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# Block components of the Lie module for the symmetric group

Roger M. Bryant and Karin Erdmann

Let  $F$  be a field of prime characteristic  $p$  and let  $B$  be a nonprincipal block of the group algebra  $FS_r$  of the symmetric group  $S_r$ . The block component  $\text{Lie}(r)_B$  of the Lie module  $\text{Lie}(r)$  is projective, by a result of Erdmann and Tan, although  $\text{Lie}(r)$  itself is projective only when  $p \nmid r$ . Write  $r = p^m k$ , where  $p \nmid k$ , and let  $S_k^*$  be the diagonal of a Young subgroup of  $S_r$  isomorphic to  $S_k \times \cdots \times S_k$ . We show that  $p^m \text{Lie}(r)_B \cong (\text{Lie}(k) \uparrow_{S_k^*}^{S_r})_B$ . Hence we obtain a formula for the multiplicities of the projective indecomposable modules in a direct sum decomposition of  $\text{Lie}(r)_B$ . Corresponding results are obtained, when  $F$  is infinite, for the  $r$ -th Lie power  $L^r(E)$  of the natural module  $E$  for the general linear group  $\text{GL}_n(F)$ .

## 1. Introduction and summary of results

Let  $r$  be a positive integer and let  $S_r$  denote the symmetric group of degree  $r$ . For any field  $F$  the Lie module  $\text{Lie}(r)$  is the  $FS_r$ -module given by the right ideal  $\omega_r FS_r$  of the group algebra  $FS_r$  where  $\omega_r$  is the *Dynkin–Specht–Wever element*, defined by  $\omega_1 = 1$  and, for  $r \geq 2$ ,

$$\omega_r = (1 - c_r)(1 - c_{r-1}) \cdots (1 - c_2), \quad (1-1)$$

where  $c_i$  is the  $i$ -cycle  $(1 \ i \ i-1 \ \dots \ 2)$ . It is known that  $\text{Lie}(r)$  has dimension  $(r-1)!$  (see [Section 2A](#)).

If  $F$  has prime characteristic  $p$  and  $p \nmid r$  then  $\text{Lie}(r)$  is a direct summand of  $FS_r$  because, as is well known,  $\omega_r^2 = r\omega_r$  (see, for example, [\[Bryant 2009, Section 3\]](#)); so in this case  $\text{Lie}(r)$  is projective. However, if  $\text{char } F = p$  and  $p \mid r$  then  $\text{Lie}(r)$  is not projective (because its dimension is not then divisible by the order of a Sylow  $p$ -subgroup of  $S_r$ ), but it was shown recently that every nonprincipal block component of  $\text{Lie}(r)$  is projective (see [\[Erdmann and Tan 2011\]](#)). Here we show that each such component can be described in a surprisingly simple way in terms of  $\text{Lie}(k)$ , where  $k$  is the  $p'$ -part of  $r$ .

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The Lie module occurs naturally in a number of contexts in algebra, algebraic topology and elsewhere (see [Erdmann and Tan 2011] for a fuller discussion). Here we shall only be concerned with the connection with free Lie algebras, where our results on the Lie module give new insight into the module structure of the homogeneous components.

Let  $G$  be a group and  $V$  an  $FG$ -module. Let  $L^r(V)$  denote the homogeneous component of degree  $r$  in the free Lie algebra  $L(V)$  freely generated by any basis of  $V$ . (Here  $L(V)$  may be regarded as the Lie subalgebra generated by  $V$  in the tensor algebra or free associative algebra on  $V$ : see Section 2A.) The vector space  $L^r(V)$  is called the  $r$ -th Lie power of  $V$  and it inherits the structure of an  $FG$ -module.

Suppose that  $F$  is infinite, let  $n$  be a positive integer, and let  $E$  denote the natural  $n$ -dimensional module over  $F$  for the general linear group  $GL_n(F)$ . Then  $L^r(E)$ , as a module for  $GL_n(F)$ , is a homogeneous polynomial module of degree  $r$ . In other words it is a module for the Schur algebra  $S_F(n, r)$  (see [Green 1980]). In the case where  $n \geq r$  the Schur functor  $f_r$  maps  $S_F(n, r)$ -modules to  $FS_r$ -modules and we have  $f_r(L^r(E)) \cong \text{Lie}(r)$  (see Section 2D).

Recall that if  $\mathcal{B}$  is the set of blocks of an algebra  $\Gamma$  and  $V$  is a  $\Gamma$ -module then we may write  $V$  as a direct sum of block components:  $V = \bigoplus_{B \in \mathcal{B}} V_B$ , where  $V_B \in B$  for all  $B$ . Our main results concern the block components of  $\text{Lie}(r)$  and  $L^r(E)$  when  $F$  has prime characteristic  $p$ . The basic results are for  $\text{Lie}(r)$ , and the results for  $L^r(E)$  are obtained from these by means of the Schur functor. To state the results we write  $r = p^m k$  where  $m \geq 0$ ,  $k \geq 1$ , and  $p \nmid k$ .

Let  $S_k^*$  be a subgroup of  $S_r$  such that  $S_k^* \cong S_k$  and  $S_k^*$  is the diagonal of a Young subgroup of  $S_r$  isomorphic to  $S_k \times \cdots \times S_k$  (with  $p^m$  factors). (See Section 2D for more details.) Since  $S_k^* \cong S_k$  we may regard  $\text{Lie}(k)$  as an  $FS_k^*$ -module and, since  $p \nmid k$ , this module is projective. Thus the induced module  $\text{Lie}(k) \uparrow_{S_k^*}^{S_r}$  is also projective. It was proved in [Erdmann and Tan 2011, Theorem 3.1] that if  $B$  is a nonprincipal block of  $FS_r$  then  $\text{Lie}(r)_B$  is projective. Here we shall prove (Theorem 3.1) that

$$\text{Lie}(r)_B \cong \frac{1}{p^m} (\text{Lie}(k) \uparrow_{S_k^*}^{S_r})_B \tag{1-2}$$

when  $B$  satisfies the condition  $\tilde{B} \neq \emptyset$  (see Section 2B): this condition is satisfied when  $B$  is nonprincipal. (The notation  $U \cong (1/q)V$  used in (1-2) means that  $qU \cong V$ , where  $qU$  denotes  $U \oplus \cdots \oplus U$  with  $q$  summands.) The special case where  $k = 1$  is of particular interest: it yields

$$\text{Lie}(p^m)_B \cong \frac{1}{p^m} (FS_{p^m})_B.$$

Since (1-2) holds for each nonprincipal block  $B$ , a comparison of dimensions gives a weaker result (Corollary 3.4) for the principal block  $B_0$ :

$$\dim \operatorname{Lie}(r)_{B_0} = \frac{1}{p^m} \dim (\operatorname{Lie}(k) \uparrow_{S_k^*}^{S_r})_{B_0}.$$

The projective indecomposable  $FS_r$ -modules may be labelled  $P^\lambda$  where  $\lambda$  ranges over the  $p$ -regular partitions of  $r$  (see [Section 2B](#)). For any  $FS_r$ -block  $B$  we write  $\lambda \in B$  when  $P^\lambda \in B$ . Since  $\operatorname{Lie}(r)_B$  is projective when  $B$  is nonprincipal, there are nonnegative integers  $m_\lambda$  such that

$$\operatorname{Lie}(r)_B \cong \bigoplus_{\lambda \in B} m_\lambda P^\lambda.$$

In [Theorem 3.5](#) we prove that

$$m_\lambda = \frac{1}{r} \sum_{d|k} \mu(d) \beta^\lambda(\tau^{k/d}), \tag{1-3}$$

where  $\mu$  is the Möbius function,  $\tau$  is an element of  $S_r$  of cycle type  $(k, k, \dots, k)$ , and  $\beta^\lambda$  denotes the Brauer character of  $D^\lambda$ , the irreducible  $FS_r$ -module isomorphic to the head of  $P^\lambda$ .

