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We define a proper moduli stack classifying covers of curves of prime degree p. The objects of this stack are torsors $Y \to \mathscr{X}$ under a finite flat $\mathscr{X}$-group scheme, with $\mathscr{X}$ a twisted curve and Y a stable curve. We also discuss embeddings of finite flat group schemes of order p into affine smooth 1-dimensional group schemes.

1. Introduction

Fix a prime number p. The study of families of Galois p-cyclic covers of curves is well understood in characteristic 0, where there is a nice smooth proper stack classifying (generically étale) covers of stable curves, with a dense open substack composed of covers of smooth curves. The reduction of this stack at a prime $\ell \neq p$ is also well understood, but the question of the reduction at p is notably much harder. For the classical modular curves, namely the unramified genus-1 case, there has been in the last years renewed intense research on this topic; see, for example, [Edixhoven 1990; Bouw and Wewers 2004; McMurdy and Coleman 2010].

The aim of the present paper is to consider the case of arbitrary genus. More precisely, we define a complete moduli stack of degree-p covers $Y \to \mathscr{X}$, with Y a stable curve which is a $\mathscr{G}$-torsor over $\mathscr{X}$, for a suitable group scheme $\mathscr{G}/\mathscr{X}$. The curve $\mathscr{X}$ is a twisted curve in the sense of [Abramovich and Vistoli 2002;

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Abramovich et al. 2011] but in general not stable. This follows the same general approach as the characteristic-0 paper [Abramovich et al. 2003], but diverges from that of [Abramovich et al. 2011], where the curve $\mathscr{X}$ is stable, the group scheme $\mathscr{G}$ is assumed linearly reductive, but Y is in general much more singular. Here the approach is based on [Raynaud 1999, Proposition 1.2.1] of Raynaud, and the more general notion of *effective model* of a group-scheme action from [Romagny 2011]. The general strategy was outlined in [Abramovich 2012] in a somewhat special case.

The ideal goal is a moduli space where, on the one hand, the object parametrized are concrete and with minimal singularities — ideally nodes, and on the other hand the singularities of the moduli space are well understood. This would allow one to easily describe objects in characteristic p and to identify their liftings in characteristic 0. In this paper we have not given a description of the singularities of the moduli space, so we fall short of this goal.

1.1. ***Rigidified group schemes.*** The group scheme $\mathscr{G}$ in our covers comes with a supplementary structure which we call a *generator*. Before we define this notion, let us briefly recall from [Katz and Mazur 1985, §1.8] the concept of a *full set of sections*. Let $Z \to S$ be a finite locally free morphism of schemes of degree N. Then for all affine S-schemes $\operatorname{Spec}(R)$, the R-algebra $\Gamma(Z_R, \mathbb{O}_{Z_R})$ is locally free of rank N and has a canonical norm mapping. We say that a set of N sections $x_1, \dots, x_N \in Z(S)$ is a *full set of sections* if and only if for any affine S-scheme $\operatorname{Spec}(R)$ and any $f \in \Gamma(Z_R, \mathbb{O}_{Z_R})$, the norm of f is equal to the product $f(x_1) \dots f(x_N)$.

Definition 1.2. Let $G \to S$ be a finite locally free group scheme of order p. A *generator* is a morphism of S-group schemes $\gamma : (\mathbb{Z}/p\mathbb{Z})_S \to G$ such that the sections $x_i = \gamma(i)$, $0 \le i \le p-1$, are a full set of sections. A *rigidified group scheme* is a group scheme of order p with a generator.

The notion of generator is easily described in terms of the Tate–Oort classification of group schemes of order p. This is explained and complemented in Appendix A.

Remark 1.3. One can define the stack of rigidified group schemes a bit more directly: consider the Artin stack $\mathfrak{G}\mathfrak{S}_p$ of group schemes of order p, and let $\mathscr{G}^u \to \mathfrak{G}\mathfrak{S}_p$ be the universal group-scheme - an object of $\mathscr{G}^u$ over a scheme S consists of a group-scheme $\mathscr{G} \to S$ with a section $S \to \mathscr{G}$. It has a unique nonzero point over $\mathbb{Q}$ corresponding to $\mathbb{Z}/p\mathbb{Z}$ with the section 1. The stack of rigidified group schemes is canonically isomorphic to the closure of this point.

Of course describing a stack as a closure of a substack is not ideal from the moduli point of view, and we find the definition using Katz–Mazur generators more satisfying.

1.4. ***Stable p-torsors.*** Fix a prime number p and integers $g, h, n \geq 0$ with $2g - 2 + n > 0$.

Definition 1.5. A *stable n-marked p-torsor* of genus g (over some base scheme S) is a triple

$$(\mathscr{X}, \mathscr{G}, Y),$$

where

(1) $(\mathscr{X}, \{\Sigma_i\}_{i=1}^n)$ is an n-marked twisted curve of genus h,

(2) $(Y, \{P_i\}_{i=1}^n)$ is a nodal curve of genus g with étale marking divisors $P_i \to S$, which is stable in the sense of Deligne, Mumford, and Knudsen,

(3) $\mathscr{G} \to \mathscr{X}$ is a rigidified group scheme of order p,

(4) $Y \to \mathscr{X}$ is a $\mathscr{G}$-torsor and $P_i = \Sigma_i \times_{\mathscr{X}} Y$ for all i.

Note that as usual the markings Σ_i (resp. P_i) are required to lie in the smooth locus of $\mathscr{X}$ (resp. Y). They split into two groups. In the first group Σ_i is twisted and $[P_i : S] = 1$, while in the second group Σ_i is a section and $[P_i : S] = p$. The number m of twisted markings is determined by $(2g-2) = p(2h-2) + m(p-1)$ and it is equivalent to fix h or m.

The notion of stable marked p-torsor makes sense over an arbitrary base scheme S. Given stable n-marked p-torsors $(\mathscr{X}, \mathscr{G}, Y)$ over S and $(\mathscr{X}', \mathscr{G}', Y')$ over S', one defines as usual a morphism $(\mathscr{X}, \mathscr{G}, Y) \to (\mathscr{X}', \mathscr{G}', Y')$ over $S \to S'$ as a fiber diagram. This defines a category fibered over $\operatorname{Spec} \mathbb{Z}$ that we denote $ST_{p,g,h,n}$.

Our main result is:

Theorem 1.6. *The category $ST_{p,g,h,n}/ \operatorname{Spec} \mathbb{Z}$ is a proper Deligne–Mumford stack with finite diagonal.*

Notice that $ST_{p,g,h,n}$ contains an open substack of étale $\mathbb{Z}/p\mathbb{Z}$-covers. Identifying the closure of this open locus remains an interesting question.

1.7. ***Organization.*** Section 2 is devoted to Proposition 2.1, in particular showing the algebraicity of $ST_{p,g,h,n}$. Section 3 completes the proof of Theorem 1.6 by showing properness. We give simple examples in Section 4. In Appendix A we discuss embeddings of group schemes of order p into smooth group schemes. In Appendix B we recall some facts about the Weil restriction of closed subschemes, and state the representability result in a form useful for us.

2. The stack $ST_{p,g,h,n}$

In this section, we review some basic facts on twisted curves and then we show:

Proposition 2.1. *The category $ST_{p,g,h,n}/ \operatorname{Spec} \mathbb{Z}$ is an algebraic stack of finite type over $\mathbb{Z}$.*

2.2. ***Twisted curves and log twisted curves.*** We review some material from Olsson's treatment in [Abramovich et al. 2011, Appendix A], with some attention to properness of the procedure of "log twisting".

Recall that a *twisted curve* over a scheme S is a tame Artin stack $\mathscr{C} \to S$ (we refer to [Abramovich et al. 2008, Definition 3.1] for this notion) with a collection of gerbes $\Sigma_i \subset \mathscr{C}$ satisfying the following conditions:

(1) The coarse moduli space C of $\mathscr{C}$ is a prestable curve over S, and the images $\bar{\Sigma}_i$ of Σ_i in C are the images of disjoint sections $\sigma_i : S \to C$ of $C \to S$ landing in the smooth locus.

(2) Étale locally on S there are positive integers r_i such that, on a neighborhood of Σ_i we can identify $\mathscr{C}$ with the root stack $C(\sqrt[r_i]{\bar{\Sigma}_i})$.

(3) Near a node z of C write $C^{\mathrm{sh}} = \operatorname{Spec}(\mathbb{O}_S^{\mathrm{sh}}[x, y]/(xy - t))^{\mathrm{sh}}$. Then there exists a positive integer a_z and an element $s \in \mathbb{O}_S^{\mathrm{sh}}$ such that $s^{a_z} = t$ and
$$\mathscr{C}^{\mathrm{sh}} = [\operatorname{Spec} \mathbb{O}_S^{\mathrm{sh}}[u, v]/(uv - s))^{\mathrm{sh}}/\mu_{a_z}],$$
where μ_{a_z} acts via $(u, v) \mapsto (\zeta u, \zeta^{-1} v)$ and where $x = u^{a_z}$ and $y = v^{a_z}$.

The *index* of a geometric point z on a twisted curve is a measure of its automorphisms: it is the integer r_i for a twisted marking or the integer a_z for a twisted node.

The purpose of [Abramovich et al. 2011, Appendix A] was to show that twisted curves form an Artin stack which is locally of finite type over $\mathbb{Z}$. There are two steps involved.

The introduction of the stack structure over the markings is a straightforward step: the stack $\mathfrak{M}^{tw}_{g,n}$ of twisted curves with genus G and n markings is the infinite disjoint union $\mathfrak{M}^{tw}_{g,n} = \sqcup \mathfrak{M}^{r}_{g,n}$, where r runs over the possible *marking indices*, namely vectors of positive integers $r = (r_1, \dots, r_n)$, and the stacks $\mathfrak{M}^{r}_{g,n}$ are all isomorphic to each other - the universal family over $\mathfrak{M}^{r}_{g,n}$ is obtained form that over $\mathfrak{M}^{(1,\dots,1)}_{g,n}$ by taking the r_i-th root of $\bar{\Sigma}_i$.

