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In earlier work, the second named author described how to extract Darmon-style $\mathscr{L}$-invariants from modular forms on Shimura curves that are special at p. In this paper, we show that these $\mathscr{L}$-invariants are preserved by the Jacquet–Langlands correspondence. As a consequence, we prove the second named author's period conjecture in the case where the base field is $\mathbb{Q}$. As a further application of our methods, we use integrals of Hida families to describe Stark–Heegner points in terms of a certain Abel–Jacobi map.

1. Introduction

Let N and p be relatively prime positive integers with p prime and let

$$f=\sum_{n=1}^{\infty} a_n(f)q^n \in S_2(\Gamma_0(Np))^{p\text{-new}}$$

be a Hecke eigenform with $a_1(f)=1$. In their study of p-adic L-functions associated to modular forms, Mazur, Tate and Teitelbaum [Mazur et al. 1986] introduce a p-adic invariant of f which they call its *$\mathscr{L}$-invariant*. Let $\mathscr{X}(f, p)$ be the set of primitive Dirichlet characters with conductor prime to p such that

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$\chi(p) = a_p(f) = \pm 1$. If $\chi \in \mathscr{X}(f, p)$ then the interpolation property forces the p-adic L-function $L_p(f, \chi, s)$ of f twisted by χ to vanish at $s = 0$. This is called an *exceptional zero* phenomenon. In this case, it is conjectured in [Mazur et al. 1986] that there is a p-adic number $\mathscr{L}^{\mathrm{MTT}}(f)$ such that for all $\chi \in \mathscr{X}(f, p)$ of conductor c,

$$L_p'(f, \chi, 0) = \mathscr{L}^{\mathrm{MTT}}(f) \frac{c}{\tau(\bar{\chi})} \frac{L(f, \bar{\chi}, 1)}{\Omega_f^{\chi(-1)}}. \tag{1-1}$$

Here, $\tau(\bar{\chi})$ is the Gauss sum associated to $\bar{\chi}$ and $\Omega_f^{\chi(-1)}$ is the real or imaginary period of f, depending on the parity of χ. Note that (1-1) makes sense after fixing embeddings $\overline{\mathbb{Q}} \subset \mathbb{C}$, $\overline{\mathbb{Q}} \subset \mathbb{C}_p$, since $L(f, \bar{\chi}, 1)/\Omega_f^{\chi(-1)}$ is algebraic by a theorem of Shimura. It follows from nonvanishing results on critical L-values that $L(f, \bar{\chi}, 1) \neq 0$ for some $\chi \in \mathscr{X}(f, p)$, making (1-1) a nontrivial statement; see [Darmon 2001, Lemma 2.17] and the following remark.

The existence of $\mathscr{L}^{\mathrm{MTT}}(f)$ was proved in the influential paper [Greenberg and Stevens 1993]. Since f is p-ordinary, that is, $a_p(f)$ is a p-adic unit, f lives in a p-adic analytic family $\boldsymbol{f}$ of eigenforms by the work of Hida [1986]. More precisely, there is a p-adic disk $U \subset \mathbb{Z}_p \times \mathbb{Z}/(p-1)\mathbb{Z}$ containing 2 and a p-adic analytic function $\boldsymbol{a}_n(\boldsymbol{f}) : U \to \mathbb{C}_p$ for each $n \geq 1$, with $\boldsymbol{a}_1(\boldsymbol{f}) = 1$, such that

(1) for all integers $k \geq 2$ with $k \in U$, $\boldsymbol{a}_n(\boldsymbol{f}, k) \in \overline{\mathbb{Q}}$ and the image of

$$\boldsymbol{f}(k) := \sum_{n=1}^{\infty} \boldsymbol{a}_n(\boldsymbol{f}, k) q^n$$

in $\mathbb{C}[\![q]\!]$ is the q-expansion of an eigenform in $S_k(\Gamma_0(Np))$,

(2) $\boldsymbol{f}(2) = f$.

Moreover, up to shrinking U around 2, $\boldsymbol{f}$ is completely determined by f. Note that $1 - \boldsymbol{a}_p(\boldsymbol{f}, k)^2$ vanishes at $k = 2$ since $a_p(f) = \pm 1$. Thus, it is natural to consider the derivative of this quantity. Greenberg and Stevens show that (1-1) holds with

$$\mathscr{L}^{\mathrm{MTT}}(f) = \frac{d}{dk}\big(1 - \boldsymbol{a}_p(\boldsymbol{f}, k)^2\big)\Big|_{k=2} =: \mathscr{L}^{\mathrm{GS}}(f). \tag{1-2}$$

Also, (1-2) extends the definition of the $\mathscr{L}$-invariant from the case $a_p(f) = 1$ originally considered in [Mazur et al. 1986] to the case $a_p(f) = \pm 1$.

Mazur, Tate, and Teitelbaum further conjecture in the same work that $\mathscr{L}^{\mathrm{MTT}}(f)$ is of local type, that is, depends only on the two-dimensional p-adic representation $\sigma_p(f)$ of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ associated to f. Greenberg and Stevens [1993] proved this by showing that $\mathscr{L}^{\mathrm{GS}}(f)$ may be described in terms of the deformation theory of $\sigma_p(f)$.

Since the $\mathscr{L}$-invariant is a local-at-p invariant of f, it is natural to attempt to extract the $\mathscr{L}$-invariant of f from its Jacquet–Langlands lift g to another indefinite quaternion algebra B split at p, that is, with $B_p \cong M_2(\mathbb{Q}_p)$, since the corresponding automorphic representations have the same local components at p. (The case of definite quaternion algebras was resolved by Bertolini, Darmon and Iovita [Bertolini et al. 2010].) Following Darmon [2001], a conjectural method for doing this was proposed in [Greenberg 2009], as follows.

We first consider a certain p-arithmetic subgroup $\Theta \subset B^\times$ of level

$$N^+ := N/\operatorname{disc} B, \tag{1-3}$$

defined precisely in (6-1). We view Θ as a subgroup of $\mathrm{GL}_2(\mathbb{Q}_p)$ using the chosen isomorphism $B_p \cong M_2(\mathbb{Q}_p)$. Let $M^0(X)$ be the space of $\mathbb{C}_p$-valued measures on $X := \mathbb{P}^1(\mathbb{Q}_p)$ with total measure zero (see Section 4). The group Θ acts on X by linear fractional transformations. This induces an action of Θ on $M^0(X)$. A Mayer–Vietoris argument, together with multiplicity one, shows that for each choice of sign $\pm$ at infinity, $\dim_{\mathbb{C}_p} H^1(\Theta, M^0(X))^{g,\pm} = 1$. Here, the superscript g indicates the eigensubspace on which the Hecke operators act according to the Hecke eigenvalues of g. The superscript $\pm$ indicates the ± 1-eigenspace for the natural conjugation action of a matrix of determinant -1 that normalizes Θ. Let $\varphi_g^\pm$ be a nonzero element of $H^1(\Theta, M^0(X))^{g,\pm}$. Our definition of the $\mathscr{L}$-invariant of g will arise by considering the image of $\varphi_g^\pm$ under a certain integration pairing that we now define.

For each $\mathscr{L} \in \mathbb{C}_p$, there is a unique branch $\log_{\mathscr{L}}$ of the p-adic logarithm such that $\log_{\mathscr{L}}(p) = \mathscr{L}$. Let $\mathscr{H}_p = \mathbb{P}^1(\mathbb{C}_p) - \mathbb{P}^1(\mathbb{Q}_p)$ be the p-adic upper half-plane. Associated to each branch of the p-adic logarithm, there is a $\mathrm{PGL}_2(\mathbb{Q}_p)$-invariant integration pairing

$$\langle\,\cdot\,,\,\cdot\,\rangle_{\mathscr{L}} : M^0(X) \times \operatorname{Div}^0 \mathscr{H}_p \to \mathbb{C}_p$$

defined by

$$\langle \mu, \{\tau'\} - \{\tau\}\rangle_{\mathscr{L}} = \int_X \log_{\mathscr{L}}\Bigl(\frac{x-\tau'}{x-\tau}\Bigr)\mu(x),$$

which, in turn, induces a pairing $H^1(\Theta, M^0(X)) \times H_1(\Theta, \operatorname{Div}^0 \mathscr{H}_p) \to \mathbb{C}_p$. Let $\partial : H_2(\Theta, \mathbb{Z}) \to H_1(\Theta, \operatorname{Div}^0 \mathscr{H}_p)$ be the boundary map in the long exact sequence in Θ-cohomology associated to the short exact sequence defining $\operatorname{Div}^0 \mathscr{H}_p$:

$$0 \to \operatorname{Div}^0 \mathscr{H}_p \to \operatorname{Div} \mathscr{H}_p \xrightarrow{\deg} \mathbb{Z} \to 0.$$

Proposition 1 [Greenberg 2009, Prop. 30]. *There are unique constants $\mathscr{L}^D(\varphi_g^\pm)$ in $\mathbb{C}_p$ such that $\langle \varphi_g^\pm, \partial H_2(\Theta, \mathbb{Z})\rangle_{-\mathscr{L}^D(\varphi_g^\pm)} = \{0\}$.*

We have chosen the notation $\mathscr{L}^{\mathrm{D}}(\varphi_g^{\pm})$ for these $\mathscr{L}$-invariants since they are defined following methods of Darmon [2001]. The goal of this paper is to relate these $\mathscr{L}$-invariants $\mathscr{L}^{\mathrm{D}}(\varphi_g^{\pm})$ arising from the cohomology of Shimura curves to those whose origins lie in the arithmetic of classical modular curves. The following is our main result:

Theorem 2.
$$\mathscr{L}^{D}(\varphi_g^{\pm}) = \mathscr{L}^{GS}(f).$$

Using Theorem 2, we deduce Conjecture 2 of [Greenberg 2009] in the case where the base field is $\mathbb{Q}$; see Section 8 for details. The proof of Theorem 2 falls into two steps. Applying a result of Hida's theory, we deform the Jacquet–Langlands lift g of f into a cohomological Hida family $\Phi_g^{\pm}$. Let $\boldsymbol{a}_p = \boldsymbol{a}_p(k)$ be the eigenvalue of U_p acting on $\Phi_g^{\pm}$. Group cohomological calculations building upon those in [Dasgupta 2005] show that

$$\mathscr{L}^{\mathrm{D}}(\varphi_g^{\pm}) = \frac{d}{dk}\bigl(1-\boldsymbol{a}_p(\boldsymbol{g},k)^2\bigr)\Big|_{k=2} =: \mathscr{L}^{\mathrm{GS}}(g).$$

It remains to show that $\mathscr{L}^{\mathrm{GS}}(g) = \mathscr{L}^{\mathrm{GS}}(f)$. We prove this in Theorem 8, which asserts a compatibility between the Jacquet–Langlands correspondence with the formation of Hida families. This result is a weak analogue of results of Chenevier [2005] for definite quaternion algebras and may be of independent interest.

In the last section of this paper, we apply our computations to the theory of *Stark–Heegner points*. Let $E/\mathbb{Q}$ be an elliptic curve of conductor Np and suppose that $\mathbb{O}$ is a real quadratic order with fraction field K such that $(\operatorname{disc}\mathbb{O}, Np) = 1$. Assume further that the sign in the functional equation of $L(E/K, s)$ is -1. Then for each character $\chi : \mathrm{Cl}_{\mathbb{O}}^{+} \to \mathbb{C}^{\times}$ of the narrow ideal class group of $\mathbb{O}$, the sign in the functional equation of $L(E/K, \chi, s)$ is also -1. Thus, the conjecture of Birch and Swinnerton-Dyer leads one to expect that

$$\operatorname{rank} E(H_{\mathbb{O}}) = \operatorname{ord}_{s=1} L(E/H_{\mathbb{O}}, s) = \operatorname{ord}_{s=1} \prod_{\chi:\mathrm{Cl}_{\mathbb{O}}^{+}\to\mathbb{C}^{\times}} L(E/K, \chi, s) \geq |\,\mathrm{Cl}_{\mathbb{O}}^{+}|, \quad (1\text{-}4)$$

where $H_{\mathbb{O}}$ is the narrow ring class field associated to the order $\mathbb{O}$. In [Greenberg 2009], a p-adic analytic construction of local *Stark–Heegner points* on E was presented, generalizing a construction of Darmon [2001] applicable when p is inert in K and the primes dividing N split in K. The local definition of Stark–Heegner points given in [Greenberg 2009] is contingent upon Conjecture 2 [ibid.] over the base field $\mathbb{Q}$; this now follows from Theorem 2. The analytically defined Stark–Heegner points are conjectured to be defined over the field $H_{\mathbb{O}}$, and are expected to generate a finite index subgroup of $E(H_{\mathbb{O}})$ when the inequality in (1-4) is an equality.

The strongest theoretical evidence presented to date for the conjectures of [Darmon 2001] on the rationality of Stark–Heegner points is the main result of [Bertolini and Darmon 2009], which proves the rationality of certain linear combinations of Stark–Heegner points. A key tool in the proof of this result is a description of the formal group logarithms of Stark–Heegner points in terms of periods of Hida families. In Section 9, we prove such a formula for the Stark–Heegner points of [Greenberg 2009]. We intend to pursue the analogue of the rationality result of [Bertolini and Darmon 2009] in future work.

2. Modular forms on quaternion algebras and the cohomology of Shimura curves

Let f be as in the introduction with level Np, $p \nmid N$. In order to ensure that f admits a Jacquet–Langlands lift to an indefinite quaternion $\mathbb{Q}$-algebra, we suppose that the tame part N of the level of f admits a factorization

$$N = N^- N^+, \quad (N^-, N^+) = 1,$$

such that f is N^--new. We work under the additional simplifying assumption that N^- is squarefree.

