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We reformulate basepoint-free theorems using notions introduced by Shokurov, such as b-divisors and saturation of linear systems. Our formulation is flexible and has some important applications. One of the main purposes of this paper is to prove a generalization of the basepoint-free theorem in Fukuda's paper "On numerically effective log canonical divisors".

1. Introduction

In this paper, we reformulate basepoint-free theorems by using Shokurov's ideas [2003] of *b-divisors* and *saturation of linear systems*. Combining the refined Kawamata–Shokurov basepoint-free theorem (quoted here as Theorem 2.1) or its generalization (Theorem 6.1) with Ambro's formulation of Kodaira's canonical bundle formula, we obtain new basepoint-free theorems (Theorems 4.4 and 6.2), which are flexible and have some important applications (Theorem 7.11). One of the main purposes of this paper is to prove the following generalization of the basepoint-free theorem given in [Fukuda 2002, Proposition 3.3]:

Theorem 1.1. *Let (X, B) be an lc pair and let $\pi : X \to S$ be a proper morphism onto a variety S. Assume the following conditions*:

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(A) *H is a π-nef $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X.*

(B) *$H - (K_X + B)$ is π-nef and π-abundant.*

(C) *$\kappa(X_\eta, (aH - (K_X + B))_\eta) \geq 0$ and*

$$\nu(X_\eta, (aH - (K_X + B))_\eta) = \nu(X_\eta, (H - (K_X + B))_\eta)$$

for some $a \in \mathbb{Q}$ with $a > 1$, where η is the generic point of S.

(D) *There is a positive integer c such that cH is Cartier and*

$$\mathcal{O}_T(cH) := \mathcal{O}_X(cH)|_T$$

is π-generated, where $T = \mathrm{Nklt}(X, B)$ is the non-klt locus of (X, B).

Then H is π-semiample.

This will be proved on page 816. As an application of Theorem 1.1, we have:

Theorem 1.2 [Fujino and Gongyo 2011, Theorem 4.12]. *Let $\pi : X \to S$ be a projective morphism between projective varieties. Let (X, B) be an lc pair such that $K_X + B$ is nef and log abundant over S. Then $K_X + B$ is f-semiample.*

We also used Theorem 1.1 to prove the finite generation of the log canonical ring for log canonical 4-folds in [Fujino 2010]; see Remark 3.4 of that paper. As we explain elsewhere [Fujino 2007b, Remark 3.10.3; 2011d, 5.1], the proof of Theorem 4.3 of [Kawamata 1985] contains a gap. Because of that gap, Theorem 5.1 of [Kawamata 1985] was also not rigorously proved, and since Proposition 3.3 of [Fukuda 2002] depends on it, our proof of Theorem 1.1 is the first rigorous proof of this important result of Fukuda.

Another purpose of this paper is to show how to use Shokurov's ideas, such as *b-divisors*, *saturation of linear systems*, various kinds of *adjunction*, and so on, by reproving some known results in our formulation. Thus one can regard this paper as Chapter $8\frac{1}{2}$ of the book [Corti et al. 2007]. It is also a complement of the paper [Fujino 2011d]. We do not use the powerful new method developed in [Ambro 2003; Fujino 2009a; 2009b; 2009c; 2011a; 2011b; 2011c]. For related topics and applications, see [Fujino 2010; Gongyo 2010, Section 6; Cacciola 2011; Fujino and Gongyo 2011].

Remark 1.3. Professor Yujiro Kawamata [2011a] has announced a correction to the error in the proof of [Kawamata 1985, Theorem 4.3]. The new proof seems to depend heavily on arguments in his preprints [2011b; 2010]. If we accept his correction, then Theorem 1.1 holds under the assumption that (X, B) is dlt and S is a point, by [Fukuda 2002, Proposition 3.3] (see Remark 6.7 (ii)). As stated in the introduction of [Kawamata 2011a], our arguments are simpler. We note that our approach is completely different from Kawamata's original one. Anyway, Theorem 1.1 plays a crucial role in our study of the log abundance conjecture for log canonical pairs; see [Fujino and Gongyo 2011, Section 4]. Therefore, this paper is very relevant for the minimal model program for log canonical pairs.

Let us explain the motivation for our formulation.

1.4. ***Motivation.*** Let (X, B) be a projective klt pair and let D be a nef Cartier divisor on X such that $D-(K_X+B)$ is nef and big. Then the Kawamata–Shokurov basepoint-free theorem means that $|mD|$ is free for every $m \gg 0$. Let $f: Y \to X$ be a projective birational morphism from a normal projective variety Y such that $K_Y+B_Y = f^*(K_X+B)$. We note that f^*D is a nef Cartier divisor on Y and that $f^*D-(K_Y+B_Y)$ is nef and big. It is obvious that $|mf^*D|$ is free for every $m \gg 0$ because $|mD|$ is free for every $m \gg 0$. In general, we cannot directly apply the Kawamata–Shokurov basepoint-free theorem to f^*D and (Y, B_Y). This is because (Y, B_Y) is sub-klt but is not always klt. Note that a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor L on X is nef, big, or semiample if and only if so is f^*L. However, the notion of klt is not stable under birational pull-backs. By adding a *saturation condition*, which is trivially satisfied for klt pairs, we can apply the Kawamata–Shokurov basepoint-free theorem for sub-klt pairs (see Theorem 2.1). By this new formulation, the basepoint-free theorem becomes more flexible and has some important applications.

1.5. ***Background.*** A key result we need is [Ambro 2004, Theorem 0.2], which is a generalization of [Fujino 2003, Section 4: Pull-back of $L^{ss}_{X/Y}$]. It originates from *Kawamata's positivity theorem* [1998] and Shokurov's idea of *adjunction*. For details, see [Ambro 2004, Introduction]. The formulation and calculation we borrow from [Ambro 2005b; 2007] grew out from Shokurov's *saturation of linear systems* [2003, 4.32].

1.6. ***Outline of the paper.*** In Section 2, we reformulate the Kawamata–Shokurov basepoint-free theorem for sub-klt pairs with a saturation condition. To state our theorem, we use the notion of b-divisors. It is very useful to discuss linear systems with some base conditions. In Section 3, we collect basic properties of b-divisors and prove some elementary properties. In Section 4, we discuss a slight generalization of the main theorem of [Kawamata 1985]. We need this generalization in Section 7. The main ingredient of our proof is Ambro's formulation of Kodaira's canonical bundle formula. By this formula and the refined Kawamata–Shokurov basepoint-free theorem obtained in Section 2, we can quickly prove Kawamata's theorem in [Kawamata 1985] and its generalization without appealing to the notion of generalized normal crossing varieties. In Section 5, we treat the basepoint-free theorem of Reid–Fukuda type. In this case, the saturation condition behaves very well for inductive arguments. It helps us understand the saturation condition of linear systems. In Section 6, we prove some variants of basepoint-free theorems, mainly due to Fukuda [2002]. We reformulate them by using b-divisors and saturation conditions. Then we use Ambro's canonical bundle formula to reduce them to the easier case instead of proving them directly by the X-method. In Section 7, we generalize the Kawamata–Shokurov basepoint-free theorem and Kawamata's main theorem in [Kawamata 1985] for *pseudo-klt pairs*. Theorem 7.11, which is new, is the main theorem of this section. It will be useful for the study of lc centers (Theorem 7.13).

Notation. Let $B = \sum b_i B_i$ be a $\mathbb{Q}$-divisor on a normal variety X such that B_i is prime for every i and that $B_i \neq B_j$ for $i \neq j$. We denote by

$$\lceil B \rceil = \sum \lceil b_i \rceil B_i, \quad \lfloor B \rfloor = \sum \lfloor b_i \rfloor B_i, \quad \text{and} \quad \{B\} = B - \lfloor B \rfloor$$

the *round-up*, the *round-down*, and the *fractional part* of B. Note that we do not use $\mathbb{R}$-divisors in this paper. We make one general remark here. Since the freeness (or semiampleness) of a Cartier divisor D on a variety X depends only on the linear equivalence class of D, we can freely replace D by a linearly equivalent divisor to prove the freeness (or semiampleness) of D.

We will work over an algebraically closed field k of characteristic zero throughout this paper.

2. Kawamata–Shokurov basepoint-free theorem revisited

Kawamata and Shokurov claimed the following theorem for klt pairs, that is, they assumed that B is effective, which implies that condition (2) is trivially satisfied. We think that our formulation is useful for some applications. Readers not familiar with the notion of b-divisors are referred to Section 3.

