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Alan Adolphson and Steven Sperber

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# On unit root formulas for toric exponential sums

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Starting from a classical generating series for Bessel functions due to Schlömilch, we use Dwork's relative dual theory to broadly generalize unit-root results of Dwork on Kloosterman sums and Sperber on hyperkloosterman sums. In particular, we express the (unique) $p$-adic unit root of an arbitrary exponential sum on the torus $\mathbb{T}^n$ in terms of special values of the $p$-adic analytic continuation of a ratio of $A$-hypergeometric functions. In contrast with the earlier works, we use noncohomological methods and obtain results that are valid for arbitrary exponential sums without any hypothesis of nondegeneracy.

## 1. Introduction

The starting point for this work is the classical generating series

$$\exp \tfrac{1}{2}(\Lambda X - \Lambda/X) = \sum_{i\in\mathbb{Z}} J_i(\Lambda)X^i$$

for the Bessel functions $\{J_i(\Lambda)\}_{i\in\mathbb{Z}}$ due to Schlömilch [1857], which was the foundation for his treatment of Bessel functions (see [Watson 1944, page 14]). Suitably normalized, it also played a fundamental role in Dwork's construction [1974] of $p$-adic cohomology for $J_0(\Lambda)$. Our realization that the series itself (suitably normalized) could be viewed as a distinguished element in Dwork's relative dual complex led us to the present generalization (which also generalizes unit-root results of Sperber [1975] on hyperkloosterman sums).

Let $A \subseteq \mathbb{Z}^n$ be a finite subset that spans $\mathbb{R}^n$ as real vector space and set

$$f_\Lambda(X) = \sum_{a\in A} \Lambda_a X^a \in \mathbb{Z}[\{\Lambda_a\}_{a\in A}][X_1^{\pm 1}, \dots, X_n^{\pm 1}],$$

where the $\Lambda_a$ and the $X_i$ are indeterminates and where $X^a = X_1^{a_1}\cdots X_n^{a_n}$ for $a = (a_1, \dots, a_n)$. Let $\mathbb{F}_q$ be the finite field of $q = p^\epsilon$ elements, $p$ a prime, and let

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$\bar{\mathbb{F}}_q$ be its algebraic closure. For each $\bar{\lambda} = (\bar{\lambda}_a)_{a\in A} \in (\bar{\mathbb{F}}_q)^{|A|}$, let

$$f_{\bar{\lambda}}(X) = \sum_{a\in A} \bar{\lambda}_a X^a \in \mathbb{F}_q(\bar{\lambda})[X_1^{\pm 1}, \dots, X_n^{\pm 1}],$$

a regular function on the $n$-torus $\mathbb{T}^n$ over $\mathbb{F}_q(\bar{\lambda})$. Fix a nontrivial additive character $\Theta : \mathbb{F}_q \to \mathbb{Q}_p(\zeta_p)$ and let $\Theta_{\bar{\lambda}}$ be the additive character $\Theta_{\bar{\lambda}} = \Theta \circ \mathrm{Tr}_{\mathbb{F}_q(\bar{\lambda})/\mathbb{F}_q}$ of the field $\mathbb{F}_q(\bar{\lambda})$. For each positive integer $l$, let $\mathbb{F}_q(\bar{\lambda}, l)$ denote the extension of degree $l$ of $\mathbb{F}_q(\bar{\lambda})$ and define an exponential sum

$$S_l = S_l(f_{\bar{\lambda}}, \Theta_{\bar{\lambda}}, \mathbb{T}^n) = \sum_{x\in\mathbb{T}^n(\mathbb{F}_q(\bar{\lambda},l))} \Theta_{\bar{\lambda}} \circ \mathrm{Tr}_{\mathbb{F}_q(\bar{\lambda},l)/\mathbb{F}_q(\bar{\lambda})}(f_{\bar{\lambda}}(x)).$$

The associated $L$-function is

$$L(f_{\bar{\lambda}}; T) = L(f_{\bar{\lambda}}, \Theta_{\bar{\lambda}}, \mathbb{T}^n; T) = \exp\left(\sum_{l=1}^{\infty} S_l \frac{T^l}{l}\right).$$

It is well-known that $L(f_{\bar{\lambda}}; T) \in \mathbb{Q}(\zeta_p)(T)$ and that its reciprocal zeros and poles are algebraic integers. We note that among these reciprocal zeros and poles there must be at least one $p$-adic unit: if $\mathbb{F}_q(\bar{\lambda})$ has cardinality $q^\kappa$, then $S_l$ is the sum of $(q^{\kappa l} - 1)^n$ $p$-th roots of unity, so $S_l$ itself is a $p$-adic unit for every $l$. On the other hand, a simple consequence of the Dwork trace formula will imply (see Section 3) that there is at most a single unit root, and it must occur amongst the reciprocal zeros (as opposed to the reciprocal poles) of $L(f_{\bar{\lambda}}; T)^{(-1)^{n+1}}$. We denote this unit root by $u(\bar{\lambda})$. It is the goal of this work to exhibit an explicit $p$-adic analytic formula for $u(\bar{\lambda})$ in terms of certain $A$-hypergeometric functions.

