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We define a new symmetry for morphisms of vector bundles, called *triality symmetry*, and compute Chern class formulas for the degeneracy loci of such morphisms. In an appendix, we show how to canonically associate an octonion algebra bundle to any rank-2 vector bundle.

1. Introduction

Let $\varphi : E \to F$ be a morphism of vector bundles on a smooth variety X, of respective ranks m and n. The r-th *degeneracy locus* of φ is the set of points of X defined by

$$D_r(\varphi) = \{x \in X \mid \operatorname{rk} \varphi(x) \leq r\},$$

where $\varphi(x) : E(x) \to F(x)$ is the corresponding linear map in the fibers over $x \in X$. Such loci are ubiquitous in algebraic geometry: many interesting varieties, from Veronese embeddings of projective spaces to Brill–Noether loci parametrizing special divisors in Jacobians, can be realized as degeneracy loci for appropriate maps of vector bundles. General geometric information about degeneracy loci is therefore often useful. In particular, one can ask for Chern class formulas for the cohomology class of $D_r(\varphi)$ in H^*X — what is $[D_r(\varphi)]$ as a polynomial in the Chern classes of E and F?

When φ is sufficiently general, so $D_r(\varphi)$ has *expected codimension* equal to $(m-r)(n-r)$, the answer is given by the Giambelli–Thom–Porteous determinantal formula. In two cases of particular interest, Chern class formulas are known for degeneracy loci where φ is not general in this sense. Taking $F = E^*$, one has the dual morphism $\varphi^* : E^{**} = E \to E^*$. Call φ *symmetric* if $\varphi^* = \varphi$, and *skew-symmetric* if $\varphi^* = -\varphi$. The codimension of $D_r(\varphi)$ is at most $\binom{m-r+1}{2}$ (in the symmetric case) or $\binom{m-r}{2}$ (in the skew-symmetric case), so such morphisms are never sufficiently general for the Giambelli–Thom–Porteous formula to apply. Formulas for these loci were given by Harris and Tu [1984] and Józefiak, Lascoux

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and Pragacz [Józefiak et al. 1981]. As explained in [Fehér et al. 2005], these formulas can also be found by computing the equivariant classes of appropriate orbit closures in the $\mathrm{GL}(E)$-representations $\mathrm{Sym}^2 E^*$ and $\bigwedge^2 E^*$, where E is a vector space. See [Fulton and Pragacz 1998, Chapter 6] for more detailed discussions of the formulas.

The primary goal of the present article is to give degeneracy locus formulas for a new class of morphisms, which we call *triality-symmetric* morphisms. Letting E be a rank-2 vector bundle, these are maps

$$\varphi : E \to \mathrm{End}(E) \oplus E^*$$

possessing a certain symmetry related to the S_3 symmetry of the D_4 Dynkin diagram. Specifically, we use the following definition:

Definition 1.1. Consider the canonical identification

$$\begin{aligned}\mathrm{Hom}(E, \mathrm{End}(E) \oplus E^*) &= (E^* \otimes E^* \otimes E) \oplus (E^* \otimes E^*)\\ &= (E^* \otimes E^* \otimes E^* \otimes {\textstyle\bigwedge}^2 E) \oplus (E^* \otimes E^*).\end{aligned}$$

A morphism $\varphi : E \to \mathrm{End}(E) \oplus E^*$ is *triality-symmetric* if the corresponding section of $\mathrm{Hom}(E, \mathrm{End}(E) \oplus E^*)$ lies in the subbundle

$$(\mathrm{Sym}^3 E^* \otimes {\textstyle\bigwedge}^2 E) \oplus {\textstyle\bigwedge}^2 E^*.$$

That is, $\varphi = \varphi_1 \oplus \varphi_2$, with φ_1 defining a symmetric trilinear form $\mathrm{Sym}^3 E \to \bigwedge^2 E$ and φ_2 defining an alternating bilinear form $\bigwedge^2 E \to \mathcal{O}_X$.

We will sometimes write $\mathrm{tSym}(E^*) = (\mathrm{Sym}^3 E^* \otimes \bigwedge^2 E) \oplus \bigwedge^2 E^*$ for the subbundle of triality-symmetric morphisms.

A few words of motivation are in order concerning this definition. For simplicity, consider the case where X is a point, and take vector spaces E and F of respective dimensions m and n. The space of all linear maps $\mathrm{Hom}(E, F)$ is also the tangent space to the Grassmannian $\mathrm{Gr}(m, m+n) = \mathrm{Gr}(m, E \oplus F) = \mathrm{GL}_{m+n}/P$ (for an appropriate maximal parabolic subgroup P) at the point corresponding to E. When $F = E^*$, there is a canonical symplectic form ω on $E \oplus E^*$, defining the Lagrangian Grassmannian $\mathrm{LG}(m, 2m) \subseteq \mathrm{Gr}(m, 2m)$, and the space of symmetric morphisms $\mathrm{Sym}^2 E^*$ is naturally identified with the tangent space to $\mathrm{LG}(m, 2m) = \mathrm{Sp}_{2m}/P$ at the point $[E]$. Moreover, $\mathrm{LG}(m, 2m)$ is the fixed locus for the involution of $\mathrm{Gr}(m, 2m)$ which sends a subspace to its orthogonal complement under ω. The situation is similar for skew-symmetric morphisms, replacing the Lagrangian Grassmannian with the orthogonal Grassmannian $\mathrm{OG}(m, 2m) = \mathrm{SO}_{2m}/P$.

From this point of view, it is natural to expect nice degeneracy locus formulas corresponding to other finite symmetries of homogeneous spaces. A particularly interesting one is the *triality* action on OG(2, 8), which we identify as

$$\mathrm{OG}(2, E \oplus \mathrm{End}(E) \oplus E^*)$$

for a two-dimensional vector space E. A concise description of this S_3 action may be found in [Anderson 2009, Appendix B]; for more details, see [van der Blij and Springer 1960; Garibaldi 1999]. For our purposes, the relevant facts are that the fixed locus is the "G_2 Grassmannian" G_2/P (for P corresponding to the long root), and the tangent space to G_2/P is naturally identified with $\mathrm{tSym}(E^*)$ at the point $[E] \in G_2/P \subseteq \mathrm{OG}(2, 8)$. (In Section 3, we will explicitly exhibit the S_3 action on the tangent space $T_{[E]}\,\mathrm{OG}(2, 8) \cong \mathrm{Hom}(E, \mathrm{End}(E)) \oplus \bigwedge^2 E^*$ fixing $\mathrm{tSym}(E^*)$.) Further motivation comes from the fact that there is a canonical *octonion algebra* structure on $E \oplus \mathrm{End}(E) \oplus E^*$, when E is a rank-2 vector bundle, just as there is a canonical symplectic structure on $E \oplus E^*$. This is the content of Proposition A.1.

