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Suppose that closed subschemes $X \subset Y \subset \mathbb{P}^N$ differ at finitely many points: when is Y a flat specialization of X union isolated points? Our main result says that this holds if X is a local complete intersection of codimension two and the multiplicity of each embedded point of Y is at most three. We show by example that no hypothesis can be weakened: the conclusion fails for embedded points of multiplicity greater than three, for local complete intersections X of codimension greater than two, and for nonlocal complete intersections of codimension two. As applications, we determine the irreducible components of Hilbert schemes of space curves with high arithmetic genus and show the smoothness of the Hilbert component whose general member is a plane curve union a point in $\mathbb{P}^3$.

1. Introduction

An attractive aspect of algebraic geometry is that moduli spaces for its objects tend themselves to be algebraic varieties. Ever since Grothendieck [1961] proved their existence, the Hilbert schemes $\mathrm{Hilb}^{p(z)}(\mathbb{P}^N)$ classifying flat families of subschemes in $\mathbb{P}^N$ with fixed Hilbert polynomial $p(z)$ have drawn great interest. One of the first major results was the connectedness of Hilbert schemes, proved in [Hartshorne 1966]. More recently Liaison theory [Peskine and Szpiro 1974; Martin-Deschamps and Perrin 1990; Migliore 1998] has focused attention on Hilbert schemes $H_{d,g}$ of degree d, arithmetic genus g, locally Cohen–Macaulay curves in $\mathbb{P}^3$. The connectedness of $H_{d,g}$ remains an open question [Nollet 1997; 2006; Hartshorne 2000; Nollet and Schlesinger 2003].

While Hilbert schemes can be quite complicated in general, Piene and Schlessinger [1985] gave a satisfying picture of $\mathrm{Hilb}^{3z+1}(\mathbb{P}^3)$: there are two smooth irreducible components of dimensions 12 and 15 which meet transversely along an 11-dimensional family. In [Chen 2008], Mori's program was applied to the 12-dimensional component of twisted cubics, working out the effective cone decomposition and the corresponding models, exhibiting it as a flip of the Kontsevich moduli space of stable maps over the Chow variety. Similarly the Hilbert scheme

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component of unions of a pair of codimension-two linear subspaces of $\mathbb{P}^N$ is a smooth Mori dream space [Chen et al. 2011].

In an effort to achieve a similar understanding of the geometry of the Hilbert scheme component H_1 of rational quartic curves in $\mathbb{P}^3$, the first obstacle is determining the other irreducible components of $\mathrm{Hilb}^{4z+1}(\mathbb{P}^3)$. There are three natural families whose general members consist of the disjoint union of a line and a plane cubic, the disjoint union of an elliptic quartic curve and a point, and the disjoint union of a plane quartic and three points, but what about a possible component whose general member has an embedded point? We show in Example 2.9 that such Hilbert scheme components exist for curves of degree four and sufficiently negative genus. This motivates the following question:

Question 1.1. If X is obtained from $Y \subset \mathbb{P}^N$ by removing the zero-dimensional components, under what conditions is Y in the Hilbert scheme closure of the family consisting of X union isolated points?

When this is the case, we say that Y is a *flat limit* of X union isolated points, or simply that one can *detach* the embedded points of Y.

Remarks 1.2. (a) From the Hilbert scheme perspective, we should allow X to vary in the flat family. On the other hand, it is clearly desirable to have results requiring no information on how X sits in its Hilbert scheme, for they will be easier to apply.

(b) Question 1.1 is already interesting when X is empty. Fogarty [1968] observed that $\mathrm{Hilb}^d(\mathbb{P}^2)$ is irreducible for all $d > 0$, but Iarrobino [1972] showed that $\mathrm{Hilb}^d(\mathbb{P}^3)$ is reducible for $d \gg 0$. The minimum such value of d is still unknown. Iarrobino and Emsalem [1978] showed that $\mathrm{Hilb}^8(\mathbb{P}^4)$ is reducible and [Mazzola 1980] showed that $\mathrm{Hilb}^d(\mathbb{P}^n)$ is irreducible for $d \leq 7$. Cartwright et al. [2009] extended this to prove that for $d \leq 8$, $\mathrm{Hilb}^d(\mathbb{P}^N)$ is reducible if and only if $d = 8$ and $N \geq 4$.

We are interested in the case $\dim X > 0$. The kernel K of the surjection $\mathbb{O}_Y \to \mathbb{O}_X$ has finite length and may be written $\bigoplus K_p$ with p in the support of K. For such p, we say that the *multiplicity* of p is length K_p. The following criterion tells when *all* subschemes obtained from X by adding an embedded point of multiplicity one at $p \in X$ are flat limits of X union an isolated point (see Theorem 2.3).

Theorem 1.3. *For $p \in X \subset \mathbb{P}^N$, the following are equivalent*:

(1) *All subschemes Y obtained from X by adding an embedded point of multiplicity one at p are flat limits of X union an isolated point.*

(2) *The ideal sheaf $\mathcal{I}_X$ has r minimal generators at p with $r \leq N$ and $\pi^{-1}(p) \cong \mathbb{P}^{r-1}$, where $\pi : \widetilde{\mathbb{P}}^N \to \mathbb{P}^N$ is the blow-up at X.*

In particular, if X is a local complete intersection, then any embedded point of multiplicity one can be detached from X.

Condition (2) makes it easy to recognize when there exist schemes Y obtained from X which are not flat limits of X union an isolated point (see Example 2.6). Sometimes an embedded point of multiplicity one cannot be detached even if X is allowed to move in the deformation (see Example 1.5). Our main result gives conditions under which embedded points of various multiplicities can be detached (see Theorem 3.9):

Theorem 1.4. *Let $X \subset \mathbb{P}^N$ be a local complete intersection of codimension two. If Y is obtained from X by adding embedded points of multiplicity at most three, then Y is a flat limit of X union isolated points.*

The hypotheses may seem restrictive, but Theorem 1.4 is sharp in all aspects, as the following examples show.

Example 1.5. For any $g \leq -15$, the Hilbert scheme $\mathrm{Hilb}^{4z+1-g}(\mathbb{P}^3)$ has an irreducible component H of dimension $9-2g$ whose general member is the union of a multiplicity 4-line containing the triple line of generic embedding dimension three and an embedded point of multiplicity one. Details are given in Example 2.9.

Example 1.6. There are local complete intersections $X \subset \mathbb{P}^N$ of codimension greater than two and Y obtained from X by adding an embedded point of multiplicity two which are not flat limits of X union two isolated points. Let X be the nonreduced curve in $\mathbb{P}^4$ with ideal $I_X = (x^2, y^2, z^2)$. The family of double point structures on X has dimension equal to eight, the same as the dimension of the family consisting of X union two isolated points, hence the former cannot lie in the closure of the latter. See Example 3.6 for details.

Example 1.7. There are local complete intersections $X \subset \mathbb{P}^N$ of codimension two and Y obtained from X by adding an embedded point of multiplicity four which are not flat limits of X union four isolated points. For X with ideal $I_X = (x^2, y^2)$ in $\mathbb{P}^N$, we give a family of such subschemes Y having dimension $5N-6$, hence the general member cannot be a flat limit of X union four isolated points for $N > 5$. See Example 3.10 for details.

Remark 1.8. (a) For Y and X as in Theorem 1.4, there is an exact sequence

$$0 \to \mathcal{I}_Y \to \mathcal{I}_X \xrightarrow{\varphi} K \to 0, \tag{1}$$

where K is a sheaf of finite length. It is clear that the sheaf K is uniquely determined by Y (it is the quotient $\mathcal{I}_X/\mathcal{I}_Y$) and that two surjections φ and φ' yield the same subscheme Y if and only if there exists an automorphism σ of K such that $\varphi' = \sigma \circ \varphi$. The technique of our proof deforms the pair (φ, K).

(b) It is *not* the case that the embedded points can be pulled away one at a time; see Example 3.5.

(c) If X is a *hypersurface* and Y is obtained from X by adding embedded points of *any* multiplicities, then Y is a flat limit of X union isolated multiple points. In particular, such Y is a flat limit of X union isolated reduced points if the multiplicities are less than eight (Proposition 2.4).

Applying Theorem 1.4 to plane curves in $\mathbb{P}^3$, we deduce the following:

Corollary 1.9. *For $d \geq 6$ and $(d-1)(d-2)/2 - 3 \leq g \leq (d-1)(d-2)/2$, the Hilbert scheme* $\mathrm{Hilb}^{dz+1-g}(\mathbb{P}^3)$ *is irreducible.*

In Section 3 we give many other applications to space curves of low degree. Letting $g = (d-1)(d-2)/2$ be the genus of a degree-d plane curve, we give the following smoothness result:

Theorem 1.10. *Let $H_d \subset$* $\mathrm{Hilb}^{dz+2-g}(\mathbb{P}^3)$ *be the closure of the family of degree-d plane curves union an isolated point. Then H_d is smooth for all $d \geq 1$, and hence isomorphic to the blow-up of* $\mathrm{Hilb}^{dz+1-g}(\mathbb{P}^3) \times \mathbb{P}^3$ *along the incidence correspondence.*

Remark 1.11. Similarly the Hilbert scheme of a hypersurface in $\mathbb{P}^N$ union an isolated point is smooth (Theorem 4.1), but the Hilbert scheme is not smooth at plane curves union certain double embedded points (Remark 4.4).

Regarding organization, we deal with the question of detaching embedded points of multiplicity one in Section 2, and with embedded points of multiplicities two or three in Section 3. Our applications to Hilbert schemes are found in Section 4.

Conventions. For a subscheme $Z \subset \mathbb{P}^N$, $\mathcal{I}_Z$ denotes its sheaf of ideals and I_Z denotes its homogeneous (saturated) ideal or sometimes the ideal of Z in an open affine chart. We often write $\mathcal{O}$ for the structure sheaf of the ambient projective space and S for the homogeneous coordinate ring. A curve is a (purely) one-dimensional scheme. We say that Y is a flat limit of X union isolated points if Y is in the Hilbert scheme closure of this family. This is equivalent to the existence of a one-parameter family $\{Y_t\}_{t \in T}$ in which Y_t is X union isolated points for t general and $Y = Y_0$, and this is typically how we exhibit such a flat limit. We sometimes speak of a flat limit of ideals (or ideal sheaves) when working with the corresponding ideals. If two schemes $X \subset Y$ differ at an embedded point supported at $p \in X$, the multiplicity of the embedded point is the length of $\mathcal{I}_{X,p}/\mathcal{I}_{Y,p}$. Throughout the paper we work over an algebraically closed field k of arbitrary characteristic, but will occasionally assume char $k \neq 2, 3$ to apply irreducibility results.

