Algebra & Number Theory

Volume 6

2012

No. 4

Block components of the Lie module for the symmetric group

Roger M. Bryant and Karin Erdmann

mathematical sciences publishers

ALGEBRA AND NUMBER THEORY 6:4(2012)

Block components of the Lie module for the symmetric group

Roger M. Bryant and Karin Erdmann

Let F be a field of prime characteristic p and let B be a nonprincipal block of the group algebra FS_r of the symmetric group S_r. The block component $\mathrm{Lie}(r)_B$ of the Lie module $\mathrm{Lie}(r)$ is projective, by a result of Erdmann and Tan, although $\mathrm{Lie}(r)$ itself is projective only when $p \nmid r$. Write $r = p^m k$, where $p \nmid k$, and let S_k^* be the diagonal of a Young subgroup of S_r isomorphic to $S_k \times \cdots \times S_k$. We show that $p^m \,\mathrm{Lie}(r)_B \cong (\mathrm{Lie}(k)\uparrow_{S_k^*}^{S_r})_B$. Hence we obtain a formula for the multiplicities of the projective indecomposable modules in a direct sum decomposition of $\mathrm{Lie}(r)_B$. Corresponding results are obtained, when F is infinite, for the r-th Lie power $L^r(E)$ of the natural module E for the general linear group $\mathrm{GL}_n(F)$.

1. Introduction and summary of results

Let r be a positive integer and let S_r denote the symmetric group of degree r. For any field F the Lie module $\mathrm{Lie}(r)$ is the FS_r-module given by the right ideal $\omega_r FS_r$ of the group algebra FS_r where ω_r is the *Dynkin–Specht–Wever element*, defined by $\omega_1 = 1$ and, for $r \geqslant 2$,

$$\omega_r = (1 - c_r)(1 - c_{r-1}) \cdots (1 - c_2), \tag{1-1}$$

where c_i is the i-cycle $(1\ i\ i{-}1\ \ldots\ 2)$. It is known that $\mathrm{Lie}(r)$ has dimension $(r-1)!$ (see Section 2A).

If F has prime characteristic p and $p \nmid r$ then $\mathrm{Lie}(r)$ is a direct summand of FS_r because, as is well known, $\omega_r^2 = r\omega_r$ (see, for example, [Bryant 2009, Section 3]); so in this case $\mathrm{Lie}(r)$ is projective. However, if $\mathrm{char}\, F = p$ and $p \mid r$ then $\mathrm{Lie}(r)$ is not projective (because its dimension is not then divisible by the order of a Sylow p-subgroup of S_r), but it was shown recently that every nonprincipal block component of $\mathrm{Lie}(r)$ *is* projective (see [Erdmann and Tan 2011]). Here we show that each such component can be described in a surprisingly simple way in terms of $\mathrm{Lie}(k)$, where k is the p'-part of r.

Supported by EPSRC Standard Research Grant EP/G025487/1.
MSC2010: primary 20C30; secondary 20G43, 20C20.
Keywords: Lie module, symmetric group, Lie power, Schur algebra, block.

The Lie module occurs naturally in a number of contexts in algebra, algebraic topology and elsewhere (see [Erdmann and Tan 2011] for a fuller discussion). Here we shall only be concerned with the connection with free Lie algebras, where our results on the Lie module give new insight into the module structure of the homogeneous components.

Let G be a group and V an FG-module. Let $L^r(V)$ denote the homogeneous component of degree r in the free Lie algebra $L(V)$ freely generated by any basis of V. (Here $L(V)$ may be regarded as the Lie subalgebra generated by V in the tensor algebra or free associative algebra on V: see Section 2A.) The vector space $L^r(V)$ is called the *r-th Lie power* of V and it inherits the structure of an FG-module.

Suppose that F is infinite, let n be a positive integer, and let E denote the natural n-dimensional module over F for the general linear group $\mathrm{GL}_n(F)$. Then $L^r(E)$, as a module for $\mathrm{GL}_n(F)$, is a homogeneous polynomial module of degree r. In other words it is a module for the Schur algebra $S_F(n, r)$ (see [Green 1980]). In the case where $n \geqslant r$ the Schur functor f_r maps $S_F(n, r)$-modules to FS_r-modules and we have $f_r(L^r(E)) \cong \mathrm{Lie}(r)$ (see Section 2D).

Recall that if $\mathcal{B}$ is the set of blocks of an algebra Γ and V is a Γ-module then we may write V as a direct sum of block components: $V = \bigoplus_{B\in\mathcal{B}} V_B$, where $V_B \in B$ for all B. Our main results concern the block components of $\mathrm{Lie}(r)$ and $L^r(E)$ when F has prime characteristic p. The basic results are for $\mathrm{Lie}(r)$, and the results for $L^r(E)$ are obtained from these by means of the Schur functor. To state the results we write $r = p^m k$ where $m \geqslant 0$, $k \geqslant 1$, and $p \nmid k$.

Let S_k^* be a subgroup of S_r such that $S_k^* \cong S_k$ and S_k^* is the diagonal of a Young subgroup of S_r isomorphic to $S_k \times \cdots \times S_k$ (with p^m factors). (See Section 2D for more details.) Since $S_k^* \cong S_k$ we may regard $\mathrm{Lie}(k)$ as an FS_k^*-module and, since $p \nmid k$, this module is projective. Thus the induced module $\mathrm{Lie}(k){\uparrow}_{S_k^*}^{S_r}$ is also projective. It was proved in [Erdmann and Tan 2011, Theorem 3.1] that if B is a nonprincipal block of FS_r then $\mathrm{Lie}(r)_B$ is projective. Here we shall prove (Theorem 3.1) that

$$\mathrm{Lie}(r)_B \cong \frac{1}{p^m}\,(\mathrm{Lie}(k){\uparrow}_{S_k^*}^{S_r})_B \tag{1-2}$$

when B satisfies the condition $\tilde{B} \neq \varnothing$ (see Section 2B): this condition is satisfied when B is nonprincipal. (The notation $U \cong (1/q)\,V$ used in (1-2) means that $q\,U \cong V$, where $q\,U$ denotes $U \oplus \cdots \oplus U$ with q summands.) The special case where $k = 1$ is of particular interest: it yields

$$\mathrm{Lie}(p^m)_B \cong \frac{1}{p^m}\,(FS_{p^m})_B.$$

Since (1-2) holds for each nonprincipal block B, a comparison of dimensions gives a weaker result (Corollary 3.4) for the principal block B_0:

$$\dim \mathrm{Lie}(r)_{B_0} = \frac{1}{p^m} \dim (\mathrm{Lie}(k)\uparrow_{S_k^*}^{S_r})_{B_0}.$$

The projective indecomposable FS_r-modules may be labelled P^λ where λ ranges over the p-regular partitions of r (see Section 2B). For any FS_r-block B we write $\lambda \in B$ when $P^\lambda \in B$. Since $\mathrm{Lie}(r)_B$ is projective when B is nonprincipal, there are nonnegative integers m_λ such that

$$\mathrm{Lie}(r)_B \cong \bigoplus_{\lambda \in B} m_\lambda\, P^\lambda.$$

In Theorem 3.5 we prove that

$$m_\lambda = \frac{1}{r} \sum_{d \mid k} \mu(d)\, \beta^\lambda(\tau^{k/d}), \tag{1-3}$$

where μ is the Möbius function, τ is an element of S_r of cycle type $(k, k, \ldots, k)$, and β^λ denotes the Brauer character of D^λ, the irreducible FS_r-module isomorphic to the head of P^λ.

