : \mathcal{D}_c(X, \Lambda_n) \rightarrow \mathcal{D}_c(Y, \Lambda_n).$$

Moreover, for $M \in \mathcal{D}(\mathcal{A}(X))$ with constructible $\mathcal{H}^j(M_n)$'s (for all j and n) and for each $i \in \mathbb{Z}$, there exist integers $a < b$ such that

$$R^i f_* M \simeq R^i f_* \tau_{[a,b]} M.$$

In particular, if M is a λ -complex on X , then $R^i f_* M$ is AR-adic for each i , and hence $f_* M = (f_* M_n)_n$ is a λ -complex on Y .

This enables us to define

$$f_* : D_c(X, \Lambda) \rightarrow D_c(Y, \Lambda)$$

to be $K \mapsto Q f_* \widehat{K}$, where $Q : \mathcal{D}_c(\mathcal{A}(Y)) \rightarrow D_c(Y, \Lambda)$ is the localization functor. It agrees with the definition in [Laszlo and Olsson 2008b, 8] when restricted to

$D_c^+(X, \Lambda)$, and for each $i \in \mathbb{Z}$ and $K \in D_c(X, \Lambda)$, there exist integers $a < b$ such that $R^i f_* K \simeq R^i f_* \tau_{[a,b]} K$.

Lemma 3.8. (i) *If $f : X \rightarrow Y$ is a morphism of S -schemes, and $K \in D_c(X, \Lambda)$ is trivialized by $(\{X\}, \mathcal{L})$ for some \mathcal{L} , then $f_* K$ is stratifiable.*

(ii) *Let \mathcal{X} be an S -algebraic stack that has a connected presentation (that is, there exists a presentation $P : X \rightarrow \mathcal{X}$ with X a connected S -scheme). Let $K_{\mathcal{X}}$ and $K'_{\mathcal{X}}$ be two Λ -dualizing complexes on \mathcal{X} , and let D and D' be the two associated dualizing functors, respectively. Let $K \in D_c(\mathcal{X}, \Lambda)$. If DK is trivialized by a pair $(\mathcal{S}, \mathcal{L})$, where all strata in \mathcal{S} are connected, then $D'K$ is trivialized by $(\mathcal{S}, \mathcal{L}')$ for some other \mathcal{L}' . In particular, for stacks with connected presentation, the property of the Verdier dual of K being stratifiable is independent of the choice of the dualizing complex.*

(iii) *Let \mathcal{X} be an S -algebraic stack that has a connected presentation, and assume that the constant sheaf Λ on \mathcal{X} is a dualizing complex. If $K \in D_c(\mathcal{X}, \Lambda)$ is trivialized by a pair $(\{\mathcal{X}\}, \mathcal{L})$, then $D_{\mathcal{X}} K$ is trivialized by $(\{\mathcal{X}\}, \mathcal{L}')$ for some \mathcal{L}' .*

Proof. (i) Since $f_* K$ is the image of $f_* \widehat{K}$, it suffices to show that $f_* \widehat{K}$ is stratifiable. Since $f_* L$ is bounded for each $L \in \mathcal{L}$, there exists a pair $(\mathcal{S}_Y, \mathcal{L}_Y)$ on Y that trivializes $f_* L$, for all $L \in \mathcal{L}$. We claim that this pair trivializes $R^i f_* \widehat{K}_n$, for each i and n .

Since $R^i f_* \widehat{K}_n = R^i f_* \tau_{[a,b]} \widehat{K}_n$ for some $a < b$, and $\tau_{[a,b]} \widehat{K}_n$ is trivialized by $(\{X\}, \mathcal{L})$, we may assume \widehat{K}_n is bounded. The claim then follows from the spectral sequence

$$R^p f_* \mathcal{H}^q((\widehat{K})_n) \implies R^{p+q} f_*((\widehat{K})_n)$$

and Remark 3.4.1.

(ii) Recall that the dualizing complex $K_{\mathcal{X}}$ (respectively $K'_{\mathcal{X}}$) is defined to be the image of a normalized complex $K_{\mathcal{X}, \bullet}$ (respectively $K'_{\mathcal{X}, \bullet}$), where each $K_{\mathcal{X}, n}$ (respectively $K'_{\mathcal{X}, n}$) is a Λ_n -dualizing complex. See [ibid., 7.2.3, 7.2.4].

Let $P : X \rightarrow \mathcal{X}$ be a presentation where X is a connected scheme. Then we have

$$P^* R\mathcal{H}om(K_{\mathcal{X}, n}, K'_{\mathcal{X}, n}) = R\mathcal{H}om(P^* K_{\mathcal{X}, n}, P^* K'_{\mathcal{X}, n}) = R\mathcal{H}om(P^1 K_{\mathcal{X}, n}, P^1 K'_{\mathcal{X}, n}).$$

Since $P^1 K_{\mathcal{X}, n}$ and $P^1 K'_{\mathcal{X}, n}$ are Λ_n -dualizing complexes on X , by [Illusie 1977, Exp. I, 2], we see that $P^* R\mathcal{H}om(K_{\mathcal{X}, n}, K'_{\mathcal{X}, n})$ (and hence $R\mathcal{H}om(K_{\mathcal{X}, n}, K'_{\mathcal{X}, n})$) is cohomologically concentrated in one degree, therefore it is quasi-isomorphic to this nontrivial cohomology sheaf, once it has been appropriately shifted. So let $R\mathcal{H}om(K_{\mathcal{X}, n}, K'_{\mathcal{X}, n}) \simeq L_n[r_n]$ for some sheaf L_n and integer r_n . Since $P^* L_n$ is invertible and hence lcc (see [Illusie 1977, p. 19]), the sheaf L_n is lcc [Olsson 2007, 9.1].

For every stratum $\mathfrak{U} \in \mathcal{S}$, let $\mathcal{L}_0(\mathfrak{U})$ be the union of $\mathcal{L}(\mathfrak{U})$ and the set of isomorphism classes of the Jordan–Hölder components of the lcc sheaf $L_0|_{\mathfrak{U}}$. Since all strata in \mathcal{S} are connected, there exists a tensor closed hull of $(\mathcal{S}, \mathcal{L}_0)$ of the form $(\mathcal{S}, \mathcal{L}')$, that is, they have the same stratification \mathcal{S} .

By Laszlo and Olsson [2008b, 4.0.8], the system

$$(L_n[r_n])_n = R\mathcal{H}om((K_{\mathcal{X},n})_n, (K'_{\mathcal{X},n})_n)$$

is normalized, so by Lemma 3.6.1, $\widehat{D'K}_{\mathcal{X}} = (L_n[r_n])_n$, and by Lemma 3.4 (vi), it is trivialized by $(\mathcal{S}, \mathcal{L}')$. Since DK is trivialized by $(\mathcal{S}, \mathcal{L}')$, so also is $D'K$, because $\widehat{D'K} \simeq \widehat{DK} \otimes^L \widehat{D'K}_{\mathcal{X}}$.

(iii) The assumption implies in particular that \mathcal{X} is connected, so by (ii), the question is independent of the choice of the dualizing complex. By definition, \widehat{K} is trivialized by $(\{\mathcal{X}\}, \mathcal{L})$, so are truncations of \widehat{K} . The essential image of $R\mathcal{H}om(\widehat{K}, \Lambda_{\bullet})$ in $D_c(\mathcal{X}, \Lambda)$ is DK , so by 3.5.1 it suffices to show that $R\mathcal{H}om(\widehat{K}, \Lambda_{\bullet}) \in \mathcal{D}_{\{\mathcal{X}\}, \mathcal{S}'}(\mathcal{A})$ for some \mathcal{S}' .

Since \mathcal{X} is quasi-compact, each Λ_n -dualizing complex is of finite quasi-injective dimension, so for each integer i , there exist integers a and b such that

$$\mathcal{H}^i R\mathcal{H}om(\widehat{K}_n, \Lambda_n) = \mathcal{H}^i R\mathcal{H}om(\tau_{[a,b]}\widehat{K}_n, \Lambda_n).$$

Using truncation triangles, we may further replace $\tau_{[a,b]}\widehat{K}_n$ by the cohomology sheaves $\mathcal{H}^j \widehat{K}_n$, and hence by their Jordan–Hölder components. Therefore, it suffices to find an \mathcal{S}' that trivializes $\mathcal{H}^i R\mathcal{H}om(L, \Lambda_0)$, for all $i \in \mathbb{Z}$ and $L \in \mathcal{L}$. Note that $R\mathcal{H}om(L, \Lambda_0) = \mathcal{H}om(L, \Lambda_0) = L^{\vee}$ is a simple Λ_0 -sheaf, so one can take $\mathcal{S}' = \{L^{\vee} | L \in \mathcal{L}\}$. \square

Remark 3.8.1. For any S -algebraic stack \mathcal{X} , the Verdier dual of a complex K in $D_c(\mathcal{X}, \Lambda)$ being stratifiable or not is independent of the choice of the dualizing complex. Let $K_{\mathcal{X}}$ and $K'_{\mathcal{X}}$ be two dualizing complexes on \mathcal{X} , defining dualizing functors D and D' , respectively. Let $P : X \rightarrow \mathcal{X}$ be a presentation, let $K_X = P^! K_{\mathcal{X}}$ and let $K'_X = P^! K'_{\mathcal{X}}$, defining dualizing functors D_X and D'_X on X , respectively. Suppose DK is stratifiable. To show $D'K$ is also stratifiable, by Corollary 3.6 (iii) it suffices to show $P^! D'K = D'_X P^* K$ is stratifiable. Since $D_X P^* K = P^! DK$ is stratifiable by assumption, we may assume $\mathcal{X} = X$ is a scheme. Since X is noetherian, it has finitely many connected components, each of which is both open and closed. Then the result follows from Corollary 3.6 (i) and Lemma 3.8 (ii).

Next we prove the main result of this section.

Theorem 3.9. *Let $f : \mathcal{X} \rightarrow \mathcal{Y}$ be a morphism of S -algebraic stacks. Then the operations f_* , $f_!$, f^* , $f^!$, $D_{\mathcal{X}}$, $-\otimes^L -$ and $R\mathcal{H}om(-, -)$ all preserve stratifiability,*

namely, they induce functors

$$\begin{aligned}
f_* &: D_c^{+, \text{stra}}(\mathcal{X}, \Lambda) \longrightarrow D_c^{+, \text{stra}}(\mathcal{Y}, \Lambda), & f_! &: D_c^{-, \text{stra}}(\mathcal{X}, \Lambda) \longrightarrow D_c^{-, \text{stra}}(\mathcal{Y}, \Lambda), \\
f^* &: D_c^{\text{stra}}(\mathcal{Y}, \Lambda) \longrightarrow D_c^{\text{stra}}(\mathcal{X}, \Lambda), & f^! &: D_c^{\text{stra}}(\mathcal{Y}, \Lambda) \longrightarrow D_c^{\text{stra}}(\mathcal{X}, \Lambda), \\
R\mathcal{H}om(-, -) &: D_c^{-, \text{stra}}(\mathcal{X}, \Lambda)^{\text{op}} \times D_c^{+, \text{stra}}(\mathcal{X}, \Lambda) \longrightarrow D_c^{+, \text{stra}}(\mathcal{X}, \Lambda), \\
\otimes^L &: D_c^{-, \text{stra}}(\mathcal{X}, \Lambda) \times D_c^{-, \text{stra}}(\mathcal{X}, \Lambda) \longrightarrow D_c^{-, \text{stra}}(\mathcal{X}, \Lambda), \\
D &: D_c^{\text{stra}}(\mathcal{X}, \Lambda) \longrightarrow D_c^{\text{stra}}(\mathcal{X}, \Lambda)^{\text{op}}.
\end{aligned}$$

Proof. We may assume all stacks involved are reduced.

We consider the Verdier dual functor D first. Let $P : X \rightarrow \mathcal{X}$ be a presentation. Since $P^*D = DP^!$, by Corollary 3.6 (iii) we can assume $\mathcal{X} = X$ is a scheme. Let K be a complex on X trivialized by $(\mathcal{S}, \mathcal{L})$. Refining if necessary, we may assume all strata in \mathcal{S} are connected and regular. Let $j : U \rightarrow X$ be the immersion of an open stratum in \mathcal{S} with complement $i : Z \rightarrow X$. Shrinking U if necessary, we may assume there is a dimension function on U [Riou 2007, Définition 2.1], hence by a result of Gabber [ibid., Théorème 0.2], the constant sheaf Λ on U is a dualizing complex. Consider the exact triangle

$$i_*D_Z(K|_Z) \longrightarrow D_X K \longrightarrow j_*D_U(K|_U) \longrightarrow .$$

By Lemma 3.8 (iii) we see that $D_U(K|_U)$ is trivialized by $(\{U\}, \mathcal{L}')$ for some \mathcal{L}' , so $j_*D_U(K|_U)$ is stratifiable by Lemma 3.8 (i). By noetherian induction we may assume $D_Z(K|_Z)$ is stratifiable, and it is clear that i_* preserves stratifiability. Therefore by Lemma 3.4 (i), $D_X K$ is stratifiable.

The case of f^* (and hence $f^!$) is proved in Lemma 3.4 (ii).

Next we prove the case of \otimes^L . For $i = 1, 2$, let $K_i \in D_c^-(\mathcal{X}, \Lambda)$, trivialized by $(\mathcal{S}_i, \mathcal{L}_i)$. Let $(\mathcal{S}, \mathcal{L})$ be a common tensor closed refinement (by Lemma 3.3) of $(\mathcal{S}_i, \mathcal{L}_i)$, $i = 1, 2$. The total tensor product $K_1 \otimes^L K_2$ is defined to be the image in $D_c(\mathcal{X}, \Lambda)$ of $\widehat{K}_1 \otimes_{\Lambda_\bullet}^L \widehat{K}_2$, which by Laszlo and Olsson [2008b, 3.0.10] is normalized, so it suffices to show (by Lemma 3.4 (vi)) that

$$\widehat{K}_{1,0} \otimes_{\Lambda_0}^L \widehat{K}_{2,0} = \widehat{K}_{1,0} \otimes_{\Lambda_0} \widehat{K}_{2,0}$$

is trivialized by $(\mathcal{S}, \mathcal{L})$. This follows from

$$\mathcal{H}^r(\widehat{K}_{1,0} \otimes_{\Lambda_0} \widehat{K}_{2,0}) = \bigoplus_{i+j=r} \mathcal{H}^i(\widehat{K}_{1,0}) \otimes_{\Lambda_0} \mathcal{H}^j(\widehat{K}_{2,0})$$

and the assumption that $(\mathcal{S}, \mathcal{L})$ is tensor closed.

The case of $R\mathcal{H}om(K_1, K_2) = D(K_1 \otimes^L DK_2)$ follows.

Finally we prove the case of f_* and $f_!$. Let $f : \mathcal{X} \rightarrow \mathcal{Y}$ be a morphism of S -algebraic stacks, and let $K \in D_{\mathcal{S}, \mathcal{L}}^-(\mathcal{X}, \Lambda)$ for some pair $(\mathcal{S}, \mathcal{L})$. We want to show

$f_!K$ is stratifiable. Let $j : \mathcal{U} \rightarrow \mathcal{X}$ be the immersion of an open stratum in \mathcal{S} , with complement $i : \mathcal{Z} \rightarrow \mathcal{X}$. From the exact triangle

$$(fj)_!j^*K \longrightarrow f_!K \longrightarrow (fi)_!i^*K \longrightarrow$$

we see that it suffices to prove the claim for fj and fi . By noetherian induction we can replace \mathcal{X} by \mathcal{U} . By Corollary 3.6 (iii) and smooth base change [Laszlo and Olsson 2008b, 12.1], we can replace \mathcal{Y} by a presentation Y , and by Corollary 3.6 (i) and [ibid., 12.3] we can shrink Y to an open subscheme. After these reductions, we assume that $\mathcal{Y} = Y$ is a regular irreducible affine S -scheme that has a dimension function on it, that K is trivialized by $(\{\mathcal{X}\}, \mathcal{L})$, and that the relative inertia stack $\mathcal{I}_f := \mathcal{X} \times_{\Delta, \mathcal{X} \times_Y \mathcal{X}, \Delta} \mathcal{X}$ is flat and has components over \mathcal{X} [Behrend 2003, 5.1.14]. Therefore by [ibid., 5.1.13], f factors as

$$\mathcal{X} \xrightarrow{g} \mathcal{Z} \xrightarrow{h} Y,$$

where g is gerbe-like and h is representable (see [ibid., 5.1.3–5.1.6] for relevant notions). So we reduce to two cases: f is representable, or f is gerbe-like.

Case when f is representable. By shrinking the S -algebraic space \mathcal{X} we can assume $\mathcal{X} = X$ is a regular connected scheme that has a dimension function, so that the constant sheaf Λ on X is a dualizing complex. By Lemma 3.8 (iii) we see that DK is trivialized by some $(\{X\}, \mathcal{L}')$, and by Lemma 3.8 (i), f_*DK is stratifiable. Therefore $f_!K = Df_*DK$ is also stratifiable.

Case when f is gerbe-like. In this case f is smooth [Behrend 2003, 5.1.5], hence étale locally on Y it has a section. Replacing Y by an étale cover, we may assume that f is a neutral gerbe, so $f : B(G/Y) \rightarrow Y$ is the structural map, for some flat group space G of finite type over Y [Laumon and Moret-Bailly 2000, 3.21]. By [ibid., 5.1.1] and Corollary 3.6 (i) we may assume G is a Y -group scheme. Next we reduce to the case when G is smooth over Y .

By assumption Y is integral. Let $k(Y)$ be the function field of Y and $\overline{k(Y)}$ an algebraic closure. Then $G_{\overline{k(Y)}, \text{red}}$ is smooth over $\overline{k(Y)}$, so there exists a finite extension L over $k(Y)$ such that $G_{L, \text{red}}$ is smooth over L . Let Y' be the normalization of Y in L , which is a scheme of finite type over S , and the natural map $Y' \rightarrow Y$ is finite surjective. It factors through $Y' \rightarrow Z \rightarrow Y$, where Z is the normalization of Y in the separable closure of $k(Y)$ in $L = k(Y')$. So $Z \rightarrow Y$ is generically étale, and $Y' \rightarrow Z$ is purely inseparable, hence a universal homeomorphism, so Y' and Z have equivalent étale sites. Replacing Y' by Z and shrinking Y we can assume $Y' \rightarrow Y$ is finite étale. Replacing Y by Y' (by Corollary 3.6 (ii)) we assume G_{red} over Y has smooth generic fiber, and by shrinking Y we assume G_{red} is smooth over Y .

G_{red} is a subgroup scheme of G [Grothendieck and Demazure 1970, Exposé VI_A, 0.2]; we write $h : G_{\text{red}} \hookrightarrow G$ for the associated closed immersion. Then

$Bh : B(G_{\text{red}}/Y) \rightarrow B(G/Y)$ is faithful and hence representable. It is also radicial: consider the diagram where the square is 2-Cartesian

$$\begin{array}{ccccc} Y & \xleftarrow{i} & G/G_{\text{red}} & \xrightarrow{g} & Y \\ & & \downarrow & & \downarrow P \\ & & B(G_{\text{red}}/Y) & \xrightarrow{Bh} & B(G/Y). \end{array}$$

The map i is a nilpotent closed embedding, so g is radicial. Since P is faithfully flat, Bh is also radicial. This shows that

$$(Bh)^* : D_c^-(B(G/Y), \Lambda) \rightarrow D_c^-(B(G_{\text{red}}/Y), \Lambda)$$

is an equivalence of categories. Replacing G by G_{red} we assume G is smooth over Y , and hence $P : Y \rightarrow B(G/Y)$ is a connected presentation.