Theorem 2.1 (Basepoint-free theorem). *Let (X, B) be a sub-klt pair, let $\pi : X \to S$ be a proper surjective morphism onto a variety S and let D be a π-nef Cartier divisor on X. Assume the following conditions*:

(1) *$rD - (K_X + B)$ is nef and big over S for some positive integer r.*

(2) (Saturation condition.) *There exists a positive integer j_0 such that*

$$\pi_* \mathcal{O}_X(\lceil \mathbf{A}(X, B) \rceil + j\overline{D}) \subseteq \pi_* \mathcal{O}_X(jD)$$

for every integer $j \geq j_0$.

Then mD is π-generated for every $m \gg 0$, that is, there exists a positive integer m_0 such that for every $m \geq m_0$ the natural homomorphism $\pi^* \pi_* \mathcal{O}_X(mD) \to \mathcal{O}_X(mD)$ is surjective.

Proof. The usual proof of the basepoint-free theorem, that is, the X-method, works without any changes if we note Lemma 3.10. For the details, see, for example, [Kawamata et al. 1987, Section 3-1]. See also Remarks 3.14–3.17. □

The assumptions in Theorem 2.1 are birational in nature. This point is indispensable in Section 4. We note that we can assume that X is nonsingular and Supp B is a simple normal crossing divisor because conditions (1) and (2) are invariant for birational pull-backs. So, it is easy to see that Theorem 2.1 is equivalent to the following theorem.

Theorem 2.2. *Let X be a nonsingular variety and let B be a $\mathbb{Q}$-divisor on X such that $\lfloor B \rfloor \leq 0$ and* Supp B *is a simple normal crossing divisor. Let $\pi : X \to S$ be a projective morphism onto a variety S and let D be a π-nef Cartier divisor on X. Assume the following conditions*:

(1) $rD - (K_X + B)$ *is nef and big over S for some positive integer r.*

(2) (Saturation condition.) *There exists a positive integer j_0 such that*

$$\pi_* \mathcal{O}_X(\lceil -B \rceil + jD) \simeq \pi_* \mathcal{O}_X(jD)$$

for every integer $j \geq j_0$.

Then mD is π-generated for every $m \gg 0$.

The following example says that the original Kawamata–Shokurov basepoint-free theorem does not necessarily hold for *sub*-klt pairs.

Example 2.3. Let $X = E$ be an elliptic curve. We take a Cartier divisor H such that $\deg H = 0$ and $lH \not\sim 0$ for every $l \in \mathbb{Z} \setminus \{0\}$. In particular, H is nef. We put $B = -P$, where P is a closed point of X. Then (X, B) is sub-klt and $H - (K_X + B)$ is ample. However, H is not semiample. In this case, $H^0(X, \mathcal{O}_X(\lceil \mathbf{A}(X, B) \rceil + j\overline{H})) \simeq H^0(X, \mathcal{O}_X(P + jH)) \simeq k$ for every j. However, $H^0(X, \mathcal{O}_X(jH)) = 0$ for all j. Therefore, the saturation condition in Theorem 2.1 does not hold.

We note that Kollár's effective basepoint-freeness holds under the same assumption as in Theorem 2.1.

Theorem 2.4 (Effective freeness). *We use the same notation and assumption as in Theorem 2.1. Then there exists a positive integer l, which depends only on* $\dim X$ *and* $\max\{r, j_0\}$, *such that lD is π-generated, that is, $\pi^* \pi_* \mathcal{O}_X(lD) \to \mathcal{O}_X(lD)$ is surjective.*

Sketch of the proof. We need no new ideas. So, we just explain how to modify the arguments in [Kollár 1993, Section 2]. From now on, we use the notation in [Kollár 1993]. In that reference, (X, Δ) is assumed to be klt, that is, (X, Δ) is sub-klt and Δ is effective. The effectivity of Δ implies that H' is f-exceptional in [ibid., (2.1.4.3)]. We need this to prove $H^0(Y, \mathcal{O}_Y(f^*N + H')) = H^0(X, \mathcal{O}_X(N))$ in [ibid., (2.1.6)].

It is not difficult to see that $0 \leq H' \leq \lceil \mathbf{A}(X, \Delta)_Y \rceil$ in our notation. Therefore, it is sufficient to assume the saturation condition Theorem 2.1(2) in the proof of Kollár's effective freeness (see [ibid., Section 2]). We make one more remark. Applying the argument in the first part of [ibid., 2.4] to $\mathbb{O}_X(j\overline{D} + \lceil \mathbf{A}(X, B) \rceil)$ on the generic fiber of $\pi : X \to S$ with the saturation condition (2) in Theorem 2.1, we obtain a positive integer l_0 that depends only on $\dim X$ and $\max\{r, j_0\}$ such that $\pi_* \mathbb{O}_X(l_0 D) \neq 0$. As explained above, the arguments in Section 2 in [ibid.] work with only minor modifications in our setting. We leave the details as an exercise for the reader. □

3. b-divisors

Let us recall the notion of singularities of pairs, referring the reader to [Fujino 2007b] for a more extended treatment.

Definition 3.1 (Singularities of pairs). Let X be a normal variety and let B be a $\mathbb{Q}$-divisor on X such that $K_X + B$ is $\mathbb{Q}$-Cartier. Let $f : Y \to X$ be a resolution of singularities such that $\mathrm{Exc}(f) \cup f_*^{-1} B$ has a simple normal crossing support, where $\mathrm{Exc}(f)$ is the exceptional locus of f. We write

$$K_Y = f^*(K_X + B) + \sum a_i A_i.$$

We note that a_i is called the *discrepancy* of A_i. Then the pair (X, B) is *sub-klt* (resp. *sub-lc*) if $a_i > -1$ (resp. $a_i \geq -1$) for every i. The pair (X, B) is *klt* (resp. *lc*) if (X, B) is sub-klt (resp. sub-lc) and B is effective. (In some literature, sub-klt and sub-lc are sometimes called klt and lc.)

Let (X, B) be an lc pair. If there exists a resolution $f : Y \to X$ such that $\mathrm{Exc}(f)$ and $\mathrm{Exc}(f) \cup f_*^{-1} B$ are simple normal crossing divisors on Y and

$$K_Y = f^*(K_X + B) + \sum a_i A_i$$

with $a_i > -1$ for all f-exceptional A_i's, then (X, B) is called *dlt*.

Remark 3.2. Let (X, B) be a klt (resp. lc) pair and let $f : Y \to X$ be a proper birational morphism of normal varieties. We put $K_Y + B_Y = f^*(K_X + B)$. Then (Y, B_Y) is not necessarily klt (resp. lc) but it is sub-klt (resp. sub-lc).

Let us recall the definition of *log canonical centers*.

Definition 3.3 (Log canonical center). Let (X, B) be a sub-lc pair. A subvariety $W \subset X$ is called a *log canonical center* or an *lc center* of (X, B) if there is a resolution $f : Y \to X$ such that $\mathrm{Exc}(f) \cup \mathrm{Supp}\, f_*^{-1} B$ is a simple normal crossing divisor on Y and a divisor E with discrepancy -1 such that $f(E) = W$. A log canonical center $W \subset X$ of (X, B) is called *exceptional* if there is a unique divisor E_W on Y with discrepancy -1 such that $f(E_W) = W$ and $f(E) \cap W = \varnothing$ for every other divisor $E \neq E_W$ on Y with discrepancy -1; see [Kollár 2007, 8.1].

3.4. ***b-divisors.*** The notion of *b-divisors*, introduced by Shokurov, plays a central role in this paper, and we now recall its definition. For details, we refer to [Ambro 2005b, 1-B] and [Corti 2007, 2.3.2]. The reader can find various examples of b-divisors in [Iskovskikh 2003].

Definition 3.5 (b-divisor). Let X be a normal variety and let $\mathrm{Div}(X)$ be the free abelian group generated by Weil divisors on X. A *b-divisor* on X is an element

$$\mathbf{D} \in \mathbf{Div}(X) = \mathrm{projlim}_{Y\to X} \mathrm{Div}(Y),$$

where the projective limit is taken over all proper birational morphisms $f : Y \to X$ of normal varieties, under the push forward homomorphism $f_* : \mathrm{Div}(Y) \to \mathrm{Div}(X)$. A $\mathbb{Q}$-*b-divisor* on X is an element of $\mathbf{Div}_{\mathbb{Q}}(X) = \mathbf{Div}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$.