2. Detaching embedded points of multiplicity one

In this section we study embedded point structures of multiplicity one on a local complete intersection $X \subset \mathbb{P}^N$ of codimension two. We also give a global result for

ACM subschemes with 3-generated ideal (Proposition 2.7). We begin by determining when an embedded point of multiplicity one can be detached from a subscheme $X \subset \mathbb{P}^N$.

Proposition 2.1. *For a proper subscheme $X \subset \mathbb{P}^N$, let $V \subset \mathrm{Hilb}^{p(z)+1}(\mathbb{P}^N)$ be the closed subset of subschemes which may be obtained from X by adding a point p (embedded or isolated). Then there is a diagram*

$$\begin{array}{ccc} \widetilde{\mathbb{P}}^N(X) & \xhookrightarrow{f} & V \\ \Big\downarrow{\scriptstyle \pi} & & \Big\downarrow{\scriptstyle h} \\ \mathbb{P}^N & = & \mathbb{P}^N, \end{array} \tag{2}$$

in which π is the blow-up at X, h sends a subscheme in V to the added point, and f extends the map $\mathbb{P}^N - X \to V$ given by $p \mapsto X \cup p$. Moreover, f is injective.

Proof. There is a uniform bound for the Castelnuovo–Mumford regularity of every ideal sheaf defining a closed subscheme with Hilbert polynomial $p(z)$, hence $h^0(\mathcal{I}_Y(m))$ is independent of $[Y] \in \mathrm{Hilb}^{p(z)+1}(\mathbb{P}^N)$ for sufficiently large m and the map

$$Y \mapsto \big(H^0(\mathbb{P}^N, \mathcal{I}_Y(m)) \subset H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(m))\big)$$

yields a closed immersion $F : \mathrm{Hilb}^{p(z)+1}(\mathbb{P}^N) \hookrightarrow \mathbb{G}$ to a suitable Grassmann variety [Harris and Morrison 1998]. Since $H^0(\mathbb{P}^N, \mathcal{I}_Y(m)) \subset H^0(\mathbb{P}^N, \mathcal{I}_X(m))$ has codimension one for $[Y] \in V$, the image $F(V)$ is contained in $\mathbb{P}(H^0(\mathbb{P}^N, \mathcal{I}_X(m)))^\vee \subset \mathbb{G}$. On the other hand, a standard construction [Peskine and Szpiro 1974, Proposition 4.1] yields a closed immersion $\widetilde{\mathbb{P}}^N(X) \xhookrightarrow{j} \mathbb{P}(H^0(\mathbb{P}^N, \mathcal{I}_X(m)))^\vee$ and for each $p \in \mathbb{P}^N - X$ we have $j(\pi^{-1}(p)) = F(h^{-1}(p))$. Since V is closed, we obtain an injective map $\widetilde{\mathbb{P}}^N(X) \hookrightarrow V$ and accompanying diagram (2). $\square$

Proposition 2.2. *In diagram* (2), *the following are equivalent*:

(a) *V is irreducible.*

(b) *For each $p \in X$, $\dim_{k(p)} \mathcal{I}_{X,p} \otimes k(p) = r \le N$ and $\pi^{-1}(p) \cong \mathbb{P}^{r-1}$.*

(c) *The map $\widetilde{\mathbb{P}}^N(X) \xrightarrow{f} V$ is bijective.*

Proof. For each $[Y] \in V$, there is an exact sequence

$$0 \to \mathcal{I}_Y \to \mathcal{I}_X \to K_p \to 0,$$

where $K_p \cong \mathcal{O}_p$ is the skyscraper sheaf of length 1 supported at p. For fixed p, the set of all such Y is given by surjections

$$\phi \in \mathrm{Hom}(\mathcal{I}_X, \mathcal{O}_p) \cong \mathrm{Hom}(\mathcal{I}_{X,p}, k(p)) \cong \mathrm{Hom}(\mathcal{I}_X \otimes k(p), k(p)) \tag{3}$$

modulo scalar. In view of Nakayama's lemma, we see that $h^{-1}(p) \cong \mathbb{P}^{r-1}_{k(p)}$, where r is the minimal number of generators for $\mathcal{I}_X$ at p.

The equivalence of (a) and (c) is clear from Proposition 2.1. Condition (c) implies that $\pi^{-1}(p) \cong h^{-1}(p) \cong \mathbb{P}^{r-1}$ for each $p \in X$ and $r \le N$ because $\pi^{-1}(p) \subset \widetilde{\mathbb{P}}^N(X)$ is a proper subset, proving (b). Conversely if (b) holds, then for $p \in X$, we have injections $\mathbb{P}^{r-1} \cong \pi^{-1}(p) \hookrightarrow h^{-1}(p) \cong \mathbb{P}^{r-1}$ which must be surjective by reason of dimension, hence $f : \widetilde{\mathbb{P}}^N(X) \to V$ is bijective on the fibers over $\mathbb{P}^N$ and is therefore bijective. □

The next result follows from the argument above. It allows one to determine when *all* embedded structures of multiplicity one supported at a fixed point can be detached.

Theorem 2.3. *For $p \in X \subset \mathbb{P}^N$, the following are equivalent*:

(1) *Every subscheme Y obtained from X by adding an embedded point of multiplicity one at p is a flat limit of X union an isolated point.*

(2) *X satisfies condition* (*b*) *of Proposition 2.2 at p.*

In particular, *these conditions hold if X is a local complete intersection.*

Proof. In the setting of Proposition 2.1, let $U = h^{-1}(\mathbb{P}^N - X) \subset V$ correspond to the subschemes obtained from X by adding an isolated point. Note that $f(\widetilde{\mathbb{P}}^N(X)) = \overline{U} \subset V$, since it is a closed subset with dense open subset U; hence for fixed $p \in X$ we have an inclusion $f(\pi^{-1}(p)) \subset h^{-1}(p) \cong \mathbb{P}^{r-1}$. Now condition (b) holds if and only if $\pi^{-1}(p) \cong \mathbb{P}^{r-1}$, if and only if $f(\pi^{-1}(p)) = h^{-1}(p)$ by reason of dimension and irreducibility of $\mathbb{P}^{r-1}$; but this equality is equivalent to $h^{-1}(p) \subset f(\widetilde{\mathbb{P}}^N(X)) = \overline{U}$, which is equivalent to condition (a). If X is a local complete intersection of codimension r, then it is well-known that $\pi^{-1}(p) \cong \mathbb{P}^{r-1}$ [Hartshorne 1977, II, Theorem 8.24(b)]; hence condition (b) from Proposition 2.2 holds. □

We can make a stronger statement when X is a hypersurface.

Proposition 2.4. *If Y is obtained from a hypersurface $X \subset \mathbb{P}^N$ by adding embedded points of any multiplicities*, *then Y is a flat limit of X union isolated multiple points. In particular*, *Y is a flat limit of X union isolated reduced points if the multiplicities are at most seven. For $N \ge 4$*, *there exist embedded structures of multiplicity eight in Y such that Y is not a flat limit of X union eight reduced points.*

Proof. Suppose that Y is defined by the surjection $\mathcal{I}_X \to K$, where K is of finite length supported at the embedded points p. Then $K \cong \bigoplus_p \mathcal{O}_{Z_p}$ for finite length subschemes Z_p supported at p ($\mathcal{I}_X$ is principal) and $\mathcal{I}_Y = \mathcal{I}_X \cdot \mathcal{I}_Z$, where Z is the union of the zero-dimensional subschemes Z_p. Use automorphisms of $\mathbb{P}^N$ to deform Z to Z_t such that the support of Z_t does not intersect X for $t \neq 0$. Then $\mathcal{I}_{X \cup Z_t} = \mathcal{I}_X \cdot \mathcal{I}_{Z_t}$ for $t \neq 0$ and in considering the associated schemes it is clear that Y is a flat limit of $X \cup Z_t$. If the length of Z_t is ≤ 7, then Z_t is a flat limit of reduced points [Mazzola 1980; Cartwright et al. 2009]; hence Y is a flat limit of X union isolated reduced points.

For $N \geq 4$, there exists a nonsmoothable, length-8 subscheme $Z \subset \mathbb{P}^N$ [Iarrobino and Emsalem 1978; Cartwright et al. 2009]. Choose an open affine $U \cong \mathbb{A}^N$ on which $\mathcal{I}_X$ is trivial, apply an automorphism of $\mathbb{P}^N$ to translate Z so that the support of Z lies in $U \cap X$, and let Y be the subscheme defined by the surjection $\mathcal{I}_X \cong \mathcal{O} \to \mathcal{O}_Z$. If $\mathcal{I}_X = (f)$ locally, then $\mathcal{I}_Y = (f)\mathcal{I}_Z$ and Y cannot be a flat limit of X union eight isolated points, for then $\mathcal{I}_Y$ would be the flat limit of ideals $(f)\mathcal{I}_{Z_t}$, where Z_t consists of eight isolated points and from the expression of $\mathcal{I}_Y$ we would obtain $\mathcal{I}_Z$ as a flat limit of $\mathcal{I}_{Z_t}$, a contradiction. □

Example 2.5. We give two examples in which Theorem 2.3 applies.

(a) If $X \subset \mathbb{P}^N$ is a local complete intersection of codimension r at p, then $\mathcal{I}_{X,p} = (f_1, \dots, f_r) \subset \mathcal{O}_{\mathbb{P}^N,p}$, where $f_1, \dots, f_r$ cut out X at p. An embedded point is determined by a surjection $\varphi : \mathcal{I}_{X,p} \to k(p)$. After changing generators, we may assume $\varphi(f_1) = 1$ and $\varphi(f_i) = 0$ for $i > 1$ so that the ideal for the corresponding subscheme Y locally at p is $(m_p f_1, f_2, \dots, f_r)$.

(b) Use $[x, y, z, w]$ to denote the coordinates of $\mathbb{P}^3$. Let $C \subset \mathbb{P}^3$ be the union of three coordinate axes with ideal $I_C = (xy, xz, yz)$. Away from the origin [0, 0, 0, 1], C is a local complete intersection. Working on the affine patch $w \neq 0$, one computes that the blow-up at C has fiber $\mathbb{P}^2$ over the origin, so condition (b) of Proposition 2.2 holds at each point. It follows from Theorem 2.3 that any subscheme D obtained from C by adding an embedded point is a flat limit of C with an isolated point.