Let B be the indefinite quaternion $\mathbb{Q}$-algebra with discriminant N^-. Let $R_{\max}$ be a maximal order in B. Let ℓ be a prime with $\ell \nmid N^-$. Since B is split at ℓ, we may choose an embedding

$$\iota_\ell : B \to M_2(\mathbb{Q}_\ell)$$

such that $\iota_\ell(R_{\max}) \subset M_2(\mathbb{Z}_\ell)$. Define

$$R = \left\{ \alpha \in R_{\max} : \iota_\ell(\alpha) \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod{N^+\mathbb{Z}_\ell} \text{ for all } \ell \nmid N^- \right\}, \tag{2-1}$$

$$R_0 = \left\{ \alpha \in R_{\max} : \iota_\ell(\alpha) \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod{pN^+\mathbb{Z}_\ell} \text{ for all } \ell \nmid N^- \right\}. \tag{2-2}$$

The rings R and R_0 are Eichler orders in B of level N^+ and pN^+, respectively. Set

$$\Gamma = R_+^\times/\{\pm 1\}, \quad \Gamma_0 = R_{0,+}^\times/\{\pm 1\},$$

where the subscript $+$ indicates elements with positive reduced norm.

Since B is split at the infinite place of $\mathbb{Q}$, we may choose an embedding

$$\iota_\infty : B \to M_2(\mathbb{R}). \tag{2-3}$$

The groups Γ and Γ_0 may be viewed as discrete groups of transformations of the complex upper half-plane $\mathscr{H}$ by identifying them with subgroups of $\mathrm{PGL}_2(\mathbb{R})$ via ι_∞. The quotients

$$Y(\mathbb{C}) := \Gamma\backslash\mathscr{H}, \qquad Y_0(\mathbb{C}) := \Gamma_0\backslash\mathscr{H}$$

are Riemann surfaces, compact exactly when $N^- \neq 1$. Let $\mathcal{H}^* = \mathcal{H} \cup \mathbb{P}^1(\mathbb{Q})$ be the extended complex upper half-plane and define

$$X(\mathbb{C}) = \begin{cases} Y(\mathbb{C}) & \text{if } N^- \neq 1, \\ \Gamma \backslash \mathcal{H}^* & \text{if } N^- = 1. \end{cases}$$

Define $X_0(\mathbb{C})$ analogously. The Riemann surfaces $X(\mathbb{C})$ and $X_0(\mathbb{C})$ are compact and may be identified with the loci of complex points of Shimura curves X and X_0 that admit canonical models over $\mathbb{Q}$. Of course, these are just the classical modular curves in the case $N^- = 1$. For the remainder of this section, we assume that $N^- \neq 1$.

Let $S_k(\Gamma)$ and $\overline{S_k(\Gamma)}$ be the spaces of holomorphic and, respectively, antiholomorphic weight k cusp forms on $X(\Gamma)$. The spaces $S_k(\Gamma_0)$ and $\overline{S_k(\Gamma_0)}$ are defined analogously. These spaces admit the action of a commutative algebra of Hecke operators, all commuting with complex conjugation (see Section 3).

Theorem 3 (Jacquet–Langlands correspondence). *Let $k \geq 2$ be an integer. There are isomorphisms*

$$S_k(\Gamma_0(N))^{N^-\text{-new}} \cong S_k(\Gamma) \quad \textit{and} \quad S_k(\Gamma_0(Np))^{N^-\text{-new}} \cong S_k(\Gamma_0).$$

Both isomorphisms are equivariant with respect to the Hecke operators T_ℓ for $\ell \nmid Np$, U_ℓ for $\ell \mid N^+$ and W_ℓ for $\ell \mid N^-$. In addition, the first isomorphism equivariant with respect to T_p, and the second is equivariant with respect to U_p.

Therefore, there is a one-dimensional subspace of $S_2(\Gamma_0)$, independent of the choice of isomorphism in the Jacquet–Langlands correspondence, on which the Hecke operators act via the eigenvalues of f. Let g be a nonzero element of this space. We call g a *Jacquet–Langlands lift of f*. Let $a_\ell(g) = a_\ell(f)$ be the eigenvalue of T_ℓ, U_ℓ, or $-W_\ell$ acting on g in the cases $\ell \nmid Np$, $\ell \mid pN^+$, and $\ell \mid pN^-$, respectively.

We are also interested in cohomological avatars of g. We have canonical isomorphisms of Betti and group cohomology

$$H^*(\Gamma, E) \cong H^*(X(\mathbb{C}), E), \qquad H^*(\Gamma_0, E) \cong H^*(X_0(\mathbb{C}), E)$$

for any characteristic zero field E endowed with the trivial action of Γ. By the de Rham theorem and the Hodge decomposition,

$$\begin{aligned} H^1(\Gamma_0, \mathbb{C}) &= H^1(X_0(\mathbb{C}), \mathbb{C}) \\ &= H^{1,0}(X_0(\mathbb{C}), \mathbb{C}) \oplus H^{0,1}(X_0(\mathbb{C}), \mathbb{C}) \;\cong\; S_2(\Gamma_0) \oplus \overline{S_2(\Gamma_0)}. \end{aligned}$$

Therefore, if E is any field containing the Hecke eigenvalues of g, we have

$$\dim_E H^1(\Gamma_0, E)^g = 2,$$

where the superscript g indicates Hecke eigenspace corresponding to the system of Hecke eigenvalues of g:

$$H^1(\Gamma_0, E)^g = \{c \in H^1(\Gamma_0, E) : T_\ell(c) = a_\ell(g)c \text{ for } \ell \nmid N,\\ U_\ell(c) = a_\ell(g)c \text{ for } \ell \mid pN^+\}.$$

(See Section 3 for a detailed description of Hecke operators acting on group cohomology.) Note that this space is stable for the Atkin–Lehner involutions $-W_\ell$ for $\ell \mid pN^-$ with eigenvalues $a_\ell(g)$. Conjugation by an element of $R_0^\times$ of reduced norm -1 induces an automorphism of $H^1(\Gamma_0, E)$ under which the subspace $H^1(\Gamma_0, E)^g$ is stable. This action corresponds to complex conjugation of cusp forms and is denoted W_∞. Therefore, $H^1(\Gamma_0, E)^g$ decomposes into one-dimensional $\pm$-eigenspaces for this action:

$$H^1(\Gamma_0, E)^g = H^1(\Gamma_0, E)^{g,+} \oplus H^1(\Gamma_0, E)^{g,-}.$$

We denote by $g^\pm$ a nonzero element of $H^1(\Gamma_0, E)^{g,\pm}$. In Section 4 we construct a cohomological Hida family $\Phi_g^\pm$ that specializes to $g^\pm$ in weight 2, and in Section 6 we use $\Phi_g^\pm$ to define the Darmon $\mathscr{L}$-invariant $\mathscr{L}^{\mathrm{D}}(g^\pm)$.

3. Hecke operators and group cohomology

In anticipation of the delicate group cohomological calculations to follow, we carefully set up notation for describing the action of Hecke operators on various cohomology groups. Let $G \subset K$ be an inclusion of groups, x an element of K, M a G-module, and M' an xGx^{-1}-module. Suppose that $\xi : M \to M'$ is a group homomorphism such that

$$\xi(gm) = xgx^{-1}\xi(m). \tag{3-1}$$

for all $g \in G$ and $m \in M$. In our applications, $M \subset M''$ for a K-module M'', and ξ is the map $m \mapsto xm$ with $M' = xM \subset M''$. The map ξ induces a homomorphism

$$\xi_* : H^*(G, M) \to H^*(xGx^{-1}, M') \tag{3-2}$$

as follows: Let $F_\bullet \to \mathbb{Z}$ be a resolution of $\mathbb{Z}$ by free K-modules. Note that F_r is also a free G-module and a free xGx^{-1}-module. In what follows, we will often take $F_r = \mathbb{Z}[K^{r+1}]$. Formally, ξ induces a map of cochain complexes relative to this resolution,

$$\xi_* : \mathrm{Hom}_G(F_r, M) \to \mathrm{Hom}_{xGx^{-1}}(F_r, M'), \quad \xi_*(\varphi)(f_r) = \xi(\varphi(x^{-1} f_r)),$$

which induces (3-2). We now use this formalism to define the Hecke operators that play a role in this paper.

• Suppose that $\ell > 0$ is a prime divisor of N^-. Then there exists an element $\lambda \in R_0$ whose reduced norm is ℓ and such that λ generates the unique two-sided ideal of R_0 with norm ℓ. The element λ normalizes R_0 by [Vignéras 1980, chapitre II, corollaire 1.7]. Take $G = \Gamma_0$ or Γ, $K = B^\times/\mathbb{Q}^\times$, $x = \lambda$. Let M be a G-module such that $M = \lambda M$ (that is, this equality holds in a K-module M'' containing M). The formalism above then yields the *Atkin–Lehner involutions*

$$W_\ell : H^r(\Gamma_0, M) \to H^r(\Gamma_0, M), \quad W_\ell : H^r(\Gamma, M) \to H^r(\Gamma, M). \tag{3-3}$$

• Let $w_p \in R_0$ be an element of reduced norm p that generates the normalizer of Γ_0 in $R[1/p]_+^\times$ and define

$$\widetilde{\Theta} = R[1/p]_+^\times \big/ \mathbb{Z}[1/p]^\times. \tag{3-4}$$

The groups Γ_0, Γ, and $\Gamma' := w_p \Gamma w_p^{-1}$ are all subgroups of $\widetilde{\Theta}$. Using the above formalism with $G = \Gamma_0$ or Γ, $K = \widetilde{\Theta}$, and $x = w_p$ yields *Atkin–Lehner maps*

$$W_p : H^r(\Gamma_0, M) \to H^r(\Gamma_0, M'), \quad W_p : H^r(\Gamma, M) \to H^r(\Gamma', M'), \tag{3-5}$$

with $M' = w_p M$. We note that these maps are isomorphisms, as applying the same formalism with w_p^{-1} instead of w_p yields inverse homomorphisms W_p^{-1}.

• Let $\ell > 0$ be a prime with $\ell \nmid N^-$. Choose an element $\lambda \in R_0$ of reduced norm ℓ. When $\ell \mid pN^+$, we insist that

$$\iota_\ell(\lambda)\mathfrak{I}_\ell \in \begin{pmatrix} 1 & 0 \\ 0 & \ell \end{pmatrix} \mathfrak{I}_\ell, \tag{3-6}$$

where $\mathfrak{I}_\ell$ is the Iwahori subgroup of $\mathrm{GL}_2(\mathbb{Z}_\ell)$ defined by

$$\mathfrak{I}_\ell = \left\{ \alpha \in \mathrm{GL}_2(\mathbb{Z}_\ell) : \alpha \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod{\ell} \right\}.$$

Consider a double coset decomposition

$$\Gamma_0 \cdot \lambda \cdot \Gamma_0 = \bigcup_i \gamma_a \Gamma_0. \tag{3-7}$$

Let Σ be the subsemigroup of $\widetilde{\Theta}$ generated by Γ_0 together with λ, and let M be a Σ-module. Let $F_\bullet \to \mathbb{Z}$ be a resolution of $\mathbb{Z}$ by free $\widetilde{\Theta}$-modules, and define an endomorphism T_ℓ of the cochain complex $\mathrm{Hom}_{\Gamma_0}(F_\bullet, M)$ by

$$(T_\ell \varphi)(f_r) = \sum_i \gamma_i \varphi(\gamma_i^{-1} f_r), \quad f_r \in F_r. \tag{3-8}$$

It is routine to check that T_ℓ does not depend on the choice of coset representatives and descends to a well defined endomorphism T_ℓ of $H^*(\Gamma_0, M)$. When $\ell \mid pN^+$, we write U_ℓ instead of T_ℓ for this operator.

• Finally, let Π denote the matrix $\lambda \in R_0$ of reduced norm p chosen above to satisfy (3-6) when $\ell = p$. Let $\Pi' = w_p \Pi w_p^{-1}$. Then

$$\iota_p(\Pi')\mathfrak{I}_p = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \mathfrak{I}_p.$$

Let U_p' be the Hecke operator associated to the double coset $\Gamma_0 \Pi' \Gamma_0$. It is easy to check that

$$U_p' = W_p \circ U_p \circ W_p^{-1}. \tag{3-9}$$

Note that this holds on the level of cochains if we choose compatible double coset decompositions:

$$\Gamma_0 \Pi \Gamma_0 = \bigcup_i \gamma_a \Gamma_0, \qquad \Gamma_0 \Pi' \Gamma_0 = \bigcup_i (w_p \gamma_a w_p^{-1}) \Gamma_0.$$

4. p-adic measures, Hida families, and Greenberg–Stevens $\mathscr{L}$-invariants

Let Y be a compact topological space with a basis of compact-open subsets and let A be a subring of $\mathbb{C}_p$. Write $C^\infty(Y) = C^\infty(Y, A)$ for the group of locally constant, A-valued functions on Y, equipped with the sup-norm. An *A-valued measure on Y* is a bounded A-linear functional on $C^\infty(Y, A)$. We write $M(Y) = M(Y, A)$ for the space of such measures, which can be identified with the space of finitely additive, A-valued functions on the set of compact-open subsets of Y whose values are bounded. For details, see [Mazur and Swinnerton-Dyer 1974, §7.1].

