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Symmetries of the transfer operator for $\Gamma_0(N)$ and a character deformation of the Selberg zeta function for $\Gamma_0(4)$

Markus Fraczek and Dieter Mayer

The transfer operator for $\Gamma_0(N)$ and trivial character χ_0 possesses a finite group of symmetries generated by permutation matrices P with $P^2 = \mathrm{id}$. Every such symmetry leads to a factorization of the Selberg zeta function in terms of Fredholm determinants of a reduced transfer operator. These symmetries are related to the group of automorphisms in $\mathrm{GL}(2, \mathbb{Z})$ of the Maass wave forms of $\Gamma_0(N)$. For the group $\Gamma_0(4)$ and Selberg's character χ_α there exists just one nontrivial symmetry operator P. The eigenfunctions of the corresponding reduced transfer operator with eigenvalue $\lambda = \pm 1$ are related to Maass forms that are even or odd, respectively, under a corresponding automorphism. It then follows from a result of Sarnak and Phillips that the zeros of the Selberg function determined by the eigenvalue $\lambda = -1$ of the reduced transfer operator stay on the critical line under deformation of the character. From numerical results we expect that, on the other hand, all the zeros corresponding to the eigenvalue $\lambda = +1$ are off this line for a nontrivial character χ_α.

1. Introduction

In the transfer operator approach to Selberg's zeta function for a Fuchsian group Γ this function gets expressed in terms of the Fredholm determinant of a transfer operator constructed from the symbolic dynamics of the geodesic flow on the corresponding surface of constant negative curvature. Though this approach has been carried out, up to now, only for certain groups, like modular subgroups of finite index [Chang and Mayer 2000; 2001a; 2001b] and Hecke triangle groups [Mayer and Strömberg 2008; Mayer et al. 2012; Mayer and Mühlenbruch 2010], it has led to new points of view on the Selberg zeta function [Zagier 2002] and the theory

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of period functions [Lewis and Zagier 2001]. Another application of this method is a precise numerical calculation of the Selberg zeta function [Strömberg 2008], which seems to be impossible by other means at the moment.

In this paper we discuss the transfer operator approach to the Selberg zeta function for Hecke congruence subgroups with a character. Of special interest is the behavior of its zeros for $\Gamma_0(4)$ under singular deformation by Selberg's character [Selberg 1990].

As found numerically in [Fraczek 2010], certain symmetries of the transfer operator for these groups play an important role in this process. These symmetries lead to a factorization of the Selberg zeta function for the full modular group $\mathrm{SL}(2,\mathbb{Z})$, as known. There it corresponds to the involution $Ju(z) = u(-z^*)$ of the Maass forms u for this group [Efrat 1993; Lewis and Zagier 2001]. Obviously the corresponding element $j = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \in \mathrm{GL}(2,\mathbb{Z})$ generates the normalizer group of $\mathrm{SL}(2,\mathbb{Z})$ in $\mathrm{GL}(2,\mathbb{Z})$. It tuns out also that the symmetries of the transfer operator for $\Gamma_0(N)$ correspond to automorphisms of the Maass forms from its normalizer group in $\mathrm{GL}(2,\mathbb{Z})$.

For the group $\Gamma_0(4)$ with a character χ_α introduced in [Selberg 1990] and discussed also in [Phillips and Sarnak 1994], there is only one such nontrivial symmetry of the transfer operator. It corresponds to the generator of $\Gamma_0(4)$'s normalizer group in $\mathrm{GL}(2,\mathbb{Z})$ leaving invariant the character χ_α. The results of Phillips and Sarnak imply that the zeros on the critical line of one factor of Selberg's function stay on this line under the deformation of the character, and hence the corresponding Maass wave forms for the trivial character remain Maass wave forms. Numerical results [Fraczek 2010], on the other hand, imply that the zeros on the critical line of the second factor of this function should all leave this line when the deformation is turned on. A detailed discussion of these numerical results and their partial proofs is in preparation [Bruggeman et al. 2012].

The paper is organized as follows: in Section 2 we recall briefly the form of the transfer operator

$$\mathbf{L}_{\beta,\rho_\pi} = \begin{pmatrix} 0 & \mathscr{L}^+_{\beta,\pi} \\ \mathscr{L}^-_{\beta,\pi} & 0 \end{pmatrix}$$

for a general finite index subgroup Γ of the modular group $\mathrm{SL}(2,\mathbb{Z})$ and unitary representation π, and introduce the symmetries

$$\tilde{P} = \begin{pmatrix} 0 & P \\ P & 0 \end{pmatrix}$$

of this operator defined by permutation matrices P. Any such symmetry leads to a factorization of the Selberg zeta function in terms of the Fredholm determinants of the reduced transfer operator $P\mathscr{L}^{+}_{\beta,\pi}$. The eigenfunctions with eigenvalues $\lambda = \pm 1$ of this reduced transfer operator then fulfill certain functional equations.

In Section 3 we discuss the generators $J_{n,-}$ of the group of automorphisms in $\mathrm{GL}(2,\mathbb{Z})$ of the Maass forms u for $\Gamma = \Gamma_0(N)$ and $\pi = \chi_0$ the trivial character. We introduce their period functions $\underline{\psi}$ and derive a formula for the period function $J_{n,-}\underline{\psi}$ of the Maass form $J_{n,-}u$.

In Section 4 we introduce Selberg's character χ_α and the nontrivial automorphism $J_{2,-}$ of the Maass forms for $\Gamma_0(4)$. We derive again a formula for the period function $J_{2,-}\underline{\psi}$ of the Maass form $J_{2,-}u$ leading to a permutation matrix $P_{2,-}$ which defines a symmetry $\tilde{P}_{2,-}$ of the transfer operator $\mathbf{L}_{\beta,\rho_{\chi_\alpha}}$. From this we conclude that the eigenfunctions with eigenvalues $\lambda = \pm 1$ of the operator $P_{2,-}\mathscr{L}^{+}_{\beta,\pi}$ correspond to Maass forms that are even or odd, respectively, under the involution $J_{2,-}$. Results of Phillips and Sarnak then imply that the zeros of the Selberg function on the critical line corresponding to the eigenfunctions with eigenvalue $\lambda = -1$ of this operator stay on this line under the deformation of the character.

2. The transfer operator and the Selberg zeta function for Hecke congruence subgroups $\Gamma_0(N)$

The starting point of the transfer operator approach to the Selberg zeta function for a subgroup Γ of the modular group $\mathrm{SL}(2,\mathbb{Z})$ of index $\mu = [\mathrm{SL}(2,\mathbb{Z}) : \Gamma] < \infty$ is the geodesic flow $\Phi_t : SM_\Gamma \to SM_\Gamma$ on the unit tangent bundle SM_Γ of the corresponding surface $M_\Gamma = \Gamma \setminus \mathbb{H}$ of constant negative curvature. Here

$$\mathbb{H} = \{z = x + iy : y > 0\}$$

denotes the hyperbolic plane with hyperbolic metric $ds^2 = (dx^2 + dy^2)/y^2$, on which the group Γ acts via Möbius transformations: $gz = (az+b)/(cz+d)$ if $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. In the present paper we mostly work with the Hecke congruence subgroup

$$\Gamma_0(N) = \left\{ g \in \mathrm{SL}(2,\mathbb{Z}) : g = \begin{pmatrix} a & b \\ cN & d \end{pmatrix} \right\},$$

with index $\mu_N = N \prod_{p|N}(1 + 1/p)$, where p is a prime number. If $\rho : \Gamma \to \mathrm{end}(\mathbb{C}^d)$ is a unitary representation of Γ then the Selberg zeta function $Z_{\Gamma,\rho}$ is defined as

$$Z_{\Gamma,\rho}(\beta) = \prod_{\gamma} \prod_{k=0}^{\infty} \det\bigl(1 - \rho(g_\gamma)\exp(-(k+\beta)l_\gamma)\bigr), \tag{2.0.1}$$

where l_γ denotes the period of the prime periodic orbit γ of Φ_t and $g_\gamma \in \Gamma$ is hyperbolic with $g_\gamma(\gamma) = \gamma$. In the dynamical approach to this function it gets expressed in terms of the so-called transfer operator, well-known from D. Ruelle's thermodynamic formalism approach to dynamical systems. For general modular groups Γ with finite index μ and finite-dimensional representation π this operator $\mathbf{L}_{\beta,\pi} : B \to B$ was determined in [Chang and Mayer 2000; 2001b] as

$$\mathbf{L}_{\beta,\pi} = \begin{pmatrix} 0 & \mathscr{L}^{+}_{\beta,\rho_\pi} \\ \mathscr{L}^{-}_{\beta,\rho_\pi} & 0 \end{pmatrix}, \tag{2.0.2}$$

where $B = B(D, \mathbb{C}^{\mu}) \bigoplus B(D, \mathbb{C}^{\mu})$ is the Banach space of holomorphic functions on the disc $D = \{z : |z-1| < \frac{3}{2}\}$, and ρ_π denotes the representation of $\mathrm{SL}(2,\mathbb{Z})$ induced from the representation π of Γ. The operator $\mathscr{L}^{\pm}_{\beta,\rho_\pi}$ is given for $\operatorname{Re}\beta > \frac{1}{2}$ by

$$(\mathscr{L}^{\pm}_{\beta,\rho_\pi}\underline{f})(z) = \sum_{n=1}^{\infty} \frac{1}{(z+n)^{2\beta}} \rho_\pi(ST^{\pm n})\underline{f}\Big(\frac{1}{z+n}\Big), \tag{2.0.3}$$

where $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. In the following we restrict ourselves to one-dimensional unitary representations π, hence unitary characters, which we denote as usual by χ. In this case the following theorem was proved in [Chang and Mayer 2001b].

Theorem 2.0.1. *The transfer operator $\mathbf{L}_{\beta,\chi} : B \to B$ with*

$$\mathbf{L}_{\beta,\chi} = \begin{pmatrix} 0 & \mathscr{L}^{+}_{\beta,\chi} \\ \mathscr{L}^{-}_{\beta,\chi} & 0 \end{pmatrix} \quad \textit{and} \quad (\mathscr{L}^{\pm}_{\beta,\chi}\underline{f})(z) = \sum_{n=1}^{\infty} \frac{1}{(z+n)^{2\beta}} \rho_\chi(ST^{\pm n})\underline{f}\Big(\frac{1}{z+n}\Big)$$

extends to a meromorphic family of nuclear operators of order zero in the entire complex β plane with possible poles at $\beta_k = (1-k)/2$, $k = 0, 1, 2, \dots$. The Selberg zeta function $Z_{\Gamma,\chi}$ for modular group Γ and character χ can be expressed as $Z_{\Gamma,\chi}(\beta) = \det(1 - \mathbf{L}_{\beta,\chi}) = \det(1 - \mathscr{L}^{+}_{\beta,\chi}\mathscr{L}^{-}_{\beta,\chi}) = \det(1 - \mathscr{L}^{-}_{\beta,\chi}\mathscr{L}^{+}_{\beta,\chi})$.