The more subtle point is the introduction of twisting at nodes. Olsson achieves this using the canonical log structure of prestable curves, and provides an equivalence between twisted curves with $r = (1, \dots, 1)$ and log-twisted curves. A *log twisted curve* over a scheme S is the data of a prestable curve C/S along with a simple extension $\mathcal{M}^S_{C/S} \hookrightarrow \mathcal{N}$, see [Abramovich et al. 2011, Definition A.3]. Here $\mathcal{M}^S_{C/S}$ is F. Kato's canonical locally free log structure of the base S of the family of prestable curves C/S, and a *simple extension* is an injective morphism $\mathcal{M}^S_{C/S} \hookrightarrow \mathcal{N}$ of locally free log structures of equal rank where an irreducible element is sent to a multiple of an irreducible element up to units. See [Abramovich et al. 2011, Definition A.1].

We now describe an aspect of this equivalence which is relevant for our main results. Consider a family of prestable curves C/S and denote by $\iota : \operatorname{Sing} C/S \to C$

the embedding of the locus where $\pi : C \to S$ fails to be smooth. A *node function* is a section a of $\pi_* \iota_* \mathbb{N}_{\mathrm{Sing}\, C/S}$. In other words it gives a positive integer a_z for each singular point z of C/S in a continuous manner. Given a morphism $T \to S$, we say that a twisted curve $\mathscr{C}/T$ with coarse moduli space C_T is *a-twisted* over C/S if the index of a node of $\mathscr{C}$ over a node z of C is precisely a_z.

Proposition 2.3. *Fix a family of prestable curves C/S of genus g with n markings over a noetherian scheme S. Further fix marking indices $r = (r_1, \ldots, r_n)$ and a node function a. Then the category of a-twisted curves over C/S with marking indices given by r is a proper and quasifinite tame stack over S.*

Proof. The problem is local on S, and further it is stable under base change in S. So it is enough to prove this when S is a versal deformation space of a prestable curve C_s of genus g with n markings, over a closed geometric point $s \in S$, in such a way that we have a chart $\mathbb{N}^k \to \mathcal{M}^S_{C/S}$ of the log structure, where k is the number of nodes of C_s. The image of the i-th generator of $\mathbb{N}^k$ in $\mathbb{O}_S$ is the defining equation of the smooth divisor D_i where the i-th node persists. Now consider an a-twisted curve over $\phi : T \to S$, corresponding to a simple extension $\phi^* \mathcal{M}^S_{C/S} \to \mathcal{N}$ where the image of the i-th generator m_i becomes an a_i-multiple up to units. This precisely means that $\mathbb{O}^*_{C_T} m_i$, the principal bundle associated to $\mathbb{O}_S(-D_i)$, is an a_i-th power. In other words, the stack of a-twisted curves over C/S is isomorphic to the stack

$$S(\sqrt[a_1]{D_1} \cdots \sqrt[a_n]{D_n}) \; = \; S(\sqrt[a_1]{D_1}) \times_S \cdots \times_S S(\sqrt[a_n]{D_n})$$

encoding a_i-th roots of $O_S(D_i)$. This is evidently a proper and quasifinite tame stack over S. □

We now turn to the indices of twisted points in a stable p-torsor.

Lemma 2.4. *Let $(\mathscr{X}, \mathscr{G}, Y)$ be a stable p-torsor. Then the index of a point $x \in \mathscr{X}$ divides p.*

Proof. Let r be the index of x and d the local degree of $Y \to \mathscr{X}$ at a point y above x. Since $Y \to \mathscr{X}$ is finite flat of degree p and $\mathscr{G}$ acts transitively on the fibers, then $d \mid p$. Let $f : \mathscr{X} \to X$ be the coarse moduli space of $\mathscr{X}$. In order to compute d, we pass to strict henselizations on S, X and Y at the relevant points. Thus S is the spectrum of a strictly henselian local ring $(R, \mathfrak{m})$, and we have two cases to consider.

If x is a smooth point,

- $X \simeq \operatorname{Spec} R[a]^{\mathrm{sh}}$,
- $Y \simeq \operatorname{Spec} R[s]^{\mathrm{sh}}$,
- $\mathscr{X} \simeq [D/\mu_r]$ with $D = \operatorname{Spec} R[u]^{\mathrm{sh}}$ and $\zeta \in \mu_r$ acting by $u \mapsto \zeta u$.

Consider the fibered product $E = Y \times_{\mathscr{X}} D$. The map $E \to Y$ is a μ_r-torsor of the form $E \simeq \operatorname{Spec} \mathbb{O}_Y[w]/(w^r - f)$ for some invertible function $f \in \mathbb{O}_Y^\times$, and $E \to D$ is a μ_r-equivariant map given by $u \mapsto \varphi w$ for some function φ on Y. Let $\tilde{x} : \operatorname{Spec} k \to D$ be a point mapping to x in $\mathscr{X}$, i.e., corresponding to $u = \mathfrak{m} = 0$, and let $\bar{\varphi}$, $\bar{f}$ be the restrictions of φ, f to $Y_{\tilde{x}}$. The preimage of $\tilde{x}$ under $E \to D$ is a finite k-scheme with algebra $k[s][w]/(\bar{\varphi}, w^r - \bar{f})$. We see that $d = r \dim_k k[s]/(\bar{\varphi})$ and hence the index r divides p.

If x is a singular point, there exist λ, μ, ν in $\mathfrak{m}$ such that

- $X \simeq \operatorname{Spec}(R[a, b]/(ab - \lambda))^{\text{sh}}$,
- $Y \simeq \operatorname{Spec}(R[s, t]/(st - \mu))^{\text{sh}}$,
- $\mathscr{X} \simeq [D/\mu_r]$, where $D = \operatorname{Spec}(R[u, v]/(uv - \nu))^{\text{sh}}$,

and $\zeta \in \mu_r$ acts by $u \mapsto \zeta u$ and $v \mapsto \zeta^{-1} v$. The scheme $E = Y \times_{\mathscr{X}} D$ is of the form $E \simeq \operatorname{Spec} \mathbb{O}_Y[w]/(w^r - f)$ for some invertible function $f \in \mathbb{O}_Y^\times$, and the map $E \to D$ is given by $u \mapsto \varphi w$, $v \mapsto \psi w^{-1}$ for some functions φ, ψ on Y satisfying $\varphi\psi = \nu$. Let $\tilde{x} : \operatorname{Spec} k \to D$ be a point mapping to x and let $\bar{\varphi}$, $\bar{\psi}$, $\bar{f}$ be the restrictions of φ, ψ, f to $Y_{\tilde{x}}$. The preimage of $\tilde{x}$ under $E \to D$ is a finite k-scheme with algebra $k[s, t][w]/(st, \bar{\varphi}, \bar{\psi}, w^r - \bar{f})$. We see that $d = r \dim_k k[s, t]/(st, \bar{\varphi}, \bar{\psi})$ and hence r divides p. □

Proof of Proposition 2.1. Let $\delta = (\delta_1, \dots, \delta_n)$ be the sequence of degrees of the markings P_i on the total space of stable p-torsors, with each δ_i equal to 1 or p. We build $ST_{p,g,h,n}$ from existing stacks: the stack $\overline{\mathcal{M}}_{g,\delta}$ of Deligne–Mumford–Knudsen stable marked curves (for the family of curves Y), the stack $\mathfrak{M}$ of twisted curves (for the family of marked twisted curves $\mathscr{X}$), and Hilbert schemes and Hom-stacks for construction of $Y \to \mathscr{X}$ and $\mathscr{G}$.

Bounding the twisted curves. We have an obvious forgetful functor $ST_{p,g,h,n} \to \overline{\mathcal{M}}_{g,\delta} \times \mathfrak{M}$. Note that the image of $ST_{p,g,h,n} \to \mathfrak{M}$ lies in an open substack $\mathfrak{M}'$ of finite type over $\mathbb{Z}$: the index of the twisted curve $\mathscr{X}$ divides p by Lemma 2.4, and its topological type is bounded by that of Y. The stack $\mathfrak{M}'$ parametrizing such twisted curves is of finite type over $\mathbb{Z}$ by [Abramovich et al. 2011, Corollary A.8].

Set $M_{Y,\mathscr{X}} = \overline{\mathcal{M}}_{g,\delta} \times \mathfrak{M}'$. This is an algebraic stack of finite type over $\mathbb{Z}$.

The map $Y \to \mathscr{X}$. Consider the universal family $Y \to M_{Y,\mathscr{X}}$ of stable curves of genus g and the universal family $\mathscr{X} \to M_{Y,\mathscr{X}}$ of twisted curves, with associated family of coarse curves $X \to M_{Y,\mathscr{X}}$. Since Hilbert schemes of fixed Hilbert polynomial are of finite type, there is an algebraic stack $\operatorname{Hom}^{\leq p}_{M_{Y,\mathscr{X}}}(Y, X)$, of finite type over $M_{Y,\mathscr{X}}$, parametrizing morphisms $Y_s \to X_s$ of degree $\leq p$ between the respective fibers. By [Abramovich et al. 2011, Corollary C.4] the stack $\operatorname{Hom}^{\leq p}_{M_{Y,\mathscr{X}}}(Y, \mathscr{X})$ corresponding to maps $Y_s \to \mathscr{X}_s$ with target the twisted curve is of finite type over $\operatorname{Hom}^{\leq p}_{M_{Y,\mathscr{X}}}(Y, X)$, hence over $M_{Y,\mathscr{X}}$. There is an open substack $M_{Y\to\mathscr{X}}$ parametrizing flat morphisms of degree precisely p. We have an obvious forgetful functor $ST_{p,g,h,n} \to M_{Y\to\mathscr{X}}$ lifting the functor $ST_{p,g,h,n} \to \overline{\mathcal{M}}_{g,\delta} \times \mathfrak{M}'$ above.