Let

$$\mathbb{X} = (\mathbb{Z}_p^2)' := \mathbb{Z}_p^2 - p(\mathbb{Z}_p^2), \quad \mathbb{X}_\infty = \mathbb{Z}_p^\times \times p\mathbb{Z}_p \subset \mathbb{X}. \tag{4-1}$$

The spaces $M(\mathbb{X})$ and $M(\mathbb{X}_\infty)$ are naturally modules for the Iwasawa algebra $\Lambda := \mathbb{Z}_p[[1 + p\mathbb{Z}_p]]$, where group-like elements act via the natural diagonal action of $1 + p\mathbb{Z}_p$ on $\mathbb{X}$; given $\ell \in 1 + p\mathbb{Z}_p$, we define $([\ell]\mu)(h(x, y)) := \mu(h(\ell x, \ell y))$.

Let

$$\epsilon : \Lambda \to \mathbb{Z}_p \tag{4-2}$$

be the augmentation map defined by $[\ell] \mapsto 1$ and let I_ϵ be the kernel of ϵ. Letting γ be a topological generator of $1 + p\mathbb{Z}_p$, it follows that I_ϵ is generated by

$$\varpi := [\gamma] - 1.$$

The group $\mathrm{GL}_2(\mathbb{Z}_p)$ acts on $\mathbb{X}$ from the left by viewing elements of $\mathbb{X}$ as column vectors. The group Γ acts on $\mathbb{X}$ via the embedding $\iota_p : R^\times \hookrightarrow \mathrm{GL}_2(\mathbb{Z}_p)$, and $\mathbb{X}_\infty$ is stable under Γ_0. Therefore, we may consider the cohomology groups $H^*(\Gamma, M(\mathbb{X}))$ and $H^*(\Gamma_0, M(\mathbb{X}_\infty))$. These cohomology groups are canonically isomorphic:

Lemma 4. *The map*

$$H^*(\Gamma, M(\mathbb{X})) \to H^*(\Gamma_0, M(\mathbb{X}_\infty))$$

induced by the Γ_0*-equivariant inclusion* $\mathbb{X}_\infty \hookrightarrow \mathbb{X}$ *is an isomorphism.*

Proof. The $p+1$ translates of $\mathbb{X}_\infty$ by Γ cover $\mathbb{X}$. It follows that

$$M(\mathbb{X}) = \text{Co-Ind}_{\Gamma_0}^{\Gamma} M(\mathbb{X}_\infty).$$

The lemma now follows from Shapiro's lemma. □

Let us assume that our measures take values in $\mathbb{Z}_p$ (so $M(\mathbb{X})$ denotes $M(\mathbb{X}, \mathbb{Z}_p)$, etc.). We set $\widetilde{\mathbb{W}} := H^1(\Gamma_0, M(\mathbb{X}_\infty)) \cong H^1(\Gamma, M(\mathbb{X}))$. View Λ as a $\mathbb{Z}_p[\![Z_p^\times]\!]$-algebra via the canonical projection

$$\mathbb{Z}_p^\times \to 1 + p\mathbb{Z}_p, \qquad \ell \mapsto \langle \ell \rangle := \ell/\omega(\ell),$$

where ω is the Teichmuller character. Define the Λ-algebra $\mathbb{W} := \widetilde{\mathbb{W}} \otimes_{\mathbb{Z}_p[\![\mathbb{Z}_p^\times]\!]} \Lambda$.

As $\Pi\mathbb{X}_\infty \subset \mathbb{X}_\infty$, the semigroup Σ of Section 3 acts on $M(\mathbb{X}_\infty)$. Therefore, the formalism of Section 3 endows $\mathbb{W}$ with an action of the U_p-operator. In addition to the U_p-action, the group $\mathbb{W}$ enjoys an action of

- Hecke operators T_ℓ for primes $\ell \nmid pN$ and U_ℓ for $\ell \mid N^+$, and
- Atkin–Lehner involutions W_ℓ for $\ell \mid N^-$.

See Section 3 for the definitions of these operators. Let $\mathbb{T}$ be the commutative Λ-subalgebra of $\mathrm{End}_\Lambda \mathbb{W}$ generated by these operators. Let $\rho : M(\mathbb{X}_\infty) \to \mathbb{Z}_p$ be the total measure map. It induces a corresponding map

$$\rho : \mathbb{W} \to H^1(\Gamma_0, \mathbb{Z}_p). \tag{4-3}$$

The map ρ respects the decomposition into $\pm$-eigenspaces:

$$\rho : \mathbb{W}^\pm \to H^1(\Gamma_0, \mathbb{Z}_p)^\pm.$$

Let $e = \lim_{n\to\infty} U_p^{n!}$ denote Hida's ordinary idempotent and, for any $\mathbb{T}$-module M, let $M^o = eM$. In particular, $\mathbb{T}^o = e\mathbb{T}$ is Hida's ordinary Hecke algebra.

Theorem 5 (Hida's control theorem). *There is an exact sequence*

$$0 \to \varpi\mathbb{W}^{\pm,o} \to \mathbb{W}^{\pm,o} \xrightarrow{\rho} H^1(\Gamma_0, \mathbb{Z}_p)^{\pm,o} \to 0. \tag{4-4}$$

The kernel of the Λ-algebra homomorphism $\mathbb{T}^o \to \overline{\mathbb{Q}}_p$ given by sending a Hecke operator to its eigenvalue on g is a prime ideal $\mathfrak{p} \subset \mathbb{T}^o$ lying above the augmentation ideal $I_\epsilon \subset \Lambda$. The following fundamental result is due to Hida in the case $N^- = 1$ (see [Greenberg and Stevens 1993]), and was extended in [Balasubramanyam and Longo 2011] to the case $N^- \neq 1$.

Theorem 6. *There is a unique minimal prime $\mathfrak{P} \subset \mathfrak{p}$, and the quotient $R := \mathbb{T}^o/\mathfrak{P}$ is a finite flat extension of Λ unramified above I_ϵ.*

Let R be as in the theorem, and let $R_\mathfrak{p}$ be the localization of R at $\mathfrak{p}$. Let E be the field of fractions of the integral closure of $\mathbb{Z}_p$ in R. It is a finite extension of $\mathbb{Q}_p$. We write $\epsilon : R_\mathfrak{p} \to R_\mathfrak{p}/\varpi R_\mathfrak{p} \cong E$ for the reduction map. This notation is justified as this map extends the augmentation $\epsilon : \Lambda \to \mathbb{Z}_p$.

Write $(\mathbb{W} \otimes_\Lambda R_\mathfrak{p})^{\pm,g}$ for the subspace of $(\mathbb{W} \otimes_\Lambda R_\mathfrak{p})^{\pm}$ on which $\mathbb{T}$ acts via the canonical map $\mathbb{T} \to R_\mathfrak{p}$. Note that

$$(\mathbb{W} \otimes_\Lambda R_\mathfrak{p})^{\pm,g} \subset (\mathbb{W} \otimes_\Lambda R_\mathfrak{p})^{\pm,o} = \mathbb{W}^{\pm,o} \otimes_\Lambda R_\mathfrak{p}$$

and that

$$H^1(\Gamma_0, \mathbb{Z}_p) \otimes_\Lambda R_\mathfrak{p} = H^1(\Gamma_0, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} E = H^1(\Gamma_0, E). \tag{4-5}$$

On the left of (4-5), we view $H^1(\Gamma_0, \mathbb{Z}_p)$ as a Λ-module via the augmentation ϵ.

Corollary 7 [Balasubramanyam and Longo 2011, §3.6]. *The sequence*

$$0 \to \varpi (\mathbb{W} \otimes_\Lambda R_\mathfrak{p})^{\pm,g} \to (\mathbb{W} \otimes_\Lambda R_\mathfrak{p})^{\pm,g} \to H^1(\Gamma_0, E)^{\pm,g} \to 0$$

obtained by tensoring (4-4) with $R_\mathfrak{p}$ over Λ and taking g-isotypic components is exact, and $\operatorname{rank}_{R_\mathfrak{p}} (\mathbb{W} \otimes_\Lambda R_\mathfrak{p})^{\pm,g} = 1$.

We now view $g^\pm$ as an element of $H^1(\Gamma_0, E)^{\pm,g}$. By Corollary 7, we may choose a lift

$$\Phi_g^\pm \in (\mathbb{W} \otimes_\Lambda R_\mathfrak{p})^{\pm,g} \tag{4-6}$$

of $g^\pm$. The element $\Phi_g^\pm$ is well defined up to multiplication by an element of $1 + \varpi R_\mathfrak{p}$. We call $\Phi_g^\pm$ a *Hida family through $g^\pm$*. We denote its U_p-eigenvalue by $\boldsymbol{a}_p(\Phi_g^\pm) \in R_\mathfrak{p}$. Since $\epsilon(\boldsymbol{a}_p(\Phi_g^\pm)) = a_p(g^\pm) = a_p(g) = a_p(f) = \pm 1$, we see that $1 - \boldsymbol{a}_p(\Phi_g^\pm)^2$ lies in $\varpi R_\mathfrak{p}$. There is a "derivative map" $d_\epsilon : \varpi R_\mathfrak{p}/(\varpi R_\mathfrak{p})^2 \to E$ that extends the map $I_\epsilon/I_\epsilon^2 \to \mathbb{Z}_p$ given by the p-adic logarithm:

$$[\ell] - 1 \mapsto \log(\ell). \tag{4-7}$$

Since $\ell \in \mathbb{Z}_p^\times$, we need not specify a branch of the p-adic logarithm. We define the *Greenberg–Stevens $\mathscr{L}$-invariant of g* by

$$\mathscr{L}^{\mathrm{GS}}(\Phi_g^\pm) = d_\epsilon\big(1 - \boldsymbol{a}_p(\Phi_g^\pm)^2\big) \in E.$$

The derivative map d_ϵ is related to the usual notion of derivative in the following way. For $0 < r \le 1$, let $\mathscr{A}_r$ be the subring of $\overline{\mathbb{Q}}[\![x]\!]$ consisting of those powers series that converge on the closed disk centered at 0 with radius r. Evidently, if $r < s$, then there is a canonical inclusion $\mathscr{A}_s \subset \mathscr{A}_r$. Therefore, we may set $\mathscr{A} = \bigcup_r \mathscr{A}_r$. Define $i : \Lambda \to \mathscr{A}_1$ by sending a group-like element $[\ell]$, for $\ell \in 1 + p\mathbb{Z}_p$, to the function $k \mapsto \ell^{k-2}$. Since R is unramified over I_ϵ and $\mathscr{A}$ is Henselian, there is a unique extension of i to a Λ-algebra homomorphism $i : R_\mathfrak{p} \to \mathscr{A}$. An element $\lambda \in R_\mathfrak{p}$ lies in $\varpi R_\mathfrak{p}$ if and only if the associated analytic function $i(\lambda)$ has a zero at $k = 2$. In this case, $d_\epsilon(\lambda) = i(\lambda)'(2)$.

Theorem 8. *We have the following equality of Greenberg–Stevens $\mathscr{L}$-invariants*:

$$\mathscr{L}^{GS}(\Phi_g^\pm) = \mathscr{L}^{GS}(f).$$

Proof. Suppose R' is a finitely generated R-subalgebra of $R_\mathfrak{p}$ such that $\Phi_g^\pm$ lies in $(\mathbb{W} \otimes_\Lambda R')^{g,\pm}$. With notation as above, there is some r_0 such that $i(R')$ is contained in $\mathscr{A}_{r_0}$.

Let $P_{k-2}(\overline{\mathbb{Q}})$ be the space of homogeneous polynomials of degree $k-2$ in indeterminates x and y, and let $V_{k-2}(\overline{\mathbb{Q}})$ be its $\overline{\mathbb{Q}}$-linear dual. Define a "specialization to weight k" map

$$\rho_k : M(\mathbb{X}_\infty) \to V_{k-2}(\overline{\mathbb{Q}})$$

by the rule

$$\rho_k(\Phi)(P) = \int_{\mathbb{X}_\infty} P(x, y)\Phi(x, y).$$

This map being Γ_0-equivariant, it induces a homomorphism

$$\rho_k : H^1(\Gamma_0, M(\mathbb{X}_\infty)) \to H^1(\Gamma_0, V_{k-2}(\overline{\mathbb{Q}})).$$

The map ρ defined in (4-3) coincides with ρ_2 in this more general notation.

If $|k-2|_p \le r$, we may extend ρ_k to a map

$$\rho_k : H^1(\Gamma_0, M(\mathbb{X}_\infty)) \otimes_\Lambda \mathscr{A}_r \to H^1(\Gamma_0, V_{k-2}(\overline{\mathbb{Q}}))$$

by setting

$$\rho_k\Bigl(\sum_i \varphi_i \otimes \alpha_i\Bigr) = \sum_i \alpha_i(k)\rho_k(\varphi_i).$$

One may verify formally that ρ_k is Hecke-equivariant.

Let $\boldsymbol{a}_\ell$ be the image in $\mathscr{A}_{r_0}$ of the eigenvalue of T_ℓ, $-\langle \ell \rangle^{(k-2)/2} W_\ell$, or U_ℓ acting on $\Phi_g^\pm$ in the cases $\ell \nmid Np$, $\ell \mid N^-$, and $\ell \mid N^+ p$, respectively. Here $\langle \ell \rangle$ denotes the projection of ℓ onto $1 + p\mathbb{Z}_p$. Set $\boldsymbol{a}_1 = 1$ and define $\boldsymbol{a}_n$ in terms of the $\boldsymbol{a}_\ell$ with $\ell \mid n$ by the usual formulas for Hecke operators.