Definition 3.6 (Discrepancy $\mathbb{Q}$-b-divisor). Let X be a normal variety and let B be a $\mathbb{Q}$-divisor on X such that $K_X + B$ is $\mathbb{Q}$-Cartier. Then the *discrepancy* $\mathbb{Q}$*-b-divisor* of the pair (X, B) is the $\mathbb{Q}$-b-divisor $\mathbf{A} = \mathbf{A}(X, B)$ with the trace $\mathbf{A}_Y$ defined by the formula

$$K_Y = f^*(K_X + B) + \mathbf{A}_Y,$$

where $f : Y \to X$ is a proper birational morphism of normal varieties.

Definition 3.7 (Cartier closure). Let D be a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on a normal variety X. Then the $\mathbb{Q}$-b-divisor $\overline{D}$ denotes the *Cartier closure* of D, whose trace on Y is $\overline{D}_Y = f^*D$, where $f : Y \to X$ is a proper birational morphism of normal varieties.

Definition 3.8. Let $\mathbf{D}$ be a $\mathbb{Q}$-b-divisor on X. The round up $\lceil \mathbf{D} \rceil \in \mathbf{Div}(X)$ is defined componentwise. The restriction of $\mathbf{D}$ to an open subset $U \subset X$ is a well-defined $\mathbb{Q}$-b-divisor on U, denoted by $\mathbf{D}|_U$. Then $\mathcal{O}_X(\mathbf{D})$ is an $\mathcal{O}_X$-module whose sections on an open subset $U \subset X$ are given by

$$H^0(U, \mathcal{O}_X(\mathbf{D})) = \{a \in k(X)^\times;\ (\overline{(a)} + \mathbf{D})|_U \geq 0\} \cup \{0\},$$

where $k(X)$ is the function field of X. Note that $\mathcal{O}_X(\mathbf{D})$ is not necessarily coherent.

3.9. ***Basic properties.*** We recall the first basic property of discrepancy $\mathbb{Q}$-b-divisors. We will treat a generalization of Lemma 3.10 for sub-lc pairs below.

Lemma 3.10. *Let (X, B) be a sub-klt pair and let D be a Cartier divisor on X. Let $f : Y \to X$ be a proper surjective morphism from a nonsingular variety Y. We write $K_Y = f^*(K_X + B) + \sum a_i A_i$. We assume that $\sum A_i$ is a simple normal crossing divisor. Then, for every integer j,*

$$\mathcal{O}_X(\lceil \mathbf{A}(X, B) \rceil + j\overline{D}) = f_*\mathcal{O}_Y(\textstyle\sum \lceil a_i \rceil A_i) \otimes \mathcal{O}_X(jD)$$

Let E be an effective divisor on Y such that $E \leq \sum \lceil a_i \rceil A_i$. Then

$$\pi_* f_* \mathcal{O}_Y(E + f^* jD) \simeq \pi_* \mathcal{O}_X(jD)$$

if

$$\pi_*\mathcal{O}_X(\lceil \mathbf{A}(X, B)\rceil + j\overline{D}) \subseteq \pi_*\mathcal{O}_X(jD),$$

where $\pi : X \to S$ *is a proper surjective morphism onto a variety* S.

Proof. For the first equality, see [Corti 2007, Lemmas 2.3.14 and 2.3.15] or their generalizations: Lemmas 3.19 and 3.20 below. Since E is effective,

$$\pi_*\mathcal{O}_X(jD) \subseteq \pi_* f_*\mathcal{O}_Y(E + f^* jD) \simeq \pi_*(f_*\mathcal{O}_Y(E) \otimes \mathcal{O}_X(jD)).$$

By the assumption and $E \leq \sum \lceil a_i \rceil A_i$,

$$\begin{aligned}\pi_*(f_*\mathcal{O}_Y(E) \otimes \mathcal{O}_X(jD)) &\subseteq \pi_*\big(f_*\mathcal{O}_Y\big(\textstyle\sum \lceil a_i\rceil A_i\big) \otimes \mathcal{O}_X(jD)\big)\\ &= \pi_*\mathcal{O}_X(\lceil \mathbf{A}(X, B)\rceil + j\overline{D})\\ &\subseteq \pi_*\mathcal{O}_X(jD).\end{aligned}$$

Therefore, we obtain $\pi_* f_*\mathcal{O}_Y(E + f^* jD) \simeq \pi_*\mathcal{O}_X(jD)$. □

We will use Lemma 3.11 in Section 4. The vanishing theorem in Lemma 3.11 is nothing but the Kawamata–Viehweg–Nadel vanishing theorem.

Lemma 3.11. *Let* X *be a normal variety and let* B *be a* $\mathbb{Q}$*-divisor on* X *such that* $K_X + B$ *is* $\mathbb{Q}$*-Cartier. Let* $f : Y \to X$ *be a proper birational morphism from a normal variety* Y*. We put* $K_Y + B_Y = f^*(K_X + B)$*. Then*

$$f_*\mathcal{O}_Y(\lceil \mathbf{A}(Y, B_Y)\rceil) = \mathcal{O}_X(\lceil \mathbf{A}(X, B)\rceil)$$

and

$$R^i f_*\mathcal{O}_Y(\lceil \mathbf{A}(Y, B_Y)\rceil) = 0$$

for every $i > 0$.

Proof. Let $g : Z \to Y$ be a resolution such that $\mathrm{Exc}(g) \cup g_*^{-1} B_Y$ has a simple normal crossing support. We put $K_Z + B_Z = g^*(K_Y + B_Y)$. Then $K_Z + B_Z = h^*(K_X + B)$, where $h = f \circ g : Z \to X$. By Lemma 3.10,

$$\mathcal{O}_Y(\lceil \mathbf{A}(Y, B_Y)\rceil) = g_*\mathcal{O}_Z(\lceil -B_Z\rceil)$$

and

$$\mathcal{O}_X(\lceil \mathbf{A}(X, B)\rceil) = h_*\mathcal{O}_Z(\lceil -B_Z\rceil).$$

Therefore, $f_*\mathcal{O}_Y(\lceil \mathbf{A}(Y, B_Y)\rceil) = \mathcal{O}_X(\lceil \mathbf{A}(X, B)\rceil)$. Since, $-B_Z = K_Z - h^*(K_X + B)$, we have

$$\lceil -B_Z\rceil = K_Z + \{B_Z\} - h^*(K_X + B).$$

Therefore, $R^i g_*\mathcal{O}_Z(\lceil -B_Z\rceil) = 0$ and $R^i h_*\mathcal{O}_Z(\lceil -B_Z\rceil) = 0$ for every $i > 0$ by the Kawamata–Viehweg vanishing theorem. Thus, $R^i f_*\mathcal{O}_Y(\lceil \mathbf{A}(Y, B_Y)\rceil) = 0$ for every $i > 0$ by Leray's spectral sequence. □

Remark 3.12. We use the same notation as in Remark 3.2. Let (X, B) be a klt pair. Let D be a Cartier divisor on X and let $\pi : X \to S$ be a proper morphism onto a variety S. We put $p = \pi \circ f : Y \to S$. Then

$$p_*\mathcal{O}_Y(jf^*D) \simeq \pi_*\mathcal{O}_X(jD) \simeq p_*\mathcal{O}_Y(\lceil \mathbf{A}(Y, B_Y)\rceil + j\overline{f^*D})$$

for every integer j. This is because $f_*\mathcal{O}_Y(\lceil \mathbf{A}(Y, B_Y)\rceil) = \mathcal{O}_X(\lceil \mathbf{A}(X, B)\rceil) \simeq \mathcal{O}_X$ by Lemma 3.11.