Example 2.6. We give two examples where Theorem 2.3 does not apply.

(a) Fix a line $L \subset \mathbb{P}^3$ and define X by $\mathcal{I}_X = \mathcal{I}_L^d$ with $d > 1$. Then $\mathcal{I}_X$ is generated by $d+1$ elements at each $p \in X$ ($I_X = I_L^d$), but $\pi^{-1}(p) \cong \mathbb{P}^1$ because the blow-ups of $\mathbb{P}^3$ at $\mathcal{I}_L$ and $\mathcal{I}_L^d$ are isomorphic [Hartshorne 1977, II, Example 7.11(a)], so condition (b) of Proposition 2.2 fails.

(b) The curve $X \subset \mathbb{P}^3$ with ideal (x^2, xy, y^3) is ACM with locally 3-generated ideal sheaf at each point $p \in X$; hence it is not possible that $\pi^{-1}(p) \cong \mathbb{P}^2$ for each $p \in C$, for then the exceptional divisor would have dimension 3. Therefore a general embedded point cannot be detached while leaving X fixed. Nevertheless, such an embedded point can be detached in the Hilbert scheme due to the following.

Proposition 2.7. *Let $X_0 \subset \mathbb{P}^N$ be ACM of codimension two with 3-generated homogeneous ideal I_{X_0}. Then each subscheme Y obtained from X_0 by adding an embedded point of multiplicity one is the flat limit of local complete intersection ACM subschemes union an isolated point.*

Proof. Write $\mathcal{O}$ for the structure sheaf of $\mathbb{P}^N$. Since X_0 is ACM and I_{X_0} is 3-generated, the ideal sheaf has minimal resolution

$$0 \to \bigoplus_{j=1}^{2} \mathcal{O}(-b_j) \xrightarrow{\psi_0} \bigoplus_{i=1}^{3} \mathcal{O}(-a_i) \xrightarrow{\pi_0} \mathcal{I}_{X_0} \to 0 \tag{4}$$

and $\mathcal{I}_Y$ is the kernel of a surjection $\varphi : \mathcal{I}_{X_0} \to \mathcal{O}_p$. Our strategy is to deform the exact sequence (4) along with φ to obtain subschemes X_t that are local complete intersections and maps $\varphi_t : \mathcal{I}_{X_t} \to \mathcal{O}_{p_t}$ to define the family Y_t. We carry this out in steps:

Claim 1. *There is a lift of* $\varphi \circ \pi_0 : \bigoplus_{i=1}^{3} \mathcal{O}(-a_i) \to \mathcal{O}_p$ *to* $\tilde{\varphi} : \bigoplus_{i=1}^{3} \mathcal{O}(-a_i) \to \mathcal{O}$ *such that the composite*

$$\tilde{\varphi} \circ \psi_0 : \bigoplus_{j=1}^{2} \mathcal{O}(-b_j) \to \mathcal{O}$$

is induced by multiplying (F, G), *where* F *and* G *are homogeneous polynomials of degrees* b_1 *and* b_2 *with no common factor.*

Claim 2. *There is a map*

$$\psi_1 : \bigoplus_{j=1}^{2} \mathcal{O}(-b_j) \to \bigoplus_{i=1}^{3} \mathcal{O}(-a_i)$$

whose cokernel is the ideal sheaf of a local complete intersection X_1.

Once we have established the claims, the rest is straightforward. Construct the linear deformation $\psi_t = t\psi_1 + (1-t)\psi_0$ for $t \in \mathbb{A}^1$ and write the composite maps $\tilde{\varphi} \circ \psi_t : \bigoplus_{j=1}^{2} \mathcal{O}(-b_j) \to \mathcal{O}$ as (F_t, G_t). Then the schemes S_t given by $F_t = G_t = 0$ are complete intersections in a neighborhood of $t = 0$ because this is true for S_0 by construction. If $S \subset \mathbb{P}^N \times \mathbb{A}^1$ is the total family, there is an integral curve T through $(p, 0)$ inside S which is not vertical at $(p, 0)$ and base extension by $T \to \mathbb{A}^1$ allows us to pick out a moving point $p_t \in S_t$ with $p_0 = p$. By abuse of notation we will use the same letter t for the parameter.

Let

$$\varphi_t : \bigoplus_{i=1}^{3} \mathcal{O}(-a_i) \xrightarrow{\tilde{\varphi}} \mathcal{O} \to \mathcal{O}_{p_t} = k(p_t)$$

be the composition. For general $t \neq 0$, $\operatorname{Coker} \psi_t$ is the ideal sheaf of an ACM local complete intersection X_t and $\varphi_t \circ \psi_t = 0$ by construction (since $p_t \in S_t$); hence we get induced maps $\mathcal{I}_{X_t} \to \mathcal{O}_{p_t}$. Since φ_0 is onto, so is φ_t for general t, the kernels giving a family of ideals $\mathcal{I}_{Y_t}$ for a family of local complete intersections X_t converging to X_0 along with a point p_t converging to $p = p_0$. If $p_t \notin X_t$, then we are done. If $p_t \in X_t$ for each t, then we have at least shown that Y is a flat limit of complete intersections having an embedded point. By Proposition 2.2, these are flat limits of local complete intersections with an isolated point and we again conclude the proof. It remains to establish the two claims above.

Proof of Claim 1. The composition $\varphi \circ \pi : \bigoplus_{i=1}^{3} \mathbb{O}(-a_i) \to \mathbb{O}_p$ lifts to

$$\widetilde{\varphi} : \bigoplus_{i=1}^{3} \mathbb{O}(-a_i) \to \mathbb{O}$$

because $H^0_*(\mathbb{O}_{\mathbb{P}^3}) \to H^0_*(\mathbb{O}_p)$ is surjective in positive degrees. Let us write this map as $\widetilde{\varphi} = (A_1, A_2, A_3) \in H^0(\bigoplus_{i=1}^{3} \mathbb{O}(a_i))$. Then the general such lift $\widetilde{\varphi}$ may be written $(A_1 + B_1, A_2 + B_2, A_3 + B_3)$ with $B_i \in I_p$. Writing

$$\psi_0 = \begin{pmatrix} f_1 & f_2 & f_3 \\ g_1 & g_2 & g_3 \end{pmatrix},$$

the desired composite map is given by $(F, G) = (\sum (A_i + B_i) f_i, \sum (A_i + B_i) g_i)$ and we need to show that F and G have no common factor. For this it suffices to show that the zero loci of F and G meet properly. Letting L be a line missing X_0 (and p), we will show that there are no common zeros along L, for general B_i. Restricting the resolution (4) to L and dualizing yields the exact sequence

$$0 \to \mathbb{O}_L \to \bigoplus_{i=1}^{3} \mathbb{O}_L(a_i) \xrightarrow{\psi_0^\vee \otimes \mathbb{O}_L} \bigoplus_{j=1}^{2} \mathbb{O}_L(b_j) \to 0.$$

Since $b_j > 0$, the rank-two bundle on the right has a nonvanishing section, which lifts to a section (r_1, r_2, r_3) of the rank-three bundle $\bigoplus_{i=1}^{3} \mathbb{O}_L(a_i)$. Since the equations in I_p of degree $d > 0$ cut out the complete linear system $H^0(\mathbb{O}_L(d))$, we can find $B_i \in I_p$ such that $(A_i + B_i)\big|_L = r_i$, for $i = 1, 2, 3$, and this choice proves our claim because the nonvanishing image of (r_1, r_2, r_3) in $\bigoplus_{j=1}^{2} \mathbb{O}_L(b_j)$ is given by the restrictions of the polynomials $\sum (A_i + B_i) f_i$ and $\sum (A_i + B_i) g_i$; hence these have no common zeros along L. □

Proof of Claim 2. It is well-known that the degeneracy locus X_1 of the general such map ψ_1 is codimension two and regular in codimension one [Chang 1989]. Here we show that X_1 is a local complete intersection as well. In the exact sequence (4) we may take $a_1 \le a_2 \le a_3$, $b_1 \le b_2$, and $b_1 > a_1$ (if $b_1 = a_1$, we can cancel off this summand and X_0 is a complete intersection, when Claim 2 is clear). Since $\sum b_j = \sum a_i$ (because $c_1(\mathcal{I}_{X_0}) = 0$), it follows that $d_1 = a_3 + a_2 - b_1 = b_2 - a_1 > 0$ and $d_2 = a_3 + a_2 - b_2 = b_1 - a_1 > 0$, so let Z be a complete intersection of two general hypersurfaces of degrees d_1 and d_2. It is easy to check $d_2 \le d_1 \le a_3$ and $d_2 \le a_2$, therefore we can link Z to X by a complete intersection $C = K_1 \cap K_2$ of hypersurfaces of degrees a_2 and a_3. The inclusion

$$\begin{array}{ccccccccc} 0 & \to & \mathbb{O}(b_1 + b_2 - 2a_2 - 2a_3) & \to & \mathbb{O}(b_2 - a_2 - a_3) \oplus \mathbb{O}(b_1 - a_2 - a_3) & \to & \mathcal{I}_Z & \to & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \to & \mathbb{O}(-a_2 - a_3) & \to & \mathbb{O}(-a_2) \oplus \mathbb{O}(-a_3) & \to & \mathcal{I}_C & \to & 0, \end{array}$$

and the cone construction from liaison theory [Migliore 1998, Proposition 5.2.10] yields the resolution

$$0 \to \mathcal{O}(-b_1) \oplus \mathcal{O}(-b_2) \to \mathcal{O}(-a_1) \oplus \mathcal{O}(-a_2) \oplus \mathcal{O}(-a_3) \to \mathcal{I}_X \to 0.$$

Hence X has the same type of resolution as X_0. By Bertini's theorem, the general hypersurface K_2 containing Z is smooth, so X is Cartier on K_2 and hence a local complete intersection. Now just take $X_1 = X$ and the claim is proved. □

Example 2.8. The easiest way to construct curves in $\mathbb{P}^3$ satisfying the hypotheses of Proposition 2.7 is by linking to a complete intersection, as in the proof of Claim 2.

(a) Any purely one-dimensional curve $C \subset \mathbb{P}^3$ of degree 3 and genus 0 is ACM [Piene and Schlessinger 1985] and has a resolution of the form

$$0 \to \mathcal{O}(-3)^2 \to \mathcal{O}(-2)^3 \to \mathcal{I}_C \to 0$$

as noted in [Ellingsrud 1975, Example 1], and links to a line by a complete intersection of two quadric surfaces. In particular, this holds for the triple line with ideal (x^2, xy, y^2).