We may shrink r_0 if necessary to ensure that $\rho_k(\Phi_g^\pm)$ is a nonzero element of $H^1(\Gamma_0, V_{k-2}(\overline{\mathbb{Q}}_p))$ for all $k \ge 2$ with $|k-2|_p \le r_0$ and $k \equiv 2 \pmod{p-1}$. The class $\rho_k(\Phi_g^\pm)$ is an eigenvector for the ℓ-th Hecke operator with eigenvalue $\boldsymbol{a}_\ell(k)$. Thus, $\{\boldsymbol{a}_\ell(k)\}$ is a system of Hecke eigenvalues occurring in $H^1(\Gamma_0, V_{k-2}(\overline{\mathbb{Q}}_p))$. In particular, $\{\boldsymbol{a}_\ell(k)\} \subset \overline{\mathbb{Q}} \subset \overline{\mathbb{Q}}_p$. By the Eichler–Shimura isomorphism [Matsushima and Shimura 1963, §4], this system of Hecke eigenvalues also occurs in $S_k(\Gamma_0)$.

By the Jacquet–Langlands correspondence, it occurs in $S_k(\Gamma_0(pN))$ as well. Thus, if we set

$$\boldsymbol{h} := \sum_{n=1}^{\infty} \boldsymbol{a}_n q^n \in \mathscr{A}_{r_0}[\![q]\!],$$

then $\boldsymbol{h}(k) = \sum \boldsymbol{a}_n(k) q^n$ is in fact the q-expansion of a classical cusp form of weight k on $\Gamma_0(Np)$ for $k \geq 2$, $|k-2|_p \leq r_0$, $k \equiv 2 \pmod{p-1}$. Furthermore, it is clear that $\boldsymbol{h}(2) = f$. Therefore, by the uniqueness of the Hida family through f [Hida 1986, Corollary 1.3, pg. 554], it follows that $\boldsymbol{a}_n(k) = \boldsymbol{a}_n(\boldsymbol{f}, k)$ for $|k-2|_p \leq r_0$. In particular, this is true for $n = p$; Theorem 8 follows. □

Finally, we record a result that will be important later. Set

$$\mathbb{W}^0 = H^1(\Gamma, M^0(\mathbb{X})) \otimes_{\mathbb{Z}_p[\![\mathbb{Z}_p^\times]\!]} \Lambda.$$

Lemma 9. *The canonical map*

$$(\mathbb{W}^0 \otimes_\Lambda R_\mathfrak{p})^{\pm,g} \to (\mathbb{W} \otimes_\Lambda R_\mathfrak{p})^{\pm,g} \tag{4-8}$$

is an isomorphism.

Proof. The map $\rho : M(\mathbb{X}) \to \mathbb{Z}_p$ gives rise to the short exact sequence

$$0 \to M^0(\mathbb{X}) \to M(\mathbb{X}) \xrightarrow{\rho} \mathbb{Z}_p \to 0.$$

Since R is Λ-flat, we may tensor $R_\mathfrak{p}$ with the associated long exact sequence in Γ-cohomology to obtain

$$\cdots \to H^0(\Gamma, E) \to \mathbb{W}^0 \otimes_\Lambda R_\mathfrak{p} \to \mathbb{W} \otimes_\Lambda R_\mathfrak{p} \to H^1(\Gamma, E) \to \cdots.$$

The space $H^0(\Gamma, E)$ is Eisenstein (that is, T_ℓ acts as $1+\ell$), so its g-isotypic component is trivial. Since the maps in the sequence above are Hecke-equivariant, it follows that the map (4-8) is injective. Similarly, if $\Phi \in (\mathbb{W} \otimes_\Lambda R_\mathfrak{p})^{\pm,g}$, then its image in $H^1(\Gamma, E)$ must be zero. This holds because g is p-new of level Γ_0, so the system of Hecke eigenvalues of g does not occur in $H^1(\Gamma, E)$. Therefore Φ is the image of an element $\widetilde{\Phi} \in \mathbb{W}^0 \otimes_\Lambda R_\mathfrak{p}$. Let ℓ be any prime such that the eigenvalue $a_\ell(g)$ of the Hecke operator T_ℓ is not equal to $\ell+1$. Let $\boldsymbol{a}_\ell(\Phi)$ denote the T_ℓ eigenvalue of Φ, that is, the image of T_ℓ in $R_\mathfrak{p}$. We claim that

$$\widetilde{\Phi}' := \frac{T_\ell - (\ell+1)}{\boldsymbol{a}_\ell(\Phi) - (\ell+1)} \widetilde{\Phi} \tag{4-9}$$

is a lift of Φ to $(\mathbb{W}^0 \otimes_\Lambda R_\mathfrak{p})^{\pm,g}$. First note that the division in (4-9) is allowed in the localization, since the image of $\boldsymbol{a}_\ell(\Phi) - (\ell+1)$ under reduction modulo $\mathfrak{p}$ is $a_\ell(g) - (\ell+1) \neq 0$. Next, it is clear that $\widetilde{\Phi}'$ maps to Φ under (4-8) since Φ has T_ℓ eigenvalue $\boldsymbol{a}_\ell(\Phi)$. Finally, let $\lambda \in \mathbb{T}^o$, and let $\boldsymbol{a}_\lambda(\Phi)$ be the corresponding eigenvalue of Φ. Then $(\lambda - \boldsymbol{a}_\lambda(\Phi))\widetilde{\Phi}$ maps to 0 in $\mathbb{W} \otimes_\Lambda R_\mathfrak{p}$ and hence arises from

$H^0(\Gamma, E)$. Since this module is Eisenstein, it is killed by $T_\ell - (\ell+1)$, and it follows that $(\lambda - \boldsymbol{a}_\lambda(\Phi))\widetilde{\Phi}' = 0$. This shows that $\widetilde{\Phi}'$ lies in $(\mathbb{W}^0 \otimes_\Lambda R_\mathfrak{p})^{\pm,g}$, and concludes the proof of the lemma. □

Using Lemma 9, we may view $\Phi_g^\pm$ an element of $(\mathbb{W}^0 \otimes_\Lambda R_\mathfrak{p})^{\pm,g}$.

5. Some commutative diagrams

In this section, we establish some commutative diagrams involving the operators U_p, U_p', and W_p acting on the group cohomology of various spaces of p-adic measures. In fact, these diagrams are so natural that they commute on the level of cochains; this fact will be used heavily in the calculations of Section 7. Recall the group $\widetilde{\Theta}$ defined in (3-4). We describe cohomology classes in terms of homogeneous cochains relative to the complex of projective $\widetilde{\Theta}$-modules

$$F_r := \mathbb{Z}[\widetilde{\Theta}^{r+1}]. \tag{5-1}$$

Thus, if G is a subgroup of $\widetilde{\Theta}$, our group of M-valued r-cochains is

$$C^r(G, M) := \mathrm{Hom}_G(F_r, M). \tag{5-2}$$

Coboundary maps $d : C^r(G, M) \to C^{r+1}(G, M)$ are defined by the usual formula

$$d\varphi(g_0, \dots, g_{r+1}) = \sum_{i=0}^{r+1} (-1)^i \varphi(g_0, \dots, \widehat{g_i}, \dots, g_{r+1}).$$

We write

$$Z^r(G, M) = \mathrm{Ker}(d : C^r(G, M) \to C^{r+1}(G, M)),$$
$$B^r(G, M) = \mathrm{Image}(d : C^{r-1}(G, M) \to C^r(G, M)),$$

and have

$$H^r(G, M) = Z^r(G, M)/B^r(G, M).$$

Defining

$$\mathbb{X}_p = \mathbb{Z}_p \times \mathbb{Z}_p^\times = w_p^{-1}\mathbb{X}_\infty, \tag{5-3}$$

we obtain Atkin–Lehner maps as in (3-5) with $M = M(\mathbb{X}_\infty)$ and $M' = M(\mathbb{X}_p)$.

Proposition 10. *The following diagrams commute*:

$$\begin{tikzcd} & C^r(\Gamma, M(\mathbb{X})) \arrow[dl, "\rho_{\mathbb{X}_\infty}"'] \arrow[dr, "\rho_{\mathbb{X}_p}"] & \\ C^r(\Gamma_0, M(\mathbb{X}_\infty)) \arrow[r, "U_p"'] & C^r(\Gamma_0, M(\mathbb{X}_\infty)) \arrow[r, "W_p^{-1}"'] & C^r(\Gamma_0, M(\mathbb{X}_p)) \end{tikzcd} \tag{5-4}$$

$$\begin{tikzcd}[column sep=small] & C^r(\Gamma', M(w_p\mathbb{X})) \arrow[dl, "\rho'_{p\mathbb{X}_p}"'] \arrow[dr, "\rho'_{\mathbb{X}_\infty}"] & \\ C^r(\Gamma_0, M(p\mathbb{X}_p)) \arrow[r, "U'_p"'] & C^r(\Gamma_0, M(p\mathbb{X}_p)) \arrow[r, "W_p^{-1}"'] & C^r(\Gamma_0, M(\mathbb{X}_\infty)) \end{tikzcd} \tag{5-5}$$

Here the maps ρ are the natural restriction maps.

Proof. Let $\varphi \in Z^r(\Gamma, M(\mathbb{X}))$. Let $g \in \widetilde{\Theta}^{r+1}$, and let h be a locally analytic function on $\mathbb{X}_p$. In the following, we will write $j_!$ for the extension-by-zero of a function j on $\mathbb{X}_\infty$ to a function on $\mathbb{X}$. We compute:

$$\begin{aligned}(W_p^{-1}U_p\rho_{\mathbb{X}_\infty}\varphi)(g)(h) &= (U_p\rho_{\mathbb{X}_\infty}\varphi)(w_pg)(h|w_p^{-1}) \\ &= \sum_{0\le i\le p-1} (\rho_{\mathbb{X}_\infty}\varphi)(\delta_i^{-1}w_pg)(h|w_p^{-1}\delta_i) \\ &= \sum_{0\le i\le p-1} \varphi(\delta_i^{-1}w_pg)((h|w_p^{-1}\delta_i)_!) \\ &= \sum_{0\le i\le p-1} \varphi(g)((h|w_p^{-1}\delta_i)_!|\delta_i^{-1}w_p) \\ &= \sum_{0\le i\le p-1} \varphi(g)(h_!\mathbf{1}_{\pi^{-1}(i+p\mathbb{Z}_p)}) \\ &= (\rho_{\mathbb{X}_p}\varphi)(g)(h).\end{aligned}$$

Essential in this calculation is that $w_p^{-1}\delta_i$ belongs to Γ and that

$$w_p^{-1}\delta_i(\mathbb{X}_\infty) = \gamma_i w_p^{-1}(\mathbb{X}_\infty) = \gamma_i(\mathbb{X}_p) = \pi^{-1}(i + p\mathbb{Z}_p).$$

The commutativity of (5-5) follows from applying the operator W_p to (5-4). $\square$

Next, we will be interested in understanding the map

$$W_pU_p : H^r(\Gamma, M(\mathbb{X})) \to H^r(\Gamma', M(w_p\mathbb{X}))$$

with respect to the decomposition $w_p\mathbb{X} = \mathbb{X}_\infty \sqcup p\mathbb{X}_p$.

Proposition 11. *The following diagram commutes*:

$$\begin{tikzcd} C^r(\Gamma, M(\mathbb{X})) \arrow[r, "\rho_{\mathbb{X}_\infty}"] \arrow[d, "W_pU_p"'] & C^r(\Gamma_0, M(\mathbb{X}_\infty)) \arrow[d, "U_p^2"] \\ C^r(\Gamma', M(w_p\mathbb{X})) \arrow[r, "\rho'_{\mathbb{X}_\infty}"'] & C^r(\Gamma_0, M(\mathbb{X}_\infty)) \end{tikzcd}$$

Proof. The result follows from the following commutative diagram and (3-9). Note that the commutativity of the triangle on the right is given by that of (5-5).

$$\begin{tikzcd}
C^r(\Gamma, M(\mathbb{X})) \arrow[r, "U_p"] \arrow[dd, "\rho_{\mathbb{X}_\infty}"'] & C^r(\Gamma, M(\mathbb{X})) \arrow[r, "W_p"] \arrow[dd, "\rho_{\mathbb{X}_\infty}"'] & C^r(\Gamma', M(w_p\mathbb{X})) \arrow[dd, "\rho'_{p\mathbb{X}_p}"'] \arrow[dr, "\rho'_{\mathbb{X}_\infty}"] & \\
& & & C^r(\Gamma_0, M(\mathbb{X}_\infty)) \\
C^r(\Gamma_0, M(\mathbb{X}_\infty)) \arrow[r, "U_p"'] & C^r(\Gamma_0, M(\mathbb{X}_\infty)) \arrow[r, "W_p"'] & C^r(\Gamma_0, M(p\mathbb{X}_p)) \arrow[ur, "W_p^{-1}U'_p"'] &
\end{tikzcd}$$

□

Proposition 12. *The following diagram commutes*:

$$\begin{tikzcd}
H^r(\Gamma, M(\mathbb{X})) \arrow[r, "\rho_{\mathbb{X}_p}"] \arrow[d, "W_pU_p"'] & H^r(\Gamma_0, M(\mathbb{X}_p)) \arrow[d, "p_*"] \\
H^r(\Gamma', M(w_p\mathbb{X})) \arrow[r, "\rho'_{p\mathbb{X}_p}"'] & H^r(\Gamma_0, M(p\mathbb{X}_p))
\end{tikzcd}$$

Here the map $p_* : H^r(\Gamma_0, M(\mathbb{X}_p)) \to H^r(\Gamma_0, M(p\mathbb{X}_p))$ *is induced by* $p_*h(x, y) = h(px, py)$ *for a locally analytic function* h *on* $p\mathbb{X}_p$.