Remark 3.13 (Multiplier ideal sheaf). Let D be an effective $\mathbb{Q}$-divisor on a nonsingular variety X. Then $\mathcal{O}_X(\lceil \mathbf{A}(X, D)\rceil)$ is nothing but the *multiplier ideal sheaf* $\mathcal{J}(X, D) \subseteq \mathcal{O}_X$ of D on X. See [Lazarsfeld 2004, Definition 9.2.1]. More generally, let X be a normal variety and let Δ be a $\mathbb{Q}$-divisor on X such that $K_X+\Delta$ is $\mathbb{Q}$-Cartier. Let D be a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X. Then $\mathcal{O}_X(\lceil \mathbf{A}(X, \Delta + D)\rceil) = \mathcal{J}((X, \Delta); D)$, where the right hand side is the *multiplier ideal sheaf* defined (but not investigated) in [Lazarsfeld 2004, Definition 9.3.56]. In general, $\mathcal{O}_X(\lceil \mathbf{A}(X, \Delta+D)\rceil)$ is a fractional ideal of $k(X)$.

The next four remarks help us understand Theorem 2.1.

Remark 3.14 (Nonvanishing theorem). By Shokurov's nonvanishing theorem (see [Kawamata et al. 1987, Theorem 2-1-1]), we have $\pi_*\mathcal{O}_X(\lceil \mathbf{A}(X, B)\rceil + j\overline{D}) \neq 0$ for every $j \gg 0$. Thus $\pi_*\mathcal{O}_X(jD) \neq 0$ for every $j \gg 0$ by condition (2) in Theorem 2.1.

Remark 3.15. We know that $\lceil \mathbf{A}(X, B)\rceil \geq 0$ since (X, B) is sub-klt. Therefore, $\pi_*\mathcal{O}_X(jD) \subseteq \pi_*\mathcal{O}_X(\lceil \mathbf{A}(X, B)\rceil + j\overline{D})$. This implies that

$$\pi_*\mathcal{O}_X(jD) \simeq \pi_*\mathcal{O}_X(\lceil \mathbf{A}(X, B)\rceil + j\overline{D})$$

for $j \geq j_0$, by condition (2) in Theorem 2.1.

Remark 3.16. If the pair (X, B) is klt, then $\lceil \mathbf{A}(X, B)\rceil$ is effective and exceptional over X. In this case, it is obvious that $\pi_*\mathcal{O}_X(jD) = \pi_*\mathcal{O}_X(\lceil \mathbf{A}(X, B)\rceil + j\overline{D})$.

Remark 3.17. Condition (2) in Theorem 2.1 is a very elementary case of *saturation of linear systems*. See [Corti 2007, 2.3.3] and [Ambro 2005b, 1-D].

We next introduce the notion of *non-klt $\mathbb{Q}$-b-divisor*, which is trivial for sub-klt pairs. We will use this in Section 5.

Definition 3.18 (Non-klt $\mathbb{Q}$-b-divisor). Let X be a normal variety and let B be a $\mathbb{Q}$-divisor on X such that $K_X + B$ is $\mathbb{Q}$-Cartier. Then the *non-klt $\mathbb{Q}$-b-divisor* of the pair (X, B) is the $\mathbb{Q}$-b-divisor $\mathbf{N} = \mathbf{N}(X, B)$ with the trace $\mathbf{N}_Y = \sum_{a_i \leq -1} a_i A_i$ for

$$K_Y = f^*(K_X + B) + \sum a_i A_i,$$

where $f : Y \to X$ is a proper birational morphism of normal varieties. It is easy to see that $\mathbf{N}(X, B)$ is a well-defined $\mathbb{Q}$-b-divisor. We put $\mathbf{A}^*(X, B) = \mathbf{A}(X, B) - \mathbf{N}(X, B)$.

Of course, $\mathbf{A}^*(X, B)$ is a well-defined $\mathbb{Q}$-b-divisor and $\lceil \mathbf{A}^*(X, B)\rceil \geq 0$. If (X, B) is sub-klt, then $\mathbf{N}(X, B) = 0$ and $\mathbf{A}(X, B) = \mathbf{A}^*(X, B)$.

The next lemma is a generalization of Lemma 3.10.

Lemma 3.19. *Let (X, B) be a sub-lc pair and let $f : Y \to X$ be a resolution such that* $\operatorname{Exc}(f) \cup \operatorname{Supp} f_*^{-1}B$ *is a simple normal crossing divisor on Y. We write $K_Y = f^*(K_X + B) + \sum a_i A_i$. Then*

$$\mathcal{O}_X(\lceil \mathbf{A}^*(X, B)\rceil) = f_*\mathcal{O}_Y\biggl(\sum_{a_i \neq -1} \lceil a_i\rceil A_i\biggr).$$

In particular, $\mathcal{O}_X(\lceil \mathbf{A}^*(X, B)\rceil)$ is a coherent $\mathcal{O}_X$-module. If (X, B) is lc, then $\mathcal{O}_X(\lceil \mathbf{A}^*(X, B)\rceil) \simeq \mathcal{O}_X$.

Let D be a Cartier divisor on X and let E be an effective divisor on Y such that $E \leq \sum_{a_i \neq -1} \lceil a_i\rceil A_i$. Then

$$\pi_* f_* \mathcal{O}_Y(E + f^* jD) \simeq \pi_*\mathcal{O}_X(jD)$$

if

$$\pi_*\mathcal{O}_X(\lceil \mathbf{A}^*(X, B)\rceil + j\overline{D}) \subseteq \pi_*\mathcal{O}_X(jD),$$

where $\pi : X \to S$ is a proper morphism onto a variety S.

Proof. By definition, $\mathbf{A}^*(X, B)_Y = \sum_{a_i \neq -1} a_i A_i$. If $g : Y' \to Y$ is a proper birational morphism from a normal variety Y', then

$$\lceil \mathbf{A}^*(X, B)_{Y'}\rceil = g^*\lceil \mathbf{A}^*(X, B)_Y\rceil + F,$$

where F is a g-exceptional effective divisor, by Lemma 3.20 below. This implies $f_*\mathcal{O}_Y(\lceil \mathbf{A}^*(X, B)_Y\rceil) = f'_*\mathcal{O}_{Y'}(\lceil \mathbf{A}^*(X, B)_{Y'}\rceil)$, where $f' = f \circ g$, from which it follows that $\mathcal{O}_X(\lceil \mathbf{A}^*(X, B)\rceil) = f_*\mathcal{O}_Y(\sum_{a_i \neq -1} \lceil a_i\rceil A_i)$ is a coherent $\mathcal{O}_X$-module. The last statement is easy to check. □

Lemma 3.20. *Let (X, B) be a sub-lc pair and let $f : Y \to X$ be a resolution as in Lemma 3.19. We consider the $\mathbb{Q}$-b-divisor $\mathbf{A}^* = \mathbf{A}^*(X, B) = \mathbf{A}(X, B) - \mathbf{N}(X, B)$. If Y' is a normal variety and $g : Y' \to Y$ is a proper birational morphism, then*

$$\lceil \mathbf{A}^*_{Y'}\rceil = g^*\lceil \mathbf{A}^*_Y\rceil + F,$$

where F is a g-exceptional effective divisor.

Proof. By definition, we have $K_Y = f^*(K_X + B) + \mathbf{A}_Y$. Therefore we may write

$$\begin{aligned} K_{Y'} &= g^* f^*(K_X + B) + \mathbf{A}_{Y'} = g^*(K_Y - \mathbf{A}_Y) + \mathbf{A}_{Y'} \\ &= g^*(K_Y + \{-\mathbf{A}^*_Y\} - \mathbf{N}_Y + \lfloor -\mathbf{A}^*_Y\rfloor) + \mathbf{A}_{Y'} \\ &= g^*(K_Y + \{-\mathbf{A}^*_Y\} - \mathbf{N}_Y) + \mathbf{A}_{Y'} - g^*\lceil \mathbf{A}^*_Y\rceil. \end{aligned}$$

We note that $(Y, \{-\mathbf{A}^*_Y\}-\mathbf{N}_Y)$ is lc and that the set of lc centers of $(Y, \{-\mathbf{A}^*_Y\}-\mathbf{N}_Y)$ coincides with that of $(Y, -\mathbf{A}^*_Y - \mathbf{N}_Y) = (Y, -\mathbf{A}_Y)$. Therefore, the round-up of $\mathbf{A}_{Y'} - g^*\lceil \mathbf{A}^*_Y \rceil - \mathbf{N}_{Y'}$ is effective and g-exceptional. Thus, we can write $\lceil \mathbf{A}^*_{Y'} \rceil = g^*\lceil \mathbf{A}^*_Y \rceil + F$, where F is a g-exceptional effective divisor. □

The next lemma is obvious by Lemma 3.19.