Consider the series

$$\exp f_\Lambda(X) = \prod_{a\in A} \exp(\Lambda_a X^a) = \sum_{i\in\mathbb{Z}^n} F_i(\Lambda) X^i \tag{1.1}$$

where the $F_i(\Lambda)$ lie in $\mathbb{Q}[\![\Lambda]\!]$. Explicitly, one has

$$F_i(\Lambda) = \sum_{\substack{u=(u_a)_{a\in A} \\ \sum_{a\in A} u_a a = i}} \frac{\Lambda^u}{\prod_{a\in A}(u_a!)}. \tag{1.2}$$

The $A$-hypergeometric system with parameter $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{C}^n$ (where $\mathbb{C}$ denotes the complex numbers) is the system of partial differential equations consisting of the operators

$$\Box_\ell = \prod_{\ell_a>0} \left(\frac{\partial}{\partial\Lambda_a}\right)^{\ell_a} - \prod_{\ell_a<0} \left(\frac{\partial}{\partial\Lambda_a}\right)^{-\ell_a}$$

for all $\ell = (\ell_a)_{a \in A} \in \mathbb{Z}^{|A|}$ satisfying $\sum_{a \in A} \ell_a a = 0$ and the operators

$$Z_j = \sum_{a \in A} a_j \Lambda_a \frac{\partial}{\partial \Lambda_a} - \alpha_j$$

for $a = (a_1, \dots, a_n) \in A$ and $j = 1, \dots, n$. Using Equations (1.1) and (1.2), it is straightforward to check that for $i \in \mathbb{Z}^n$, $F_i(\Lambda)$ satisfies the $A$-hypergeometric system with parameter $i$.

Fix $\pi$ satisfying $\pi^{p-1} = -p$ and $\Theta(1) \equiv \pi \pmod{\pi^2}$. It follows from (1.2) that the $F_i(\pi\Lambda)$ converge $p$-adically for all $\Lambda$ satisfying $|\Lambda_a| < 1$ for all $a \in A$. Let

$$\mathcal{F}(\Lambda) = F_0(\pi\Lambda)/F_0(\pi\Lambda^p).$$

The main result of this paper is the following statement. Note that we make no restriction (such as nondegeneracy) on the choice of $\bar{\lambda} \in (\bar{\mathbb{F}}_q)^{|A|}$.

**Theorem 1.3.** *The series $\mathcal{F}(\Lambda)$ converges $p$-adically for $|\Lambda_a| \leq 1$ for all $a \in A$ and the unit root of $L(f_{\bar{\lambda}}; T)$ is given by*

$$u(\bar{\lambda}) = \mathcal{F}(\lambda)\mathcal{F}(\lambda^p)\mathcal{F}(\lambda^{p^2}) \cdots \mathcal{F}(\lambda^{p^{\epsilon d(\bar{\lambda})-1}}),$$

*where $\lambda$ denotes the Teichmüller lifting of $\bar{\lambda}$ and $d(\bar{\lambda}) = [\mathbb{F}_q(\bar{\lambda}) : \mathbb{F}_q]$.*

**Remark.** Historically, expressing the unit root of a zeta- or $L$-function in terms of special values of $p$-adic hypergeometric functions has been accomplished by studying the action of Frobenius on the associated $p$-adic cohomology. Hypergeometric functions arise because the variation of $p$-adic cohomology of a parametrized family of varieties or of exponential sums is described by ($p$-adic) hypergeometric differential equations. A systematic listing of the correspondence between such parametrized families and classical hypergeometric equations is given in the appendix to [Dwork and Loeser 1993].

The first result of this type was Dwork's formula [1969] for the unit root of a nonsupersingular elliptic curve $y^2 = x(x-1)(x-\bar{\lambda})$ in terms of the Gaussian hypergeometric function $F(\frac{1}{2}, \frac{1}{2}, 1; \lambda)$. Later he established the corresponding result for the unit root of the family of Kloosterman sums $x + \bar{\lambda}/x$ using the $p$-adic Bessel function [Dwork 1974]. Since then, a number of authors have proved similar results.

We have systematically avoided the use of cohomology in this article. The cohomology spaces associated to the exponential sums $S_l(f_{\bar{\lambda}}, \Theta_{\bar{\lambda}}, \mathbb{T}^n)$ may not be well behaved for all $\bar{\lambda}$. In any case, it would require substantially more work to describe the action of Frobenius on cohomology (although, of course, this would give information about more than just the unit root).

## 2. Analytic continuation

We begin by proving the analytic continuation of the function $\mathcal{F}$ defined in the introduction.

Let $C \subseteq \mathbb{R}^n$ be the real cone generated by the elements of $A$ and let $\Delta \subseteq \mathbb{R}^n$ be the convex hull of the set $A \cup \{(0, \dots, 0)\}$. Put $M = C \cap \mathbb{Z}^n$. For $\nu \in M$, define the *weight* of $\nu$, $w(\nu)$, to be the least nonnegative real (hence rational) number such that $\nu \in w(\nu)\Delta$. There exists $D \in \mathbb{Z}_{>0}$ such that $w(\nu) \in \mathbb{Q}_{\geq 0} \cap \mathbb{Z}[1/D]$. The weight function $w$ is easily seen to have the following properties:

(i) $w(\nu) \geq 0$, and $w(\nu) = 0$ if and only if $\nu = 0$.

(ii) $w(c\nu) = cw(\nu)$ for $c \in \mathbb{Z}_{\geq 0}$.

(iii) $w(\nu + \mu) \leq w(\nu) + w(\mu)$, with equality holding if and only if $\nu$ and $\mu$ are cofacial, that is, lie in a cone over the same closed face of $\Delta$.