The rigidified group scheme $\mathscr{G}$. The scheme $Y_2 = Y \times_{\mathscr{X}} Y$ is flat of degree p over Y. Giving it the structure of a group scheme over Y with unit section equal to the diagonal $Y \to Y_2$ is tantamount to choosing structure Y-arrows $m : Y_2 \times_Y Y_2 \to Y_2$ and $i' : Y_2 \to Y_2$, which are parametrized by a Hom-scheme, and passing to the closed subscheme where these give a group-scheme structure (that this condition is closed follows from representability of the Weil restriction; see the discussion in Appendix B and in particular Corollary B.4). Giving a group scheme $\mathscr{G}$ over $\mathscr{X}$ with isomorphism $\mathscr{G} \times_{\mathscr{X}} Y \simeq Y_2$ is tantamount to giving descent data for Y_2 with its chosen group-scheme structure. This is again parametrized by a suitable Hom-scheme. Finally requiring that the projection $Y_2 \to Y$ correspond to an action of $\mathscr{G}$ on Y is a closed condition (again by Weil restriction, see Corollary B.4).

Passing to a suitable Hom-stack we can add a homomorphism $\mathbb{Z}/p\mathbb{Z} \to \mathscr{G}$, giving a section $\mathscr{X} \to \mathscr{G}$ (equivalently a morphism $\mathscr{X} \to \mathscr{G}^u$, see Remark 1.3). By [Katz and Mazur 1985, corollary 1.3.5], the locus of the base where this section is a generator is closed. Since $Y_2 \to Y$ and $Y \to \mathscr{X}$ are finite, all the necessary Hom-stacks are in fact of finite type.

The resulting stack is clearly isomorphic to $ST_{p,g,h,n}$. □

3. Properness

Since $ST_{p,g,h,n} \to \operatorname{Spec} \mathbb{Z}$ is of finite type, we need to prove the valuative criterion for properness.

We have the following situation:

(1) R is a discrete valuation ring with spectrum $S = \operatorname{Spec} R$, fraction field K with corresponding generic point $\eta = \operatorname{Spec} K$, and residue field κ with corresponding special point $s = \operatorname{Spec} \kappa$.

(2) $(\mathscr{X}_\eta, \mathscr{G}_\eta, Y_\eta)$ a stable marked p-torsor of genus g over η.

By an *extension* of $(\mathscr{X}_\eta, \mathscr{G}_\eta, Y_\eta)$ across s we mean

(1) a local extension $R \to R'$ with K'/K finite,

(2) a stable marked p-torsor $(\mathscr{X}', \mathscr{G}', Y')$ of genus g over $S' = \operatorname{Spec} R'$, and

(3) an isomorphism $(\mathscr{X}', \mathscr{G}', Y')'_\eta \simeq (\mathscr{X}_\eta, \mathscr{G}_\eta, Y_\eta) \times_\eta \eta'$.

Proposition 3.1. *An extension exists. When extension over S' exists, it is unique up to a unique isomorphism.*

Proof. We proceed in three steps.

Extension of Y_η. Since $\overline{\mathcal{M}}_{g,\delta}$ is proper, there is a stable marked curve Y' extending Y_η over some S', and this extension is unique up to a unique isomorphism. We replace S by S', and assume that there is Y over S with generic fiber Y_η.

Coarse extension of $\mathscr{X}_\eta$. By uniqueness, the action of $G = \mathbb{Z}/p\mathbb{Z}$ on Y_η induced by the map $G_{\mathscr{X}_\eta} \to \mathscr{G}_\eta$ extends to Y. There is a finite extension K'/K such that the intersection points of the orbits of geometric irreducible components of Y_η under the action of G are all K'-rational. We may and do replace S by the spectrum of the integral closure of R in K'. Let us call $Y_1, \dots, Y_m$ the orbits of irreducible components of Y and $\{y_{i,j}\}_{1\le i,j\le m}$ their intersections, which is a set of disjoint sections of Y. For each $i = 1, \dots, m$ we define a morphism $\pi_i : Y_i \to X_i$ as follows. If the action of G on Y_i is nontrivial we put $X_i := Y_i/G$ and π_i equal to the quotient morphism. If the action of G on Y_i is trivial, note that we must have $\mathrm{char}(K) = p$, since the map from Y_i to its image in $\mathscr{X}$ is a $\mathscr{G}$-torsor while $G_{\mathscr{X}} \to \mathscr{G}$ is an isomorphism in characteristic 0. Then we consider the Frobenius twist $X_i := Y_i^{(p)}$ and we define $\pi_i : Y_i \to X_i$ to be the relative Frobenius. Finally we let X be the scheme obtained by gluing the X_i along the sections $x_{i,j} = \pi_i(y_{i,j}) \in X_i$ and $x_{j,i} = \pi_j(y_{i,j}) \in X_j$. There are markings $\Sigma_i^X \subset X$ given by the closures in X of the generic markings $\Sigma_i^{X_\eta}$. It is clear that the morphisms π_i glue to a morphism $\pi : Y \to X$.

Extension of $\mathscr{X}_\eta$ and $Y_\eta \to \mathscr{X}_\eta$ along generic nodes and markings. In the following two lemmas we extend the stack structure of $\mathscr{X}_\eta$, and then the map $Y_\eta \to \mathscr{X}_\eta$, along the generic nodes and the markings:

Lemma 3.2. *There is a unique extension $\bar{\mathscr{X}}$ of the twisted curve $\mathscr{X}_\eta$ over X, such that $\bar{\mathscr{X}} \to X$ is an isomorphism away from the generic nodes and the markings.*

Proof. We follow [Abramovich et al. 2011, proof of Proposition 4.3]. First, let $\Sigma_{i,\eta}^{\mathscr{X}_\eta}$ be a marking on $\mathscr{X}_\eta$. There is an extension $\Sigma_i^X \subset X$. Let r be the index of $\mathscr{X}_\eta$ at $\Sigma_{i,\eta}^{\mathscr{X}_\eta}$. Then we define $\mathscr{X}$ to be the stack of r-th roots of Σ_i^X on X. This extension is unique by the separatedness of stacks of r-th roots.

Now let $x_\eta \in X_\eta$ be a node with index r and let $x \in X_s$ be its reduction. Locally in the étale topology, around x the curve X looks like the spectrum of $R[u, v]/(uv)$. Let B_u resp. B_v be the branches at x in X. The stacks of r-th roots of the divisor $u = 0$ in B_u and of the divisor $v = 0$ in B_v are isomorphic and glue to give a stack $\bar{\mathscr{X}}$. By definition of r we have $\bar{\mathscr{X}}_\eta \simeq \mathscr{X}_\eta$. This extension is unique by the separatedness of stacks of r-th roots, so the construction of $\bar{\mathscr{X}}$ descends to X. ☐

Lemma 3.3. *There is a unique lifting $Y \to \bar{\mathscr{X}}$.*

Proof. We need to check that there is a lifting at any point $y \in Y_s$ which either lies on a marking or is the reduction of a generic node. We can apply the purity lemma [Abramovich et al. 2011, Lemma 4.4] provided that the local fundamental group of Y at y is trivial and the local Picard group of Y at y is torsion-free. In order to see this, we replace R by its strict henselization and Y by the spectrum of the strict henselization of the local ring at y. We let $U = Y \setminus \{y\}$.

If y lies on a marking then Y is isomorphic to the spectrum of $R[a]^{\mathrm{sh}}$. Since this ring is local regular of dimension 2, the scheme U has trivial fundamental group by the Zariski–Nagata purity theorem, and trivial Picard group by Auslander–Buchsbaum. Hence the purity lemma applies.

If y is the reduction of a generic node, then Y is isomorphic to the strict henselization of $R[a, b]/(ab)$. Let $B_a = \mathrm{Spec}(R[a]^{\mathrm{sh}})$ resp. $B_b = \mathrm{Spec}(R[b]^{\mathrm{sh}})$ be the branches at y and $U_a = U \cap B_a$, $U_b = U \cap B_b$.

The schemes U_a and U_b have trivial fundamental group by Zariski–Nagata, and they intersect in Y in a single point of the generic fiber. Moreover the map $U_a \sqcup U_b \to U$, being finite surjective and finitely presented, is of effective descent for finite étale coverings [Grothendieck 1971, corollaire 4.12]. It then follows from the van Kampen theorem [ibid., théorème 5.1] that $\pi_1(U) = 1$.

For the computation of the local Picard group, first notice that since B_a, B_b are local regular of dimension 2 we have $\mathrm{Pic}(U_a) = \mathrm{Pic}(U_b) = 0$, and moreover it is easy to see that $H^0(U_a, \mathcal{O}_{U_a}^\times) = H^0(U_b, \mathcal{O}_{U_b}^\times) = R^\times$. Now we consider the long exact sequence in cohomology associated to the short exact sequence

$$0 \to \mathcal{O}_U^\times \to i_{a,*}\mathcal{O}_{U_a}^\times \oplus i_{b,*}\mathcal{O}_{U_b}^\times \to i_{ab,*}\mathcal{O}_{U_{ab}}^\times \to 0,$$

where the symbols $i_?$ stand for the obvious closed immersions. We obtain

$$\begin{aligned}\mathrm{Pic}(U) &= \mathrm{coker}\big(H^0(U_a, \mathcal{O}_{U_a}^\times) \oplus H^0(U_b, \mathcal{O}_{U_a}^\times) \to H^0(U_{ab}, \mathcal{O}_{U_{ab}}^\times)\big) \\ &= K^\times / R^\times = \mathbb{Z},\end{aligned}$$

which is torsion-free as desired. □

Note that we still need to introduce stack structure over special nodes of $\bar{\mathcal{X}}$.