Since E is required to have rank-2, a triality-symmetric morphism may have rank 0, 1, or 2. Write $D_r(\varphi) \subseteq X$ for the locus of points where φ has rank at most r. For a triality-symmetric morphism φ, define the *expected codimension* of $D_r(\varphi)$ to be 5, 3, or 0 if $r = 0$, $r = 1$, or $r = 2$, respectively. With this understood, we may state our main theorem:

Theorem 1.2. *Let c_1, c_2 be the Chern classes of E^*, and let x_1, x_2 be Chern roots. Let $\varphi : E \to \mathrm{End}(E) \oplus E^*$ be a triality-symmetric morphism. If $D_r(\varphi)$ has expected codimension and X is Cohen–Macaulay, then we have $[D_r(\varphi)] = P_r(c_1, c_2)$ in H^*X, where*

$$\begin{aligned}
P_2 &= 1,\\
P_1 &= 3\,c_2\,c_1 = 3x_1x_2(x_1 + x_2),\\
P_0 &= c_2\,c_1\,(9\,c_2 - 2\,c_1^2) = x_1x_2(x_1 + x_2)(2x_1 - x_2)(-x_1 + 2x_2).
\end{aligned}$$

A secondary goal of this article is to illustrate two points of view on degeneracy loci. In this spirit, we will give two proofs of the main theorem, both involving the simple Lie group of type G_2, but using substantially different approaches. The first relates degeneracy loci for triality-symmetric morphisms to certain Schubert loci in a G_2 flag bundle, just as Fulton's generalization of the Harris–Tu formulas relates symmetric morphisms to type C flag bundles [Fulton 1996]. One then applies the formulas for G_2 Schubert loci developed in [Anderson 2011] to derive the formulas of Theorem 1.2.

The second proof uses equivariant cohomology in the spirit of [Fehér and Rimányi 2004; Fehér et al. 2005] (but see Remark 5.3). More precisely, when P is the maximal parabolic subgroup of G_2 which omits the long root and E is a two-dimensional vector space, we consider $(\mathrm{Sym}^3 E^* \otimes \bigwedge^2 E) \oplus \bigwedge^2 E^*$ as a P-module and compute the equivariant classes of the P-orbit closures in this vector space.

Certain of these orbit closures correspond to degeneracy loci, and one can deduce Theorem 1.2 from the equivariant formulas. Along the way, we explicitly identify the P-orbit closures in $(\mathrm{Sym}^3 E^* \otimes \bigwedge^2 E) \oplus \bigwedge^2 E^*$, and compute all their equivariant classes (Proposition 5.1 and Theorem 5.2).

Triality symmetry is the G_2 case of a general notion of symmetry for morphisms of vector bundles. In fact, two types of symmetry for morphisms can be naturally associated to any maximal parabolic subgroup P of a complex reductive group G, as described in [Anderson 2009, Appendix C]. The "orbit" approach used in the second proof of Theorem 1.2 generalizes to the following problem: *Compute the equivariant classes of P-orbit (or B-orbit) closures for the adjoint action on* $\mathfrak{g}/\mathfrak{p}$. Solutions to this problem account for many of the known degeneracy locus formulas; see, for example, [Fehér and Rimányi 2003; Knutson and Miller 2005].

A related problem is to classify situations where there are finitely many orbits. In the case of P acting on $\mathfrak{g}/\mathfrak{p}$, this problem was investigated by Popov and Röhrle [1997], and such parabolic actions have been classified [Bürgstein and Hesselink 1987; Hille and Röhrle 1999; Jürgens and Röhrle 2002]. The classification of Borel or Levi subgroup actions on $\mathfrak{g}/\mathfrak{p}$ with finitely many orbits appears to be unknown.

We have endeavored to make our perspective on triality and G_2 accessible to general algebraic geometers, and in this spirit, the ingredients of the first proof of Theorem 1.2 are spelled out quite explicitly. The second proof is more streamlined, but requires a little more specialized background; we hope that the reader versed in Lie theory will appreciate both points of view.

2. Preliminaries

All varieties are over $\mathbb{C}$. We will write X for the base variety. If E is a vector bundle on X, we write $E(x)$ for the fiber over $x \in X$. We often suppress notation for pullback of vector bundles.

2.1. ***Octonions.*** An *octonion algebra* over $\mathbb{C}$ is an 8-dimensional complex vector space C, equipped with

- a nondegenerate quadratic norm N, and
- a bilinear multiplication with unit e, written $u \otimes v \mapsto uv$,

such that $N(uv) = N(u)N(v)$ for all $u, v \in C$. Recall that any quadratic norm N corresponds to a symmetric bilinear form $\langle \cdot\,, \cdot \rangle$, by $N(v) = \frac{1}{2}\langle v, v\rangle$ and $\langle u, v\rangle = N(u+v) - N(u) - N(v)$, and a norm is called nondegenerate if the corresponding bilinear form is nondegenerate.

Up to isomorphism, there is only one octonion algebra over $\mathbb{C}$ (or over any algebraically closed field). The multiplication is only required to be bilinear, and indeed it is neither commutative nor associative.

The notion of an octonion algebra globalizes easily to *octonion bundles*, where C is a rank-8 vector bundle on a variety X, the multiplication is a vector bundle map $C \otimes C \to C$, and for simplicity we assume the norm takes values in $\mathcal{O}_X$. For more on octonions and octonion bundles, see [Springer and Veldkamp 2000, Sections 1–2; Petersson 1993; Anderson 2009, Section 2].

The group of algebra automorphisms of an octonion algebra (that is, linear automorphisms preserving multiplication) is the simple complex Lie group of type G_2 [Springer and Veldkamp 2000, Section 2]; abusing notation, we will write G_2 to denote this group.[1]

Let E be a rank-2 vector bundle on X. Then $C = E \oplus \mathrm{End}(E) \oplus E^*$ has a canonical structure of an octonion bundle, which is described in Proposition A.1. In the case where X is a point, so E is a 2-dimensional vector space, the same formulas (1) and (2) define an octonion algebra. It will be convenient to use a basis adapted to this construction. Let v_1, v_2 be a basis for E, with dual basis v_1^*, v_2^* for E^*, and extend to a basis for $C = E \oplus \mathrm{End}(E) \oplus E^*$ by setting

$$\begin{aligned} v_3 &= v_2^* \otimes v_1, & v_4 &= v_1^* \otimes v_1, & v_5 &= v_2^* \otimes v_2, \\ v_6 &= v_1^* \otimes v_2, & v_7 &= v_2^*, & v_8 &= v_1^*. \end{aligned} \tag{1}$$

One checks that the identity element of C is $e = v_4 + v_5$.