(iv) If $\dim \Delta = n$, let $\{\ell_i\}_{i=1}^N$ be linear forms such that the codimension-one faces of $\Delta$ not containing the origin lie in the hyperplanes $\{\ell_i = 1\}_{i=1}^N$. Then
$$w(\nu) = \max\{\ell_i(\nu)\}_{i=1}^N.$$

Let $\Omega$ be a finite extension of $\mathbb{Q}_p$ containing $\pi$ and an element $\tilde{\pi}$ satisfying $\operatorname{ord} \tilde{\pi} = (p-1)/p^2$ (we always normalize the valuation so that $\operatorname{ord} p = 1$). Put
$$R = \Big\{\xi(\Lambda) = \sum_{\nu \in (\mathbb{Z}_{\geq 0})^{|A|}} c_\nu \Lambda^\nu \ \Big|\ c_\nu \in \Omega \text{ and } \{|c_\nu|\}_\nu \text{ is bounded}\Big\},$$
$$R' = \Big\{\xi(\Lambda) = \sum_{\nu \in (\mathbb{Z}_{\geq 0})^{|A|}} c_\nu \Lambda^\nu \ \Big|\ c_\nu \in \Omega \text{ and } c_\nu \to 0 \text{ as } \nu \to \infty\Big\}.$$

Equivalently, $R$ is the ring of formal power series in $\{\Lambda_a\}_{a \in A}$ that converge on the open unit polydisk in $\Omega^{|A|}$, and $R'$ the ring of those that converge on the closed unit polydisk. Define a norm on $R$ by setting $|\xi(\Lambda)| = \sup_\nu\{|c_\nu|\}$. Both $R$ and $R'$ are complete in this norm. Note that (1.2) implies that the coefficients $F_i(\pi\Lambda)$ of $\exp \pi f_\Lambda(X)$ belong to $R$.

Let $S$ be the set
$$S = \Big\{\xi(\Lambda, X) = \sum_{\mu \in M} \xi_\mu(\Lambda)\tilde{\pi}^{-w(\mu)} X^{-\mu} \ \Big|\ \xi_\mu(\Lambda) \in R \text{ and } \{|\xi_\mu(\Lambda)|\}_\mu \text{ is bounded}\Big\}.$$

Let $S'$ be defined analogously with $R$ replaced by $R'$. Define a norm on $S$ by setting
$$|\xi(\Lambda, X)| = \sup_\mu\{|\xi_\mu(\Lambda)|\}.$$

Both $S$ and $S'$ are complete under this norm.

Define $\theta(t)=\exp(\pi(t-t^p))=\sum_{i=0}^{\infty} b_i t^i$. By [Dwork 1962, Section 4a],

$$\operatorname{ord} b_i \geq \frac{i(p-1)}{p^2}. \tag{2.1}$$

Let

$$F(\Lambda, X)=\prod_{a\in A}\theta(\Lambda_a X^a)=\sum_{\mu\in M} B_\mu(\Lambda)X^\mu.$$

**Lemma 2.2.** *One has $B_\mu(\Lambda)\in R'$ and $|B_\mu(\Lambda)|\leq|\tilde{\pi}|^{w(\mu)}$.*

*Proof.* From the definition,

$$B_\mu(\Lambda)=\sum_{\nu\in(\mathbb{Z}_{\geq 0})^{|A|}} B_\nu^{(\mu)}\Lambda^\nu,$$

where

$$B_\nu^{(\mu)}=\begin{cases}\prod_{a\in A} b_{\nu_a} & \text{if } \sum_{a\in A}\nu_a a=\mu,\\ 0 & \text{if } \sum_{a\in A}\nu_a a\neq\mu.\end{cases}$$

It follows from (2.1) that $B_\nu^{(\mu)}\to 0$ as $\nu\to\infty$, which shows that $B_\mu(\Lambda)\in R'$. We have

$$\operatorname{ord} B_\nu^{(\mu)}\geq\sum_{a\in A}\operatorname{ord} b_{\nu_a}\geq\sum_{a\in A}\frac{\nu_a(p-1)}{p^2}\geq w(\mu)\frac{p-1}{p^2},$$

which implies $|B_\mu(\Lambda)|\leq|\tilde{\pi}|^{w(\mu)}$. □

By the proof of Lemma 2.2, we may write $B_\nu^{(\mu)}=\tilde{\pi}^{w(\mu)}\tilde{B}_\nu^{(\mu)}$ with $|\tilde{B}_\nu^{(\mu)}|\leq 1$. We may then write $B_\mu(\Lambda)=\tilde{\pi}^{w(\mu)}\tilde{B}_\mu(\Lambda)$ with $\tilde{B}_\mu(\Lambda)=\sum_\nu \tilde{B}_\nu^{(\mu)}\Lambda^\nu$ and $|\tilde{B}_\mu(\Lambda)|\leq 1$. Let

$$\xi(\Lambda, X)=\sum_{\nu\in M}\xi_\nu(\Lambda)\tilde{\pi}^{-w(\nu)}X^{-\nu}\in S.$$

We claim that the product $F(\Lambda, X)\xi(\Lambda^p, X^p)$ is well-defined. Formally we have

$$F(\Lambda, X)\xi(\Lambda^p, X^p)=\sum_{\rho\in\mathbb{Z}^n}\zeta_\rho(\Lambda)X^{-\rho},$$

where

$$\zeta_\rho(\Lambda)=\sum_{\substack{\mu,\nu\in M\\ \mu-p\nu=-\rho}}\tilde{\pi}^{w(\mu)-w(\nu)}\tilde{B}_\mu(\Lambda)\xi_\nu(\Lambda^p). \tag{2.3}$$