Extension of $\mathcal{G}_\eta$ over generic points of $\bar{\mathcal{X}}_s$. Let ξ be the generic point of a component of $\bar{\mathcal{X}}_s$. Let U be the localization of $\bar{\mathcal{X}}$ at ξ and V be its inverse image in Y. Consider the closure $\mathcal{G}_\xi$ of $\mathcal{G}_\eta$ in $\mathrm{Aut}_U V$.

Proposition 3.4. *The scheme $\mathcal{G}_\xi \to U$ is a finite flat group scheme of order p, and $V \to U$ is a $\mathcal{G}_\xi$-torsor.*

Proof. This is a generalization of [Raynaud 1999, Proposition 1.2.1], see [Romagny 2011, Theorem 4.3.5]. □

Extension of $\mathcal{G}_\eta$ over the smooth locus of $\bar{\mathcal{X}}/S$. Quite generally, for a stable p-torsor $(\mathcal{X}, \mathcal{G}, Y)$ over a scheme T, by $\mathrm{Aut}_{\mathcal{X}} Y$ we denote the algebraic stack whose objects over an T-scheme U are pairs (u, f) with $u \in \mathcal{X}(U)$ and f a U-automorphism of

$Y \times_{\mathcal{X}} U$. Now consider $\bar{\mathcal{X}}^{\mathrm{sm}}$, the smooth locus of $\bar{\mathcal{X}}/S$, and its inverse image Y^{sm} in Y. Then $Y^{\mathrm{sm}} \to \bar{\mathcal{X}}^{\mathrm{sm}}$ is flat. Let $\mathcal{G}^{\mathrm{sm}}$ be the closure of $\mathcal{G}_\eta$ in $\mathrm{Aut}_{\bar{\mathcal{X}}^{\mathrm{sm}}} Y^{\mathrm{sm}}$.

Proposition 3.5. *The scheme $\mathcal{G}^{\mathrm{sm}} \to \bar{\mathcal{X}}^{\mathrm{sm}}$ is a finite flat group scheme of order p, and $Y^{\mathrm{sm}} \to \bar{\mathcal{X}}^{\mathrm{sm}}$ is a $\mathcal{G}^{\mathrm{sm}}$-torsor.*

Proof. Given Proposition 3.4, and since $\bar{\mathcal{X}}^{\mathrm{sm}}$ has local charts $U \to \bar{\mathcal{X}}^{\mathrm{sm}}$ with U regular 2-dimensional, this follows from [Abramovich 2012, Propositions 2.2.2 and 2.2.3]. □

Extension of $\mathcal{G}^{\mathrm{sm}}$ over generic nodes of $\mathcal{X}/S$. Consider the complement $\bar{\mathcal{X}}^0$ of the isolated nodes of $\bar{\mathcal{X}}_s$, and its inverse image Y^0 in Y.

Lemma 3.6. *The morphism $Y^0 \to \bar{\mathcal{X}}^0$ is flat.*

Proof. It is enough to verify the claim at the reduction x_s of an arbitrary generic node $x_\eta \in X_\eta$. Since generic nodes remain distinct in reduction, it is enough to prove that $Y \to \mathcal{X}$ is flat at a chosen point $y_s \in Y$ above x_s. Since the branches at y_s are not exchanged by $\mathcal{G}$, étale locally Y and $\mathcal{X}$ are the union of two branches which are flat over S and the restriction of $Y \to \mathcal{X}$ to each of the branches at x_s is flat. Since proper morphisms descend flatness [EGA IV$_3$ 1966, IV.11.5.3, p. 152], it follows that $Y \to \mathcal{X}$ is flat at y_s. □

Let $\mathcal{G}^0$ be the closure of $\mathcal{G}^{\mathrm{sm}}$ in $\mathrm{Aut}_{\bar{\mathcal{X}}^0} Y^0$.

Proposition 3.7. *The stack $\mathcal{G}^0 \to \bar{\mathcal{X}}^0$ is a finite flat group scheme of order p, and $Y^0 \to \bar{\mathcal{X}}^0$ is a $\mathcal{G}^0$ torsor.*

Proof. We only have to look around the closure of a generic node. Again since proper morphisms descend flatness, it is enough to prove the claim separately on the two branches. Then the result follows again from [Abramovich 2012, Propositions 2.2.2 and 2.2.3] by the same reason as in the proof of 3.5. □

Twisted structure at special nodes. Let P be a special node of X. By [Abramovich 2012, Section 3.2] there is a canonical twisted structure $\mathcal{X}$ at P determined by the local degree of Y/X. If near a given node Y_η/X_η is inseparable, then this degree is p. Otherwise Y/X has an action of $\mathbb{Z}/p\mathbb{Z}$ which is nontrivial near P, and therefore the local degree is either 1 or p. Then $\mathcal{X}$ is twisted with index p at P whenever this local degree is p. These twisted structures at the various nodes P glue to give a twisted curve $\mathcal{X}$.

We claim that this $\mathcal{X}$ is unique up to a unique isomorphism. This follows from Proposition 2.3 above. Indeed, let a be the node function which to a node P of X gives the local degree of Y/X at Y, and let r_i be the fixed indices at the sections. Then the stack of a-twisted curves over X/S with markings of indices r_i is proper over S, hence $\mathcal{X}$ is uniquely determined by $\mathcal{X}_\eta$ up to unique isomorphism.

By Lemma 3.2.1 of [Abramovich 2012], there is a unique lifting $Y \to \mathscr{X}$, and by Theorem 3.2.2 in the same reference the group scheme $\mathscr{G}^0$ extends uniquely to $\mathscr{G} \to \mathscr{X}$ such that Y is a $\mathscr{G}$-torsor. The rigidification extends immediately by taking the closure, since $\mathscr{G} \to \mathscr{X}$ is finite. □

4. Examples

4.1. ***First, some nonexamples.*** Consider a smooth projective curve X of genus $h > 1$ in characteristic p and a p-torsion point in its Jacobian, corresponding to a μ_p-torsor $Y' \to X$. This is *not* a stable p-torsor in the sense of Definition 1.5: the curve Y' is necessarily unstable, with singularities which are not even nodal. In fact, $Y' \to X$ may be described by a locally logarithmic differential form ω on X, such that if locally $\omega = df/f$ for some $f \in \mathscr{O}_X^\times$ then Y' is given by an equation $z^p = f$. Since the genus $h > 1$, all differentials on X have zeroes, and each zero of ω (i.e., a zero of the derivative of f with respect to a coordinate) contributes to a unibranch singularity on Y'.

Now consider a ramified $\mathbb{Z}/p\mathbb{Z}$-cover $Y \to X$ of smooth projective curves over a field. Let $y \in Y$ be a fixed point for the action of $\mathbb{Z}/p\mathbb{Z}$ and let $x \in X$ be its image. In characteristic 0, since the stabilizer of y is a multiplicative group, the curve X may be twisted at x to yield a stable $\mathbb{Z}/p\mathbb{Z}$-torsor $Y \to \mathscr{X}$. However in characteristic p the stabilizer is additive and the result is not a $\mathbb{Z}/p\mathbb{Z}$-torsor. Hence ramified covers of smooth curves in characteristic p do not provide stable $\mathbb{Z}/p\mathbb{Z}$-torsors.

However something else does occur in both examples: the torsor $Y' \to X$ of the first example, and the branched cover $Y \to X$ in the second, lift to characteristic 0. The reduction back to characteristic p of the corresponding stable torsor "contains the original cover" in the following sense: there is a unique component $\mathscr{X}$ whose coarse moduli space is isomorphic to X. In particular that component $\mathscr{X}$ is necessarily a twisted curve, and the group scheme over it has to degenerate to α_p over the twisted points. We see a manifestation of this in the next example.

4.2. ***Limit of a p-isogeny of elliptic curves.*** Now consider the case where X is an elliptic curve, with a marked point x, over a discrete valuation ring R of characteristic 0 and residue characteristic p. For simplicity assume that R contains μ_p; let η be the generic point of $\operatorname{Spec} R$ and s the closed point of $\operatorname{Spec} R$. Given a p-torsion point on X with nontrivial reduction, we obtain a corresponding nontrivial μ_p-isogeny $Y' \to X$. Over the generic point η we can make Y'_η stable by marking the fiber P_η over x_η. But note that the reduction of P_η in Y' is not étale, hence something must modified. Since our stack is proper, a stable p-torsor $Y \to \mathscr{X}$ limiting $Y'_\eta \to X_\eta$ exists, at least over a base change of R. Here is how to describe it.

Consider the completed local ring $\hat{\mathcal{O}}_{Y',O} \simeq R[\![Z]\!]$ at the origin $O \in Y'_s$ and its spectrum $\mathbb{D}$. Then $\mathbb{D}_\eta$ is identified with an open p-adic disk modulo Galois action. Write $P_\eta = \{P_{\eta,1}, \ldots, P_{\eta,p}\}$ as a sum of points permuted by the μ_p-action. Then the $P_{\eta,i}$ induce K-rational points of $\mathbb{D}_\eta$ which moreover are π-adically equidistant; i.e., the valuation $v = v_\pi(P_{\eta,i} - P_{\eta,j})$ is independent of i, j. It follows that after blowing-up the closed subscheme with ideal (π^v, Z) these points reduce to p distinct points in the exceptional divisor. Thus after twisting at the node, the fiber $Y_s \to \mathcal{X}_s$ over the special point s of R is described as follows:

$$\begin{array}{ccccc}
Y_s & = & Y'_s \cup \mathbb{P}^1 & \hookleftarrow & P \\
\downarrow & & \downarrow & & \downarrow \\
X'_s & = & E \cup Q & \hookleftarrow & \{0\}
\end{array}$$

Here

- Y_s is a union of two components $Y'_s \cup \mathbb{P}^1$, attached at the origin of Y'_s,
- $\mathcal{X}_s$ is a twisted curve with two components $E \cup Q$,
- $E = X_s(\sqrt[p]{x})$ and $Q = \mathbb{P}^1(\sqrt[p]{\infty})$, with the twisted points attached,
- the map $Y_s \to \mathcal{X}_s$ decomposes into $Y'_s \to E$ and $\mathbb{P}^1 \to Q$,
- $\mathbb{P}^1 \to Q$ is an Artin–Schreier cover ramified at ∞,
- the curve is marked by the inverse image of $0 \in Q$ in $\mathbb{P}^1$, which is a $\mathbb{Z}/p\mathbb{Z}$-torsor $P \subset \mathbb{P}^1$,
- the map $Y'_s \to E$ is a lift of $Y'_s \to X_s$, and
- the group scheme $\mathcal{G} \to \mathcal{X}$ is generically étale on Q and generically μ_p on E, but the fiber over the node is α_p.