Proof. The result follows from the following commutative diagram.

$$\begin{tikzcd}
C^r(\Gamma, M(\mathbb{X})) \arrow[r, "U_p"] \arrow[dd, "\rho_{\mathbb{X}_\infty}"'] \arrow[drrr, "\rho_{\mathbb{X}_p}"] & C^r(\Gamma, M(\mathbb{X})) \arrow[r, "W_p"] \arrow[dd, "\rho_{\mathbb{X}_\infty}"'] & C^r(\Gamma', M(w_p\mathbb{X})) \arrow[dd, "\rho'_{p\mathbb{X}_p}"'] & \\
& & & C^r(\Gamma_0, M(\mathbb{X}_p)) \\
C^r(\Gamma_0, M(\mathbb{X}_\infty)) \arrow[r, "U_p"'] & C^r(\Gamma_0, M(\mathbb{X}_\infty)) \arrow[r, "W_p"'] & C^r(\Gamma_0, M(p\mathbb{X}_p)) \arrow[ur, "W_p^{-2}=p_*^{-1}"'] &
\end{tikzcd}$$

The commutativity of the diagonal map $\rho_{\mathbb{X}_p}$ with the arrows that lie below it follows from that of (5-4). The fact that $W_p^2 = p_*$ follows from the fact that $w_p^2 \in p\Gamma_0$ and hence induces the same map on Γ_0-cohomology as multiplication by p. □

6. p-arithmetic cohomology classes and Darmon $\mathscr{L}$-invariants

Let

$$\Theta = \ker\left(\mathrm{ord}_p \circ \mathrm{nrd} : \widetilde{\Theta} \to \mathbb{Z}/2\mathbb{Z}\right), \tag{6-1}$$

where $\mathrm{nrd} : B^\times \to \mathbb{Q}^\times$ is the reduced norm map. Thus, Θ is a normal subgroup of $\widetilde{\Theta}$ of index two and $\widetilde{\Theta}/\Theta$ is generated by the image of w_p. By analyzing its action on the Bruhat–Tits tree of $\mathrm{PGL}_2(\mathbb{Q}_p)$, the group Θ can be expressed as an amalgamation (free product) $\Theta \cong \Gamma *_{\Gamma_0} \Gamma'$ [Greenberg 2009]. Associated to such an amalgamation and a Θ-module M, there is a Mayer–Vietoris sequence

$$\cdots \to H^{r-1}(\Gamma_0, M) \xrightarrow{\delta} H^r(\Theta, M) \xrightarrow{(\mathrm{res}^{\Theta}_{\Gamma}, \mathrm{res}^{\Theta}_{\Gamma'})} H^r(\Gamma, M) \oplus H^r(\Gamma', M)$$
$$\xrightarrow{(\mathrm{res}^{\Gamma}_{\Gamma_0} - \mathrm{res}^{\Gamma'}_{\Gamma_0})} H^r(\Gamma_0, M) \to \cdots . \tag{6-2}$$

Recall that we defined $X = \mathbb{P}^1(\mathbb{Q}_p)$. View $\mathbb{Q}_p$ as a subspace of $\mathbb{P}^1(\mathbb{Q}_p)$ via the inclusion $z \mapsto (z : 1)$. Thus, $(x : y)$ can be identified with the fraction x/y. Set $\infty = (1 : 0)$. We view $\mathbb{Z}_p \subset \mathbb{Q}_p$ as a subspace of X and set

$$X_\infty = X - \mathbb{Z}_p = w_p \mathbb{Z}_p.$$

Our first goal in this section is to use (6-2) in order to construct a cohomology class in $H^1(\Theta, M^0(X))^\pm$ associated to $g^\pm$. (Such a class is constructed in [Greenberg 2009] using different methods.) The map

$$\pi : \mathbb{X} \to X, \qquad \pi(x, y) = (x : y)$$

and the induced pushforward of measures $\pi_* : M(\mathbb{X}) \to M(X)$ can be described via the following isomorphism, a consequence of the fact that π is a $\mathbb{Z}_p^\times$-fibration:

$$M(X) \cong M(\mathbb{X}) \otimes_{\mathbb{Z}_p[\![\mathbb{Z}_p^\times]\!]} \mathbb{Z}_p. \tag{6-3}$$

Here, $\mathbb{Z}_p$ is given the structure of a $\mathbb{Z}_p[\![\mathbb{Z}_p^\times]\!]$-algebra via the augmentation map defined in (4-2). Recall that by Lemma 9, we may assume that the cohomological Hida family $\Phi_g^\pm$ associated to $g^\pm$ belongs to $H^1(\Gamma, M^0(\mathbb{X})) \otimes_{\mathbb{Z}_p[\![\mathbb{Z}_p^\times]\!]} R_{\mathfrak{p}}$. For notational simplicity, we suppress the $\otimes_{\mathbb{Z}_p[\![\mathbb{Z}_p^\times]\!]} R_{\mathfrak{p}}$ in the sequel and write $g^\pm \in H^1(\Gamma, M^0(\mathbb{X}))$; this does not affect any subsequent arguments in a substantive way, though our measures now take values in E.

Proposition 13. *There is a unique cohomology class $\varphi_g^\pm \in H^1(\Theta, M^0(X))$ such that*

$$\mathrm{res}^{\Theta}_{\Gamma} \varphi_g^\pm = \pi_* \Phi_g^\pm, \quad \mathrm{res}^{\Theta}_{\Gamma'} \varphi_g^\pm = \pi_* W_p U_p \Phi_g^\pm.$$

Proof. The uniqueness follows from (6-2) as $H^0(\Gamma_0, M^0(X)) = 0$. We must show the existence of $\varphi_g^\pm$. To this end, let

$$\varphi_g^\pm = \pi_* \Phi_g^\pm \in H^1(\Gamma, M^0(X)), \quad \varphi_g'^\pm = \pi_* W_p U_p \Phi_g^\pm \in H^1(\Gamma', M^0(X)).$$

From (6-2), we must show that $\mathrm{res}^{\Gamma}_{\Gamma_0} \varphi_g^\pm = \mathrm{res}^{\Gamma}_{\Gamma_0} \varphi_g'^\pm$ in $H^1(\Gamma_0, M^0(X))$. Since the kernel of $H^1(\Gamma_0, M^0(X)) \to H^1(\Gamma_0, M(X))$ is Eisenstein, it suffices to prove this equality after viewing $\varphi_g^{\pm}$ and $\varphi_g'^{\pm}$ as taking values in $M(X)$. Let

$$\begin{aligned}
\rho_{\mathbb{Z}_p} &: H^1(\Gamma, M(X)) \to H^1(\Gamma_0, M(\mathbb{Z}_p)) \\
\rho'_{X_\infty} &: H^1(\Gamma', M(X)) \to H^1(\Gamma_0, M(\mathbb{Z}_p)) \\
\rho_{X_\infty} &: H^1(\Gamma, M(X)) \to H^1(\Gamma_0, M(X_\infty)) \\
\rho'_{X_\infty} &: H^1(\Gamma', M(X)) \to H^1(\Gamma_0, M(X_\infty))
\end{aligned}$$

be the maps induced by the inclusions $\mathbb{Z}_p \hookrightarrow X$ and $X_\infty \hookrightarrow X$ and restriction of groups to Γ_0. From the decomposition

$$H^1(\Gamma_0, M(X)) = H^1(\Gamma_0, M(\mathbb{Z}_p)) \oplus H^1(\Gamma_0, M(X_\infty)),$$

we must show that $\rho_{\mathbb{Z}_p}\varphi_g^{\pm} = \rho_{\mathbb{Z}_p}\varphi_g'^{\pm}$ and $\rho_{X_\infty}\varphi_g^{\pm} = \rho'_{X_\infty}\varphi_g'^{\pm}$. By Propositions 11 and 12, the following diagrams commute:

$$\begin{tikzcd}
H^1(\Gamma, M(X)) \arrow[d, "W_pU_p"'] \arrow[dr, "\rho_{\mathbb{Z}_p}"] & \\
H^1(\Gamma', M(X) \arrow[r, "\rho'_{\mathbb{Z}_p}"'] & H^1(\Gamma_0, M(\mathbb{Z}_p))
\end{tikzcd}
\qquad
\begin{tikzcd}
H^1(\Gamma, M(X)) \arrow[r, "\rho_{X_\infty}"] \arrow[d, "W_pU_p"'] & H^1(\Gamma_0, M(X_\infty)) \arrow[d, "U_p^2"] \\
H^1(\Gamma', M(X)) \arrow[r, "\rho'_{X_\infty}"'] & H^1(\Gamma_0, M(X_\infty))
\end{tikzcd}$$

The diagram on the left proves $\rho_{\mathbb{Z}_p}\varphi_g^{\pm} = \rho_{\mathbb{Z}_p}\varphi_g'^{\pm}$, one of the desired identities. The one on the right says $\rho'_{X_\infty}\varphi_g'^{\pm} = U_p^2\rho_{X_\infty}\varphi_g^{\pm}$. By (6-3),

$$U_p^2\rho_{X_\infty}\varphi_g^{\pm} = \epsilon(\boldsymbol{a}_p(\Phi_g^{\pm}))^2\rho_{X_\infty}\varphi_g^{\pm} = \rho_{X_\infty}\varphi_g^{\pm},$$

completing the proof. □

For each choice of $\mathscr{L} \in \mathbb{P}^1(E)$, we define an integration map

$$\kappa_{\mathscr{L}} : H^r(\Theta, M^0(X)) \to H^{r+1}(\Theta, E)$$

as follows: Let $C(X)$ denote the space of continuous E-valued functions on X. Choose a base-point $\tau \in \mathscr{H}_p(E) = \mathbb{P}^1(E) - \mathbb{P}^1(\mathbb{Q}_p)$ and define

$$\xi_{\mathscr{L},\tau} \in C^1(\widetilde{\Theta}, C(X)/E)$$

by

$$\xi_{\mathscr{L},\tau}(g_0, g_1) = \begin{cases} \log_{\mathscr{L}}\left(\dfrac{z-g_1\tau}{z-g_0\tau}\right) & \text{if } \mathscr{L} \in E, \\[2ex] \operatorname{ord}_p\left(\dfrac{z-g_1\tau}{z-g_0\tau}\right) & \text{if } \mathscr{L} = \infty. \end{cases}$$

It is easy to see that $d\xi_{\mathscr{L},\tau} = 0$ and that the cohomology class represented by $\xi_{\mathscr{L},\tau}$ does not depend on τ.

Let G be any subgroup of $\widetilde{\Theta}$, let $\varphi \in C^r(G, M^0(X))$, and consider the cup product

$$\xi_{\mathscr{L},\tau} \cup \varphi \in C^{r+1}(G, (C(X)/E) \otimes_E M^0(X)).$$

The $\widetilde{\Theta}$-invariant integration pairing $(C(X)/E) \otimes_E M^0(X) \to E$ induces a map

$$I : C^{r+1}(G, (C(X)/E) \otimes_E M^0(X)) \to C^{r+1}(G, E).$$

Set $\kappa_{\mathscr{L},\tau}(\varphi) = I(\xi_{\mathscr{L},\tau} \cup \varphi) \in C^{r+1}(G, E)$, i.e.,

$$\kappa_{\mathscr{L},\tau}(\varphi)(g_0, \dots, g_{r+1}) = \int_X \log_{\mathscr{L}}\Big(\frac{z-g_1\tau}{z-g_0\tau}\Big)\varphi(g_1, \dots, g_{r+1}). \tag{6-4}$$

One may compute directly that

$$d\kappa_{\mathscr{L},\tau}(\varphi) = \kappa_{\mathscr{L},\tau}(d\varphi). \tag{6-5}$$

Therefore, the correspondence $\varphi \mapsto \kappa_{\mathscr{L},\tau}(\varphi)$ induces a map

$$\kappa_{\mathscr{L}} : H^r(G, M^0(X)) \to H^{r+1}(G, E),$$

which, as our notation suggests, does not depend on the choice of τ. Define

$$H^1(\Gamma_0, E)_{p\text{-new}} := H^1(\Gamma_0, E)\big/ \operatorname{Image}\big(H^1(\Gamma, E) \oplus H^1(\Gamma', E) \to H^1(\Gamma_0, E)\big),$$

and let

$$\delta : H^1(\Gamma_0, E)_{p\text{-new}} \hookrightarrow H^2(\Theta, E) \tag{6-6}$$

be the injective map induced by the connecting homomorphism in the Mayer–Vietoris sequence (6-2).