This shows that the zeros of the Selberg function are given by those β-values for which $\lambda = 1$ belongs to the spectrum $\sigma(\mathbf{L}_{\beta,\chi})$, or equivalently to the spectrum $\sigma(\mathscr{L}^{-}_{\beta,\chi}\mathscr{L}^{+}_{\beta,\chi}) = \sigma(\mathscr{L}^{+}_{\beta,\chi}\mathscr{L}^{-}_{\beta,\chi})$. From Selberg's trace formula one knows that there are two kinds of such zeros: the trivial zeros at $\beta = -k$, $k = 1, 2, \dots$, and the so-called spectral zeros. The former correspond to eigenvalues $\lambda = \beta(1-\beta)$ of the automorphic Laplacian with $\operatorname{Re}\beta = \frac{1}{2}$ or $\frac{1}{2} \le \beta \le 1$, and the latter to resonances of the Laplacian, that is, poles of the scattering determinant with $\operatorname{Re}\beta < \frac{1}{2}$ and $\operatorname{Im}\beta > 0$ [Hejhal 1983; Venkov 1990]. For arithmetic groups like the congruence subgroups with trivial or congruent character χ one knows that these resonances lie on the line $\operatorname{Re}\beta = \frac{1}{4}$, corresponding to the nontrivial zeros $\zeta_R(2\beta) = 0$ of the Riemann zeta function ζ_R in the trivial case and to the zeros $L(2\beta, \chi_\alpha) = 0$ of other Dirichlet L-functions in the congruent case, assuming the generalized Riemann hypothesis, as well as on the line $\operatorname{Re}\beta = 0$. For general Fuchsian groups and congruence subgroups with noncongruent character, however, these resonances can be anywhere in the half-plane $\operatorname{Re}\beta < \frac{1}{2}$.

2.1. ***Symmetries of the transfer operator for $\mathbf{\Gamma_0(N)}$.*** It turns out that there exists for any N a finite number h_N of $\mu_N \times \mu_N$ permutation matrices P with $P^2 = \mathrm{id}_{\mu_N}$ such that the matrix

$$\tilde{P} = \begin{pmatrix} 0 & P \\ P & 0 \end{pmatrix}$$

commutes with the transfer operator $\mathbf{L}_{\beta,\chi}$ and hence

$$P \mathscr{L}^{+}_{\beta,\chi} = \mathscr{L}^{-}_{\beta,\chi} P. \tag{2.1.1}$$

Thereby $P = (P_{ij})_{1 \le i,j \le \mu_N}$ acts in the Banach space $B(D, \mathbb{C}^{\mu_N})$ as $(P\underline{f})_i(z) = \sum_{j=1}^{\mu_N} P_{ij} f_j(z)$ if $\underline{f}(z) = (f_i(z))_{1 \le i \le \mu_N}$. We call such a matrix $\tilde{P}$ a symmetry of the transfer operator. As an example consider the group $\Gamma_0(4)$ and Selberg's character χ_α, $0 \le \alpha \le 1$, which will be described later. Its transfer operator $\mathbf{L}_{\beta,\chi_\alpha}$ has the following form:

$$\mathbf{L}_{\beta,\chi_\alpha} \tilde{f}_{+1} = \sum_{q=0}^{\infty} f_{-3}\big|_{2\beta} \tilde{S} T^{1+4q} + f_{-4}\big|_{2\beta} \tilde{S} T^{2+4q} + f_{-5}\big|_{2\beta} \tilde{S} T^{3+4q} + f_{-2}\big|_{2\beta} \tilde{S} T^{4+4q},$$

$$\begin{aligned}\mathbf{L}_{\beta,\chi_\alpha} \tilde{f}_{+2} = \sum_{q=0}^{\infty} & e^{2\pi i(1+4q)\alpha} f_{-1}\big|_{2\beta} \tilde{S} T^{1+4q} + e^{2\pi i(2+4q)\alpha} f_{-1}\big|_{2\beta} \tilde{S} T^{2+4q} \\ & + e^{2\pi i(3+4q)\alpha} f_{-1}\big|_{2\beta} \tilde{S} T^{3+4q} + e^{2\pi i(4+4q)\alpha} f_{-1}\big|_{2\beta} \tilde{S} T^{4+4q},\end{aligned}$$

$$\begin{aligned}\mathbf{L}_{\beta,\chi_\alpha} \tilde{f}_{+3} = \sum_{q=0}^{\infty} & e^{-2\pi i\alpha} f_{-2}\big|_{2\beta} \tilde{S} T^{1+4q} + e^{-2\pi i\alpha} f_{-3}\big|_{2\beta} \tilde{S} T^{2+4q} \\ & + e^{-2\pi i\alpha} f_{-4}\big|_{2\beta} \tilde{S} T^{3+4q} + e^{-2\pi i\alpha} f_{-5}\big|_{2\beta} \tilde{S} T^{4+4q},\end{aligned}$$

$$\begin{aligned}\mathbf{L}_{\beta,\chi_\alpha} \tilde{f}_{+4} = \sum_{q=0}^{\infty} & e^{-2\pi i\alpha(1+4q)} f_{-6}\big|_{2\beta} \tilde{S} T^{1+4q} + e^{-2\pi i\alpha(2+4q)} f_{-6}\big|_{2\beta} \tilde{S} T^{2+4q} \\ & + e^{-2\pi i\alpha(3+4q)} f_{-6}\big|_{2\beta} \tilde{S} T^{3+4q} + e^{-2\pi i\alpha(4+4q)} f_{-6}\big|_{2\beta} \tilde{S} T^{4+4q},\end{aligned}$$

$$\begin{aligned}\mathbf{L}_{\beta,\chi_\alpha} \tilde{f}_{+5} = \sum_{q=0}^{\infty} & e^{2\pi i\alpha} f_{-4}\big|_{2\beta} \tilde{S} T^{1+4q} + e^{2\pi i\alpha} f_{-5}\big|_{2\beta} \tilde{S} T^{2+4q} \\ & + e^{2\pi i\alpha} f_{-2}\big|_{2\beta} \tilde{S} T^{3+4q} + e^{2\pi i\alpha} f_{-3}\big|_{2\beta} \tilde{S} T^{4+4q},\end{aligned}$$

$$\mathbf{L}_{\beta,\chi_\alpha} \tilde{f}_{+6} = \sum_{q=0}^{\infty} f_{-5}\big|_{2\beta} \tilde{S} T^{1+4q} + f_{-2}\big|_{2\beta} \tilde{S} T^{2+4q} + f_{-3}\big|_{2\beta} \tilde{S} T^{3+4q} + f_{-4}\big|_{2\beta} \tilde{S} T^{4+4q},$$

$$\mathbf{L}_{\beta,\chi_\alpha} \tilde{f}_{-1} = \sum_{q=0}^{\infty} f_{+5}\big|_{2\beta} \tilde{S} T^{1+4q} + f_{+4}\big|_{2\beta} \tilde{S} T^{2+4q} + f_{+3}\big|_{2\beta} \tilde{S} T^{3+4q} + f_{+2}\big|_{2\beta} \tilde{S} T^{4+4q},$$

$$\mathbf{L}_{\beta,\chi_\alpha}\tilde{f}_{-2} = \sum_{q=0}^{\infty} e^{-2\pi i\alpha(1+4q)} f_{+1}\big|_{2\beta}\tilde{S}T^{1+4q} + e^{-2\pi i\alpha(2+4q)} f_{+1}\big|_{2\beta}\tilde{S}T^{2+4q} + e^{-2\pi i\alpha(3+4q)} f_{+1}\big|_{2\beta}\tilde{S}T^{3+4q} + e^{-2\pi i\alpha(4+4q)} f_{+1}\big|_{2\beta}\tilde{S}T^{4+4q},$$

$$\mathbf{L}_{\beta,\chi_\alpha}\tilde{f}_{-3} = \sum_{q=0}^{\infty} e^{-2\pi i\alpha} f_{+4}\big|_{2\beta}\tilde{S}T^{1+4q} + e^{-2\pi i\alpha} f_{+3}\big|_{2\beta}\tilde{S}T^{2+4q} + e^{-2\pi i\alpha} f_{+2}\big|_{2\beta}\tilde{S}T^{3+4q} + e^{-2\pi i\alpha} f_{+5}\big|_{2\beta}\tilde{S}T^{4+4q},$$

$$\mathbf{L}_{\beta,\chi_\alpha}\tilde{f}_{-4} = \sum_{q=0}^{\infty} e^{2\pi i\alpha(1+4q)} f_{+6}\big|_{2\beta}\tilde{S}T^{1+4q} + e^{2\pi i\alpha(2+4q)} f_{+6}\big|_{2\beta}\tilde{S}T^{2+4q} + e^{2\pi i\alpha(3+4q)} f_{+6}\big|_{2\beta}\tilde{S}T^{3+4q} + e^{2\pi i\alpha(4+4q)} f_{+6}\big|_{2\beta}\tilde{S}T^{4+4q},$$

$$\mathbf{L}_{\beta,\chi_\alpha}\tilde{f}_{-5} = \sum_{q=0}^{\infty} e^{2\pi i\alpha} f_{+2}\big|_{2\beta}\tilde{S}T^{1+4q} + e^{2\pi i\alpha} f_{+5}\big|_{2\beta}\tilde{S}T^{2+4q} + e^{2\pi i\alpha} f_{+4}\big|_{2\beta}\tilde{S}T^{3+4q} + e^{2\pi i\alpha} f_{+3}\big|_{2\beta}\tilde{S}T^{4+4q},$$

$$\mathbf{L}_{\beta,\chi_\alpha}\tilde{f}_{-6} = \sum_{q=0}^{\infty} f_{+3}\big|_{2\beta}\tilde{S}T^{1+4q} + f_{+2}\big|_{2\beta}\tilde{S}T^{2+4q} + f_{+5}\big|_{2\beta}\tilde{S}T^{3+4q} + f_{+4}\big|_{2\beta}\tilde{S}T^{4+4q},$$

where $\tilde{f} \in B(D,\mathbb{C}^\mu) \bigoplus B(D,\mathbb{C}^\mu)$ is given by $\tilde{f} = (\underline{f}_+, \underline{f}_-)$, $\underline{f}_\pm = (f_{\pm i})_{1\le i\le 6}$, and $\tilde{S}z = 1/z$. The induced representation ρ_{χ_α} of the character χ_α on $\Gamma_0(4)$ is defined in terms of the coset decomposition of $\mathrm{SL}(2,\mathbb{Z})$

$$\mathrm{SL}(2,\mathbb{Z}) = \bigcup_{i=1}^{6} \Gamma_0(4) R_i \tag{2.1.2}$$

as

$$\rho_\chi(g)_{ij} = \delta_{\Gamma_0(4)}(R_i g R_j^{-1})\chi_\alpha(R_i g R_j^{-1}), \quad 1 \le i, \quad j \le 6. \tag{2.1.3}$$

Thereby we have chosen the following representatives $R_i \in \mathrm{SL}(2,\mathbb{Z})$ of the cosets $\Gamma_0(4)R_i$

$$R_1 = \mathrm{id}_2, \qquad R_i = ST^{i-2}, \quad 2 \le i \le 5, \quad \text{and} \quad R_6 = ST^2S. \tag{2.1.4}$$

It turns out that the two permutation matrices P_1 and P_2 corresponding to the permutations

$$\sigma_1 = \frac{1\ 2\ 3\ 4\ 5\ 6}{1\ 2\ 5\ 4\ 3\ 6} \tag{2.1.5}$$

and

$$\sigma_2 = \frac{1\ 2\ 3\ 4\ 5\ 6}{6\ 4\ 3\ 2\ 5\ 1} \tag{2.1.6}$$

fulfill (2.1.1) for $\alpha = 0$ and hence the corresponding matrices $\tilde{P}_i$, $i = 1, 2$, commute with the transfer operator $\mathbf{L}_{\beta,\chi_0}$ where χ_0 is the trivial character. The matrix $\tilde{P}_2$, on the other hand, commutes even with the operator $\mathbf{L}_{\beta,\chi_\alpha}$ for all α. The matrix $\rho_{\chi_0}(S)$ is given by the permutation σ_S where

$$\sigma_S = \frac{1\ 2\ 3\ 4\ 5\ 6}{2\ 1\ 5\ 6\ 3\ 4}, \tag{2.1.7}$$

and an easy calculation shows that $P_i\rho_{\chi_0}(S) = \rho_{\chi_0}(S)P_i$, $i = 1, 2$. The matrix $\rho_{\chi_0}(T)$, on the other hand, is given by the permutation σ_T with

$$\sigma_T = \frac{1\ 2\ 3\ 4\ 5\ 6}{1\ 3\ 4\ 5\ 2\ 6}. \tag{2.1.8}$$

One then checks that $P_i\rho_{\chi_0}(T) = \rho_{\chi_0}(T^{-1})P_i$, $i = 1, 2$. Therefore $P_i\rho_{\chi_0}(ST^n) = \rho_{\chi_0}(ST^{-n})P_i$ for all $n \in \mathbb{N}$ and $i = 1, 2$. For the character χ_α analogous relations hold for P_2.