With respect to this basis, the symmetric bilinear form $\langle \cdot, \cdot \rangle$ is given by

$$\begin{aligned} \langle v_p, v_{9-q} \rangle &= -\delta_{pq}, \ \text{ for } \{p, q\} \neq \{4, 5\}; \\ \langle v_4, v_5 \rangle &= 1. \end{aligned} \tag{2}$$

Write $V = e^\perp \subset C$ for the orthogonal complement of the identity element with respect to $\langle \cdot, \cdot \rangle$. Thus V is defined by $v_4^* + v_5^* = 0$.

Let the torus $T = (\mathbb{C}^*)^2$ act on C in this basis via the matrix

$$\mathrm{diag}(z_1,\ z_2,\ z_1 z_2^{-1},\ 1,\ 1,\ z_1^{-1} z_2,\ z_2^{-1},\ z_1^{-1}), \tag{3}$$

with weights

$$\{t_1,\ t_2,\ t_1 - t_2,\ 0,\ 0,\ -t_1 + t_2,\ -t_2,\ -t_1\}. \tag{4}$$

This is induced from the standard action on $E = \mathrm{span}\,\{v_1, v_2\}$. The algebra structure of C is preserved by this action, so $T \subseteq G_2$; in fact, T is a maximal torus.

2.2. ***Roots and weights.*** For general Lie-theoretic notions, we refer to [Humphreys 1975]; here we explain the relevant facts for type G_2. Let G_2 be the automorphism group of an octonion algebra C, as above, so G_2 is presented as a subgroup of

[1] In fact, one can show that $u^2 = \langle u, e \rangle u - N(u) e$ for any element $u \in C$, so any algebra automorphism also preserves the norm.

$\mathrm{GL}(C) \cong \mathrm{GL}_8$. Let $T \subset B \subset G_2$ be a maximal torus and Borel subgroup, and let $\mathfrak{t} \subset \mathfrak{b} \subset \mathfrak{g}_2$ be the corresponding Lie algebras. Once a basis for C has been chosen as in (1), we will always take T to be the torus acting as in (3), and we may take B to be the intersection of the upper-triangular matrices in GL_8 with the subgroup G_2. Write α_1 and α_2 for the two *simple roots*, with α_2 the long root. In terms of the weights t_1, t_2 of (4), we have

$$\alpha_1 = t_1 - t_2, \qquad \alpha_2 = -t_1 + 2\,t_2. \tag{5}$$

The *positive roots* are $\alpha_1, \alpha_2, \alpha_1+\alpha_2, 2\,\alpha_1+\alpha_2, 3\alpha_1+\alpha_2, 3\alpha_1+2\,\alpha_2$; the *negative roots* are $-\alpha$, for α a positive root.

Let $P \subset G_2$ be the standard maximal parabolic subgroup omitting the long root, with Lie algebra $\mathfrak{p} \subset \mathfrak{g}_2$. Thus $\mathfrak{p} = \mathfrak{b} \oplus \mathfrak{g}_{-\alpha_1}$, where $\mathfrak{g}_{-\alpha_1} \subset \mathfrak{g}_2$ is the weight space for the negative root $-\alpha_1$.

The *Weyl group* is $W = N(T)/T$, where $N(T)$ is the normalizer of T in G_2. It is isomorphic to the dihedral group with 12 elements, and is generated by the *simple reflections* $s = s_{\alpha_1}$ and $t = s_{\alpha_2}$, and is defined by the relations $s^2 = t^2 = (st)^6 = 1$. There is an embedding $W \hookrightarrow S_7$ coming from the action of G_2 on $V \subset C$ given by

$$s \mapsto 2\,1\,5\,4\,3\,7\,6,$$
$$t \mapsto 1\,3\,2\,4\,6\,5\,7;$$

see [Anderson 2009, Section A.3]. We will sometimes treat elements of W as permutations via this embedding.

2.3. ***Flag bundles and Schubert loci.*** We refer to [Anderson 2009; 2011] for proofs of the following facts with more details. (There the term "γ-isotropic" is used instead of "G_2-isotropic" in reference to a trilinear form γ.)

Let C be an octonion algebra, and let $V = e^{\perp} \subset C$ be as before. A subspace $E \subseteq C$ is called *G_2-isotropic* if $E \subseteq V$ and $uv = 0$ for all $u, v \in E$. A maximal G_2-isotropic subspace has dimension 2, and a *G_2-isotropic flag* is a chain $E_1 \subset E_2 \subset V$ (with $\dim E_i = i$), where E_2 is G_2-isotropic. Such a flag can be canonically extended to a complete flag $E_1 \subset E_2 \subset E_3 \subset \cdots \subset E_7 = V$: When $E_1 \subset E_2$ is G_2-isotropic, with E_1 spanned by a vector u, then $E_u := \{v \in V \mid uv = 0\}$ is a three-dimensional subspace containing E_2. To get a complete flag, set $E_3 = E_u$, and then take orthogonal complements with respect to the norm N for the rest, so $E_4 = E_3^{\perp}$, etc. (See [Anderson 2011, Section 2.2] for this construction.)

The *G_2 flag variety* Fl_{G_2} parametrizes all G_2-isotropic flags in $V \subset C$. It is a six-dimensional projective homogeneous space, isomorphic to G_2/B for a Borel subgroup $B \subset G_2$. The *G_2 Grassmannian* Gr_{G_2} parametrizes two-dimensional G_2-isotropic subspaces of V; this is isomorphic to the five-dimensional homogeneous space G_2/P. The construction of a complete G_2-isotropic flag gives an embedding $\mathrm{Fl}_{G_2} \hookrightarrow \mathrm{Fl}(\mathbb{C}^7) = \mathrm{SL}_7/B$.

For an octonion bundle C on X with its rank-7 subbundle V, there is an associated *G_2-isotropic flag bundle* $\mathbf{Fl}_{G_2}(V) \to X$, as well as a *$G_2$-isotropic Grassmann bundle* $\mathbf{Gr}_{G_2}(V) \to X$. These are (étale-)locally trivial fiber bundles, with fibers Fl_{G_2} and Gr_{G_2}, respectively. The flag bundle $\mathbf{Fl}_{G_2}$ comes with a tautological flag of subbundles $\widetilde{E}_\bullet$ of V.

Given a complete G_2-isotropic flag of subbundles $F_1 \subset F_2 \subset \cdots \subset F_7 = V$ on X, the *Schubert loci* in $\mathbf{Fl}_{G_2}(V)$ are defined by

$$\Omega_w(F_\bullet) = \{x \in \mathbf{Fl}_{G_2} \mid \dim(\widetilde{E}_p(x) \cap F_q(x)) \geq r_w(q,p) \text{ for } 1 \leq p, q \leq 7\}, \quad (6)$$

where for $w \in W$, $r_w(q,p)$ is $\#\{i \leq q \mid w(8-i) \leq p\}$, and $\widetilde{E}_\bullet$ is the tautological flag on $\mathbf{Fl}_{G_2}$. Here we are using the embedding $W \hookrightarrow S_7$ discussed above.[2] The codimension of Ω_w is the *length* of w, i.e., the least number of simple transpositions needed to write w as a word in s and t.