Notice that we can view $Y'_s \to E$, marked by the origin on Y'_s, as a twisted torsor as well, but this twisted torsor does not lift to characteristic 0 simply because the marked point on Y'_s can not be lifted to an invariant divisor. This is an example of the phenomenon described at the end of Section 4.1 above.

A very similar picture occurs when the cover $Y'_\eta \to X_\eta$ degenerates to an α_p-torsor. If, however, the reduction of the cover is a $\mathbb{Z}/p\mathbb{Z}$-torsor, then $Y' \to X$, marked by the fiber over the origin, is already stable and new components do not appear.

4.3. ***The double cover of $\mathbb{P}^1$ branched over 4 points.*** Consider an elliptic double cover Y over $\mathbb{P}^1$ in characteristic 0 given by the equation $y^2 = x(x-1)(x-\lambda)$. Marked by the four branched points, it becomes a stable μ_2-torsor over the twisted curve $Q = \mathbb{P}^1(\sqrt{0, 1, \infty, \lambda})$. What is its reduction in characteristic 2? We describe here one case, the others can be described in a similar way.

If the elliptic curve Y has good ordinary reduction E_s, the picture is as follows: Y_s has three components $\mathbb{P}^1 \cup E_s \cup \mathbb{P}^1$. The twisted curve $\mathscr{X}_s$ also has three rational components $Q_1 \cup Q_2 \cup Q_3$. The map splits as $\mathbb{P}^1 \to Q_1$, $E_s \to Q_2$ and $\mathbb{P}^1 \to Q_3$, where the first and last are generically μ_2-covers, and $E_s \to Q_2$ is a lift of the hyperelliptic cover $E_s \to \mathbb{P}^1$. The fibers of $\mathscr{G}$ at the nodes of X_s are both α_2. The points 0, 1, ∞, λ reduce to two pairs, one pair on each of the two $\mathbb{P}^1$ components, for instance:

$$\begin{array}{ccccc} & & \mathbb{P}^1 \cup E_s \cup \mathbb{P}^1 & & \\ & & \downarrow & & \\ \{0, 1\} & \hookrightarrow & Q_1 \cup Q_2 \cup Q_3 & \hookleftarrow & \{\lambda, \infty\}. \end{array}$$

Appendix A. Group schemes of order p

In this appendix, we give some complements on group schemes of order p. The main topic is the construction of an embedding of a given group scheme of order p into an affine smooth one-dimensional group scheme (an analogue of Kummer or Artin–Schreier theory). Although not strictly necessary in the paper, this result highlights the nature of our stable torsors in two respects: firstly because the original definition of generators in [Katz and Mazur 1985, §1.4] involves a smooth ambient group scheme, and secondly because the short exact sequence given by this embedding induces a long exact sequence in cohomology that may be useful for computations of torsors.

Anyway, let us now state the result.

Definition A.1. Let $G \to S$ be a finite locally free group scheme of order p.

(1) A *generator* is a morphism of S-group schemes $\gamma : (\mathbb{Z}/p\mathbb{Z})_S \to G$ such that the sections $x_i = \gamma(i)$, $0 \le i \le p-1$, are a full set of sections.

(2) A *cogenerator* is a morphism of S-group schemes $\kappa : G \to \mu_{p,S}$ such that the Cartier dual $(\mathbb{Z}/p\mathbb{Z})_S \to G^\vee$ is a generator.

We will prove the following.

Theorem A.2. *Let S be a scheme and let $G \to S$ be a finite locally free group scheme of order p. Let $\kappa : G \to \mu_{p,S}$ be a cogenerator. Then κ can be canonically inserted into a commutative diagram with exact rows*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & G & \longrightarrow & \mathscr{G} & \xrightarrow{\varphi_\kappa} & \mathscr{G}' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\kappa} & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mu_{p,S} & \longrightarrow & \mathbb{G}_{m,S} & \xrightarrow{p} & \mathbb{G}_{m,S} & \longrightarrow & 0 \end{array}$$

where $\varphi_\kappa : \mathscr{G} \to \mathscr{G}'$ is an isogeny between affine smooth one-dimensional S-group schemes with geometrically connected fibers.

In order to obtain this, we introduce two categories of invertible sheaves with sections: one related to groups with a cogenerator and one related to groups defined as kernels of isogenies, and we compare these categories.

Remark A.3. Not all group schemes of order p can be embedded into an affine smooth group scheme as in the theorem. For example, assume that there exists a closed immersion from $G = (\mathbb{Z}/p\mathbb{Z})_{\mathbb{Q}}$ to some affine smooth one-dimensional geometrically connected $\mathbb{Q}$-group scheme $\mathscr{G}$. Then $\mathscr{G}$ is a form of $\mathbb{G}_{m,\mathbb{Q}}$ and G is its p-torsion subgroup. Since $\mathscr{G}$ is trivialized by a quadratic field extension $K/\mathbb{Q}$, we obtain $G_K \simeq \mu_{p,K}$. This implies that K contains the p-th roots of unity, which is impossible for $p > 3$. Similar examples can be given for $\mathbb{Z}/p\mathbb{Z}$ over the Tate–Oort ring $\Lambda \otimes \mathbb{Q}$.

A.4. ***Tate–Oort group schemes.*** We recall the notations and results of the Tate–Oort classification of group schemes of order p over the ring Λ [Tate and Oort 1970, Section 2]. We introduce two fibered categories:

- a Λ-category TG of *triples* encoding *groups*, and
- a Λ-category TGC of *triples* encoding *groups with a cogenerator*.

Let $\chi : \mathbb{F}_p \to \mathbb{Z}_p$ be the unique multiplicative section of the reduction map, that is $\chi(0) = 0$ and if $m \in \mathbb{F}_p^\times$ then $\chi(m)$ is the $(p-1)$-st root of unity with residue equal to m. Set

$$\Lambda = \mathbb{Z}[\chi(\mathbb{F}_p), \frac{1}{p(p-1)}] \cap \mathbb{Z}_p.$$

There is in Λ a particular element w_p equal to p times a unit.

Definition A.5. The category TG is the category fibered over Spec Λ whose fiber categories over a Λ-scheme S are as follows.

- Its objects are the triples (L, a, b), where L is an invertible sheaf and $a \in \Gamma(S, L^{\otimes(p-1)})$, $b \in \Gamma(S, L^{\otimes(1-p)})$ satisfy $a \otimes b = w_p 1_{\mathcal{O}_S}$.
- Morphisms between (L, a, b) and (L', a', b') are the morphisms of invertible sheaves $f : L \to L'$, viewed as global sections of $L^{\otimes -1} \otimes L'$, such that $a \otimes f^{\otimes p} = f \otimes a'$ and $b' \otimes f^{\otimes p} = f \otimes b$.

The main result of [Tate and Oort 1970] is an explicit description of a covariant equivalence of fibered categories between TG and the category of finite locally free group schemes of order p. The group scheme associated to a triple (L, a, b) is denoted $G^L_{a,b}$. Its Cartier dual is isomorphic to $G^{L^{-1}}_{b,a}$.

Examples A.6. We have $(\mathbb{Z}/p\mathbb{Z})_S = G^{\mathbb{O}_S}_{1,w_p}$ and $\mu_{p,S} = G^{\mathbb{O}_S}_{w_p,1}$. Moreover if $G = G^L_{a,b}$ then a morphism $(\mathbb{Z}/p\mathbb{Z})_S \to G$ is given by a global section $u \in \Gamma(S, L)$ such that $u^{\otimes p} = u \otimes a$ and a morphism $G \to \mu_{p,S}$ is given by a global section $v \in \Gamma(S, L^{-1})$ such that $v^{\otimes p} = v \otimes b$.

Lemma A.7. *Let S be a Λ-scheme and let $G = G^L_{a,b}$ be a finite locally free group scheme of rank p over S. Then*:

(1) *Let $\gamma : (\mathbb{Z}/p\mathbb{Z})_S \to G$ be a morphism of S-group schemes given by a section $u \in \Gamma(S, L)$ such that $u^{\otimes p} = u \otimes a$. Then γ is a generator if and only if $u^{\otimes(p-1)} = a$.*

(2) *Let $\kappa : G \to \mu_{p,S}$ be a morphism of S-group schemes given by a section $v \in \Gamma(S, L^{-1})$ such that $v^{\otimes p} = v \otimes b$. Then κ is a cogenerator if and only if $v^{\otimes(p-1)} = b$.*

Proof. The proof of (2) follows from (1) by Cartier duality so we only deal with (1). The claim is local on S so we may assume that S is affine equal to $\mathrm{Spec}(R)$ and L is trivial. It follows from [Tate and Oort 1970] that $G = \mathrm{Spec}\, R[x]/(x^p - ax)$ and the section $\gamma(i) : \mathrm{Spec}(R) \xrightarrow{i} (\mathbb{Z}/p\mathbb{Z})_R \to G$ is given by the morphism of algebras $R[x]/(x^p - ax) \to R$, $x \mapsto \chi(i)u$. Thus γ is a generator if and only if $\mathrm{Norm}(f) = \prod f(\chi(i)u)$ for all functions $f = f(x)$. In particular for $f = 1+x$ one finds $\mathrm{Norm}(f) = (-1)^p a + 1$ and $\prod(1 + \chi(i)u) = (-1)^p u^{p-1} + 1$. Therefore if γ is a generator then $u^{p-1} = a$. Conversely, assuming that $u^{p-1} = a$ we want to prove that $\mathrm{Norm}(f) = \prod f(\chi(i)u)$ for all f. It is enough to prove this in the universal case where $R = \Lambda[a, b, u]/(ab - w_p, u^p - u)$. Since a is not a zerodivisor in R, it is in turn enough to prove the equality after base change to $K = R[1/a]$. Then G_K is étale and the morphism

$$K[x]/(x^p - ax) = K[x]/\prod(x - \chi(i)u) \to K^p$$

taking f to the tuple $(f(\chi(i)u))_{0\le i\le p-1}$ is an isomorphism of algebras. Since the norm in K^p is the product of the coordinates, the result follows. □

Definition A.8. The category TGC is the category fibered over $\mathrm{Spec}\,\Lambda$ whose fibers over a Λ-scheme S are as follows.