For the trivial character χ_0 one can determine for the group $\Gamma_0(N)$ the number h_N of matrices P_i with the above properties and hence the defining symmetries of the transfer operator as follows:

Theorem 2.1.1. *For the Hecke congruence subgroup $\Gamma_0(N)$ and trivial character $\chi_0 \equiv 1$ there exist h_N matrices $\tilde{P} = \left(\begin{smallmatrix} 0 & P \\ P & 0 \end{smallmatrix}\right)$ commuting with the transfer operator $\mathbf{L}_{\beta,\chi_0}$ where P is a $\mu_N \times \mu_N$ permutation matrix satisfying $P^2 = \mathbf{1}_{\mu_N}$,*

$$P\rho_{\chi_0}(S) = \rho_{\chi_0}(S)P \quad \textit{and} \quad P\rho_{\chi_0}(T) = \rho_{\chi_0}(T^{-1})P,$$

and hence

$$P\mathcal{L}^+_{\beta,\chi_0} = \mathcal{L}^-_{\beta,\chi_0}P.$$

Thereby $h_N = \max\{k : k|24$ and $k^2|N\}$. The permutation matrices P are determined by the h_N generators j of the normalizer group $\mathcal{N}_N$ of $\Gamma_0(N)$ in $\mathrm{GL}(2, \mathbb{Z})$. The Selberg zeta function Z_{Γ,χ_0} can be written as

$$Z_{\Gamma,\chi_0} = \det(1 - P\mathcal{L}^+_{\beta,\chi_0})\det(1 + P\mathcal{L}^+_{\beta,\chi_0}).$$

Remark 2.1.2. For $\Gamma_0(4)$, obviously $h_N = 2$. According to Theorem 2.1.1, there exist two permutation matrices P_1 and P_2 given by the permutations σ_1 and σ_2 above. Since $P_1P_2 = P_2P_1$ and $P_i\mathcal{L}^+_{\beta,\chi_0} = \mathcal{L}^-_{\beta,\chi_0}P_i$, $i = 1, 2$, we find

$$P_1P_2P_1\mathcal{L}^+_{\beta,\chi_0} = P_1P_2\mathcal{L}^-_{\beta,\chi_0}P_1 = P_1\mathcal{L}^+_{\beta,\chi_0}P_2P_1 = P_1\mathcal{L}^+_{\beta,\chi_0}P_1P_2,$$

and the operators P_1P_2 and $P_1\mathcal{L}^+_{\beta,\chi_0}$ commute, where the operator P_1P_2 corresponds to the permutation

$$\sigma = \frac{1\ 2\ 3\ 4\ 5\ 6}{6\ 4\ 5\ 2\ 3\ 1}. \tag{2.1.9}$$

We find also $P_1 P_2 \mathscr{L}^+_{\beta,\chi_0} = \mathscr{L}^+_{\beta,\chi_0} P_1 P_2$. But $(P_1 P_2)^2 = \mathrm{id}_6$, hence this operator has only the eigenvalues $\lambda = \pm 1$ and the Banach space $B(D,\mathbb{C}^6)$ decomposes as $B(D,\mathbb{C}^6) = B(D,\mathbb{C}^6)_+ \oplus B(D,\mathbb{C}^6)_-$ with $P_1 P_2 \underline{f}_\pm = \pm \underline{f}_\pm$ for $\underline{f}_\pm \in B(D,\mathbb{C}^6)_\pm$. Therefore the elements $\underline{f}_\epsilon \in B(D,\mathbb{C}^6)_\epsilon$, $\epsilon = \pm$ have the form $(\underline{f}_\epsilon)_i = f_i$, $1 \le i \le 3$ and $(\underline{f}_\epsilon)_{\sigma(i)} = \epsilon f_i$, $1 \le i \le 3$. Denote by

$$\mathscr{L}^+_{\beta,\chi_0,\pm} : B(D,\mathbb{C}^6)_\pm \to B(D,\mathbb{C}^6)_\pm \quad \text{and} \quad P_1\mathscr{L}^+_{\beta,\chi_0,\pm} : B(D,\mathbb{C}^6)_\pm \to B(D,\mathbb{C}^6)_\pm,$$

the restrictions of the operators $\mathscr{L}^+_{\beta,\chi_0}$ and $P_1\mathscr{L}^+_{\beta,\chi_0}$, respectively, to the subspace $B(D,\mathbb{C}^6)_\pm$, which obviously is isomorphic to the space $B(D,\mathbb{C}^3)$. Then

$$\det(1 \pm P_1\mathscr{L}^+_{\beta,\chi_0}) = \det(1 \pm P_1\mathscr{L}^+_{\beta,\chi_0,+}) \det(1 \pm P_1\mathscr{L}^+_{\beta,\chi_0,-}),$$

where the operator $P_1\mathscr{L}^+_{\beta,\chi_0,\epsilon} : B(D,\mathbb{C}^3) \to B(D,\mathbb{C}^3)$ can be written as

$$P_1\mathscr{L}^+_{\beta,\chi_0,\epsilon} = \begin{pmatrix} 0 & \epsilon\mathscr{L}_{\beta,2}+\mathscr{L}_{\beta,4} & \epsilon\mathscr{L}_{\beta,1}+\mathscr{L}_{\beta,3} \\ \mathscr{L}_\beta & 0 & 0 \\ 0 & \mathscr{L}_{\beta,1}+\epsilon\mathscr{L}_{\beta,3} & \epsilon\mathscr{L}_{\beta,2}+\mathscr{L}_{\beta,4} \end{pmatrix}, \tag{2.1.10}$$

with $\mathscr{L}_{\beta,k} f = \sum_{q=0}^\infty f\big|_{2\beta} \tilde{S} T^{1+kq}$, $1 \le k \le 4$, and $\mathscr{L}_\beta = \sum_{k=1}^4 \mathscr{L}_{\beta,k}$. The operator $\mathscr{L}^+_{\beta,\chi_0,\epsilon}$ in the space $B(D,\mathbb{C}^3)$, on the other hand, has the form

$$\mathscr{L}^+_{\beta,\chi_0,\epsilon} = \begin{pmatrix} 0 & \epsilon\mathscr{L}_{\beta,2}+\mathscr{L}_{\beta,4} & \epsilon\mathscr{L}_{\beta,1}+\mathscr{L}_{\beta,3} \\ \mathscr{L}_\beta & 0 & 0 \\ 0 & \epsilon\mathscr{L}_{\beta,1}+\mathscr{L}_{\beta,3} & \mathscr{L}_{\beta,2}+\epsilon\mathscr{L}_{\beta,4} \end{pmatrix}. \tag{2.1.11}$$

To relate the Fredholm determinants of the operators $(P_1\mathscr{L}^+_{\beta,\chi_0,\epsilon})^2$ and $(\mathscr{L}^+_{\beta,\chi_0,\epsilon})^2$ we use the following simple lemma:

Lemma 2.1.3. *Let α, β, and γ be complex numbers and $\epsilon = \pm 1$. Then λ is an eigenvalue of the matrix*

$$\mathbb{L}_1 = \begin{pmatrix} 0 & \alpha & \beta \\ \gamma & 0 & 0 \\ 0 & \beta & \epsilon\alpha \end{pmatrix}$$

if and only if $\epsilon\lambda$ is an eigenvalue of the matrix

$$\mathbb{L}_2 = \begin{pmatrix} 0 & \alpha & \beta \\ \gamma & 0 & 0 \\ 0 & \epsilon\beta & \alpha \end{pmatrix}.$$

Proof. The proof follows from comparing the characteristic polynomials of the two matrices. □

This shows that, for all $n \in \mathbb{N}$,

$$\operatorname{trace} \mathbb{L}_1^n = \sum_{k=1}^3 (\mathbb{L}_1^n)_{k,k} = \epsilon^n \operatorname{trace} \mathbb{L}_2^n = \epsilon^n \sum_{k=1}^3 (\mathbb{L}_2^n)_{k,k}.$$

But then is not too difficult to see that also $\operatorname{trace}(\mathcal{L}^+_{\beta,\chi_0,\epsilon})^n = \epsilon^n \operatorname{trace}(P_1\mathcal{L}^+_{\beta,\chi_0,\epsilon})^n$ for all $n \in \mathbb{N}$ and hence $\det(1-(P_1\mathcal{L}^+_{\beta,\chi_0,\epsilon})^2) = \det(1-(\mathcal{L}^+_{\beta,\chi_0,\epsilon})^2)$ for $\epsilon = \pm$. Therefore the Selberg zeta function $Z_{\Gamma_0(4),\chi_0}(\beta)$ for the group $\Gamma_0(4)$ with trivial character χ_0 can be written as

$$\begin{aligned} Z_{\Gamma_0(4),\chi_0}(\beta) &= \det(1-(P_1\mathcal{L}^+_{\beta,\chi_0})^2) = \det(1-(\mathcal{L}^+_{\beta,\chi_0})^2) \\ &= \det(1-\mathcal{L}^+_{\beta,\chi_0})\det(1+\mathcal{L}^+_{\beta,\chi_0}). \end{aligned} \tag{2.1.12}$$

Furthermore, this function factorizes in this case also as

$$\begin{aligned} Z_{\Gamma_0(4),\chi_0}(\beta) = \det(1-P_1\mathcal{L}^+_{\beta,\chi_0,+})\det(1-P_1\mathcal{L}^+_{\beta,\chi_0,-}) \\ \times \det(1+P_1\mathcal{L}^+_{\beta,\chi_0,+})\det(1+P_1\mathcal{L}^+_{\beta,\chi_0-}). \end{aligned} \tag{2.1.13}$$

To prove Theorem 2.1.1 we relate the matrices P to the generating automorphisms in $\mathrm{GL}(2,\mathbb{Z})$ of the Maass wave forms for $\Gamma_0(N)$. We can determine this way the explicit form of these matrices P. For this we derive, in a first step, a Lewis-type functional equation for the eigenfunctions of the operator $P\mathcal{L}^+_{\beta,\chi}$ with eigenvalues $\lambda = \pm 1$.