Proposition 14. *The cohomology class $\varphi_g^{\pm}$ defined in Proposition 13 satisfies the following*:

(1) *The identity $\kappa_\infty(\varphi_g^{\pm}) = \delta(g^{\pm})$ holds in $H^2(\Theta, E)$.*

(2) *There is a unique $\mathscr{L} \in E$, denoted $-\mathscr{L}^{\mathrm{D}}(g^{\pm})$, such that $\kappa_{\mathscr{L}}(\varphi_g^{\pm}) = 0$.*

Proof. The first statement is argued in the proof of [Greenberg 2009, Lemma 32]. By [ibid., Lemmas 32 and 33], the eigenspace of $H^2(\Theta, E)^{\pm}$ on which the Hecke operators away from p act via the eigenvalues of g is 1-dimensional and is spanned by $\kappa_\infty(\varphi_g^{\pm}) = \delta(g^{\pm})$, where δ is as in (6-6). The class $\delta(g^{\pm})$ is nonzero as $g^{\pm}$ is a nonzero p-new form and δ is injective on such classes. Since the map κ_0 (the one corresponding to $\mathscr{L} = 0$) is Hecke-equivariant, there is a unique constant $\mathscr{L}^{\mathrm{D}}(g^{\pm}) \in E$ such that $\kappa_0(\varphi_g^{\pm}) = \mathscr{L}^{\mathrm{D}}(g^{\pm})\kappa_\infty(\varphi_g^{\pm})$. But the identity $\log_{\mathscr{L}} = \log_0 + \mathscr{L}\operatorname{ord}_p$ implies that $\kappa_{\mathscr{L}} = \kappa_0 + \mathscr{L}\kappa_\infty$, and the second statement of the proposition follows with $\mathscr{L} = -\mathscr{L}^{\mathrm{D}}(g^{\pm})$. □

Definition 15. The quantity $\mathscr{L}^{\mathrm{D}}(g^{\pm})$ is called the *Darmon $\mathscr{L}$-invariant of $g^{\pm}$*.

7. Equality of the Greenberg–Stevens and Darmon $\mathscr{L}$-invariants

Let $\mathscr{L} \in E$. The goal of this section is to prove the following:

Theorem 16. *We have*

$$\kappa_{\mathscr{L}}(\varphi_g^{\pm}) = (\mathscr{L}^{GS}(g) + \mathscr{L})\delta(g^{\pm})$$

in $H^2(\Theta, E)$. Therefore, $\mathscr{L}^D(g^{\pm}) = \mathscr{L}^{GS}(g)$.

Since the Riemann surfaces $\Gamma\backslash\mathscr{H}$ and $\Gamma'\backslash\mathscr{H}$ are compact if and only if $N^- \neq 1$, we have

$$H^2(\Gamma, E) \cong \begin{cases} E & \text{if } N^- \neq 1, \\ \{0\} & \text{if } N^- = 1. \end{cases}$$

In either case, this space is Eisenstein for the Hecke operators. Since the restriction maps are Hecke-equivariant, $\operatorname{res}_{\Gamma}^{\Theta} \kappa_{\mathscr{L}}(\varphi_g^{\pm}) = 0$ and $\operatorname{res}_{\Gamma'}^{\Theta} \kappa_{\mathscr{L}}(\varphi_g^{\pm}) = 0$.

Fix a base point $\tau \in \mathscr{H}_p(E)$ and a representative $\varphi \in C^1(\Theta, M^0(X))$ for the cohomology class $\varphi_g^{\pm} \in H^1(\Theta, M^0(X))$. Let $\psi \in C^1(\Gamma, E)$ and $\psi' \in C^1(\Gamma', E)$ be 1-cochains such that $d\psi = \kappa_{\mathscr{L},\tau}(\varphi)|_{\Gamma}$ and $d\psi' = \kappa_{\mathscr{L},\tau}(\varphi)|_{\Gamma'}$. Then $\psi - \psi'$ is a 1-cocycle on $\Gamma_0 = \Gamma \cap \Gamma'$ and, tracing through the construction of the connecting homomorphism in the long exact sequence in cohomology associated to (6-6), one finds that

$$\delta([\psi - \psi']) = \kappa_{\mathscr{L}}(\varphi_g^{\pm}) \tag{7-1}$$

in $H^2(\Theta, E)$. Through a general cohomological calculation, we will find explicit formulas for ψ and ψ' and show that

$$[\psi - \psi'] = (\mathscr{L}^{\mathrm{GS}}(g) + \mathscr{L})g^{\pm}. \tag{7-2}$$

Equations (7-1) and (7-2) prove Theorem 16.

Let $\varphi \in C^1(\Theta, M^0(X))$ be a cocycle representing the class $\varphi_g^{\pm}$. Let

$$\Phi = \Phi_g^{\pm} \in H^1(\Gamma, M^0(\mathbb{X}))$$

denote the Hida family defined in (4-6) that lifts $\operatorname{res}_{\Gamma}^{\Theta}[\varphi]$ with respect to the push-forward map $\pi_* : M^0(\mathbb{X}) \to M^0(X)$. Let $\widetilde{\varphi}_0 \in C^1(\Gamma, M^0(\mathbb{X}))$ be a cocycle representing Φ. Then there exists a cochain $m \in Z^0(\Gamma, M^0(X))$ such that $\pi_*\widetilde{\varphi}_0 = \varphi + dm$. Since $F_0 = \mathbb{Z}[\widetilde{\Theta}]$ is Θ-projective and thus Γ-projective, we may lift m to a cochain $\widetilde{m} \in C^0(\Gamma, M^0(\mathbb{X}))$. Setting $\widetilde{\varphi} = \widetilde{\varphi}_0 - d\widetilde{m} \in C^1(\Gamma, M^0(\mathbb{X}))$, we obtain a cocycle representing Φ that satisfies

$$\pi_*\widetilde{\varphi} = \varphi. \tag{7-3}$$

For any $\sigma \in C^r(\Gamma, M^0(\mathbb{X}))$ and $\sigma' \in C^r(\Gamma', M^0(w_p\mathbb{X}))$, define $\lambda_{\mathscr{L}}(\sigma) \in C^r(\Gamma, E)$ and $\lambda'_{\mathscr{L}}(\sigma') \in C^r(\Gamma', E)$ by the formulas

$$\lambda_{\mathscr{L}}(\sigma)(g_0, g_1, \ldots, g_r) = \int_{\mathbb{X}} \log_{\mathscr{L}}(x - (g_0\tau)y)\, \sigma(g_0, g_1, \ldots, g_r)(x, y), \tag{7-4}$$
$$\lambda'_{\mathscr{L}}(\sigma')(g_0, g_1, \ldots, g_r) = \int_{w_p\mathbb{X}} \log_{\mathscr{L}}(x - (g_0\tau)y)\, \sigma'(g_0, g_1, \ldots, g_r)(x, y).$$

These maps are Γ and Γ'-invariant, respectively, because the values of σ and σ' have total measure zero.

Lemma 17. *For any $\sigma \in C^r(\Gamma, M^0(\mathbb{X}))$ and $\sigma' \in C^r(\Gamma', M^0(w_p\mathbb{X}))$, we have*

$$d\lambda_{\mathscr{L}}(\sigma) = \kappa_{\mathscr{L}}(\pi_*\sigma) + \lambda_{\mathscr{L}}(d\sigma), \qquad d\lambda'_{\mathscr{L}}(\sigma') = \kappa_{\mathscr{L}}(\pi_*\sigma') + \lambda'_{\mathscr{L}}(d\sigma').$$

Proof. Letting $h = (g_0, \ldots, g_{r+1})$ and $h_i = (g_0, \ldots, \widehat{g_i}, \ldots, g_r)$, we have

$$\begin{aligned}
d\lambda(\sigma)(h) &= \int_{\mathbb{X}} \log_{\mathscr{L}}(x - (g_1\tau)y)\sigma(h_0)(x, y) \\
&\qquad\qquad + \sum_{i=1}^{r+1}(-1)^i \int_{\mathbb{X}} \log_{\mathscr{L}}(x - (g_0\tau)y)\sigma(h_i)(x, y) \\
&= \int_{\mathbb{X}} \log_{\mathscr{L}}\Big(\frac{x-(g_1\tau)y}{x-(g_0\tau)y}\Big)\sigma(h_0)(x, y) + \int_{\mathbb{X}} \log_{\mathscr{L}}(x-(g_0\tau)y)\, d\sigma(h)(x, y) \\
&= \int_X \log_{\mathscr{L}}\Big(\frac{z-g_1\tau}{z-g_0\tau}\Big)\pi_*\sigma(h_0)(z) + \lambda_{\mathscr{L}}(d\sigma)(h) \\
&= \kappa_{\mathscr{L}}(\pi_*\sigma)(h) + \lambda_{\mathscr{L}}(d\sigma)(h),
\end{aligned}$$

as desired. The second equality is proved in a similar manner. □

Lemma 17 implies that if we define

$$\psi = \lambda_{\mathscr{L}}(\widetilde{\varphi}) \in C^1(\Gamma, E), \tag{7-5}$$

then $d\psi = \kappa_{\mathscr{L}}(\varphi)$. Similarly, define

$$\psi' = \lambda'_{\mathscr{L}}(W_pU_p\widetilde{\varphi}) \in C^1(\Gamma', E). \tag{7-6}$$

Then

$$\begin{aligned}
d\psi' &= \kappa_{\mathscr{L}}(\pi_* W_pU_p\widetilde{\varphi}) + d\lambda'_{\mathscr{L}}(dW_pU_p\widetilde{\varphi}) \\
&= \kappa_{\mathscr{L}}(W_pU_p\varphi) + 0 \\
&= \kappa_{\mathscr{L}}(\varphi),
\end{aligned}$$

where the last equality is justified by the following lemma:

Lemma 18. *We have the identity of Θ-cochains $W_pU_p\varphi = \varphi$.*

Proof. Consider the diagram

$$\begin{tikzcd}
 & C^r(\Gamma, M(X)) \arrow[dl, "\rho_{X_\infty}"'] \arrow[dr, "\rho_{\mathbb{Z}_p}"] & \\
C^r(\Gamma_0, M(X_\infty)) \arrow[r, "U_p"'] \arrow[d, "\rho_{X_\infty}^{-1}"'] & C^r(\Gamma_0, M(X_\infty)) \arrow[r, "W_p"'] \arrow[d, "\rho_{X_\infty}^{-1}"'] & C^r(\Gamma_0, M(\mathbb{Z}_p)) \arrow[d, "\rho_{\mathbb{Z}_p}^{-1}"] \\
C^r(\Gamma, M(X)) \arrow[r, "U_p"'] & C^r(\Gamma, M(X)) \arrow[r, "W_p"'] & C^r(\Gamma', M(X))
\end{tikzcd}$$

The maps ρ_{X_∞} and $\rho_{\mathbb{Z}_p}$ are isomorphisms by Shapiro's lemma. The bottom squares of the diagram commute by definition and the upper triangle commutes as it is the pushforward via π_* in (5-4). The lemma follows. $\square$

Having found explicit formulas for ψ and ψ' in (7-5) and (7-6), respectively, we now turn towards proving (7-2). Recall that $\Phi = [\widetilde{\varphi}]$ is a U_p-eigenvector with eigenvalue $\boldsymbol{a}_p(\Phi)$ satisfying $\epsilon(\boldsymbol{a}_p(\Phi)) = \pm 1$. We defined

$$\mathscr{L}^{\mathrm{GS}}(\Phi) = d_\epsilon(1 - \boldsymbol{a}_p(\Phi)^2).$$

Proposition 19. *The class of the cocycle $\psi - \psi'$ in $H^1(\Gamma_0, E)$ is equal to*

$$(\mathscr{L}^{GS}(\Phi) + \mathscr{L})\rho_*[\varphi],$$

where $\rho_* : H^1(\Theta, M^0(X)) \to H^1(\Gamma_0, M(X_\infty)) \to H^1(\Gamma_0, E)$ is the composition of the canonical restriction map ρ_{X_∞} with the total measure on X_∞ map (as in (4-3)*).*

Proof. We use the decompositions $\mathbb{X} = \mathbb{X}_\infty \sqcup \mathbb{X}_p$ and $w_p\mathbb{X} = \mathbb{X}_\infty \sqcup p\mathbb{X}_p$ to study the integrals defining ψ and ψ' (see (4-1) and (5-3)). Writing $h = (g_0, g_1)$, we find:

$$\begin{aligned}(\psi - \psi')(h) = \int_{\mathbb{X}_\infty} \log_{\mathscr{L}}(x - (g_0\tau)y)\widetilde{\varphi}(h) + \int_{\mathbb{X}_p} \log_{\mathscr{L}}(x - (g_0\tau)y)\widetilde{\varphi}(h) \\ - \int_{\mathbb{X}_\infty} \log_{\mathscr{L}}(x - (g_0\tau)y)W_pU_p\widetilde{\varphi}(h) \\ - \int_{p\mathbb{X}_p} \log_{\mathscr{L}}(x - (g_0\tau)y)W_pU_p\widetilde{\varphi}(h). \quad (7\text{-}7)\end{aligned}$$

Propositions 11 and 12 allow us to rewrite these last two integrals as

$$\int_{\mathbb{X}_\infty} \log_{\mathscr{L}}(x - (g_0\tau)y)W_pU_p\widetilde{\varphi}(h) = \int_{\mathbb{X}_\infty} \log_{\mathscr{L}}(x - (g_0\tau)y)U_p^2\widetilde{\varphi}(h) \qquad (7\text{-}8)$$

and

$$\int_{p\mathbb{X}_p} \log_{\mathscr{L}}(x-(g_0\tau)y) W_p U_p \widetilde{\varphi}(h)$$
$$= \int_{p\mathbb{X}_p} \log_{\mathscr{L}}(x-(g_0\tau)y) p_* \widetilde{\varphi}(h) = \int_{\mathbb{X}_p} \log_{\mathscr{L}}(px-(g_0\tau)py) \widetilde{\varphi}(h)$$
$$= \int_{\mathbb{X}_p} \log_{\mathscr{L}}(x-(g_0\tau)y) \widetilde{\varphi}(h) + \mathscr{L}\widetilde{\varphi}(h)(\mathbb{X}_p). \tag{7-9}$$

Combining (7-7), (7-8), and (7-9), we obtain

$$(\psi-\psi')(h) = \int_{\mathbb{X}_\infty} \log_{\mathscr{L}}(x-(g_0\tau)y)(1-U_p^2)\widetilde{\varphi}(h) - \mathscr{L}\widetilde{\varphi}(h)(\mathbb{X}_p). \tag{7-10}$$

We now view $\widetilde{\varphi}$ as an element of $Z^r(\Gamma_0, M^0(\mathbb{X}_\infty))$ and calculate the class in $H^r(\Gamma_0, E)$ represented by the right side of (7-10). We have that

$$\widetilde{\varphi}(h)(\mathbb{X}_p) = \varphi(h)(\mathbb{Z}_p) = -\varphi(h)(X_\infty),$$

and hence represents the class $-\rho_*[\varphi]$ in $H^r(\Gamma_0, E)$. Therefore the last term in (7-10) represents the class $\mathscr{L}\rho_*[\varphi]$.