Let d be the relative dimension of G over Y . By assumption, the constant sheaf Λ on Y is a dualizing complex, and so $f^! \Lambda = \Lambda(-d)$ (and hence the constant sheaf Λ on \mathcal{X} is a dualizing complex on \mathcal{X}). By Lemma 3.8 (iii), we see that DK is trivialized by a pair of the form $(\{\mathcal{X}\}, \mathcal{L}')$. To show $f_! K$ is stratifiable is equivalent to showing that $Df_! K = f_* DK$ is stratifiable. So replacing K by DK , it suffices to show that $f_* K$ is stratifiable, where $K \in D_{\{\mathcal{X}\}, \mathcal{L}}^+(\mathcal{X}, \Lambda)$ for some \mathcal{L} .

Consider the strictly simplicial smooth hypercover associated to the presentation $P : Y \rightarrow B(G/Y)$, and let $f_i : \prod_i G \rightarrow Y$ be the structural map. As in the proof of Lemma 3.8 (i), it suffices to show the existence of a pair $(\mathcal{S}_Y, \mathcal{L}_Y)$ on Y that trivializes $R^n f_* L$, for all $L \in \mathcal{L}$ and $n \in \mathbb{Z}$. From the spectral sequence [Laszlo and Olsson 2008b, 10.0.9]

$$E_1^{ij} = R^j f_{i*} f_i^* P^* L \implies R^{i+j} f_* L,$$

we see that it suffices for the pair $(\mathcal{S}_Y, \mathcal{L}_Y)$ to trivialize all the E_1^{ij} -terms. Assume $i \geq 1$. If we regard the map $f_i : \prod_i G \rightarrow Y$ as the product map

$$\prod_i f_i : \prod_i G \rightarrow \prod_i Y,$$

where the products are fiber products over Y , then we can write $f_i^* P^* L$ as

$$f_1^* P^* L \boxtimes_{\Lambda_0} \Lambda_0 \boxtimes_{\Lambda_0} \cdots \boxtimes_{\Lambda_0} \Lambda_0.$$

Note that the scheme Y satisfies the condition (LO). By Künneth formula [Laszlo and Olsson 2008b, 11.0.14] we have

$$f_{i*} f_i^* P^* L = f_{1*} f_1^* P^* L \otimes_{\Lambda_0} f_{1*} \Lambda_0 \otimes_{\Lambda_0} \cdots \otimes_{\Lambda_0} f_{1*} \Lambda_0.$$

Since $f_{1*}f_1^*P^*L$ and $f_{1*}\Lambda_0$ are bounded complexes (by a theorem of Gabber [Illusie et al. 2008]), there exists a tensor closed pair $(\mathcal{Y}, \mathcal{L}_Y)$ that trivializes them, for all $L \in \mathcal{L}$. The proof is finished. \square

Consequently, the theorem also holds for $\overline{\mathbb{Q}}_\ell$ -coefficients.

Finally we give a lemma which will be used in the next section. This will play the same role as [Behrend 2003, 6.3.16].

Lemma 3.10. *Let X be a connected variety over an algebraically closed field k of characteristic not equal to ℓ , and let \mathcal{L} be a finite set of isomorphism classes of simple lcc Λ_0 -sheaves on X . Then there exists an integer d (depending only on \mathcal{L}) such that, for every lisse Λ -adic sheaf \mathcal{F} on X trivialized by \mathcal{L} , and for every integer i , we have*

$$\dim_E H_c^i(X, \mathcal{F} \otimes_\Lambda E) \leq d \cdot \text{rank}_E(\mathcal{F} \otimes_\Lambda E),$$

where E is the fraction field of Λ .

Proof. Since \mathcal{L} is finite and $0 \leq i \leq 2 \dim X$, there exists an integer $d > 0$ such that $\dim_{\Lambda_0} H_c^i(X, L) \leq d \cdot \text{rank}_{\Lambda_0} L$, for every i and every $L \in \mathcal{L}$. For a short exact sequence of lcc Λ_0 -sheaves

$$0 \longrightarrow \mathcal{G}' \longrightarrow \mathcal{G} \longrightarrow \mathcal{G}'' \longrightarrow 0$$

on X , the cohomological sequence

$$\dots \longrightarrow H_c^i(X, \mathcal{G}') \longrightarrow H_c^i(X, \mathcal{G}) \longrightarrow H_c^i(X, \mathcal{G}'') \longrightarrow \dots$$

implies that $\dim_{\Lambda_0} H_c^i(X, \mathcal{G}) \leq \dim_{\Lambda_0} H_c^i(X, \mathcal{G}') + \dim_{\Lambda_0} H_c^i(X, \mathcal{G}'')$. So it is clear that if \mathcal{G} is trivialized by \mathcal{L} , then $\dim_{\Lambda_0} H_c^i(X, \mathcal{G}) \leq d \cdot \text{rank}_{\Lambda_0} \mathcal{G}$, for every i .

Since we only consider $\mathcal{F} \otimes_\Lambda E$, we may assume $\mathcal{F} = (\mathcal{F}_n)_n$ is flat, of some constant rank over Λ (since X is connected), and this Λ -rank is equal to

$$\text{rank}_{\Lambda_0} \mathcal{F}_0 = \text{rank}_E(\mathcal{F} \otimes_\Lambda E).$$

Recall that $H_c^i(X, \mathcal{F})$ is a finitely generated Λ -module [Illusie 1977, VI, 2.2.2], so by Nakayama's lemma the minimal number of generators of the module is at most $\dim_{\Lambda_0}(\Lambda_0 \otimes_\Lambda H_c^i(X, \mathcal{F}))$. Similar to ordinary cohomology groups [Milne 2008, 19.2], we have an injection

$$\Lambda_0 \otimes_\Lambda H_c^i(X, \mathcal{F}) \hookrightarrow H_c^i(X, \mathcal{F}_0)$$

of Λ_0 -vector spaces. Therefore, $\dim_E H_c^i(X, \mathcal{F} \otimes_\Lambda E)$ is less than or equal to the minimal number of generators of $H_c^i(X, \mathcal{F})$ over Λ , which is at most

$$\begin{aligned} \dim_{\Lambda_0}(\Lambda_0 \otimes_\Lambda H_c^i(X, \mathcal{F})) &\leq \dim_{\Lambda_0} H_c^i(X, \mathcal{F}_0) \\ &\leq d \cdot \text{rank}_{\Lambda_0} \mathcal{F}_0 = d \cdot \text{rank}_E(\mathcal{F} \otimes_\Lambda E). \end{aligned} \quad \square$$

4. Convergent complexes and finiteness

We return to \mathbb{F}_q -algebraic stacks $\mathcal{X}_0, \mathcal{Y}_0, \dots$ of finite type. A complex K_0 in $W(\mathcal{X}_0, \overline{\mathbb{Q}}_\ell)$ is said to be *stratifiable* if K on \mathcal{X} is stratifiable, and we denote by $W^{\text{stra}}(\mathcal{X}_0, \overline{\mathbb{Q}}_\ell)$ the full subcategory of such complexes. Note that if K_0 is a lisse-étale complex, and it is stratifiable on \mathcal{X}_0 , then it is geometrically stratifiable (that is, K on \mathcal{X} is stratifiable). It turns out that in order for the trace formula to hold, it suffices to make this weaker assumption of geometric stratifiability. So we will only discuss stratifiable Weil complexes. Again, by a sheaf we mean a Weil $\overline{\mathbb{Q}}_\ell$ -sheaf.

Definition 4.1. (i) Let $K \in D_c(\overline{\mathbb{Q}}_\ell)$ and $\varphi : K \rightarrow K$ an endomorphism. The pair (K, φ) is said to be an ι -convergent complex (or just a convergent complex, since we fixed ι) if the complex series in two directions

$$\sum_{n \in \mathbb{Z}} \sum_{H^n(K), H^n(\varphi)} |\alpha|^s$$

is convergent, for every real number $s > 0$. In this case let $\text{Tr}(\varphi, K)$ be the absolutely convergent complex series

$$\sum_n (-1)^n \iota \text{Tr}(H^n(\varphi), H^n(K))$$

or its limit.

(ii) Let $K_0 \in W^-(\mathcal{X}_0, \overline{\mathbb{Q}}_\ell)$. We call K_0 an ι -convergent complex of sheaves (or just a convergent complex of sheaves), if for every integer $v \geq 1$ and every point $x \in \mathcal{X}_0(\mathbb{F}_{q^v})$, the pair $(K_{\bar{x}}, F_x)$ is a convergent complex. In particular, all bounded complexes are convergent.

(iii) Let $K_0 \in W^-(\mathcal{X}_0, \overline{\mathbb{Q}}_\ell)$ be a convergent complex of sheaves. Define

$$c_v(\mathcal{X}_0, K_0) = \sum_{x \in [\mathcal{X}_0(\mathbb{F}_{q^v})]} \frac{1}{\#\text{Aut}_x \mathbb{F}_{q^v}} \text{Tr}(F_x, K_{\bar{x}}) \in \mathbb{C},$$

and define the L -series of K_0 to be the formal power series

$$L(\mathcal{X}_0, K_0, t) = \exp \left(\sum_{v \geq 1} c_v(\mathcal{X}_0, K_0) \frac{t^v}{v} \right) \in \mathbb{C}[[t]].$$

The zeta function $Z(\mathcal{X}_0, t)$ in Definition 1.2 is a special case of this definition as $Z(\mathcal{X}_0, t) = L(\mathcal{X}_0, \overline{\mathbb{Q}}_\ell, t)$. It has rational coefficients.

Notation 4.1.1. We sometimes write $c_v(K_0)$ for $c_v(\mathcal{X}_0, K_0)$, if it is clear that K_0 is on \mathcal{X}_0 . We also write $c_v(\mathcal{X}_0)$ for $c_v(\mathcal{X}_0, \overline{\mathbb{Q}}_\ell)$.

Remark 4.1.2. (i) Behrend [2003, 6.2.3] defined convergent complexes with respect to arithmetic Frobenius elements, and our definition is for geometric Frobenius, and it is essentially the same as Behrend's definition, except we work with ι -mixed Weil complexes (which means *all* Weil complexes, by Remark 2.8.1) for an arbitrary isomorphism $\iota : \overline{\mathbb{Q}}_\ell \rightarrow \mathbb{C}$, while Behrend [2003] works with pure or mixed lisse-étale sheaves with integer weights. Also our definition is a little different from that in [Olsson 2008a]; the condition there is weaker.

(ii) Recall that Aut_x is defined to be the fiber over x of the inertia stack $\mathcal{I}_0 \rightarrow \mathcal{X}_0$. It is a group scheme of finite type [Laumon and Moret-Bailly 2000, 4.2] over $k(x)$, so $\text{Aut}_x k(x)$ is a finite group.

(iii) If we have the following commutative diagram

$$\begin{array}{ccc} \text{Spec } \mathbb{F}_{q^{vd}} & \longrightarrow & \text{Spec } \mathbb{F}_{q^v} \\ & \searrow x' & \downarrow x \\ & & \mathcal{X}_0, \end{array}$$

then $(K_{\overline{x}}, F_x)$ is convergent if and only if $(K_{\overline{x}'}, F_{x'})$ is convergent, because we have $F_{x'} = F_x^d$ and $s \mapsto sd : \mathbb{R}^{>0} \rightarrow \mathbb{R}^{>0}$ is a bijection. In particular, for a lisse-étale complex of sheaves, the property of being a convergent complex is independent of q and the structural morphism $\mathcal{X}_0 \rightarrow \text{Spec } \mathbb{F}_q$. Also note that, for every integer $v \geq 1$, a complex K_0 on \mathcal{X}_0 is convergent if and only if $K_0 \otimes \mathbb{F}_{q^v}$ on $\mathcal{X}_0 \otimes \mathbb{F}_{q^v}$ is convergent.

We restate the main theorem in [Behrend 2003] using compactly supported cohomology as follows. It generalizes Theorem 1.1. We will prove it in this section and the next.

Theorem 4.2. *Let $f : \mathcal{X}_0 \rightarrow \mathcal{Y}_0$ be a morphism of \mathbb{F}_q -algebraic stacks, and let $K_0 \in W_m^{-, \text{stra}}(\mathcal{X}_0, \overline{\mathbb{Q}}_\ell)$ be a convergent complex of sheaves. Then:*

- (i) (Finiteness) $f_! K_0$ is a convergent complex of sheaves on \mathcal{Y}_0 .
- (ii) (Trace formula) $c_v(\mathcal{X}_0, K_0) = c_v(\mathcal{Y}_0, f_! K_0)$ for every integer $v \geq 1$.

First we give a few lemmas.

Lemma 4.3. *Let*

$$\begin{array}{ccccccc} K' & \longrightarrow & K & \longrightarrow & K'' & \longrightarrow & K'[1] \\ \varphi' \downarrow & & \varphi \downarrow & & \downarrow \varphi'' & & \downarrow \varphi'[1] \\ K' & \longrightarrow & K & \longrightarrow & K'' & \longrightarrow & K'[1]. \end{array}$$

be an endomorphism of an exact triangle $K' \rightarrow K \rightarrow K'' \rightarrow K'[1]$ in $D_c^-(\overline{\mathbb{Q}}_\ell)$. If any two of the three pairs (K', φ') , (K'', φ'') and (K, φ) are convergent, then so is

the third, and

$$\mathrm{Tr}(\varphi, K) = \mathrm{Tr}(\varphi', K') + \mathrm{Tr}(\varphi'', K'').$$

Proof. By the rotation axiom we can assume (K', φ') and (K'', φ'') are convergent. We have the exact sequence

$$\cdots \longrightarrow H^n(K') \longrightarrow H^n(K) \longrightarrow H^n(K'') \longrightarrow H^{n+1}(K') \longrightarrow \cdots.$$

Since $H^n(K)$ is an extension of a subobject of $H^n(K'')$ by a quotient object of $H^n(K')$, we have

$$\sum_{H^n(K), \varphi} |\alpha|^s \leq \sum_{H^n(K'), \varphi'} |\alpha|^s + \sum_{H^n(K''), \varphi''} |\alpha|^s$$

for every real $s > 0$, so (K, φ) is convergent.

Since the series $\sum_{n \in \mathbb{Z}} (-1)^n \sum_{H^n(K), \varphi} \iota \alpha$ converges absolutely, we can change the order of summation, and the second assertion follows if we split the long exact sequence above into short exact sequences. \square

Corollary 4.4. *If $K'_0 \rightarrow K_0 \rightarrow K''_0 \rightarrow K'_0[1]$ is an exact triangle in $W^-(\mathcal{X}_0, \overline{\mathbb{Q}}_\ell)$, and two of the three complexes K'_0 , K''_0 and K_0 are convergent complexes, then so is the third, and $c_v(K_0) = c_v(K'_0) + c_v(K''_0)$.*

Proof. For every $x \in \mathcal{X}_0(\mathbb{F}_{q^v})$, we have an exact triangle

$$K'_x \longrightarrow K_x \longrightarrow K''_x \longrightarrow$$

in $D_c^-(\overline{\mathbb{Q}}_\ell)$, equivariant under the action of F_x . Then apply Lemma 4.3. \square

Lemma 4.5. *Theorem 4.2 holds for $f : \mathrm{Spec} \mathbb{F}_{q^d} \rightarrow \mathrm{Spec} \mathbb{F}_q$.*

Proof. We have an equivalence of triangulated categories

$$W^-(\mathrm{Spec} \mathbb{F}_q, \overline{\mathbb{Q}}_\ell) \xrightarrow{\sim} D_c^-(\mathrm{Rep}_{\overline{\mathbb{Q}}_\ell} G),$$

where G is the Weil group $F^{\mathbb{Z}}$ of \mathbb{F}_q . Let H be the subgroup $F^{d\mathbb{Z}}$, the Weil group of \mathbb{F}_{q^d} . Since $f : \mathrm{Spec} \mathbb{F}_{q^d} \rightarrow \mathrm{Spec} \mathbb{F}_q$ is finite, we have $f_! = f_*$, and it is the induced-module functor

$$\mathrm{Hom}_{\overline{\mathbb{Q}}_\ell[H]}(\overline{\mathbb{Q}}_\ell[G], -) : D_c^-(\mathrm{Rep}_{\overline{\mathbb{Q}}_\ell} H) \longrightarrow D_c^-(\mathrm{Rep}_{\overline{\mathbb{Q}}_\ell} G),$$

which is isomorphic to the coinduced-module functor $\overline{\mathbb{Q}}_\ell[G] \otimes_{\overline{\mathbb{Q}}_\ell[H]} -$. In particular, $f_!$ is exact on the level of sheaves.

Let A be a $\overline{\mathbb{Q}}_\ell$ -representation of H , and $B = \overline{\mathbb{Q}}_\ell[G] \otimes_{\overline{\mathbb{Q}}_\ell[H]} A$. Let x_1, \dots, x_m be an ordered basis for A with respect to which F^d is an upper triangular matrix

$$\begin{bmatrix} \alpha_1 & * & * \\ & \ddots & * \\ & & \alpha_m \end{bmatrix}$$

with eigenvalues $\alpha_1, \dots, \alpha_m$. Then B has an ordered basis

$$\begin{array}{cccc} 1 \otimes x_1, & F \otimes x_1, & \cdots, & F^{d-1} \otimes x_1, \\ 1 \otimes x_2, & F \otimes x_2, & \cdots, & F^{d-1} \otimes x_2, \\ \vdots & \vdots & \ddots & \vdots \\ 1 \otimes x_m, & F \otimes x_m, & \cdots, & F^{d-1} \otimes x_m, \end{array}$$

with respect to which F is the matrix

$$\begin{bmatrix} M_1 & * & * \\ & \ddots & * \\ & & M_m \end{bmatrix}, \quad \text{where } M_i = \begin{bmatrix} 0 & \cdots & 0 & \alpha_i \\ 1 & \ddots & 0 & 0 \\ & \ddots & \ddots & \vdots \\ & & & 1 & 0 \end{bmatrix}.$$

The characteristic polynomial of F on B is $\prod_{i=1}^m (t^d - \alpha_i)$.

Let K_0 be a complex of sheaves on $\text{Spec } \mathbb{F}_{q^d}$. The eigenvalues of the Frobenius F on $\mathcal{H}^n(f_!K) = f_!\mathcal{H}^n(K)$ are all the d -th roots of the eigenvalues of F^d on $\mathcal{H}^n(K)$, so for every $s > 0$ we have

$$\sum_n \sum_{\mathcal{H}^n(f_!K), F} |\alpha|^s = d \sum_n \sum_{\mathcal{H}^n(K), F^d} |\alpha|^{s/d}.$$

This shows that $f_!K_0$ is a convergent complex on $\text{Spec } \mathbb{F}_q$ if and only if K_0 is a convergent complex on $\text{Spec } \mathbb{F}_{q^d}$.

Next we prove

$$c_v(\text{Spec } \mathbb{F}_{q^d}, K_0) = c_v(\text{Spec } \mathbb{F}_q, f_!K_0)$$

for every $v \geq 1$. Since $H^n(f_!K) = f_!H^n(K)$, and both sides are absolutely convergent series so that one can change the order of summation without changing the limit, it suffices to prove it when $K = A$ is a single representation concentrated in degree 0. Let us review this classical calculation. Use the notation above. For each i , fix a d -th root $\alpha_i^{1/d}$ of α_i , and let ζ_d be a primitive d -th root of unity. Then the eigenvalues of F on B are $\zeta_d^k \alpha_i^{1/d}$, for $i = 1, \dots, m$ and $k = 0, \dots, d-1$.