- Its objects are the triples (L, a, v), where L is an invertible sheaf and $a \in \Gamma(S, L^{\otimes(p-1)})$, $v \in \Gamma(S, L^{\otimes -1})$ satisfy $a \otimes v^{\otimes(p-1)} = w_p 1_{\mathbb{O}_S}$.
- Morphisms between (L, a, v) and (L', a', v') are the morphisms of invertible sheaves $f : L \to L'$, viewed as global sections of $L^{\otimes -1} \otimes L'$, such that $a \otimes f^{\otimes p} = f \otimes a'$ and $v' \otimes f = v$.

By Lemma A.7, the category TGC is equivalent to the category of group schemes with a cogenerator. The functor from group schemes with a cogenerator to group schemes that forgets the cogenerator is described in terms of categories of invertible sheaves by the functor $\omega : TGC \to TG$ given by $\omega(L, a, v) = (L, a, v^{\otimes(p-1)})$.

Note also that Lemma A.7 tells us that for any locally free group scheme G over a Λ-scheme S, there exists a finite locally free morphism $S' \to S$ of degree $p-1$ such that $G \times_S S'$ admits a generator or a cogenerator.

A.9. ***Congruence group schemes.*** Here, we introduce and describe a $\mathbb{Z}$-category TCG of *triples* encoding *congruence groups*.

Let R be ring with a discrete valuation v and let $\lambda \in R$ be such that $(p-1)v(\lambda) \leq v(p)$. In [Sekiguchi et al. 1989] are introduced some group schemes $H_\lambda = \operatorname{Spec} R[x]/(((1+\lambda x)^p - 1)/\lambda^p)$ with multiplication $x_1 \star x_2 = x_1 + x_2 + \lambda x_1 x_2$. (The notation in *loc. cit.* is $\mathcal{N}$.) Later Raynaud called them *congruence groups of level* λ and we will follow his terminology. We now define the analogues of these group schemes over a general base. The objects that are the input of the construction constitute the following category.

Definition A.10. The category TCG is the category fibered over $\operatorname{Spec}\mathbb{Z}$ whose fibers over a scheme S are as follows.

- Its objects are the triples (M, λ, μ), where M is an invertible sheaf over S and the global sections $\lambda \in \Gamma(S, M^{-1})$ and $\mu \in \Gamma(S, M^{p-1})$ are subject to the condition $\lambda^{\otimes(p-1)} \otimes \mu = p1_{\mathcal{O}_S}$.
- Morphisms between (M, λ, μ) and (M', λ', μ') are morphisms of invertible sheaves $f : M \to M'$ viewed as sections of $M^{-1} \otimes M'$ such that $f \otimes \lambda' = \lambda$ and $f^{\otimes(p-1)} \otimes \mu = \mu'$.

We will exhibit a functor $(M, \lambda, \mu) \rightsquigarrow H^M_{\lambda,\mu}$ from TCG to the category of group schemes, with $H^M_{\lambda,\mu}$ defined as the kernel of a suitable isogeny.

First, starting from (M, λ) we construct a smooth affine one-dimensional group scheme denoted $\mathcal{G}^{(M,\lambda)}$, or simply $\mathcal{G}^{(\lambda)}$. We see λ as a morphism $\lambda : \mathbb{V}(M) \to \mathbb{G}_{a,S}$ of (geometric) line bundles over S, where $\mathbb{V}(M) = \operatorname{Spec}\operatorname{Sym}(M^{-1})$ is the (geometric) line bundle associated to M. We define $\mathcal{G}^{(\lambda)}$ as a scheme by the fibered product

$$\begin{array}{ccc} \mathcal{G}^{(\lambda)} & \xrightarrow{1+\lambda} & \mathbb{G}_{m,S} \\ \downarrow & & \downarrow \\ \mathbb{V}(M) & \xrightarrow{1+\lambda} & \mathbb{G}_{a,S} \end{array} .$$

The points of $\mathcal{G}^{(\lambda)}$ with values in an S-scheme T are the global sections $u \in \Gamma(T, M \otimes \mathcal{O}_T)$ such that $1 + \lambda \otimes u$ is invertible. We endow $\mathcal{G}^{(\lambda)}$ with a multiplication given on the T-points by

$$u_1 \star u_2 = u_1 + u_2 + \lambda \otimes u_1 \otimes u_2 .$$

The zero section of $\mathbb{V}(M)$ sits in $\mathcal{G}^{(\lambda)}$ and is the unit section for the law just defined. The formula

$$(1+\lambda\otimes u_1)(1+\lambda\otimes u_2)=1+\lambda\otimes(u_1\star u_2)$$

shows that $1+\lambda:\mathcal{G}^{(\lambda)}\to\mathbb{G}_{m,S}$ is a morphism of group schemes. Moreover, if the locus where $\lambda:\mathbb{V}(M)\to\mathbb{G}_{a,S}$ is an isomorphism is scheme-theoretically dense, then $\star$ is the unique group law on $\mathcal{G}^{(\lambda)}$ for which this holds. This construction is functorial in (M,λ): given a morphism of invertible sheaves $f:M\to M'$, in other words a global section of $M^{-1}\otimes M'$, such that $f\otimes\lambda'=\lambda$, there is a morphism $f:\mathcal{G}^{(\lambda)}\to\mathcal{G}^{(\lambda')}$ making the diagram

$$\begin{array}{ccc}\mathcal{G}^{(\lambda)} & \xrightarrow{1+\lambda} & \mathbb{G}_{m,S}\\ {\scriptstyle f}\downarrow & \nearrow{\scriptstyle 1+\lambda'} & \\ \mathcal{G}^{(\lambda')} & & \end{array}$$

commutative. The notation is coherent since that morphism is indeed induced by the extension of f to the sheaves of symmetric algebras.

Then, we use the section $\mu\in\Gamma(S,M^{p-1})$ and the relation $\lambda^{\otimes(p-1)}\otimes\mu=p1_{\mathcal{O}_S}$ to define an isogeny φ fitting into a commutative diagram

$$\begin{array}{ccc}\mathcal{G}^{(\lambda)} & \xrightarrow{\varphi} & \mathcal{G}^{(\lambda^{\otimes p})}\\ {\scriptstyle 1+\lambda}\downarrow & & \downarrow{\scriptstyle 1+\lambda^{\otimes p}}\\ \mathbb{G}_{m,S} & \xrightarrow{\wedge p} & \mathbb{G}_{m,S}\end{array}.$$

The formula for φ is given on the T-points $u\in\Gamma(T,M\otimes\mathcal{O}_T)$ by

$$\varphi(u)=u^{\otimes p}+\sum_{i=1}^{p-1}\left\{{p \atop i}\right\}\lambda^{\otimes(i-1)}\otimes\mu\otimes u^{\otimes i},$$

where $\left\{{p \atop i}\right\}=\frac{1}{p}\binom{p}{i}$ is the binomial coefficient divided by p. In order to check that the diagram is commutative and that φ is an isogeny, we may work locally on S hence we may assume that S is affine and that $M=\mathcal{O}_S$. In this case, the two claims follow from the universal case; i.e., from points (1) and (2) in the following lemma.

Lemma A.11. *Let $\mathcal{O}=\mathbb{Z}[E,F]/(E^{p-1}F-p)$ and let $\lambda,\mu\in\mathcal{O}$ be the images of the indeterminates E, F. Then, the polynomial*

$$P(X)=X^p+\sum_{i=1}^{p-1}\left\{{p \atop i}\right\}\lambda^{i-1}\mu X^i\ \in\mathcal{O}[X]$$

satisfies:

(1) $1+\lambda^p P(X) = (1+\lambda X)^p$, *and*

(2) $P(X+Y+\lambda XY) = P(X)+P(Y)+\lambda^p P(X)P(Y)$.

Proof. Point (1) follows by expanding $(1+\lambda X)^p$ and using the fact that $p=\lambda^{p-1}\mu$ in $\mathcal{O}$. Then we compute:

$$\begin{aligned} 1+\lambda^p P(X+Y+\lambda XY) &= (1+\lambda(X+Y+\lambda XY))^p \\ &= (1+\lambda X)^p(1+\lambda Y)^p \\ &= (1+\lambda^p P(X))(1+\lambda^p P(Y)) \\ &= 1+\lambda^p(P(X)+P(Y)+\lambda^p P(X)P(Y)). \end{aligned}$$

Since λ is a nonzerodivisor in $\mathcal{O}$, point (2) follows. □

Definition A.12. We denote by $H^M_{\lambda,\mu}$ the kernel of φ, and call it the *congruence group scheme* associated to (M,λ,μ).