It remains to prove that the first term in (7-10) represents the class $\mathscr{L}^{\mathrm{GS}}(\widetilde{\varphi})\rho_*[\varphi]$ in $H^1(\Gamma_0, E)$. Since $(1-U_p^2)\Phi = \alpha\Phi$ with $\alpha = 1-\boldsymbol{a}_p(\Phi)^2$, we may write

$$(1-U_p^2)\widetilde{\varphi} = \alpha\widetilde{\varphi} + d\nu \tag{7-11}$$

for some $\nu \in C^0(\Gamma_0, M(\mathbb{X}_\infty))$. Pushing forward via π_*, we obtain

$$(1-U_p^2)\varphi = 0 + \pi_*(d\nu).$$

Since the term on the left is zero, we obtain $d\pi_*(\nu) = 0$. Thus $\pi_*\nu$ represents a class in $H^0(\Gamma_0, M(X_\infty))$.

Lemma 20. *The cohomology group $H^0(\Gamma_0, M(X_\infty))$ is zero.*

Proof. It is easy to see that

$$\mathfrak{I}_p = \{g \in \mathrm{GL}_2(\mathbb{Z}_p) : g \text{ is upper-triangular modulo } p\}$$

acts transitively on the set of balls in X_∞ of radius p^{-n} for any $n \geq 1$. Since Γ_0 is p-adically dense in $\mathfrak{I}_p$, Γ_0 acts transitively on this set as well. It follows that if μ is a Γ_0-invariant measure on X_∞, then $\mu(B) = p^{-n+1}\mu(X_\infty)$ for all compact-open balls $B \subset X_\infty$ of radius p^{-n}. Since the values of μ are assumed to be p-adically bounded, it follows that $\mu = 0$. □

By the lemma, we conclude that $\pi_*\nu$ is a coboundary. Arguing above as in the definition of the cocycle $\widetilde{\varphi}$ satisfying (7-3), we may alter ν by a coboundary to assume that $\pi_*\nu = 0$.

We may now calculate the cohomology class represented by (7-10). Substituting (7-11) into (7-10), the term from $\alpha\widetilde{\varphi}$ yields

$$\int_{\mathbb{X}_\infty} \log_{\mathscr{L}}(x-(g_0\tau)y)\alpha\widetilde{\varphi}(h). \tag{7-12}$$

By Proposition 21 below, the expression in (7-12) represents the class $\mathscr{L}^{\mathrm{GS}}(\widetilde{\varphi})\rho_*[\varphi]$ in $H^1(\Gamma_0, E)$. It remains to prove that the term arising from $d\nu$ is trivial in cohomology, i.e., that

$$h \mapsto \int_{\mathbb{X}_\infty} \log_{\mathscr{L}}(x-(g_0\tau)y)d\nu(h) \tag{7-13}$$

is a coboundary. Note that the right side of (7-13) is equal to

$$\int_{\mathbb{X}_\infty} \log_{\mathscr{L}}(x)d\nu(h) + \int_{X_\infty} \log_{\mathscr{L}}(1-(g_0\tau)/z)\pi_* d\nu(h). \tag{7-14}$$

The last term of (7-14) is zero since $\pi_* d\nu = 0$. The first term of (7-14) is equal to the coboundary of the 0-cochain given by

$$g_0 \mapsto \int_{\mathbb{X}_\infty} \log_{\mathscr{L}}(x)\nu(g_0). \tag{7-15}$$

We leave to the reader the exercise of using the equation $\pi_*\nu = 0$ to show that the 0-cochain in (7-15) is Γ_0-invariant. This proves that (7-13) is a coboundary and completes the proof of the proposition. □

The following proposition, applied with $\alpha = 1 - \boldsymbol{a}_p(\Phi)^2$, was used above to extract the invariant $\mathscr{L}^{\mathrm{GS}}(\Phi)$ from the cohomology class $[\Phi]$.

Proposition 21. *Let $\sigma \in Z^r(\Gamma_0, M(\mathbb{X}_\infty))$, let $\alpha \in I_\epsilon \subset \Lambda$ and define*

$$\eta(g_0,\dots,g_r) = \int_{\mathbb{X}_\infty} \log_{\mathscr{L}}(x-(g_0\tau)y)\alpha\sigma(g_0,\dots,g_r).$$

Then $\eta \in Z^r(\Gamma_0, E)$ and represents the class

$$[\eta] = d_\epsilon(\alpha)\rho_*[\sigma] \in H^r(\Gamma_0, E).$$

Proof. Since $\alpha \in I_\epsilon$, we have $\pi_*(\alpha\sigma) = 0$; in particular, $\alpha\sigma$ has total measure 0. It follows from this fact and a routine calculation that η is a cochain. That η is a cocycle follows from the equations $d(\alpha\sigma) = \alpha\, d\sigma = 0$.

To evaluate the class $[\eta] \in H^r(\Gamma_0, E)$, we consider α of the form $[\ell]-1$ for $\ell \in 1+p\mathbb{Z}_p$. Writing $h=(g_0,\dots,g_r)$, we have

$$\begin{aligned}\eta(h) &= \int_{\mathbb{X}_\infty} \log_{\mathscr{L}}(x-(g_0\tau)y)([\ell]\sigma-\sigma)(h)\\ &= \int_{\mathbb{X}_\infty} \bigl(\log_{\mathscr{L}}(\ell x-(g_0\tau)\ell y)-\log_{\mathscr{L}}(x-(g_0\tau)y)\bigr)\sigma(h)\\ &= \int_{\mathbb{X}_\infty} \log_{\mathscr{L}}(\ell)\sigma(h) = \log(\ell)\cdot\sigma(h)(\mathbb{X}_\infty) = d_\epsilon([\ell]-1)\rho_*\sigma(h).\end{aligned}$$

This proves the result for $\alpha=[\ell]-1$, and hence gives the result for general $\alpha \in I_\epsilon$ as the ideal I_ϵ is generated over Λ by such elements. □

This concludes the proof of Proposition 19, and since $\rho_*\varphi_g^\pm = g^\pm$, we deduce (7-2) and hence Theorem 16. Combining with Theorem 8, we also complete the proof of Theorem 2.

8. Multiplicative integrals and period lattices

In this section, we suppose that the Hecke eigenvalues of g belong to $\mathbb{Z}$. In this case, it is shown in [Greenberg 2009, §8] that we may take

$$\varphi_g^\pm \in H^1(\Theta, M^0(X,\mathbb{Z}))^{g,\pm}.$$

That is, we may find an element $\varphi_g^\pm \in H^1(\Theta, M^0(X,\mathbb{Z}))^{g,\pm}$ whose image in $H^1(\Theta, M^0(X,E))$ is a basis for $H^1(\Theta, M^0(X,E))^{g,\pm}$. Using this integral cohomology class, we may define multiplicative versions of many of the objects considered in previous sections.

Following Darmon [2001], we consider the *multiplicative integration pairing*

$$C(X)^\times/E^\times \times M^0(X,\mathbb{Z}) \to E^\times, \quad (f,\mu)\mapsto \times\!\!\!\!\!\!\int_X f\mu \tag{8-1}$$

defined by

$$\times\!\!\!\!\!\!\int_X f\mu = \lim_{||\mathcal{U}||\to 0} \prod_{U\in\mathcal{U}} f(z_U)^{\mu(U)}.$$

Here, $\mathcal{U}$ is a finite cover of X by compact open sets and z_U is an arbitrary point of U. The limit is taken over uniformly finer covers $\mathcal{U}$. It is clear that

$$\log_{\mathscr{L}} \times\!\!\!\!\!\!\int f\mu = \int \log_{\mathscr{L}}(f)\mu \quad \text{for any } \mathscr{L}.$$

The pairing (8-1) is easily seen to be $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant and thus induces a corresponding pairing

$$\langle\cdot,\cdot\rangle^\times : H_1(\Theta, C(X)^\times/E^\times) \times H^1(\Theta, M^0(X,\mathbb{Z})) \to E^\times. \tag{8-2}$$

Let $\Delta = \mathrm{Div}\,\mathcal{H}_p$ and let $\Delta^0 = \mathrm{Div}^0\,\mathcal{H}_p$. From the long exact sequence associated to the short exact sequence of $\mathrm{GL}_2(\mathbb{Q}_p)$-modules $0 \to \Delta^0 \to \Delta \to \mathbb{Z} \to 0$, we extract a connecting homomorphism $\partial : H_2(\Theta, \mathbb{Z}) \to H_1(\Theta, \Delta^0)$. Let $j : \Delta^0 \to C(X)^\times/E^\times$ be the map sending a divisor D to a rational function on X with divisor D. (Note that such a function is only well-defined up to multiplication by a nonzero scalar.) The map j being $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant, it induces a corresponding map

$$j_* : H_1(\Theta, \Delta^0) \to H_1(\Theta, C(X)^\times/E^\times).$$

We may also define multiplicative refinements of the cocycles $\kappa_{\mathcal{L},\tau}(\varphi)$ as follows. Let $\tau \in \mathcal{H}_p$, let $\varphi \in C^r(\Theta, M^0(X, \mathbb{Z}))$, and define $\kappa_\tau(\varphi) \in C^{r+1}(\Theta, E^\times)$ by the rule

$$\kappa_\tau(\varphi)(g_0, \ldots, g_{r+1}) = \times\!\!\!\!\!\!\int_X \left(\frac{z - g_1\tau}{z - g_0\tau}\right)\varphi(g_1, \ldots, g_{r+1}) \in E^\times.$$

As with $\kappa_{\mathcal{L},\tau}$, the homomorphism κ_τ induces a map

$$\kappa : H^r(\Theta, M^0(X, \mathbb{Z})) \to H^{r+1}(\Theta, E^\times)$$

that does not depend on τ.

By the universal coefficients theorem, there is a natural surjective map

$$H^{r+1}(\Theta, E^\times) \to \mathrm{Hom}(H_{r+1}(\Theta, \mathbb{Z}), E^\times).$$

Lemma 22. *The image of $\kappa(\varphi_g^\pm)$ in $\mathrm{Hom}(H_2(\Theta, \mathbb{Z}), E^\times)$ is given by*

$$\xi \mapsto \langle -j_*\partial\xi, \varphi_g^\pm\rangle^\times.$$

Proof. Suppose

$$\xi = \sum_i 1 \otimes (\gamma_i, \delta_i, \epsilon_i) \in Z_2(\Theta, \mathbb{Z}) = \mathbb{Z} \otimes_\Theta \mathbb{Z}[\Theta^3]$$

is a 2-cycle on Θ with values in $\mathbb{Z}$. Tracing through the construction of the connecting homomorphism, one computes that $\partial[\xi]$ is represented by the cycle

$$\sum_i (\gamma_i\tau - \delta_i\tau) \otimes (\delta_i, \epsilon_i).$$

Therefore,

$$\langle j_*\partial\xi, \varphi_g^\pm\rangle^\times = \prod_i \times\!\!\!\!\!\!\int_X \left(\frac{z - \gamma_i\tau}{z - \delta_i\tau}\right) \varphi_g^\pm(\delta_i, \epsilon_i).$$

By the definition of the map in the universal coefficients theorem, the image of $\kappa(\varphi_g^\pm)$ in $\mathrm{Hom}(H_2(\Theta, \mathbb{Z}), E^\times)$ sends ξ to

$$\prod_i \kappa(\varphi_g^\pm)(\gamma_i, \delta_i, \epsilon_i) = \prod_i \times\!\!\!\!\!\!\int_X \left(\frac{z - \delta_i\tau}{z - \gamma_i\tau}\right)\varphi_g^\pm(\delta_i, \epsilon_i).$$

The result follows. □

In view of Lemma 22, we set

$$L_g^{\pm} = \langle j_* \partial H_2(\Theta, \mathbb{Z}), \varphi_g^{\pm} \rangle^{\times} = \langle H_2(\Theta, \mathbb{Z}), \kappa(\varphi_g^{\pm}) \rangle \subset E^{\times}.$$

Proposition 23 [Greenberg 2009, Proposition 30]. *$L_g^{\pm}$ is a lattice in $E^{\times}$.*

Therefore, there is a unique $\mathscr{L} \in E$ such that $\log_{\mathscr{L}}(L_g^{\pm}) = 0$. We define the *$\mathscr{L}$-invariant of the lattice $L_g^{\pm}$*, denoted $\mathscr{L}(L_g^{\pm})$, to be the negative of this constant $\mathscr{L}$.