If $d \nmid v$, then $\text{Hom}_{\mathbb{F}_q}(\mathbb{F}_{q^d}, \mathbb{F}_{q^v}) = \emptyset$, so $c_v(\text{Spec } \mathbb{F}_{q^d}, A) = 0$. On the other hand,

$$c_v(\text{Spec } \mathbb{F}_q, f_!A) = \text{Tr}(F^v, B) = \sum_{i,k} \zeta_d^{vk} \alpha_i^{v/d} = \sum_i \alpha_i^{v/d} \sum_{k=0}^{d-1} \zeta_d^{vk} = 0.$$

If $d \mid v$, then $\text{Hom}_{\mathbb{F}_q}(\mathbb{F}_{q^d}, \mathbb{F}_{q^v}) = \text{Hom}_{\mathbb{F}_q}(\mathbb{F}_{q^d}, \mathbb{F}_{q^d}) = \mathbb{Z}/d\mathbb{Z}$. So

$$c_v(\mathbb{F}_{q^d}, A) = d \text{Tr}(F^v, A) = d \sum_i \alpha_i^{v/d}.$$

On the other hand,

$$c_v(\mathbb{F}_q, B) = \text{Tr}(F^v, B) = \sum_{i,k} \zeta_d^{vk} \alpha_i^{v/d} = \sum_{i,k} \alpha_i^{v/d} = d \sum_i \alpha_i^{v/d}. \quad \square$$

Next, we consider BG_0 , for a finite group scheme G_0 over \mathbb{F}_q .

Lemma 4.6. *Let G_0 be a finite \mathbb{F}_q -group scheme, and let \mathcal{F}_0 be a sheaf on BG_0 . Then $H_c^r(BG, \mathcal{F}) = 0$ for all $r \neq 0$, and $H_c^0(BG, \mathcal{F}) \simeq H^0(BG, \mathcal{F})$ has dimension at most $\text{rank}(\mathcal{F})$. Moreover, the set of ι -weights of $H_c^0(BG, \mathcal{F})$ is a subset of the ι -weights of \mathcal{F}_0 .*

Proof. By [Olsson 2008a, 7.12–7.14] we can replace G_0 by its maximal reduced closed subscheme, and assume G_0 is reduced, hence étale. Then G_0 is the same as a finite group $G(\mathbb{F})$ with a continuous action of $\text{Gal}(\mathbb{F}_q)$ [Milne 2012, XII, 2.11]. We will also write G for the group $G(\mathbb{F})$, if there is no confusion. Since $\text{Spec } \mathbb{F} \rightarrow BG$ is surjective, we see that there is no nontrivial stratification on BG . In particular, all sheaves on BG are lisse, and they are just $\overline{\mathbb{Q}}_\ell$ -representations of G .

BG is quasi-finite and proper over \mathbb{F} , with finite diagonal, so by [Olsson 2008a, 5.8], $H_c^r(BG, \mathcal{F}) = 0$ for all $r \neq 0$. From [ibid., 5.1], we see that if \mathcal{F} is a sheaf on BG corresponding to the representation V of G , then $H_c^0(BG, \mathcal{F}) = V_G$ and $H^0(BG, \mathcal{F}) = V^G$, and there is a natural isomorphism

$$v \mapsto \sum_{g \in G} gv : V_G \longrightarrow V^G.$$

Therefore

$$h_c^0(BG, \mathcal{F}) = \dim V_G \leq \dim V = \text{rank}(\mathcal{F}),$$

and the weights of V_G form a subset of the weights of V (counted with multiplicities). \square

4.7. (i) If k is a field, by a k -algebraic group G we mean a smooth k -group scheme of finite type. If G is connected, then it is geometrically connected [Grothendieck and Demazure 1970, Exposé VI_A, 2.1.1].

(ii) For a connected k -algebraic group G , let $a : BG \rightarrow \text{Spec } k$ be the structural map. Then

$$a^* : \Lambda\text{-Sh}(\text{Spec } k) \longrightarrow \Lambda\text{-Sh}(BG)$$

is an equivalence of categories. This is because

- BG has no nontrivial stratifications (it is covered by $\text{Spec } k$ which has no nontrivial stratifications), and therefore

- any constructible Λ -adic sheaf on BG is lisse, given by an adic system $(M_n)_n$ of sheaves on $\text{Spec } k$ with G -actions, and these actions are trivial since G is connected, see [Behrend 2003, 5.2.9].

(iii) Let G_0 be a connected \mathbb{F}_q -algebraic group. By a theorem of Lang [1956, Theorem 2], every G_0 -torsor over $\text{Spec } \mathbb{F}_q$ is trivial, with automorphism group G_0 , therefore

$$c_v(BG_0) = \frac{1}{c_v(G_0)} = \frac{1}{\#G_0(\mathbb{F}_{q^v})}.$$

Recall the following theorem of Borel as in [Behrend 2003, 6.1.6].

Theorem 4.8. *Let k be a field and G a connected k -algebraic group. Consider the Leray spectral sequence given by the projection $f : \text{Spec } k \rightarrow BG$,*

$$E_2^{rs} = H^r(BG_{\bar{k}}) \otimes H^s(G_{\bar{k}}) \implies \overline{\mathbb{Q}}_\ell.$$

Let $N^s = E_{s+1}^{0,s} \subset H^s(G_{\bar{k}})$ be the transgressive subspaces, for $s \geq 1$, and let N be the graded $\overline{\mathbb{Q}}_\ell$ -vector space $\bigoplus_{s \geq 1} N^s$. We have:

- (a) $N^s = 0$ if s is even.
- (b) The canonical map $\bigwedge N \rightarrow H^*(G_{\bar{k}})$ is an isomorphism of graded $\overline{\mathbb{Q}}_\ell$ -algebras.
- (c) The spectral sequence above induces an epimorphism of graded $\overline{\mathbb{Q}}_\ell$ -vector spaces $H^*(BG_{\bar{k}}) \twoheadrightarrow N[-1]$. Any section induces an isomorphism

$$\text{Sym}^*(N[-1]) \xrightarrow{\sim} H^*(BG_{\bar{k}}).$$

Remark 4.8.1. (i) The E_2^{rs} -term in Theorem 4.8 should be $H^r(BG_{\bar{k}}, R^s f_* \overline{\mathbb{Q}}_\ell)$, and $R^s f_* \overline{\mathbb{Q}}_\ell$ is a constructible sheaf on BG . By 4.7 (ii), the sheaf $R^s f_* \overline{\mathbb{Q}}_\ell$ is isomorphic to $a^* f^* R^s f_* \overline{\mathbb{Q}}_\ell = a^* H^s(G_{\bar{k}})$, where $a : BG \rightarrow \text{Spec } k$ is the structural map and $H^s(G_{\bar{k}})$ is the $\text{Gal}(k)$ -module regarded as a sheaf on $\text{Spec } k$. Therefore by projection formula, $E_2^{rs} = H^r(BG_{\bar{k}}) \otimes H^s(G_{\bar{k}})$.

(ii) Since the spectral sequence converges to $\overline{\mathbb{Q}}_\ell$ sitting in degree 0, all E_∞^{rs} are zero, except E_∞^{00} . For each $s \geq 1$, consider the differential map

$$d_{s+1}^{0,s} : E_{s+1}^{0,s} \longrightarrow E_{s+1}^{s+1,0}$$

on the $(s+1)$ st page. This map must be injective (respectively surjective) because it is the last possibly nonzero map from $E_*^{0,s}$ (respectively into $E_*^{s+1,0}$). Therefore, it is an isomorphism. Here $N^s = E_{s+1}^{0,s}$ is a subspace of $E_2^{0,s} = H^s(G_{\bar{k}})$, and $E_{s+1}^{s+1,0}$ is a quotient of $E_2^{s+1,0} = H^{s+1}(BG_{\bar{k}})$. We get the surjection $H^{s+1}(BG_{\bar{k}}) \rightarrow N^s$ by using the isomorphism $d_{s+1}^{0,s}$.

4.8.2. Let G_0 be a connected \mathbb{F}_q -algebraic group of dimension d . We intend to apply Theorem 4.8 to investigate the compact support cohomology groups $H_c^*(BG)$.

We have graded Galois-invariant subspaces $N = \bigoplus_{r \geq 1} N^r \subset \bigoplus_{r \geq 0} H^r(G)$ concentrated in odd degrees, such that the induced map

$$\bigwedge N \longrightarrow H^*(G)$$

is an isomorphism, and such that $H^*(BG) \cong \text{Sym}^* N[-1]$. Let $n_r = \dim N^r$, and let v_{r1}, \dots, v_{rn_r} be a basis for N^r with respect to which the Frobenius acting on N^r is upper triangular

$$\begin{bmatrix} \alpha_{r1} & * & * \\ & \ddots & * \\ & & \alpha_{rn_r} \end{bmatrix}$$

with eigenvalues $\alpha_{r1}, \dots, \alpha_{rn_r}$. By Deligne [1980, 3.3.5], the weights of $H^r(G)$ are $\geq r$, so $|\alpha_{ri}| \geq q^{r/2} > 1$. We have

$$H^*(BG) = \text{Sym}^* \overline{\mathbb{Q}}_\ell[v_{ij} \mid \text{for all } i, j] = \overline{\mathbb{Q}}_\ell[v_{ij}],$$

with $\deg(v_{ij}) = i + 1$. Note that all $i + 1$ are even. In particular, $H^{2r-1}(BG) = 0$ and

$$\begin{aligned} H^{2r}(BG) &= \{\text{homogeneous polynomials of degree } 2r \text{ in } v_{ij}\} \\ &= \overline{\mathbb{Q}}_\ell \left\langle \prod_{i,j} v_{ij}^{m_{ij}} ; \sum_{i,j} m_{ij}(i+1) = 2r \right\rangle. \end{aligned}$$

With respect to an appropriate order of the basis, the matrix representing F acting on $H^{2r}(BG)$ is upper triangular, with eigenvalues

$$\prod_{i,j} \alpha_{ij}^{m_{ij}}, \quad \text{for } \sum_{i,j} m_{ij}(i+1) = 2r.$$

By Poincaré duality, the eigenvalues of F acting on $H_c^{-2r-2d}(BG)$ are

$$q^{-d} \prod_{i,j} \alpha_{ij}^{-m_{ij}}, \quad \text{where } \sum_{i,j} m_{ij}(i+1) = 2r.$$

Here $(m_{ij})_{i,j}$ are tuples of nonnegative integers. Therefore the reciprocal characteristic polynomial of F on $H_c^{-2r-2d}(BG)$ is

$$P_{-2r-2d}(BG_0, t) = \prod_{\substack{m_{ij} \geq 0 \\ \sum_{i,j} m_{ij}(i+1) = 2r}} \left(1 - q^{-d} \prod_{i,j} \alpha_{ij}^{-m_{ij}} \cdot t \right).$$

In the following two lemmas we prove two key cases of Theorem 4.2 (i).

Lemma 4.9. *Let G_0 be an \mathbb{F}_q -group scheme of finite type. Then Theorem 4.2 (i) holds for the structural map $f : BG_0 \rightarrow \text{Spec } \mathbb{F}_q$ and any convergent complex $K_0 \in W^-(BG_0, \overline{\mathbb{Q}}_\ell)$.*

Proof. By Olsson [2008a, 7.12–7.14] we may assume that G_0 is reduced (hence smooth), so that the natural projection $\text{Spec } \mathbb{F}_q \rightarrow BG_0$ is a presentation. Note that

then the assumptions of ι -mixedness and stratifiability on K_0 are verified automatically, by Proposition 2.8 and Corollary 3.6 (iii), even though we will not use them explicitly in the proof.

Let G_0^0 be the identity component of G_0 and consider the exact sequence of algebraic groups

$$1 \longrightarrow G_0^0 \longrightarrow G_0 \longrightarrow \pi_0(G_0) \longrightarrow 1.$$

The fibers of the induced map $BG_0 \rightarrow B\pi_0(G_0)$ are isomorphic to BG_0^0 , so we reduce to prove two cases: G_0 is finite étale (or even a finite constant group scheme, by Remark 4.1.2 (iii)), or G_0 is connected and smooth.

Case of G_0 finite constant. Let G_0/\mathbb{F}_q be the finite constant group scheme associated with a finite group G , and let $K_0 \in W^-(BG_0, \overline{\mathbb{Q}}_\ell)$. Again we denote by G both the group scheme $G_0 \otimes \mathbb{F}$ and the finite group $G_0(\mathbb{F})$, if no confusion arises. Let y be the unique point in $\text{Spec } \mathbb{F}_q$,

$$\begin{array}{ccc} BG & \longrightarrow & BG_0 \\ f_{\bar{y}} \downarrow & & \downarrow f \\ \text{Spec } \mathbb{F} & \xrightarrow{\bar{y}} & \text{Spec } \mathbb{F}_q \end{array}$$

is identified with the coinvariance functor

$$(\)_G : D_c^-(\text{Rep}_{\overline{\mathbb{Q}}_\ell} G) \longrightarrow D_c^-(\overline{\mathbb{Q}}_\ell),$$

which is exact on the level of modules, since the category $\text{Rep}_{\overline{\mathbb{Q}}_\ell} G$ is semisimple. So $(f_! K_0)_{\bar{y}} = (f_{\bar{y}})_! K = K_G$ and $\mathcal{H}^n(K_G) = \mathcal{H}^n(K)_G$. Therefore

$$\sum_{\mathcal{H}^n((f_{\bar{y}})_! K), F} |\alpha|^s \leq \sum_{\mathcal{H}^n(K), F} |\alpha|^s$$

for every $n \in \mathbb{Z}$ and $s > 0$. Therefore, if K_0 is a convergent complex, so is $f_! K_0$.

Case of G_0 smooth and connected. In this case

$$f^* : \overline{\mathbb{Q}}_\ell\text{-Sh}(\text{Spec } \mathbb{F}_q) \longrightarrow \overline{\mathbb{Q}}_\ell\text{-Sh}(BG_0)$$

is an equivalence of categories by 4.7 (ii). Let $d = \dim G_0$, and let \mathcal{F}_0 be a sheaf on BG_0 , corresponding to a representation V of the Weil group $W(\mathbb{F}_q)$, with β_1, \dots, β_m as eigenvalues of F . By the projection formula [Laszlo and Olsson 2008b, 9.1.i] we have $H_c^n(BG, \mathcal{F}) \simeq H_c^n(BG) \otimes V$, and by 4.8.2 the eigenvalues of F on $H_c^{-2r-2d}(BG) \otimes V$ are (using the notation in 4.8.2)

$$q^{-d} \beta_k \prod_{i,j} \alpha_{ij}^{-m_{ij}},$$

for $k = 1, \dots, m$ and tuples (m_{ij}) such that $\sum_{i,j} m_{ij}(i+1) = 2r$. For every $s > 0$,

$$\begin{aligned} \sum_{n \in \mathbb{Z}} \sum_{H_c^n(BG) \otimes V, F} |\alpha|^s &= \sum_{m_{ij}, k} q^{-ds} |\beta_k|^s \prod_{i,j} |\alpha_{ij}^{-m_{ij}}|^s \\ &= \left(\sum_{k=1}^m |\beta_k|^s \right) \left(\sum_{m_{ij}} q^{-ds} \prod_{i,j} |\alpha_{ij}^{-m_{ij} s}| \right), \end{aligned}$$

which converges to

$$q^{-ds} \left(\sum_{k=1}^m |\beta_k|^s \right) \prod_{i,j} \frac{1}{1 - |\alpha_{ij}|^{-s}},$$

since $|\alpha_{ij}|^{-s} < 1$ and the product above is taken over finitely many indices.

Let K_0 be a convergent complex on BG_0 , and for each $k \in \mathbb{Z}$, let V_k be a $W(\mathbb{F}_q)$ -module corresponding to $\mathcal{H}^k K_0$. For every $x \in BG_0(\mathbb{F}_q)$ (for instance the trivial G_0 -torsor), the pair $(\mathcal{H}^k(K)_{\bar{x}}, F_x)$ is isomorphic to (V_k, F) . Consider the $W(\mathbb{F}_q)$ -equivariant spectral sequence

$$H_c^r(BG, \mathcal{H}^k(K)) \implies H_c^{r+k}(BG, K).$$

We have

$$\begin{aligned} \sum_{n \in \mathbb{Z}} \sum_{H_c^n(BG, K), F} |\alpha|^s &\leq \sum_{n \in \mathbb{Z}} \sum_{r+k=n} \sum_{H_c^r(BG, \mathcal{H}^k K), F} |\alpha|^s \\ &= \sum_{r, k \in \mathbb{Z}} \sum_{H_c^r(BG) \otimes V_k, F} |\alpha|^s \\ &= \sum_{k \in \mathbb{Z}} \sum_{r \in \mathbb{Z}} \sum_{H_c^r(BG) \otimes V_k, F} |\alpha|^s \\ &= \sum_{k \in \mathbb{Z}} q^{-ds} \left(\sum_{V_k, F} |\alpha|^s \right) \prod_{i,j} \frac{1}{1 - |\alpha_{ij}|^{-s}} \\ &= \left(\sum_{k \in \mathbb{Z}} \sum_{V_k, F} |\alpha|^s \right) \left(q^{-ds} \prod_{i,j} \frac{1}{1 - |\alpha_{ij}|^{-s}} \right), \end{aligned}$$

where the first factor is convergent by assumption, and the product in the second factor is taken over finitely many indices. This shows that $f_! K_0$ is a convergent complex. \square

Let E_λ be a finite subextension of $\overline{\mathbb{Q}_\ell}/\mathbb{Q}_\ell$ with ring of integers Λ and residue field Λ_0 , and let $(\mathcal{S}, \mathcal{L})$ be a pair on \mathcal{X} defined by simple lcc Λ_0 -sheaves on strata. A complex $K_0 \in W(\mathcal{X}_0, \overline{\mathbb{Q}_\ell})$ is said to be $(\mathcal{S}, \mathcal{L})$ -stratifiable (or trivialized by $(\mathcal{S}, \mathcal{L})$), if K is defined over E_λ , with an integral model over Λ trivialized by $(\mathcal{S}, \mathcal{L})$.

Lemma 4.10. *Let X_0/\mathbb{F}_q be a geometrically connected variety, E_λ a finite subextension of $\overline{\mathbb{Q}_\ell}/\mathbb{Q}_\ell$ with ring of integers Λ , and let \mathcal{L} be a finite set of simple lcc Λ_0 -sheaves on X . Then Theorem 4.2 (i) holds for the structural morphism $f : X_0 \rightarrow \text{Spec } \mathbb{F}_q$ and all lisse ι -mixed convergent complexes K_0 on X_0 that are trivialized by $(\{X\}, \mathcal{L})$.*

Proof. Let $N = \dim X_0$. From the spectral sequence

$$E_2^{r,k} = H_c^r(X, \mathcal{H}^k K) \implies H_c^{r+k}(X, K)$$

we see that

$$\sum_{n \in \mathbb{Z}} \sum_{H_c^n(X, K), F} |\alpha|^s \leq \sum_{n \in \mathbb{Z}} \sum_{r+k=n} \sum_{H_c^r(X, \mathcal{H}^k K), F} |\alpha|^s = \sum_{\substack{0 \leq r \leq 2N \\ k \in \mathbb{Z}}} \sum_{H_c^r(X, \mathcal{H}^k K), F} |\alpha|^s,$$

therefore it suffices to show that the series $\sum_{k \in \mathbb{Z}} \sum_{H_c^r(X, \mathcal{H}^k K), F} |\alpha|^s$ converges for each $0 \leq r \leq 2N$.