(b) If $C \subset \mathbb{P}^3$ is any locally Cohen–Macaulay curve of degree 4 and genus 1, then C is nonplanar, so $h^1(\mathcal{I}_{C_0}(n)) \leq (d-2)(d-3)/2 - g = 0$ for all n [Martin-Deschamps and Perrin 1993, Theorem 1.3] and therefore C is ACM. Now $\chi(\mathcal{I}_C(1)) = 0$, so $H^2(\mathcal{I}_C(1)) = 0$. Furthermore $H^1(\mathcal{I}_C(2)) = 0$ (C is ACM) and $H^3(\mathcal{I}_C(0)) = 0$ so $\mathcal{I}_C$ is Mumford 3-regular. In particular $\mathcal{I}_C(3)$ is generated by global sections, and we can link C by the complete intersection of a quadric and cubic to a curve D of degree 2 and genus 0. Since D is planar, it is a complete intersection, so using the method of the proof of Claim 2, above, we see that C has resolution

$$0 \to \mathcal{O}(-4) \oplus \mathcal{O}(-3) \to \mathcal{O}(-3) \oplus \mathcal{O}(-2)^2 \to \mathcal{I}_C \to 0$$

and again Proposition 2.7 applies to C. The quadruple line with ideal (x^2, xy, y^3) is such an example, explaining Example 2.6(b).

Sometimes a one-dimensional subscheme D with embedded points is not a flat limit of curves C union isolated points even if one allows C to deform. In other words, the Hilbert scheme can have irreducible components whose general member has an embedded point.

Example 2.9. We exhibit an irreducible component of $\mathrm{Hilb}^{4z+1-g}(\mathbb{P}^3)$ whose general member has an embedded point for any $g \leq -15$. The irreducible components of the Hilbert schemes $H_{4,g}$ of locally Cohen–Macaulay curves of degree 4 and arithmetic genus g are known [Nollet and Schlesinger 2003, Table III]. We note two typographical errors in the table, namely the family G_5 of double conics has dimension $13 - 2g$ instead of $13 - 3g$ [Nollet and Schlesinger 2003, p. 189] and the general member of family $G_{7,a}$ should be $W \cup_{3p} L$ instead of just W. Now consider the irreducible component G_4 of dimension $9 - 3g$, consisting of thick quadruple lines. Each curve $[C] \in G_4$ has a supporting line L and there is an exact sequence

$$0 \to \mathcal{I}_C \to \mathcal{I}_W \xrightarrow{\phi} \mathcal{O}_L(-g-1) \to 0,$$

where W is the triple line given by $\mathcal{I}_W = \mathcal{I}_L^2$ [Nollet and Schlesinger 2003, Proposition 2.1]. The surjection ϕ factors through $\mathcal{I}_W \otimes \mathcal{O}_L \cong \mathcal{O}_L(-2)^2$, hence is given by $\phi(x^2) = a$, $\phi(xy) = b$, and $\phi(y^2) = c$ for three homogeneous polynomials a, b, and c of degree $-g+1$. Writing the ideal of C as

$$I_C = (x^3, x^2y, xy^2, y^3, axy - bx^2, by^2 - cxy),$$

we see that at general point $p \in L$, a, b, and c are units in the local ring $\mathcal{O}_{\mathbb{P}^3,p}$, therefore $I_{C,p} = (x^3, axy - bx^2, by^2 - cxy)$ and I_C is generically 3-generated for general ϕ.

Now consider the locus $V \subset \mathrm{Hilb}^{4z+2-g}(\mathbb{P}^3)$ obtained by adding an isolated or embedded point to C as above, as in Proposition 2.1. The closure of the component corresponding to C along with isolated points has dimension three. Since I_C is generically 3-generated, the set of embedded point structures at general $p \in C$ is parametrized by $\mathbb{P}^2$ and we obtain a second three-dimensional family. Thus V is reducible with at least these two three-dimensional components (conceivably the locus where I_C is generated by more elements could generate another family). Varying the curve $[C] \in G_4$, we obtain at least two corresponding families of dimension $12 - 3g$ (because $\dim G_4 = 9 - 3g$). Let F be the closure of the family whose general curve has an embedded point.

We claim that F is an irreducible component of $\mathrm{Hilb}^{4z+2-g}(\mathbb{P}^3)$. The general member $[D] \in F$ cannot be a flat limit of curves possessing more than one isolated or embedded point (counted with multiplicity). Since G_4 is an irreducible component of $H_{4,g}$ for $g \leq -2$, D is not a flat limit of another family of curves with an isolated or embedded point of multiplicity one, because the underlying locally Cohen–Macaulay curve $C \subset D$ is not. Finally D cannot be a flat limit of locally Cohen–Macaulay curves of genus $g-1$ because the maximal dimension of such a family for $g \leq -15$ is $12 - 3g = \dim F$.

3. Detaching embedded points of multiplicity two or three

In this section we prove that if Y has embedded points of multiplicity two (see Proposition 3.3) or three (see Proposition 3.7) and the underlying subscheme $X \subset \mathbb{P}^N$ is a local complete intersection of codimension two, then Y is a flat limit of X union isolated points. Along with Theorem 2.3, this shows that an embedded point of multiplicity at most three can be detached from X, from which we deduce our main result, Theorem 1.4.

We begin with several propositions that take care of the easier cases, leaving the more difficult cases to Proposition 3.7. We also show that these results may fail for local complete intersections of codimension greater than two (Example 3.6) and for embedded points of multiplicity greater than three (Example 3.10).

Proposition 3.1. *Let $X \subset \mathbb{P}^N$ be a local complete intersection of codimension two, Z a zero-dimensional subscheme of embedding dimension at most one and suppose that Y is defined by the exact sequence*

$$0 \to \mathcal{I}_Y \to \mathcal{I}_X \xrightarrow{\varphi} \mathcal{O}_Z \to 0.$$

Then Y is a flat limit of X union isolated points.

Proof. Since the result is local, we may assume that Z is supported at a point p and has length d. Since Z has embedding dimension ≤ 1, we can choose a smooth connected curve C_0 of high degree containing Z and not entirely in X. If $p \notin X$, the result is clear because Z is a flat limit of isolated points in C_0. In the interesting case $p \in X$, our idea is to take a deformation C_t of C_0 and use d isolated points in C_t to perform the detaching process.

Let C be a translation of C_0 by $\mathrm{PGL}(N+1)$ which misses X. Now for $m \gg 0$, the general pair $F, G \in H^0(\mathcal{I}_X(m))$ give hypersurfaces which cut out X in an open neighborhood of p. Write $\mathcal{O}$ for the structure sheaf of $\mathbb{P}^N$. For the purposes of this proof we may assume that X is equal to the complete intersection defined by F and G, giving the Koszul resolution

$$0 \to \mathcal{O}(-2m) \xrightarrow{\psi} \mathcal{O}(-m) \oplus \mathcal{O}(-m) \xrightarrow{\pi} \mathcal{I}_X \to 0. \tag{5}$$

Because the restriction map $H^0(\mathcal{O}(m)) \to H^0(\mathcal{O}_Z(m))$ is surjective for $m \gg 0$, we can lift the images of F, G to $\mathcal{O}$, hence the composition $\varphi \circ \pi : \mathcal{O}(-m)^2 \to \mathcal{O}_Z$ factors through $\mathcal{O}$ and we obtain $\widetilde{\varphi} : \mathcal{O}(-m)^2 \to \mathcal{O}$ inducing φ. The composition $\widetilde{\varphi} \circ \psi$ vanishes on a hypersurface S of degree $2m$ containing both X and Z.

By Bertini's theorem, we could have chosen the equations F and G cutting out X near p to be smooth away from X, meeting C in disjoint reduced sets of points, so the restrictions to C induce a sheaf surjection $\mathcal{O}_C^2 \to \mathcal{O}_C(m)$. If $\widetilde{\varphi}$ is given by $A_0, B_0 \in H^0(\mathcal{O}(m))$, then S has equation $FA_0 + GB_0 = 0$, but A_0 and B_0 are only determined up to elements of $H^0(\mathcal{I}_Z(m))$. Since the natural map $H^0(\mathcal{I}_Z(m))^2 \to H^0(\mathcal{O}_C(m))$ is surjective, given by $(A, B) \mapsto FA + GB$, we may choose A_0 and B_0 to assume that $S \cap C$ is a reduced set of $2m(\deg C)$ points.

Now consider a family of translations C_t from C to C_0, parametrized by $t \in \mathbb{A}^1$. Now $C_0 \cap S$ contains Z at p and $C_t \cap S$ consists of $2m(\deg C)$ reduced points for general $t \neq 0$. Possibly after a base extension, we may pick d distinct points

$p_{1,t}, p_{2,t}, \dots, p_{d,t}$ in $S \cap C_t$ near p. Letting $Z_t = \{p_{1,t}, p_{2,t}, \dots, p_{d,t}\}$, the flat limit of Z_t is exactly Z, because the ideal of the limit contains the equations of the curve C_0 by construction, and Z is the unique length-d subscheme of C_0 at p.

Letting φ_t be the composition

$$\mathcal{O}(-m)^2 \xrightarrow{\widetilde{\varphi}} \mathcal{O} \to \mathcal{O}_{Z_t},$$

we have $\varphi_t \circ \psi = 0$ by construction, so these maps factor through $\mathcal{I}_X$ and we obtain a family of maps $\varphi_t : \mathcal{I}_X \to \mathcal{O}_{Z_t}$. Since $\varphi_0 = \varphi$ is surjective, so are φ_t for t near 0 and the family $\mathcal{I}_{Y_t} = \ker \varphi_t$ gives the desired family. □

Proposition 3.2. *Let $X \subset \mathbb{P}^N$ be a local complete intersection, K a sheaf of finite length supported at p, and Y and Y^1 defined by the commutative diagram of short exact sequences*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{I}_Y & \longrightarrow & \mathcal{I}_X & \xrightarrow{\varphi=(\alpha,\beta)} & K \oplus \mathcal{O}_p & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \pi_1} & & \\
0 & \longrightarrow & \mathcal{I}_{Y^1} & \longrightarrow & \mathcal{I}_X & \xrightarrow{\alpha} & K & \longrightarrow & 0.
\end{array}$$

Then Y is a flat limit of Y^1 union an isolated point. In particular, if Y^1 is a flat limit of X union isolated points, then so is Y.