2.2. ***A Lewis-type functional equation.*** Consider any finite index modular subgroup Γ and any unitary character $\chi : \Gamma \to \mathbb{C}^\star$, together with the induced representation ρ_χ of $\mathrm{SL}(2,\mathbb{Z})$. Assume there exists a symmetry $\tilde{P} = \left(\begin{smallmatrix} 0 & P \\ P & 0 \end{smallmatrix}\right)$, with P a permutation matrix with properties analogous to Theorem 2.1.1, and commuting with the transfer operator

$$\mathbf{L}_{\beta,\chi} = \begin{pmatrix} 0 & \mathcal{L}^+_{\beta,\rho_\chi} \\ \mathcal{L}^-_{\beta,\rho_\chi} & 0 \end{pmatrix}$$

of Γ. If $\underline{f}$ is an eigenfunction of the operator $P\mathcal{L}^+_{\beta,\chi}$ with eigenvalues $\lambda = \pm 1$ then one can show:

Proposition 2.2.1. *If $P\mathcal{L}^+_{\beta,\chi}\underline{f}(\zeta) = \lambda \underline{f}(\zeta)$ with $\lambda = \pm 1$ then the function $\underline{\Psi}(\zeta) := P\rho_\chi(T^{-1}S)P\underline{f}(\zeta-1)$ fulfills the functional equations*

$$\underline{\Psi}(\zeta) = \lambda\zeta^{-2\beta}P\rho_\chi(S)\underline{\Psi}\Big(\frac{1}{\zeta}\Big) \tag{2.2.1}$$

and

$$\underline{\Psi}(\zeta) - \rho_\chi(T^{-1})\underline{\Psi}(\zeta+1) - (\zeta+1)^{-2\beta}\rho_\chi(T'^{-1})\underline{\Psi}\Big(\frac{\zeta}{\zeta+1}\Big) = \underline{0}, \tag{2.2.2}$$

where $T' = ST^{-1}S$. On the other hand, every solution $\underline{\Psi}$ of (2.2.1) and (2.2.2) holomorphic in the cut β-plane $(-\infty, 0]$ satisfying $\Psi_i(z) = o(z^{-\min\{1,2\operatorname{Re}s\}})$ as $z \downarrow 0$ and $\Psi_i(z) = o(z^{-\min\{0,2\operatorname{Re}s-1\}})$ as $z \to \infty$ determines an eigenfunction $\underline{f}$ with eigenvalues $\lambda = \pm 1$ of the operator $P\mathcal{L}^+_{\beta,\chi}$.

Proof. Let $\operatorname{Re}\beta > \frac{1}{2}$. If $P\mathscr{L}_\beta^+ \underline{f}(\zeta) = \lambda \underline{f}(\zeta)$, $\lambda = \pm 1$, then obviously

$$P\rho_\chi(STS)PP\mathscr{L}_\beta^+ \underline{f}(\zeta+1) = \lambda P\rho_\chi(STS)P\underline{f}(\zeta+1).$$

Subtracting the two equations leads to

$$\lambda \underline{f}(\zeta) - \lambda P\rho_\chi(STS)P\underline{f}(\zeta+1) - (\zeta+1)^{-2\beta}P\rho_\chi(ST)\underline{f}\Big(\frac{1}{\zeta+1}\Big) = \underline{0},$$

and hence the function $\underline{\psi}(\zeta) := P\underline{f}(\zeta-1)$ fulfills the equation

$$\underline{\psi}(\zeta) - \rho_\chi(STS)\underline{\psi}(\zeta+1) - \lambda\zeta^{-2\beta}\rho_\chi(ST)P\underline{\psi}\Big(\frac{\zeta+1}{\zeta}\Big) = \underline{0}. \tag{2.2.3}$$

Replacing ζ by $\dfrac{1}{\zeta}$ and multiplying the resulting equation by $\zeta^{-2\beta}\rho_\chi(STS)P\rho_\chi(T^{-1}S)$ gives

$$\zeta^{-2\beta}\rho_\chi(STS)P\rho_\chi(T^{-1}S)\underline{\psi}\Big(\frac{1}{\zeta}\Big) - \zeta^{-2\beta}\rho_\chi(STS)P\rho_\chi(S)\underline{\psi}\Big(\frac{\zeta+1}{\zeta}\Big) \\ - \lambda\rho_\chi(STS)\underline{\psi}(\zeta+1) = \underline{0}.$$

Since $\rho_\chi(S)P = P\rho_\chi(S)$, one finds, comparing with (2.2.3),

$$\underline{\psi}(\zeta) = \lambda\zeta^{-2\beta}\rho_\chi(STS)P\rho_\chi(T^{-1}S)\underline{\psi}\Big(\frac{1}{\zeta}\Big).$$

Hence the function $\underline{\tilde{\psi}} := \rho_\chi(T^{-1}S)\underline{\psi}$ fulfills (2.2.1). The same equation is then fulfilled also by the function

$$\underline{\Psi}(\zeta) := P\underline{\tilde{\psi}}(\zeta) = P\rho_\chi(T^{-1}S)P\underline{f}(\zeta-1), \tag{2.2.4}$$

that is,

$$\underline{\Psi}(\zeta) = \lambda\zeta^{-2\beta}P\rho_\chi(S)\underline{\Psi}\Big(\frac{1}{\zeta}\Big). \tag{2.2.5}$$

Inserting finally $\underline{\psi}(\zeta) = \rho_\chi(ST)P\underline{\Psi}(\zeta)$ into (2.2.3) and using (2.2.1) leads to

$$\underline{\Psi}(\zeta) - P\rho_\chi(T)P\underline{\Psi}(\zeta+1) - (\zeta+1)^{-2\beta}P\rho_\chi(T')P\underline{\Psi}\Big(\frac{\zeta}{\zeta+1}\Big) = \underline{0}.$$

But by assumption $P\rho_\chi(T)P = \rho_\chi(T^{-1})$; hence $P\rho_\chi(T')P = \rho_\chi(T'^{-1})$ and thus

$$\underline{\Psi}(\zeta) - \rho_\chi(T^{-1})\underline{\Psi}(\zeta+1) - (\zeta+1)^{-2\beta}\rho_\chi(T'^{-1})\underline{\Psi}\Big(\frac{\zeta}{\zeta+1}\Big) = \underline{0}. \tag{2.2.6}$$

Hence for $\operatorname{Re}\beta > \frac{1}{2}$ the first part of the proposition holds. By analytic continuation in β one proves the general case.

To prove the second part we follow the arguments of [Deitmar and Hilgert 2007, Lemma 4.1]: if $\underline{\Psi}(\zeta)$ is a solution of the Lewis equation (2.2.2) with $\beta \notin \mathbb{Z}$ then

$\underline{\Psi}$ has the asymptotic expansions

$$\underline{\Psi}(\zeta) \sim_{\zeta\to 0} \zeta^{2\beta} Q_0\Big(\frac{1}{\zeta}\Big) + \sum_{l=-1}^{\infty} \underline{C}_l^* \zeta^l, \quad \underline{\Psi}(\zeta) \sim_{\zeta\to\infty} Q_\infty(\zeta) + \sum_{l=-1}^{\infty} \underline{C}_l^{*\prime} \zeta^{-l-2\beta},$$

where $Q_0, Q_\infty : \mathbb{C} \to \mathbb{C}^\mu$ are smooth functions such that $Q_0(\zeta+1) = \rho_\chi(T')Q_0(\zeta)$ and $Q_\infty(\zeta+1) = \rho_\chi(T)Q_\infty(\zeta)$, and the constants $\underline{C}_l^*$ and $\underline{C}_l^{*\prime}$ are determined by the Taylor coefficients $\underline{C}_m = (1/m!)\underline{\Psi}^{(m)}(1)$. The functions Q_0 and Q_∞ are defined as follows for general β with $-2\operatorname{Re}\beta < M \in \mathbb{N}$:

$$\begin{aligned} Q_0(\zeta) := \zeta^{-2\beta}\underline{\Psi}\Big(\frac{1}{\zeta}\Big) - \sum_{m=0}^{M} \zeta_{\rho_\chi}(m+2\beta, z)\underline{C}_m \\ - \sum_{n=0}^{\infty} (n+\zeta)^{-2\beta} \rho_\chi(T'^{-n}T^{-1}) \bigg(\underline{\Psi}\Big(1+\frac{1}{n+\zeta}\Big) - \sum_{m=0}^{M} \frac{\underline{C}_m}{(n+\zeta)^m}\bigg), \end{aligned}$$

$$\begin{aligned} Q_\infty(\zeta) := \underline{\Psi}(\zeta) - \sum_{m=0}^{M} \zeta'_{\rho_\chi}(m+2\beta, \zeta+1)\underline{C}_m \\ - \sum_{n=0}^{\infty} (n+\zeta)^{-2\beta} \rho_\chi(T^{-(n-1)}T'^{-1}) \bigg(\underline{\Psi}\Big(1-\frac{1}{n+\zeta}\Big) - \sum_{m=0}^{M} \frac{\underline{C}_m}{(n+\zeta)^m}\bigg), \end{aligned}$$

where

$$\zeta_{\rho_\chi}(a, \zeta) = \frac{1}{N^a} \sum_{k=0}^{N-1} \rho_\chi(T'^{-k}T^{-1})\zeta\Big(a, \frac{k+\zeta}{N}\Big)$$

and

$$\zeta'_{\rho_\chi}(a, \zeta) = \frac{1}{N^a} \sum_{k=0}^{N-1} \rho_\chi(T^{-k}T'^{-1})\zeta_H\Big(a, \frac{k+\zeta}{N}\Big),$$

with $\zeta_H(a, \zeta)$ the Hurwitz zeta function. According to [Deitmar and Hilgert 2007, Remark 4.2] any solution $\underline{\Psi}$ of (2.2.2) with $\underline{\Psi}(\zeta) = \underline{o}(\zeta^{-\min\{1,2\beta\}})$ for $\zeta \to 0$ fulfills the equation

$$\underline{\Psi}(\zeta) = \zeta^{-2\beta} \sum_{n=0}^{\infty} (n+\zeta^{-1})^{-2\beta} \rho_\chi(T'^{-n}T^{-1})\underline{\Psi}\Big(1+\frac{1}{n+\zeta^{-1}}\Big)$$

and moreover $\underline{C}_{-1}^* = 0$. But if $\underline{\Psi}(\zeta)$ fulfills also (2.2.1) then one finds

$$\lambda\zeta^{-2\beta} P\rho_\chi(S)\underline{\Psi}\Big(\frac{1}{\zeta}\Big) = \zeta^{-2\beta} \sum_{n=0}^{\infty} (n+\zeta^{-1})^{-2\beta} \rho_\chi(T'^{-n}T^{-1})\underline{\Psi}\Big(1+\frac{1}{n+\zeta^{-1}}\Big),$$

and hence

$$\lambda P\rho_\chi(S)\underline{\Psi}(\zeta+1) = \sum_{n=1}^{\infty}(n+\zeta)^{-2\beta}\rho_\chi(T'^{-(n-1)}T^{-1})\underline{\Psi}\left(1+\frac{1}{n+\zeta}\right). \qquad (2.2.7)$$

According to (2.2.4) $\underline{\Psi}(\zeta+1) = P\rho_\chi(T^{-1}S)P\underline{f}(\zeta)$, and hence we get

$$\lambda\rho_\chi(ST^{-1}S)P\underline{f}(\zeta) = \sum_{n=1}^{\infty}(n+\zeta)^{-2\beta}\rho_\chi(T'^{-(n-1)}T^{-1})P\rho_\chi(T^{-1}S)P\underline{f}\left(\frac{1}{\zeta+n}\right).$$

Inserting $T'^{-(n-1)} = ST^{(n-1)}S$ one arrives at

$$\lambda\underline{f}(\zeta) = \sum_{n=1}^{\infty}(n+\zeta)^{-2\beta}P\rho_\chi(ST^n)\rho_\chi(ST^{-1})P\rho_\chi(T^{-1}S)P\underline{f}\left(\frac{1}{\zeta+n}\right).$$

Since $\rho_\chi(ST^{-1})P = P\rho_\chi(ST)$ we get finally

$$\lambda\underline{f}(\zeta) = \sum_{n=1}^{\infty}\frac{1}{(n+\zeta)^{2\beta}}P\rho_\chi(ST^n)\underline{f}\left(\frac{1}{n+\zeta}\right).$$

Hence any solution $\underline{\Psi}$ of the Lewis equations (2.2.1) and (2.2.2) with the asymptotics at the cut $\zeta=0$ determines an eigenfunction $\underline{f}$ of the transfer operator $P\mathscr{L}^{+}_{\beta,\chi}$ with eigenvalues $\lambda = \pm 1$. □

3. Automorphism of the Maass forms and their period functions for $\Gamma_0(N)$

The Maass forms $u = u(z)$ of a cofinite Fuchsian group Γ and unitary character χ are real analytic functions $u : \mathbb{H} \to \mathbb{C}$ with

- $\Delta u(z) = \lambda u(z)$,
- $u(gz) = \chi(g)u(z)$ for all $g \in \Gamma$, and
- $u(g_j z) = O(y^C)$ as $y \to \infty$ for some constant $C \in \mathbb{R}$ and all cusps $z_j = g_j(i\infty)$ of Γ.