Let d be the number in Lemma 3.10 for \mathcal{L} . Each cohomology sheaf $\mathcal{H}^n K_0$ is ι -mixed and lisse on X_0 , so by Theorem 2.6 (i) we have the decomposition

$$\mathcal{H}^n K_0 = \bigoplus_{b \in \mathbb{R}/\mathbb{Z}} (\mathcal{H}^n K_0)(b)$$

according to the weights mod \mathbb{Z} , defined over E_λ , see Remark 2.6.1 (ii). For each coset b , we choose a representative $b_0 \in b$, and take a $b_1 \in \overline{\mathbb{Q}_\ell}^*$ such that $w_q(b_1) = -b_0$. Then the sheaf $(\mathcal{H}^n K_0)(b)^{(b_1)}$ deduced by twist is lisse with integer punctual weights. Let W be the filtration by punctual weights Theorem 2.6 (ii) of $(\mathcal{H}^n K_0)(b)^{(b_1)}$. For every $v \geq 1$ and $x \in X_0(\mathbb{F}_{q^v})$, and every real $s > 0$, we have

$$\begin{aligned} \sum_{n \in \mathbb{Z}} \sum_{(\mathcal{H}^n K_0)_{\bar{x}}, F_x} |\alpha|^{s/v} &= \sum_{\substack{n \in \mathbb{Z} \\ b \in \mathbb{R}/\mathbb{Z}}} \sum_{(\mathcal{H}^n K_0)(b)_{\bar{x}}, F_x} |\alpha|^{s/v} \\ &= \sum_{\substack{n \in \mathbb{Z} \\ b \in \mathbb{R}/\mathbb{Z}}} \sum_{(\mathcal{H}^n K_0)(b)_{\bar{x}}^{(b_1)}, F_x} |\alpha/b_1^v|^{s/v} \\ &= \sum_{\substack{n \in \mathbb{Z} \\ b \in \mathbb{R}/\mathbb{Z}}} q^{b_0 s/2} \sum_{(\mathcal{H}^n K_0)(b)_{\bar{x}}^{(b_1)}, F_x} |\alpha|^{s/v} \\ &= \sum_{\substack{n \in \mathbb{Z} \\ b \in \mathbb{R}/\mathbb{Z}}} q^{b_0 s/2} \sum_{i \in \mathbb{Z}} \text{Gr}_i^W((\mathcal{H}^n K_0)(b)_{\bar{x}}^{(b_1)}, F_x) |\alpha|^{s/v} \\ &= \sum_{\substack{n \in \mathbb{Z} \\ b \in \mathbb{R}/\mathbb{Z}}} q^{b_0 s/2} \sum_{i \in \mathbb{Z}} q^{i s/2} \cdot \text{rank Gr}_i^W((\mathcal{H}^n K_0)(b)_{\bar{x}}^{(b_1)}, F_x). \end{aligned}$$

Since K_0 is a convergent complex, this series is convergent.

For each $n \in \mathbb{Z}$, every direct summand $(\mathcal{H}^n K_0)(b)$ of $\mathcal{H}^n K_0$ is trivialized by $(\{X\}, \mathcal{L})$. The filtration W of each $(\mathcal{H}^n K_0)(b)^{(b_1)}$ gives a filtration of $(\mathcal{H}^n K_0)(b)$ (also denoted W) by twisting back, and it is clear that this latter filtration is defined over E_λ . We have $\mathrm{Gr}_i^W((\mathcal{H}^n K_0)(b)^{(b_1)}) = (\mathrm{Gr}_i^W(\mathcal{H}^n K_0)(b))^{(b_1)}$, and each $\mathrm{Gr}_i^W(\mathcal{H}^n K_0)(b)$ is trivialized by $(\{X\}, \mathcal{L})$. By Lemma 3.10,

$$\begin{aligned} h_c^r(X, \mathrm{Gr}_i^W((\mathcal{H}^n K)(b)^{(b_1)})) &= h_c^r(X, \mathrm{Gr}_i^W(\mathcal{H}^n K)(b)) \\ &\leq d \cdot \mathrm{rank} \mathrm{Gr}_i^W(\mathcal{H}^n K)(b) \\ &= d \cdot \mathrm{rank} \mathrm{Gr}_i^W((\mathcal{H}^n K)(b)^{(b_1)}), \end{aligned}$$

where the first equality follows from [Laszlo and Olsson 2008b, 9.1.i]. Therefore

$$\begin{aligned} \sum_{n \in \mathbb{Z}} \sum_{H_c^r(X, \mathcal{H}^n K), F} |\alpha|^s &= \sum_{\substack{n \in \mathbb{Z} \\ b \in \mathbb{R}/\mathbb{Z}}} \sum_{H_c^r(X, (\mathcal{H}^n K)(b)), F} |\alpha|^s \\ &= \sum_{\substack{n \in \mathbb{Z} \\ b \in \mathbb{R}/\mathbb{Z}}} \sum_{H_c^r(X, (\mathcal{H}^n K)(b)^{(b_1)}, F} |b_1^{-1} \alpha|^s \\ &\leq \sum_{\substack{n \in \mathbb{Z} \\ b \in \mathbb{R}/\mathbb{Z}}} q^{b_0 s/2} \sum_{i \in \mathbb{Z}} \sum_{H_c^r(X, \mathrm{Gr}_i^W((\mathcal{H}^n K)(b)^{(b_1)}), F} |\alpha|^s \\ &\leq \sum_{\substack{n \in \mathbb{Z} \\ b \in \mathbb{R}/\mathbb{Z}}} q^{b_0 s/2} \sum_{i \in \mathbb{Z}} q^{(i+r)s/2} \cdot h_c^r(X, \mathrm{Gr}_i^W((\mathcal{H}^n K)(b)^{(b_1)})) \\ &\leq q^{r s/2} d \sum_{\substack{n \in \mathbb{Z} \\ b \in \mathbb{R}/\mathbb{Z}}} q^{b_0 s/2} \sum_{i \in \mathbb{Z}} q^{i s/2} \cdot \mathrm{rank} \mathrm{Gr}_i^W((\mathcal{H}^n K)(b)^{(b_1)}), \end{aligned}$$

and it converges. □

Now we prove Theorem 4.2 (i) in general.

Proof. We may assume all stacks involved are reduced. From Theorem 2.11 and Theorem 3.9 we know that $f_! K_0 \in W_m^{-, \mathrm{stra}}(\mathcal{Y}_0, \overline{\mathbb{Q}}_\ell)$.

Let $y \in \mathcal{Y}_0(\mathbb{F}_{q^v})$, we want to show that $((f_! K_0)_{\overline{y}}, F_y)$ is a convergent complex. Since the property of being convergent depends only on the cohomology sheaves, by base change [Laszlo and Olsson 2008b, 12.5.3] we reduce to the case when $\mathcal{Y}_0 = \mathrm{Spec} \mathbb{F}_{q^v}$. Replacing q by q^v , we may assume $v = 1$. By Remark 4.1.2 (iii) we only need to show that $(R\Gamma_c(\mathcal{X}, K), F)$ is convergent.

If $j : \mathcal{U}_0 \hookrightarrow \mathcal{X}_0$ is an open substack with complement $i : \mathcal{Z}_0 \hookrightarrow \mathcal{X}_0$, then we have an exact triangle

$$j_! j^* K_0 \longrightarrow K_0 \longrightarrow i_* i^* K_0 \longrightarrow$$

in $W^-(\mathcal{X}_0, \overline{\mathbb{Q}}_\ell)$, which induces an exact triangle

$$R\Gamma_c(\mathcal{U}_0, j^*K_0) \longrightarrow R\Gamma_c(\mathcal{X}_0, K_0) \longrightarrow R\Gamma_c(\mathcal{Z}_0, i^*K_0) \longrightarrow$$

in $W^-(\text{Spec } \mathbb{F}_q, \overline{\mathbb{Q}}_\ell)$. So by Corollary 4.4 and noetherian induction, it suffices to prove Theorem 4.2 (i) for a nonempty open substack. By [Behrend 2003, 5.1.14] we may assume that the inertia stack \mathcal{I}_0 is flat over \mathcal{X}_0 . Then we can form the rigidification $\pi : \mathcal{X}_0 \rightarrow X_0$ with respect to \mathcal{I}_0 [Olsson 2008b, §1.5], where X_0 is an \mathbb{F}_q -algebraic space of quasi-compact diagonal. X_0 contains an open dense subscheme [Knutson 1971, II, 6.7]. Replacing \mathcal{X}_0 by the inverse image of this scheme, we can assume X_0 is a scheme.

If Theorem 4.2 (i) holds for two composable morphisms f and g , then it holds for their composition $g \circ f$. Since $R\Gamma_c(\mathcal{X}_0, -) = R\Gamma_c(X_0, -) \circ \pi_!$, we reduce to proving Theorem 4.2 (i) for these two morphisms. For every $x \in X_0(\mathbb{F}_{q^v})$, the fiber of π over x is a gerbe over $\text{Spec } k(x)$. Extending the base $k(x)$ (see Remark 4.1.2 (iii)) one can assume it is a neutral gerbe (in fact all gerbes over a finite field are neutral; see [Behrend 2003, 6.4.2]). This means the following diagram is 2-Cartesian:

$$\begin{array}{ccc} B \text{Aut}_x & \longrightarrow & \mathcal{X}_0 \\ \downarrow & & \downarrow \pi \\ \text{Spec } \mathbb{F}_{q^v} & \xrightarrow{x} & X_0. \end{array}$$

So we reduce to two cases: $\mathcal{X}_0 = BG_0$ for an \mathbb{F}_q -algebraic group G_0 , or $\mathcal{X}_0 = X_0$ is an \mathbb{F}_q -scheme. The first case is proved in Lemma 4.9.

For the second case, given a convergent complex $K_0 \in W_m^{-, \text{strat}}(X_0, \overline{\mathbb{Q}}_\ell)$, defined over some E_λ with ring of integers Λ , and trivialized by a pair $(\mathcal{S}, \mathcal{L})$ (\mathcal{L} being defined over Λ_0) on X , we can refine this pair so that every stratum is connected, and replace X_0 by models of the strata defined over some finite extension of \mathbb{F}_q Remark 4.1.2 (iii). This case is proved in Lemma 4.10. \square

5. Trace formula for stacks

We prove two special cases of Theorem 4.2 (ii) in the following two propositions.

Proposition 5.1. *Let G_0 be a finite étale group scheme over \mathbb{F}_q , and \mathcal{F}_0 a sheaf on BG_0 . Then*

$$c_1(BG_0, \mathcal{F}_0) = c_1(\text{Spec } \mathbb{F}_q, R\Gamma_c(BG_0, \mathcal{F}_0)).$$

Proof. This is a special case of [Olsson 2008a, 8.6] on correspondences given by group homomorphisms. \square

Proposition 5.2. *Let G_0 be a connected \mathbb{F}_q -algebraic group, and let \mathcal{F}_0 be a sheaf on BG_0 . Then*

$$c_1(BG_0, \mathcal{F}_0) = c_1(\text{Spec } \mathbb{F}_q, R\Gamma_c(BG_0, \mathcal{F}_0)).$$

Proof. Let $f : BG_0 \rightarrow \text{Spec } \mathbb{F}_q$ be the structural map and $d = \dim G_0$. By 4.7 (ii), the sheaf \mathcal{F}_0 on BG_0 takes the form f^*V , for some sheaf V on $\text{Spec } \mathbb{F}_q$. By 4.7 (iii), we have

$$c_1(BG_0, \mathcal{F}_0) = \frac{1}{\#G_0(\mathbb{F}_q)} \text{Tr}(F_x, \mathcal{F}_x) = \frac{\text{Tr}(F, V)}{\#G_0(\mathbb{F}_q)}.$$

By the projection formula we have $H_c^n(BG, \mathcal{F}) \simeq H_c^n(BG) \otimes V$, so

$$\text{Tr}(F, H_c^n(BG, \mathcal{F})) = \text{Tr}(F, H_c^n(BG)) \cdot \text{Tr}(F, V).$$

Then

$$\begin{aligned} c_1(\text{Spec } \mathbb{F}_q, R\Gamma_c(BG_0, \mathcal{F}_0)) &= \sum_n (-1)^n \text{Tr}(F, H_c^n(BG, \mathcal{F})) \\ &= \text{Tr}(F, V) \sum_n (-1)^n \text{Tr}(F, H_c^n(BG)), \end{aligned}$$

so we can assume $\mathcal{F}_0 = \overline{\mathbb{Q}}_\ell$. Using the notations in 4.8.2 we have

$$\begin{aligned} \sum_n (-1)^n \text{Tr}(F, H_c^n(BG)) &= \sum_{r \geq 0} \text{Tr}(F, H_c^{-2r-2d}(BG)) \\ &= \sum_{r \geq 0} \sum_{\substack{m_{ij}(i+1)=2r \\ m_{ij} \geq 0}} q^{-d} \prod_{i,j} \alpha_{ij}^{-m_{ij}} = q^{-d} \sum_{m_{ij} \geq 0} \prod_{i,j} \alpha_{ij}^{-m_{ij}} \\ &= q^{-d} \prod_{i,j} (1 + \alpha_{ij}^{-1} + \alpha_{ij}^{-2} + \cdots) = q^{-d} \prod_{i,j} \frac{1}{1 - \alpha_{ij}^{-1}}. \end{aligned}$$

It remains to show that

$$\#G_0(\mathbb{F}_q) = q^d \prod_{i,j} (1 - \alpha_{ij}^{-1}).$$

In 4.8.2, we saw that if each N^i has an ordered basis $v_{i_1}, \dots, v_{i_{n_i}}$ with respect to which F is upper triangular, then since $H^*(G) = \bigwedge N$, $H^i(G)$ has a basis

$$v_{i_1 j_1} \wedge v_{i_2 j_2} \wedge \cdots \wedge v_{i_m j_m},$$

such that $\sum_{r=1}^m i_r = i$, $i_r \leq i_{r+1}$, and if $i_r = i_{r+1}$, then $j_r < j_{r+1}$. The eigenvalues of F on $H^i(G)$ are $\alpha_{i_1 j_1} \cdots \alpha_{i_m j_m}$ for such indices. By Poincaré duality, the eigenvalues of F on $H_c^{2d-i}(G)$ are $q^d (\alpha_{i_1 j_1} \cdots \alpha_{i_m j_m})^{-1}$. Note that all the i_r are odd, so

$$2d - i \equiv i = \sum_{r=1}^m i_r \equiv m \pmod{2}.$$

Applying the classical trace formula to G_0 , we have

$$\#G_0(\mathbb{F}_q) = \sum (-1)^m q^d \alpha_{i_1 j_1}^{-1} \cdots \alpha_{i_m j_m}^{-1} = q^d \prod_{i,j} (1 - \alpha_{ij}^{-1}).$$

This finishes the proof. \square

5.2.1. Note that in Propositions 5.1 and 5.2 we did not make explicit use of the fact that \mathcal{F}_0 is ι -mixed.

Now we prove Theorem 4.2 (ii) in general.

Proof. Since $c_v(\mathcal{X}_0, K_0) = c_1(\mathcal{X}_0 \otimes \mathbb{F}_{q^v}, K_0 \otimes \mathbb{F}_{q^v})$, we can assume $v = 1$. We shall reduce to proving Theorem 4.2 (ii) for all fibers of f over \mathbb{F}_q -points of \mathcal{Y}_0 , following the approach of Behrend [2003, 6.4.9].

Let $y \in \mathcal{Y}_0(\mathbb{F}_q)$ and $(\mathcal{X}_0)_y$ be the fiber over y . Then $(\mathcal{X}_0)_y(\mathbb{F}_q)$ is the groupoid of pairs (x, α) , where $x \in \mathcal{X}_0(\mathbb{F}_q)$ and $\alpha : f(x) \rightarrow y$ is an isomorphism in $\mathcal{Y}_0(\mathbb{F}_q)$. Suppose $(\mathcal{X}_0)_y(\mathbb{F}_q) \neq \emptyset$, and fix an $x \in (\mathcal{X}_0)_y(\mathbb{F}_q)$. Then $\text{Isom}(f(x), y)(\mathbb{F}_q)$ is a trivial left $\text{Aut}_y(\mathbb{F}_q)$ -torsor. There is also a natural right action of $\text{Aut}_x(\mathbb{F}_q)$ on $\text{Isom}(f(x), y)(\mathbb{F}_q)$, namely $\varphi \in \text{Aut}_x(\mathbb{F}_q)$ takes α to $\alpha \circ f(\varphi)$. Under this action, for α and α' to be in the same orbit, there should be a $\varphi \in \text{Aut}_x(\mathbb{F}_q)$ such that the diagram

$$\begin{array}{ccc} f(x) & \xrightarrow{f(\varphi)} & f(x) \\ & \searrow \alpha' & \swarrow \alpha \\ & y & \end{array}$$

commutes; by definition this means (x, α) is isomorphic to (x, α') in $(\mathcal{X}_0)_y(\mathbb{F}_q)$. So the set of orbits $\text{Isom}(f(x), y)(\mathbb{F}_q) / \text{Aut}_x(\mathbb{F}_q)$ is identified with the inverse image of the class of x along the map $[(\mathcal{X}_0)_y(\mathbb{F}_q)] \rightarrow [\mathcal{X}_0(\mathbb{F}_q)]$. The stabilizer group of $\alpha \in \text{Isom}(f(x), y)(\mathbb{F}_q)$ is $\text{Aut}_{(x,\alpha)}(\mathbb{F}_q)$, the automorphism group of (x, α) in $(\mathcal{X}_0)_y(\mathbb{F}_q)$.

In general, if a finite group G acts on a finite set S , then we have

$$\sum_{[x] \in S/G} \frac{\#G}{\#\text{Stab}_G(x)} = \sum_{[x] \in S/G} \#\text{Orb}_G(x) = \#S.$$

Now for $S = \text{Isom}(f(x), y)(\mathbb{F}_q)$ and $G = \text{Aut}_x(\mathbb{F}_q)$, we have

$$\sum_{\substack{(x,\alpha) \in [(\mathcal{X}_0)_y(\mathbb{F}_q)] \\ (x,\alpha) \mapsto x}} \frac{\#\text{Aut}_x(\mathbb{F}_q)}{\#\text{Aut}_{(x,\alpha)}(\mathbb{F}_q)} = \#\text{Isom}(f(x), y)(\mathbb{F}_q) = \#\text{Aut}_y(\mathbb{F}_q);$$

the last equality follows from the fact that S is a trivial $\text{Aut}_y(\mathbb{F}_q)$ -torsor.