Proof. The result is local at p. The direct sum allows us to write $\varphi = (\alpha, \beta)$, where α defines Y^1 as above. The surjection $\beta : \mathcal{I}_X \to \mathcal{O}_p$ defines an embedded point structure Y^2 on X. Since X is a local complete intersection, Y^2 is a flat limit of X union an isolated point by Theorem 2.3, meaning that there is a flat family Y_t^2 for $t \in T$ with $Y_0^2 = Y^2$ and $Y_t^2 = X \cup p_t$ with $p_t \notin X$ for $t \neq 0$. This gives a family of surjections $\beta_t : \mathcal{I}_X \to \mathcal{O}_{p_t}$ with $\mathcal{I}_{Y_t^2} = \ker \beta_t$ and $\beta_0 = \beta$.

Let $\gamma_t : \mathcal{I}_{Y^1} \subset \mathcal{I}_X \xrightarrow{\beta_t} \mathcal{O}_{p_t}$ be the composite map. Clearly γ_t is surjective for $t \neq 0$, because $p_t \notin \mathcal{I}_X$, so the inclusion $\mathcal{I}_{Y^1} \subset \mathcal{I}_X$ is an equality at these points. The map γ_0 is also a surjection, since, locally at p, if we choose $f \in \mathcal{I}_X$ such that $\varphi(f) = (0, 1)$, then $\alpha(f) = 0 \Rightarrow f \in \mathcal{I}_{Y^1}$ and $\gamma_0(f) = 1$. This family of maps gives a flat family Y_t, and for $t \neq 0$ Y_t consists of Y^1 union an isolated point. Finally, the kernel of $\gamma_0 : \mathcal{I}_{Y^1} \to \mathcal{O}_p$ is exactly $\mathcal{I}_Y$, for $g \in \mathcal{I}_{Y^1} \Rightarrow g \in \mathcal{I}_X$ and $\alpha(g) = 0$. Now $\gamma_0(g) = 0 \iff \beta(g) = 0 \iff \varphi(g) = 0 \iff g \in \mathcal{I}_Y$. □

Proposition 3.3. *Let $X \subset \mathbb{P}^N$ be a local complete intersection of codimension two and obtain Y by adding an embedded point of multiplicity two with associated exact sequence*

$$0 \to \mathcal{I}_Y \to \mathcal{I}_X \xrightarrow{\varphi} K_p \to 0,$$

where K_p is a sheaf of length 2 supported at p. Then either (a) *$K_p \cong \mathcal{O}_p \oplus \mathcal{O}_p$, or* (b) *$K_p \cong \mathcal{O}_Z$, where $Z \subset \mathbb{P}^N$ has length 2. In either case, Y is a flat limit of X union two isolated points.*

Proof. If $K_p \cong \mathcal{O}_p \oplus \mathcal{O}_p$, apply Proposition 3.2. Since Y^1 is obtained from X by adding an embedded point of multiplicity one, it is a flat limit of X union an isolated point, hence Y is a flat limit of X union two isolated points.

Now suppose $K_p \ncong \mathcal{O}_p \oplus \mathcal{O}_p$. Then the surjection $K_p \to K_p \otimes k(p)$ is not an isomorphism, thus $K_p \otimes \mathcal{O}_p$ is one-dimensional as an $\mathcal{O}_p = k(p)$ vector space. Therefore K_p is principal by Nakayama's lemma, so there is a surjection $\mathcal{O} \to K_p$ whose kernel is the ideal sheaf $\mathcal{I}_Z$ of a length-2 subscheme, which is contained in a unique line and has embedding dimension one. We apply Proposition 3.1 to see that Y is a flat limit of X union two isolated points. □

Remark 3.4. We give the local equations of the embedded point structures for cases (a) and (b) of Proposition 3.3 for $X \subset \mathbb{P}^N$:

(a) If $I_{X,p} = (f, g)$, then $I_{Y,p} = m_p \cdot I_{X,p}$.

(b) Replacing generators so that $\varphi(f) = 1$ and $\varphi(g) = 0$, we obtain $I_{Y,p} = (g, f \cdot I_Z)$, Z being the length-2 subscheme.

Example 3.5. In case (b) of Proposition 3.3, there is a unique subscheme $X \subset E \subset Y$ with an embedded point of multiplicity one, because the unique length-1 quotient of $\mathcal{O}_Z$ is $\mathcal{O}_p$, obtained by modding out by the maximal ideal. Using such subschemes E, we explain why it was necessary to prove case (b) by pulling away two points simultaneously. For example, let $X \subset \mathbb{A}^3$ have ideal $I_X = (x^2, y^2)$ and let $p = (0, 0, 0)$, where $[x, y, z]$ denotes the coordinates of $\mathbb{A}^3$. Add an embedded point to X at p using the map $I_X \to k$ by $x^2 \mapsto 1$, $y^2 \mapsto 0$ to obtain E with $I_E = (y^2, x^3, x^2y, x^2z)$ being 4-generated. By Proposition 2.2, one can add a second point at p to obtain Y with an embedded point of multiplicity two, which is not a flat limit of E union an isolated point.

Example 3.6. Proposition 3.3 may fail for local complete intersections of codimension greater than two. For example, suppose that $C \subset \mathbb{P}^4$ is the complete intersection with ideal $I_C = (x^2, y^2, z^2)$, where $[x, y, z, u, w]$ denotes the projective coordinates. Consider the double point structures D on C given by surjections $\phi : (x^2, y^2, z^2) \to K = \mathcal{O}_Z$, where Z is the double point with ideal $I_Z = (x, y, z, u^2)$. An arbitrary map $\phi : I_C \to S/I_Z$ is given by

$$\phi(x^2) = a + bu, \quad \phi(y^2) = c + du, \quad \phi(z^2) = e + fu,$$

where S is the coordinate ring of $\mathbb{P}^4$, $a, b, c, d, e, f \in k$, and any tuple (a, b, c, d, e, f) is possible because $I_C \subset I_Z$. The automorphisms of $K = \mathcal{O}_Z$ are given by multiplication by $A + Bu$ with $A \neq 0$. Thus the maps for which $\phi(x^2)$ generates K (that is, $a \neq 0$) are uniquely determined up to automorphisms of K by the quotients $\phi(y^2)/\phi(x^2) = (c + du)/(a + bu)$ and $\phi(z^2)/\phi(x^2) = (e + fu)/(a + bu)$. By Remark 1.8(a), these quantities uniquely determine the corresponding subschemes D. In other words, if we compose the map above by the automorphism of $\mathcal{O}_Z$ given by multiplication by $(a+bu)^{-1}=(a-bu)/a^2$, we may assume that $\phi(x^2)=1$ when the corresponding ideal of D is given by

$$\begin{aligned}(x^2(I_Z),\, y^2-(c+du)x^2,\, z^2-(e+fu)x^2)&\\ =(x^3, x^2y, x^2z, x^2w^2, y^2-(c+du)x^2, z^2-(e+fu)x^2)&\end{aligned}$$

and each tuple $(c,d,e,f)\in k^4$ yields a distinct subscheme D, so we obtain a four-dimensional family of such double point structures D.

Finally, the same argument applies to any double point structure D on C. Since there is a choice of any point $p\in C$ for the support of $K=\mathcal{O}_Z$ and the structure of Z is uniquely determined by a line through p (parametrized by a hyperplane $\mathbb{P}^3$), the family of such double point structures has dimension $1+3+4=8$. The general such structure cannot be a flat limit of C union two isolated points, for this family also has dimension eight.

Now we turn to the case of multiplicity three.

Proposition 3.7. *Let $X\subset\mathbb{P}^N$ be a local complete intersection of codimension two. Let Y be the subscheme obtained from X by an exact sequence*

$$0\to\mathcal{I}_Y\to\mathcal{I}_X\xrightarrow{\varphi}K\to 0,$$

where K is a length-3 sheaf supported at p. Then one of the following holds:

(a) *$K\cong\mathcal{O}_p\oplus\mathcal{O}_Z$, where $Z\subset\mathbb{P}^N$ is a double point on a line.*

(b) *$K\cong\mathcal{O}_Z$, where $Z\subset\mathbb{P}^N$ is a triple point on a line.*

(c) *$K\cong\mathcal{O}_Z$, where $Z\subset\mathbb{P}^N$ is a triple point on a smooth conic.*

(d) *$K\cong\mathcal{O}_Z$, where Z is contained in a plane H and $\mathcal{I}_{Z,H}=\mathcal{I}^2_{p,H}$.*

(e) *$K\cong\mathrm{Hom}_{\mathcal{O}_p}(\mathcal{O}_Z,\mathcal{O}_p)$ with Z as in case* (d).

In each case, Y is a flat limit of X union three isolated points.

Proof. If K is a direct summand, one summand is $\mathcal{O}_p$ and the other is $\mathcal{O}_p^2$ or $\mathcal{O}_Z$ for a double point Z. The former is not possible as a quotient of the locally 2-generated ideal $\mathcal{I}_C$, leading to case (a). If K is principal, then the surjection $\mathcal{O}\to K$ shows that $K\cong\mathcal{O}_Z$ for a length-3 subscheme supported at p. Since $h^0(\mathcal{O}_Z(1))=3$ and $h^0(\mathcal{O}(1))=N+1$, Z is a planar triple point. It is easy to classify planar triple points, leading to cases (b), (c), and (d). If K is not principal and not a direct summand, then it is 2-generated as a quotient of $\mathcal{I}_X$ via φ. The two generators have a common nonzero multiple, otherwise they would express K as a direct sum of two principal modules. The common nonzero multiple is therefore a generator of the dual $\mathrm{Hom}(\mathcal{O}_Z,\mathcal{O}_p)$, where $\mathcal{O}_Z$ must be one of cases (b), (c), or (d). However, the duals to cases (b) and (c) are principal and we are left with the dual of case (d), which is case (e).

That Y is a flat limit of X union three isolated points follows from Propositions 3.2 and 3.3 in case (a) and from Proposition 3.1 in cases (b) and (c). Cases (d) and (e) require new ideas.