Now suppose that F is infinite and n is a positive integer. We have observed that $L^r(E)$ is an $S_F(n,r)$-module. Similarly, $L^k(E^{\otimes p^m})$ is an $S_F(n,r)$-module, and (by the argument in [Donkin and Erdmann 1998, Section 3.1]) it is isomorphic to a direct summand of $E^{\otimes r}$. It is a consequence of [Erdmann and Tan 2011, Theorem 3.2] that if B is a block of $S_F(n,r)$ satisfying the condition $\tilde{B} \neq \varnothing$ (see Section 2C) then $L^r(E)_B$ is isomorphic to a direct sum of summands of $E^{\otimes r}$. Here we shall prove (Theorem 3.6) that

$$L^r(E)_B \cong \frac{1}{p^m} L^k(E^{\otimes p^m})_B.$$

The indecomposable summands of $E^{\otimes r}$ are tilting modules $T(\lambda)$, where λ is a p-regular partition of r with at most n parts (see Section 2C). For any $S_F(n,r)$-block B we write $\lambda \in B$ when $T(\lambda) \in B$. In Theorem 3.7 we prove that if $\tilde{B} \neq \varnothing$ then

$$L^r(E)_B \cong \bigoplus_{\lambda \in B} m_\lambda\, T(\lambda),$$

where the multiplicities m_λ are given by (1-3). This extends [Donkin and Erdmann 1998, Section 3.3, Theorem], which gives the same result in the case where $p \nmid r$.

All modules over fields in this paper will be assumed to be finite-dimensional, and modules for algebras are right modules unless otherwise specified.

2. Preliminaries

2A. ***The Lie module.*** Let r and n be positive integers where $n \geqslant r$. Let Δ be the free associative ring ($\mathbb{Z}$-algebra) on free generators $x_1, \ldots, x_n$ and let L be the

Lie subring of Δ generated by $x_1, \dots, x_n$. By [Bourbaki 1972, chapitre II, §3, théorème 1], L is free on $x_1, \dots, x_n$. Let Δ_r denote the homogeneous component of Δ of degree r. Then S_r has a left action by "place permutations" on Δ_r, given by $\alpha(y_1 \cdots y_r) = y_{1\alpha} \cdots y_{r\alpha}$ for all $\alpha \in S_r$ and all $y_1, \dots, y_r \in \{x_1, \dots, x_n\}$. (Note that we write multiplication in S_r from left to right.) Hence Δ_r is a left $\mathbb{Z}S_r$-module. Let ω_r be the element of $\mathbb{Z}S_r$ given by (1-1). Then it is well known and easily verified that $\omega_r(y_1 \cdots y_r) = [y_1, \dots, y_r]$ where $[y_1, \dots, y_r]$ denotes the left-normed Lie product $[\cdots[[y_1, y_2], y_3], \dots, y_r]$.

The group S_r also has a right action on Δ by automorphisms, where $x_i\alpha = x_{i\alpha}$ for $i = 1, \dots, r$ and $x_i\alpha = x_i$ for $i > r$. Thus Δ_r becomes a $(\mathbb{Z}S_r, \mathbb{Z}S_r)$-bimodule. Let Δ_r^0 be the $\mathbb{Z}$-subspace of Δ_r spanned by the monomials $x_{1\alpha} \cdots x_{r\alpha}$ with $\alpha \in S_r$. Thus Δ_r^0 is a subbimodule of Δ_r and the map $\xi : \mathbb{Z}S_r \to \Delta_r^0$ defined by $\xi(\alpha) = x_{1\alpha} \cdots x_{r\alpha}$ is an isomorphism of bimodules.

Let $L_r^0 = L \cap \Delta_r^0$. Then L_r^0 is spanned over $\mathbb{Z}$ by all elements of the form $[x_{1\alpha}, \dots, x_{r\alpha}]$. Also, by [Bourbaki 1972, chapitre II, §3, théorème 2], L_r^0 is free of rank $(r-1)!$ as a $\mathbb{Z}$-module. We have $\omega_r\Delta_r^0 = L_r^0$. Thus the isomorphism ξ maps $\omega_r\mathbb{Z}S_r$ to L_r^0, and so $\omega_r\mathbb{Z}S_r$ is isomorphic to L_r^0 as a right $\mathbb{Z}S_r$-module.

All of the above still applies if $\mathbb{Z}$ is replaced by R, where R is an arbitrary commutative ring with unity. Also (using subscripts to show coefficient rings) we have $R \otimes_{\mathbb{Z}} L_{r,\mathbb{Z}}^0 \cong L_{r,R}^0$. The Lie module $\mathrm{Lie}_R(r)$ is the RS_r-module defined by $\mathrm{Lie}_R(r) = \omega_r RS_r$. Thus $\mathrm{Lie}_R(r) \cong L_{r,R}^0$. It follows that $\mathrm{Lie}_R(r) \cong R \otimes_{\mathbb{Z}} \mathrm{Lie}_{\mathbb{Z}}(r)$ and $\mathrm{Lie}_R(r)$ is free of rank $(r-1)!$ as an R-module. If F is a field we have

$$\dim \mathrm{Lie}_F(r) = (r-1)! \tag{2-1}$$

and, if F is understood, we write $\mathrm{Lie}_F(r)$ as $\mathrm{Lie}(r)$.