If we assume Theorem 4.2 (ii) holds for the fibers $f_y : (\mathcal{X}_0)_y \rightarrow \text{Spec } \mathbb{F}_q$ of f , for all $y \in \mathcal{Y}_0(\mathbb{F}_q)$, then

$$\begin{aligned}
c_1(\mathcal{Y}_0, f!K_0) &= \sum_{y \in [\mathcal{Y}_0(\mathbb{F}_q)]} \frac{\text{Tr}(F_y, (f!K)_{\bar{y}})}{\#\text{Aut}_y(\mathbb{F}_q)} \\
&= \sum_{y \in [\mathcal{Y}_0(\mathbb{F}_q)]} \frac{\text{Tr}(F_y, (f_y!K)_{\bar{y}})}{\#\text{Aut}_y(\mathbb{F}_q)} \\
&= \sum_{y \in [\mathcal{Y}_0(\mathbb{F}_q)]} \frac{1}{\#\text{Aut}_y(\mathbb{F}_q)} \sum_{(x, \alpha) \in [(\mathcal{X}_0)_y(\mathbb{F}_q)]} \frac{\text{Tr}(F_x, K_{\bar{x}})}{\#\text{Aut}_{(x, \alpha)}(\mathbb{F}_q)} \\
&= \sum_{y \in [\mathcal{Y}_0(\mathbb{F}_q)]} \frac{1}{\#\text{Aut}_y(\mathbb{F}_q)} \sum_{\substack{x \in [\mathcal{X}_0(\mathbb{F}_q)] \\ x \mapsto y}} \left(\sum_{\substack{(x, \alpha) \in [(\mathcal{X}_0)_y(\mathbb{F}_q)] \\ (x, \alpha) \mapsto x}} \frac{\text{Tr}(F_x, K_{\bar{x}})}{\#\text{Aut}_{(x, \alpha)}(\mathbb{F}_q)} \right) \\
&= \sum_{y \in [\mathcal{Y}_0(\mathbb{F}_q)]} \frac{1}{\#\text{Aut}_y(\mathbb{F}_q)} \sum_{\substack{x \in [\mathcal{X}_0(\mathbb{F}_q)] \\ x \mapsto y}} \frac{1}{\#\text{Aut}_x(\mathbb{F}_q)} \\
&\quad \left(\sum_{\substack{(x, \alpha) \in [(\mathcal{X}_0)_y(\mathbb{F}_q)] \\ (x, \alpha) \mapsto x}} \frac{\#\text{Aut}_x(\mathbb{F}_q)}{\#\text{Aut}_{(x, \alpha)}(\mathbb{F}_q)} \right) \text{Tr}(F_x, K_{\bar{x}}) \\
&= \sum_{y \in [\mathcal{Y}_0(\mathbb{F}_q)]} \frac{1}{\#\text{Aut}_y(\mathbb{F}_q)} \sum_{\substack{x \in [\mathcal{X}_0(\mathbb{F}_q)] \\ x \mapsto y}} \frac{\text{Tr}(F_x, K_{\bar{x}})}{\#\text{Aut}_x(\mathbb{F}_q)} \#\text{Aut}_y(\mathbb{F}_q) \\
&= \sum_{x \in [\mathcal{X}_0(\mathbb{F}_q)]} \frac{\text{Tr}(F_x, K_{\bar{x}})}{\#\text{Aut}_x(\mathbb{F}_q)} =: c_1(\mathcal{X}_0, K_0).
\end{aligned}$$

Here the second equality follows from [Laszlo and Olsson 2008b, 12.5.3]. Thus we reduce to the case when $\mathcal{Y}_0 = \text{Spec } \mathbb{F}_q$.

If $K'_0 \rightarrow K_0 \rightarrow K''_0 \rightarrow K'_0[1]$ is an exact triangle of convergent complexes in $W_m^{-, \text{stra}}(\mathcal{X}_0, \overline{\mathbb{Q}}_\ell)$, then by Corollary 4.4 and Theorem 4.2 (i) we have

$$\begin{aligned}
c_1(\mathcal{X}_0, K_0) &= c_1(\mathcal{X}_0, K'_0) + c_1(\mathcal{X}_0, K''_0) \quad \text{and} \\
c_1(\mathcal{Y}_0, f!K_0) &= c_1(\mathcal{Y}_0, f!K'_0) + c_1(\mathcal{Y}_0, f!K''_0).
\end{aligned}$$

If $j : \mathcal{U}_0 \rightarrow \mathcal{X}_0$ is an open substack with complement $i : \mathcal{Z}_0 \rightarrow \mathcal{X}_0$, then

$$c_1(\mathcal{X}_0, j!j^*K_0) = c_1(\mathcal{U}_0, j^*K_0) \quad \text{and} \quad c_1(\mathcal{X}_0, i_*i^*K_0) = c_1(\mathcal{Z}_0, i^*K_0).$$

By noetherian induction we can shrink \mathcal{X}_0 to a nonempty open substack. So we may assume the inertia stack \mathcal{I}_0 is flat over \mathcal{X}_0 , with rigidification $\pi : \mathcal{X}_0 \rightarrow X_0$, where X_0 is a scheme. If Theorem 4.2 (ii) holds for two composable morphisms f and g , then it holds for $g \circ f$. So we reduce to two cases as before: $\mathcal{X}_0 = X_0$

is a scheme, or $\mathcal{X}_0 = BG_0$, where G_0 is either a connected algebraic group, or a finite étale algebraic group over \mathbb{F}_q . We may assume X_0 is separated, by further shrinking (for instance to an affine open subscheme).

For a complex of sheaves K_0 and an integer n , we have an exact triangle

$$\tau_{<n}K_0 \longrightarrow \tau_{<n+1}K_0 \longrightarrow \mathcal{H}^n(K_0)[-n] \longrightarrow,$$

so

$$\begin{aligned} c_1(\tau_{<n+1}K_0) &= c_1(\tau_{<n}K_0) + c_1(\mathcal{H}^n(K_0)[-n]) \\ &= c_1(\tau_{<n}K_0) + (-1)^n c_1(\mathcal{H}^n(K_0)). \end{aligned}$$

Since K_0 is bounded above, $\tau_{<N}K_0 \simeq K_0$ for $N \gg 0$. Since K_0 is convergent, $c_1(\tau_{<n}K_0) \rightarrow 0$ absolutely as $n \rightarrow -\infty$, so the series $\sum_{n \in \mathbb{Z}} (-1)^n c_1(\mathcal{H}^n(K_0))$ converges absolutely to $c_1(K_0)$.

Applying $R\Gamma_c$ we get an exact triangle

$$R\Gamma_c(\mathcal{X}_0, \tau_{<n}K_0) \longrightarrow R\Gamma_c(\mathcal{X}_0, \tau_{<n+1}K_0) \longrightarrow R\Gamma_c(\mathcal{X}_0, \mathcal{H}^n K_0)[-n] \longrightarrow$$

in $W^-(\text{Spec } \mathbb{F}_q, \overline{\mathbb{Q}}_\ell)$. We claim that, for $\mathcal{X}_0 = X_0$ a scheme, or BG_0 , we have

$$\lim_{n \rightarrow -\infty} c_1(\text{Spec } \mathbb{F}_q, R\Gamma_c(\mathcal{X}_0, \tau_{<n}K_0)) = 0$$

absolutely. Recall that $c_1(R\Gamma_c(\tau_{<n}K_0)) = \sum_{i \in \mathbb{Z}} (-1)^i \text{Tr}(F, H_c^i(\mathcal{X}, \tau_{<n}K))$, so we need to show that

$$\sum_{i \in \mathbb{Z}} \sum_{H_c^i(\mathcal{X}, \tau_{<n}K), F} |\alpha| \rightarrow 0 \quad \text{as } n \rightarrow -\infty.$$

From the spectral sequence

$$H_c^r(\mathcal{X}, \mathcal{H}^k \tau_{<n}K) \implies H_c^{r+k}(\mathcal{X}, \tau_{<n}K)$$

we see that

$$\sum_{i \in \mathbb{Z}} \sum_{H_c^i(\mathcal{X}, \tau_{<n}K), F} |\alpha| \leq \sum_{i \in \mathbb{Z}} \sum_{r+k=i} \sum_{H_c^r(\mathcal{X}, \mathcal{H}^k \tau_{<n}K), F} |\alpha| = \sum_{i \in \mathbb{Z}} \sum_{\substack{r+k=i \\ k < n}} \sum_{H_c^r(\mathcal{X}, \mathcal{H}^k K), F} |\alpha|.$$

Let $d = \dim \mathcal{X}_0$ (see 9.1). In the cases where \mathcal{X}_0 is a scheme or BG_0 , we have $H_c^r(\mathcal{X}, \mathcal{F}) = 0$ for every sheaf \mathcal{F} unless $r \leq 2d$ (see 4.8.2 and Lemma 4.6). Therefore

$$\sum_{\substack{r+k=i \\ k < n}} \sum_{H_c^r(\mathcal{X}, \mathcal{H}^k K), F} |\alpha| \leq \sum_{i < n+2d} \sum_{r+k=i} \sum_{H_c^r(\mathcal{X}, \mathcal{H}^k K), F} |\alpha|,$$

and it suffices to show that the series

$$\sum_{i \in \mathbb{Z}} \sum_{r+k=i} \sum_{H_c^r(\mathcal{X}, \mathcal{H}^k K), F} |\alpha|$$

converges. We already proved this for BG_0 in Lemma 4.9, and for schemes X_0 in Lemma 4.10 (we may shrink X_0 so that the assumption in Lemma 4.10 is satisfied).

Note that in the two cases $\mathcal{X}_0 = X_0$ or BG_0 , Theorem 4.2 (ii) holds when K_0 is a sheaf concentrated in degree 0. For separated schemes X_0 , this is a classical result of Grothendieck [1965] and Verdier [1967]; for BG_0 , this is done in Propositions 5.1 and 5.2. Therefore, for a general convergent complex K_0 , we have

$$\begin{aligned} c_1(R\Gamma_c(\tau_{<n+1}K_0)) &= c_1(R\Gamma_c(\tau_{<n}K_0)) + c_1(R\Gamma_c(\mathcal{H}^n K_0)[-n]) \\ &= c_1(R\Gamma_c(\tau_{<n}K_0)) + (-1)^n c_1(\mathcal{H}^n K_0), \end{aligned}$$

and so

$$c_1(R\Gamma_c(K_0)) = \sum_{n \in \mathbb{Z}} (-1)^n c_1(\mathcal{H}^n K_0) + \lim_{n \rightarrow -\infty} c_1(R\Gamma_c(\tau_{<n}K_0)) = c_1(K_0). \quad \square$$

Corollary 5.3. *Let $f : \mathcal{X}_0 \rightarrow \mathcal{Y}_0$ be a morphism of \mathbb{F}_q -algebraic stacks, and let $K_0 \in W_m^{-, \text{stra}}(\mathcal{X}_0, \overline{\mathbb{Q}}_\ell)$ be a convergent complex of sheaves. Then*

$$L(\mathcal{X}_0, K_0, t) = L(\mathcal{Y}_0, f_! K_0, t).$$

6. Infinite products

For a convergent complex K_0 on \mathcal{X}_0 , the series $\sum_{v \geq 1} c_v(K_0)t^v/v$ (and hence the L -series $L(\mathcal{X}_0, K_0, t)$) usually has a finite radius of convergence. For instance, we have the following lemma.

Lemma 6.1. *Let X_0/\mathbb{F}_q be a variety of dimension d . Then the radius of convergence of $\sum_{v \geq 1} c_v(X_0)t^v/v$ is $1/q^d$.*

Proof. Let $f_{X_0}(t) = \sum_{v \geq 1} c_v(X_0)t^v/v$. Let Y_0 be an irreducible component of X_0 with complement U_0 . Then $c_v(X_0) = c_v(Y_0) + c_v(U_0)$, and since all the c_v -terms are nonnegative, we see that the radius of convergence of $f_{X_0}(t)$ is the minimum of that of $f_{Y_0}(t)$ and that of $f_{U_0}(t)$. Since $\max\{\dim(Y_0), \dim(U_0)\} = d$, and U_0 has fewer irreducible component than X_0 , by induction we can assume X_0 is irreducible.

Then there exists an open dense subscheme $U_0 \subset X_0$ that is smooth over $\text{Spec } \mathbb{F}_q$. Let $Z_0 = X_0 - U_0$, then $\dim(Z_0) < \dim(X_0) = d$. From the cohomology sequence

$$H_c^{2d-1}(Z) \longrightarrow H_c^{2d}(U) \longrightarrow H_c^{2d}(X) \longrightarrow H_c^{2d}(Z)$$

we see that $H_c^{2d}(X) = H_c^{2d}(U) = \overline{\mathbb{Q}}_\ell(-d)$. The Frobenius eigenvalues $\{\alpha_{ij}\}_j$ on $H_c^i(X)$ have ι -weights $\leq i$, for $0 \leq i < 2d$ [Deligne 1980, 3.3.4]. By the fixed point

formula,

$$\begin{aligned} \frac{c_v(X_0)}{c_{v+1}(X_0)} &= \frac{q^{vd} + \sum_{0 \leq i < 2d} (-1)^i \sum_j \alpha_{ij}^v}{q^{(v+1)d} + \sum_{0 \leq i < 2d} (-1)^i \sum_j \alpha_{ij}^{v+1}} \\ &= \frac{\frac{1}{q^d} + \frac{1}{q^d} \sum_{0 \leq i < 2d} (-1)^i \sum_j \left(\frac{\alpha_{ij}}{q^d}\right)^v}{1 + \sum_{0 \leq i < 2d} (-1)^i \sum_j \left(\frac{\alpha_{ij}}{q^d}\right)^{v+1}}, \end{aligned}$$

which converges to $1/q^d$ as $v \rightarrow \infty$, therefore the radius of convergence of $f_{X_0}(t)$ is

$$\lim_{v \rightarrow \infty} \frac{c_v(X_0)/v}{c_{v+1}(X_0)/(v+1)} = \frac{1}{q^d}. \quad \square$$

In order to prove the meromorphic continuation of Theorem 8.1, we want to express the L -series as a possibly infinite product. For schemes, if we consider only bounded complexes, the L -series can be expressed as a finite alternating product of polynomials $P_n(X_0, K_0, t)$, so it is rational [Grothendieck 1965]. In the stack case, even for the sheaf \mathbb{Q}_ℓ , there might be infinitely many nonzero compact cohomology groups, and we need to consider the issue of convergence of the coefficients in an infinite products.

Definition 6.2. Let $f_n(t) = \sum_{k \geq 0} a_{nk} t^k \in \mathbb{C}[[t]]$ be a sequence of power series over \mathbb{C} . The sequence is said to be convergent term by term, if for each k , the sequence $(a_{nk})_n$ converges, and the series

$$\lim_{n \rightarrow \infty} f_n(t) := \sum_{k \geq 0} t^k \lim_{n \rightarrow \infty} a_{nk}$$

is called the limit of the sequence $(f_n(t))_n$.

6.2.1. Strictly speaking, a series (respectively infinite product) is defined to be a sequence $(a_n)_n$, usually written as an “infinite sum” (respectively “infinite product”) so that $(a_n)_n$ is the sequence of finite partial sums (respectively finite partial products) of it. So the definition above applies to series and infinite products.

Recall that $\log(1 + g) = \sum_{m \geq 1} (-1)^{m+1} g^m / m$ for $g \in t\mathbb{C}[[t]]$.

Lemma 6.3. (i) Let $f_n(t) = 1 + \sum_{k \geq 1} a_{nk} t^k \in \mathbb{C}[[t]]$ be a sequence of power series. Then $(f_n(t))_n$ is convergent term by term if and only if $(\log f_n(t))_n$ is convergent term by term, and

$$\lim_{n \rightarrow \infty} \log f_n(t) = \log \lim_{n \rightarrow \infty} f_n(t).$$

(ii) Let f and g be two power series with constant term 1. Then

$$\log(fg) = \log(f) + \log(g).$$

(iii) Let $f_n(t) \in 1 + t\mathbb{C}[[t]]$ be a sequence as in (i). Then the infinite product $\prod_{n \geq 1} f_n(t)$ converges term by term if and only if the series $\sum_{n \geq 1} \log f_n(t)$ converges term by term, and

$$\sum_{n \geq 1} \log f_n(t) = \log \prod_{n \geq 1} f_n(t).$$

Proof. (i) We have

$$\begin{aligned} \log f_n(t) &= \sum_{m \geq 1} (-1)^{m+1} \left(\sum_{k \geq 1} a_{nk} t^k \right)^m / m \\ &= t \cdot a_{n1} + t^2 \left(a_{n2} - \frac{1}{2} a_{n1}^2 \right) + t^3 \left(a_{n3} - a_{n1} a_{n2} + \frac{1}{3} a_{n1}^3 \right) \\ &\quad + t^4 \left(a_{n4} - a_{n1} a_{n3} - \frac{1}{2} a_{n2}^2 + a_{n1}^2 a_{n2} \right) + \cdots =: \sum_{k \geq 1} A_{nk} t^k. \end{aligned}$$

In particular, for each k , the function $A_{nk} - a_{nk} = h(a_{n1}, \dots, a_{n,k-1})$ is a polynomial in $a_{n1}, \dots, a_{n,k-1}$ with rational coefficients. So if $(a_{nk})_n$ converges for each k , then $(A_{nk})_n$ also converges, and by induction the converse also holds. If we have $\lim_{n \rightarrow \infty} a_{nk} = a_k$, then $\lim_{n \rightarrow \infty} A_{nk} = a_k + h(a_1, \dots, a_{k-1})$, and

$$\log \lim_{n \rightarrow \infty} f_n(t) = \log \left(1 + \sum_{k \geq 1} a_k t^k \right) = \sum_{k \geq 1} (a_k + h(a_1, \dots, a_{k-1})) t^k = \lim_{n \rightarrow \infty} \log f_n(t).$$

(ii) \log and \exp are inverse to each other on power series, so it suffices to prove that for f and $g \in t\mathbb{C}[[t]]$, we have

$$\exp(f + g) = \exp(f) \exp(g).$$

This follows from the binomial formula:

$$\begin{aligned} \exp(f + g) &= \sum_{n \geq 0} (f + g)^n / n! = \sum_{n \geq 0} \frac{1}{n!} \sum_{k=0}^n \binom{n}{k} f^k g^{n-k} = \sum_{n \geq 0} \sum_{k=0}^n \frac{f^k}{k!} \cdot \frac{g^{n-k}}{(n-k)!} \\ &= \sum_{i, j \geq 0} \frac{f^i}{i!} \cdot \frac{g^j}{j!} = \left(\sum_{i \geq 0} f^i / i! \right) \left(\sum_{j \geq 0} g^j / j! \right) = \exp(f) \exp(g). \end{aligned}$$

(iii) Let $F_N(t) = \prod_{n=1}^N f_n(t)$. Applying (i) to the sequence $(F_N(t))_N$, we see that the infinite product $\prod_{n \geq 1} f_n(t)$ converges term by term if and only if (by definition) $(F_N(t))_N$ converges term by term, if and only if the sequence $(\log F_N(t))_N$ converges term by term, if and only if (by definition) the series $\sum_{n \geq 1} \log f_n(t)$ converges term by term, since by (ii)

$$\log \prod_{n=1}^N f_n(t) = \sum_{n=1}^N \log f_n(t)$$

Also

$$\begin{aligned} \log \prod_{n \geq 1} f_n(t) &= \log \lim_{N \rightarrow \infty} F_N(t) = \lim_{N \rightarrow \infty} \log F_N(t) \\ &= \lim_{N \rightarrow \infty} \sum_{n=1}^N \log f_n(t) =: \sum_{n \geq 1} \log f_n(t). \quad \square \end{aligned}$$

6.4. For a complex of sheaves K_0 on \mathcal{X}_0 and $n \in \mathbb{Z}$, define

$$P_n(\mathcal{X}_0, K_0, t) := \det(1 - Ft, H_c^n(\mathcal{X}, K)).$$

We regard $P_n(\mathcal{X}_0, K_0, t)^{\pm 1}$ as a complex power series with constant term 1 via ι .

Proposition 6.5. *For every convergent complex of sheaves $K_0 \in W_m^{-, \text{stra}}(\mathcal{X}_0, \overline{\mathbb{Q}}_\ell)$, the infinite product*

$$\prod_{n \in \mathbb{Z}} P_n(\mathcal{X}_0, K_0, t)^{(-1)^{n+1}}$$

is convergent term by term to the L-series $L(\mathcal{X}_0, K_0, t)$.

Proof. The complex $R\Gamma_c(\mathcal{X}, K)$ is bounded above, so $P_n(\mathcal{X}_0, K_0, t) = 1$ for $n \gg 0$, and the infinite product is a one-direction limit, namely $n \rightarrow -\infty$.