Now suppose that  $F$  is infinite and  $n$  is a positive integer. We have observed that  $L^r(E)$  is an  $S_F(n, r)$ -module. Similarly,  $L^k(E^{\otimes p^m})$  is an  $S_F(n, r)$ -module, and (by the argument in [\[Donkin and Erdmann 1998, Section 3.1\]](#)) it is isomorphic to a direct summand of  $E^{\otimes r}$ . It is a consequence of [\[Erdmann and Tan 2011, Theorem 3.2\]](#) that if  $B$  is a block of  $S_F(n, r)$  satisfying the condition  $\tilde{B} \neq \emptyset$  (see [Section 2C](#)) then  $L^r(E)_B$  is isomorphic to a direct sum of summands of  $E^{\otimes r}$ . Here we shall prove ([Theorem 3.6](#)) that

$$L^r(E)_B \cong \frac{1}{p^m} L^k(E^{\otimes p^m})_B.$$

The indecomposable summands of  $E^{\otimes r}$  are tilting modules  $T(\lambda)$ , where  $\lambda$  is a  $p$ -regular partition of  $r$  with at most  $n$  parts (see [Section 2C](#)). For any  $S_F(n, r)$ -block  $B$  we write  $\lambda \in B$  when  $T(\lambda) \in B$ . In [Theorem 3.7](#) we prove that if  $\tilde{B} \neq \emptyset$  then

$$L^r(E)_B \cong \bigoplus_{\lambda \in B} m_\lambda T(\lambda),$$

where the multiplicities  $m_\lambda$  are given by (1-3). This extends [\[Donkin and Erdmann 1998, Section 3.3, Theorem\]](#), which gives the same result in the case where  $p \nmid r$ .

All modules over fields in this paper will be assumed to be finite-dimensional, and modules for algebras are right modules unless otherwise specified.

## 2. Preliminaries

**2A. The Lie module.** Let  $r$  and  $n$  be positive integers where  $n \geq r$ . Let  $\Delta$  be the free associative ring ( $\mathbb{Z}$ -algebra) on free generators  $x_1, \dots, x_n$  and let  $L$  be the

Lie subring of  $\Delta$  generated by  $x_1, \dots, x_n$ . By [Bourbaki 1972, chapitre II, §3, théorème 1],  $L$  is free on  $x_1, \dots, x_n$ . Let  $\Delta_r$  denote the homogeneous component of  $\Delta$  of degree  $r$ . Then  $S_r$  has a left action by “place permutations” on  $\Delta_r$ , given by  $\alpha(y_1 \cdots y_r) = y_{1\alpha} \cdots y_{r\alpha}$  for all  $\alpha \in S_r$  and all  $y_1, \dots, y_r \in \{x_1, \dots, x_n\}$ . (Note that we write multiplication in  $S_r$  from left to right.) Hence  $\Delta_r$  is a left  $\mathbb{Z}S_r$ -module. Let  $\omega_r$  be the element of  $\mathbb{Z}S_r$  given by (1-1). Then it is well known and easily verified that  $\omega_r(y_1 \cdots y_r) = [y_1, \dots, y_r]$  where  $[y_1, \dots, y_r]$  denotes the left-normed Lie product  $[\cdots [[y_1, y_2], y_3], \dots, y_r]$ .

The group  $S_r$  also has a right action on  $\Delta$  by automorphisms, where  $x_i\alpha = x_{i\alpha}$  for  $i = 1, \dots, r$  and  $x_i\alpha = x_i$  for  $i > r$ . Thus  $\Delta_r$  becomes a  $(\mathbb{Z}S_r, \mathbb{Z}S_r)$ -bimodule. Let  $\Delta_r^0$  be the  $\mathbb{Z}$ -subspace of  $\Delta_r$  spanned by the monomials  $x_{1\alpha} \cdots x_{r\alpha}$  with  $\alpha \in S_r$ . Thus  $\Delta_r^0$  is a subbimodule of  $\Delta_r$  and the map  $\xi : \mathbb{Z}S_r \rightarrow \Delta_r^0$  defined by  $\xi(\alpha) = x_{1\alpha} \cdots x_{r\alpha}$  is an isomorphism of bimodules.

Let  $L_r^0 = L \cap \Delta_r^0$ . Then  $L_r^0$  is spanned over  $\mathbb{Z}$  by all elements of the form  $[x_{1\alpha}, \dots, x_{r\alpha}]$ . Also, by [Bourbaki 1972, chapitre II, §3, théorème 2],  $L_r^0$  is free of rank  $(r - 1)!$  as a  $\mathbb{Z}$ -module. We have  $\omega_r \Delta_r^0 = L_r^0$ . Thus the isomorphism  $\xi$  maps  $\omega_r \mathbb{Z}S_r$  to  $L_r^0$ , and so  $\omega_r \mathbb{Z}S_r$  is isomorphic to  $L_r^0$  as a right  $\mathbb{Z}S_r$ -module.

All of the above still applies if  $\mathbb{Z}$  is replaced by  $R$ , where  $R$  is an arbitrary commutative ring with unity. Also (using subscripts to show coefficient rings) we have  $R \otimes_{\mathbb{Z}} L_{r,\mathbb{Z}}^0 \cong L_{r,R}^0$ . The Lie module  $\text{Lie}_R(r)$  is the  $RS_r$ -module defined by  $\text{Lie}_R(r) = \omega_r RS_r$ . Thus  $\text{Lie}_R(r) \cong L_{r,R}^0$ . It follows that  $\text{Lie}_R(r) \cong R \otimes_{\mathbb{Z}} \text{Lie}_{\mathbb{Z}}(r)$  and  $\text{Lie}_R(r)$  is free of rank  $(r - 1)!$  as an  $R$ -module. If  $F$  is a field we have

$$\dim \text{Lie}_F(r) = (r - 1)! \tag{2-1}$$

and, if  $F$  is understood, we write  $\text{Lie}_F(r)$  as  $\text{Lie}(r)$ .