When K is a field of characteristic zero there is a formula for the character ψ_r of $\mathrm{Lie}_K(r)$. Let μ denote the Möbius function and let σ be an r-cycle of S_r. For each divisor d of r let C_d denote the conjugacy class of $\sigma^{r/d}$ in S_r. Then, for $g \in S_r$,

$$\psi_r(g) = \begin{cases} \mu(d)(r-1)!/|C_d| & \text{if } g \in C_d, \\ 0 & \text{if } g \notin C_d \text{ for all } d. \end{cases} \tag{2-2}$$

(See, for example, [Donkin and Erdmann 1998, Section 3.2].) Hence, if θ is any class function on S_r with values in K and we write

$$(\theta, \psi_r)_{S_r} = \frac{1}{|S_r|} \sum_{g \in S_r} \theta(g)\psi_r(g^{-1}), \tag{2-3}$$

we have

$$(\theta, \psi_r)_{S_r} = \frac{1}{r} \sum_{d|r} \mu(d)\theta(\sigma^{r/d}). \tag{2-4}$$

2B. *Representations of S_r.* By a partition of a nonnegative integer r we mean, as usual, a finite sequence $\lambda = (\lambda_1, \dots, \lambda_s)$ of integers satisfying $\lambda_1 \geqslant \dots \geqslant \lambda_s > 0$ and $\lambda_1 + \dots + \lambda_s = r$. We call $\lambda_1, \dots, \lambda_s$ the *parts* of λ. We write $\Lambda^+(r)$ for the set of all partitions of r. If $r = 0$ then $\Lambda^+(r)$ contains only the empty partition, which we denote by $\varnothing$. Let p be a prime number. A partition λ is *p-regular* if λ does not have p or more equal parts, and we write $\Lambda^+_p(r)$ for the set of all p-regular partitions of r. The *p-core* of a partition λ of r is the partition $\tilde{\lambda}$ of r', for some $r' \leqslant r$, obtained from (the diagram of) λ by the removal of as many "rim p-hooks" as possible: see [James and Kerber 1981, Section 2.7]. We write $\mathscr{C}_r$ for the set of all p-cores of elements of $\Lambda^+(r)$.

Let r be a positive integer and let F be a field of prime characteristic p. The irreducible FS_r-modules may be labelled (up to isomorphism) as D^λ with $\lambda \in \Lambda^+_p(r)$, where D^λ is a quotient module of the Specht module S^λ (see [James and Kerber 1981, 7.1.14]). Here $D^{(r)}$ is isomorphic to the trivial FS_r-module F because $S^{(r)} \cong F$. For each λ we write P^λ for the projective cover of D^λ. Thus the projective indecomposable FS_r-modules are the P^λ with $\lambda \in \Lambda^+_p(r)$. If F' is an extension field of F then (using subscripts to show coefficient fields) we have $D^\lambda_{F'} \cong F' \otimes_F D^\lambda_F$ and $P^\lambda_{F'} \cong F' \otimes_F P^\lambda_F$.

We recall a few general facts about blocks. If Γ is a finite-dimensional F-algebra we may write Γ uniquely as a finite direct sum of indecomposable two-sided ideals, $\Gamma = \bigoplus_{B \in \mathscr{B}} \Gamma_B$. These ideals are the blocks of Γ, but it is convenient also to refer to the labels B as the blocks. The identity element of Γ may be written as $\sum_{B \in \mathscr{B}} e_B$, with $e_B \in \Gamma_B$ for all B. The elements e_B are the block idempotents: they are primitive central idempotents of Γ (see, for example, [Benson 1995]). Any Γ-module V satisfying $Ve_B = V$ is said to belong to B, and we write $V \in B$. Every Γ-module V may be written uniquely in the form $V = \bigoplus_{B \in \mathscr{B}} V_B$, where $V_B \in B$ for all B (indeed, $V_B = Ve_B$). We call V_B the block component of V corresponding to B.

By the Nakayama conjecture (see [James and Kerber 1981, 6.1.21]), the blocks of FS_r may be labelled $B(\nu)$ with $\nu \in \mathscr{C}_r$ in such a way that $S^\lambda \in B(\tilde{\lambda})$ for all $\lambda \in \Lambda^+_p(r)$. Since D^λ is a quotient of S^λ and P^λ is indecomposable with D^λ as a quotient, we have $D^\lambda, P^\lambda \in B(\tilde{\lambda})$. We use the same notation for the blocks of FS_r for every field F of characteristic p. By consideration of composition factors we see that, for any FS_r-module V, any extension field F' of F, and any ν, we have

$$(F' \otimes_F V)_{B(\nu)} \cong F' \otimes_F V_{B(\nu)}. \tag{2-5}$$

If B is a block and $B = B(\nu)$ we write $\tilde{B} = \nu$. Also, for $\lambda \in \Lambda^+_p(r)$, we write $\lambda \in B$ if $D^\lambda \in B$ (or equivalently $P^\lambda \in B$). The *principal block* is the block B_0 containing the trivial irreducible $D^{(r)}$. Thus $\tilde{B}_0 = (\bar{r})$, where $\bar{r}$ denotes the remainder on dividing r by p. If $\tilde{B} = \varnothing$ then $p \mid r$ and $\bar{r} = 0$ so that $B = B_0$. Hence if B is nonprincipal we have $\tilde{B} \neq \varnothing$.

If $p \nmid r$ then $\mathrm{Lie}(r)$ is projective (see Section 1). But if $p \mid r$ and $\tilde{B} \neq \varnothing$ then $B \neq B_0$ and so $\mathrm{Lie}(r)_B$ is projective by [Erdmann and Tan 2011, Theorem 3.1]. Hence we have the following result.

Theorem 2.1 [Erdmann and Tan 2011]. *If B is a block of FS_r such that $\tilde{B} \neq \varnothing$ then* $\mathrm{Lie}(r)_B$ *is projective.*

As is well known, Brauer characters of FS_r-modules have integer values: this follows, for example, from [Nagao and Tsushima 1989, Chapter 3, Lemma 6.13]. (Consequently Brauer characters of FS_r-modules are uniquely defined and do not depend upon choices of roots of unity.) We regard Brauer characters as maps from S_r to $\mathbb{Z}$ by assigning the value zero to p-singular elements of S_r. For each $\lambda \in \Lambda_p^+(r)$ we write β^λ and ζ^λ for the Brauer characters of D^λ and P^λ, respectively. By the orthogonality relations for Brauer characters (see [Nagao and Tsushima 1989, Chapter 3, Theorem 6.10]) we have

$$(\beta^\lambda, \zeta^\rho)_{S_r} = \begin{cases} 1 & \text{if } \lambda = \rho, \\ 0 & \text{if } \lambda \neq \rho, \end{cases} \tag{2-6}$$

where $(\beta^\lambda, \zeta^\rho)_{S_r}$ is defined as in (2-3).

2C. *Polynomial representations of* $\mathbf{GL}_n(F)$. Suppose now that F is an infinite field of prime characteristic p and let n and r be positive integers. We refer to [Green 1980] and [Donkin and Erdmann 1998] for background concerning polynomial $\mathrm{GL}_n(F)$-modules and the Schur algebra $S_F(n,r)$. Let E denote the natural $\mathrm{GL}_n(F)$-module. Thus $E^{\otimes r}$ is an $S_F(n,r)$-module. If k and t are positive integers such that $r = kt$ and if V is an $S_F(n,t)$-module then $V^{\otimes k}$ and $L^k(V)$ are $S_F(n,r)$-modules.