In case (d) we have $K_p \cong \mathcal{O}_Z$, where $Z \subset H$ is the planar triple point supported at p of embedding dimension two. As in Proposition 3.1, X is contained in hypersurfaces with equations F and G of degree $m \gg 0$, giving a Koszul resolution (5), $\varphi : \mathcal{I}_X \to \mathcal{O}_Z$ lifts to $\tilde{\varphi} : \mathcal{O}(-m)^2 \to \mathcal{O}$ and there is a hypersurface S of degree $2m$, where $\tilde{\varphi} \circ \psi = 0$. The intersection $H \cap S$ contains an integral curve T passing through p. Our idea is to realize this triple embedded structure as the flat limit of a fixed double embedded structure at p union a single point varying in T.

Let $\tilde{T} \xrightarrow{f} T \subset H \cong \mathbb{P}^2$ be the normalization of T and choose a point $0 \in T$ such that $f(0) = p$. For $t \neq 0$, let $L_t \subset H$ be the line through p and $f(t)$. As $t \to 0$, $f(t) \to p$ and the line L_t has a unique limit L_0 (complete the associated map $T - \{0\} \to (\mathbb{P}^2)^\vee$ to obtain this limiting line). Choose local coordinates x, y on $\mathbb{A}^2 \subset \mathbb{P}^2$ so that $p = (0, 0)$ and $L_0 = \{x = 0\}$. The double point W at p with ideal (x^2, y) is a closed subscheme of Z (which has ideal (x^2, xy, y^2)). We now show that $\lim_{t \to 0} f(t) \cup W = Z$ in the Hilbert scheme of length-3 subschemes of H. If $f(t) = (a(t), b(t))$ in the local coordinates above, then the ideal for $W \cup f(t)$ is

$$I_t = (x^2, y) \cap (x - a(t), y - b(t)),$$

which contains the product of the two ideals. Since $\lim_{t \to 0}(a(t), b(t)) = (0, 0)$, the limiting ideal contains (x^3, xy, y^2). If the line L_t has equation $l_t = 0$, then $l_t^2 \in I_t$ and by choice of coordinates we have $\lim_{t \to 0} l_t^2 = x^2$, so the limiting ideal also contains x^2 and hence (x^2, xy, y^2), which defines Z.

The rest is analogous to Proposition 3.1. The composite map

$$\mathcal{O}(-2m) \xrightarrow{\psi} \mathcal{O}(-m)^2 \xrightarrow{\tilde{\varphi}} \mathcal{O} \to \mathcal{O}_S \to \mathcal{O}_{W \cup f(t)}$$

is zero, inducing a family of maps $\varphi_t : \mathcal{I}_X \to \mathcal{O}_{W \cup f(t)}$. Since φ_0 is onto, so is φ_t for t near 0. Therefore the kernels $\mathcal{I}_{Y_t}$ give a flat family whose limit is Y as $t \to 0$. Using our earlier results, for $t \neq 0$ each Y_t is a flat limit of X union isolated points, and therefore so is Y.

Finally consider case (e), where $K_p \cong \operatorname{Hom}_{\mathcal{O}_p}(\mathcal{O}_Z, \mathcal{O}_p)$ with $\mathcal{O}_p = k(p)$ the residue field at p and $Z \subset H \subset \mathbb{P}^N$, with H a plane and $\mathcal{I}_{Z,H} = \mathcal{I}_{p,H}^2$. Choose affine coordinates $x, y, z_1, \dots, z_{N-2}$ centered at p so that $I_H = (z_1, \dots, z_{N-2})$ and x, y are coordinates for $H \cong \mathbb{A}^2$. Let f, g be the restrictions of F, G to this affine patch, so that $I_X = (f, g)$. If $g - uf = h \in I_H$ for some unit u in the local ring, replace g with h as a generator for I_X. In this way we may assume $g \in I_H$ or (f, g) is not principal modulo I_H locally around p. Now $\mathcal{O}_Z$ is generated by $1, x, y$ as an $\mathcal{O}_p$-vector space, so K_p is generated by dual basis $x^*, y^*, 1^*$ as a vector space and by x^*, y^* as an $\mathcal{O}_H$-module with structure given by $xx^* = 1^* = yy^*$, $xy^* = yx^* = 0$.

Since φ is surjective, $\varphi(f) = ax^* + by^* + c1^*$ and $\varphi(g) = dx^* + ey^* + f1^*$ are also module generators for K_p, and in particular $ae - bd \neq 0$. Now consider the new coordinates $X = ay - bx$, $Y = ex - dy$ for H. With these one sees that $\mathrm{Ann}(\varphi(f)) = (I_H, X)$, $\mathrm{Ann}(\varphi(g)) = (I_H, Y)$, and $Y\varphi(f) = (ae - bd)1^* = X\varphi(g)$. It follows that $I_Y = (I_H(f, g), Xf, Yg, Yf - Xg)$. So by replacing the coordinates, we can present the ideal of Y as

$$I_Y = (I_H(f, g), xf, yg, yf - xg).$$

We will directly deform this ideal to obtain the result. The locus

$$S = \{(A, B, C, D) : \\ f(A, B) = 0, g(C, D) = 0, (B - D)g(C, B) - (C - A)f(C, B) = 0\}$$

contains $(0, 0, 0, 0)$ and each component has dimension ≥ 1; hence S contains an integral curve T through the origin. Let $\sigma : T \to H \times H \cong \mathbb{A}^4$ be the inclusion with coordinate functions $\sigma(t) = (a(t), b(t), c(t), d(t))$ and $0 \in T$ chosen so that $\sigma(0) = (0, 0, 0, 0)$. We claim that T can be chosen with $(a(t), b(t)) \neq (c(t), d(t))$. This is clear if $g \in I_H$, for then the second equation $g(C, D) = 0$ puts no restriction on C and D, and S is defined by only two equations: on a surface there are many integral curves T through the origin. The other possibility by our assumption is that $g \neq uf$ modulo I_H for any invertible u in an affine neighborhood of the origin. Here the restrictions of f and g to H have a greatest common divisor h so that $f = hf_1$ and $g = hg_1$ with f_1 and g_1 vanishing at the origin and relatively prime modulo I_H locally around the origin. If we look at the sublocus of S defined as above with f_1 and g_1 in place of f and g, the condition of the claim holds and we obtain the desired integral curve T.

Now consider the family of ideals

$$I_t = \big(I_H(f, g),\ (x - c(t))f,\ (y - b(t))g,\ (y - d(t))f - (x - a(t))g\big).$$

We claim that the ideal I_t scheme-theoretically cuts out exactly X and the three points $(a(t), b(t))$, $(c(t), b(t))$, and $(c(t), d(t))$ (which may be isolated or embedded, two may coincide if $a(t) = c(t)$ or $b(t) = d(t)$) for generic t near 0.

The claim holds away from H via the generators $I_H(f, g)$. At points $(x, y) \in H$ away from (a, b), (c, b), and (c, d) (we suppress the variable t) the claim also holds. If $x \neq c$, then $x - c$ is a unit, $f \in I_t$ and there are two cases: if $x = a$, then $y \neq b$, hence $y - b$ is a unit and $g \in I_t$; otherwise $x \neq a$ and the last equation shows that $g \in I_t$. If $x = c$, then $y \neq b, d$, so $g \in I_t$ and $f \in I_t$ by the last equation.

Finally we consider $(x, y) \in \{(a, b), (c, b), (c, d)\}$. The claim is easily checked if these points are distinct ($a \neq c$ and $b \neq d$) by checking that length $I_X/I_t = 1$. For example, at $(x, y) = (a, b)$ we have $x \neq c$ so $f \in I_t$, when I_X/I_t is generated by g alone, and since $I_H g, (y-b)g, (x-a)g \in I_t$, we have $I_X/I_t \cong k$. The other points $(x, y) = (c, b), (c, d)$ are similar. In the degenerate case $a = c$, we need to show that length $I_X/I_t = 2$ at $(x, y) = (a, b) = (c, b)$. Here $y \neq d$ so $u = (y-d)$ is a unit and $uf - (x-a)g \in I_t$, showing that I_X/I_t is generated by g. Further I_t contains $I_H g$, $(y-b)g$, and $(x-c)^2 g$ (use $(x-c)f$ and $uf - (x-c)g$), so the quotient has length 2. The other degenerate case $b = d$ can be verified similarly. This proves the claim.

With the claim, the ideal I_t cuts out X and three other points (possibly embedded in X, but not all supported at the same point). Using our earlier results, these schemes are flat limits of X and isolated points. Since $\lim_{t\to 0}(a(t), b(t), c(t), d(t)) = (0, 0, 0, 0)$ by construction, we also have $\lim_{t\to 0} I_t = I_Y$, and we conclude. □

Remark 3.8. For $I_{X,p} = (f, g)$ locally at p in Proposition 3.7, we write local equations for the embedded point structure Y according to the various cases:

(a) If $K_p = \mathbb{O}_p \oplus \mathbb{O}_Z$ and $\varphi(f) = (1, 0)$, $\varphi(g) = (0, 1)$, then $I_{Y,p} = (f m_p, g I_Z)$ with $f \in I_Z$.

(b) If $K_p = \mathbb{O}_Z$ and $\varphi(f) = 1$, $\varphi(g) = 0$, then $I_{Y,p} = (f I_Z, g)$ with $g \in I_Z$.

(c) Similarly we have $I_{Y,p} = (f I_Z, g)$ with $g \in I_Z$.

(d) Again we have $I_{Y,p} = (f I_Z, g)$ with $g \in I_Z$.

(e) This is the most interesting structure. As shown in the proof, $I_{Y,p} = (xf - yg, yf, zf, xg, zg)$ for suitable coordinates x, y, z.

Putting these results together, we obtain our main theorem.