When  $K$  is a field of characteristic zero there is a formula for the character  $\psi_r$  of  $\text{Lie}_K(r)$ . Let  $\mu$  denote the Möbius function and let  $\sigma$  be an  $r$ -cycle of  $S_r$ . For each divisor  $d$  of  $r$  let  $C_d$  denote the conjugacy class of  $\sigma^{r/d}$  in  $S_r$ . Then, for  $g \in S_r$ ,

$$\psi_r(g) = \begin{cases} \mu(d)(r - 1)!/|C_d| & \text{if } g \in C_d, \\ 0 & \text{if } g \notin C_d \text{ for all } d. \end{cases} \tag{2-2}$$

(See, for example, [Donkin and Erdmann 1998, Section 3.2].) Hence, if  $\theta$  is any class function on  $S_r$  with values in  $K$  and we write

$$(\theta, \psi_r)_{S_r} = \frac{1}{|S_r|} \sum_{g \in S_r} \theta(g) \psi_r(g^{-1}), \tag{2-3}$$

we have

$$(\theta, \psi_r)_{S_r} = \frac{1}{r} \sum_{d|r} \mu(d) \theta(\sigma^{r/d}). \tag{2-4}$$

**2B. Representations of  $S_r$ .** By a partition of a nonnegative integer  $r$  we mean, as usual, a finite sequence  $\lambda = (\lambda_1, \dots, \lambda_s)$  of integers satisfying  $\lambda_1 \geq \dots \geq \lambda_s > 0$  and  $\lambda_1 + \dots + \lambda_s = r$ . We call  $\lambda_1, \dots, \lambda_s$  the *parts* of  $\lambda$ . We write  $\Lambda^+(r)$  for the set of all partitions of  $r$ . If  $r = 0$  then  $\Lambda^+(r)$  contains only the empty partition, which we denote by  $\emptyset$ . Let  $p$  be a prime number. A partition  $\lambda$  is  *$p$ -regular* if  $\lambda$  does not have  $p$  or more equal parts, and we write  $\Lambda_p^+(r)$  for the set of all  $p$ -regular partitions of  $r$ . The  *$p$ -core* of a partition  $\lambda$  of  $r$  is the partition  $\tilde{\lambda}$  of  $r'$ , for some  $r' \leq r$ , obtained from (the diagram of)  $\lambda$  by the removal of as many “rim  $p$ -hooks” as possible: see [James and Kerber 1981, Section 2.7]. We write  $\mathcal{C}_r$  for the set of all  $p$ -cores of elements of  $\Lambda^+(r)$ .

Let  $r$  be a positive integer and let  $F$  be a field of prime characteristic  $p$ . The irreducible  $FS_r$ -modules may be labelled (up to isomorphism) as  $D^\lambda$  with  $\lambda \in \Lambda_p^+(r)$ , where  $D^\lambda$  is a quotient module of the Specht module  $S^\lambda$  (see [James and Kerber 1981, 7.1.14]). Here  $D^{(r)}$  is isomorphic to the trivial  $FS_r$ -module  $F$  because  $S^{(r)} \cong F$ . For each  $\lambda$  we write  $P^\lambda$  for the projective cover of  $D^\lambda$ . Thus the projective indecomposable  $FS_r$ -modules are the  $P^\lambda$  with  $\lambda \in \Lambda_p^+(r)$ . If  $F'$  is an extension field of  $F$  then (using subscripts to show coefficient fields) we have  $D_{F'}^\lambda \cong F' \otimes_F D_F^\lambda$  and  $P_{F'}^\lambda \cong F' \otimes_F P_F^\lambda$ .

We recall a few general facts about blocks. If  $\Gamma$  is a finite-dimensional  $F$ -algebra we may write  $\Gamma$  uniquely as a finite direct sum of indecomposable two-sided ideals,  $\Gamma = \bigoplus_{B \in \mathcal{B}} \Gamma_B$ . These ideals are the blocks of  $\Gamma$ , but it is convenient also to refer to the labels  $B$  as the blocks. The identity element of  $\Gamma$  may be written as  $\sum_{B \in \mathcal{B}} e_B$ , with  $e_B \in \Gamma_B$  for all  $B$ . The elements  $e_B$  are the block idempotents: they are primitive central idempotents of  $\Gamma$  (see, for example, [Benson 1995]). Any  $\Gamma$ -module  $V$  satisfying  $Ve_B = V$  is said to belong to  $B$ , and we write  $V \in B$ . Every  $\Gamma$ -module  $V$  may be written uniquely in the form  $V = \bigoplus_{B \in \mathcal{B}} V_B$ , where  $V_B \in B$  for all  $B$  (indeed,  $V_B = Ve_B$ ). We call  $V_B$  the block component of  $V$  corresponding to  $B$ .

By the Nakayama conjecture (see [James and Kerber 1981, 6.1.21]), the blocks of  $FS_r$  may be labelled  $B(\nu)$  with  $\nu \in \mathcal{C}_r$  in such a way that  $S^\lambda \in B(\tilde{\lambda})$  for all  $\lambda \in \Lambda_p^+(r)$ . Since  $D^\lambda$  is a quotient of  $S^\lambda$  and  $P^\lambda$  is indecomposable with  $D^\lambda$  as a quotient, we have  $D^\lambda, P^\lambda \in B(\tilde{\lambda})$ . We use the same notation for the blocks of  $FS_r$  for every field  $F$  of characteristic  $p$ . By consideration of composition factors we see that, for any  $FS_r$ -module  $V$ , any extension field  $F'$  of  $F$ , and any  $\nu$ , we have

$$(F' \otimes_F V)_{B(\nu)} \cong F' \otimes_F V_{B(\nu)}. \quad (2-5)$$

If  $B$  is a block and  $B = B(\nu)$  we write  $\tilde{B} = \nu$ . Also, for  $\lambda \in \Lambda_p^+(r)$ , we write  $\lambda \in B$  if  $D^\lambda \in B$  (or equivalently  $P^\lambda \in B$ ). The *principal block* is the block  $B_0$  containing the trivial irreducible  $D^{(r)}$ . Thus  $\tilde{B}_0 = (\bar{r})$ , where  $\bar{r}$  denotes the remainder on

dividing  $r$  by  $p$ . If  $\tilde{B} = \emptyset$  then  $p|r$  and  $\bar{r} = 0$  so that  $B = B_0$ . Hence if  $B$  is nonprincipal we have  $\tilde{B} \neq \emptyset$ .

If  $p \nmid r$  then  $\text{Lie}(r)$  is projective (see [Section 1](#)). But if  $p|r$  and  $\tilde{B} \neq \emptyset$  then  $B \neq B_0$  and so  $\text{Lie}(r)_B$  is projective by [[Erdmann and Tan 2011](#), Theorem 3.1]. Hence we have the following result.