Let $\Lambda^+(n,r)$ denote the set of all partitions of r with at most n parts and let $\Lambda_p^+(n,r)$ denote the set of all p-regular partitions in $\Lambda^+(n,r)$. The irreducible $S_F(n,r)$-modules may be labelled $L(\lambda)$ with $\lambda \in \Lambda^+(n,r)$. For each such λ there is also an indecomposable $S_F(n,r)$-module $T(\lambda)$ called a "tilting module", and (see [Donkin and Erdmann 1998, Section 1.3]) there are nonnegative integers n_λ such that

$$E^{\otimes r} \cong \bigoplus_{\lambda \in \Lambda_p^+(n,r)} n_\lambda \, T(\lambda). \tag{2-7}$$

The main facts about the blocks of $S_F(n,r)$ were obtained in [Donkin 1994] and summarised in [Erdmann and Tan 2011]. When $n \geqslant r$ the blocks may be labelled $B(\nu)$ with $\nu \in \mathscr{C}_r$ in such a way that $L(\lambda) \in B(\tilde{\lambda})$. If B is a block and $B = B(\nu)$ we write $\tilde{B} = \nu$. When $n < r$, p-cores do not necessarily label unique blocks, but if $L(\lambda)$ and $L(\rho)$ are in the same block then $\tilde{\lambda} = \tilde{\rho}$. Thus, for each block B, there is an element $\tilde{B}$ of $\mathscr{C}_r$ (where $\tilde{B}$ has at most n parts) with the property that $\tilde{\lambda} = \tilde{B}$ whenever $L(\lambda) \in B$. For each $\nu \in \mathscr{C}_r$ we write $B(\nu)$ for the set of blocks B such that $\tilde{B} = \nu$. (Thus $B(\nu)$ is empty if ν has more than n parts.) If V is an $S_F(n,r)$-module we write $V_{B(\nu)}$ for the direct sum of the block components V_B of V corresponding to blocks B in $B(\nu)$. For all n and all $\lambda \in \Lambda^+(n,r)$, $T(\lambda)$ is indecomposable and has $L(\lambda)$ as a composition factor (see [Erdmann 1994, Section 1.3]); thus $T(\lambda)$ and $L(\lambda)$ belong to the same block. For a block B and $\lambda \in \Lambda^+(n,r)$ we write $\lambda \in B$ if $L(\lambda) \in B$ (or equivalently $T(\lambda) \in B$). We define the principal block to be the block B_0 containing $L(\lambda)$ where $\lambda = (r)$. Thus $\tilde{B_0} = (\bar{r})$, with $\bar{r}$ as before. As in the case of FS_r, if $n \geqslant r$ and B is nonprincipal then $\tilde{B} \neq \varnothing$.

Let $\mathscr{T}$ denote the class of all $S_F(n,r)$-modules that are isomorphic to direct sums of tilting modules $T(\lambda)$ where $\lambda \in \Lambda_p^+(n,r)$. Thus $E^{\otimes r} \in \mathscr{T}$ by (2-7). If $p \nmid r$ then $L^r(E)$ is isomorphic to a direct summand of $E^{\otimes r}$ (see Section 1) and so $L^r(E) \in \mathscr{T}$. But if $p \mid r$ and $\tilde{B} \neq \varnothing$ then $L^r(E)_B \in \mathscr{T}$ by [Erdmann and Tan 2011, Theorem 3.2]. Hence we have the following result.

Theorem 2.2 [Erdmann and Tan 2011]. *If B is a block of $S_F(n,r)$ such that $\tilde{B} \neq \varnothing$ then $L^r(E)_B \in \mathscr{T}$.*

Suppose now that n_1 and n_2 are positive integers with $n_1 \geqslant n_2$ and let d_{n_1,n_2} denote the functor from the category of $S_F(n_1,r)$-modules to the category of $S_F(n_2,r)$-modules described in [Green 1980, Section 6.5]. This functor is exact (in particular it preserves direct sums) and we call it *truncation*. Note that $\Lambda^+(n_2,r) \subseteq \Lambda^+(n_1,r)$. We temporarily use subscripts to distinguish between modules for $S_F(n_1,r)$ and $S_F(n_2,r)$. Then, if $\lambda \in \Lambda^+(n_1,r)$ and $M(\lambda)$ denotes either $L(\lambda)$ or $T(\lambda)$, we have

$$d_{n_1,n_2}(M_{n_1}(\lambda)) \cong \begin{cases} M_{n_2}(\lambda) & \text{if } \lambda \in \Lambda^+(n_2,r), \\ 0 & \text{otherwise.} \end{cases} \tag{2-8}$$

(For the case of $L(\lambda)$ see [Green 1980, Section 6.5] and for $T(\lambda)$ see [Erdmann 1994, Section 1.7].)

Write $d = d_{n_1,n_2}$ and use the same notation for arbitrary r. Then, if k and t are positive integers and V is an $S_F(n_1,t)$-module, it is easy to check that $d(V^{\otimes k}) \cong d(V)^{\otimes k}$ and $d(L^k(V)) \cong L^k(d(V))$. Furthermore $d(E_{n_1}^{\otimes t}) \cong E_{n_2}^{\otimes t}$. Also, if V is an $S_F(n_1,r)$-module and $\nu \in \mathscr{C}_r$, it follows from (2-8) that

$$d(V_{B(\nu)}) \cong d(V)_{B(\nu)}. \tag{2-9}$$

2D. ***The Schur functor.*** We continue with the notation of the previous subsection but now assume that $n \geqslant r$. The Schur functor f_r is an exact functor from the category of $S_F(n,r)$-modules to the category of FS_r-modules (see [Green 1980, Chapter 6]). If U is an $S_F(n,r)$-module then $f_r(U)$ may be thought of as the weight space of U corresponding to the weight $(1,\dots,1,0,\dots,0)$, with r coordinates equal to 1, and the action of S_r on $f_r(U)$ comes by taking S_r as a group of permutation matrices in $\mathrm{GL}_n(F)$ (see, for example, [Donkin and Erdmann 1998, Section 1.2]). It is easily seen that

$$f_r(E^{\otimes r}) \cong FS_r. \tag{2-10}$$

Let $\{e_1,\dots,e_n\}$ be the standard basis of E. Then $f_r(L^r(E))$ is the subspace of $L^r(E)$ spanned by the left-normed Lie products $[e_{1\alpha},\dots,e_{r\alpha}]$ with $\alpha \in S_r$. In the notation of Section 2A, $f_r(L^r(E)) \cong L^0_{r,F}$. Thus, since $L^0_{r,F} \cong \mathrm{Lie}(r)$, we obtain

$$f_r(L^r(E)) \cong \mathrm{Lie}(r). \tag{2-11}$$

For all $\lambda \in \Lambda^+_p(n,r) = \Lambda^+_p(r)$, we have (see [Donkin and Erdmann 1998, Section 1.3])

$$f_r(T(\lambda)) \cong P^\lambda. \tag{2-12}$$

As observed in [Erdmann and Tan 2011], f_r sends modules in the $S_F(n,r)$-block $B(\nu)$ to modules in the FS_r-block $B(\nu)$ labelled by the same p-core. Thus, if V is any $S_F(n,r)$-module, we have