Theorem 3.9. *Let $X \subset \mathbb{P}^N$ be a local complete intersection of codimension two. If Y is obtained from X by adding embedded points of multiplicity at most three, then Y is a flat limit of X union isolated points.*

Proof. Suppose the embedded points are supported at $p_1, \dots, p_r$ with respective multiplicities $m_1, \dots, m_r \le 3$. If Y_1 is the scheme which is isomorphic to Y near p_1 and equal to X away from p_1, it follows from Theorem 2.3 and Propositions 3.3 and 3.7 that Y_1 is in the Hilbert scheme closure of the family consisting of X union m_1 isolated points. Similarly if Y_2 is locally isomorphic to Y near p_2 and equal to X away from p_2, Y_2 is in the closure of the family of X union m_2 points. It follows that $Y_1 \cup Y_2$ is in the closure of the family of Y_1 union m_2 isolated points, the fixed embedded point at p_1 not affecting the relevant deformations. Since Y_1 union m_2 isolated points is in the closure of the family of X union $m_1 + m_2$ isolated points, we see that $Y_1 \cup Y_2$ is in this closure as well. Adding one point at a time in this way we find that Y is in the closure of the family of X union $m_1 + \cdots + m_r$ isolated points. □

Example 3.10. Here we show that it is not always possible to detach embedded points of multiplicity four. For linearly independent variables x, y, z, w, consider the $R = k[x, y, z, w]$-module K given by

$$K = \langle a, b\rangle/(za, wa, xb, yb, xa - zb, ya - wb).$$

In changing the choice of vector space basis for the linear forms x, y, z, w, we obtain a family of such modules on which the group GL(4) acts. It's easily checked that the R-module automorphisms of any fixed K have dimension five (one can write them down explicitly). For another K' determined by basis x', y', z', w' and an isomorphism $\psi : K \to K'$, the map ψ uniquely determines x', y', z', w' in terms of x, y, z, w, because the relations yield 16 equations in 16 unknowns. One can check that the family of candidate isomorphisms ψ has dimension 12 and a five-dimensional subspace corresponds to the identity coordinate change. Hence, we find that the isomorphism classes of such modules K has dimension $16 - (12 - 5) = 9$.

Now for $X \subset \mathbb{P}^N$ given by $I_X = (x^2, y^2)$, the family of embedded point structures on X given by such K has dimension $5N - 6$. The choice of the support of K has dimension equal to $\dim X = N - 2$; choosing the linear subspace $\langle x, y, z, w\rangle$ at p is given by $G(4, N)$ of dimension $4N - 16$; choosing the isomorphism class of K has dimension nine (see above); the choice of map $\varphi : \mathcal{I}_X \to K$ depends on eight parameters, but the resulting family of ideals $\mathcal{I}_Y$ given by the kernels has dimension three because the automorphisms of K have dimension five. All in all, the family has dimension $(N-2)+(4N-16)+9+(8-5) = 5N-6$. For $N \geq 6$, we have $5N - 6 \geq 4N$, so the family cannot lie in the $4N$-dimensional closure of those obtained by unions of X with isolated points.

4. Applications to Hilbert schemes

In the previous section we proved various results about when a local complete intersection X with embedded points are flat limits of X union isolated points. In this section we apply these results to describe the irreducible components of certain Hilbert schemes. In view of Proposition 2.4, we deduce the following:

Theorem 4.1. *Let $p(z)$ be the Hilbert polynomial of a degree-d hypersurface in $\mathbb{P}^N$. Then*:

(a) *The Hilbert schemes* $\mathrm{Hilb}^{p(z)+e}(\mathbb{P}^N)$ *are irreducible for* $0 \leq e \leq 7$.

(b) *The Hilbert scheme* $\mathrm{Hilb}^{p(z)+1}(\mathbb{P}^N)$ *is smooth, isomorphic to* $\mathrm{Hilb}^{p(z)}(\mathbb{P}^N) \times \mathbb{P}^N$.

Proof. It follows from Proposition 2.4 that any (multiple) embedded point can be detached from a hypersurface, and for $e \leq 7$ we also know that any subscheme of length $e \leq 7$ is a flat limit of reduced points [Mazzola 1980; Cartwright et al. 2009]. Therefore $\mathrm{Hilb}^{p(z)+e}(\mathbb{P}^N)$ is the closure of the open subset formed by a degree-d

hypersurface and e isolated points and $\mathrm{Hilb}^{p(z)+e}(\mathbb{P}^N)$ is irreducible of dimension $\binom{d+N}{d} - 1 + Ne$.

Now take $e = 1$. It is easily checked that the Hilbert scheme is smooth at points corresponding to a hypersurface and an isolated point. Write $[x_0, x_1, \dots, x_N]$ for the coordinates of $\mathbb{P}^N$. If $X \subset \mathbb{P}^N$ is a degree-d hypersurface and Y is obtained from X by adding an embedded point located at $x_1 = x_2 = \cdots = x_N = 0$, then the ideal of Y is simply $I_Y = (x_1, x_2, \dots, x_N) \cdot I_X$, so I_Y is generated in degree $d + 1$. Since the generator of $\mathcal{I}_X \to K$ is onto, $H^1(\mathcal{I}_Y(n)) = 0$ for $n \geq d$ and so the comparison theorem [Piene and Schlessinger 1985] applies (see also [Ellingsrud 1975; Kleppe 1979]). Now the argument of [Piene and Schlessinger 1985, Lemma 4, Case (iii)] goes through, which we include for self-containment: $H^0(\mathcal{N}_Y) = \mathrm{Hom}(I_Y, S/I_Y)_0$, where S is the coordinate ring of $\mathbb{P}^N$ and $\mathcal{N}_Y$ is the normal sheaf to Y. Given the dimension of $\mathrm{Hilb}^{p(z)+1}(\mathbb{P}^N)$, it suffices to prove that $\dim \mathrm{Hom}(I_Y, S/I_Y)_0 \leq \binom{d+N}{d} - 1 + N$. Setting $A = S/I_Y$ and $K = I_X/I_Y$, the S-module K has Koszul resolution of the form

$$0 \to S(-d-N) \to \cdots \to S(-d-2)^{N(N-1)/2} \to S(-d-1)^N \to S(-d) \to K \to 0.$$

Applying $\mathrm{Hom}(-, A)$ to this resolution shows $\mathrm{Hom}(K, A) = K(d)$ and $\mathrm{Ext}^1(K, A)$ is generated by the vectors $(f x_0)e_i$ with $1 \leq i \leq N$, where f is the defining equation of X. Applying $\mathrm{Hom}(-, A)$ to the short exact sequence $I_Y \to I_X \to K$ gives

$$0 \to \mathrm{Hom}(K, A) \to \mathrm{Hom}(I_X, A) \to \mathrm{Hom}(I_Y, A) \to \mathrm{Ext}^1(K, A) \to \cdots$$

but $\dim \mathrm{Hom}(K, A)_0 = \dim K(d)_0 = 1$ and $\mathrm{Hom}(I_X, A) \cong A(d)$, hence we have $\dim \mathrm{Hom}(I_X, A)_0 = \binom{d+N}{d}$. Since $\dim \mathrm{Ext}^1(K, A)_0 \leq N$ by the above, we conclude that the Hilbert scheme is smooth. The natural rational map $\mathrm{Hilb}^{p(z)}(\mathbb{P}^N) \times \mathbb{P}^N \to \mathrm{Hilb}^{p(z)+1}(\mathbb{P}^N)$ is actually a bijective morphism in view of the unique form of the ideal, and hence is an isomorphism by Zariski's main theorem. □

We are also interested in Hilbert schemes of space curves and obtain the following irreducibility result for one-dimensional subschemes of high genus. Recall that if C is a space curve of degree d, then $g = p_a(C) \leq \binom{d-1}{2}$ with equality for plane curves.

Theorem 4.2. *The Hilbert scheme* $\mathrm{Hilb}^{dz+1-g}(\mathbb{P}^3)$ *is irreducible for* (d, g) *satisfying* $d \geq 3$, $\binom{d-1}{2} - 4 < g \leq \binom{d-1}{2}$, *and* $g > \binom{d-2}{2}$, *with a general member consisting of a plane curve of degree d union isolated points.*

Proof. The Hilbert scheme is nonempty for all $g \leq \binom{d-1}{2}$ due to plane curves union isolated points. For $d \geq 3$, the genus of a nonplane curve satisfies $g \leq \binom{d-2}{2}$ [Hartshorne 1994], so if $[C] \in \mathrm{Hilb}^{dz+1-g}(\mathbb{P}^3)$ and $C_0 \subset C$ is the curve remaining after removing embedded or isolated points, then C_0 is planar, hence a complete intersection. Since C is obtained by adding at most three embedded or isolated points, it is a flat limit of those with isolated points by Propositions 2.2, 3.3, and 3.7, and we conclude that the corresponding Hilbert scheme is irreducible. □

When just one isolated or embedded point is added to a plane curve of degree d and genus $g=(d-1)(d-2)/2$, the resulting Hilbert scheme component is smooth:

Theorem 4.3. *For $g=(d-1)(d-2)/2$, the component $H_d \subset \mathrm{Hilb}^{dz+2-g}(\mathbb{P}^3)$ of the Hilbert scheme whose general member is a degree-d plane curve union an isolated point is smooth for all $d \geq 1$. Moreover, H_d is isomorphic to the blow-up of $\mathrm{Hilb}^{dz+1-g}(\mathbb{P}^3) \times \mathbb{P}^3$ along the incidence correspondence.*

Proof. For $d=2$ and 3, this was proved in [Chen et al. 2011] and [Piene and Schlessinger 1985], respectively, even though H_d is not the full Hilbert scheme in these cases. For $d=1$ and $d \geq 4$, H_d is the full Hilbert scheme, and it suffices to compute the global sections $H^0(\mathcal{N}_D)$ of the normal sheaf associated to a point $[D] \in H_d$; so let D be the union of a plane curve C and the point $p=(0,0,0,1)$. If $p \notin C$, smoothness follows from $\mathcal{N}_D \cong \mathcal{N}_C \oplus \mathcal{N}_p$. If $p \in C$ is an embedded point, write $I_C=(z,f)$ with $f \in (x,y)$ and $z=0$ the equation of the plane H containing C. Consider the exact sequence (1):

$$0 \to \mathcal{I}_D \to \mathcal{I}_C \xrightarrow{\varphi} \mathcal{O}_p \to 0.$$

If $\varphi(z)=0$, then $D \subset H$ and $h^0(\mathcal{N}_{D,H})=\binom{d+2}{2}+1$ from Theorem 4.1, so the exact sequence

$$0 \to \mathcal{N}_{D,H} \to \mathcal{N}_{D,\mathbb{P}^3} \to \mathcal{O}_D(1) \to 0$$

yields $h^0(\mathcal{N}_{D,\mathbb{P}^3}) \leq \binom{d+2}{2}+1+h^0(\mathcal{O}_D(1))$. If $d \geq 4$, then $h^0(\mathcal{O}_D(1))=4$ and we have $h^0(\mathcal{N}_{D,\mathbb{P}^3}) \leq \dim H_d$, so H_d is smooth at $[D]$. Similarly, $h^0(\mathcal{O}_D(1))=3$ if $d=1$ and we obtain $h^0(\mathcal{N}_D) \leq 7 = \dim H_1$.