Let $\alpha_{n1}, \dots, \alpha_{nm_n}$ be the eigenvalues (counted with multiplicity) of the morphism F on $H_c^n(\mathcal{X}, K)$, regarded as complex numbers via ι , so that

$$P_n(t) = P_n(\mathcal{X}_0, K_0, t) = (1 - \alpha_{n1}t) \cdots (1 - \alpha_{nm_n}t).$$

By Lemma 6.3 (iii) it suffices to show that the series

$$\sum_{n \in \mathbb{Z}} (-1)^{n+1} \log P_n(t)$$

converges term by term to $\sum_{v \geq 1} c_v(K_0)t^v/v$.

We have

$$\begin{aligned} \sum_{n \in \mathbb{Z}} (-1)^{n+1} \log P_n(t) &= \sum_{n \in \mathbb{Z}} (-1)^{n+1} \log \prod_i (1 - \alpha_{ni}t) \\ &= \sum_{n \in \mathbb{Z}} (-1)^n \sum_i \sum_{v \geq 1} \frac{\alpha_{ni}^v t^v}{v} = \sum_{v \geq 1} \frac{t^v}{v} \sum_{n \in \mathbb{Z}} (-1)^n \sum_i \alpha_{ni}^v = \sum_{v \geq 1} \frac{t^v}{v} c_v(R\Gamma_c(K_0)), \end{aligned}$$

which converges term by term by Theorem 4.2 (i), and is equal to $\sum_{v \geq 1} c_v(K_0)t^v/v$ by Theorem 4.2 (ii). \square

Remark 6.5.1. In particular we have

$$Z(\mathcal{X}_0, t) = \prod_{n \in \mathbb{Z}} P_n(\mathcal{X}_0, t)^{(-1)^{n+1}},$$

where $P_n(\mathcal{X}_0, t) = P_n(\mathcal{X}_0, \overline{\mathbb{Q}}_\ell, t)$. This generalizes the classical result for schemes [Grothendieck 1965, 5.1]. When we want to emphasize the dependence on the prime ℓ , we will write $P_{n,\ell}(\mathcal{X}_0, t)$.

If G_0 is a connected \mathbb{F}_q -algebraic group, 4.8.2 shows that the zeta function of BG_0 is given by

$$Z(BG_0, t) = \prod_{r \geq 0} \prod_{\substack{m_{ij} \geq 0 \\ \sum_{i,j} m_{ij}(i+1) = 2r}} \left(1 - q^{-d} \prod_{i,j} \alpha_{ij}^{-m_{ij}} \cdot t\right)^{-1} = \prod_{m_{ij} \geq 0} \left(1 - q^{-d} \prod_{i,j} \alpha_{ij}^{-m_{ij}} \cdot t\right)^{-1}.$$

7. Examples of zeta functions

In this section we compute the zeta functions of some stacks, and in each example we do it in two ways: by counting rational points and by computing cohomology groups. Also we investigate some analytic properties.

Example 7.1. $B\mathbb{G}_m$. By 4.7 (iii) we have $c_v(B\mathbb{G}_m) = 1/c_v(\mathbb{G}_m)$, so the zeta function is

$$Z(B\mathbb{G}_m, t) = \exp\left(\sum_{v \geq 1} c_v(B\mathbb{G}_m) \frac{t^v}{v}\right) = \exp\left(\sum_{v \geq 1} \frac{1}{q^v - 1} \frac{t^v}{v}\right).$$

Using Borel's theorem 4.8 one can show (or see [Laumon and Moret-Bailly 2000, 19.3.2]) that the cohomology ring $H^*(B\mathbb{G}_m)$ is a polynomial ring $\overline{\mathbb{Q}}_\ell[T]$, generated by a variable T of degree 2, and that the Frobenius action on cohomology is given by $FT^n = q^n T^n$. So by Poincaré duality, we have

$$\begin{aligned} \mathrm{Tr}(F, H_c^{-2n-2}(B\mathbb{G}_m)) &= \mathrm{Tr}(F, H_c^{-2n-2}(B\mathbb{G}_m, \overline{\mathbb{Q}}_\ell(-1)))/q \\ &= \mathrm{Tr}(F^{-1}, H^{2n}(B\mathbb{G}_m))/q = q^{-n-1}. \end{aligned}$$

This gives

$$\prod_{n \in \mathbb{Z}} P_n(B\mathbb{G}_m, t)^{(-1)^{n+1}} = \prod_{n \geq 1} (1 - q^{-n}t)^{-1}.$$

It is easy to verify the result in Remark 6.5.1 directly:

$$\begin{aligned} \exp\left(\sum_{v \geq 1} \frac{1}{q^v - 1} \frac{t^v}{v}\right) &= \exp\left(\sum_{v \geq 1} \frac{1/q^v}{1 - 1/q^v} \frac{t^v}{v}\right) = \exp\left(\sum_{v \geq 1} \frac{t^v}{v} \sum_{n \geq 1} \frac{1}{q^{nv}}\right) \\ &= \prod_{n \geq 1} \exp\left(\sum_{v \geq 1} \frac{(t/q^n)^v}{v}\right) = \prod_{n \geq 1} (1 - t/q^n)^{-1}. \end{aligned}$$

There is also a functional equation

$$Z(B\mathbb{G}_m, qt) = \frac{1}{1-t} Z(B\mathbb{G}_m, t),$$

which implies that $Z(B\mathbb{G}_m, t)$ has a meromorphic continuation to the whole complex plane, with simple poles at $t = q^n$, for $n \geq 1$.

$H_c^{-2n-2}(B\mathbb{G}_m)$ is pure of weight $-2n - 2$. A natural question is if Deligne's theorem of weights [Deligne 1980, 3.3.4] still holds for algebraic stacks. Olsson told me that it does not hold in general, as the following example shows.

Example 7.2. BE , where E is an elliptic curve over \mathbb{F}_q . Again by 4.7 (iii) we have

$$c_v(BE) = 1/\#E(\mathbb{F}_{q^v}).$$

Let α and β be the roots of the reciprocal characteristic polynomial of the Frobenius on $H^1(E)$:

$$x^2 - (1 + q - c_1(E))x + q = 0. \tag{7.2.1}$$

Then for every $v \geq 1$, we have $c_v(E) = 1 - \alpha^v - \beta^v + q^v = (1 - \alpha^v)(1 - \beta^v)$. So

$$\begin{aligned} c_v(BE) &= \frac{1}{(1 - \alpha^v)(1 - \beta^v)} = \frac{\alpha^{-v}}{1 - \alpha^{-v}} \cdot \frac{\beta^{-v}}{1 - \beta^{-v}} \\ &= \left(\sum_{n \geq 1} \alpha^{-nv} \right) \left(\sum_{m \geq 1} \beta^{-mv} \right) = \sum_{n, m \geq 1} \left(\frac{1}{\alpha^n \beta^m} \right)^v, \end{aligned}$$

and the zeta function is

$$\begin{aligned} Z(BE, t) &= \exp \left(\sum_{v \geq 1} c_v(BE) \frac{t^v}{v} \right) \\ &= \exp \left(\sum_{\substack{n, m \geq 1 \\ v \geq 1}} \left(\frac{t}{\alpha^n \beta^m} \right)^v / v \right) = \prod_{n, m \geq 1} \left(1 - \frac{t}{\alpha^n \beta^m} \right)^{-1}. \end{aligned}$$

To compute its cohomology, one can apply Borel's theorem 4.8 to E , and we have $N = N^1 = H^1(E)$, so $N[-1]$ is a 2-dimensional vector space sitting in degree 2, on which F has eigenvalues α and β . Then $H^*(BE)$ is a polynomial ring $\overline{\mathbb{Q}}_\ell[a, b]$ in two variables, both sitting in degree 2, and the basis a, b can be chosen so that the Frobenius action F on $H^2(BE)$ is upper triangular (or even diagonal)

$$\begin{bmatrix} \alpha & \gamma \\ & \beta \end{bmatrix}.$$

Then F acting on

$$H^{2n}(BE) = \text{Sym}^n N[-1] = \overline{\mathbb{Q}}_\ell \langle a^n, a^{n-1}b, \dots, b^n \rangle$$

can be represented by

$$\begin{bmatrix} \alpha^n & * & * & * \\ & \alpha^{n-1}\beta & * & * \\ & & \ddots & * \\ & & & \beta^n \end{bmatrix},$$

with eigenvalues $\alpha^n, \alpha^{n-1}\beta, \dots, \beta^n$. So the eigenvalues of F on $H_c^{-2-2n}(BE)$ are $q^{-1}\alpha^{-n}, q^{-1}\alpha^{1-n}\beta^{-1}, \dots, q^{-1}\beta^{-n}$ and

$$\prod_{n \in \mathbb{Z}} P_n(BE, t)^{(-1)^{n+1}} = 1/(1-q^{-1}t) \cdot [(1-q^{-1}\alpha^{-1}t)(1-q^{-1}\beta^{-1}t)] \\ \cdot [(1-q^{-1}\alpha^{-2}t)(1-q^{-1}\alpha^{-1}\beta^{-1}t)(1-q^{-1}\beta^{-2}t)] \cdots$$

Note that the right hand side is the same as $Z(BE, t)$ above (since $\alpha\beta = q$).

Let $Z_1(t) := Z(BE, qt)$. Its radius of convergence is 1, since by Lemma 6.1

$$\lim_{v \rightarrow \infty} \frac{c_v(BE)}{c_{v+1}(BE)} = \lim_{v \rightarrow \infty} \frac{c_{v+1}(E)}{c_v(E)} = q.$$

There is also a functional equation

$$Z_1(\alpha t) = \frac{1}{1-\alpha t} Z_1(t) Z_2(t),$$

where

$$Z_2(t) = \frac{1}{(1-\alpha\beta^{-1}t)(1-\alpha\beta^{-2}t)(1-\alpha\beta^{-3}t) \cdots}.$$

$Z_2(t)$ is holomorphic in the open unit disk and satisfies the functional equation

$$Z_2(\beta t) = \frac{1}{1-\alpha t} Z_2(t).$$

Therefore $Z_2(t)$, and hence $Z(BE, t)$, has a meromorphic continuation to the whole complex t -plane with the obvious poles.

Remark 7.2.1. $H_c^{-2-2n}(BE)$ is pure of weight $-2-n$, which is not $\leq -2-2n$ unless $n=0$. So the upper bound of weights for schemes fails for BE . This also leads to the failure of the decomposition theorem for BE ; see [Sun 2012, §1], for the example of a pure complex on BE which is not geometrically semisimple.

Also note that, the Equation (7.2.1) is independent of ℓ , so the polynomials $P_{n,\ell}(BE, t)$ are independent of ℓ .

Example 7.3. BG_0 , where G_0 is a finite étale \mathbb{F}_q -group scheme, corresponding to a finite group G and a Frobenius automorphism σ on it. Then $BG_0(\mathbb{F}_{q^v}) \simeq G/\rho^{(v)}$, where $\rho^{(v)}$ is the right action of G on the set G given by $h : g \mapsto \sigma^v(h^{-1})gh$. So

$$c_v(BG_0) = \sum_{[g] \in G/\rho^{(v)}} \frac{1}{\#\text{Stab}_{\rho^{(v)}}(g)} = \frac{\#G}{\#G} = 1,$$

and the zeta function is

$$Z(BG_0, t) = \frac{1}{1-t}.$$

Its cohomology groups are given in Lemma 4.6: $H_c^0(BG) = \overline{\mathbb{Q}}_\ell$, and other $H_c^i = 0$. This verifies Remark 6.5.1.

Note that $Z(BG_0, t)$ is the same as the zeta function of its coarse moduli space $\text{Spec } \mathbb{F}_q$. As a consequence, for every \mathbb{F}_q -algebraic stack \mathcal{X}_0 , with finite inertia $\mathcal{I}_0 \rightarrow \mathcal{X}_0$ and coarse moduli space $\pi : \mathcal{X}_0 \rightarrow X_0$ [Conrad 2005, 1.1], we have $Z(\mathcal{X}_0, t) = Z(X_0, t)$, and hence it is a rational function. This is because for every $x \in X_0(\mathbb{F}_{q^v})$, the fiber $\pi^{-1}(x)$ is a neutral gerbe over $\text{Spec } k(x)$, and from the above we see that $c_v(\pi^{-1}(x)) = 1$, and hence $c_v(\mathcal{X}_0) = c_v(X_0)$. The fact that $Z(X_0, t)$ is a rational function follows from [Knutson 1971, II, 6.7] and noetherian induction. More generally, we have the following.

Proposition 7.3.1. *Let \mathcal{X}_0 be an \mathbb{F}_q -algebraic stack. Suppose that \mathcal{X}_0 either has finite inertia, or is Deligne–Mumford (not necessarily separated). Then for every $K_0 \in W^b(\mathcal{X}_0, \overline{\mathbb{Q}}_\ell)$, the L-series $L(\mathcal{X}_0, K_0, t)$ is a rational function.*

Proof. It suffices to show that Theorem 4.2 holds for the structural morphism $\mathcal{X}_0 \rightarrow \text{Spec } \mathbb{F}_q$ and $K_0 \in W^b(\mathcal{X}_0, \overline{\mathbb{Q}}_\ell)$ in these two cases. We will not make explicit use of the fact from Remark 2.8.1 that K_0 is ι -mixed.

Case when \mathcal{X}_0 has finite inertia. Let $\pi : \mathcal{X}_0 \rightarrow X_0$ be its coarse moduli space. For any sheaf \mathcal{F}_0 on \mathcal{X}_0 , we have isomorphisms $H_c^r(X, R^0\pi_!\mathcal{F}) \simeq H_c^r(\mathcal{X}, \mathcal{F})$ by Lemma 4.6, so $R\Gamma_c(\mathcal{X}_0, \mathcal{F}_0)$ is a bounded complex, hence a convergent complex. To prove the trace formula for $\mathcal{X}_0 \rightarrow \text{Spec } \mathbb{F}_q$ and the sheaf \mathcal{F}_0 , it suffices to prove it for $\mathcal{X}_0 \rightarrow X_0$ and $X_0 \rightarrow \text{Spec } \mathbb{F}_q$. The first case, when passing to fibers, is reduced to BG_0 , and when passing to fibers again, it is reduced to the two subcases: when G_0 is finite, or when G_0 is connected. In both of these two cases as well as the case of an algebraic space $X_0 \rightarrow \text{Spec } \mathbb{F}_q$, the trace formula can be proved without using ι -mixedness 5.2.1. Therefore, Theorem 4.2 holds for $\mathcal{X}_0 \rightarrow \text{Spec } \mathbb{F}_q$ and any sheaf, hence any bounded complex, on \mathcal{X}_0 .

The trace formula is equivalent to the equality of power series

$$L(\mathcal{X}_0, K_0, t) = \prod_{i \in \mathbb{Z}} P_i(\mathcal{X}_0, K_0, t)^{(-1)^{i+1}},$$

and the right-hand side is a finite product, because $R\Gamma_c(\mathcal{X}_0, K_0)$ is bounded. Therefore, $L(\mathcal{X}_0, K_0, t)$ is rational.

Case when \mathcal{X}_0 is Deligne–Mumford. For both (i) and (ii) of Theorem 4.2, we may replace \mathcal{X}_0 by a nonempty open substack, hence by [Laumon and Moret-Bailly 2000, 6.1.1] we may assume \mathcal{X}_0 is the quotient stack $[X'_0/G]$, where X'_0 is an affine \mathbb{F}_q -scheme of finite type and G is a finite group acting on X'_0 . This stack has finite diagonal, and hence finite inertia, so by the previous case we are done. Also, we know that $R\Gamma_c(\mathcal{X}_0, K_0)$ is bounded, therefore $L(\mathcal{X}_0, K_0, t)$ is rational. \square

If one wants to use Poincaré duality to get a functional equation for the zeta function, [Olsson 2008a, 5.17] and [Laszlo and Olsson 2008b, 9.1.2] suggest that we should assume \mathcal{X}_0 to be proper smooth and of finite diagonal. Under these

assumptions, one gets the expected functional equation for the zeta function, as well as the independence of ℓ for the coarse moduli space, which is proper but possibly singular. Examples of such stacks include the moduli stack of pointed stable curves $\overline{\mathcal{M}}_{g,n}$ over \mathbb{F}_q .

Proposition 7.3.2. *Let \mathcal{X}_0 be a proper smooth \mathbb{F}_q -algebraic stack of equidimension d , with finite diagonal, and let $\pi : \mathcal{X}_0 \rightarrow X_0$ be its coarse moduli space. Then $Z(X_0, t)$ satisfies the usual functional equation*

$$Z\left(X_0, \frac{1}{q^d t}\right) = \pm q^{d\chi/2} t^\chi Z(X_0, t),$$

where $\chi := \sum_{i=0}^{2d} (-1)^i \deg P_{i,\ell}(X_0, t)$. Moreover, $H^i(X)$ is pure of weight i , for every $0 \leq i \leq 2d$, and the reciprocal roots of each $P_{i,\ell}(X_0, t)$ are algebraic integers independent of ℓ .

Proof. First we show that the adjunction map $\overline{\mathbb{Q}}_\ell \rightarrow \pi_* \pi^* \overline{\mathbb{Q}}_\ell = \pi_* \overline{\mathbb{Q}}_\ell$ is an isomorphism. Since π is quasi-finite and proper [Conrad 2005, 1.1], we have $\pi_* = \pi_!$ [Olsson 2008a, 5.1] and $R^r \pi_! \overline{\mathbb{Q}}_\ell = 0$ for $r \neq 0$ [Olsson 2008a, 5.8]. The natural map $\overline{\mathbb{Q}}_\ell \rightarrow R^0 \pi_* \overline{\mathbb{Q}}_\ell$ is an isomorphism, since the geometric fibers of π are connected.

Therefore $R\Gamma(\mathcal{X}_0, \overline{\mathbb{Q}}_\ell) = R\Gamma(X_0, \pi_* \overline{\mathbb{Q}}_\ell) = R\Gamma(X_0, \overline{\mathbb{Q}}_\ell)$, and hence

$$H^i(\mathcal{X}) \simeq H_c^i(\mathcal{X}) \simeq H^i(X) \simeq H_c^i(X)$$

for all i [Olsson 2008a, 5.17]. Let $P_i(t) = P_i(\mathcal{X}_0, t) = P_i(X_0, t)$. Since X_0 is an algebraic space of dimension d , $P_i(t) = 1$ if $i \notin [0, 2d]$. Since \mathcal{X}_0 is proper and smooth, Poincaré duality gives a perfect pairing

$$H^i(\mathcal{X}) \times H^{2d-i}(\mathcal{X}) \longrightarrow \overline{\mathbb{Q}}_\ell(-d).$$

Following the standard proof for proper smooth varieties, see [Milne 2008, 27.12], we get the expected functional equation for $Z(\mathcal{X}_0, t) = Z(X_0, t)$.

$H^i(X)$ is mixed of weights $\leq i$ [Deligne 1980, 3.3.4], so by Poincaré duality, it is pure of weight i . Following the proof in [Deligne 1974a, p. 276], this purity implies that the polynomials $P_{i,\ell}(X_0, t)$ have integer coefficients independent of ℓ . \square

Remark 7.3.3. Weizhe Zheng suggested Proposition 7.3.1 to me. He also suggested that we give a functional equation relating $L(\mathcal{X}_0, DK_0, t)$ and $L(\mathcal{X}_0, K_0, t)$, for $K_0 \in W^b(\mathcal{X}_0, \overline{\mathbb{Q}}_\ell)$, where \mathcal{X}_0 is a proper \mathbb{F}_q -algebraic stack with finite diagonal, of equidimension d , but not necessarily smooth. Here is the functional equation:

$$L(\mathcal{X}_0, K_0, t^{-1}) = t^{\chi_c} \cdot Q \cdot L(\mathcal{X}_0, DK_0, t),$$

where $\chi_c = \sum_{i=0}^{2d} (-1)^i h_c^i(\mathcal{X}, K)$ and $Q = (t^{\chi_c} L(\mathcal{X}_0, K_0, t))|_{t=\infty}$. Note that the rational function $L(\mathcal{X}_0, K_0, t)$ has degree $-\chi_c$, hence Q is well-defined. The proof is similar to the above.