**Theorem 2.1** [[Erdmann and Tan 2011](#)]. *If  $B$  is a block of  $FS_r$  such that  $\tilde{B} \neq \emptyset$  then  $\text{Lie}(r)_B$  is projective.*

As is well known, Brauer characters of  $FS_r$ -modules have integer values: this follows, for example, from [[Nagao and Tsushima 1989](#), Chapter 3, Lemma 6.13]. (Consequently Brauer characters of  $FS_r$ -modules are uniquely defined and do not depend upon choices of roots of unity.) We regard Brauer characters as maps from  $S_r$  to  $\mathbb{Z}$  by assigning the value zero to  $p$ -singular elements of  $S_r$ . For each  $\lambda \in \Lambda_p^+(r)$  we write  $\beta^\lambda$  and  $\zeta^\lambda$  for the Brauer characters of  $D^\lambda$  and  $P^\lambda$ , respectively. By the orthogonality relations for Brauer characters (see [[Nagao and Tsushima 1989](#), Chapter 3, Theorem 6.10]) we have

$$(\beta^\lambda, \zeta^\rho)_{S_r} = \begin{cases} 1 & \text{if } \lambda = \rho, \\ 0 & \text{if } \lambda \neq \rho, \end{cases} \quad (2-6)$$

where  $(\beta^\lambda, \zeta^\rho)_{S_r}$  is defined as in (2-3).

**2C. Polynomial representations of  $\text{GL}_n(F)$ .** Suppose now that  $F$  is an infinite field of prime characteristic  $p$  and let  $n$  and  $r$  be positive integers. We refer to [[Green 1980](#)] and [[Donkin and Erdmann 1998](#)] for background concerning polynomial  $\text{GL}_n(F)$ -modules and the Schur algebra  $S_F(n, r)$ . Let  $E$  denote the natural  $\text{GL}_n(F)$ -module. Thus  $E^{\otimes r}$  is an  $S_F(n, r)$ -module. If  $k$  and  $t$  are positive integers such that  $r = kt$  and if  $V$  is an  $S_F(n, t)$ -module then  $V^{\otimes k}$  and  $L^k(V)$  are  $S_F(n, r)$ -modules.

Let  $\Lambda^+(n, r)$  denote the set of all partitions of  $r$  with at most  $n$  parts and let  $\Lambda_p^+(n, r)$  denote the set of all  $p$ -regular partitions in  $\Lambda^+(n, r)$ . The irreducible  $S_F(n, r)$ -modules may be labelled  $L(\lambda)$  with  $\lambda \in \Lambda^+(n, r)$ . For each such  $\lambda$  there is also an indecomposable  $S_F(n, r)$ -module  $T(\lambda)$  called a ‘‘tilting module’’, and (see [[Donkin and Erdmann 1998](#), Section 1.3]) there are nonnegative integers  $n_\lambda$  such that

$$E^{\otimes r} \cong \bigoplus_{\lambda \in \Lambda_p^+(n, r)} n_\lambda T(\lambda). \quad (2-7)$$

The main facts about the blocks of  $S_F(n, r)$  were obtained in [[Donkin 1994](#)] and summarised in [[Erdmann and Tan 2011](#)]. When  $n \geq r$  the blocks may be labelled  $B(\nu)$  with  $\nu \in \mathcal{C}_r$  in such a way that  $L(\lambda) \in B(\tilde{\lambda})$ . If  $B$  is a block and  $B = B(\nu)$  we write  $\tilde{B} = \nu$ . When  $n < r$ ,  $p$ -cores do not necessarily label unique blocks, but

if  $L(\lambda)$  and  $L(\rho)$  are in the same block then  $\tilde{\lambda} = \tilde{\rho}$ . Thus, for each block  $B$ , there is an element  $\tilde{B}$  of  $\mathcal{C}_r$  (where  $\tilde{B}$  has at most  $n$  parts) with the property that  $\tilde{\lambda} = \tilde{B}$  whenever  $L(\lambda) \in B$ . For each  $\nu \in \mathcal{C}_r$  we write  $B(\nu)$  for the set of blocks  $B$  such that  $\tilde{B} = \nu$ . (Thus  $B(\nu)$  is empty if  $\nu$  has more than  $n$  parts.) If  $V$  is an  $S_F(n, r)$ -module we write  $V_{B(\nu)}$  for the direct sum of the block components  $V_B$  of  $V$  corresponding to blocks  $B$  in  $B(\nu)$ . For all  $n$  and all  $\lambda \in \Lambda^+(n, r)$ ,  $T(\lambda)$  is indecomposable and has  $L(\lambda)$  as a composition factor (see [Erdmann 1994, Section 1.3]); thus  $T(\lambda)$  and  $L(\lambda)$  belong to the same block. For a block  $B$  and  $\lambda \in \Lambda^+(n, r)$  we write  $\lambda \in B$  if  $L(\lambda) \in B$  (or equivalently  $T(\lambda) \in B$ ). We define the principal block to be the block  $B_0$  containing  $L(\lambda)$  where  $\lambda = (r)$ . Thus  $\tilde{B}_0 = (\bar{r})$ , with  $\bar{r}$  as before. As in the case of  $FS_r$ , if  $n \geq r$  and  $B$  is nonprincipal then  $\tilde{B} \neq \emptyset$ .

Let  $\mathcal{T}$  denote the class of all  $S_F(n, r)$ -modules that are isomorphic to direct sums of tilting modules  $T(\lambda)$  where  $\lambda \in \Lambda_p^+(n, r)$ . Thus  $E^{\otimes r} \in \mathcal{T}$  by (2-7). If  $p \nmid r$  then  $L^r(E)$  is isomorphic to a direct summand of  $E^{\otimes r}$  (see Section 1) and so  $L^r(E) \in \mathcal{T}$ . But if  $p \mid r$  and  $\tilde{B} \neq \emptyset$  then  $L^r(E)_B \in \mathcal{T}$  by [Erdmann and Tan 2011, Theorem 3.2]. Hence we have the following result.

**Theorem 2.2** [Erdmann and Tan 2011]. *If  $B$  is a block of  $S_F(n, r)$  such that  $\tilde{B} \neq \emptyset$  then  $L^r(E)_B \in \mathcal{T}$ .*

Suppose now that  $n_1$  and  $n_2$  are positive integers with  $n_1 \geq n_2$  and let  $d_{n_1, n_2}$  denote the functor from the category of  $S_F(n_1, r)$ -modules to the category of  $S_F(n_2, r)$ -modules described in [Green 1980, Section 6.5]. This functor is exact (in particular it preserves direct sums) and we call it *truncation*. Note that  $\Lambda^+(n_2, r) \subseteq \Lambda^+(n_1, r)$ . We temporarily use subscripts to distinguish between modules for  $S_F(n_1, r)$  and  $S_F(n_2, r)$ . Then, if  $\lambda \in \Lambda^+(n_1, r)$  and  $M(\lambda)$  denotes either  $L(\lambda)$  or  $T(\lambda)$ , we have

$$d_{n_1, n_2}(M_{n_1}(\lambda)) \cong \begin{cases} M_{n_2}(\lambda) & \text{if } \lambda \in \Lambda^+(n_2, r), \\ 0 & \text{otherwise.} \end{cases} \tag{2-8}$$

(For the case of  $L(\lambda)$  see [Green 1980, Section 6.5] and for  $T(\lambda)$  see [Erdmann 1994, Section 1.7].)