Lemma 3.21. *Let (X, B) be a sub-lc pair and let $f : Y \to X$ be a proper birational morphism from a normal variety Y. We put $K_Y + B_Y = f^*(K_X + B)$. Then $f_*\mathcal{O}_Y(\lceil \mathbf{A}^*(Y, B_Y)\rceil) = \mathcal{O}_X(\lceil \mathbf{A}^*(X, B)\rceil)$.*

4. Basepoint-free theorem; nef and abundant case

We recall the definition of *abundant* divisors, which are called *good* divisors in [Kawamata 1985]. See [Kawamata et al. 1987, Section 6-1].

Definition 4.1 (Abundant divisor). Let X be a complete normal variety and let D be a $\mathbb{Q}$-Cartier nef $\mathbb{Q}$-divisor on X. We define the *numerical Iitaka dimension* to be

$$\nu(X, D) = \max\{e;\ D^e \not\equiv 0\}.$$

This means that $D^{e'} \cdot S = 0$ for any e'-dimensional subvarieties S of X with $e' > e$ and there exists an e-dimensional subvariety T of X such that $D^e \cdot T > 0$. Then it is easy to see that $\kappa(X, D) \le \nu(X, D)$, where $\kappa(X, D)$ denotes *Iitaka's D-dimension*. A nef $\mathbb{Q}$-divisor D is said to be *abundant* if the equality $\kappa(X, D) = \nu(X, D)$ holds. Let $\pi : X \to S$ be a proper surjective morphism of normal varieties and let D be a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X. Then D is said to be π-abundant if $D|_{X_\eta}$ is abundant, where X_η is the generic fiber of π.

The next theorem is the main theorem of [Kawamata 1985]. For the relative statement, see [Nakayama 1986, Theorem 5]. We reduced Theorem 4.2 to Theorem 2.1 by using Ambro's results in [Ambro 2004] and [Ambro 2007], which is the main theme of [Fujino 2011d]. For the details, see [Fujino 2011d, Section 2].

Theorem 4.2 cf. [Kawamata et al. 1987, Theorem 6-1-11]. *Let (X, B) be a klt pair and let $\pi : X \to S$ be a proper morphism onto a variety S. Assume the following conditions*:

(a) *H is a π-nef $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X.*

(b) *$H - (K_X + B)$ is π-nef and π-abundant.*

(c) *$\kappa(X_\eta, (aH - (K_X + B))_\eta) \ge 0$ and*

$$\nu(X_\eta, (aH - (K_X + B))_\eta) = \nu(X_\eta, (H - (K_X + B))_\eta)$$

for some $a \in \mathbb{Q}$ with $a > 1$, where η is the generic point of S.

Then H is π-semiample.

Definition 4.3 (Iitaka fibration). Let $\pi : X \to S$ be a proper surjective morphism of normal varieties. Let D be a $\mathbb{Q}$-Cartier $\mathbb{Q}$-Weil divisor on X such that $\kappa(X_\eta, D_\eta) \geq 0$, where η is the generic point of S. Let $X \dashrightarrow W$ be the rational map over S induced by $\pi^*\pi_*\mathcal{O}_X(mD) \to \mathcal{O}_X(mD)$ for a sufficiently large and divisible integer m. We consider a projective surjective morphism $f : Y \to Z$ of nonsingular varieties that is birational to $X \dashrightarrow W$. We call $f : Y \to Z$ the *Iitaka fibration* with respect to D over S.

We now state the main result of this section, which will be used in the proof of Theorem 7.11. It is a slight generalization of Theorem 4.2.

Theorem 4.4. *Let (X, B) be a sub-klt pair and let $\pi : X \to S$ be a proper morphism onto a variety S. Assume the following conditions*:

(a) *H is a π-nef $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X.*

(b) *$H - (K_X + B)$ is π-nef and π-abundant.*

(c) *$\kappa(X_\eta, (aH - (K_X + B))_\eta) \geq 0$ and*

$$\nu(X_\eta, (aH - (K_X + B))_\eta) = \nu(X_\eta, (H - (K_X + B))_\eta)$$

for some $a \in \mathbb{Q}$ with $a > 1$, where η is the generic point of S.

(d) *Let $f : Y \to Z$ be the Iitaka fibration with respect to $H - (K_X + B)$ over S. We assume that there exists a proper birational morphism $\mu : Y \to X$ and put $K_Y + B_Y = \mu^*(K_X + B)$. In this setting, we assume* rank $f_*\mathcal{O}_Y(\lceil \mathbf{A}(Y, B_Y)\rceil) = 1$.

(e) (Saturation condition.) *There exist positive integers b and j_0 such that bH is Cartier and $\pi_*\mathcal{O}_X(\lceil \mathbf{A}(X, B)\rceil + jb\overline{H}) \subseteq \pi_*\mathcal{O}_X(jbH)$ for every positive integer $j \geq j_0$.*

Then H is π-semiample.

Proof. The proof of Theorem 4.2 given in [Fujino 2011d, Section 2] works without any changes. We note that condition (d) implies [ibid., Lemma 2.3] and that we can use condition (e) in the proof of [ibid., Lemma 2.4]. □

Remark 4.5. The rank of $f_*\mathcal{O}_Y(\lceil \mathbf{A}(Y, B_Y)\rceil)$ is a birational invariant for $f : Y \to Z$ by Lemma 3.11.

Remark 4.6. If (X, B) is klt and bH is Cartier, it is obvious that

$$\pi_*\mathcal{O}_X(\lceil \mathbf{A}(X, B)\rceil + jb\overline{H}) \simeq \pi_*\mathcal{O}_X(jbH)$$

for every positive integer j (see Remark 3.16).

Remark 4.7. We can easily generalize Theorem 4.4 to varieties in class $\mathcal{C}$ by suitable modifications. For details, see [Fujino 2011d, Section 4].

The following examples help us understand condition (d).

Example 4.8. Let $X = E$ be an elliptic curve and let $P \in X$ be a closed point. Take a general member $P_1 + P_2 + P_3 \in |3P|$. We put $B = \frac{1}{3}(P_1 + P_2 + P_3) - P$. Then (X, B) is sub-klt and $K_X + B \sim_{\mathbb{Q}} 0$. In this case, $\mathcal{O}_X(\lceil \mathbf{A}(X, B) \rceil) \simeq \mathcal{O}_X(P)$ and $H^0(X, \mathcal{O}_X(\lceil \mathbf{A}(X, B) \rceil)) \simeq k$.

Example 4.9. Let $f : X = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)) \to Z = \mathbb{P}^1$ be the Hirzebruch surface and let C (resp. E) be the positive (resp. negative) section of f. We take a general member $B_0 \in |5C|$. Note that $|5C|$ is a free linear system on X. We put $B = -\frac{1}{2}E + \frac{1}{2}B_0$ and consider the pair (X, B). Then (X, B) is sub-klt. We put $H = 0$. Then H is a nef Cartier divisor on X and $aH - (K_X + B) \sim_{\mathbb{Q}} \frac{1}{2}F$ for every rational number a, where F is a fiber of f. Therefore, $aH - (K_X + B)$ is nef and abundant for every rational number a. In this case, $\mathcal{O}_X(\lceil \mathbf{A}(X, B) \rceil) \simeq \mathcal{O}_X(E)$. Thus

$$\begin{aligned} H^0(X, \mathcal{O}_X(\lceil \mathbf{A}(X, B) \rceil + j\overline{H})) &\simeq H^0(X, \mathcal{O}_X(E)) \simeq k \\ &\simeq H^0(X, \mathcal{O}_X) \simeq H^0(X, \mathcal{O}_X(jH)) \end{aligned}$$

for every integer j. Therefore, $\pi : X \to \operatorname{Spec} k$, H, and (X, B) satisfy conditions (a), (b), (c), and (e) in Theorem 4.4. However, (d) is not satisfied. In our case, it is easy to see that $f : X \to Z$ is the Iitaka fibration with respect to $H - (K_X + B)$. Since $f_*\mathcal{O}_X(\lceil \mathbf{A}(X, B) \rceil) \simeq f_*\mathcal{O}_X(E)$, we have $\operatorname{rank} f_*\mathcal{O}_X(\lceil \mathbf{A}(X, B) \rceil) = 2$.