The cusp forms are those forms which decay exponentially fast at the cusps. If $u \in L_2(M_\Gamma)$ we call u a Maass wave form.

Definition 3.0.1. An element $j \in \mathrm{GL}(2,\mathbb{Z})$ defines an automorphism J of the Maass wave form u for the group Γ and character χ if Ju with $Ju(z) := u(jz)$ is a Maass form for Γ and character χ.

Obviously j defines an automorphism J if and only if j is a normalizer of the group Γ and the character χ is invariant under j, that is, $\chi(jgj^{-1}) = \chi(g)$ for all $g \in \Gamma$. Thereby $jz = (az^*+b)/(cz^*+d)$ if $\det g = ad - bd = -1$. We have to show that the function $Ju(z) = u(jz)$ has at most polynomial growth at the cusps $z_i = \tau_i(i\infty)$ of Γ, where $\tau_i \in \mathrm{SL}(2,\mathbb{Z})$. If $\det j = -1$, one has $u(j\tau_i(z)) = u(j\tau_i j_{0,-} j_{0,-}(z))$ where $j_{0,-} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. Then $j\tau_i j_{0,-} \in \mathrm{SL}(2,\mathbb{Z})$ and hence $j\tau_i j_{0,-} = \gamma_i R_i$ for some $\gamma_i \in \Gamma$ and some representative R_i of the cosets $\Gamma \setminus \mathrm{SL}(2,\mathbb{Z})$. But $R_i = \eta\tau_{\sigma(i)}$ for some $\eta \in \Gamma$ and some index $\sigma(i)$. Hence $u(j\tau_i(z)) = u(\tau_{\sigma(i)}(-z^*))$ which has at most polynomial growth at the cusps. The same argument applies if $\det j = 1$, and it shows also that Ju is a Maass wave form or a cusp form if u is one.

3.1. ***The group of automorphisms of Maass forms for $\Gamma_0(N)$ and trivial character χ_0.*** We restrict ourselves now to the case $\Gamma = \Gamma_0(N)$ and assume $\chi = \chi_0$. Denote by $\mathcal{N}_N$ the normalizer group $\{\Gamma_0(N)j : j \text{ normalizer of } \Gamma_0(N) \text{ in } \mathrm{GL}(2,\mathbb{Z})\}$. Using results of [Lehner and Newman 1964; Conway and Norton 1979], we find:

Proposition 3.1.1. *For $h_N = \max\{r : r|24$ and $r^2|N\}$ and $k_N := N/h_N$ the normalizer group $\mathcal{N}_N$ is given by*

$$\mathcal{N}_N = \left\{\Gamma_0(N)j_{n,\pm},\ j_{n,\pm} = \begin{pmatrix} 1 & 0 \\ nk_N & \pm 1 \end{pmatrix},\ 0 \le n \le h_N - 1\right\}.$$

Proof. Using the fact that the divisors k of 24 are exactly the numbers for which $a \cdot d = 1 \bmod k$ implies $a = d \bmod k$ one shows that the normalizer group of $\Gamma_0(N)$ in $\mathrm{SL}(2,\mathbb{Z})$ is $\Gamma_0(N) \setminus \Gamma_0(N/\nu)$ [Lehner and Newman 1964] with $\nu = 2^{\min\{3,[\epsilon_2/2]\}} \cdot 3^{\min\{1,[\epsilon_3/2]\}}$, $\epsilon_2 = \max\{l : 2^l|N\}$, and $\epsilon_3 = \max\{l : 3^l|N\}$. But obviously $\nu = h_N$ and $[\Gamma_0(k_N) : \Gamma_0(N)] = h_N$ and hence $\mathcal{N}_N = \Gamma_0(N) \setminus \left(\Gamma_0(k_N) \bigcup \Gamma_0(k_N) j_{0,-}\right)$. Since $j_{n,\pm} \ne j_{m,\pm} \bmod \Gamma_0(N)$ for $n \ne m$, this group has just the $2h_N$ elements $\Gamma_0(N)j_{n,\pm}$, $0 \le n \le h_N - 1$. The normalizer group $\mathcal{N}_N$ is therefore generated by the h_N generators $\{\Gamma_0(N)j_{n,-}, 0 \le n \le h_N - 1\}$. ☐

3.2. ***The period functions of $\Gamma_0(N)$ and character χ.*** For u a Maass form with $\Delta u = \beta(1-\beta)u$ and $\Gamma_0(N) \setminus \mathrm{SL}(2,\mathbb{Z}) = \{\Gamma_0(N)R_i, 1 \le i \le \mu_N\}$ its vector-valued period function $\underline{u}$ is defined by

$$\underline{u} = (u_i(z))_{1 \le i \le \mu_N} \text{ where } u_i(z) = u(R_i z). \tag{3.2.1}$$

Then one has, as shown for instance in [Mühlenbruch 2006]:

- $\underline{u}(gz) = \rho_\chi(g)\underline{u}(z)$ for all $g \in \mathrm{SL}(2,\mathbb{Z})$ and ρ_χ the representation of $\mathrm{SL}(2,\mathbb{Z})$ induced from the character χ on $\Gamma_0(N)$ and
- $\Delta u_i(z) = \beta(1-\beta)u_i(z)$, $1 \le i \le \mu_N$.

Given two eigenfunctions $u = u(z)$ and $v = v(z)$ of the hyperbolic Laplacian with identical eigenvalue $\lambda = \beta(1-\beta)$, one knows [Lewis and Zagier 2001] that the 1-form $\eta = \eta(u, v)$, with

$$\eta(u,v)(z) := \left[v(z)\partial_y u(z) - u(z)\partial_y v(z)\right]dx + \left[u(z)\partial_x v(z) - v(z)\partial_x u(z)\right]dy$$

is closed. If $u = u(z)$ is a Maass wave form for $\Gamma_0(N)$ with eigenvalue $\lambda = \beta(1-\beta)$ and $R_\zeta(z) = y/((\zeta - x)^2 + y^2)$ denotes the Poisson kernel, the vector-valued period function $\underline{\psi} = (\psi_j(\zeta))_{1\le j\le \mu_N}$ is defined as

$$\psi_j(\zeta) := \int_0^\infty \eta(u_j, R_\zeta^\beta)(z). \tag{3.2.2}$$

The following result has been shown for trivial character χ_0 in [Mühlenbruch 2006]. That proof can be extended, however, immediately to the case of a nontrivial character χ.

Proposition 3.2.1. *The period function $\underline{\psi} = \underline{\psi}(\zeta)$ of a Maass wave form $u = u(z)$ for $\Gamma_0(N)$ and unitary character χ is holomorphic in the cut ζ-plane $\mathbb{C} \setminus (-\infty, 0]$ and fulfills there the Lewis functional equation (2.2.2):*

$$\underline{\psi}(\zeta) - \rho_\chi(T^{-1})\underline{\psi}(\zeta+1) - (\zeta+1)^{-2\beta}\rho_\chi(T'^{-1})\underline{\psi}\left(\frac{\zeta}{\zeta+1}\right) = \underline{0},$$

where ρ_χ denotes the representation of $\mathrm{SL}(2,\mathbb{Z})$ *induced from the character χ of* $\Gamma_0(N)$.

On the other hand, it follows from [Deitmar and Hilgert 2007] that the solutions of the above equation holomorphic in the cut ζ-plane with certain asymptotic behavior at the cut 0 and at ∞ are in one-to-one correspondence with the Maass wave forms. That paper treats only the trivial character but it can be extended also to the case of the nontrivial character χ. Since the function $\underline{\Psi}(\zeta) = P\rho_\chi(T^{-1}S)P\underline{f}(\zeta-1)$ with $\underline{f}$ an eigenfunction of the operator $P\mathcal{L}^+_{\beta,\chi}$ with eigenvalues $\lambda = \pm 1$ is such a solution of (2.2.2), these eigenfunctions are in one-to-one correspondence with the Maass wave forms. As in the case of the full modular group $\mathrm{SL}(2,\mathbb{Z})$ treated in [Chang and Mayer 1998; Lewis and Zagier 2001] one can extend this result to arbitrary Maass forms, that is, also to the real analytic Eisenstein series for $\Gamma_0(N)$ and unitary character χ.

3.3. ***Automorphisms of the period functions.*** We have seen that the group of automorphisms in $\mathrm{GL}(2,\mathbb{Z})$ of the Maass forms u of $\Gamma_0(N)$ and trivial character χ_0 is generated by the matrices

$$j_{n,-} = \begin{pmatrix} 1 & 0 \\ nk_N & -1 \end{pmatrix}, \quad 0 \le n \le \mu_N - 1.$$

Denote by $J_{n,-}u$ the Maass form $J_{n,-}u(z) := u(j_{n,-}z)$ and by $J_{n,-}\underline{\psi}$ its period function. Then one shows

Theorem 3.3.1. *The period function $J_{n,-}\underline{\psi} = (J_{n,-}\psi_j(\zeta))_{1\le j\le\mu_N}$ is given by*

$$J_{n,-}\psi_j(\zeta) = \zeta^{-2\beta}\psi_{\lambda_{n\_}\circ\sigma\circ\delta(j)}\left(\frac{1}{\zeta}\right), \tag{3.3.1}$$

where the permutations λ_{n_-}, σ and δ are determined through the coset representatives R_j of $\Gamma_0(N) \setminus \mathrm{SL}(2, \mathbb{Z})$ *as follows*:

$$j_{n,+} R_j = \theta_j R_{\lambda_{n,-}(j)}, \quad j_{0,-} R_j j_{0,-} = \gamma_j R_{\sigma(j)}, \quad \textit{and} \quad R_j S = \eta_j R_{\delta(j)},$$

with $\theta_j, \gamma_j, \eta_j \in \Gamma_0(N)$ *for* $1 \le j \le \mu_N$.