$$f_r(V_{B(\nu)}) \cong f_r(V)_{B(\nu)}. \tag{2-13}$$

Let k be a divisor of r, and write $t = r/k$. (We do not at present assume that $p \nmid k$.) For each $\alpha \in S_k$ we may define $\alpha^* \in S_r$ by $((i-1)t+j)\alpha^* = (i\alpha - 1)t + j$ for $i = 1,\dots,k$ and $j = 1,\dots,t$. The set $\{\alpha^* : \alpha \in S_k\}$ is a subgroup S^*_k of S_r isomorphic to S_k. The subgroup of S_r consisting of all permutations fixing $\{(i-1)t+j : i = 1,\dots,k\}$ setwise for $j = 1,\dots,t$ is a Young subgroup of S_r isomorphic to $S_k \times \cdots \times S_k$, and S^*_k may be thought of as the diagonal of this subgroup. The diagonal of any other Young subgroup isomorphic to $S_k \times \cdots \times S_k$ is a conjugate of S^*_k in S_r. Note that if σ is the r-cycle $(1\ 2\ \dots\ r)$ of S_r and σ_k is the k-cycle $(1\ 2\ \dots\ k)$ of S_k then $\sigma^t = \sigma^*_k \in S^*_k$. For $i = 1,\dots,k$, write $\Omega_i = \{(i-1)t+j : j = 1,\dots,t\}$. The subgroup $S^{(k)}_t$ of S_r consisting of all permutations fixing each Ω_i setwise is a Young subgroup isomorphic to $S_t \times \cdots \times S_t$. For each $\alpha \in S_k$ we have $\Omega_i\alpha^* = \Omega_{i\alpha}$ for $i = 1,\dots,k$. The subgroup $S^{(k)}_t S^*_k$ of S_r is isomorphic to the wreath product $S_t \operatorname{wr} S_k$.

Let V be an $S_F(n,t)$-module. Then $f_t(V)$ is an FS_t-module, so $f_t(V)^{\otimes k}$ is an $FS^{(k)}_t$-module. Indeed, $f_t(V)^{\otimes k}$ may be regarded as an $FS^{(k)}_t S^*_k$-module, where the action of S^*_k is to permute the tensor factors. We regard $\mathrm{Lie}(k)$ as an FS^*_k-module by means of the isomorphism $\alpha \mapsto \alpha^*$ from S_k to S^*_k. Then $\mathrm{Lie}(k)$ may also be regarded as an $FS^{(k)}_t S^*_k$-module, by taking trivial action of $S^{(k)}_t$. The following result is part of [Lim and Tan 2012, Corollary 3.2].

Lemma 2.3 [Lim and Tan 2012]. *In the above notation,*

$$f_r(L^k(V)) \cong (f_t(V)^{\otimes k} \otimes \mathrm{Lie}(k)){\uparrow}_{S_t^{(k)} S_k^*}^{S_r}.$$

Corollary 2.4. *In the above notation,*

$$f_r(L^k(E^{\otimes t})) \cong \mathrm{Lie}(k){\uparrow}_{S_k^*}^{S_r}.$$

Proof. By (2-10) and Lemma 2.3,

$$f_r(L^k(E^{\otimes t})) \cong ((FS_t)^{\otimes k} \otimes \mathrm{Lie}(k)){\uparrow}_{S_t^{(k)} S_k^*}^{S_r}.$$

Clearly $(FS_t)^{\otimes k}$ is a transitive permutation module under the action of $S_t^{(k)} S_k^*$ and the stabiliser of the basis element $1 \otimes \cdots \otimes 1$ is S_k^*. Thus $(FS_t)^{\otimes k}$ is induced from a one-dimensional trivial module for S_k^* and (by [Benson 1995, Proposition 3.3.3(i)]) we have

$$(FS_t)^{\otimes k} \otimes \mathrm{Lie}(k) \cong \mathrm{Lie}(k){\uparrow}_{S_k^*}^{S_t^{(k)} S_k^*}.$$

The result follows. □

3. Main results

Recall from Section 2B that if B is a nonprincipal block of FS_r then $\tilde{B} \neq \varnothing$. Our main result on the Lie module is as follows. We use the notation of Section 2D, regarding $\mathrm{Lie}(k)$ as an FS_k^*-module.

Theorem 3.1. *Let F be a field of prime characteristic p. Let r be a positive integer and write $r = p^m k$ where $m \geqslant 0$, $k \geqslant 1$, and $p \nmid k$. Let B be a block of FS_r such that $\tilde{B} \neq \varnothing$ and let S_k^* be the diagonal of a Young subgroup $S_k \times \cdots \times S_k$ of S_r. Then*

$$\mathrm{Lie}(r)_B \cong \frac{1}{p^m} (\mathrm{Lie}(k){\uparrow}_{S_k^*}^{S_r})_B.$$

Note that $\mathrm{Lie}(k){\uparrow}_{S_k^*}^{S_r}$ is projective since $\mathrm{Lie}(k)$ is projective (see Section 1).

We commence the proof of Theorem 3.1. If F' is an extension field of F then, by the description of the Lie module in Section 2A, $\mathrm{Lie}_{F'}(r) \cong F' \otimes \mathrm{Lie}_F(r)$ and $\mathrm{Lie}_{F'}(k) \cong F' \otimes \mathrm{Lie}_F(k)$. Thus, if B is any block of FS_r, we have $\mathrm{Lie}_{F'}(r)_B \cong F' \otimes \mathrm{Lie}_F(r)_B$ and $(\mathrm{Lie}_{F'}(k){\uparrow}_{S_k^*}^{S_r})_B \cong F' \otimes (\mathrm{Lie}_F(k){\uparrow}_{S_k^*}^{S_r})_B$ by (2-5). Hence it suffices to prove Theorem 3.1 for the prime field $\mathbb{F}_p$ and then, by the Noether–Deuring theorem, it suffices to prove the theorem for any chosen field F of characteristic p. We choose F so that there is a p-modular system (K, R, F) with the properties specified in [Nagao and Tsushima 1989, Chapter 3, Section 6]. Note, in particular, that K has characteristic 0 and contains sufficient roots of unity, K is the field of fractions of R, and $F = R/(\pi)$ where (π) is the maximal ideal of R.

We state some standard facts associated with p-modular systems in order to establish terminology and notation. If G is any finite group then the natural epimorphism $R \to F$ yields an epimorphism $RG \to FG$. If this epimorphism maps u to v, where $u \in RG$ and $v \in FG$, we say that v *lifts* to u. By an *RG-lattice* we mean an RG-module that is free of finite rank as an R-module. If M is an RG-lattice we write $\overline{M} = M/\pi M$. Thus $\overline{M} \cong F \otimes_R M$ and $\overline{M}$ has the structure of an FG-module. An FG-module V is said to be *liftable* if there exists M such that $\overline{M} \cong V$, in which case we say that V lifts to M. If M is an RG-lattice then $K \otimes_R M$ is a KG-module. If U is any KG-module then there is an RG-lattice M such that $U \cong K \otimes_R M$ (see [Benson 1995, Lemma 1.9.1]) and we say that $\overline{M}$ is obtained from U by *modular reduction*.