Remark 4.10. In Theorem 4.4, assumptions (a)–(c) are the same as in Theorem 4.2. Condition (e) is indispensable by Example 2.3 for sub-klt pairs. By using the nonvanishing theorem for generalized normal crossing varieties in [Kawamata 1985, Theorem 5.1], which is the hardest part to prove in [Kawamata 1985], the semiampleness of H seems to follow from conditions (a), (b), (c), and (e). However, we need (d) to apply Ambro's canonical bundle formula to the Iitaka fibration $f : Y \to Z$. See, for example, [Fujino 2011d, Section 3]. Unfortunately, as we saw in Example 4.9, condition (d) does not follow from the other assumptions. Anyway, condition (d) is automatically satisfied if (X, B) is klt; see [Fujino 2011d, Lemma 2.3].

The following two examples show that the effective version of Theorem 4.2 does not necessarily hold. The first one is an obvious example.

Example 4.11. Let $X = E$ be an elliptic curve and let m be an arbitrary positive integer. Then there is a Cartier divisor H on X such that $mH \sim 0$ and $lH \not\sim 0$ for $0 < l < m$. Therefore, the effective version of Theorem 4.2 does not necessarily hold.

The next one shows the reason why Theorem 2.4 does not imply the effective version of Theorem 4.2.

Example 4.12. Let E be an elliptic curve and $G = \mathbb{Z}/m\mathbb{Z} = \langle \zeta \rangle$, where ζ is a primitive m-th root of unity. We take an m-torsion point $a \in E$. The cyclic group

G acts on $E \times \mathbb{P}^1$ as follows:

$$E \times \mathbb{P}^1 \ni (x, [X_0 : X_1]) \mapsto (x + a, [\zeta X_0 : X_1]) \in E \times \mathbb{P}^1.$$

We put $X = (E \times \mathbb{P}^1)/G$. Then X has a structure of elliptic surface $p : X \to \mathbb{P}^1$. In this setting,

$$K_X = p^* \left(K_{\mathbb{P}^1} + \frac{m-1}{m}[0] + \frac{m-1}{m}[\infty] \right).$$

We put $H = p^{-1}(0)_{\mathrm{red}}$. Then H is a Cartier divisor on X. It is easy to see that H is nef and $H - K_X$ is nef and abundant. Moreover, $\kappa(X, aH - K_X) = \nu(X, aH - K_X) = 1$ for every rational number $a > 0$. It is obvious that $|mH|$ is free. However, $|lH|$ is not free for $0 < l < m$. Thus, the effective version of Theorem 4.2 does not hold.

5. Basepoint-free theorem of Reid–Fukuda type

The following result is a reformulation of the main theorem of [Fujino 2000].

Theorem 5.1 (Basepoint-free theorem of Reid–Fukuda type). *Let X be a nonsingular variety and let B be a $\mathbb{Q}$-divisor on X such that* Supp B *is a simple normal crossing divisor and (X, B) is sub-lc. Let $\pi : X \to S$ be a proper morphism onto a variety S and let D be a π-nef Cartier divisor on X. Assume the following conditions*:

(1) *$rD - (K_X + B)$ is nef and log big over S for some positive integer r.*

(2) (Saturation condition.) *There exists a positive integer j_0 such that*

$$\pi_* \mathcal{O}_X(\lceil \mathbf{A}^*(X, B) \rceil + j\overline{D}) \subseteq \pi_* \mathcal{O}_X(jD)$$

for every integer $j \geq j_0$.

Then mD is π-generated for every $m \gg 0$, that is, there exists a positive integer m_0 such that for every $m \geq m_0$ the natural homomorphism $\pi^* \pi_* \mathcal{O}_X(mD) \to \mathcal{O}_X(mD)$ is surjective.

Definition 5.2. Let (X, B) be a sub-lc pair and let $\pi : X \to S$ be a proper morphism onto a variety S. Let $\mathcal{L}$ be a line bundle on X. We say that $\mathcal{L}$ is *nef and log big* over S if and only if $\mathcal{L}$ is π-nef and π-big and the restriction $\mathcal{L}|_W$ is big over $\pi(W)$ for every lc center W of the pair (X, B). A $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor H on X is said to be nef and log big over S if and only if so is $\mathcal{O}_X(cH)$, where c is a positive integer such that cH is Cartier.

Proof of Theorem 5.1. We write $B = T + B_+ - B_-$ such that T, B_+, and B_- are effective divisors, they have no common irreducible components, $\lfloor B_+ \rfloor = 0$, and

$\lfloor T \rfloor = T$. If $T = 0$, then (X, B) is sub-klt. So, theorem follows from Theorem 2.1. Thus, we assume $T \neq 0$. Let T_0 be an irreducible component of T. If $m \geq r$, then

$$mD + \lceil B_- \rceil - T_0 - (K_X + B + \lceil B_- \rceil - T_0) = mD - (K_X + B)$$

is nef and log big over S for the pair $(X, B + \lceil B_- \rceil - T_0)$. We note that $B + \lceil B_- \rceil - T_0$ is effective. Therefore, $R^1 \pi_* \mathcal{O}_X(\lceil B_- \rceil - T_0 + mD) = 0$ for $m \geq r$ by the vanishing theorem: Lemma 5.3. Thus, we obtain the following commutative diagram for $m \geq \max\{r, j_0\}$:

$$\begin{array}{ccccc} \pi_* \mathcal{O}_X(\lceil B_- \rceil + mD) & \longrightarrow & \pi_* \mathcal{O}_{T_0}(\lceil B_-|_{T_0} \rceil + mD|_{T_0}) & \longrightarrow & 0 \\ \uparrow \cong & & \uparrow \iota & & \\ \pi_* \mathcal{O}_X(mD) & \xrightarrow{\alpha} & \pi_* \mathcal{O}_{T_0}(mD|_{T_0}). & & \end{array}$$

Here, we used

$$\begin{aligned} \pi_* \mathcal{O}_X(mD) &\subseteq \pi_* \mathcal{O}_X(\lceil B_- \rceil + mD) \\ &\simeq \pi_* \mathcal{O}_X(\lceil \mathbf{A}^*(X, B) \rceil + m\overline{D}) \\ &\subseteq \pi_* \mathcal{O}_X(mD) \end{aligned}$$

for $m \geq j_0$ (see Lemma 3.19). We put $K_{T_0} + B_{T_0} = (K_X + B)|_{T_0}$ and $D_{T_0} = D|_{T_0}$. Then (T_0, B_{T_0}) is sub-lc and it is easy to see that $rD_{T_0} - (K_{T_0} + B_{T_0})$ is nef and log big over $\pi(T_0)$. It is obvious that T_0 is nonsingular and $\mathrm{Supp}\, B_{T_0}$ is a simple normal crossing divisor. We note that $\pi_* \mathcal{O}_{T_0}(\lceil \mathbf{A}^*(T_0, B_{T_0}) \rceil + j\overline{D_{T_0}}) \simeq \pi_* \mathcal{O}_{T_0}(jD_{T_0})$ for every $j \geq \max\{r, j_0\}$ follows from the above diagram, that is, the natural inclusion ι is isomorphism for $m \geq \max\{r, j_0\}$. Thus, α is surjective for $m \geq \max\{r, j_0\}$. By induction, mD_{T_0} is π-generated for every $m \gg 0$. We can apply the same argument to every irreducible component of T. Therefore, the relative base locus of mD is disjoint from T for every $m \gg 0$ since the restriction map $\alpha : \pi_* \mathcal{O}_X(mD) \to \pi_* \mathcal{O}_{T_0}(mD_{T_0})$ is surjective for every irreducible component T_0 of T. The arguments in [Fukuda 1996, Proof of Theorem 3], which is a variant of the X-method, work without any changes (cf. Theorem 6.1). So, we obtain that mD is π-generated for every $m \gg 0$. □

The following vanishing theorem was already used in the proof of Theorem 5.1. The proof is an easy exercise by induction on $\dim X$ and on the number of the irreducible components of $\lfloor \Delta \rfloor$.

Lemma 5.3. *Let $\pi : X \to S$ be a proper morphism from a nonsingular variety X. Let $\Delta = \sum d_i \Delta_i$ be a sum of distinct prime divisors such that* $\mathrm{Supp}\, \Delta$ *is a simple normal crossing divisor and d_i is a rational number with $0 \leq d_i \leq 1$ for every i. Let D be a Cartier divisor on X. Assume that $D - (K_X + \Delta)$ is nef and log big over S for the pair (X, Δ). Then $R^i \pi_* \mathcal{O}_X(D) = 0$ for every $i > 0$.*

As in Theorem 2.4, effective freeness holds under the same assumption as in Theorem 5.1.