Proof. For $u = u(z)$ a Maass form for $\Gamma_0(N)$ and trivial character χ_0 and $\underline{u} = \underline{u}(z)$ its vector-valued Maass form consider the Maass forms $J_{n,\pm} u(z) = u(j_{n,\pm} z)$ and $J_{n,\pm} \underline{u}(z) = (J_{n,\pm} u_j(z))_{1 \le j \le \mu_N}$, with $J_{n,\pm} u_j(z) = u(j_{n,\pm} R_j z)$. Since $j_{n,+} R_j = \theta_j R_{\lambda_{n.-}(j)}$ for some unique $\theta_j \in \Gamma_0(N)$ and permutation $\lambda_{n,-}$ of $\{1, 2, \ldots, \mu_N\}$ one gets for $J_{n,+} u_j$

$$J_{n,+} u_j(z) = u(R_{\lambda_{n,-}(j)} z) = u_{\lambda_{n,-}(j)}(z). \tag{3.3.2}$$

For $J_{n,+} u_j(-z^*) = u(j_{n,+} R_j(-z^*)) = u(j_{n,+} R_j j_{0,-} z)$, on the other hand, one finds

$$J_{n,+} u_j(-z^*) = u(j_{n,-} j_{0,-} R_j j_{0,-} z) = u(j_{n,-} R_{\sigma(j)} z),$$

since $j_{0,-} R_j j_{0,-} = \gamma_j R_{\sigma(j)}$ for some unique $\gamma_j \in \Gamma_0(N)$ and permutation σ of $\{1, 2, \ldots, \mu_N\}$. Hence

$$J_{n,+} u_j(-z^*) = J_{n,-} u_{\sigma(j)}(z). \tag{3.3.3}$$

Consider next $J_{n,+} u_j(Sz) = J_{n,+} u(R_j S z)$. Since $R_j S = \eta_j R_{\delta(j)}$ for unique $\eta_j \in \Gamma_0(N)$ and permutation δ of $\{1, 2, \ldots, \mu_N\}$, one has

$$J_{n,+} u_j(Sz) = J_{n,+} u(R_{\delta(j)} z) = J_{n,+} u_{\delta(j)}(z).$$

Hence by (3.3.2)

$$J_{n,+} u_j(Sz) = u_{\lambda_{n,-} \circ \delta(j)}(z). \tag{3.3.4}$$

On the other hand, one gets for $J_{n,+} u_j(S(-z^*)) = J_{n,+} u_j(-S z^*)$ by using (3.3.3):

$$J_{n,+} u_j(S(-z^*)) = J_{n,-} u_{\sigma(j)}(Sz) = u(j_{n,-} R_{\sigma(j)} S z),$$

and therefore

$$J_{n,+} u_j(-S z^*) = u(j_{n,-} \eta_{\sigma(j)} R_{\delta \circ \sigma(j)}(z)) = J_{n,-} u_{\delta \circ \sigma(j)}(z).$$

But $\sigma \circ \delta = \delta \circ \sigma$ and therefore

$$J_{n,+} u_j(S(-z^*)) = J_{n,-} u_{\sigma \circ \delta(j)}(z). \tag{3.3.5}$$

Define next

$$v_{\pm,j}(z) := J_{n,+} u_j(z) \pm J_{n,+} u_j(-z^*).$$

Then by (3.3.2) and (3.3.3) one has

$$v_{\pm,j}(z) = u_{\lambda(j)}(z) \pm J_{n,-}u_{\sigma(j)}(z)$$

and hence, if $\Delta u(z) = \beta(1-\beta)u(z)$,

$$\Delta v_{\pm,j}(z) = \beta(1-\beta)v_{\pm,j}(z) \tag{3.3.6}$$

and

$$v_{\pm,j}(-z^*) = \pm v_{\pm,j}(z). \tag{3.3.7}$$

Equations (3.3.4) and (3.3.5), on the other hand, show

$$v_{\pm,j}(Sz) = v_{\pm,\delta(j)}(z). \tag{3.3.8}$$

Set $\underline{\psi}'_{\pm}(\zeta) := \int_0^{i\infty} \eta(\underline{v}_{\pm}, R_{\zeta}^{\beta})(z)$. Then, since $v_{\pm,j}(-z^*) = \pm v_{\pm,j}(z)$, one finds (see [Lewis and Zagier 2001])

$$\psi'_{+,j}(\zeta) = 2\beta \int_0^{\infty} \frac{t^{\beta} v_{+,j}(it)}{(\zeta^2+t^2)^{\beta+1}}\, dt, \tag{3.3.9}$$

$$\psi'_{-,j}(\zeta) = -\int_0^{\infty} \frac{t^{\beta}\partial_x v_{-,j}(it)}{(\zeta^2+t^2)^{\beta}}\, dt. \tag{3.3.10}$$

Using next the identity (3.3.8) one easily shows

$$\psi'_{\pm,j}(\zeta) = \pm\zeta^{-2\beta}\psi'_{\pm,\delta(j)}\Big(\frac{1}{\zeta}\Big). \tag{3.3.11}$$

But $v_{\pm,j}(z) = u_{\lambda(j)}(z) \pm J_{n,-}u_{\sigma(j)}(z)$ and hence

$$\psi'_{\pm,j}(\zeta) = \psi_{\lambda_{n,-}(j)}(\zeta) \pm J_{n,-}\psi_{\sigma(j)}(\zeta).$$

Therefore

$$\psi_{\lambda_{n,-}(j)}(\zeta) \pm J_{n,-}\psi_{\sigma(j)}(\zeta) = \pm\zeta^{-2\beta}\Big(\psi_{\lambda_{n,-}\circ\delta(j)}\Big(\frac{1}{\zeta}\Big) \pm J_{n,-}\psi_{\sigma\circ\delta(j)}\Big(\frac{1}{\zeta}\Big)\Big). \tag{3.3.12}$$

Adding these two equations leads finally to

$$\psi_{\lambda_{n,-}(j)}(\zeta) = \zeta^{-2\beta} J_{n,-}\psi_{\sigma\circ\delta(j)}\Big(\frac{1}{\zeta}\Big), \tag{3.3.13}$$

and therefore to the equation

$$J_{n,-}\psi_j(\zeta) = \zeta^{-2\beta}\psi_{\lambda_{n,-}\circ\sigma\circ\delta(j)}\Big(\frac{1}{\zeta}\Big), \tag{3.3.14}$$

which was to be proven. □

Remark 3.3.2. As can be seen from their action on the coset representatives R_j the permutation δ commutes with the permutations $\lambda_{n,-}$ and σ. Furthermore one has $\sigma^2 = \delta^2 = (\lambda_{n,-} \circ \sigma)^2 = \mathrm{id}$, where id denotes the identity permutation. This shows also that the automorphisms $J_{n,-}$ are involutions both of the Maass forms and the period functions, a special case of these involutions for all groups $\Gamma_0(N)$ being $J_{0,-}u(z) = u(-z^*)$.

Denote by $Q_{n,-}$, $0 \le n \le h_N - 1$, the $\mu_N \times \mu_N$ permutation matrix corresponding to the permutation $\lambda_{n,-} \circ \sigma \circ \delta$.

Theorem 3.3.3. *The permutation matrices $P_{n,-} := \rho_{\chi_0}(S) Q_{n,-}$, $0 \le n \le h_N - 1$, define symmetries*

$$\tilde{P}_{n,-} = \begin{pmatrix} 0 & P_{n,-} \\ P_{n,-} & 0 \end{pmatrix}$$

for the transfer operator

$$\mathbf{L}_{\beta,\chi_0} = \begin{pmatrix} 0 & \mathcal{L}^+_{\beta,\chi_0} \\ \mathcal{L}^+_{\beta,\chi_0} & 0 \end{pmatrix}$$

for $\Gamma_0(N)$ and trivial character $\chi_0 \equiv 1$, with $P^2_{n,-} = \mathrm{id}_{\mu_N}$, $P_{n,-}\rho_{\chi_0}(S) = \rho_{\chi_0}(S) P_{n,-}$, and $P_{n,-}\rho_{\chi_0}(T) = \rho_{\chi_0}(T^{-1}) P_{n,-}$, and therefore $P_{n,-}\mathcal{L}^+_{\beta,\chi_0} = \mathcal{L}^+_{\beta,\chi_0} P_{n,-}$. The permutation matrix $P_{n,-}$ is determined by the permutation $\lambda_{n,-} \circ \sigma$ and hence by the coset representatives $j_{n,-} R_j j_{0,-}$.

Proof. The matrix $P_{n,-}\rho_{\chi_0}(S)$ is determined by the coset representatives $j_{n,-} R_j S j_{0,-}$ whereas $\rho_{\chi_0}(S) P_{n,-}$ is determined by the coset representatives $j_{n,-} R_j j_{0,-} S$ and $j_{0,-} S = S j_{0,-}$. Hence $P_{n,-}\rho_{\chi_0}(S) = \rho_{\chi_0}(S) P_{n,-}$. On the other hand $T j_{0,-} = j_{0,-} T^{-1}$ and therefore $P_{n,-}\rho_{\chi_0}(T) = \rho_{\chi_0}(T^{-1}) P_{n,-}$. $\square$

Obviously Theorem 2.1.1 follows from Theorem 3.3.3. For the automorphisms $j_{n,+} = j_{n,-} j_{0,-}$ one gets the symmetry

$$\tilde{P}_{n,+} = \begin{pmatrix} P_{n,+} & 0 \\ 0 & P_{n,+} \end{pmatrix},$$

with $P_{n,+}$ the permutation matrix corresponding to the permutation $\lambda_{n,-} \circ \sigma \circ \lambda_{0,-} \circ \sigma$ determined by the coset representatives $j_{n,+} R_j$.

Remark 3.3.4. The symmetry $P_{0,-}$ is given by $\rho_{\chi_0}(SM)$ where $M = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and ρ_{χ_0} denotes the representation of $\mathrm{GL}(2,\mathbb{Z})$ induced from the trivial character χ_0 of $\Gamma_0(N)$. The transfer operator $\mathcal{L}^{MM}_\beta$ of Manin and Marcolli [2002] for $\Gamma_0(N)$ turns out to coincide with the operator $\rho_{\chi_0}(S) P_{0,-} \mathcal{L}^+_{\beta,\chi_0} \rho_{\chi_0}(S)$ and appears as a special case of our operators $P_{n,-}\mathcal{L}^+_{\beta,\chi_0}$.

Corollary 3.3.5. *The permutation matrices $P_{n,-}$, $0 \le n \le h_N - 1$, generate a finite group consisting of the permutation matrices $\{P_{n,\pm}, 0 \le n \le h_N - 1\}$ and isomorphic to the normalizer group $\mathcal{N}_N$ of $\Gamma_0(N)$ in $\mathrm{GL}(2, \mathbb{Z})$. The symmetries $\{\tilde{P}_{n,\pm}, 0 \le n \le h_N - 1\}$ of the transfer operator $\mathbf{L}_{\beta,\chi_0}$ for $\Gamma_0(N)$ and trivial character χ_0 define a finite group isomorphic to the group $\mathcal{N}_N$.*

4. Selberg's character χ_α for $\Gamma_0(4)$

The group $\Gamma_0(4)$ is freely generated by the two elements $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ -4 & 1 \end{pmatrix}$. Hence any $g \in \Gamma_0(4)$ can be written as $g = \prod_{i=1}^{N_g} T^{m_i} B^{n_i}$. If $\Omega(g) = \sum_{i=1}^{N_g} m_i$ then Selberg's character χ_α [Selberg 1990] is defined as

$$\chi_\alpha(g) = \exp(2\pi i \alpha \Omega(g)), \quad 0 \le \alpha \le 1. \tag{4.1}$$

Denote by z_i, $1 \le i \le 3$, the inequivalent cusps of $\Gamma_0(4)$ and by T_i the generators of their stabilizer groups Γ_{z_i} with $T_i z_i = z_i$. They can be taken as $z_1 = i\infty$, $z_2 = 0$, $z_3 = -\frac{1}{2}$ and $T_1 = T$, $T_2 = B$, $T_3 = T^{-1}B^{-1}$. The character χ_α is singular in the cusp z_i if and only if $\chi_\alpha(T_i) = 1$. Otherwise the character is nonsingular in z_i. It is well known that the multiplicity $\kappa(\chi_\alpha)$ of the continuous spectrum of the automorphic Laplacian Δ with character χ_α is given by $\kappa(\chi_\alpha) = \#\{i : \chi_\alpha(T_i) = 1\}$. Therefore $\kappa(\chi_\alpha) = 3$ for $\alpha = 0$ whereas $\kappa(\chi_\alpha) = 1$ for $\alpha \ne 0$ and hence the multiplicity of the continuous spectrum of the Laplacian changes from 3 to 1 when the trivial character is deformed to χ_α with $\alpha \ne 0$. It is known [Phillips and Sarnak 1994] that the character χ_α is congruent (or arithmetic) if and only if $\alpha \in \{k\frac{1}{8}, 0 \le k \le 4\}$. Since the Selberg zeta function given in (2.0.1) has the property $Z_{\Gamma_0(4),\chi_\alpha} = Z_{\Gamma_0(4),\chi_{-\alpha}}$ and obviously $\chi_\alpha = \chi_{\alpha+1}$ we can restrict the deformation parameter α to the range $0 \le \alpha \le \frac{1}{2}$.