By a standard result (see [ibid., Theorem 1.9.4]), each block idempotent e_B of FS_r can be lifted to an element $\widehat{e}_B$ of RS_r to obtain pairwise-orthogonal primitive central idempotents of RS_r summing to the identity. If M is an RS_r-lattice such that $M\widehat{e}_B = M$ we write $M \in B$. Every RS_r-lattice M may be written uniquely in the form $M = \bigoplus_B M_B$ where, for each B, M_B is an RS_r-lattice belonging to B. Similar facts and notation apply to KS_r-modules, using the same idempotents $\widehat{e}_B$. If U is a KS_r-module then, since K has characteristic zero, U_B is a direct sum of irreducible KS_r-modules belonging to B.

It is easily verified that if M is an RS_r-lattice and B is a block then

$$(K \otimes_R M)_B \cong K \otimes_R M_B \quad \text{and} \quad \overline{M_B} \cong \overline{M}_B. \tag{3-1}$$

We let σ be an r-cycle of S_r chosen as in Section 2D (with $t = p^m$) so that $\sigma^{p^m} = \sigma_k^* \in S_k^*$, where σ_k is a k-cycle of S_k.

Lemma 3.2. *If g is an element of the cyclic subgroup $\langle\sigma\rangle$ such that g has order divisible by p and if χ is the character of an irreducible KS_r-module U belonging to a block B such that $\tilde{B} \neq \varnothing$ then $\chi(g) = 0$.*

Proof. Let M be an RS_r-lattice such that $U \cong K \otimes_R M$. Since U belongs to B it follows from (3-1) that M belongs to B. Let D be the defect group of the FS_r-block B (see [Benson 1995, Section 6.1]). (Thus D is a p-group, determined up to conjugacy in S_r.) By [ibid., Corollary 6.1.3], D is also the defect group of B regarded as a block of RS_r. Thus, by [ibid., Proposition 6.1.2], M is projective relative to D.

Let $B = B(\nu)$ where $\nu \in \mathscr{C}_r$. Thus $\nu \neq \varnothing$ and so ν is a partition of r' for some r' satisfying $0 < r' \leqslant r$. It follows from [James and Kerber 1981, 6.2.45] that D can be taken to be a Sylow p-subgroup of a subgroup $S_{r-r'}$ of S_r fixing r' points of $\{1, \dots, r\}$. Hence every element of D fixes some point of $\{1, \dots, r\}$.

Let g be as in the statement of the lemma. The p-part of g is a nontrivial element of $\langle\sigma\rangle$ and hence has no fixed points in $\{1, \dots, r\}$. It follows that the p-part of g is not conjugate in S_r to an element of D. Therefore, by [Nagao and Tsushima 1989, Chapter 4, Theorem 7.4], we have $\chi(g) = 0$, as required. $\square$

Lemma 3.3. *If B is a block such that $\tilde{B} \neq \varnothing$ then* $p^m \operatorname{Lie}_K(r)_B \cong (\operatorname{Lie}_K(k){\uparrow}_{S_k^*}^{S_r})_B$.

Proof. The result is trivial if $r = k$. Thus we may assume that $p \mid r$. Let ψ_r denote the character of the KS_r-module $\operatorname{Lie}_K(r)$ and let ψ_k denote the character of the KS_k^*-module $\operatorname{Lie}_K(k)$. In order to prove the lemma it suffices to show that the multiplicity of each irreducible KS_r-module U belonging to B is the same in $p^m \operatorname{Lie}_K(r)$ as in $\operatorname{Lie}_K(k){\uparrow}_{S_k^*}^{S_r}$. Let χ be the character of U. By the orthogonality relations and Frobenius reciprocity for ordinary characters, it suffices to prove

$$p^m\,(\chi, \psi_r)_{S_r} = (\chi{\downarrow}_{S_k^*}^{S_r}, \psi_k)_{S_k^*}. \tag{3-2}$$

By (2-4) we have

$$\begin{aligned} r\,(\chi, \psi_r)_{S_r} &= \sum_{d \mid r} \mu(d)\chi(\sigma^{r/d}) \\ &= \sum_{d \mid k} \mu(d)\chi(\sigma^{r/d}) - \sum_{d \mid k} \mu(d)\chi(\sigma^{r/pd}). \end{aligned}$$

However, for $d \mid k$, we have $\chi(\sigma^{r/pd}) = 0$ by Lemma 3.2. Thus

$$r\,(\chi, \psi_r)_{S_r} = \sum_{d \mid k} \mu(d)\chi((\sigma^{p^m})^{k/d}).$$

Recall that $\sigma^{p^m} = \sigma_k^* \in S_k^*$ where σ_k is a k-cycle of S_k. Hence, by (2-4) applied to S_k^*,

$$k\,(\chi{\downarrow}_{S_k^*}^{S_r}, \psi_k)_{S_k^*} = \sum_{d \mid k} \mu(d)\chi((\sigma^{p^m})^{k/d}).$$

This gives (3-2). $\square$

We can now prove Theorem 3.1. Let B be a block of FS_r such that $\tilde{B} \neq \varnothing$. By the description of the Lie module in Section 2A, $\operatorname{Lie}(r)$ lifts to the RS_r-lattice $\operatorname{Lie}_R(r)$ and $\operatorname{Lie}(k)$ lifts to the RS_k^*-lattice $\operatorname{Lie}_R(k)$. Thus $p^m \operatorname{Lie}(r)$ and $\operatorname{Lie}(k){\uparrow}_{S_k^*}^{S_r}$ lift to $p^m \operatorname{Lie}_R(r)$ and $\operatorname{Lie}_R(k){\uparrow}_{S_k^*}^{S_r}$, respectively. Also, $K \otimes p^m \operatorname{Lie}_R(r) \cong p^m \operatorname{Lie}_K(r)$ and $K \otimes \operatorname{Lie}_R(k){\uparrow}_{S_k^*}^{S_r} \cong \operatorname{Lie}_K(k){\uparrow}_{S_k^*}^{S_r}$. Hence $p^m \operatorname{Lie}(r)$ and $\operatorname{Lie}(k){\uparrow}_{S_k^*}^{S_r}$ are modular reductions of $p^m \operatorname{Lie}_K(r)$ and $\operatorname{Lie}_K(k){\uparrow}_{S_k^*}^{S_r}$, respectively. It follows by (3-1) that $p^m \operatorname{Lie}(r)_B$ and $(\operatorname{Lie}(k){\uparrow}_{S_k^*}^{S_r})_B$ are modular reductions of $p^m \operatorname{Lie}_K(r)_B$ and $(\operatorname{Lie}_K(k){\uparrow}_{S_k^*}^{S_r})_B$, respectively. However, by Lemma 3.3, these two last modules are isomorphic. Therefore, by [Nagao and Tsushima 1989, Chapter 3, Lemma 6.4], $p^m \operatorname{Lie}(r)_B$ and $(\operatorname{Lie}(k){\uparrow}_{S_k^*}^{S_r})_B$ have the same Brauer character.