To prove convergence of this series, we need to show that $w(\mu)-w(\nu)\to\infty$ as $\nu\to\infty$. By property (iv) of the weight function, for a given $\nu\in M$ we may choose a linear form $\ell$ (depending on $\nu$) for which $w(\nu) = \ell(\nu)$ while $w(\mu) \geq \ell(\mu)$. Since $\mu = p\nu - \rho$, we get

$$w(\mu) - w(\nu) \geq \ell(\mu - \nu) = \ell((p-1)\nu) - \ell(\rho) = (p-1)w(\nu) - \ell(\rho). \tag{2.4}$$

As $\nu \to \infty$, $(p-1)w(\nu) \to \infty$ while $\ell(\rho)$ takes values in a finite set of rational numbers (there are only finitely many possibilities for $\ell$). This gives the desired result.

For a formal series $\sum_{\rho \in \mathbb{Z}^n} \zeta_\rho(\Lambda) X^{-\rho}$ with $\zeta_\rho(\Lambda) \in \Omega[\![\Lambda]\!]$, define

$$\gamma'\biggl(\sum_{\rho \in \mathbb{Z}^n} \zeta_\rho(\Lambda) X^{-\rho}\biggr) = \sum_{\rho \in M} \zeta_\rho(\Lambda) X^{-\rho}$$

and define for $\xi(\Lambda, X) \in S$

$$\begin{aligned} \alpha^*(\xi(\Lambda, X)) &= \gamma'(F(\Lambda, X)\xi(\Lambda^p, X^p)) \\ &= \sum_{\rho \in M} \zeta_\rho(\Lambda) X^{-\rho}. \end{aligned}$$

For $\rho \in M$ put $\eta_\rho(\Lambda) = \tilde{\pi}^{w(\rho)} \zeta_\rho(\Lambda)$, so that

$$\alpha^*(\xi(\Lambda, X)) = \sum_{\rho \in M} \eta_\rho(\Lambda) \tilde{\pi}^{-w(\rho)} X^{-\rho} \tag{2.5}$$

with

$$\eta_\rho(\Lambda) = \sum_{\substack{\mu, \nu \in M \\ \mu - p\nu = \rho}} \tilde{\pi}^{w(\rho) + w(\mu) - w(\nu)} \tilde{B}_\mu(\Lambda) \xi_\nu(\Lambda^p). \tag{2.6}$$

Since $w(\rho) \geq \ell(\rho)$ for $\rho \in M$, (2.4) implies that

$$w(\rho) + w(\mu) - w(\nu) \geq (p-1)w(\nu), \tag{2.7}$$

so by (2.6), $|\eta_\rho(\Lambda)| \leq |\xi(\Lambda, X)|$ for all $\rho \in M$. This shows $\alpha^*(\xi(\Lambda, X)) \in S$ and

$$|\alpha^*(\xi(\Lambda, X))| \leq |\xi(\Lambda, X)|.$$

Furthermore, this argument also shows that $\alpha^*(S') \subseteq S'$.

**Lemma 2.8.** *If* $\xi_0(\Lambda) = 0$, *then* $|\alpha^*(\xi(\Lambda, X))| \leq |\tilde{\pi}|^{(p-1)/D} |\xi(\Lambda, X)|$.

*Proof.* This follows immediately from (2.6) and (2.7) since $w(\nu) \geq 1/D$ for $\nu \neq 0$. □

From (2.6), we have

$$\eta_0(\Lambda) = \sum_{\nu \in M} \tilde{B}_{p\nu}(\Lambda) \xi_\nu(\Lambda^p) \tilde{\pi}^{(p-1)w(\nu)}. \tag{2.9}$$

Note that $\tilde{B}_0(\Lambda) = B_0(\Lambda) \equiv 1 \pmod{\tilde{\pi}}$ since $\operatorname{ord} b_i > 0$ for all $i > 0$ implies $\operatorname{ord} B_\nu^{(0)} > 0$ for all $\nu \neq 0$. Thus $B_0(\Lambda)$ is an invertible element of $R'$. The following lemma is then immediate from (2.9).

**Lemma 2.10.** *If $\xi_0(\Lambda)$ is an invertible element of $R$ (resp. $R'$), then so is $\eta_0(\Lambda)$.*

Put

$$T = \{\xi(\Lambda, X) \in S \mid |\xi(\Lambda, X)| \leq 1 \text{ and } \xi_0(\Lambda) = 1\}$$

and put $T' = T \cap S'$. Using the notation of (2.5), define $\beta : T \to T$ by

$$\beta(\xi(\Lambda, X)) = \frac{\alpha^*(\xi(\Lambda, X))}{\eta_0(\Lambda)}.$$

Note that $\beta(T') \subseteq T'$.