Write  $d = d_{n_1, n_2}$  and use the same notation for arbitrary  $r$ . Then, if  $k$  and  $t$  are positive integers and  $V$  is an  $S_F(n_1, t)$ -module, it is easy to check that  $d(V^{\otimes k}) \cong d(V)^{\otimes k}$  and  $d(L^k(V)) \cong L^k(d(V))$ . Furthermore  $d(E_{n_1}^{\otimes t}) \cong E_{n_2}^{\otimes t}$ . Also, if  $V$  is an  $S_F(n_1, r)$ -module and  $\nu \in \mathcal{C}_r$ , it follows from (2-8) that

$$d(V_{B(\nu)}) \cong d(V)_{B(\nu)}. \tag{2-9}$$

**2D. The Schur functor.** We continue with the notation of the previous subsection but now assume that  $n \geq r$ . The Schur functor  $f_r$  is an exact functor from the category of  $S_F(n, r)$ -modules to the category of  $FS_r$ -modules (see [Green 1980,



Chapter 6]). If  $U$  is an  $S_F(n, r)$ -module then  $f_r(U)$  may be thought of as the weight space of  $U$  corresponding to the weight  $(1, \dots, 1, 0, \dots, 0)$ , with  $r$  coordinates equal to 1, and the action of  $S_r$  on  $f_r(U)$  comes by taking  $S_r$  as a group of permutation matrices in  $\text{GL}_n(F)$  (see, for example, [Donkin and Erdmann 1998, Section 1.2]). It is easily seen that

$$f_r(E^{\otimes r}) \cong FS_r. \quad (2-10)$$

Let  $\{e_1, \dots, e_n\}$  be the standard basis of  $E$ . Then  $f_r(L^r(E))$  is the subspace of  $L^r(E)$  spanned by the left-normed Lie products  $[e_{1\alpha}, \dots, e_{r\alpha}]$  with  $\alpha \in S_r$ . In the notation of Section 2A,  $f_r(L^r(E)) \cong L_{r,F}^0$ . Thus, since  $L_{r,F}^0 \cong \text{Lie}(r)$ , we obtain

$$f_r(L^r(E)) \cong \text{Lie}(r). \quad (2-11)$$

For all  $\lambda \in \Lambda_p^+(n, r) = \Lambda_p^+(r)$ , we have (see [Donkin and Erdmann 1998, Section 1.3])

$$f_r(T(\lambda)) \cong P^\lambda. \quad (2-12)$$

As observed in [Erdmann and Tan 2011],  $f_r$  sends modules in the  $S_F(n, r)$ -block  $B(\nu)$  to modules in the  $FS_r$ -block  $B(\nu)$  labelled by the same  $p$ -core. Thus, if  $V$  is any  $S_F(n, r)$ -module, we have

$$f_r(V_{B(\nu)}) \cong f_r(V)_{B(\nu)}. \quad (2-13)$$

Let  $k$  be a divisor of  $r$ , and write  $t = r/k$ . (We do not at present assume that  $p \nmid k$ .) For each  $\alpha \in S_k$  we may define  $\alpha^* \in S_r$  by  $((i-1)t + j)\alpha^* = (i\alpha - 1)t + j$  for  $i = 1, \dots, k$  and  $j = 1, \dots, t$ . The set  $\{\alpha^* : \alpha \in S_k\}$  is a subgroup  $S_k^*$  of  $S_r$  isomorphic to  $S_k$ . The subgroup of  $S_r$  consisting of all permutations fixing  $\{(i-1)t + j : i = 1, \dots, k\}$  setwise for  $j = 1, \dots, t$  is a Young subgroup of  $S_r$  isomorphic to  $S_k \times \dots \times S_k$ , and  $S_k^*$  may be thought of as the diagonal of this subgroup. The diagonal of any other Young subgroup isomorphic to  $S_k \times \dots \times S_k$  is a conjugate of  $S_k^*$  in  $S_r$ . Note that if  $\sigma$  is the  $r$ -cycle  $(1\ 2 \dots r)$  of  $S_r$  and  $\sigma_k$  is the  $k$ -cycle  $(1\ 2 \dots k)$  of  $S_k$  then  $\sigma^t = \sigma_k^* \in S_k^*$ . For  $i = 1, \dots, k$ , write  $\Omega_i = \{(i-1)t + j : j = 1, \dots, t\}$ . The subgroup  $S_t^{(k)}$  of  $S_r$  consisting of all permutations fixing each  $\Omega_i$  setwise is a Young subgroup isomorphic to  $S_t \times \dots \times S_t$ . For each  $\alpha \in S_k$  we have  $\Omega_i \alpha^* = \Omega_i \alpha$  for  $i = 1, \dots, k$ . The subgroup  $S_t^{(k)} S_k^*$  of  $S_r$  is isomorphic to the wreath product  $S_t \text{ wr } S_k$ .

Let  $V$  be an  $S_F(n, t)$ -module. Then  $f_t(V)$  is an  $FS_t$ -module, so  $f_t(V)^{\otimes k}$  is an  $FS_t^{(k)}$ -module. Indeed,  $f_t(V)^{\otimes k}$  may be regarded as an  $FS_t^{(k)} S_k^*$ -module, where the action of  $S_k^*$  is to permute the tensor factors. We regard  $\text{Lie}(k)$  as an  $FS_k^*$ -module by means of the isomorphism  $\alpha \mapsto \alpha^*$  from  $S_k$  to  $S_k^*$ . Then  $\text{Lie}(k)$  may also be regarded as an  $FS_t^{(k)} S_k^*$ -module, by taking trivial action of  $S_t^{(k)}$ . The following result is part of [Lim and Tan 2012, Corollary 3.2].

**Lemma 2.3** [Lim and Tan 2012]. *In the above notation,*

$$f_r(L^k(V)) \cong (f_t(V)^{\otimes k} \otimes \text{Lie}(k)) \uparrow_{S_t^{(k)} S_k^*}^{S_r}.$$

**Corollary 2.4.** *In the above notation,*

$$f_r(L^k(E^{\otimes t})) \cong \text{Lie}(k) \uparrow_{S_k^*}^{S_r}.$$

*Proof.* By (2-10) and Lemma 2.3,

$$f_r(L^k(E^{\otimes t})) \cong ((FS_t)^{\otimes k} \otimes \text{Lie}(k)) \uparrow_{S_t^{(k)} S_k^*}^{S_r}.$$

Clearly  $(FS_t)^{\otimes k}$  is a transitive permutation module under the action of  $S_t^{(k)} S_k^*$  and the stabiliser of the basis element  $1 \otimes \cdots \otimes 1$  is  $S_k^*$ . Thus  $(FS_t)^{\otimes k}$  is induced from a one-dimensional trivial module for  $S_k^*$  and (by [Benson 1995, Proposition 3.3.3(i)]) we have

$$(FS_t)^{\otimes k} \otimes \text{Lie}(k) \cong \text{Lie}(k) \uparrow_{S_k^*}^{S_t^{(k)} S_k^*}.$$

The result follows. □

### 3. Main results

Recall from Section 2B that if  $B$  is a nonprincipal block of  $FS_r$  then  $\tilde{B} \neq \emptyset$ . Our main result on the Lie module is as follows. We use the notation of Section 2D, regarding  $\text{Lie}(k)$  as an  $FS_k^*$ -module.