This construction is functorial in (M,λ,μ). Precisely, consider two triples (M,λ,μ) and (M',λ',μ') and a morphism of invertible sheaves $f: M\to M'$ viewed as a section of $M^{-1}\otimes M'$ such that $f\otimes\lambda'=\lambda$ and $f^{\otimes(p-1)}\otimes\mu=\mu'$. Then we have morphisms $f:\mathcal{G}^{(\lambda)}\to\mathcal{G}^{(\lambda')}$ and $f^{\otimes p}:\mathcal{G}^{(\lambda^{\otimes p})}\to\mathcal{G}^{(\lambda'^{\otimes p})}$ compatible with the isogenies φ and φ', and f induces a morphism $H^M_{\lambda,\mu}\to H^{M'}_{\lambda',\mu'}$. Note also that the image of $H^M_{\lambda,\mu}$ under $1+\lambda:\mathcal{G}^{(\lambda)}\to\mathbb{G}_{m,S}$ factors through $\mu_{p,S}$, so that by construction $H^M_{\lambda,\mu}$ comes embedded into a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^M_{\lambda,\mu} & \longrightarrow & \mathcal{G}^{(\lambda)} & \longrightarrow & \mathcal{G}^{(\lambda^{\otimes p})} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\kappa} & & \downarrow{\scriptstyle 1+\lambda} & & \downarrow{\scriptstyle 1+\lambda^{\otimes p}} & & \\ 0 & \longrightarrow & \mu_{p,S} & \longrightarrow & \mathbb{G}_{m,S} & \longrightarrow & \mathbb{G}_{m,S} & \longrightarrow & 0. \end{array}$$

The formation of this diagram is also functorial.

Lemma A.13. *The morphism $\kappa: H^M_{\lambda,\mu}\to\mu_{p,S}$ is a cogenerator.*

Proof. We have to show that the dual map $(\mathbb{Z}/p\mathbb{Z})_S\to(H^M_{\lambda,\mu})^\vee$ is a generator. This means verifying locally on S certain equalities of norms. Hence we may assume that S is affine and that M is trivial, then reduce to the universal case where S is the spectrum of the ring $\mathcal{O}$ with elements λ,μ satisfying $\lambda^{p-1}\mu=p$ as in Lemma A.11, and finally restrict to the schematically dense open subscheme $S'=D(\lambda)\subset S$. Since $\mathcal{G}^{(\lambda)}\times_S S'\to\mathbb{G}_{m,S'}$ is an isomorphism, then $H^M_{\lambda,\mu}\times_S S'\to\mu_{p,S'}$ and the dual morphism also are. The claim follows immediately. □

A.14. ***Equivalence between TGC and $TCG \otimes_{\mathbb{Z}} \Lambda$.*** The results of the previous subsection imply that for a Λ-scheme S, a triple $(M, \lambda, \mu) \in TCG(S)$ gives rise in a functorial way to a finite locally free group scheme with cogenerator $\kappa : H^M_{\lambda,\mu} \to \mu_{p,S}$, that is, an object of $TGC(S)$.

Theorem A.15. *The functor*

$$F : TCG \otimes_{\mathbb{Z}} \Lambda \to TGC$$

defined above is an equivalence of fibered categories over Λ. If (M, λ, μ) has image (L, a, v) then $H^M_{\lambda,\mu} \simeq G^L_{a,v^{\otimes(p-1)}}$.

Proof. The main point is to describe F in detail using the Tate–Oort classification, and to see that it is essentially surjective. The description of the action of F on morphisms and the verification that it is fully faithful offers no difficulty and will be omitted.

Let (M, λ, μ) be a triple in $TCG(S)$ and let $G = H^M_{\lambda,\mu}$. We use the notations of Section 2 of [Tate and Oort 1970], in particular the structure of the group μ_p is described by a function z, the sheaf of χ-eigensections $J = y\mathbb{O}_S \subset \mathbb{O}_{\mu_p}$ with distinguished generator $y = (p-1)e_1(1-z)$, and constants

$$w_1 = 1,\ w_2,\ \dots,\ w_{p-1},\ w_p = pw_{p-1} \in \Lambda.$$

The augmentation ideal of the algebra $\mathbb{O}_G$ is the sheaf I generated by M^{-1}, and by Tate and Oort's results the subsheaf of χ-eigensections is the sheaf $I_1 = e_1(I)$, where e_1 is the $\mathbb{O}_S$-linear map defined in [Tate and Oort 1970]. It is an invertible sheaf and L is (by definition) its inverse.

We claim that in fact $I_1 = e_1(M^{-1})$. In order to see this, we may work locally. Let x be a local generator for M^{-1} and let

$$t := (p-1)e_1(-x) \in I_1.$$

Let us write $\lambda = \lambda_0 x$ for some local function λ_0. We first prove that

$$x = \frac{1}{1-p}\left(t + \frac{\lambda_0 t^2}{w_2} + \cdots + \frac{\lambda_0^{p-2} t^{p-1}}{w_{p-1}}\right). \tag{$\star$}$$

In fact, by construction the map $\mathbb{O}_{\mu_p} \to \mathbb{O}_G$ is given by $z = 1 + \lambda_0 x$, so we get $y = (p-1)e_1(1-z) = \lambda_0 t$. In order to check the expression for x in terms of t, we can reduce to the universal case (Lemma A.11). Then λ_0 is not a zerodivisor and we can harmlessly multiply both sides by λ_0. In this form, the equality to be proven is nothing else than the identity (16) in [Tate and Oort 1970]. Now write $t = \alpha t^*$ with t^* a local generator for I_1 and α a local function. Using ($\star$) we find that $x = \alpha x^*$ for some $x^* \in \mathbb{O}_G$. Since x generates M^{-1} in the fibers over S, this proves that α is invertible. Finally t is a local generator for I_1 and this finishes the proof that $I_1 = e_1(M^{-1})$.

Let $x^\vee$ be the local generator for M dual to x and write $\mu = \mu_0(x^\vee)^{\otimes(p-1)}$ for some local function μ_0 such that $(\lambda_0)^{p-1}\mu_0 = p$. Let $t^\vee$ be the local generator for L dual to t. We define a local section a of $L^{\otimes(p-1)}$ by

$$a = w_{p-1}\mu_0(t^\vee)^{\otimes(p-1)}$$

and a local section v of L^{-1} by

$$v = \lambda_0 t.$$

These sections are independent of the choice of the local generator x, because if $x' = \alpha x$ then

$$(x')^\vee = \alpha^{-1}x^\vee \ ; \ t' = \alpha t \ ; \ (t')^\vee = \alpha^{-1}t^\vee \ ; \ \lambda_0' = \alpha^{-1}\lambda_0 \ ; \ \mu_0' = \alpha^{p-1}\mu_0$$

so that

$$a' = w_{p-1}\mu_0'(t'^\vee)^{\otimes(p-1)} = w_{p-1}\alpha^{p-1}\mu_0\alpha^{1-p}(t^\vee)^{\otimes(p-1)} = a$$

and

$$v' = \lambda_0' t' = \alpha^{-1}\lambda_0\alpha t = v.$$

They glue to global sections a and v satisfying

$$a \otimes v^{\otimes(p-1)} = w_p 1_{\mathbb{O}_S}.$$

Let us prove that a and v are indeed the sections defining G and the cogenerator in the Tate–Oort classification. The verification for a amounts to checking that the relation

$$t^p = w_{p-1}\mu_0 t$$

holds in the algebra $\mathbb{O}_G$. This may be seen in the universal case where λ_0 is not a zerodivisor, hence after multiplying by $(\lambda_0)^p$ this follows from the equality $y^p = w_p y$ from [Tate and Oort 1970]. The verification for v amounts to noting that the cogenerator $G \to \mu_{p,S}$ is indeed given by $y \mapsto v$.

This completes the description of F on objects. Finally we prove that F is essentially surjective. Assume given (L, a, v) and let t be a local generator for $I_1 = L^{-1}$. Write $a = w_{p-1}\mu_0(t^\vee)^{\otimes(p-1)}$, $v = \lambda_0 t$ and define an element $x \in \mathbb{O}_G$ by the expression ($\star$) above. If we change the generator t to another $t' = \alpha t$, then $\lambda_0' = \alpha^{-1}\lambda_0$ and $x' = \alpha x$. It follows that the subsheaf of $\mathbb{O}_G$ generated by x does not depend on the choice of the generator for I_1, call it N. Reducing to the universal case as before, we prove that $t = (p-1)e_1(-x)$. This shows that in fact N is an invertible sheaf and we take M to be its inverse. Finally we define sections $\lambda \in \Gamma(S, M^{-1})$ and $\mu \in \Gamma(S, M^{\otimes(p-1)})$ by the local expressions $\lambda = \lambda_0 x$ and

$\mu = \mu_0(x^\vee)^{\otimes(p-1)}$. It is verified like in the case of a, v before that they do not depend on the choice of t and hence are well-defined global sections. The equality $\lambda^{\otimes(p-1)} \otimes \mu = p1_{\mathbb{O}_S}$ holds true and the proof is now complete. □

Proof of Theorem A.2. We keep the notation of the theorem. Since the construction of the isogeny φ_κ and the whole commutative diagram is canonical, if we perform it after fppf base change $S' \to S$ then it will descend to S. We choose $S' = S_1 \amalg S_2$, where $S_1 = S \otimes_{\mathbb{Z}} \mathbb{Z}[1/p]$ and $S_2 = S \otimes_{\mathbb{Z}} \Lambda$. Over S_1 the group scheme G is étale and the cogenerator is an isomorphism by [Katz and Mazur 1985, Lemma 1.8.3]. We take $\mathcal{G} = \mathcal{G}' = \mathbb{G}_{m,S}$ and φ_κ is the p-th power map. Over S_2 we use Theorem A.15 which provides a canonical isomorphism between κ and $H^M_{\lambda,\mu}$ with its canonical cogenerator, embedded into a diagram of the desired form. This completes the proof. □

Appendix B. Weil restriction of closed subschemes

Let $Z \to X$ be a morphism of S-schemes (or algebraic spaces) and denote by $h : X \to S$ the structure map. The Weil restriction h_*Z of Z along h is the functor on S-schemes defined by $(h_*Z)(T) = \mathrm{Hom}_X(X \times_S T, Z)$. It may be seen as a left adjoint to the pullback along h, or as the functor of sections of $Z \to X$.