**Proposition 2.11.** *The operator $\beta$ is a contraction mapping on the complete metric space $T$. More precisely, if $\xi^{(1)}(\Lambda, X), \xi^{(2)}(\Lambda, X) \in T$, then*

$$|\beta(\xi^{(1)}(\Lambda, X)) - \beta(\xi^{(2)}(\Lambda, X))| \leq |\tilde{\pi}|^{(p-1)/D} |\xi^{(1)}(\Lambda, X) - \xi^{(2)}(\Lambda, X)|.$$

*Proof.* We have (in the obvious notation)

$$\begin{aligned}
\beta(\xi^{(1)}&(\Lambda, X)) - \beta(\xi^{(2)}(\Lambda, X)) \\
&= \frac{\alpha^*(\xi^{(1)}(\Lambda, X))}{\eta_0^{(1)}(\Lambda)} - \frac{\alpha^*(\xi^{(2)}(\Lambda, X))}{\eta_0^{(2)}(\Lambda)} \\
&= \frac{\alpha^*(\xi^{(1)}(\Lambda, X) - \xi^{(2)}(\Lambda, X))}{\eta_0^{(1)}(\Lambda)} - \alpha^*(\xi^{(2)}(\Lambda, X)) \frac{\eta_0^{(1)}(\Lambda) - \eta_0^{(2)}(\Lambda)}{\eta_0^{(1)}(\Lambda)\eta_0^{(2)}(\Lambda)}.
\end{aligned}$$

Since $\eta_0^{(1)}(\Lambda) - \eta_0^{(2)}(\Lambda)$ is the coefficient of $X^0$ in $\alpha^*(\xi^{(1)}(\Lambda, X) - \xi^{(2)}(\Lambda, X))$, we have

$$|\eta_0^{(1)}(\Lambda) - \eta_0^{(2)}(\Lambda)| \leq |\alpha^*(\xi^{(1)}(\Lambda, X) - \xi^{(2)}(\Lambda, X))|.$$

And since the coefficient of $X^0$ in $\xi^{(1)}(\Lambda, X) - \xi^{(2)}(\Lambda, X)$ equals 0, the proposition follows from Lemma 2.8. □

**Remark.** Proposition 2.11 implies that $\beta$ has a unique fixed point in $T$. And since $\beta$ is stable on $T'$, that fixed point must lie in $T'$. Let $\xi(\Lambda, X) \in T'$ be the unique fixed point of $\beta$. The equation $\beta(\xi(\Lambda, X)) = \xi(\Lambda, X)$ is equivalent to the equation

$$\alpha^*(\xi(\Lambda, X)) = \eta_0(\Lambda)\xi(\Lambda, X).$$

Since $\alpha^*$ is stable on $S'$, it follows that

$$\eta_0(\Lambda)\xi_\mu(\Lambda) \in R' \quad \text{for all } \mu \in M. \tag{2.12}$$

In particular, since $\xi_0(\Lambda) = 1$, we have $\eta_0(\Lambda) \in R'$.

Put $C_0 = C \cap (-C)$, the largest subspace of $\mathbb{R}^n$ contained in $C$, and put $M_0 = \mathbb{Z}^n \cap C_0$, a subgroup of $M$. For a formal series $\sum_{\mu\in\mathbb{Z}^n} c_\mu(\Lambda)X^\mu$ with $c_\mu(\Lambda)\in\Omega[\![\Lambda]\!]$ we define

$$\gamma\biggl(\sum_{\mu\in\mathbb{Z}^n} c_\mu(\Lambda)X^\mu\biggr) = \sum_{\mu\in M_0} c_\mu(\Lambda)X^\mu$$

and set

$$\zeta(\Lambda, X) = \gamma(\exp(\pi f_\Lambda(X))).$$

Of course, when the origin is an interior point of $\Delta$, then $M_0 = \mathbb{Z}^n$ and $\zeta(\Lambda, X) = \exp(\pi f_\Lambda(X))$. In any case, the coefficients of $\zeta(\Lambda, X)$ belong to $R$.

Since $\exp(\pi f_\Lambda(X)) = \prod_{a\in A} \exp(\pi\Lambda_a X^a)$, we can expand this product to get

$$\zeta(\Lambda, X) = \gamma\biggl(\prod_{a\in A}\sum_{\nu_a=0}^{\infty}\frac{(\pi\Lambda_a X^a)^{\nu_a}}{\nu_a!}\biggr) = \sum_{\mu\in M_0} G_\mu(\Lambda)\tilde{\pi}^{-w(\mu)}X^{-\mu},$$

where

$$G_\mu(\Lambda) = \sum_{\nu\in(\mathbb{Z}_{\geq 0})^{|A|}} G_\nu^{(\mu)}\Lambda^\nu,$$

with

$$G_\nu^{(\mu)} = \begin{cases} \tilde{\pi}^{w(\mu)}\displaystyle\prod_{a\in A}\frac{\pi^{\nu_a}}{\nu_a!} & \text{if } \displaystyle\sum_{a\in A}\nu_a a = -\mu, \\ 0 & \text{if } \displaystyle\sum_{a\in A}\nu_a a \neq -\mu. \end{cases}$$

Since $\operatorname{ord}\pi^i/i! > 0$ for all $i > 0$, it follows that $G_\mu(\Lambda)\in R$, $|G_\mu(\Lambda)| \leq |\tilde{\pi}|^{w(\mu)}$, and $G_0(\Lambda)$ is invertible in $R$. This implies that $\zeta(\Lambda, X)/G_0(\Lambda)\in T$. Note also that since $F(\Lambda, X) = \exp(\pi f_\Lambda(X))/\exp(\pi f_{\Lambda^p}(X^p))$, it is straightforward to check that

$$\gamma'(F(\Lambda, X)) = \gamma(F(\Lambda, X)) = \gamma\biggl(\frac{\exp\pi f_\Lambda(X)}{\exp\pi f_{\Lambda^p}(X^p)}\biggr) = \frac{\zeta(\Lambda, X)}{\zeta(\Lambda^p, X^p)}.$$