Proposition 24. *The $\mathscr{L}$-invariant of the lattice $L_g^{\pm}$ is equal to $\mathscr{L}^D(\varphi_g^{\pm})$.*

Proof. By the universal coefficients theorem,

$$\begin{aligned} \log_{\mathscr{L}}(L_g^{\pm}) &= \log_{\mathscr{L}} \langle H_2(\Theta, \mathbb{Z}), \kappa(\varphi_g^{\pm}) \rangle \\ &= \langle H_2(\Theta, \mathbb{Z}), \kappa_{\mathscr{L}}(\varphi_g^{\pm}) \rangle \end{aligned}$$

is equal to 0 if and only if $\kappa_{\mathscr{L}}(\varphi_g^{\pm}) = 0$. By definition, this occurs if and only if $\mathscr{L} = -\mathscr{L}^{\mathrm{D}}(\varphi_g^{\pm})$. □

Corollary 25 [Greenberg 2009, Conjecture 2]. *Let q be the Tate period of the elliptic curve $\mathscr{E}/\mathbb{Q}$ associated to f. Then*

$$\mathscr{L}(L_g^{\pm}) = \log_p(q)/\operatorname{ord}_p(q).$$

Proof. By Proposition 23 and Theorem 2, $\mathscr{L}(L_g^{\pm}) = \mathscr{L}^{\mathrm{D}}(\varphi_g^{\pm}) = \mathscr{L}^{\mathrm{GS}}(f)$. By the Galois-theoretic portion of the proof of the Greenberg–Stevens theorem [Greenberg and Stevens 1993, Theorem 3.18], we have $\mathscr{L}^{\mathrm{GS}}(f) = \log_p(q)/\operatorname{ord}_p(q)$. □

In [Greenberg 2009], a construction was given for local *Stark–Heegner points* on $E^{\times}/L_g^{\pm}$. We conjectured that the elliptic curve $E^{\times}/L_g^{\pm}$ is isogenous to $\mathscr{E}/E$, yielding a construction of local points on $\mathscr{E}$. Corollary 25 proves this conjecture and makes the construction unconditional. In the following section, we apply the above techniques further to obtain a formula for the formal group logarithms of these Stark–Heegner points in terms of Hida families.

9. Abel–Jacobi maps and Stark–Heegner points

In this section we recall the definition of Stark–Heegner points and give a formula for the formal group logarithms of these points in terms of Hida families. This formula will be used in [Greenberg and Shahabi ≥ 2012] to prove partial results towards the rationality of the Stark–Heegner points following the methods of [Bertolini and Darmon 2009].

Let $\mathscr{H}_{p,\mathrm{ur}}$ denote the *unramified p-adic upper half-plane*:

$$\mathscr{H}_{p,\mathrm{ur}} = \mathbb{P}^1(\mathbb{C}_p) - r^{-1}(\mathbb{P}^1(\mathbb{F}_p)) \subset \mathscr{H}_p,$$

where $r : \mathbb{P}^1(\mathbb{C}_p) \to \mathbb{P}^1(\overline{\mathbb{F}}_p)$ is the reduction map. The action of $\mathrm{GL}_2(\mathbb{Z}_p)$ on $\mathcal{H}_p$ preserves $\mathcal{H}_{p,\mathrm{ur}}$. We set $\Delta_{\mathrm{ur}} = \mathrm{Div}\,\mathcal{H}_{p,\mathrm{ur}}$ and $\Delta^0_{\mathrm{ur}} = \mathrm{Div}^0\,\mathcal{H}_{p,\mathrm{ur}}$. If $\tau_1, \tau_2 \in \mathcal{H}_{p,\mathrm{ur}}$, $z \in X$ and $(x, y) \in \mathbb{X}$, then the quantities

$$\log_{\mathcal{L}}\Big(\frac{z-\tau}{z-\tau'}\Big), \quad \log_{\mathcal{L}}(x - y\tau)$$

do not depend on $\mathcal{L}$ because the arguments are p-adic units. For this reason, we do not specify a branch of the p-adic logarithm and simply write log. The natural $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant pairing

$$\langle\,\cdot\,,\,\cdot\,\rangle : M^0(X) \times C(X)/E \to \mathbb{C}_p$$

induces a pairing

$$\langle\,\cdot\,,\,\cdot\,\rangle : H^1(\Gamma, M^0(X)) \times H_1(\Gamma, C(X)/E) \to \mathbb{C}_p. \tag{9-1}$$

Define $j : \Delta^0_{\mathrm{ur}} \to C(X)/E$ by

$$j(\{\tau_2\} - \{\tau_1\})(z) = \log\Big(\frac{z-\tau_2}{z-\tau_1}\Big).$$

Since it is Γ-equivariant, j induces a homomorphism

$$j_* : H_1(\Gamma, \Delta^0_{\mathrm{ur}}) \to H_1(\Gamma, C(X)/E).$$

We define one more pairing

$$\langle\,\cdot,\cdot\,\rangle : H^1(\Gamma, M^0(X)) \times H_1(\Gamma, \Delta^0_{\mathrm{ur}}) \to \mathbb{C}_p$$

by $\langle\varphi, \xi\rangle = \langle\varphi, j_*\xi\rangle$.

Let $\mathbb{T}^{(p)}$ be the Hecke generated by the operators away from p, that is, the operators T_ℓ for $\ell \nmid pN$, U_ℓ for $\ell \mid N^+$, and the involutions W_ℓ for $\ell \mid N^-$ (see §3). There is a natural action of $\mathbb{T}^{(p)}$ on $H_1(\Gamma, \Delta^0_{\mathrm{ur}})$ described by double cosets such that, endowing $\mathrm{Hom}(H_1(\Gamma, \Delta^0_{\mathrm{ur}}), E)$ with the corresponding dual action, the map

$$\begin{aligned} A : H^1(\Gamma, M^0(X)) &\to \mathrm{Hom}(H_1(\Gamma, \Delta^0_{\mathrm{ur}}), E), \\ \varphi &\mapsto \big(\xi \mapsto \langle\varphi, \xi\rangle\big) \end{aligned}$$

induced by the pairing (9-1) is $\mathbb{T}^{(p)}$-equivariant. For g as in the previous sections, define

$$A_g^{\pm} = A(\mathrm{Res}^{\Theta}_{\Gamma}\,\varphi_g^{\pm}).$$

We have $A_g^{\pm} \in \mathrm{Hom}(H_1(\Gamma, \Delta^0_{\mathrm{ur}}), E)^{g,\pm}$, where $\mathrm{Hom}(H_1(\Gamma, \Delta^0_{\mathrm{ur}}), E)^{g,\pm}$ is the eigenspace on which $\mathbb{T}^{(p)}$ acts via the Hecke eigenvalues of g and W_∞ acts as ± 1.

Proposition 26. *There is a unique homomorphism* $\mathrm{AJ}_g^\pm \in \mathrm{Hom}(H_1(\Gamma, \Delta_{ur}), E)^{g,\pm}$ *such that the diagram*

$$\begin{tikzcd} H_1(\Gamma, \Delta_{ur}^0) \arrow[rr] \arrow[dr, "A_g^\pm"'] & & H_1(\Gamma, \Delta_{ur}) \arrow[dl, "\mathrm{AJ}_g^\pm"] \\ & E & \end{tikzcd}$$

commutes, where the horizontal map is induced by the inclusion $\Delta_{ur}^0 \hookrightarrow \Delta_{ur}$.

The proof of Proposition 26 is given in [Greenberg 2009, §10] and is very similar to the first half of the proof of Lemma 9.

Remark 27. We have chosen the notation $\mathrm{AJ}_g^\pm$ for this map because it formally resembles an Abel–Jacobi map.

Define $J : \Delta_{\mathrm{ur}} \to C(\mathbb{X})/E$ by $J(\{\tau\})(x, y) = \log(x - y\tau)$. Since it is Γ-equivariant, J induces a homomorphism $J_* : H_1(\Gamma, \Delta) \to H_1(\Gamma, C(\mathbb{X})/E)$. The natural Γ-equivariant pairing $M^0(\mathbb{X}) \times C(\mathbb{X})/E \to E$ induces a corresponding pairing $H^1(\Gamma, M^0(\mathbb{X})) \times H_1(\Gamma, C(\mathbb{X})/E) \to E$.

Corollary 28. *The map* $\mathrm{AJ}_g^\pm : H_1(\Gamma, \Delta_{ur}) \to E$ *is given by* $\mathrm{AJ}_g^\pm(\xi) = \langle \Phi_g^\pm, J_*\xi \rangle$.

Proof. It is easy to see that the element $\widetilde{\mathrm{AJ}}_g^\pm$ of $\mathrm{Hom}(H_1(\Gamma, \Delta_{\mathrm{ur}}), E)$ defined by $\xi \mapsto \langle \Phi_g^\pm, J_*\xi \rangle$ belongs to the $(g, \pm)$-eigenspace. Since $\pi_*\Phi_g^\pm = \mathrm{Res}_\Gamma^\Theta \varphi_g^\pm$, the diagram

$$\begin{tikzcd} H_1(\Gamma, \Delta_{\mathrm{ur}}^0) \arrow[r, "j_*"] \arrow[dd] & H_1(\Gamma, C(X)/E) \arrow[dd, "\pi^*"'] \arrow[dr, "\langle \mathrm{Res}_\Gamma^\Theta \varphi_g^\pm, \cdot \rangle"] & \\ & & E \\ H_1(\Gamma, \Delta_{\mathrm{ur}}) \arrow[r, "J_*"'] & H_1(\Gamma, C(\mathbb{X})/E) \arrow[ur, "\langle \Phi_g^\pm, \cdot \rangle"'] & \end{tikzcd}$$

commutes, implying that

$$\begin{tikzcd} H_1(\Gamma, \Delta_{\mathrm{ur}}^0) \arrow[rr] \arrow[dr, "A_g^\pm"'] & & H_1(\Gamma, \Delta_{\mathrm{ur}}) \arrow[dl, "\widetilde{\mathrm{AJ}}_g^\pm"] \\ & E & \end{tikzcd}$$

commutes as well. Therefore, by Proposition 26, $\mathrm{AJ}_g^\pm = \widetilde{\mathrm{AJ}}_g^\pm$. $\square$

Let K be a real quadratic field and let $\mathbb{O} \subset K$ be an order such that $\operatorname{disc} \mathbb{O}$ is relatively prime to N and p. There is an embedding

$$\psi : K \to B$$

such that $\psi(\mathbb{O}) = \psi(K) \cap R$. For details regarding this point, see [Vignéras 1980, chapitre III, 5C]. Suppose further that p is inert in K. Then $\psi(K^\times)$ acts on $\mathbb{P}^1(E)$ via ι_p with two fixed points τ_ψ and $\overline{\tau}_\psi$ in $\mathscr{H}_{p,\mathrm{ur}}$, conjugate under the action of $\mathrm{Gal}(K_p/\mathbb{Q}_p)$. Let ϵ be a generator of the unit group of $\mathbb{O}$. Then since $\psi(\epsilon)\tau_\psi = \tau_\psi$, we have

$$\{\tau_\psi\} \otimes (1, \psi(\epsilon)) \in Z_1(\Gamma, \Delta_{\mathrm{ur}}).$$

Let $C_{[\psi]}$ be the corresponding class in $H_1(\Gamma, \Delta_{\mathrm{ur}})$. The brackets around ψ indicate that $C_{[\psi]}$ depends only on the Γ-conjugacy class of the embedding ψ. Assuming that the Hecke eigenvalues of g lie in $\mathbb{Z}$, we may associate an elliptic curve $\mathscr{E}/\mathbb{Q}$ to g by the Eichler–Shimura construction. Let $\log_\omega$ be the logarithm of the formal group law on $\mathscr{E}$ associated to the differential dq/q on $E^\times/q^{\mathbb{Z}}$. Note that $\log_\omega$ factorizes as

$$\mathscr{E}(E) \to E^\times/q^{\mathbb{Z}} \to E,$$

where the left arrow is the inverse of the Tate uniformization of $\mathscr{E}$ and the right arrow is $\log_{\mathscr{L}}$ with

$$-\mathscr{L} = \mathscr{L}^{\mathrm{GS}}(g) = \mathscr{L}^{\mathrm{D}}(\varphi_g^\pm) = \mathscr{L}^{\mathrm{MTT}}(g) = \frac{\log_p(q)}{\operatorname{ord}_p(q)}.$$

The points

$$\mathrm{AJ}_g^\pm(C_{[\psi]}) \in E = \log_\omega \mathscr{E}(E)$$

are called *Stark–Heegner points* on $\mathscr{E}$. We conjecture in [Greenberg 2009, §10] that the locally defined points $\mathrm{AJ}_g^\pm(C_{[\psi]})$ in fact belong to $\log_{\mathscr{E}}(\mathscr{E}(H_{\mathbb{O}}))$, where $H_{\mathbb{O}}$ is the ring class field of K associated to the order $\mathbb{O}$. By the results of this section, we have the following formula for $\mathrm{AJ}_g^\pm(C_{[\psi]})$ in terms of the Hida family $\Phi_g^\pm$:

Corollary 29. $$\mathrm{AJ}_g^\pm(C_{[\psi]}) = \langle \Phi_g^\pm, J_* C_{[\psi]} \rangle.$$

In [Greenberg and Shahabi ≥ 2012], we apply this formula with the methods of [Bertolini and Darmon 2009] to prove partial results towards the rationality of the Stark–Heegner points $\mathrm{AJ}_g^\pm(C_{[\psi]})$ over $H_{\mathbb{O}}$.

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