If $E_\bullet$ is a second G_2-isotropic flag on X, it defines a section $s_{E_\bullet} : X \to \mathbf{Fl}_{G_2}$ such that $s_{E_\bullet}^* \widetilde{E}_\bullet = E_\bullet$. We define degeneracy loci in X as the scheme-theoretic inverse images of Schubert loci:

$$\Omega_w(E_\bullet, F_\bullet) = s_{E_\bullet}^{-1} \Omega_w(F_\bullet).$$

3. Triality symmetry

Triality symmetry is described in terms of coordinates as follows. Assume X is a point, so E is a two-dimensional vector space. Choose a basis $\{v_1, v_2\}$ for E, and let $\{v_3, \dots, v_8\}$ be a basis for $\mathrm{End}(E) \oplus E^*$ as in (1). Suppose $\varphi : E \to \mathrm{End}(E) \oplus E^*$ is given by $\varphi = \varphi_1 \oplus \varphi_2$, with

$$\varphi_1(v_1) = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \quad \varphi_1(v_2) = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix},$$

and $\varphi_2(v_1) = z\, v_2^*$, $\varphi_2(v_2) = -z\, v_1^*$. In terms of the chosen bases for E and $\mathrm{End}(E) \oplus E^*$, φ has matrix A_φ^t, whose transpose is

$$A_\varphi = \begin{pmatrix} b_1 & a_1 & d_1 & c_1 & z & 0 \\ b_2 & a_2 & d_2 & c_2 & 0 & -z \end{pmatrix}. \quad (7)$$

[2]This definition of r_w differs slightly from that of [Anderson 2011]; there the assignment $(q,p) \mapsto \#\{i \leq q \mid w(i) \leq p\}$ is called r_w. The two are related by replacing w with $ww_\circ$, where $w_\circ$ is the longest element of W.

Identify $\mathrm{Hom}(E, \mathrm{End}(E)) = E^* \otimes E^* \otimes E$ with $E^* \otimes E^* \otimes E^*$ by mapping

$$v_i^* \otimes v_j^* \otimes v_1 \mapsto v_{ij2}^*,$$
$$v_i^* \otimes v_j^* \otimes v_2 \mapsto -v_{ij1}^*,$$

where $v_{ijk}^* = v_i^* \otimes v_j^* \otimes v_k^*$ for $1 \le i, j, k \le 2$. (The sign appears because of the canonical isomorphism $E^* \otimes E^* \otimes E \cong E^* \otimes E^* \otimes E^* \otimes \bigwedge^2 E$; we are using $v_1 \wedge v_2$ to identify $E \cong E^* \otimes \bigwedge^2 E$ with E^*.) Thus φ is triality-symmetric if and only if the corresponding coordinates of v_{ijk}^* are invariant under permutations of the indices. Explicitly, there is an S_3-action on $\mathrm{Hom}(E, \mathrm{End}(E)) \oplus \bigwedge^2 E^*$ generated by elements τ and σ whose action on matrices A_φ is given by

$$\tau \begin{pmatrix} b_1 & a_1 & d_1 & c_1 & z & 0 \\ b_2 & a_2 & d_2 & c_2 & 0 & -z \end{pmatrix} = \begin{pmatrix} -d_2 & -d_1 & c_2 & c_1 & z & 0 \\ b_2 & b_1 & -a_2 & -a_1 & 0 & -z \end{pmatrix}$$
$$\text{and} \quad \sigma \begin{pmatrix} b_1 & a_1 & d_1 & c_1 & z & 0 \\ b_2 & a_2 & d_2 & c_2 & 0 & -z \end{pmatrix} = \begin{pmatrix} a_2 & a_1 & c_2 & c_1 & z & 0 \\ b_2 & b_1 & d_2 & d_1 & 0 & -z \end{pmatrix}.$$

This means that the triality-symmetric maps are those whose (transposed) matrix is of the form

$$A_\varphi = \begin{pmatrix} a & -d & d & c & z & 0 \\ b & a & -a & d & 0 & -z \end{pmatrix}. \tag{8}$$

Here a is also the coordinate of v_{122}^*, b is the coordinate of v_{222}^*, $-c$ is the coordinate of v_{111}^*, and $-d$ is the coordinate of v_{112}^*. Note that the S_3-invariants coincide with the τ-invariants.

Remark 3.1. "Triality" usually refers to several phenomena related to the S_3 symmetry of the D_4 Dynkin diagram. It was first described by Cartan [1925]; see [Knus et al. 1998] for a thorough discussion. The connection with our context can be explained briefly as follows. Automorphisms of the D_4 Dynkin diagram correspond to outer automorphisms of the simply connected group Spin_8; these all fix a parabolic subgroup P, and therefore define automorphisms of $\mathrm{Spin}_8 / P \cong \mathrm{OG}(2, 8)$ and the tangent space $T_{eP} \mathrm{Spin}_8 / P$. The tangent space can be identified with matrices as in (7), and under this identification the automorphism group S_3 acts as described above.

4. Graphs

For any morphism $\varphi : E \to F$, let $E_\varphi \subset E \oplus F$ be its graph, that is, the subbundle whose fiber over x is $E_\varphi(x) = \{(v, \varphi(v)) \mid v \in E(x)\}$. If $\varphi : E \to E^*$ is symmetric, then its graph is isotropic for the *canonical skew-symmetric form* on $E \oplus E^*$ defined by $(v_1 \oplus f_1, v_2 \oplus f_2) = f_1(v_2) - f_2(v_1)$. Thus one obtains a map to the Lagrangian bundle of isotropic flags in $E \oplus E^*$, and formulas for the degeneracy loci of φ are deduced from formulas for Schubert loci; see [Fulton 1996; Fulton and Pragacz 1998].

In this section, we consider morphisms $\varphi : E \to \mathrm{End}(E) \oplus E^*$. There is, by Proposition A.1, a canonical octonion algebra structure on $E \oplus \mathrm{End}(E) \oplus E^*$. We give formulas for degeneracy loci of morphisms whose graphs are G_2-isotropic with respect to this structure. In general such morphisms are not triality-symmetric (nor vice versa). For rank-1 maps, however, the two notions agree.

After a suitable change of coordinates (including a switch to opposite Schubert cells), the parametrization of the open Schubert cell in G_2/P given in [Anderson 2009, Section D.1] becomes

$$\widetilde{\Omega}^\circ = \begin{pmatrix} 1 & 0 & a & -d & d & c & z & -X \\ 0 & 1 & b & a & -a & d & -Z & -Y \end{pmatrix}, \tag{9}$$

where $X = -ac - d^2$, $Y = z + ad - bc$, and $Z = -a^2 - bd$. Morphisms $E \to \mathrm{End}(E) \oplus E^*$ with G_2-isotropic graph are exactly those whose (transposed) matrix has the form of the last six columns of (9).