It follows that if $\xi(\Lambda, X)$ is a series satisfying $\gamma(\xi(\Lambda, X))\in S$, then

$$\begin{aligned}\alpha^*(\gamma(\xi(\Lambda, X))) &= \gamma'(F(\Lambda, X)\gamma(\xi(\Lambda^p, X^p))) = \gamma(F(\Lambda, X))\gamma(\xi(\Lambda^p, X^p)) \\ &= \frac{\zeta(\Lambda, X)\gamma(\xi(\Lambda^p, X^p))}{\zeta(\Lambda^p, X^p)}. \end{aligned} \tag{2.13}$$

**Remark.** In terms of the $A$-hypergeometric functions $\{F_i(\Lambda)\}_{i\in M}$ defined in (1.1), we have $\exp(\pi f_\Lambda(X)) = \sum_{i\in M} F_i(\pi\Lambda)X^i$, so for $i\in M_0$ we have the relation

$$F_i(\pi\Lambda) = \tilde{\pi}^{-w(-i)}G_{-i}(\Lambda). \tag{2.14}$$

**Proposition 2.15.** *The unique fixed point of $\beta$ is $\zeta(\Lambda, X)/G_0(\Lambda)$.*

*Proof.* By (2.13), we have

$$\alpha^*\left(\frac{\zeta(\Lambda, X)}{G_0(\Lambda)}\right) = \frac{G_0(\Lambda)}{G_0(\Lambda^p)}\frac{\zeta(\Lambda, X)}{G_0(\Lambda)}, \tag{2.16}$$

which is equivalent to the assertion of the proposition. □

By the remark following Proposition 2.11, $\zeta(\Lambda, X)/G_0(\Lambda) \in T'$. This gives the following result.

**Corollary 2.17.** *For all $\mu \in M_0$, $G_\mu(\Lambda)/G_0(\Lambda) \in R'$.*

In the notation of the remark following Proposition 2.11, one has $\xi(\Lambda, X) = \zeta(\Lambda, X)/G_0(\Lambda)$ and $\eta_0(\Lambda) = G_0(\Lambda)/G_0(\Lambda^p)$, so (2.12) implies the following result.

**Corollary 2.18.** *For all $\mu \in M_0$, $G_\mu(\Lambda)/G_0(\Lambda^p) \in R'$.*

In view of (2.14), this implies that the function $\mathcal{F}(\Lambda) = F_0(\pi\Lambda)/F_0(\pi\Lambda^p)$ converges on the closed unit polydisk, which was the first assertion of Theorem 1.3.

## 3. *p*-adic theory

Fix $\bar\lambda = (\bar\lambda_a)_{a\in A} \in (\bar{\mathbb{F}}_q)^{|A|}$ and let $\lambda = (\lambda_a)_{a\in A} \in (\bar{\mathbb{Q}}_p)^{|A|}$, where $\lambda_a$ is the Teichmüller lifting of $\bar\lambda_a$. We recall Dwork's description of $L(f_{\bar\lambda}; T)$. Let $\Omega_0 = \mathbb{Q}_p(\lambda, \zeta_p, \tilde\pi)$ $(= \mathbb{Q}_p(\lambda, \pi, \tilde\pi))$ and let $\mathcal{O}_0$ be the ring of integers of $\Omega_0$.

We consider certain spaces of functions with support in $M$. We will assume that $\Omega_0$ has been extended by a finite totally ramified extension so that there is an element $\tilde\pi_0$ in $\Omega_0$ satisfying $\tilde\pi_0^D = \tilde\pi$. We shall write $\tilde\pi^{w(\nu)}$ and mean by it $\tilde\pi_0^{Dw(\nu)}$ for $\nu \in M$. Using this convention to simplify notation, we define

$$B = \left\{ \sum_{\nu\in M} A_\nu \tilde\pi^{w(\nu)} X^\nu \;\middle|\; A_\nu \in \Omega_0,\ A_\nu \to 0 \text{ as } \nu \to \infty \right\}. \tag{3.1}$$

Then $B$ is an $\Omega_0$-algebra which is complete under the norm

$$\left| \sum_{\nu\in M} A_\nu \tilde\pi^{w(\nu)} X^\nu \right| = \sup_{\nu\in M} |A_\nu|.$$

We construct a Frobenius map with arithmetic import in the usual way. Let

$$F(\lambda, X) = \prod_{a\in A} \theta(\lambda_a X^a) = \sum_{\mu\in M} B_\mu(\lambda) X^\mu,$$

i.e., $F(\lambda, X)$ is the specialization of $F(\Lambda, X)$ at $\Lambda = \lambda$, which is permissible by

Lemma 2.2. Note also that Lemma 2.2 implies

$$\operatorname{ord} B_\mu(\lambda) \geq \frac{w(\mu)(p-1)}{p^2},$$

so we may write $B_\mu(\lambda) = \tilde{\pi}^{w(\mu)} \tilde{B}_\mu(\lambda)$ with $\tilde{B}_\mu(\lambda)$ $p$-integral.