Example 7.4. BGL_N . We have

$$\# GL_N(\mathbb{F}_{q^v}) = (q^{vN} - 1)(q^{vN} - q^v) \cdots (q^{vN} - q^{v(N-1)}),$$

so one can use $c_v(BGL_N) = 1/c_v(GL_N)$ to compute $Z(BGL_N, t)$. One can also compute the cohomology groups of BGL_N using Borel's theorem 4.8. We refer to [Behrend 1993, 2.3.2] for the result. Let us consider the case $N = 2$ only. The general case is similar.

We have

$$c_v(BGL_2) = \frac{1}{q^{4v}} \left(1 + \frac{1}{q^v} + \frac{2}{q^{2v}} + \frac{2}{q^{3v}} + \frac{3}{q^{4v}} + \frac{3}{q^{5v}} + \cdots \right),$$

and therefore

$$\begin{aligned} Z(BGL_2, t) &= \exp\left(\sum_v \frac{(t/q^4)^v}{v}\right) \cdot \exp\left(\sum_v \frac{(t/q^5)^v}{v}\right) \cdot \exp\left(\sum_v \frac{2(t/q^6)^v}{v}\right) \cdots \\ &= \frac{1}{1-t/q^4} \cdot \frac{1}{1-t/q^5} \cdot \left(\frac{1}{1-t/q^6}\right)^2 \cdot \left(\frac{1}{1-t/q^7}\right)^2 \cdot \left(\frac{1}{1-t/q^8}\right)^3 \cdots \end{aligned}$$

So $Z(BGL_2, qt) = Z(BGL_2, t) \cdot Z_1(t)$, where

$$Z_1(t) = \frac{1}{(1-t/q^3)(1-t/q^5)(1-t/q^7)(1-t/q^9) \cdots}.$$

$Z_1(t)$ satisfies the functional equation

$$Z_1(q^2 t) = \frac{1}{1-t/q} \cdot Z_1(t),$$

So $Z_1(t)$, and hence $Z(BGL_2, t)$, has a meromorphic continuation with the obvious poles.

The nonzero compactly supported cohomology groups of BGL_2 are given as follows:

$$H_c^{-8-2n}(BGL_2) = \overline{\mathbb{Q}}_\ell(n+4)^{\oplus \lfloor \frac{n}{2} \rfloor + 1}, \quad n \geq 0.$$

This gives

$$\prod_{n \in \mathbb{Z}} P_n(BGL_2, t)^{(-1)^{n+1}} = \frac{1}{(1-t/q^4)(1-t/q^5)(1-t/q^6)^2(1-t/q^7)^2 \cdots},$$

and Remark 6.5.1 is verified. Note that the eigenvalues are $1/q^{n+4}$, which are independent of ℓ .

8. Analytic continuation

We state and prove a generalized version of Theorem 1.3.

Theorem 8.1. *Let \mathcal{X}_0 be an \mathbb{F}_q -algebraic stack, and let $K_0 \in W_m^{-, \text{stra}}(\mathcal{X}_0, \overline{\mathbb{Q}}_\ell)$ be a convergent complex. Then $L(\mathcal{X}_0, K_0, t)$ has a meromorphic continuation to the whole complex t -plane, and its poles can only be zeros of the polynomials $P_{2n}(\mathcal{X}_0, K_0, t)$ for some integers n .*

We need a preliminary lemma. For an open subset $U \subset \mathbb{C}$, let $\mathcal{O}(U)$ be the set of analytic functions on U . There exists a sequence $\{K_n\}_{n \geq 1}$ of compact subsets of U such that $U = \bigcup_n K_n$ and $K_n \subset (K_{n+1})^\circ$. For f and g in $\mathcal{O}(U)$, define

$$\rho_n(f, g) = \sup\{|f(z) - g(z)|; z \in K_n\} \quad \text{and} \quad \rho(f, g) = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \frac{\rho_n(f, g)}{1 + \rho_n(f, g)}.$$

Then ρ is a metric on $\mathcal{O}(U)$ and the topology it induces is independent of the subsets $\{K_n\}_n$ chosen (see [Conway 1973, VII, §1]).

The following lemma is from [Conway 1973, p. 167, 5.9].

Lemma 8.2. *Let $U \subset \mathbb{C}$ be connected and open and let $(f_n)_n$ be a sequence in $\mathcal{O}(U)$ such that no f_n is identically zero. If $\sum_n (f_n(z) - 1)$ converges absolutely and uniformly on compact subsets of U , then $\prod_{n \geq 1} f_n(z)$ converges in $\mathcal{O}(U)$ to an analytic function $f(z)$. If z_0 is a zero of f , then z_0 is a zero of only a finite number of the functions f_n , and the multiplicity of the zero of f at z_0 is the sum of the multiplicities of the zeros of the functions f_n at z_0 .*

Now we prove Theorem 8.1.

Proof. Factorize $P_n(\mathcal{X}_0, K_0, t)$ as $\prod_{j=1}^{m_n} (1 - \alpha_{nj}t)$ in \mathbb{C} . Since $R\Gamma_c(\mathcal{X}_0, K_0)$ is a convergent complex by Theorem 4.2 (i), the series $\sum_{n,j} |\alpha_{nj}|$ converges.

By Proposition 6.5 we have

$$L(\mathcal{X}_0, K_0, t) = \prod_{n \in \mathbb{Z}} \left(\prod_{j=1}^{m_n} (1 - \alpha_{nj}t) \right)^{(-1)^{n+1}}$$

as formal power series. Take U to be the region $\mathbb{C} - \{\alpha_{nj}^{-1}; n \text{ even}\}$ with the intention of applying Lemma 8.2. Take the lexicographical order on the set of all factors

$$1 - \alpha_{nj}t, \text{ for } n \text{ odd}; \quad \frac{1}{1 - \alpha_{nj}t}, \text{ for } n \text{ even}.$$

Each factor is an analytic function on U . The sum $\sum_n (f_n(z) - 1)$ here is equal to

$$\sum_{n \text{ odd}, j} (-\alpha_{nj}t) + \sum_{n \text{ even}, j} \frac{\alpha_{nj}t}{1 - \alpha_{nj}t}.$$

Let

$$g_n(t) = \begin{cases} \sum_{j=1}^{m_n} |\alpha_{nj}t|, & n \text{ odd,} \\ \sum_{j=1}^{m_n} \frac{|\alpha_{nj}t|}{|1 - \alpha_{nj}t|}, & n \text{ even.} \end{cases}$$

We need to show that $\sum_n g_n(t)$ is pointwise convergent, uniformly on compact subsets of U . Precisely, we want to show that for any compact subset $B \subset U$, and for any $\varepsilon > 0$, there exists a constant $N_B \in \mathbb{Z}$ such that

$$\sum_{n \leq N} g_n(t) < \varepsilon$$

for all $N \leq N_B$ and $t \in B$. Since $g_n(t)$ are nonnegative, it suffices to do this for $N = N_B$. There exists a constant M_B such that $|t| < M_B$ for all $t \in B$. Since $\sum_{n,j} |\alpha_{nj}|$ converges, $|\alpha_{nj}| \rightarrow 0$ as $n \rightarrow -\infty$, and there exists a constant $L_B \in \mathbb{Z}$ such that $|\alpha_{nj}| < 1/(2M_B)$ for all $n < L_B$. So

$$g_n(t) \leq 2 \sum_{j=1}^{m_n} |\alpha_{nj}t|$$

for all $n < L_B$ and $t \in B$. There exists a constant $N_B < L_B$ such that

$$\sum_{n \leq N_B} \sum_j |\alpha_{nj}| < \varepsilon/(2M_B)$$

and hence

$$\sum_{n \leq N_B} g_n(t) \leq 2 \sum_{n \leq N_B} \sum_j |\alpha_{nj}t| \leq 2M_B \sum_{n \leq N_B} \sum_j |\alpha_{nj}| < \varepsilon.$$

By Lemma 8.2, $L(\mathcal{X}_0, K_0, t)$ extends to an analytic function on U . By the second part of Lemma 8.2, the α_{nj}^{-1} 's, for n even, are at worst poles rather than essential singularities, therefore the L -series is meromorphic on \mathbb{C} . \square

Now $L(\mathcal{X}_0, K_0, t)$ can be called an “ L -function”.

9. Weight theorem for algebraic stacks

9.1. We prove Theorem 1.4 in this section. For the reader's convenience, we briefly review the definition of the *dimension* of a locally noetherian S -algebraic stack \mathcal{X} from [Laumon and Moret-Bailly 2000, Chapter 11].

If X is a locally noetherian S -algebraic space and x is a point of X , the dimension $\dim_x X$ of X at x is defined to be $\dim_{x'} X'$, for any pair (X', x') where X' is an S -scheme étale over X and $x' \in X'$ maps to x . This is independent of the choice of

the pair. If $f : X \rightarrow Y$ is a morphism of S -algebraic spaces, locally of finite type, and x is a point of X with image y in Y , then the relative dimension $\dim_x f$ of f at x is defined to be $\dim_x X_y$.

Let $P : X \rightarrow \mathcal{X}$ be a presentation of an S -algebraic stack \mathcal{X} , and let x be a point of X . Then the relative dimension $\dim_x P$ of P at x is defined to be the relative dimension at (x, x) of the smooth morphism $\text{pr}_1 : X \times_{\mathcal{X}} X \rightarrow X$ of S -algebraic spaces.

If \mathcal{X} is a locally noetherian S -algebraic stack and if ξ is a point of \mathcal{X} , the dimension of \mathcal{X} at ξ is defined to be $\dim_{\xi} \mathcal{X} = \dim_x X - \dim_x P$, where $P : X \rightarrow \mathcal{X}$ is an arbitrary presentation of \mathcal{X} and x is an arbitrary point of X lying over ξ . This definition is independent of all the choices made. At last one defines the dimension of \mathcal{X} by $\dim \mathcal{X} = \sup_{\xi} \dim_{\xi} \mathcal{X}$. For quotient stacks we have $\dim [X/G] = \dim X - \dim G$.

Now we prove Theorem 1.4.

Proof. If $j : \mathcal{U}_0 \rightarrow \mathcal{X}_0$ is an open substack with complement $i : \mathcal{Z}_0 \rightarrow \mathcal{X}_0$, then we have an exact sequence

$$\dots \longrightarrow H_c^n(\mathcal{U}, j^*\mathcal{F}) \longrightarrow H_c^n(\mathcal{X}, \mathcal{F}) \longrightarrow H_c^n(\mathcal{Z}, i^*\mathcal{F}) \longrightarrow \dots$$

If both $H_c^n(\mathcal{U}, j^*\mathcal{F})$ and $H_c^n(\mathcal{Z}, i^*\mathcal{F})$ are zero (respectively have all ι -weights $\leq m$ for some number m), then so is $H_c^n(\mathcal{X}, \mathcal{F})$. Since the dimensions of \mathcal{U}_0 and \mathcal{Z}_0 are no more than that of \mathcal{X}_0 , and the set of punctual ι -weights of $i^*\mathcal{F}_0$ and of $j^*\mathcal{F}_0$ is the same as that of \mathcal{F}_0 , we may shrink \mathcal{X}_0 to a nonempty open substack. We can also make any finite base change on \mathbb{F}_q . To simplify notation, we may use twist (see 2.4) and projection formula to assume $w = 0$. As before, we reduce to the case when \mathcal{X}_0 is geometrically connected, and the inertia $f : \mathcal{I}_0 \rightarrow \mathcal{X}_0$ is flat, with rigidification $\pi : \mathcal{X}_0 \rightarrow X_0$, where X_0 is a scheme. The squares in the following diagram are 2-Cartesian:

$$\begin{array}{ccccc}
 & & \mathcal{I}_0 & \longleftarrow & \text{Aut}_y \\
 & & f \downarrow & & \downarrow \\
 B \text{Aut}_{\bar{x}} & \longrightarrow & B \text{Aut}_x & \longrightarrow & \mathcal{X}_0 \xleftarrow{y} \text{Spec } \mathbb{F}_{q^v} \\
 \pi_{\bar{x}} \downarrow & & \pi_x \downarrow & & \downarrow \pi \\
 \text{Spec } \mathbb{F} & \xrightarrow{\bar{x}} & \text{Spec } \mathbb{F}_{q^v} & \xrightarrow{x} & X_0
 \end{array}$$

We have $(R^k \pi_! \mathcal{F}_0)_{\bar{x}} = H_c^k(B \text{Aut}_{\bar{x}}, \mathcal{F})$. Since f is representable and flat, and \mathcal{X}_0 is connected, all automorphism groups Aut_x have the same dimension, say d .

Assume Theorem 1.4 holds for all BG_0 , where G_0 are \mathbb{F}_q -algebraic groups. Then $R^k \pi_! \mathcal{F}_0 = 0$ for $k > -2d$, and for $k \leq -2d$, the punctual ι -weights of

$R^k \pi_! \mathcal{F}_0$ are $\leq \frac{k}{2} - d$, hence by [Deligne 1980, 3.3.4], the punctual ι -weights of $H_c^r(X, R^k \pi_! \mathcal{F})$ are $\leq \frac{k}{2} - d + r$. Consider the Leray spectral sequence

$$E_2^{rk} = H_c^r(X, R^k \pi_! \mathcal{F}) \implies H_c^{r+k}(\mathcal{X}, \mathcal{F}).$$

If we maximize $\frac{k}{2} - d + r$ under the constraints

$$r + k = n, \quad 0 \leq r \leq 2 \dim X_0 \quad \text{and} \quad k \leq -2d,$$

we find that $H_c^n(\mathcal{X}, \mathcal{F}) = 0$ for $n > 2 \dim X_0 - 2d = 2 \dim \mathcal{X}_0$, and for $n \leq 2 \dim \mathcal{X}_0$, the punctual ι -weights of $H_c^n(\mathcal{X}, \mathcal{F})$ are $\leq \dim X_0 + \frac{n}{2} - d = \dim \mathcal{X}_0 + \frac{n}{2}$.

So we reduce to the case $\mathcal{X}_0 = BG_0$. By Lemma 4.6 the Leray spectral sequence for $h : BG_0 \rightarrow B\pi_0(G_0)$ degenerates to isomorphisms

$$H_c^0(B\pi_0(G), R^n h_! \mathcal{F}) \simeq H_c^n(BG, \mathcal{F}).$$

The fibers of h are isomorphic to BG_0^0 , so by base change and Lemma 4.6 we reduce to the case when G_0 is connected. Let $d = \dim G_0$ and $f : BG_0 \rightarrow \text{Spec } \mathbb{F}_q$ be the structural map. In this case, $\mathcal{F}_0 \cong f^* V$ for some $\overline{\mathbb{Q}}_\ell$ -representation V of $W(\mathbb{F}_q)$, and hence \mathcal{F}_0 and V have the same punctual ι -weights. Using the natural isomorphism $H_c^n(BG) \otimes V \simeq H_c^n(BG, \mathcal{F})$, we reduce to the case when $\mathcal{F}_0 = \overline{\mathbb{Q}}_\ell$. In 4.8.2 we see that, if $\alpha_{i1}, \dots, \alpha_{in_i}$ are the eigenvalues of F on N^i , $i \geq 1$ odd, then the eigenvalues of F on $H_c^{-2k-2d}(BG)$ are

$$q^{-d} \prod_{i,j} \alpha_{ij}^{-m_{ij}}, \quad \text{where} \quad \sum_{i,j} m_{ij}(i+1) = 2k.$$

Since $i \geq 1$, we have $\sum i m_{ij} \geq k$; together with $|\alpha_{ij}| \geq q^{i/2}$ one deduces

$$\left| q^{-d} \prod_{i,j} \alpha_{ij}^{-m_{ij}} \right| \leq q^{(-k-2d)/2},$$

so the punctual ι -weights of $H_c^{-2k-2d}(BG)$ are $\leq -k - 2d$ for $k \geq 0$, and the other compactly supported cohomology groups are zero.

It is clear from the proof and [Deligne 1980, 3.3.10] that the weights of $H_c^n(\mathcal{X}, \mathcal{F})$ differ from the weights of \mathcal{F}_0 by integers.

Recall that $H^{2k}(BG)$ is pure of weight $2k$, for a linear algebraic group G_0 over \mathbb{F}_q [Deligne 1974b, 9.1.4]. Therefore, $H_c^{-2k-2d}(BG)$ is pure of weight $-2k - 2d$, and following the same proof as above, we are done. \square

Remark 9.2. When $\mathcal{X}_0 = X_0$ is a scheme, and $n \leq 2 \dim X_0$, we have

$$\dim X_0 + \frac{n}{2} + w \geq n + w,$$

so our bound for weights is worse than the bound in [Deligne 1980, 3.3.4]. For an \mathbb{F}_q -abelian variety A , our bound for the weights of $H_c^n(BA)$ is sharp: the weights are exactly $\dim(BA) + \frac{n}{2}$, whenever the cohomology group is nonzero.

We hope Theorem 1.4 has useful and interesting applications, for instance for generalizing the decomposition theorem of Beilinson–Bernstein–Deligne–Gabber (see [Sun 2012]) to stacks with affine stabilizers, and for studying the Hasse–Weil zeta functions of Artin stacks over number fields. For instance, it implies that the Hasse–Weil zeta function is analytic in some right half complex s -plane.

Using Theorem 1.4 we can show certain stacks have \mathbb{F}_q -points.

Example 9.3. Let \mathcal{X}_0 be a form of $B\mathbb{G}_m$, that is, $\mathcal{X} \cong B\mathbb{G}_{m,\mathbb{F}}$ over \mathbb{F} . Then all the automorphism group schemes in \mathcal{X}_0 are affine, and for any $n \geq 0$ we have $h_c^{-2-2n}(\mathcal{X}) = h_c^{-2-2n}(B\mathbb{G}_m) = 1$. Let α_{-2-2n} be the eigenvalue of F on $H_c^{-2-2n}(\mathcal{X})$. Then by Theorem 1.4 we have $|\alpha_{-2-2n}| \leq q^{-1-n}$. Smoothness is fppf local on the base, so \mathcal{X}_0 is smooth and connected, hence $H_c^{-2}(\mathcal{X}) = \overline{\mathbb{Q}}_\ell(1)$ and $\alpha_{-2} = q^{-1}$. So

$$\begin{aligned} \#\mathcal{X}_0(\mathbb{F}_q) &= \sum_{n \geq 0} \text{Tr}(F, H_c^{-2-2n}(\mathcal{X})) = q^{-1} + \alpha_{-4} + \alpha_{-6} + \cdots \\ &\geq q^{-1} - q^{-2} - q^{-3} + \cdots = q^{-1} - \frac{q^{-1}}{q-1} > 0 \end{aligned}$$

when $q \neq 2$. In fact, since there exists an integer $r \geq 1$ such that $\mathcal{X}_0 \otimes_{\mathbb{F}_q} \mathbb{F}_{q^r} \cong B\mathbb{G}_{m,\mathbb{F}_{q^r}}$, we see that all cohomology groups $H_c^{-2-2n}(\mathcal{X})$ are pure, that is, $|\alpha_{-2-2n}| = q^{-1-n}$.