Lemma 4.1. *Suppose X is a point and $\varphi : E \to \mathrm{End}(E) \oplus E^*$ is a triality-symmetric map with matrix A_φ^t as in (8). Then the graph E_φ is contained in $V \subset C$, and is G_2-isotropic if and only if*

$$a^2 + bd = ac + d^2 = ad - bc = 0. \tag{10}$$

Conversely, suppose $\varphi : E \to \mathrm{End}(E) \oplus E^*$ has G_2-isotropic graph as in (9). Then φ is triality-symmetric if and only if the equations (10) hold.

Proof. This is a straightforward verification, using the basis $\{v_i\}$ as in (1). It is clear that the row span of (9) is always in $V \subset C$ since the fourth and fifth columns add to zero. The condition that the row span be the graph E_φ means $X = Z = 0$ and $Y = z$, which are precisely the equations (10). □

Corollary 4.2. *Let $\varphi : E \to \mathrm{End}(E) \oplus E^*$ be a morphism of rank at most 1 such that the component $\varphi_2 : E \to E^*$ is zero. Then φ is triality-symmetric if and only if $E_\varphi \subset C$ is G_2-isotropic. (This holds scheme-theoretically, that is, the equations locally defining these two subsets of* $\mathrm{Hom}(E, \mathrm{End}(E))$ *are the same.)*

Proof. This is a local statement, so we may assume X is a point and compute in coordinates. In this case, it follows from Lemma 4.1 by adding the equation $z = 0$. (In fact, for a morphism with G_2-isotropic graph, the rank condition is forced by $\varphi_2 \equiv 0$.) □

Corollary 4.2 implies that the formulas of Theorem 1.2 (for triality-symmetric morphisms) will agree with formulas for morphisms with G_2-isotropic graphs. Before proceeding with the proof of Theorem 1.2, we will describe the connection between triality-symmetry and G_2-isotropic graphs more precisely.

Let $\mathbf{Gr}_{G_2} \subseteq \mathbf{Gr}(2, C)$ be the G_2-Grassmannian bundle on X, and let $\mathbf{Gr}^\circ$ be the open subset parametrizing subbundles of $C = E \oplus \mathrm{End}(E) \oplus E^*$ whose projection onto E is an isomorphism; locally on X, coordinates for $\mathbf{Gr}^\circ$ are given as in (9). Identifying a morphism $E \to \mathrm{End}(E) \oplus E^*$ with its graph, note that $\mathrm{Hom}(E, \mathrm{End}(E) \oplus E^*)$ is identified with the corresponding open subset of $\mathbf{Gr}(2, C)$, so $\mathbf{Gr}^\circ = \mathbf{Gr}_{G_2} \cap \mathrm{Hom}(E, \mathrm{End}(E) \oplus E^*)$ parametrizes morphisms with G_2-isotropic graph.

When X is a point, we have remarked that the space of triality-symmetric maps $\mathrm{tSym}(E^*)$ is naturally isomorphic to the tangent space $T_{[E]}\,\mathrm{Gr}_{G_2}$. For general X, this globalizes to an identification of the vector bundle $\mathrm{tSym}(E^*)$ with the normal bundle $N_{X/\mathbf{Gr}} = N_{X/\mathbf{Gr}^\circ}$, where X is embedded in $\mathbf{Gr}^\circ \subset \mathbf{Gr}$ by the section corresponding to the subbundle $E \subset C$.

Now let $\mathbf{D}_1 \subseteq \mathrm{tSym}(E^*)$ be the locus of triality symmetric morphisms of rank at most 1, and let $\Omega_1^\circ \subset \mathbf{Gr}^\circ$ be the locus of morphisms with G_2-isotropic graph of rank at most 1 such that the component φ_2 is zero. The next lemma identifies $\mathbf{D}_1$ with the normal cone to X in Ω_1°.

Lemma 4.3. *Inside* $\mathrm{tSym}(E^*) = N_{X/\mathbf{Gr}^\circ}$, *we have* $\mathbf{D}_1 = C_X\Omega_1^\circ$.

Proof. This can be checked locally on X, so assume X is a point. Note that both $\mathrm{tSym}(E^*)$ and $\mathbf{Gr}^\circ$ are isomorphic to $\mathbb{A}^5$. By Corollary 4.2, $\mathbf{D}_1$ and Ω_1° are defined by the same equations, namely (10) together with $z = 0$; since these equations are already homogeneous, we have that Ω_1° is equal to its tangent cone at the origin. □

Corollary 4.4. *Let φ be any triality-symmetric morphism and let ψ be any morphism with G_2-isotropic graph. Let $s_\varphi : X \to \mathrm{tSym}(E^*)$ and $s_\psi : X \to \mathbf{Gr}^\circ$ be the sections determined by φ and ψ. Then $s_\varphi^*[\mathbf{D}_1] = s_\psi^*[\Omega_1]$ in H^*X.*

Proof. Let s_0 be the zero section of $\mathrm{tSym}(E^*)$, and let s_E be the section of $\mathbf{Gr}^\circ$ corresponding to $E \subset C$. By Lemma 4.3 and the basic construction of intersection theory (see [Fulton 1998, Section 6]), we have $s_0^*[\mathbf{D}_1] = s_E^*[\Omega_1]$. On the other hand, both $\mathrm{tSym}(E^*)$ and $\mathbf{Gr}^\circ$ are affine bundles on X, so every section determines the same pullback on cohomology. (In fact, $\mathbf{Gr}^\circ$ is isomorphic to $\mathrm{tSym}(E^*)$, as one sees from the parametrization in (9), although it is not a vector subbundle of $\mathrm{Hom}(E, \mathrm{End}(E) \oplus E^*)$.) □

Consequently, for morphisms φ and ψ as in Corollary 4.4, we have $[D_1(\varphi)] = [D_1(\psi)]$ whenever

$$s_\varphi^*[\mathbf{D}_1] = [s_\varphi^{-1}\mathbf{D}_1] \quad \text{and} \quad s_\psi^*[\Omega_1] = [s_\psi^{-1}\Omega_1]. \tag{11}$$

(Indeed, $D_1(\varphi) = s_\varphi^{-1}\mathbf{D}_1$ and $D_1(\psi) = s_\psi^{-1}\Omega_1$ by definition.) When X is Cohen–Macaulay, so are $\mathbf{D}_1$ and Ω_1; this can be seen directly from the equations, or by using the fact that Schubert varieties are Cohen–Macaulay. The conditions (11) are therefore equivalent to the condition that $D_1(\varphi)$ and $D_1(\psi)$ have expected codimension in X, by [Fulton and Pragacz 1998, Lemma, p. 108].