If $Z \to X$ is a closed immersion of schemes (or algebraic spaces) of finite presentation over S, there are two main cases where h_*Z is known to be representable by a closed subscheme of S. As is well-known, this has applications to representability of various equalizers, kernels, centralizers, normalizers, etc. These two cases are:

(i) if $X \to S$ is proper flat and $Z \to S$ is separated, by the Grothendieck–Artin theory of the Hilbert scheme,

(ii) if $X \to S$ is essentially free, by [Grothendieck 1970, théorème 6.4].

In this appendix, we want to prove that h_*Z is representable by a closed subscheme of S in a case that includes both situations and is often easier to check in practice, namely the case where $X \to S$ is flat and pure.

B.1. ***Essentially free and pure morphisms.*** We recall the notions of essentially free and pure morphisms and check that essentially free morphisms and proper morphisms are pure.

In [Grothendieck 1970, §6], a morphism $X \to S$ is called *essentially free* if and only if there exists a covering of S by open affine subschemes S_i, and for each i an affine faithfully flat morphism $S'_i \to S_i$ and a covering of $X'_i = X \times_S S'_i$ by open affine subschemes $X'_{i,j}$ such that the function ring of $X'_{i,j}$ is free as a module over the function ring of S'_i.

In fact, the proof of [Grothendieck 1970, théorème 6.4] works just as well with a slightly weaker notion than freeness of modules. Namely, for a module M over a ring A, let us say that M is *quasireflexive* if the canonical map $M \to M^{\vee\vee}$ from M to its linear bidual is injective after any change of base ring $A \to A'$. It is a simple exercise to see that this is equivalent to M being a submodule of a product module A^I for some set I, over A and after any base change $A \to A'$. For instance, free modules, projective modules, product modules are quasireflexive. This gives rise to a notion of *essentially quasireflexive* morphism, and in particular *essentially projective* morphism. Then inspection of the proof of [Grothendieck 1970, théorème 6.4] shows that it remains valid for these morphisms.

In [Raynaud and Gruson 1971, 3.3.3], a morphism locally of finite type $X \to S$ is called *pure* if and only if for all points $s \in S$, with henselization $(\tilde{S}, \tilde{s})$, and all points $\tilde{x} \in \tilde{X}$ where $\tilde{X} = X \times_S \tilde{S}$, if $\tilde{x}$ is an associated point in its fiber then its closure in $\tilde{X}$ meets the special fiber. Examples of pure morphisms include proper morphisms (by the valuative criterion for properness) and morphisms locally of finite type and flat, with geometrically irreducible fibers without embedded components [ibid., 3.3.4].

Finally if $X \to S$ is locally of finite presentation and essentially free, then it is pure. Indeed, with the notations above for an essentially free morphism, one sees using [ibid., 3.3.7] that it is enough to see that for each i, j the scheme $X'_{i,j}$ is pure over S'_i. But since the function ring of $X'_{i,j}$ is free over the function ring of S'_i, this follows from [ibid., 3.3.5].

B.2. *Representability of h_*Z.*

Proposition B.3. *Let $h : X \to S$ be a morphism of finite presentation, flat and pure, and let $Z \to X$ be a closed immersion. Then the Weil restriction h_*Z is representable by a closed subscheme of S.*

Proof. The question is local for the étale topology on S. Let $s \in S$ be a point and let $\mathbb{O}^h$ be the henselization of the local ring at s. By [Raynaud and Gruson 1971, 3.3.13], for each $x \in X$ lying over s, there exists an open affine subscheme U^h_x of $X \times_S \operatorname{Spec}(\mathbb{O}^h)$ containing x and whose function ring is free as an $\mathbb{O}^h$-module. Since X_s is quasi-compact, there is a finite number of points $x_1, \dots, x_n$ such that the open affines $U^h_i = U^h_{x_i}$ cover it. Since X is locally of finite presentation, after restricting to an étale neighborhood $S' \to S$ of s, there exist affine open subschemes U_i of X inducing the U^h_i. According to [ibid., 3.3.8], the locus of the base scheme S where $U_i \to S$ is pure is open, so after shrinking S we may assume that for each i the affine U_i is flat and pure. This means that its function ring is projective by [ibid., 3.3.5]. In other words, the union $U = U_1 \cup \cdots \cup U_n$ is essentially projective over S in the terms of the comments in B.1. If $k : U \to X$ denotes the structure map, it follows from [Grothendieck 1970, théorème 6.4] that $k_*(Z \cap U)$ is representable by a closed subscheme of S. On the other hand, according to [Romagny 2011, 3.1.8], replacing S again by a smaller neighborhood of s, the open immersion $U \to X$

is S-universally schematically dense. One deduces immediately that the natural morphism $h_* Z \to k_*(Z \cap U)$ is an isomorphism. This finishes the proof. □

This proposition has a long list of corollaries and applications, listed in [Grothendieck 1970, §6]. In particular:

Corollary B.4. *Let $X \to S$ be a morphism of finite presentation, flat and pure and $Y \to S$ a separated morphism. Consider two morphisms $f, g : X \to Y$. Then the condition $f = g$ is represented by a closed subscheme of S.*

Proof. Apply the previous proposition to the pullback of the diagonal of Y along $(f, g) : X \to Y \times_S Y$. □

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References

[Abramovich 2012] D. Abramovich, "Raynaud's group-scheme and reduction of coverings", pp. 1–18 in *Number theory, analysis and geometry*, edited by D. Goldfeld et al., Springer, New York, 2012. MR 2867909

[Abramovich and Vistoli 2002] D. Abramovich and A. Vistoli, "Compactifying the space of stable maps", *J. Amer. Math. Soc.* **15**:1 (2002), 27–75. MR 2002i:14030 Zbl 0991.14007

[Abramovich et al. 2003] D. Abramovich, A. Corti, and A. Vistoli, "Twisted bundles and admissible covers", *Comm. Algebra* **31**:8 (2003), 3547–3618. MR 2005b:14049 Zbl 1077.14034

[Abramovich et al. 2008] D. Abramovich, M. Olsson, and A. Vistoli, "Tame stacks in positive characteristic", *Ann. Inst. Fourier (Grenoble)* **58**:4 (2008), 1057–1091. MR 2009c:14002 Zbl 1222.14004

[Abramovich et al. 2011] D. Abramovich, M. Olsson, and A. Vistoli, "Twisted stable maps to tame Artin stacks", *J. Algebraic Geom.* **20**:3 (2011), 399–477. MR 2012c:14024 Zbl 1225.14020

[Bouw and Wewers 2004] I. I. Bouw and S. Wewers, "Stable reduction of modular curves", pp. 1–22 in *Modular curves and Abelian varieties*, edited by J. Cremona et al., Progr. Math. **224**, Birkhäuser, Basel, 2004. MR 2005b:11079 Zbl 1147.11316

[Edixhoven 1990] B. Edixhoven, "Minimal resolution and stable reduction of $X_0(N)$", *Ann. Inst. Fourier (Grenoble)* **40**:1 (1990), 31–67. MR 92f:11080 Zbl 0679.14009

[EGA IV$_3$ 1966] A. Grothendieck, *Eléments de géométrie algébrique, IV: Étude locale des schémas et des morphismes de schémas, III*, Publications Mathématiques de l'I.H.E.S. **28**, Inst. Hautes Études Sci., Paris, 1966. MR 0217086 Zbl 0144.19904

[Grothendieck 1970] A. Grothendieck, "Groupes diagonalisables", exposé VIII in *Schémas en groupes, II: Groupes de type multiplicatif, et structure des schémas en groupes généraux* (SGA 3), Lecture Notes in Mathematics **152**, Springer, Berlin, 1970. MR 43 #223b Zbl 0209.24201

[Grothendieck 1971] A. Grothendieck, "Descente des morphismes étales; application au groupe fondamental", exposé IX in *Revêtements étales et groupe fondamental* (SGA 1), Lecture Notes in Mathematics **224**, Springer-Verlag, Berlin, 1971. Reprinted Soc. Math. de France, Paris, 2003. MR 0354651 (50 #7129)

[Katz and Mazur 1985] N. M. Katz and B. Mazur, *Arithmetic moduli of elliptic curves*, Annals of Mathematics Studies **108**, Princeton University Press, 1985. MR 86i:11024 Zbl 0576.14026

[McMurdy and Coleman 2010] K. McMurdy and R. Coleman, "Stable reduction of $X_0(p^3)$", *Algebra Number Theory* **4**:4 (2010), 357–431. MR 2011k:14022 Zbl 1215.11060

[Raynaud 1999] M. Raynaud, "Spécialisation des revêtements en caractéristique $p > 0$", *Ann. Sci. École Norm. Sup.* (4) **32**:1 (1999), 87–126. MR 2000e:14016 Zbl 0999.14004

[Raynaud and Gruson 1971] M. Raynaud and L. Gruson, "Critères de platitude et de projectivité: Techniques de "platification" d'un module", *Invent. Math.* **13** (1971), 1–89. MR 46 #7219

[Romagny 2011] M. Romagny, "Effective models of group schemes", preprint, 2011. To appear in *J. Algebraic Geom.*

[Sekiguchi et al. 1989] T. Sekiguchi, F. Oort, and N. Suwa, "On the deformation of Artin–Schreier to Kummer", *Ann. Sci. École Norm. Sup.* (4) **22**:3 (1989), 345–375. MR 91g:14041 Zbl 0714.14024

[Tate and Oort 1970] J. Tate and F. Oort, "Group schemes of prime order", *Ann. Sci. École Norm. Sup.* (4) **3** (1970), 1–21. MR 42 #278 Zbl 0195.50801

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