Lemma 4.1. *The Selberg character χ_α is invariant under the map defined by $j_{2,-}z = z^*/(2z^* - 1)$, and $J_{2,-}u(z) := u(j_{2,-}z)$ is a Maass form for $\Gamma_0(4)$ and character χ_α if $u = u(z)$ is such a Maass form.*

Proof. We only have to show that χ_α is invariant under the map $j_{2,-}z = z^*/(2z^* - 1)$. For $g = T$ we find $j_{2,-}Tj_{2,-} = TB$ and hence

$$\chi_\alpha(j_{2,-}Tj_{2,-}) = \chi_\alpha(TB) = \chi_\alpha(T),$$

whereas for $g = B$ one finds $j_{2,-}Bj_{2,-} = B^{-1}$ and hence

$$\chi_\alpha(j_{2,-}Bj_{2,-}) = \chi_\alpha(B^{-1}) = \chi_\alpha(B).$$

Therefore $\chi_\alpha(j_{2,-}gj_{2,-}) = \chi_\alpha(g)$ for all $g \in \Gamma_0(4)$. □

If $u = u(z)$ is a Maass form for $\Gamma_0(4)$ with character χ_α and $\underline{\psi} = (\psi_j(\zeta))_{1 \le j \le 6}$ is its period function, denote by $J_- u$ the Maass form given by $J_- u(z) := u(j_{2,-}z)$, and by $J_- \underline{\psi} = (J_- \psi_j(\zeta))_{1 \le j \le 6}$ its period function. Then one shows:

Theorem 4.2. *The period function $J_-\underline{\psi}$ of the Maass form J_-u is given by*

$$J_-\psi_j(\zeta) = \zeta^{-2\beta}\chi_\alpha(\eta_{\sigma\circ\delta(j)})\psi_{\lambda_{2,-}\circ\sigma\circ\delta(j)}\Big(\frac{1}{\zeta}\Big), \tag{4.2}$$

where the permutations $\lambda_{2,-}$, σ, and δ, as well as the $\eta_j \in \Gamma_0(4)$, are determined through the coset representatives R_j by

$$j_{2,+}R_j = \theta_j R_{\lambda_{2,-}(j)}, \quad j_{0,-}R_j j_{0,-} = \gamma_j R_{\sigma(j)}, \quad R_j S = \eta_j R_{\delta(j)},$$

with θ_j, γ_j, and $\eta_j \in \Gamma_0(4)$ for $1 \le j \le 6$.

Proof. Set $j_\pm := j_{2,\pm}$ and $J_\pm u(z) := u(j_\pm z)$. Then J_-u is a Maass form for $\Gamma_0(4)$ and character χ_α whereas J_+u is a Maass form for $\Gamma_0(4)$ and character $\chi_{-\alpha}$. The vector-valued Maass form $J_+\underline{u} = (J_+u_j)_{1\le j\le 6}$ is given by $J_+u_j(z) = u(j_+R_jz)$. We have chosen the representatives R_j of the cosets in $\mathrm{SL}(2,\mathbb{Z}) = \bigcup_{1\le j\le 6}\Gamma_0(4)R_j$ as follows:

$$R_1 = \mathrm{id}_2, \qquad R_j = ST^{j-2}, \quad 2\le j\le 5, \qquad R_6 = ST^2S.$$

But $j_+R_j = \theta_j R_{\lambda_{2,-}(j)}$ for some $\theta_j \in \Gamma_0(4)$ and some permutation $\lambda_{2,-}$ of the set $\{1,2,\dots,6\}$ and hence $J_+u_j(z) = \chi_\alpha(\theta_j)u(R_{\lambda_{2,-}(j)}z)$. It turns out that $\theta_j = B^{-1}$ for $1\le j\le 3$ and $\theta_j = \mathrm{id}_2$ for $4\le j\le 6$. Hence $\chi_\alpha(\theta_j) = 1$ and

$$J_+u_j(z) = u_{\lambda_{2,-}(j)}(z), \quad 1\le j\le 6, \tag{4.3}$$

with $\lambda_{2,-}$ the permutation

$$\lambda_{2,-} = \frac{1\ 2\ 3\ 4\ 5\ 6}{6\ 4\ 5\ 2\ 3\ 1}. \tag{4.4}$$

Consider next $J_+u_j(-z^*) = J_+u_j(j_{0,-}z)$. Then

$$J_+u_j(j_{0,-}z) = u(j_+R_jj_{0,-}z) = u(j_+j_{0,-}j_{0,-}R_jj_{0,-}z).$$

If $j_{0,-}R_jj_{0,-} = \gamma_jR_{\sigma(j)}$ then $J_+u_j(j_{0,-}z) = u(j_-\gamma_jj_-j_-R_{\sigma(j)}z)$. But it turns out that $j_-\gamma_jj_- = \mathrm{id}_2$ for $j = 1,2,6$ and $j_-\gamma_jj_- = B$ for $j = 3,4,5$, hence $\chi_\alpha(j_-\gamma_jj_-) = 1$ and therefore

$$J_+u_j(-z^*) = J_+u_j(j_{0,-}z) = J_-u_{\sigma(j)}(z). \tag{4.5}$$

Since, furthermore, $J_+u_j(Sz) = u(j_+R_jSz) = u(j_+\eta_jR_{\delta(j)}z)$, one finds

$$J_+u_j(Sz) = \chi_{-\alpha}(\eta_j)u_{\lambda_{2,-}\circ\delta(j)}(z), \tag{4.6}$$

where δ is the permutation

$$\delta = \frac{1\ 2\ 3\ 4\ 5\ 6}{2\ 1\ 5\ 6\ 3\ 4}, \tag{4.7}$$

$\eta_j = \mathrm{id}_2$ for $j = 1, 2, 4, 6$, and $\eta_3 = \eta_5^{-1} = T^{-1}B^{-1}$. For $J_+u_j(-Sz^*)$ one gets with (4.5) $J_+u_j(-Sz^*) = J_-u_{\sigma(j)}(Sz) = u(j_{2,-}R_{\sigma(j)}Sz)$ and

$$J_+u_j(-Sz^*) = u(j_{2,-}\eta_{\sigma(j)}R_{\delta\circ\sigma(j)}z) = \chi_\alpha(\eta_{\sigma(j)})J_-u_{\delta\circ\sigma(j)}(z).$$

Using the explicit form of the η_j one shows $\chi_\alpha(\eta_{\sigma(j)}) = \chi_{-\alpha}(\eta_j)$ and therefore

$$J_+u_j(-Sz^*) = \chi_{-\alpha}(\eta_j)J_-u_{\delta\circ\sigma(j)}(z). \tag{4.8}$$

Define next $v_{\pm,j} = v_{\pm,j}(z)$ as

$$v_{\pm,j}(z) := J_+u_j(z) \pm J_+u_j(-z^*). \tag{4.9}$$

Then $v_{\pm,j}(-z^*) = \pm v_{\pm,j}(z)$, and by (4.6) and (4.8) we have

$$v_{\pm,j}(Sz) = \chi_{-\alpha}(\eta_j)v_{\pm,\delta(j)}(z). \tag{4.10}$$

If therefore $\psi'_{\pm,j}(\zeta) := \int_0^{i\infty} \eta(v_{\pm,j}, R_\zeta^\beta)(z)$ one gets from relation (4.10)

$$\psi'_{\pm,j}(\zeta) = \pm\zeta^{-2\beta}\chi_{-\alpha}(\eta_j)\psi'_{\pm,\delta(j)}\left(\frac{1}{\zeta}\right) \tag{4.11}$$

and using the identity (4.9)

$$\begin{aligned}\psi_{\lambda_{2,-}(j)}(\zeta) &\pm J_-\psi_{\sigma(j)}(\zeta) \\ &= \pm\zeta^{-2\beta}\chi_{-\alpha}(\eta_j)\left(\psi_{\lambda_{2,-}\circ\delta(j)}\left(\frac{1}{\zeta}\right) \pm J_-\psi_{\sigma\circ\delta(j)}\left(\frac{1}{\zeta}\right)\right).\end{aligned} \tag{4.12}$$

Adding these two equations leads finally to

$$J_-\psi_j(\zeta) = \zeta^{-2\beta}\chi_\alpha(\eta_{\sigma\circ\delta(j)})\psi_{\lambda_{2,-}\circ\sigma\circ\delta(j)}\left(\frac{1}{\zeta}\right), \qquad \square$$

Inserting the explicit form of the permutations

$$\sigma \circ \delta = \frac{1\ 2\ 3\ 4\ 5\ 6}{2\ 1\ 3\ 6\ 5\ 4} \tag{4.13}$$

and

$$\lambda_{2,-} \circ \sigma \circ \delta = \frac{1\ 2\ 3\ 4\ 5\ 6}{4\ 6\ 5\ 1\ 3\ 2} \tag{4.14}$$

and the character values

$$\chi_\alpha(\eta_1) = \chi_\alpha(\eta_2) = \chi_\alpha(\eta_4) = \chi_\alpha(\eta_6) = 1 \quad \text{and} \quad \chi_\alpha(\eta_3) = \chi_\alpha(\eta_5)^{-1} = e^{-2\pi i\alpha},$$

one finds

$$
\begin{aligned}
J_-\psi_1(\zeta) &= \zeta^{-2\beta}\psi_4\Big(\frac{1}{\zeta}\Big), & J_-\psi_2(\zeta) &= \zeta^{-2\beta}\psi_6\Big(\frac{1}{\zeta}\Big),\\
J_-\psi_3(\zeta) &= \zeta^{-2\beta}e^{-2\pi i\alpha}\psi_4\Big(\frac{1}{\zeta}\Big), & J_-\psi_4(\zeta) &= \zeta^{-2\beta}\psi_1\Big(\frac{1}{\zeta}\Big),\\
J_-\psi_5(\zeta) &= \zeta^{-2\beta}e^{2\pi i\alpha}\psi_3\Big(\frac{1}{\zeta}\Big), & J_-\psi_6(\zeta) &= \zeta^{-2\beta}\psi_2\Big(\frac{1}{\zeta}\Big).
\end{aligned}
\tag{4.15}
$$

Define the matrix $Q_{2,-}$ through the equation $J_{2,-}\underline{\psi}(\zeta) = \zeta^{-2\beta}Q_{2,-}\underline{\psi}(1/\zeta)$.