First proof of Theorem 1.2. Let $\varphi : E \to \mathrm{End}(E) \oplus E^*$ have G_2-isotropic graph E_φ. Suppose E has a rank-1 subbundle, so E_φ also does. (One can always arrange for this, by passing to a $\mathbb{P}^1$-bundle if necessary.) Write $E_1 \subset E_2 = E$ and $F_1 \subset F_2 = E_\varphi$, and extend these to complete G_2-isotropic flags $E_\bullet$ and $F_\bullet$. For $w \in W$, set $\Omega_w(\varphi) = \Omega_w(E_\bullet, F_\bullet)$. Since $E_\varphi \cong E$, the Chern classes are the same. Let $-x_1, -x_2$ be Chern roots of E, so x_1, x_2 are Chern roots of E^*. Then, by [Anderson 2011, Theorem 2.4 and Section 2.5], we have

$$[\Omega_w(\varphi)] = \mathfrak{G}_w(x_1, x_2; -x_1, -x_2) \tag{12}$$

in H^*X, where $\mathfrak{G}_w(x_1, x_2; y_1, y_2)$ is the "G_2 double Schubert polynomial" defined in the same reference.

It remains to determine the w for which $D_r(\varphi) = \Omega_w(\varphi)$. We have

$$D_r(\varphi) = \{x \in X \mid \dim(E(x) \cap E_\varphi(x)) \geq 2 - r\},$$

and it is easy to check that

$$D_2(\varphi) = \Omega_{\mathrm{id}}(\varphi) = X, \quad D_1(\varphi) = \Omega_{tst}(\varphi), \quad \text{and} \quad D_0(\varphi) = \Omega_{tstst}(\varphi). \tag{13}$$

Indeed, the element $tst \in W$ corresponds to the permutation 3 6 1 4 7 2 5 (see [Anderson 2011, Section A.3]), so the condition defining Ω_{tst} is $\dim(E_2 \cap F_2) \geq r_{tst}(2, 2) = 1$. The other two identities are clear. This also justifies our definition of expected codimension for triality-symmetric degeneracy loci: the expected codimension of $D_r(\varphi)$ is the length of the corresponding element of W.

Specializing the polynomials $\mathfrak{G}_w$ given in [Anderson 2009, Section D.2] for these three w, we obtain the desired formulas for P_r. □

Remark 4.5. The twelve polynomials $\mathfrak{G}_w(x_1, x_2; -x_1, -x_2)$ become the equivariant localizations of Schubert classes in G_2/B at the point eB after the substitution $x_i = -t_i$; see [Anderson 2009, Section D.3].

Remark 4.6. We defined the scheme structure on $D_1(\varphi)$ for a triality-symmetric morphism by taking the equations (10) together with $z = 0$. In fact, the ideal generated by 2×2 minors of the matrix (8) is the same as the one generated by minors of (9), but this ideal is not radical. (It is generated by (10) together with az, bz, cz, dz, z^2.) The requirement $\varphi_2 \equiv 0$ for rank-1 maps is transparent on the triality-symmetric side; for G_2-isotropic graphs, the scheme structure is defined by pullback from the Schubert locus Ω_{tst}, and one sees $\varphi_2 \equiv 0$ from a parametrization of Schubert cells [Anderson 2009, Section D.1].

5. Orbits

Another approach to the computation of triality-symmetric degeneracy loci is as follows. Inside the vector bundle

$$\big(\mathrm{Sym}^3 E^* \otimes \textstyle\bigwedge^2 E\big) \oplus \textstyle\bigwedge^2 E^* \subset \mathrm{Hom}\big(E, \mathrm{End}(E) \oplus E^*\big),$$

there is a locus $\mathbf{D}_r$ consisting of morphisms of rank at most r. By definition, a triality-symmetric morphism φ defines a section s_φ of $\big(\mathrm{Sym}^3 E^* \otimes \bigwedge^2 E\big) \oplus \bigwedge^2 E^*$, and $D_r(\varphi) = s_\varphi^{-1}\mathbf{D}_r$ is the scheme-theoretic preimage.

It suffices to solve this problem on the classifying space for the vector bundle E (or on algebraic approximations thereof), so let $X = \mathrm{BGL}_2$.[3] Replace E with the standard representation of GL_2, and write

$$U = \big(\mathrm{Sym}^3 E^* \otimes \textstyle\bigwedge^2 E\big) \oplus \textstyle\bigwedge^2 E^*.$$

The relevant vector bundle on BGL_2 is $U \times^{\mathrm{GL}_2} \mathrm{EGL}_2$, where $\mathrm{EGL}_2 \to \mathrm{BGL}_2$ is the universal principal GL_2-bundle. Letting $D_r \subseteq U \subset \mathrm{Hom}(E, \mathrm{End}(E) \oplus E^*)$ be the locus of maps of rank at most r, we have

$$\mathbf{D}_r = D_r \times^{\mathrm{GL}_2} \mathrm{EGL}_2 \subseteq U \times^{\mathrm{GL}_2} \mathrm{EGL}_2\,.$$

Therefore $[\mathbf{D}_r] = [D_r]^{\mathrm{GL}_2}$ in $H^*(U \times^{\mathrm{GL}_2} \mathrm{EGL}_2) = H^*_{\mathrm{GL}_2}(U)$, and the problem becomes a computation in the equivariant cohomology of the vector space U.

Moreover, as we shall see below, D_r is an orbit closure for the action of GL_2 on U. In fact, we will use a larger group action. As discussed in Section 1, U may be identified with the tangent space

$$T_{[E]} G_2/P \cong \mathfrak{g}_2/\mathfrak{p},$$

so P acts on U via the adjoint action on $\mathfrak{g}_2/\mathfrak{p}$. Let $P = L \cdot P_u$ be the Levi decomposition, with P_u the unipotent radical and L a Levi subgroup. We will be interested in P-orbit closures in $\mathfrak{g}_2/\mathfrak{p}$.

First observe that L is isomorphic to GL_2. Here is one way to see this. Since E defines a point in G_2/P, the parabolic P may be identified with the subgroup of G_2 stabilizing E. Every linear automorphism of E induces an algebra automorphism of $C = E \oplus \mathrm{End}(E) \oplus E^*$; therefore $\mathrm{GL}(E) \cong \mathrm{GL}_2$ is a (reductive) subgroup of P, and we have $\mathrm{GL}(E) \subseteq L$. On the other hand, L is connected (since P is) and four-dimensional (by the root decomposition), so this inclusion must be an equality.

[3] Topologically, we may assume E is pulled back from the tautological bundle on $\mathrm{Gr}(2, n)$, for $n \gg 0$, so one can take a Grassmannian for an approximation to BGL_2.