By Theorem 2.1, $\operatorname{Lie}(r)_B$ is projective. Since $\operatorname{Lie}(k)$ is a projective FS_k^*-module, $\operatorname{Lie}(k){\uparrow}_{S_k^*}^{S_r}$ is a projective FS_r-module and so $(\operatorname{Lie}(k){\uparrow}_{S_k^*}^{S_r})_B$ is projective. Thus

$p^m \operatorname{Lie}(r)_B$ and $(\operatorname{Lie}(k)\uparrow_{S_k^*}^{S_r})_B$ are projective modules with the same Brauer characters. Therefore, by [Benson 1995, Corollary 5.3.6], these modules are isomorphic. This proves Theorem 3.1.

Corollary 3.4. *If B_0 is the principal block of FS_r then*

$$\dim \operatorname{Lie}(r)_{B_0} = \frac{1}{p^m} \dim (\operatorname{Lie}(k)\uparrow_{S_k^*}^{S_r})_{B_0} .$$

Proof. For each nonprincipal block B of FS_r we have

$$\dim \operatorname{Lie}(r)_B = \frac{1}{p^m} \dim (\operatorname{Lie}(k)\uparrow_{S_k^*}^{S_r})_B ,$$

by Theorem 3.1. However, by (2-1),

$$\dim \operatorname{Lie}(k)\uparrow_{S_k^*}^{S_r} = (k-1)!\, r!/k! = p^m (r-1)! = p^m \dim \operatorname{Lie}(r).$$

The result follows. □

Theorem 3.5. *In the notation of* Theorem 3.1, *we have*

$$\operatorname{Lie}(r)_B \cong \bigoplus_{\substack{\lambda \in \Lambda_p^+(r) \\ \lambda \in B}} m_\lambda P^\lambda, \tag{3-3}$$

where, for each λ,

$$m_\lambda = \frac{1}{r} \sum_{d \mid k} \mu(d)\, \beta^\lambda(\tau^{k/d}), \tag{3-4}$$

where τ is an element of S_r of cycle type $(k, k, \dots, k)$ and β^λ denotes the Brauer character of D^λ.

Proof. By Theorem 2.1, $\operatorname{Lie}(r)_B$ is projective. Thus it satisfies (3-3) for suitable nonnegative integers m_λ. It remains to prove (3-4). If F' is an extension field of F then $\operatorname{Lie}_{F'}(r) \cong F' \otimes \operatorname{Lie}_F(r)$ and $P^\lambda_{F'} \cong F' \otimes P^\lambda_F$. Also, block components are preserved under field extensions, by (2-5). Hence it suffices to prove the result for the field $\mathbb{F}_p$ and then, by a similar argument, it suffices to prove the result for any chosen field F of characteristic p. We take F from the p-modular system (K, R, F) used in the proof of Theorem 3.1.

Since $\operatorname{Lie}(k)\uparrow_{S_k^*}^{S_r}$ is projective we have

$$\operatorname{Lie}(k)\uparrow_{S_k^*}^{S_r} \cong \bigoplus_{\rho \in \Lambda_p^+(r)} m'_\rho P^\rho$$

for suitable nonnegative integers m'_ρ. Let $\lambda \in \Lambda_p^+(r)$ where $\lambda \in B$. By Theorem 3.1 we have $m_\lambda = (1/p^m) m'_\lambda$. Let ϕ denote the Brauer character of $\operatorname{Lie}(k)\uparrow_{S_k^*}^{S_r}$. By the orthogonality relation (2-6) we have

$$m_\lambda = \frac{1}{p^m} m'_\lambda = \frac{1}{p^m} (\beta^\lambda, \phi)_{S_r}.$$

As observed in the proof of Theorem 3.1, $\mathrm{Lie}(k)\uparrow_{S_k^*}^{S_r}$ is a modular reduction of $\mathrm{Lie}_K(k)\uparrow_{S_k^*}^{S_r}$. The character of $\mathrm{Lie}_K(k)\uparrow_{S_k^*}^{S_r}$ is $\psi_k\uparrow_{S_k^*}^{S_r}$, where ψ_k denotes the character of $\mathrm{Lie}_K(k)$ as a KS_k^*-module. By [Nagao and Tsushima 1989, Chapter 3, Lemma 6.4], ϕ and $\psi_k\uparrow_{S_k^*}^{S_r}$ take the same value on p'-elements of S_r. Thus, by Frobenius reciprocity,

$$m_\lambda = \frac{1}{p^m} (\beta^\lambda, \psi_k\uparrow_{S_k^*}^{S_r})_{S_r} = \frac{1}{p^m} (\beta^\lambda\downarrow_{S_k^*}^{S_r}, \psi_k)_{S_k^*}.$$

Let τ be as in the statement of the theorem. Then τ is conjugate to, and therefore can be taken to be, an element σ_k^* of S_k^* corresponding to a k-cycle σ_k of S_k. Thus, by (2-4), we have

$$\frac{1}{p^m} (\beta^\lambda\downarrow_{S_k^*}^{S_r}, \psi_k)_{S_k^*} = \frac{1}{p^m k} \sum_{d \mid k} \mu(d)\, \beta^\lambda(\tau^{k/d}).$$

The result follows. □

We now turn to Lie powers and, for the rest of this section, we assume that F is infinite. As before, let n be a positive integer and let E be the natural $\mathrm{GL}_n(F)$-module.

Theorem 3.6. *Let F be an infinite field of prime characteristic p. Let r be a positive integer and write $r = p^m k$ where $m \geqslant 0$, $k \geqslant 1$, and $p \nmid k$. Let B be a block of $S_F(n, r)$ such that $\tilde{B} \neq \varnothing$. Then*

$$L^r(E)_B \cong \frac{1}{p^m} L^k(E^{\otimes p^m})_B.$$

Proof. Let $\mathcal{T}$ be as defined in Section 2C. Thus, by Theorem 2.2, $L^r(E)_B \in \mathcal{T}$. Also, since $L^k(E^{\otimes p^m})$ is a direct summand of $E^{\otimes r}$, we have $L^k(E^{\otimes p^m}) \in \mathcal{T}$, by (2-7).