**Theorem 3.1.** *Let  $F$  be a field of prime characteristic  $p$ . Let  $r$  be a positive integer and write  $r = p^m k$  where  $m \geq 0$ ,  $k \geq 1$ , and  $p \nmid k$ . Let  $B$  be a block of  $FS_r$  such that  $\tilde{B} \neq \emptyset$  and let  $S_k^*$  be the diagonal of a Young subgroup  $S_k \times \cdots \times S_k$  of  $S_r$ . Then*

$$\text{Lie}(r)_B \cong \frac{1}{p^m} (\text{Lie}(k) \uparrow_{S_k^*}^{S_r})_B.$$

Note that  $\text{Lie}(k) \uparrow_{S_k^*}^{S_r}$  is projective since  $\text{Lie}(k)$  is projective (see Section 1).

We commence the proof of Theorem 3.1. If  $F'$  is an extension field of  $F$  then, by the description of the Lie module in Section 2A,  $\text{Lie}_{F'}(r) \cong F' \otimes \text{Lie}_F(r)$  and  $\text{Lie}_{F'}(k) \cong F' \otimes \text{Lie}_F(k)$ . Thus, if  $B$  is any block of  $FS_r$ , we have  $\text{Lie}_{F'}(r)_B \cong F' \otimes \text{Lie}_F(r)_B$  and  $(\text{Lie}_{F'}(k) \uparrow_{S_k^*}^{S_r})_B \cong F' \otimes (\text{Lie}_F(k) \uparrow_{S_k^*}^{S_r})_B$  by (2-5). Hence it suffices to prove Theorem 3.1 for the prime field  $\mathbb{F}_p$  and then, by the Noether–Deuring theorem, it suffices to prove the theorem for any chosen field  $F$  of characteristic  $p$ . We choose  $F$  so that there is a  $p$ -modular system  $(K, R, F)$  with the properties specified in [Nagao and Tsushima 1989, Chapter 3, Section 6]. Note, in particular, that  $K$  has characteristic 0 and contains sufficient roots of unity,  $K$  is the field of fractions of  $R$ , and  $F = R/(\pi)$  where  $(\pi)$  is the maximal ideal of  $R$ .

We state some standard facts associated with  $p$ -modular systems in order to establish terminology and notation. If  $G$  is any finite group then the natural epimorphism  $R \rightarrow F$  yields an epimorphism  $RG \rightarrow FG$ . If this epimorphism maps  $u$  to  $v$ , where  $u \in RG$  and  $v \in FG$ , we say that  $v$  *lifts* to  $u$ . By an  $RG$ -lattice we mean an  $RG$ -module that is free of finite rank as an  $R$ -module. If  $M$  is an  $RG$ -lattice we write  $\bar{M} = M/\pi M$ . Thus  $\bar{M} \cong F \otimes_R M$  and  $\bar{M}$  has the structure of an  $FG$ -module. An  $FG$ -module  $V$  is said to be *liftable* if there exists  $M$  such that  $\bar{M} \cong V$ , in which case we say that  $V$  lifts to  $M$ . If  $M$  is an  $RG$ -lattice then  $K \otimes_R M$  is a  $KG$ -module. If  $U$  is any  $KG$ -module then there is an  $RG$ -lattice  $M$  such that  $U \cong K \otimes_R M$  (see [Benson 1995, Lemma 1.9.1]) and we say that  $\bar{M}$  is obtained from  $U$  by *modular reduction*.

By a standard result (see [ibid., Theorem 1.9.4]), each block idempotent  $e_B$  of  $FS_r$  can be lifted to an element  $\hat{e}_B$  of  $RS_r$  to obtain pairwise-orthogonal primitive central idempotents of  $RS_r$  summing to the identity. If  $M$  is an  $RS_r$ -lattice such that  $M\hat{e}_B = M$  we write  $M \in B$ . Every  $RS_r$ -lattice  $M$  may be written uniquely in the form  $M = \bigoplus_B M_B$  where, for each  $B$ ,  $M_B$  is an  $RS_r$ -lattice belonging to  $B$ . Similar facts and notation apply to  $KS_r$ -modules, using the same idempotents  $\hat{e}_B$ . If  $U$  is a  $KS_r$ -module then, since  $K$  has characteristic zero,  $U_B$  is a direct sum of irreducible  $KS_r$ -modules belonging to  $B$ .

It is easily verified that if  $M$  is an  $RS_r$ -lattice and  $B$  is a block then

$$(K \otimes_R M)_B \cong K \otimes_R M_B \quad \text{and} \quad \overline{M}_B \cong \bar{M}_B. \tag{3-1}$$

We let  $\sigma$  be an  $r$ -cycle of  $S_r$  chosen as in Section 2D (with  $t = p^m$ ) so that  $\sigma^{p^m} = \sigma_k^* \in S_k^*$ , where  $\sigma_k$  is a  $k$ -cycle of  $S_k$ .

**Lemma 3.2.** *If  $g$  is an element of the cyclic subgroup  $\langle \sigma \rangle$  such that  $g$  has order divisible by  $p$  and if  $\chi$  is the character of an irreducible  $KS_r$ -module  $U$  belonging to a block  $B$  such that  $\tilde{B} \neq \emptyset$  then  $\chi(g) = 0$ .*

*Proof.* Let  $M$  be an  $RS_r$ -lattice such that  $U \cong K \otimes_R M$ . Since  $U$  belongs to  $B$  it follows from (3-1) that  $M$  belongs to  $B$ . Let  $D$  be the defect group of the  $FS_r$ -block  $B$  (see [Benson 1995, Section 6.1]). (Thus  $D$  is a  $p$ -group, determined up to conjugacy in  $S_r$ .) By [ibid., Corollary 6.1.3],  $D$  is also the defect group of  $B$  regarded as a block of  $RS_r$ . Thus, by [ibid., Proposition 6.1.2],  $M$  is projective relative to  $D$ .

Let  $B = B(v)$  where  $v \in \mathcal{C}_r$ . Thus  $v \neq \emptyset$  and so  $v$  is a partition of  $r'$  for some  $r'$  satisfying  $0 < r' \leq r$ . It follows from [James and Kerber 1981, 6.2.45] that  $D$  can be taken to be a Sylow  $p$ -subgroup of a subgroup  $S_{r-r'}$  of  $S_r$  fixing  $r'$  points of  $\{1, \dots, r\}$ . Hence every element of  $D$  fixes some point of  $\{1, \dots, r\}$ .