Theorem 5.4 (Effective freeness). *We use the same notation and assumption as in Theorem 5.1. Then there exists a positive integer l, which depends only on* $\dim X$ *and* $\max\{r, j_0\}$, *such that lD is π-generated, that is, $\pi^*\pi_*\mathcal{O}_X(lD) \to \mathcal{O}_X(lD)$ is surjective.*

Sketch of the proof. If (X, B) is sub-klt, then this theorem is nothing but Theorem 2.4. So, we can assume that (X, B) is not sub-klt. In this case, the arguments in [Fukuda 1996, Section 4] work with only minor modifications. From now on, we use the notation in [Fukuda 1996, Section 4]. By minor modifications, the proof in [Fukuda 1996, Section 4] works under the following weaker assumptions: X is nonsingular and Δ is a $\mathbb{Q}$-divisor on X such that $\operatorname{Supp}\Delta$ is a simple normal crossing divisor and (X, Δ) is sub-lc. In [Fukuda 1996, Claim 5], E_i is f-exceptional. In our setting, this is not true. However, the bound

$$0 \le \sum_{cb_i - e_i + p_i < 0} \lceil -(cb_i - e_i + p_i) \rceil E_i \le \lceil \mathbf{A}^*(X, \Delta)_Y \rceil,$$

which always holds even when Δ is not effective, is sufficient for us. It is because we can use the saturation condition (2) in Theorem 5.1. We leave the details as an exercise for the reader since all we have to do is to repeat the arguments in [Kollár 1993, Section 2] and [Fukuda 1996, Section 4]. □

The final statement in this section is the (effective) basepoint-free theorem of Reid–Fukuda type for dlt pairs.

Corollary 5.5. *Let (X, B) be a dlt pair and let $\pi : X \to S$ be a proper morphism onto a variety S. Let D be a π-nef Cartier divisor on X. Assume that $rD-(K_X+B)$ is nef and log big over S for some positive integer r. Then there exists a positive integer m_0 such that mD is π-generated for every $m \ge m_0$ and we can find a positive integer l, which depends only on* $\dim X$ *and r, such that lD is π-generated.*

Proof. Let $f : Y \to X$ be a resolution such that $\operatorname{Exc}(f)$ and $\operatorname{Exc}(f) \cup \operatorname{Supp} f_*^{-1}B$ are simple normal crossing divisors, $K_Y + B_Y = f^*(K_X + B)$, and f is an isomorphism over all the generic points of lc centers of the pair (X, B). Then (Y, B_Y) is sub-lc, and $rD_Y - (K_Y + B_Y)$ is nef and log big over S, where $D_Y = f^*D$. Since $\lceil \mathbf{A}^*(X, B) \rceil$ is effective and exceptional over X, $p_*\mathcal{O}_Y(\lceil \mathbf{A}^*(Y, B_Y) \rceil + j\overline{D_Y}) \simeq p_*\mathcal{O}_Y(jD_Y)$ for every j, where $p = \pi \circ f$. So, we can apply Theorems 5.1 and 5.4 to D_Y and (Y, B_Y). This concludes the proof. □

For the (effective) basepoint-freeness for lc pairs, see [Fujino 2009a; Fujino 2009b, 3.3.1 Base Point Free Theorem; Fujino 2011a, Theorem 13.1; Fujino 2011c, Theorem 1.2].

6. Variants of basepoint-free theorems due to Fukuda

The starting point of this section is a slight generalization of Theorem 2.1. It is essentially the same as [Fukuda 1996, Theorem 3].

Theorem 6.1. *Let X be a nonsingular variety and let B be a $\mathbb{Q}$-divisor on X such that (X, B) is sub-lc and* Supp B *is a simple normal crossing divisor. Let $\pi : X \to S$ be a proper morphism onto a variety S and let H be a π-nef $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X. Assume the following conditions*:

(1) *$H - (K_X + B)$ is nef and big over S.*

(2) (Saturation condition.) *There exist positive integers b and j_0 such that*

$$\pi_*\mathcal{O}_X(\lceil \mathbf{A}^*(X, B)\rceil + jb\overline{H}) \subseteq \pi_*\mathcal{O}_X(jbH)$$

for every integer $j \geq j_0$.

(3) *There is a positive integer c such that cH is Cartier and*

$$\mathcal{O}_T(cH) := \mathcal{O}_X(cH)|_T$$

is π-generated, where $T = -\mathbf{N}(X, B)_X$.

Then H is π-semiample.

Proof. If (X, B) is sub-klt, then this follows from Theorem 2.1. By replacing H by a multiple, we can assume that $b = 1$, $j_0 = 1$, and $c = 1$. Since

$$lH + \lceil \mathbf{A}^*_X \rceil - T - (K_X + \{B\}) = lH - (K_X + B)$$

is nef and big over S for every positive integer l, we have the following commutative diagram by the Kawamata–Viehweg vanishing theorem:

$$\begin{array}{ccccc} \pi_*\mathcal{O}_X(lH + \lceil \mathbf{A}^*_X \rceil) & \longrightarrow & \pi_*(\mathcal{O}_T(lH) \otimes \mathcal{O}_T(\lceil \mathbf{A}^*_X|_T \rceil)) & \longrightarrow & 0 \\ \uparrow \cong & & \uparrow \iota & & \\ \pi_*\mathcal{O}_X(lH) & \xrightarrow[\alpha]{} & \pi_*\mathcal{O}_T(lH). & & \end{array}$$

Thus, the natural inclusion ι is an isomorphism and α is surjective for every $l \geq 1$. In particular, $\pi_*\mathcal{O}_X(lH) \neq 0$ for every $l \geq 1$. The same arguments as in [Fukuda 1996, Proof of Theorem 3] show that H is π-semiample. □

The main purpose of this section is to prove Theorem 6.2 below, which is a generalization of Theorem 4.4 and Theorem 6.1. The basic strategy of the proof is the same as that of Theorem 4.4. That is, by using Ambro's canonical bundle formula, we reduce it to the case when $H - (K_X + B)$ is nef and big. This is nothing but Theorem 6.1.

Theorem 6.2. *Let X be a nonsingular variety and let B be a $\mathbb{Q}$-divisor on X such that (X, B) is sub-lc and* Supp B *is a simple normal crossing divisor. Let $\pi : X \to S$ be a proper morphism onto a variety S. Assume the following conditions*:

(a) *H is a π-nef $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X.*

(b) *$H - (K_X + B)$ is π-nef and π-abundant.*

(c) *$\kappa(X_\eta, (aH - (K_X + B))_\eta) \geq 0$ and $\nu(X_\eta, (aH - (K_X + B))_\eta) = \nu(X_\eta, (H - (K_X + B))_\eta)$ for some $a \in \mathbb{Q}$ with $a > 1$, where η is the generic point of S.*

(d) *Let $f : Y \to Z$ be the Iitaka fibration with respect to $H - (K_X + B)$ over S. We assume that there exists a proper birational morphism $\mu : Y \to X$ and put $K_Y + B_Y = \mu^*(K_X + B)$. We also assume* rank $f_*\mathcal{O}_Y(\lceil \mathbf{A}^*(Y, B_Y)\rceil) = 1$.

(e) (Saturation condition.) *There exist positive integers b and j_0 such that bH is Cartier and $\pi_*\mathcal{O}_X(\lceil \mathbf{A}^*(X, B)\rceil + jb\overline{H}) \subseteq \pi_*\mathcal{O}_X(jbH)$ for every positive integer $j \geq j_0$.*

(f) *There is a positive integer c such that cH is Cartier and*

$$\mathcal{O}_T(cH) := \mathcal{O}_X(cH)|_T$$

is π-generated, where $T = -\mathbf{N}(X, B)_X$.

Then H is π-semiample.