Now suppose that $\varphi(z) \neq 0$ and $d \geq 4$, since the case $d=1$ is straightforward. Write $[x,y,z,w]$ for the coordinates of $\mathbb{P}^3$ and S for the coordinate ring. The exact sequence (1) shows that $h^1(\mathcal{I}_D(n))=0$ for all $n>0$; hence the map $(S/I_D)_n \to H^0(\mathcal{O}_D(n))$ is an isomorphism for all $n>0$. It follows that the comparison theorem [Piene and Schlessinger 1985] (see also [Ellingsrud 1975; Kleppe 1979]) applies to D so that $H^0(\mathcal{N}_D) \cong \mathrm{Hom}(I_D, S/I_D)_0$. Since $\varphi(f)=\lambda\varphi(zw^{d-1})$ for some $\lambda \in k$, $\varphi(f-\lambda zw^{d-1})=0$ and we may write $I_D=(xz,yz,z^2,f-\lambda zw^{d-1})$. For smoothness at $[D]$, it suffices to show this when $\lambda=0$, because the members of the family parametrized by λ are projectively equivalent for $\lambda \neq 0$. Thus we may assume $I_D=(xz,yz,z^2,f)$ with $f \in (x,y)$ and write $f=xg+yh$ for $g,h \in S_{d-1}$.

Now consider $\rho \in \mathrm{Hom}(I_D, S/I_D)_0$. Observe that a basis for $(S/I_D)_2$ consists of $\{x^2,xy,y^2,zw,w^2,wx,wy\}$ and there is a similar basis for $(S/I_D)_3$ consisting of eleven monomials because $\deg f>3$. In terms of these bases, the Koszul relations

$$z\rho(xz)=x\rho(z^2), \quad z\rho(xy)=y\rho(z^2), \quad x\rho(yz)=y\rho(xz)$$

require that

$$\rho(z^2) = a_1 wz, \quad \rho(xz) = a_2 wz + a_3 xz + a_4 x^2 + a_5 xy,$$
$$\rho(yz) = a_6 wz + a_3 zy + a_4 xy + a_5 y^2.$$

Modulo (xz, yz, z^2) we may write

$$\rho(f) = a_7 z w^{d-1} + G,$$

with $G \in k[x, y, w]_d$. Now $g\rho(zx) + h\rho(zy) = z\rho(f) = zG$ modulo I_D gives a linear relation between the coefficient of w^d in G and a_2 and a_6. Since $\rho(f)$ is only determined modulo f, there are $\binom{d+2}{2} - 2$ degrees of freedom in choosing $\rho(f)$, so that

$$\dim \operatorname{Hom}(I_D, S/I_D)_0 \le 7 + \binom{d+2}{2} - 2 = \binom{d+2}{2} + 5 = \dim H_d.$$

The second statement follows from Proposition 2.1 by varying C. Indeed, the rational map $M = \operatorname{Hilb}^{dz+1-g}(\mathbb{P}^3) \times \mathbb{P}^3 \dashrightarrow H_d \subset \operatorname{Hilb}^{dz+2-g}(\mathbb{P}^3)$ given by $(C, p) \mapsto C \cup p$ has indeterminacy locus equal to the incidence correspondence $\Delta = \{(C, p) : p \in C\}$. For fixed $[C] \in \operatorname{Hilb}^{dz+1-g}(\mathbb{P}^3)$, the fiber is isomorphic to $\mathbb{P}^3$ and via this isomorphism the intersection with Δ is identified with $C \subset \mathbb{P}^3$. Thus when Δ is blown up, the fiber over C is identified with $\widetilde{\mathbb{P}}^3(C)$, which according to Proposition 2.1 is in bijective correspondence with $V \subset H_d$ (using the notation in Proposition 2.1). It follows that after blowing up the indeterminacy locus $\Delta \subset M$ we obtain a bijective morphism $\widetilde{M}(\Delta) \to H_d$, which is an isomorphism by Zariski's main theorem. □

Remark 4.4. One can verify by similar tangent space calculations that the Hilbert scheme of plane curves with two isolated or embedded points is singular exactly along the plane curves with the double embedded points of type (a) in Proposition 3.3. It is interesting that the Hilbert scheme is smooth along curves with the double embedded points of type (b).

Example 4.5. The only locally Cohen–Macaulay curve of degree 1 is a line. By our results, any curve obtained from a line L by adding ≤ 3 embedded points is a flat limit of L union the right number of isolated points. It follows that $\operatorname{Hilb}^{z+1-g}(\mathbb{P}^3)$ is irreducible of dimension $4-3g$ for $-3 \le g \le 0$. On the other hand, it is reducible for $g \ll 0$ because the Hilbert scheme of sufficiently many points in $\mathbb{P}^3$ is not irreducible [Iarrobino 1972].

Example 4.6. For one-dimensional subschemes of degree 2 and high genus the irreducible components of $\operatorname{Hilb}^{2z+1-g}(\mathbb{P}^3)$ are as follows:

(a) If $g = 0$, the Hilbert scheme is irreducible, consisting of plane curves.

(b) If $g = -1$, there are two irreducible components. The first component H_1 has general member a pair of skew lines and has dimension eight. The second component H_2 has general member a plane conic union an isolated point and has dimension 11. There are also plane curves with embedded points, but these lie in H_2 by Proposition 2.2. Both components H_1 and H_2 are smooth [Chen et al. 2011].

(c) Similarly if $g = -2$, there are three irreducible components. There is the family H_1 of double lines of genus $g = -2$ with no embedded points of dimension nine, the family H_2 of two skew lines union an isolated point of dimension 11, and the family H_3 of conics union two isolated points of dimension 14. Because all the underlying locally Cohen–Macaulay curves in question are local complete intersections, we know from Proposition 2.2 and Proposition 3.3 that we have not missed any possibilities.

(d) For $g = -3$ we can write down four irreducible components following the same pattern as above and our results show that we have not missed any irreducible components. However when $g = -4$ we cannot be sure that there is not an irreducible component whose general member consists of a plane curve with some horrible quadruple point.

Example 4.7. For one-dimensional subschemes of degree 3 and high genus, we can make similar lists of the irreducible components of $\mathrm{Hilb}^{3z+1-g}(\mathbb{P}^3)$:

(a) If $g = 1$, the Hilbert scheme is irreducible and consists of plane curves.

(b) If $g = 0$, the Hilbert scheme has two irreducible components. The family H_1 has general member a twisted cubic and has dimension 12. The family H_2 has general member a plane cubic union an isolated point and has dimension 15. This example has been well-studied in [Piene and Schlessinger 1985].

(c) If $g = -1$, there are three irreducible components. The component H_1 whose general member is a line and a disjoint conic has dimension 12. The component H_2 whose general member is a twisted cubic union an isolated point has dimension 15. The component H_3 whose general member is a plane cubic union two isolated points has dimension 18. To see that these are all, we need to show that degenerations of a twisted cubic curve union an embedded point cannot form an irreducible component of their own, something which is not clear in view of Example 2.6(a). However all ACM curves of degree 3 and genus 0 have resolution

$$0 \to \mathcal{O}(-3)^2 \to \mathcal{O}(-2)^3 \to \mathcal{I}_C \to 0$$

[Ellingsrud 1975, Example 1] and we can apply Proposition 2.7.

Example 4.8. Consider the Hilbert schemes $\mathrm{Hilb}^{4z+1-g}(\mathbb{P}^3)$:

(a) If $g = 3$ or 2, the Hilbert scheme is irreducible by Theorem 4.2.

(b) If $g = 1$, the Hilbert scheme has two irreducible components. One component H_1 has general member a plane quartic union two isolated points and has dimension 23. Any subschemes not parametrized by H_1 have no isolated or embedded points (any nonplanar locally Cohen–Macaulay curve satisfies the genus bound $g \leq (d-2)(d-3)/2$ [Martin-Deschamps and Perrin 1993]), so we are looking at the Hilbert scheme $H_{4,1}$ of locally Cohen–Macaulay curves, which we described in Example 2.8.

This brings us to the last example, which might be known to experts, though we have not seen a rigorous proof in the literature.

Theorem 4.9. *The Hilbert scheme* $\mathrm{Hilb}^{4z+1}(\mathbb{P}^3)$ *has four irreducible components*:

H_1: *The closure of the family of rational quartic curves has dimension* 16.

H_2: *The family whose general member is a disjoint union of a plane cubic and a line has dimension* 16.

H_3: *The family whose general member is a disjoint union of an elliptic quartic curve and a point has dimension* 19.

H_4: *The family whose general member is a disjoint union of a plane quartic curve and three distinct points has dimension* 26.

Proof. The dimension counts are standard, so we only need to show that every subscheme parametrized by $\mathrm{Hilb}^{4z+1}(\mathbb{P}^3)$ is contained in one of these families and no family is contained in another. The second part is easy: the family H_4 has the largest dimension, but none of the others lie in its closure due to the three isolated or embedded points. Similarly H_3 has larger dimension than H_1 and H_2, but H_1 and H_2 are not in its closure due to the isolated or embedded point. Since families H_1 and H_2 have the same dimension, neither lies in the closure of the other.

To complete the proof, we show that each $[C] \in \mathrm{Hilb}^{4z+1}(\mathbb{P}^3)$ lies in one of the families H_i listed above. Fixing such $C \subset \mathbb{P}^3$, let $C_0 \subset C$ be the purely one-dimensional part. There is no such curve of genus $g = 2$ [Hartshorne 1994], leaving three cases. If $g(C_0) = 0$, then $C = C_0$ is locally Cohen–Macaulay, and it is known that the Hilbert scheme $H_{4,0}$ of locally Cohen–Macaulay curves has two irreducible components, described in H_1 and H_2 above [Nollet and Schlesinger 2003]. If $g(C_0) = 3$, then C_0 is a plane quartic and hence a complete intersection. It follows from Propositions 2.2, 3.3, and 3.7 that C is a flat limit of subschemes which are plane quartics union three isolated points, so $[C] \in H_4$. If $g(C_0) = 1$, then $[C] \in H_3$ by Example 2.8. $\square$

It would be interesting to describe more precisely the intersection of the components H_1 through H_4 in $\mathrm{Hilb}^{4z+1}(\mathbb{P}^3)$ as done in [Piene and Schlessinger 1985] for $\mathrm{Hilb}^{3z+1}(\mathbb{P}^3)$, though this will require a classification of all curves of degree 4 and genus 0 up to projective equivalence. It would also be interesting to determine the birational geometry of the component H_1, as done in [Chen 2008] for $\mathrm{Hilb}^{3z+1}(P^3)$.

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