In fact, one can classify the forms of $B\mathbb{G}_{m,\mathbb{F}_q}$ as follows. If \mathcal{X}_0 is a form, then it is also a gerbe over $\text{Spec } \mathbb{F}_q$, hence a neutral gerbe BG_0 for some algebraic group G_0 by Behrend [2003, 6.4.2]. By comparing the automorphism groups, we see that G_0 is a form of $\mathbb{G}_{m,\mathbb{F}_q}$. There is only one nontrivial form of $\mathbb{G}_{m,\mathbb{F}_q}$, because

$$H^1(\mathbb{F}_q, \text{Aut}(\mathbb{G}_m)) = H^1(\mathbb{F}_q, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z},$$

and this form is the kernel $R_{\mathbb{F}_{q^2}/\mathbb{F}_q}^1 \mathbb{G}_{m,\mathbb{F}_{q^2}}$ of the norm map

$$R_{\mathbb{F}_{q^2}/\mathbb{F}_q} \mathbb{G}_{m,\mathbb{F}_{q^2}} \xrightarrow{\text{Nm}} \mathbb{G}_{m,\mathbb{F}_q},$$

where $R_{\mathbb{F}_{q^2}/\mathbb{F}_q}$ is the operation of Weil’s restriction of scalars. Therefore, the only nontrivial form of $B\mathbb{G}_{m,\mathbb{F}_q}$ is $B(R_{\mathbb{F}_{q^2}/\mathbb{F}_q}^1 \mathbb{G}_{m,\mathbb{F}_{q^2}})$. In particular, they all have \mathbb{F}_q -points, even when $q = 2$.

Example 9.4. Consider the projective line \mathbb{P}^1 with the following action of \mathbb{G}_m : it acts by multiplication on the open part $\mathbb{A}^1 \subset \mathbb{P}^1$, and leaves the point ∞ fixed. So we get a quotient stack $[\mathbb{P}^1/\mathbb{G}_m]$ over \mathbb{F}_q . Let \mathcal{X}_0 be a form of $[\mathbb{P}^1/\mathbb{G}_m]$. We want to find an \mathbb{F}_q -point on \mathcal{X}_0 , or even better, an \mathbb{F}_q -point on \mathcal{X}_0 which, when considered as a point in $\mathcal{X}(\mathbb{F}) \cong [\mathbb{P}^1/\mathbb{G}_m](\mathbb{F})$, lies in the open dense orbit $[\mathbb{G}_m/\mathbb{G}_m](\mathbb{F})$.

9.4.1. Consider the following general situation. Let G_0 be a connected \mathbb{F}_q -algebraic group, and let X_0 be a proper smooth variety with a G_0 -action over \mathbb{F}_q . Let

$$[X_0/G_0] \xrightarrow{f} BG_0 \xrightarrow{g} \text{Spec } \mathbb{F}_q$$

be the natural maps, and let \mathcal{X}_0 be a form of $[X_0/G_0]$. Then f is representable and proper. For every k , $R^k f_* \overline{\mathbb{Q}}_\ell$ is a lisse sheaf, and takes the form $g^* V_k$ for some sheaf V_k on $\text{Spec } \mathbb{F}_q$. Consider the Leray spectral sequence

$$E_2^{rk} = R^r g_* R^k f_* \overline{\mathbb{Q}}_\ell \implies R^{r+k} (gf)_* \overline{\mathbb{Q}}_\ell.$$

Since $R^r g_* R^k f_* \overline{\mathbb{Q}}_\ell = R^r g_*(g^* V_k) = (R^r g_* \overline{\mathbb{Q}}_\ell) \otimes V_k$, we have

$$h_c^n(\mathcal{X}) = h_c^n([X/G]) \leq \sum_{r+k=n} h_c^r(BG) \cdot \dim V_k = \sum_{r+k=n} h_c^r(BG) \cdot h^k(X).$$

Now we return to $[\mathbb{P}^1/\mathbb{G}_m]$. Since $h^0(\mathbb{P}^1) = h^2(\mathbb{P}^1) = 1$ and $h_c^{-2i}(B\mathbb{G}_m) = 1$ for $i \geq 1$, we see that $h_c^n(\mathcal{X}) = 0$ for n odd and

$$h_c^{2n}(\mathcal{X}) \leq h^0(\mathbb{P}^1) h_c^{2n}(B\mathbb{G}_m) + h^2(\mathbb{P}^1) h_c^{2n-2}(B\mathbb{G}_m) = \begin{cases} 0, & n \geq 1, \\ 1, & n = 0, \\ 2, & n < 0. \end{cases}$$

Since \mathcal{X}_0 is connected and smooth of dimension 0, we have $H_c^0(\mathcal{X}) = \overline{\mathbb{Q}}_\ell$. By Theorem 1.4, the ι -weights of $H_c^{2n}(\mathcal{X})$ are $\leq 2n$. The trace formula gives

$$\begin{aligned} \#\mathcal{X}_0(\mathbb{F}_q) &= \sum_{n \leq 0} \text{Tr}(F, H_c^{2n}(\mathcal{X})) = 1 + \sum_{n < 0} \text{Tr}(F, H_c^{2n}(\mathcal{X})) \\ &\geq 1 - 2 \sum_{n < 0} q^n = 1 - \frac{2}{q-1} > 0 \end{aligned}$$

when $q \geq 4$.

In order for the rational point to be in the open dense orbit, we need an upper bound for the number of \mathbb{F}_q -points on the closed orbits. When passing to \mathbb{F} , there are 2 closed orbits, both having stabilizer $\mathbb{G}_{m,\mathbb{F}}$. So in $[\mathcal{X}_0(\mathbb{F}_q)]$ there are at most 2 points whose automorphism groups are forms of the algebraic group $\mathbb{G}_{m,\mathbb{F}_q}$. From the cohomology sequence

$$1 \longrightarrow (R_{\mathbb{F}_{q^2}/\mathbb{F}_q}^1 \mathbb{G}_{m,\mathbb{F}_{q^2}})(\mathbb{F}_q) \longrightarrow \mathbb{F}_{q^2}^* \xrightarrow{\text{Nm}} \mathbb{F}_q^*$$

we see that

$$\#(R_{\mathbb{F}_{q^2}/\mathbb{F}_q}^1 \mathbb{G}_{m,\mathbb{F}_{q^2}})(\mathbb{F}_q) = q + 1.$$

Since $1/(q+1) \leq 1/(q-1)$, the space that the closed orbits can take is at most $2/(q-1)$, and equality holds only when the two closed orbits are both defined

over \mathbb{F}_q with stabilizer \mathbb{G}_m . In order for there to exist an \mathbb{F}_q -point in the open dense orbit, we need

$$1 - \frac{2}{q-1} > \frac{2}{q-1},$$

and this is so when $q \geq 7$.

10. About independence of ℓ

The coefficients of the expansion of the infinite product

$$Z(\mathcal{X}_0, t) = \prod_{i \in \mathbb{Z}} P_{i,\ell}(\mathcal{X}_0, t)^{(-1)^{i+1}}$$

are rational numbers and are independent of ℓ , because the $c_v(\mathcal{X}_0)$'s are rational numbers independent of ℓ . A famous conjecture is that this is also true for each $P_{i,\ell}(\mathcal{X}_0, t)$. First we show that the roots of $P_{i,\ell}(\mathcal{X}_0, t)$ are Weil q -numbers. Note that $P_{i,\ell}(\mathcal{X}_0, t) \in \mathbb{Q}_\ell[t]$.

Definition 10.1. *An algebraic number is called a Weil q -number if all of its conjugates have the same weight relative to q , and this weight is a rational integer. It is called a Weil q -integer if in addition it is an algebraic integer. A number in $\overline{\mathbb{Q}}_\ell$ is called a Weil q -number if it is a Weil q -number via ι .*

For $\alpha \in \overline{\mathbb{Q}}_\ell$, being a Weil q -number or not is independent of ι ; in fact the images in \mathbb{C} under various ι 's are conjugate.

For an \mathbb{F}_q -variety X_0 , not necessarily smooth or proper, [Deligne 1980, 3.3.4] implies all Frobenius eigenvalues of $H_c^i(X)$ are Weil q -integers. The following lemma generalizes this.

Lemma 10.2. *For every \mathbb{F}_q -algebraic stack \mathcal{X}_0 , and a prime number $\ell \neq p$, the roots of each $P_{i,\ell}(\mathcal{X}_0, t)$ are Weil q -numbers. In particular, the coefficients of $P_{i,\ell}(\mathcal{X}_0, t)$ are algebraic numbers in \mathbb{Q}_ℓ (that is, algebraic over \mathbb{Q}).*

Proof. For an open immersion $j : \mathcal{U}_0 \rightarrow \mathcal{X}_0$ with complement $i : \mathcal{Z}_0 \rightarrow \mathcal{X}_0$, we have an exact sequence

$$\cdots \longrightarrow H_c^i(\mathcal{U}) \longrightarrow H_c^i(\mathcal{X}) \longrightarrow H_c^i(\mathcal{Z}) \longrightarrow \cdots,$$

thus we may shrink to a nonempty open substack. In particular, Lemma 10.2 holds for algebraic spaces, by Knutson [1971, II 6.7] and Deligne [1980, 3.3.4].

We may assume \mathcal{X}_0 is smooth and connected. By Poincaré duality, it suffices to show that the Frobenius eigenvalues of $H^i(\mathcal{X})$ are Weil q -numbers, for all i . Take a presentation $X_0 \rightarrow \mathcal{X}_0$ and consider the associated strictly simplicial smooth covering $X_0^\bullet \rightarrow \mathcal{X}_0$ by algebraic spaces. Then there is a spectral sequence [Laszlo

and Olsson 2008b, 10.0.9]

$$E_1^{rk} = H^k(X^r) \implies H^{r+k}(\mathcal{X}),$$

and the assertion for \mathcal{X}_0 follows from the assertion for algebraic spaces. \square

Problem 10.3. Is each

$$P_{i,\ell}(\mathcal{X}_0, t) = \det(1 - Ft, H_c^i(\mathcal{X}, \mathbb{Q}_\ell))$$

a polynomial with coefficients in \mathbb{Q} , and the coefficients are independent of ℓ ?

Remark 10.3.1. (i) Note that, unlike the case for varieties, we cannot expect the coefficients to be integers (for instance, for $B\mathbb{G}_m$, the coefficients are $1/q^i$).

(ii) Problem 10.3 is known to be true for smooth proper varieties [Deligne 1980, 3.3.9], and (coarse moduli spaces of) proper smooth algebraic stacks of finite diagonal (see Proposition 7.3.2). It remains open for general varieties. Even the Betti numbers are not known to be independent of ℓ for a general variety, see [Illusie 2006].

Let us give positive answer to Problem 10.3 in some special cases of algebraic stacks. In Section 7 we see that it holds for BE and BGL_N . We can generalize these two cases as follows.

Lemma 10.4. *Problem 10.3 has a positive answer for:*

- (i) BA , where A is an \mathbb{F}_q -abelian variety.
- (ii) BG_0 , where G_0 is a linear algebraic group over \mathbb{F}_q .

Proof. (i) Let $g = \dim A$. Then $N = H^1(A)$ is a $2g$ -dimensional vector space, with eigenvalues $\alpha_1, \dots, \alpha_{2g}$ for the Frobenius action F , and N is pure of weight 1. Let a_1, \dots, a_{2g} be a basis for N so that F is given by the upper-triangular matrix

$$\begin{bmatrix} \alpha_1 & * & * \\ & \ddots & * \\ & & \alpha_{2g} \end{bmatrix}.$$

Then $H^*(BA) = \text{Sym}^* N[-1] = \overline{\mathbb{Q}}_\ell[a_1, \dots, a_{2g}]$, where each a_i sits in degree 2. In degree $2n$, $H^{2n}(BA) = \overline{\mathbb{Q}}_\ell\langle a_{i_1} \cdots a_{i_n} \mid 1 \leq i_1, \dots, i_n \leq 2g \rangle$, and the eigenvalues are $\alpha_{i_1} \cdots \alpha_{i_n}$. By Poincaré duality

$$H_c^{-2n-2g}(BA) = H^{2n}(BA)^\vee \otimes \overline{\mathbb{Q}}_\ell(g)$$

we see that the eigenvalues of F on $H_c^{-2g-2n}(BA)$ are

$$q^{-g} \cdot \alpha_{i_1}^{-1} \cdots \alpha_{i_n}^{-1}.$$

Each factor

$$P_{-2g-2n}(q^g t) = \prod_{1 \leq i_1, \dots, i_n \leq 2g} (1 - (\alpha_{i_1} \cdots \alpha_{i_n})^{-1} t)$$

stays unchanged if we permute the α_i 's arbitrarily, so the coefficients are symmetric polynomials in the α_i^{-1} 's with integer coefficients, hence are polynomials in the elementary symmetric functions, which are coefficients of $\prod_{i=1}^{2g} (t - \alpha_i^{-1})$. The polynomial

$$\prod_{i=1}^{2g} (1 - \alpha_i t) = \det (1 - Ft, H^1(A, \mathbb{Q}_\ell))$$

also has roots α_i^{-1} , and this is a polynomial with integer coefficients, independent of ℓ , since A is smooth and proper. Let $m = \pm q^g$ be leading coefficient of it. Then

$$\prod_{i=1}^{2g} (t - \alpha_i^{-1}) = \frac{1}{m} \prod_{i=1}^{2g} (1 - \alpha_i t).$$

This verifies Problem 10.3 for BA .

(ii) Let $d = \dim G_0$. For every $k \geq 0$, $H^{2k}(BG)$ is pure of weight $2k$ [Deligne 1974b, 9.1.4], hence $H_c^{-2d-2k}(BG)$ is pure of weight $-2d-2k$ by Poincaré duality. The entire function

$$\frac{1}{Z(BG_0, t)} = \prod_{k \geq 0} P_{-2d-2k}(BG_0, t) \in \mathbb{Q}[[t]]$$

is independent of ℓ , and invariant under the action of $\text{Gal}(\mathbb{Q})$ on the coefficients of the Taylor expansion. Therefore the roots of $P_{-2d-2k}(BG_0, t)$ can be described as

“zeros of $\frac{1}{Z(BG_0, t)}$ that have weight $2d + 2k$ relative to q ,”

which is a description independent of ℓ , and these roots (which are algebraic numbers) are permuted under $\text{Gal}(\mathbb{Q})$. Hence $P_{-2d-2k}(BG_0, t)$ has rational coefficients. □

The following proposition generalizes Proposition 7.3.2 and Lemma 10.4 (ii).

Proposition 10.5. *Let X_0 be the coarse moduli space of a proper smooth \mathbb{F}_q -algebraic stack of finite diagonal, and let G_0 be a linear \mathbb{F}_q -algebraic group that acts on X_0 , and let \mathcal{X}_0 be a form of the quotient stack $[X_0/G_0]$. Then Problem 10.3 is verified for \mathcal{X}_0 .*

Proof. It suffices to show that $H_c^n(\mathcal{X})$ is pure of weight n , for every n . To show this, we can make a finite extension of the base field \mathbb{F}_q , so we may assume that $\mathcal{X}_0 = [X_0/G_0]$. Let

$$\mathcal{X}_0 \xrightarrow{f} BG_0 \xrightarrow{h} B\pi_0(G_0)$$

be the natural maps.

Let $d = \dim G_0$. Consider the spectral sequence

$$H_c^{-2d-2r}(BG, R^k f_! \overline{\mathbb{Q}}_\ell) \implies H_c^{-2d-2r+k}(\mathcal{X}).$$

The E_2 -terms can be computed from the degenerate Leray spectral sequence for h :

$$H_c^{-2d-2r}(BG, R^k f_! \overline{\mathbb{Q}}_\ell) \simeq H_c^0(B\pi_0(G), R^{-2d-2r} h_! R^k f_! \overline{\mathbb{Q}}_\ell).$$

We remark that the restriction of $R^{-2d-2r} h_! R^k f_! \overline{\mathbb{Q}}_\ell$ along the natural projection $\text{Spec } \mathbb{F}_q \rightarrow B\pi_0(G_0)$ is isomorphic to the Galois module $H_c^{-2d-2r}(BG^0, R^k f_! \overline{\mathbb{Q}}_\ell)$, and since G_0^0 is connected, $(R^k f_! \overline{\mathbb{Q}}_\ell)|_{BG_0^0}$ is the inverse image of some sheaf V_k via the structural map $BG_0^0 \rightarrow \text{Spec } \mathbb{F}_q$. By base change, we see that the sheaf V_k , regarded as a $\text{Gal}(\mathbb{F}_q)$ -module, is $H^k(X)$. By projection formula we have

$$H_c^{-2d-2r}(BG^0, R^k f_! \overline{\mathbb{Q}}_\ell) \simeq H_c^{-2d-2r}(BG^0) \otimes H^k(X)$$

as representations of $\text{Gal}(\mathbb{F}_q)$, and by Proposition 7.3.2, the right hand side is pure of weight $-2d - 2r + k$. By Lemma 4.6, $H_c^{-2d-2r}(BG, R^k f_! \overline{\mathbb{Q}}_\ell)$ is also pure of weight $-2d - 2r + k$, therefore $H_c^n(\mathcal{X})$ is pure of weight n , for every n . \square

10.6. Finally, let us consider the following much weaker version of independence of ℓ . For \mathcal{X}_0 and $i \in \mathbb{Z}$, let $\Psi(\mathcal{X}_0, i)$ be the following property: the Frobenius eigenvalues of $H_c^i(\mathcal{X}, \overline{\mathbb{Q}}_\ell)$, counted with multiplicity, for all $\ell \neq p$, are contained in a finite set of algebraic numbers with multiplicities assigned, and this set together with the assignment of multiplicity, depends only on \mathcal{X}_0 and i . In particular it is independent of ℓ . In other words, there is a finite decomposition of the set of all prime numbers $\ell \neq p$ into disjoint union of some subsets, such that the Frobenius eigenvalues of $H_c^i(\mathcal{X}, \overline{\mathbb{Q}}_\ell)$ depends only on the subset that ℓ belongs to. If this property holds, we also denote such a finite set of algebraic numbers (which is not unique) by $\Psi(\mathcal{X}_0, i)$, if there is no confusion.

Proposition 10.6.1. *The property $\Psi(\mathcal{X}_0, i)$ holds for every \mathcal{X}_0 and i .*

Proof. If \mathcal{U}_0 is an open substack of \mathcal{X}_0 with complement \mathcal{Z}_0 , and properties $\Psi(\mathcal{U}_0, i)$ and $\Psi(\mathcal{Z}_0, i)$ hold, then $\Psi(\mathcal{X}_0, i)$ also holds, and the finite set $\Psi(\mathcal{X}_0, i)$ a subset of $\Psi(\mathcal{U}_0, i) \cup \Psi(\mathcal{Z}_0, i)$.

Firstly we prove this for schemes X_0 . By shrinking X_0 we can assume it is a connected smooth variety. By Poincaré duality it suffices to prove the similar statement $\Psi^*(X_0, i)$ for ordinary cohomology, that is, with H_c^i replaced by H^i , for all i . This follows from [de Jong 1996] and [Deligne 1980, 3.3.9]. Therefore it also holds for all algebraic spaces.

For a general algebraic stack \mathcal{X}_0 , by shrinking it we can assume it is connected smooth. By Poincaré duality, it suffices to prove $\Psi^*(\mathcal{X}_0, i)$ for all i . This can be

done by taking a hypercover by simplicial algebraic spaces, and considering the associated spectral sequence. \square

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