Let  $g$  be as in the statement of the lemma. The  $p$ -part of  $g$  is a nontrivial element of  $\langle \sigma \rangle$  and hence has no fixed points in  $\{1, \dots, r\}$ . It follows that the  $p$ -part of  $g$  is

not conjugate in  $S_r$  to an element of  $D$ . Therefore, by [Nagao and Tsushima 1989, Chapter 4, Theorem 7.4], we have  $\chi(g) = 0$ , as required.  $\square$

**Lemma 3.3.** *If  $B$  is a block such that  $\tilde{B} \neq \emptyset$  then  $p^m \text{Lie}_K(r)_B \cong (\text{Lie}_K(k) \uparrow_{S_k^*}^{S_r})_B$ .*

*Proof.* The result is trivial if  $r = k$ . Thus we may assume that  $p|r$ . Let  $\psi_r$  denote the character of the  $KS_r$ -module  $\text{Lie}_K(r)$  and let  $\psi_k$  denote the character of the  $KS_k^*$ -module  $\text{Lie}_K(k)$ . In order to prove the lemma it suffices to show that the multiplicity of each irreducible  $KS_r$ -module  $U$  belonging to  $B$  is the same in  $p^m \text{Lie}_K(r)$  as in  $\text{Lie}_K(k) \uparrow_{S_k^*}^{S_r}$ . Let  $\chi$  be the character of  $U$ . By the orthogonality relations and Frobenius reciprocity for ordinary characters, it suffices to prove

$$p^m (\chi, \psi_r)_{S_r} = (\chi \downarrow_{S_k^*}^{S_r}, \psi_k)_{S_k^*}. \quad (3-2)$$

By (2-4) we have

$$\begin{aligned} r (\chi, \psi_r)_{S_r} &= \sum_{d|r} \mu(d) \chi(\sigma^{r/d}) \\ &= \sum_{d|k} \mu(d) \chi(\sigma^{r/d}) - \sum_{d|k} \mu(d) \chi(\sigma^{r/pd}). \end{aligned}$$

However, for  $d|k$ , we have  $\chi(\sigma^{r/pd}) = 0$  by Lemma 3.2. Thus

$$r (\chi, \psi_r)_{S_r} = \sum_{d|k} \mu(d) \chi((\sigma^{p^m})^{k/d}).$$

Recall that  $\sigma^{p^m} = \sigma_k^* \in S_k^*$  where  $\sigma_k$  is a  $k$ -cycle of  $S_k$ . Hence, by (2-4) applied to  $S_k^*$ ,

$$k (\chi \downarrow_{S_k^*}^{S_r}, \psi_k)_{S_k^*} = \sum_{d|k} \mu(d) \chi((\sigma^{p^m})^{k/d}).$$

This gives (3-2).  $\square$

We can now prove Theorem 3.1. Let  $B$  be a block of  $FS_r$  such that  $\tilde{B} \neq \emptyset$ . By the description of the Lie module in Section 2A,  $\text{Lie}(r)$  lifts to the  $RS_r$ -lattice  $\text{Lie}_R(r)$  and  $\text{Lie}(k)$  lifts to the  $RS_k^*$ -lattice  $\text{Lie}_R(k)$ . Thus  $p^m \text{Lie}(r)$  and  $\text{Lie}(k) \uparrow_{S_k^*}^{S_r}$  lift to  $p^m \text{Lie}_R(r)$  and  $\text{Lie}_R(k) \uparrow_{S_k^*}^{S_r}$ , respectively. Also,  $K \otimes p^m \text{Lie}_R(r) \cong p^m \text{Lie}_K(r)$  and  $K \otimes \text{Lie}_R(k) \uparrow_{S_k^*}^{S_r} \cong \text{Lie}_K(k) \uparrow_{S_k^*}^{S_r}$ . Hence  $p^m \text{Lie}(r)$  and  $\text{Lie}(k) \uparrow_{S_k^*}^{S_r}$  are modular reductions of  $p^m \text{Lie}_K(r)$  and  $\text{Lie}_K(k) \uparrow_{S_k^*}^{S_r}$ , respectively. It follows by (3-1) that  $p^m \text{Lie}(r)_B$  and  $(\text{Lie}(k) \uparrow_{S_k^*}^{S_r})_B$  are modular reductions of  $p^m \text{Lie}_K(r)_B$  and  $(\text{Lie}_K(k) \uparrow_{S_k^*}^{S_r})_B$ , respectively. However, by Lemma 3.3, these two last modules are isomorphic. Therefore, by [Nagao and Tsushima 1989, Chapter 3, Lemma 6.4],  $p^m \text{Lie}(r)_B$  and  $(\text{Lie}(k) \uparrow_{S_k^*}^{S_r})_B$  have the same Brauer character.

By Theorem 2.1,  $\text{Lie}(r)_B$  is projective. Since  $\text{Lie}(k)$  is a projective  $FS_k^*$ -module,  $\text{Lie}(k) \uparrow_{S_k^*}^{S_r}$  is a projective  $FS_r$ -module and so  $(\text{Lie}(k) \uparrow_{S_k^*}^{S_r})_B$  is projective. Thus

$p^m \text{Lie}(r)_B$  and  $(\text{Lie}(k) \uparrow_{S_k^*}^{S_r})_B$  are projective modules with the same Brauer characters. Therefore, by [Benson 1995, Corollary 5.3.6], these modules are isomorphic. This proves [Theorem 3.1](#).

**Corollary 3.4.** *If  $B_0$  is the principal block of  $FS_r$  then*

$$\dim \text{Lie}(r)_{B_0} = \frac{1}{p^m} \dim (\text{Lie}(k) \uparrow_{S_k^*}^{S_r})_{B_0}.$$

*Proof.* For each nonprincipal block  $B$  of  $FS_r$  we have

$$\dim \text{Lie}(r)_B = \frac{1}{p^m} \dim (\text{Lie}(k) \uparrow_{S_k^*}^{S_r})_B,$$

by [Theorem 3.1](#). However, by (2-1),

$$\dim \text{Lie}(k) \uparrow_{S_k^*}^{S_r} = (k-1)! r! / k! = p^m (r-1)! = p^m \dim \text{Lie}(r).$$

The result follows. □

**Theorem 3.5.** *In the notation of [Theorem 3.1](#), we have*

$$\text{Lie}(r)_B \cong \bigoplus_{\substack{\lambda \in \Lambda_p^+(r) \\ \lambda \in B}} m_\lambda P^\lambda, \tag{3-3}$$

where, for each  $\lambda$ ,

$$m_\lambda = \frac{1}{r} \sum_{d|k} \mu(d) \beta^\lambda(\tau^{k/d}), \tag{3-4}$$

where  $\tau$  is an element of  $S_r$  of cycle type  $(k, k, \dots, k)$  and  $\beta^\lambda$  denotes the Brauer character of  $D^\lambda$ .