Suppose first that $n \geqslant r$. Then we may write $B = B(\nu)$ where $\nu \neq \varnothing$. By (2-11) and (2-13),

$$f_r(p^m L^r(E)_{B(\nu)}) \cong p^m \,\mathrm{Lie}(r)_{B(\nu)}.$$

Similarly, by Corollary 2.4 and (2-13),

$$f_r(L^k(E^{\otimes p^m})_{B(\nu)}) \cong (\mathrm{Lie}(k)\uparrow_{S_k^*}^{S_r})_{B(\nu)}.$$

Also, by Theorem 3.1, $p^m \,\mathrm{Lie}(r)_{B(\nu)} \cong (\mathrm{Lie}(k)\uparrow_{S_k^*}^{S_r})_{B(\nu)}$. It follows from (2-12) that if $U, V \in \mathcal{T}$ and $f_r(U) \cong f_r(V)$ then $U \cong V$. Hence the isomorphism in Theorem 3.6 holds when $n \geqslant r$.

Now suppose that $n < r$ and let $\tilde{B} = \nu$. Thus $B \in B(\nu)$. Consider the $S_F(r,r)$-block $B(\nu)$. By the first case, there is an isomorphism of $S_F(r,r)$-modules,

$$L^r(E)_{B(\nu)} \cong \frac{1}{p^m} L^k(E^{\otimes p^m})_{B(\nu)}. \tag{3-5}$$

We apply truncation $d_{r,n}$ to (3-5). By (2-9) and the other properties of truncation given in Section 2C, we obtain (3-5) for $S_F(n,r)$-modules. Hence the corresponding block components are isomorphic for all $S_F(n,r)$-blocks in $B(\nu)$ and we obtain the isomorphism of Theorem 3.6. □

Theorem 3.7. *In the notation of Theorem 3.6, we have*

$$L^r(E)_B \cong \bigoplus_{\substack{\lambda \in \Lambda_p^+(n,r) \\ \lambda \in B}} m_\lambda T(\lambda),$$

where m_λ is given by (3-4).

Proof. By Theorem 2.2, $L^r(E)_B \in \mathcal{T}$. Thus $L^r(E)_B$ is isomorphic to a direct sum of tilting modules $T(\lambda)$ with $\lambda \in \Lambda_p^+(n,r)$ and $\lambda \in B$. Let $\tilde{B} = \nu$. Then, for $n \geqslant r$, we have $f_r(L^r(E)_{B(\nu)}) \cong \mathrm{Lie}(r)_{B(\nu)}$, by (2-11) and (2-13), and $f_r(T(\lambda)) \cong P^\lambda$ for all $\lambda \in \Lambda_p^+(n,r)$, by (2-12). Thus, for $n \geqslant r$, the result is given by Theorem 3.5. For $n < r$ the result follows by truncation, as in the proof of Theorem 3.6. (Note that the effect of truncation on $T(\lambda)$ is given by (2-8).) □

References

[Benson 1995] D. J. Benson, *Representations and cohomology, vol. 1: Basic representation theory of finite groups and associative algebras*, Cambridge Studies in Advanced Mathematics **30**, Cambridge University Press, 1995. MR 1644252 Zbl 0908.20001

[Bourbaki 1972] N. Bourbaki, *Groupes et algèbres de Lie: Chapitre II: Algèbres de Lie libres; Chapitre III: Groupes de Lie*, Actualités Scientifiques et Industrielles **1349**, Hermann, Paris, 1972. MR 58 #28083a Zbl 0244.22007

[Bryant 2009] R. M. Bryant, "Lie powers of infinite-dimensional modules", *Beiträge Algebra Geom.* **50**:1 (2009), 179–193. MR 2010f:20004 Zbl 1185.17003

[Donkin 1994] S. Donkin, "On Schur algebras and related algebras, IV: The blocks of the Schur algebras", *J. Algebra* **168**:2 (1994), 400–429. MR 95j:20037 Zbl 0832.20013

[Donkin and Erdmann 1998] S. Donkin and K. Erdmann, "Tilting modules, symmetric functions, and the module structure of the free Lie algebra", *J. Algebra* **203**:1 (1998), 69–90. MR 99e:20056 Zbl 0984.20029

[Erdmann 1994] K. Erdmann, "Symmetric groups and quasi-hereditary algebras", pp. 123–161 in *Finite-dimensional algebras and related topics* (Ottawa, 1992), edited by V. Dlab and L. L. Scott, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci. **424**, Kluwer, Dordrecht, 1994. MR 95k:20012 Zbl 0831.20013

[Erdmann and Tan 2011] K. Erdmann and K. M. Tan, "The non-projective part of the Lie module for the symmetric group", *Arch. Math.* (*Basel*) **96**:6 (2011), 513–518. MR 2012h:20025 Zbl 1234.20013

[Green 1980] J. A. Green, *Polynomial representations of* GL_n, Lecture Notes in Mathematics **830**, Springer-Verlag, Berlin, 1980. MR 606556 (83j:20003) Zbl 0451.20037

[James and Kerber 1981] G. James and A. Kerber, *The representation theory of the symmetric group*, Encyclopedia of Mathematics and its Applications **16**, Addison-Wesley, Reading, MA, 1981. MR 83k:20003 Zbl 0491.20010

[Lim and Tan 2012] K. J. Lim and K. M. Tan, "The Schur functor on tensor powers", *Arch. Math.* (*Basel*) **98**:2 (2012), 99–104. Zbl 06018039

[Nagao and Tsushima 1989] H. Nagao and Y. Tsushima, *Representations of finite groups*, Academic Press, Boston, MA, 1989. MR 90h:20008 Zbl 0673.20002

Communicated by David Benson
Received 2011-03-10 Revised 2011-06-08 Accepted 2011-07-06

roger.bryant@manchester.ac.uk
School of Mathematics, University of Manchester, Oxford Road, Manchester, M13 9PL, United Kingdom

erdmann@maths.ox.ac.uk *Mathematical Institute, University of Oxford, 24-29 St. Giles, Oxford, OX1 3LB, United Kingdom*
http://www.maths.ox.ac.uk/~erdmann

Algebra & Number Theory

msp.berkeley.edu/ant

See inside back cover or www.jant.org for submission instructions.

The subscription price for 2012 is US $175/year for the electronic version, and $275/year (+$40 shipping outside the US) for print and electronic. Subscriptions, requests for back issues from the last three years and changes of subscribers address should be sent to Mathematical Sciences Publishers, Department of Mathematics, University of California, Berkeley, CA 94720-3840, USA.

Algebra & Number Theory (ISSN 1937-0652) at Mathematical Sciences Publishers, Department of Mathematics, University of California, Berkeley, CA 94720-3840 is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

ANT peer review and production are managed by EditFLOW® from Mathematical Sciences Publishers.

PUBLISHED BY
mathematical sciences publishers
http://msp.org/

A NON-PROFIT CORPORATION

Typeset in LaTeX

Algebra & Number Theory

Volume 6 No. 4 2012

1937-0652(2012)6:4;1-B