Proposition 4.3. *The permutation matrix $P_{2,-} := \rho_{\chi_\alpha}(S)Q_{2,-}$ defines a symmetry*

$$\tilde{P}_{2,-} = \begin{pmatrix} 0 & P_{2,-} \\ P_{2,-} & 0 \end{pmatrix}$$

of the transfer operator

$$\mathbf{L}_{\beta,\chi_\alpha} = \begin{pmatrix} 0 & \mathscr{L}^+_{\beta,\chi_\alpha} \\ \mathscr{L}^+_{\beta,\chi_\alpha} & 0 \end{pmatrix}$$

for $\Gamma_0(4)$ and character χ_α with $P^2_{2,-} = \mathrm{id}_6$,

$$P_{2,-}\rho_{\chi_\alpha}(S) = \rho_{\chi_\alpha}(S)P_{2,-} \quad \textit{and} \quad P_{2,-}\rho_{\chi_\alpha}(T) = \rho_{\chi_\alpha}(T^{-1})P_{2,-},$$

and therefore $P_{2,-}\mathscr{L}^+_{\beta,\chi_\alpha} = \mathscr{L}^-_{\beta,\chi_\alpha}P_{2,-}$. The permutation matrix $P_{2,-}$ corresponds to the permutation $\lambda_{2,-}\circ\sigma$ and hence is determined by the coset representatives $J_{2,-}R_j j_{0,-}$.

Proof. For our choice of coset representatives R_j as given in (2.1.4) one finds for $\rho_{\chi_\alpha}(S)$

$$\rho_{\chi_\alpha}(S) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{-2\pi i\alpha} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & e^{2\pi i\alpha} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \tag{4.16}$$

and hence the matrix $Q_{2,-}$ is given by

$$Q_{2,-} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & e^{-2\pi i\alpha} & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{2\pi i\alpha} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{4.17}$$

For $\rho_{\chi_\alpha}(T)$ one finds

$$\rho_{\chi_\alpha}(T) = \begin{pmatrix} e^{2\pi i\alpha} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{-2\pi i\alpha} \end{pmatrix}. \tag{4.18}$$

A simple calculation then confirms that $P_{2,-}\rho_{\chi_\alpha}(S) = \rho_{\chi_\alpha}(S)P_{2,-}$ and $P_{2,-}\rho_{\chi_\alpha}(T) = \rho_{\chi_\alpha}(T^{-1})P_{2,-}$, with

$$P_{2,-} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{4.19}$$

and hence defines a symmetry of the transfer operator $\mathbf{L}_{\beta,\chi_\alpha}$. The matrix $P_{2,-}$ coincides with the permutation matrix P_2 corresponding to the permutation σ_2 in (2.1.6). □

Remark 4.4. For the trivial character χ_0, the map $j_{0,-}z = -z^*$ defines an automorphism of the Maass forms for the group $\Gamma_0(4)$. Indeed, this is an automorphism for all Hecke congruence subgroups $\Gamma_0(N)$. In this case the permutation $\lambda_{0,-}$ is the trivial permutation and the matrix $Q_{0,-}$ is determined by the permutation $\sigma \circ \delta$. For $\Gamma_0(4)$ this is given by (4.13). Using (4.16) with $\alpha = 0$ one obtains for $P_{0,-} = \rho_{\chi_0}(S)Q_{0,-}$ the permutation σ_1 as given in (2.1.5). The symmetry $\tilde{P}_1$ for $\Gamma_0(4)$ and trivial character χ_0 hence corresponds to the automorphism $z \to -z^*$ of the Maass forms for this group.

We have seen that for every eigenfunction $\underline{f} = \underline{f}(\zeta)$ of the operator $P_2\mathscr{L}^+_{\beta,\chi_\alpha}$ with eigenvalues $\lambda = \pm 1$ the function $\underline{\Psi} = \underline{\Psi}(\zeta) = P_2\rho_{\chi_\alpha}(T^{-1}S)P_2\underline{f}(\zeta - 1)$ fulfills the functional equation

$$\underline{\Psi}(\zeta) = \lambda\zeta^{-2\beta}\rho_{\chi_\alpha}(S)P_2\underline{\Psi}\left(\frac{1}{\zeta}\right) = \lambda J_-\underline{\Psi}(\zeta) \tag{4.20}$$

and hence is an eigenfunction of the involution J_- corresponding to the automorphism $j_- = j_{2,-}$ of the Maass forms for $\Gamma_0(4)$ and character χ_α. Hence this shows:

Proposition 4.5. *The eigenfunctions $\underline{f} = \underline{f}(\zeta)$ of the operator $P_2\mathscr{L}^+_{\beta,\chi_\alpha}$ with eigenvalues $\lambda = \pm 1$ correspond to Maass forms which are even or odd, respectively, under the involution $J_- = J_{2,-}$.*

For a conjugate character $\hat{\chi}_\alpha$, Phillips and Sarnak [1994] have shown that the Maass cusp forms that are odd under the corresponding conjugate involution $\hat{J}$ are still cusp forms under the deformation of this character. Hence:

Corollary 4.6. *The zeros of the Selberg zeta function for the group* $\Gamma_0(4)$ *and character* χ_α *corresponding to eigenfunctions of the operator* $P_2\mathscr{L}^+_{\beta,\chi_\alpha}$ *with eigenvalue* $\lambda = -1$ *which for* $\alpha = 0$ *are on the critical line* $\operatorname{Re}\beta = \frac{1}{2}$ *stay, for all* α, *on this line.*

Remark 4.7. The operator $P_2\mathscr{L}^+_{\beta,\chi_\alpha}$ can be used to calculate numerically the Selberg zeta function for small values of $\operatorname{Im}\beta$ and arbitrary $0 \le \alpha \le \frac{1}{2}$. These numerical calculations confirm the above corollary and let us expect that all the zeros of the Selberg function corresponding to the eigenvalue $\lambda = 1$ of the operator $P_2\mathscr{L}^+_{\beta,\chi_\alpha}$ for $\alpha = 0$ leave the critical line when α becomes positive. A detailed discussion of the numerical treatment of the behavior of the zeros of the Selberg function under character deformation will appear elsewhere [Bruggeman et al. 2012].

References

[Bruggeman et al. 2012] R. W. Bruggeman, M. Fraczek, and D. Mayer, "Perturbation of zeros of the Selberg zeta function for $\Gamma_0(4)$", preprint, 2012. arXiv 1201.2324v1

[Chang and Mayer 1998] C.-H. Chang and D. Mayer, "The period function of the nonholomorphic Eisenstein series for PSL(2, **Z**)", *Math. Phys. Electron. J.* **4** (1998), no. 6. MR 99k:11070 Zbl 0927.11027

[Chang and Mayer 2000] C.-H. Chang and D. Mayer, "Thermodynamic formalism and Selberg's zeta function for modular groups", *Regul. Chaotic Dyn.* **5**:3 (2000), 281–312. MR 2001k:37045 Zbl 0979.37014

[Chang and Mayer 2001a] C.-H. Chang and D. H. Mayer, "Eigenfunctions of the transfer operators and the period functions for modular groups", pp. 1–40 in *Dynamical, spectral, and arithmetic zeta functions* (San Antonio, TX, 1999), edited by M. L. Lapidus and M. van Frankenhuysen, Contemp. Math. **290**, Amer. Math. Soc., Providence, RI, 2001. MR 2003h:11056 Zbl 1037.11032

[Chang and Mayer 2001b] C.-H. Chang and D. H. Mayer, "An extension of the thermodynamic formalism approach to Selberg's zeta function for general modular groups", pp. 523–562 in *Ergodic theory, analysis, and efficient simulation of dynamical systems*, edited by B. Fiedler, Springer, Berlin, 2001. MR 2002k:37046 Zbl 1211.37030

[Conway and Norton 1979] J. H. Conway and S. P. Norton, "Monstrous moonshine", *Bull. London Math. Soc.* **11**:3 (1979), 308–339. MR 81j:20028 Zbl 0424.20010

[Deitmar and Hilgert 2007] A. Deitmar and J. Hilgert, "A Lewis correspondence for submodular groups", *Forum Math.* **19**:6 (2007), 1075–1099. MR 2008j:11058 Zbl 1211.11063

[Efrat 1993] I. Efrat, "Dynamics of the continued fraction map and the spectral theory of SL(2, $\mathbb{Z}$)", *Invent. Math.* **114**:1 (1993), 207–218. MR 94h:11052 Zbl 0811.11037

[Fraczek 2010] M. Fraczek, *Character deformation of the Selberg zeta function for congruence subgroups via the transfer operator*, thesis, Clausthal Institute of Technology, 2010.

[Hejhal 1983] D. A. Hejhal, *The Selberg trace formula for* PSL(2, **R**), vol. 2, Lecture Notes in Mathematics **1001**, Springer, Berlin, 1983. MR 86e:11040 Zbl 0543.10020

[Lehner and Newman 1964] J. Lehner and M. Newman, "Weierstrass points of $\Gamma_0\,(n)$", *Ann. of Math. (2)* **79** (1964), 360–368. MR 28 #5045 Zbl 0124.29203

[Lewis and Zagier 2001] J. Lewis and D. Zagier, "Period functions for Maass wave forms, I", *Ann. of Math. (2)* **153**:1 (2001), 191–258. MR 2003d:11068 Zbl 1061.11021

[Manin and Marcolli 2002] Y. I. Manin and M. Marcolli, "Continued fractions, modular symbols, and noncommutative geometry", *Selecta Math. (N.S.)* **8**:3 (2002), 475–521. MR 2004a:11039 Zbl 1116.11033

[Mayer and Mühlenbruch 2010] D. Mayer and T. Mühlenbruch, "Nearest λ_q-multiple fractions", pp. 147–184 in *Spectrum and dynamics*, edited by D. Jakobson et al., CRM Proc. Lecture Notes **52**, Amer. Math. Soc., Providence, RI, 2010. MR 2011m:11148 Zbl 05907544

[Mayer and Strömberg 2008] D. Mayer and F. Strömberg, "Symbolic dynamics for the geodesic flow on Hecke surfaces", *J. Mod. Dyn.* **2**:4 (2008), 581–627. MR 2009j:37049 Zbl 1161.37031

[Mayer et al. 2012] D. Mayer, T. Mühlenbruch, and F. Strömberg, "The transfer operator for the Hecke triangle groups", *Discrete Contin. Dyn. Syst. A* **32**:7 (2012), 2453–2484. arXiv 0912.2236

[Mühlenbruch 2006] T. Mühlenbruch, "Hecke operators on period functions for $\Gamma_0(n)$", *J. Number Theory* **118**:2 (2006), 208–235. MR 2007i:11062 Zbl 1122.11022

[Phillips and Sarnak 1994] R. Phillips and P. Sarnak, "Cusp forms for character varieties", *Geom. Funct. Anal.* **4**:1 (1994), 93–118. MR 94k:11061 Zbl 0804.11038

[Selberg 1990] A. Selberg, "Remarks on the distribution of poles of Eisenstein series", pp. 251–278 in *Festschrift in honor of I. I. Piatetski-Shapiro on the occasion of his sixtieth birthday, Part 2* (Ramat Aviv, 1989), edited by S. Gelbart et al., Israel Math. Conf. Proc. **3**, Weizmann, Jerusalem, 1990. MR 93c:11035 Zbl 0712.11034

[Strömberg 2008] F. Strömberg, "Computation of Selberg's zeta functions on Hecke triangle groups", preprint, 2008. arXiv 0804.4837

[Venkov 1990] A. Venkov, "The theory of the Selberg zeta-function", pp. 47–54 in *Spectral theory of automorphic functions and its applications*, Math. Appl. (Soviet Ser.) **51**, Kluwer, Dordrecht, 1990.

[Zagier 2002] D. Zagier, "New points of view on the Selberg zeta function", in *Proceedings of Japanese–German Seminar: explicit structures of modular forms and zeta functions* (Hakuba, 2001), edited by T. Ibukiyama, Ryushi-Do, Nagano, 2002.

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