The L-action on $\mathfrak{g}_2/\mathfrak{p}$ is identified with the natural GL_2-action on U: as an L-module, we have

$$\mathfrak{g}_2/\mathfrak{p} \cong \left(\mathrm{Sym}^3 E^* \otimes \textstyle\bigwedge^2 E\right) \oplus \textstyle\bigwedge^2 E^*,$$

where $E \cong \mathbb{C}^2$ is the standard representation of $L \cong \mathrm{GL}_2$ (with weights $t_1 = 2\alpha_1 + \alpha_2$ and $t_2 = \alpha_1 + \alpha_2$). As a P-module, $\mathfrak{g}_2/\mathfrak{p}$ is indecomposable, but there is an exact sequence

$$0 \to \mathrm{Sym}^3 E^* \otimes \textstyle\bigwedge^2 E \to \mathfrak{g}_2/\mathfrak{p} \to \textstyle\bigwedge^2 E^* \to 0.$$

These identifications of L- and P-modules follow directly from the weight decomposition of $\mathfrak{g}_2/\mathfrak{p}$: the T-weights are

$$-\alpha_2, \quad -\alpha_1 - \alpha_2, \quad -2\alpha_1 - \alpha_2, \quad -3\alpha_1 - \alpha_2, \quad -3\alpha_1 - 2\alpha_2. \tag{14}$$

As a first step to computing the classes of P-orbits in $H_T^*(\mathfrak{g}/\mathfrak{p})$, we give explicit descriptions of these orbits.

By the classification given in [Jürgens and Röhrle 2002], there are finitely many P-orbits on $\mathfrak{g}/\mathfrak{p}$. In fact, there are five orbits. To describe them, let

$$U' = \mathrm{Sym}^3 E^* \otimes \textstyle\bigwedge^2 E \subset U = \mathfrak{g}_2/\mathfrak{p}.$$

Let b, a, d, c be coordinates on U' with weights $-\alpha_2$, $-\alpha_1-\alpha_2$, $-2\alpha_1-\alpha_2$, $-3\alpha_1-\alpha_2$, respectively. The five orbits are O_c, with $c = 0, 1, 2, 3, 5$ giving the codimension; their closures are nested and described by the following proposition:

Proposition 5.1. *The P-orbit closures in $U = \mathfrak{g}_2/\mathfrak{p}$ are as follows*:

- $\overline{O}_0 = U$.
- $\overline{O}_1 = U'$.
- *$\overline{O}_2$ is the discriminant locus in U' defined by the vanishing of the quartic polynomial $a^2d^2 + 4a^3c + 4bd^3 - 27b^2c^2 + 18abcd$.*
- *$\overline{O}_3$ is the (affine) cone over the twisted cubic curve in $\mathbb{P}^3 = \mathbb{P}U'$ defined by the condition that the matrix*

  $$\begin{pmatrix} a & -d & c \\ b & a & d \end{pmatrix}$$

  have rank 1.
- $\overline{O}_5 = O_5 = \{0\}$.

The proof is straightforward, using the orbit classification of [Bürgstein and Hesselink 1987, Table 2]. See [Anderson 2009, Section 5.2] for details. Each of these orbit closures is Cohen–Macaulay, as may be checked easily from the equations.

From the description in terms of cubic polynomials, it is easy to find representatives for orbits in U'. Here we give representatives as weight vectors in $\mathfrak{g}/\mathfrak{p}$. Let $Y_\alpha \in \mathfrak{g}_2/\mathfrak{p}$ be a weight vector for α. We have

$$\begin{aligned}
O_0 &= P \cdot Y_{-3\alpha_1-2\alpha_2} = U \smallsetminus U',\\
O_1 &= P \cdot (Y_{-3\alpha_1-\alpha_2} + Y_{-\alpha_2}) \cong P/P_u \cong \mathrm{GL}_2,\\
O_2 &= P \cdot Y_{-\alpha_1-\alpha_2},\\
O_3 &= P \cdot Y_{-\alpha_2},\\
O_5 &= \{0\}.
\end{aligned}$$

Using Proposition 5.1, it is a simple matter to compute the equivariant classes.

Theorem 5.2. *In $H_T^*(U) = \mathbb{Z}[\alpha_1, \alpha_2] = \mathbb{Z}[t_1, t_2]$, we have*

$$\begin{aligned}
[\overline{O}_0] &= 1,\\
[\overline{O}_1] &= -3\alpha_1 - 2\alpha_2 = -t_1 - t_2,\\
[\overline{O}_2] &= 2(-3\alpha_1 - 2\alpha_2)^2 = 2(t_1+t_2)^2,\\
[\overline{O}_3] &= -3(\alpha_1+\alpha_2)(2\alpha_1+\alpha_2)(3\alpha_1+2\alpha_2)\\
&= -3t_1t_2(t_1+t_2),\\
[\overline{O}_5] &= -\alpha_2(\alpha_1+\alpha_2)(2\alpha_1+\alpha_2)(3\alpha_1+\alpha_2)(3\alpha_1+2\alpha_2)\\
&= t_1t_2(t_1+t_2)(2t_1-t_2)(t_1-2t_2).
\end{aligned}$$

Proof. The normal space to $U' = \overline{O}_1 \subset U$ has weight $-3\alpha_1 - 2\alpha_2$, so the formula for $[\overline{O}_1]$ is clear. Since the restriction $H_T^*(U) \to H_T^*(U')$ is an isomorphism, the Gysin pushforward $H_T^*(U') \to H_T^*(U)$ is multiplication by $[U']$. Therefore it suffices to compute the remaining classes in $H_T^*(U')$. The locus $\overline{O}_2$ is a hypersurface in U' defined by an equation of weight $-6\alpha_1 - 4\alpha_2$, so its class in $H_T^*(U)$ is $(-6\alpha_1 - 4\alpha_2)\cdot[U']$. The class of $[\overline{O}_3]$ in $H_T^*(U')$ is found by the classical Giambelli (or Salmon–Roberts) formula; see for example [Fulton and Pragacz 1998, Section 1.1]. Finally, the class of the origin is the product of all the T-weights on U. □

Remark 5.3. These classes cannot be computed using the "restriction equation" method of Fehér and Rimányi [2004] because the stabilizer of $O_1 = P/P_u$ is unipotent. This means the restriction map $H_P^*(U) \to H_P^*(O_1) \cong H_{P_u}^*(\mathrm{pt}) = H^*(\mathrm{pt})$ is zero in positive degrees, and all the restriction equations are of the form $0 = 0$. The problem persists for the other orbits.

Lemma 5.4. *The orbit $O_3 \subset \mathfrak{g}/\mathfrak{p} \subset \mathrm{Hom}(E, \mathrm{End}(E) \oplus E^*)$ consists of the triality-symmetric morphisms of rank* 1.