*Proof.* By [Theorem 2.1](#),  $\text{Lie}(r)_B$  is projective. Thus it satisfies (3-3) for suitable nonnegative integers  $m_\lambda$ . It remains to prove (3-4). If  $F'$  is an extension field of  $F$  then  $\text{Lie}_{F'}(r) \cong F' \otimes \text{Lie}_F(r)$  and  $P_{F'}^\lambda \cong F' \otimes P_F^\lambda$ . Also, block components are preserved under field extensions, by (2-5). Hence it suffices to prove the result for the field  $\mathbb{F}_p$  and then, by a similar argument, it suffices to prove the result for any chosen field  $F$  of characteristic  $p$ . We take  $F$  from the  $p$ -modular system  $(K, R, F)$  used in the proof of [Theorem 3.1](#).

Since  $\text{Lie}(k) \uparrow_{S_k^*}^{S_r}$  is projective we have

$$\text{Lie}(k) \uparrow_{S_k^*}^{S_r} \cong \bigoplus_{\rho \in \Lambda_p^+(r)} m'_\rho P^\rho$$

for suitable nonnegative integers  $m'_\rho$ . Let  $\lambda \in \Lambda_p^+(r)$  where  $\lambda \in B$ . By [Theorem 3.1](#) we have  $m_\lambda = (1/p^m)m'_\lambda$ . Let  $\phi$  denote the Brauer character of  $\text{Lie}(k) \uparrow_{S_k^*}^{S_r}$ . By the

orthogonality relation (2-6) we have

$$m_\lambda = \frac{1}{p^m} m'_\lambda = \frac{1}{p^m} (\beta^\lambda, \phi)_{S_r}.$$

As observed in the proof of [Theorem 3.1](#),  $\text{Lie}(k) \uparrow_{S_k^*}^{S_r}$  is a modular reduction of  $\text{Lie}_K(k) \uparrow_{S_k^*}^{S_r}$ . The character of  $\text{Lie}_K(k) \uparrow_{S_k^*}^{S_r}$  is  $\psi_k \uparrow_{S_k^*}^{S_r}$ , where  $\psi_k$  denotes the character of  $\text{Lie}_K(k)$  as a  $KS_k^*$ -module. By [[Nagao and Tsushima 1989](#), Chapter 3, Lemma 6.4],  $\phi$  and  $\psi_k \uparrow_{S_k^*}^{S_r}$  take the same value on  $p'$ -elements of  $S_r$ . Thus, by Frobenius reciprocity,

$$m_\lambda = \frac{1}{p^m} (\beta^\lambda, \psi_k \uparrow_{S_k^*}^{S_r})_{S_r} = \frac{1}{p^m} (\beta^\lambda \downarrow_{S_k^*}^{S_r}, \psi_k)_{S_k^*}.$$

Let  $\tau$  be as in the statement of the theorem. Then  $\tau$  is conjugate to, and therefore can be taken to be, an element  $\sigma_k^*$  of  $S_k^*$  corresponding to a  $k$ -cycle  $\sigma_k$  of  $S_k$ . Thus, by (2-4), we have

$$\frac{1}{p^m} (\beta^\lambda \downarrow_{S_k^*}^{S_r}, \psi_k)_{S_k^*} = \frac{1}{p^m k} \sum_{d|k} \mu(d) \beta^\lambda(\tau^{k/d}).$$

The result follows. □

We now turn to Lie powers and, for the rest of this section, we assume that  $F$  is infinite. As before, let  $n$  be a positive integer and let  $E$  be the natural  $\text{GL}_n(F)$ -module.

**Theorem 3.6.** *Let  $F$  be an infinite field of prime characteristic  $p$ . Let  $r$  be a positive integer and write  $r = p^m k$  where  $m \geq 0, k \geq 1$ , and  $p \nmid k$ . Let  $B$  be a block of  $S_F(n, r)$  such that  $\tilde{B} \neq \emptyset$ . Then*

$$L^r(E)_B \cong \frac{1}{p^m} L^k(E^{\otimes p^m})_B.$$

*Proof.* Let  $\mathcal{T}$  be as defined in [Section 2C](#). Thus, by [Theorem 2.2](#),  $L^r(E)_B \in \mathcal{T}$ . Also, since  $L^k(E^{\otimes p^m})$  is a direct summand of  $E^{\otimes r}$ , we have  $L^k(E^{\otimes p^m}) \in \mathcal{T}$ , by (2-7).

Suppose first that  $n \geq r$ . Then we may write  $B = B(v)$  where  $v \neq \emptyset$ . By (2-11) and (2-13),

$$f_r(p^m L^r(E)_{B(v)}) \cong p^m \text{Lie}(r)_{B(v)}.$$

Similarly, by [Corollary 2.4](#) and (2-13),

$$f_r(L^k(E^{\otimes p^m})_{B(v)}) \cong (\text{Lie}(k) \uparrow_{S_k^*}^{S_r})_{B(v)}.$$

Also, by [Theorem 3.1](#),  $p^m \text{Lie}(r)_{B(v)} \cong (\text{Lie}(k) \uparrow_{S_k^*}^{S_r})_{B(v)}$ . It follows from (2-12) that if  $U, V \in \mathcal{T}$  and  $f_r(U) \cong f_r(V)$  then  $U \cong V$ . Hence the isomorphism in [Theorem 3.6](#) holds when  $n \geq r$ .

Now suppose that  $n < r$  and let  $\tilde{B} = v$ . Thus  $B \in B(v)$ . Consider the  $S_F(r, r)$ -block  $B(v)$ . By the first case, there is an isomorphism of  $S_F(r, r)$ -modules,

$$L^r(E)_{B(v)} \cong \frac{1}{p^m} L^k(E^{\otimes p^m})_{B(v)}. \quad (3-5)$$

We apply truncation  $d_{r,n}$  to (3-5). By (2-9) and the other properties of truncation given in Section 2C, we obtain (3-5) for  $S_F(n, r)$ -modules. Hence the corresponding block components are isomorphic for all  $S_F(n, r)$ -blocks in  $B(v)$  and we obtain the isomorphism of Theorem 3.6.  $\square$

**Theorem 3.7.** *In the notation of Theorem 3.6, we have*

$$L^r(E)_B \cong \bigoplus_{\substack{\lambda \in \Lambda_p^+(n,r) \\ \lambda \in B}} m_\lambda T(\lambda),$$

where  $m_\lambda$  is given by (3-4).

*Proof.* By Theorem 2.2,  $L^r(E)_B \in \mathcal{T}$ . Thus  $L^r(E)_B$  is isomorphic to a direct sum of tilting modules  $T(\lambda)$  with  $\lambda \in \Lambda_p^+(n, r)$  and  $\lambda \in B$ . Let  $\tilde{B} = v$ . Then, for  $n \geq r$ , we have  $f_r(L^r(E)_{B(v)}) \cong \text{Lie}(r)_{B(v)}$ , by (2-11) and (2-13), and  $f_r(T(\lambda)) \cong P^\lambda$  for all  $\lambda \in \Lambda_p^+(n, r)$ , by (2-12). Thus, for  $n \geq r$ , the result is given by Theorem 3.5. For  $n < r$  the result follows by truncation, as in the proof of Theorem 3.6. (Note that the effect of truncation on  $T(\lambda)$  is given by (2-8).)  $\square$

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