Let

$$\Psi(X^\mu) = \begin{cases} X^{\mu/p} & \text{if } p \,|\, \mu_i \text{ for all } i, \\ 0 & \text{otherwise.} \end{cases}$$

We show that $\Psi \circ F(\lambda, X)$ acts on $B$. If $\xi = \sum_{\nu \in M} A_\nu \tilde{\pi}^{w(\nu)} X^\nu \in B$, then

$$\Psi\biggl(\biggl(\sum_{\mu \in M} \tilde{\pi}^{w(\mu)} \tilde{B}_\mu(\lambda) X^\mu\biggr)\biggl(\sum_{\nu \in M} A_\nu \tilde{\pi}^{w(\nu)} X^\nu\biggr)\biggr) = \sum_{\omega \in M} C_\omega(\lambda) \tilde{\pi}^{w(\omega)} X^\omega$$

where

$$C_\omega(\lambda) = \sum_\nu \tilde{\pi}^{w(p\omega-\nu)+w(\nu)-w(\omega)} \tilde{B}_{p\omega-\nu}(\lambda) A_\nu.$$

For any positive constant $K$ there are only finitely many values of $\nu$ for which $w(\nu)$ and $w(p\omega - \nu)$ are $< K$. This implies that the series $C_\omega(\lambda)$ converges.

We have $pw(\omega) = w(p\omega) \leq w(p\omega - \nu) + w(\nu)$, so that

$$\operatorname{ord} C_\omega(\lambda) \geq \inf_\nu \{\operatorname{ord} \tilde{\pi}^{(p-1)w(\omega)} A_\nu\} = \frac{(p-1)^2 w(\omega)}{p^2} + \inf_\nu \{\operatorname{ord} A_\nu\}. \tag{3.2}$$

This implies that $\Psi(F(\lambda, X)\xi) \in B$.

Let $d(\bar{\lambda}) = [\mathbb{F}_q(\bar{\lambda}) : \mathbb{F}_q]$, so that $\lambda^{p^{\epsilon d(\bar{\lambda})}} = \lambda$. Put

$$\alpha_\lambda = \Psi^{\epsilon d(\bar{\lambda})} \circ \biggl(\prod_{i=0}^{\epsilon d(\bar{\lambda})-1} F(\lambda^{p^i}, X^{p^i})\biggr).$$

For any power series $P(T)$ in the variable $T$ with constant term 1, define

$$P(T)^{\delta_{\bar{\lambda}}} = P(T)/P(p^{\epsilon d(\bar{\lambda})} T).$$

Then $\alpha_\lambda$ is a completely continuous operator on $B$ and the Dwork trace formula [Dwork 1962; Serre 1962] gives

$$L(f_{\bar{\lambda}}, \Theta_{\bar{\lambda}}, \mathbb{T}^n; T)^{(-1)^{n+1}} = \det(I - T\alpha_\lambda \,|\, B)^{\delta_{\bar{\lambda}}^n}. \tag{3.3}$$

By Equation (3.2), the $(\omega, \nu)$-entry of the matrix of $\alpha_\lambda$ [Serre 1962, Section 2] has ord $> 0$ unless $\omega = \nu = 0$. The formula for $\det(I - T\alpha_\lambda)$ [Serre 1962, Proposition 7a)] then shows that this Fredholm determinant can have at most a single unit root. Since $L(f_{\bar{\lambda}}; T)$ has at least one unit root (Section 1), it follows from (3.3) that $L(f_{\bar{\lambda}}; T)$ has exactly one unit root.

## 4. Dual theory

It will be important to consider the trace formula in the dual theory as well. The basis for this construction goes back to [Dwork 1964] and [Serre 1962]. We define

$$B^* = \Big\{\xi^* = \sum_{\mu\in M} A^*_\mu \tilde{\pi}^{-w(\mu)} X^{-\mu} \;\Big|\; \{A^*_\mu\}_{\mu\in M} \text{ is a bounded subset of } \Omega_0\Big\},$$

a $p$-adic Banach space with the norm $|\xi^*| = \sup_{\mu\in M}\{|A^*_\mu|\}$. We define a pairing $\langle\ ,\ \rangle : B^* \times B \to \Omega_0$: if $\xi = \sum_{\mu\in M} A_\mu \tilde{\pi}^{w(\mu)} X^\mu$, $\xi^* = \sum_{\mu\in M} A^*_\mu \tilde{\pi}^{-w(\mu)} X^{-\mu}$, set

$$\langle \xi^*, \xi\rangle = \sum_{\mu\in M} A_\mu A^*_\mu \in \Omega_0.$$

The series on the right converges since $A_\mu \to 0$ as $\mu\to\infty$ and $\{A^*_\mu\}_{\mu\in M}$ is bounded. This pairing identifies $B^*$ with the dual space of $B$, i.e., the space of continuous linear mappings from $B$ to $\Omega_0$ [Serre 1962, Proposition 3].

Let $\Phi$ be the endomorphism of the space of formal series defined by

$$\Phi\bigg(\sum_{\mu\in\mathbb{Z}^n} c_\mu X^{-\mu}\bigg) = \sum_{\mu\in\mathbb{Z}^n} c_\mu X^{-p\mu},$$

and let $\gamma'$ be the endomorphism

$$\gamma'\bigg(\sum_{\mu\in\mathbb{Z}^n} c_\mu X^{-\mu}\bigg) = \sum_{\mu\in M} c_\mu X^{-\mu}.$$

Consider the formal composition $\alpha^*_\lambda = \gamma' \circ \Big(\prod_{i=0}^{\epsilon d(\bar\lambda)-1} F(\lambda^{p^i}, X^{p^i})\Big) \circ \Phi^{\epsilon d(\bar\lambda)}$.