Proof. Any rank-1 map φ must correspond to an element $\varphi_1 \oplus \varphi_2 \in U = U' \oplus \bigwedge^2 E^*$ with $\varphi_2 = 0$, that is, φ lies in U'. (If $\varphi_2 \neq 0$, then φ surjects onto E^*.)

The action of P on U' is the same as that of its Levi subgroup GL_2. Let $P_{\hat{2}} \subset \mathrm{GL}_8$ be the parabolic which stabilizes E. The inclusion $P \hookrightarrow P_{\hat{2}} \subset \mathrm{GL}_8$ induces an inclusion of Levi subgroups $\mathrm{GL}_2 = \mathrm{GL}_2 \times 1 \hookrightarrow \mathrm{GL}_2 \times \mathrm{GL}_6$, and the latter group acts on $\mathrm{Hom}(E, \mathrm{End}(E) \oplus E^*)$ by left-right matrix multiplication,[4] so it preserves ranks of morphisms. Therefore it will suffice to check that a representative for O_2 has rank 2, and a representative from O_3 has rank 1.

For these, we use the coordinate description given in Section 3. Under the identification of U' with the space of cubic polynomials, the monomial xy^2 corresponds to the basis vector v^*_{122}. The orbit is O_2 (since xy^2 has two distinct zeroes), and the corresponding matrix A_φ has $b = c = d = 0$ and $a \neq 0$; it is easy to see this means φ has rank 2. Similarly, x^3 corresponds to v^*_{111}, and the corresponding A_φ has $a = b = d = 0$ and $c \neq 0$, so φ has rank 1. $\square$

The formulas of Theorem 1.2 now follow from those of Theorem 5.2.

Second proof of Theorem 1.2. Let $f : X \to \mathrm{BGL}_2$ be the map defined (up to homotopy) by the given vector bundle E on X. The corresponding map

$$f^* : H^* \mathrm{BGL}_2 = H^*_{\mathrm{GL}_2}(\mathrm{pt}) = \mathbb{Z}[c_1, c_2] \to H^* X$$

is given by $c_i \mapsto c_i(E) = (-1)^i c_i(E^*)$. Equivalently, using the inclusion $H^*_{\mathrm{GL}_2}(\mathrm{pt}) \subset H^*_T(\mathrm{pt}) = \mathbb{Z}[t_1, t_2]$ and Chern roots x_1, x_2 for E^*, the map is given by $t_i \mapsto -x_i$.

Using Lemma 5.4, we have $f^{-1}\overline{O}_3 = D_1(\varphi)$, so by [Fulton and Pragacz 1998, p. 108] and the fact that $\overline{O}_3$ is Cohen–Macaulay, we obtain $f^*[\overline{O}_3] = [D_1(\varphi)]$ when $D_1(\varphi)$ has expected codimension. $\square$

Remark 5.5. The proof of Theorem 1.2 given in Section 4 works verbatim for Chow cohomology. The proof in this section also works, though to apply equivariant techniques, one needs to take extra care to ensure that the bundle E is pulled back from an algebraic approximation to the classifying space. To achieve this, one can replace X with an appropriate composition of an affine bundle and a Chow envelope; see [Graham 1997, p. 486] for the argument.

Appendix: Octonion bundles

There is a G_2 analogue of the well-known fact that for any vector bundle E, the direct sum $E \oplus E^*$ carries canonical symplectic (type C) and symmetric (type D) forms; see for example [Fulton and Pragacz 1998, p. 71]. The intrinsic construction

[4] Identifying $\mathrm{Hom}(E, \mathrm{End}(E) \oplus E^*)$ with 6×2 matrices, the action is by $(g, h) \cdot A = hAg^{-1}$. This is the action induced by restricting the conjugation action of GL_8 on 8×8 matrices when the subspace of 6×2 matrices is placed in the lower-left corner.

presented here seems to appear first in [Landsberg and Manivel 2006, p. 151]; it is closely related to the Cayley–Dickson doubling construction [Petersson 1993].

We fix some notation. For any vector bundle E, let

$$\mathrm{Tr} : \mathrm{End}(E) = E^* \otimes E \to \mathcal{O}_X$$

be the canonical contraction map, and let

$$\mathrm{End}^0(E) = \ker(\mathrm{Tr}) \subset \mathrm{End}(E)$$

be the subbundle of trace-zero endomorphisms. Let $e : \mathcal{O}_X \to \mathrm{End}(E)$ be the identity section. Thus the composition $\mathrm{Tr} \circ e : \mathcal{O}_X \to \mathcal{O}_X$ is multiplication by $\mathrm{rk}(E)$. Also, when E has rank 2, the *conjugation* map $\mathrm{End}(E) \to \mathrm{End}(E)$ is given by $e \circ \mathrm{Tr} - \mathrm{id}$. (Here id is the identity morphism, as opposed to the identity section e.) Conjugation is an involution; locally, it is $\xi \mapsto \overline{\xi} := \mathrm{Tr}(\xi)e - \xi$.

The norm on an octonion bundle C corresponds to a nondegenerate symmetric bilinear form $\langle \cdot , \cdot \rangle$. Let $V \subset C$ be the orthogonal complement to the identity subbundle defined by e. A subbundle $E \subset C$ is *G_2-isotropic* if it is contained in V and the multiplication map $E \otimes E \to C$ is the zero map.

Proposition A.1 (cf. [Landsberg and Manivel 2006, p. 151]). *Let E be a rank-2 vector bundle on a variety X. Then $C = E \oplus \mathrm{End}(E) \oplus E^*$ has a canonical octonion bundle structure with identity section $e : \mathcal{O}_X \to \mathrm{End}(E) \subset C$. The subbundle $E = E \oplus 0 \oplus 0 \subset C$ is G_2-isotropic.*

More specifically, there is a quadratic norm $N : C \to \mathcal{O}_X$ and bilinear multiplication $m : C \otimes C \to C$ for $C = E \oplus \mathrm{End}(E) \oplus E^*$ which are compatible. The norm corresponds to the bilinear form $\langle \cdot , \cdot \rangle$ defined by

$$\langle x \oplus \xi \oplus f,\ y \oplus \eta \oplus g \rangle = \mathrm{Tr}(\xi)\,\mathrm{Tr}(\eta) - \mathrm{Tr}(\xi\eta) - f(y) - g(x). \tag{1}$$

The multiplication is given by

$$(x \oplus \xi \oplus f) \cdot (y \oplus \eta \oplus g) = (\eta x + \overline{\xi} y) \oplus (\overline{g \otimes x} + \xi\eta + f \otimes y) \oplus (g\xi + f\overline{\eta}). \tag{2}$$

One only needs to verify compatibility of the norm with multiplication; see [Anderson 2009, Section 2.4] for details.

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