**Proposition 4.1.** *The operator $\alpha^*_\lambda$ is an endomorphism of $B^*$ which is adjoint to $\alpha_\lambda : B \to B$.*

*Proof.* As $\alpha^*_\lambda$ is the composition of the operators $\gamma'\circ F(\lambda^{p^i}, X)\circ\Phi$ and $\alpha_\lambda$ is the composition of the operators $\Psi\circ F(\lambda^{p^i}, X)$, $i=0,\dots,\epsilon d(\bar\lambda)-1$, it suffices to check that $\gamma'\circ F(\lambda, X)\circ\Phi$ is an endomorphism of $B^*$ adjoint to $\Psi\circ F(\lambda, X) : B\to B$. Let $\xi^*(X) = \sum_{\mu\in M} A^*_\mu \tilde{\pi}^{-w(\mu)} X^{-\mu} \in B^*$. The proof that the product $F(\lambda, X)\xi^*(X^p)$ is well-defined is analogous to the proof of convergence of the series (2.3). We have

$$\gamma'(F(\lambda, X)\xi^*(X^p)) = \sum_{\omega\in M} C_\omega(\lambda)\tilde{\pi}^{-w(\omega)} X^{-\omega},$$

where

$$C_\omega(\lambda) = \sum_{\mu - p\nu = -\omega} \tilde{B}_\mu(\lambda) A^*_\nu \tilde{\pi}^{w(\omega)+w(\mu)-w(\nu)}. \tag{4.2}$$

Note that

$$pw(\nu) = w(p\nu) \le w(\omega) + w(\mu)$$

since $p\nu = \omega + \mu$. Thus

$$(p-1)w(\nu) \le w(\omega) + w(\mu) - w(\nu),$$

which implies that the series on the right-hand side of (4.2) converges and that $|C_\omega(\lambda)| \le |\xi^*|$ for all $\omega \in M$. It follows that $\gamma'(F(\lambda, X)\xi^*(X^p)) \in B^*$. It is straightforward to check that $\langle \Phi(X^{-\mu}), X^\nu \rangle = \langle X^{-\mu}, \Psi(X^\nu)\rangle$ and that

$$\langle \gamma'(F(\lambda, X)X^{-\mu}), X^\nu \rangle = \langle X^{-\mu}, F(\lambda, X)X^\nu \rangle$$

for all $\mu, \nu \in M$, which implies the maps are adjoint. □

By [Serre 1962, Proposition 15] we have $\det(I - T\alpha_\lambda^* \mid B^*) = \det(I - T\alpha_\lambda \mid B)$, so (3.3) implies

$$L(f_{\bar\lambda}, \Theta_{\bar\lambda}, \mathbb{T}^n; T)^{(-1)^{n+1}} = \det(I - T\alpha_\lambda^* \mid B^*)^{\delta_{\bar\lambda}^n}. \tag{4.3}$$

From Equations (2.14) and (2.16), we have

$$\alpha^*\left(\frac{\zeta(\Lambda, X)}{G_0(\Lambda)}\right) = \mathcal{F}(\Lambda)\frac{\zeta(\Lambda, X)}{G_0(\Lambda)}.$$

It follows by iteration that for $m \ge 0$,

$$(\alpha^*)^m\left(\frac{\zeta(\Lambda, X)}{G_0(\Lambda)}\right) = \left(\prod_{i=0}^{m-1} \mathcal{F}(\Lambda^{p^i})\right)\frac{\zeta(\Lambda, X)}{G_0(\Lambda)}. \tag{4.4}$$

We have

$$\frac{\zeta(\Lambda, X)}{G_0(\Lambda)} = \sum_{\mu \in M_0} \frac{G_\mu(\Lambda)}{G_0(\Lambda)} \tilde\pi^{-w(\mu)} X^{-\mu},$$

so by Corollary 2.17 we may evaluate at $\Lambda = \lambda$ to get an element of $B^*$:

$$\frac{\zeta(\Lambda, X)}{G_0(\Lambda)}\bigg|_{\Lambda=\lambda} = \sum_{\mu \in M_0} \frac{G_\mu(\Lambda)}{G_0(\Lambda)}\bigg|_{\Lambda=\lambda} \tilde\pi^{-w(\mu)} X^{-\mu} \in B^*.$$

It is straightforward to check that the specialization of the left-hand side of (4.4) with $m = \epsilon d(\bar\lambda)$ at $\Lambda = \lambda$ is exactly $\alpha_\lambda^*((\zeta(\Lambda, X)/G_0(\Lambda))|_{\Lambda=\lambda})$, so specializing (4.4) with $m = \epsilon d(\bar\lambda)$ at $\Lambda = \lambda$ gives

$$\alpha_\lambda^*\left(\frac{\zeta(\Lambda, X)}{G_0(\Lambda)}\bigg|_{\Lambda=\lambda}\right) = \left(\prod_{i=0}^{\epsilon d(\bar\lambda)-1} \mathcal{F}(\lambda^{p^i})\right)\frac{\zeta(\Lambda, X)}{G_0(\Lambda)}\bigg|_{\Lambda=\lambda}. \tag{4.5}$$

Equation (4.5) shows that $\prod_{i=0}^{\epsilon d(\bar\lambda)-1} \mathcal{F}(\lambda^{p^i})$ is a (unit) eigenvalue of $\alpha_\lambda^*$, hence by (4.3) it is the unique unit eigenvalue of $L(f_{\bar\lambda}; T)$.

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adolphs@math.okstate.edu *Department of Mathematics, Oklahoma State University, Stillwater, OK 74078, United States*

sperber@math.umn.edu *School of Mathematics, University of Minnesota, Minneapolis, MN 55455, United States*

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