Proof. If $H - (K_X + B)$ is big, this follows from Theorem 6.1. So, we can assume that $H - (K_X + B)$ is not big. Form now on, we use the notation from the proof of Theorem 4.2, which is given in [Fujino 2011d, Section 2]. We just explain how to modify that proof. Let us recall the commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{\ f\ } & Z \\ {\scriptstyle\mu}\big\downarrow & & \big\downarrow{\scriptstyle\varphi} \\ X & \xrightarrow[\ \pi\ ]{} & S \end{array}$$

from the proof of [Fujino 2011d, Theorem 1.1], where $f : Y \to Z$ is the Iitaka fibration with respect to $H - (K_X + B)$ over S. For the details, see [Fujino 2011d, Section 2]. We note that $\mu^*H = H_Y$ and $H_Y \sim f^*D$. Here, we replaced H with a multiple and assumed that H and D are Cartier (see [Fujino 2011d, page 307]). We can also assume that $b = j_0 = 1$ in (e) and $c = 1$ in (f) by replacing H with a multiple. We start with the following obvious lemma.

Lemma 6.3. *We put $T' = -\mathbf{N}(X, B)_Y$. Then $\mu(T') \subset T$. Therefore, $\mathcal{O}_{T'}(H_Y) := \mathcal{O}_Y(H_Y)|_{T'}$ is p-generated, where $p = \pi \circ \mu$.*

Lemma 6.4. *If $f(T') = Z$, then H_Y is p-semiample. In particular, H is π-semiample.*

Proof. There exists an irreducible component T'_0 of T' such that $f(T'_0) = Z$. Since $(H_Y)|_{T'_0} \sim (f^*D)|_{T'_0}$ is p-semiample, D is φ-semiample. This implies that H_Y is p-semiample and H is π-semiample. □

Therefore, we can assume that T' is not dominant onto Z. Thus $\mathbf{A}(Y, B_Y) = \mathbf{A}^*(Y, B_Y)$ over the generic point of Z. Equivalently, (Y, B_Y) is sub-klt over the generic point of Z. As in [Fujino 2011d, Proof of Theorem 1.1], we have these properties:

(1) $K_Y + B_Y \sim_{\mathbb{Q}} f^*(K_Z + B_Z + M)$, where B_Z is the *discriminant* $\mathbb{Q}$-divisor of (Y, B_Y) on Z and M is the *moduli* $\mathbb{Q}$-divisor on Z.

(2′) (Z, B_Z) is sub-lc.

(3) M is a φ-nef $\mathbb{Q}$-divisor on Z.

(4′) $\varphi_*\mathcal{O}_Z(\lceil \mathbf{A}^*(Z, B_Z)\rceil + j\overline{D}) \subseteq \varphi_*\mathcal{O}_Z(jD)$ for every positive integer j.

(5) $D - (K_Z + B_Z)$ is φ-nef and φ-big.

(6) Y and Z are nonsingular and Supp B_Y and Supp B_Z are simple normal crossing divisors.

(7) $\mathcal{O}_{T''}(D) := \mathcal{O}_Z(D)|_{T''}$ is φ-generated where $T'' = -\mathbf{N}(Z, B_Z)_Z$.

Once the conditions above were satisfied, D is φ-semiample by Theorem 6.1. Therefore, H is π-semiample. So, all we have to do is check the conditions. Conditions (1), (2′), (3), (5), (6) are satisfied by a result of Ambro; see [Fujino 2011d, Proof of Theorem 1.1]. We note that $f_*\mathcal{O}_Y(\lceil \mathbf{A}(Y, B_Y)\rceil)$ and $f_*\mathcal{O}_Y(\lceil \mathbf{A}^*(Y, B_Y)\rceil)$ both have rank 1. By the same computation as in [Ambro 2007, Lemma 9.2.2 and Proposition 9.2.3], we have the following lemma.

Lemma 6.5. $\mathcal{O}_Z(\lceil \mathbf{A}^*(Z, B_Z)\rceil + j\overline{D}) \subseteq f_*\mathcal{O}_Y(\lceil \mathbf{A}^*(Y, B_Y)\rceil + j\overline{H_Y})$ *for every integer* j.

Thus, we have (4′) by the saturation condition (e) (for details, see [Fujino 2011d, Proof of Theorem 1.1], and Lemma 3.21). By definition, we have

$$lH_Y + \lceil \mathbf{A}^*_Y \rceil - T' - (K_Y + \{B_Y\}) \sim_{\mathbb{Q}} f^*((l-1)D + M_0),$$

where

$$H_Y - (K_Y + B_Y) = \mu^*(H - (K_X + B)) \sim_{\mathbb{Q}} f^*M_0.$$

Note that $(l-1)D + M_0$ is φ-nef and φ-big for $l \geq 1$. By the Kollár type injectivity theorem,

$$R^1 p_*\mathcal{O}_Y(lH_Y + \lceil \mathbf{A}^*_Y \rceil - T') \to R^1 p_*\mathcal{O}_Y(lH_Y + \lceil \mathbf{A}^*_Y \rceil)$$

is injective for $l \geq 1$. Note that the above injectivity can be checked easily by [Fujino 2007a, Theorem 1.1]. Here, we used the fact that $f(T') \subsetneq Z$. So, we have the commutative diagram

$$\begin{array}{ccccc} p_*\mathcal{O}_Y(lH_Y+\ulcorner \mathbf{A}^*_Y\urcorner) & \longrightarrow & p_*(\mathcal{O}_{T'}(lH_Y)\otimes \mathcal{O}_{T'}(\ulcorner \mathbf{A}^*_Y|_{T'}\urcorner)) & \longrightarrow & 0 \\ \uparrow{\cong} & & \uparrow{\iota} & & \\ p_*\mathcal{O}_Y(lH_Y) & \underset{\alpha}{\longrightarrow} & p_*\mathcal{O}_{T'}(lH_Y) & & \end{array}$$

The isomorphism of the left vertical arrow follows from the saturation condition (e). Thus, the natural inclusion ι is an isomorphism and α is surjective for $l \geq 1$. In particular, the relative base locus of lH_Y is disjoint from T' since $\mathcal{O}_{T'}(lH_Y)$ is p-generated (cf. Lemma 6.3). On the other hand, $H_Y \sim f^*D$. Therefore, $\mathcal{O}_{T''}(D)$ is φ-generated since $T'' \subset f(T')$. So, we obtain condition (7). This completes the proof of Theorem 6.2. □

As a corollary of Theorem 6.2, we obtain the generalization of [Fukuda 2002, Proposition 3.3] stated in the introduction (Theorem 1.1). Before we derive it, we recall the definition of *non-klt loci*.

Definition 6.6 (Non-klt locus). Let (X, B) be an lc pair. We consider the closed subset

$$\mathrm{Nklt}(X, B) = \{x \in X \mid (X, B) \text{ is not klt at } x\}$$

of X. We call $\mathrm{Nklt}(X, B)$ the *non-klt locus* of (X, B).

Proof of Theorem 1.1. Let $h : X' \to X$ be a resolution such that $\mathrm{Exc}(h) \cup \mathrm{Supp}\, h_*^{-1}B$ is a simple normal crossing divisor and $K_{X'} + B_{X'} = h^*(K_X + B)$. Then $H_{X'} = h^*H$, $(X', B_{X'})$, and $\pi' = \pi \circ h : X' \to S$ satisfy assumptions (a), (b), and (c) in Theorem 6.2. By the same argument as in the proof of [Fujino 2011d, Lemma 2.3], we obtain rank $f_*\mathcal{O}_Y(\ulcorner \mathbf{A}^*(Y, B_Y)\urcorner) = 1$, where $f : Y \to Z$ is the Iitaka fibration as in (d) in Theorem 6.2. Note that $\ulcorner \mathbf{A}^*(Y, B_Y)\urcorner$ is effective and exceptional over X. Since B is effective, $\ulcorner \mathbf{A}^*(X, B)\urcorner$ is effective and exceptional over X,

$$\pi'_*\mathcal{O}_{X'}(\ulcorner \mathbf{A}^*(X', B_{X'})\urcorner + jb\overline{H_{X'}}) \subseteq \pi'_*\mathcal{O}_{X'}(jbH_{X'})$$

for every integer j, where b is a positive integer such that bH is Cartier. So, the saturation condition (e) in Theorem 6.2 is satisfied. Finally, $\mathcal{O}_{T'}(cH_{X'}) := \mathcal{O}_{X'}(cH_{X'})|_{T'}$ is π'-generated, where $T' = -\mathbf{N}(X, B)_{X'}$, by assumption (D) and the fact that $h(T') \subset T$. So, condition (f) in Theorem 6.2 for $H_{X'}$ and $(X', B_{X'})$ is satisfied. Therefore, $H_{X'}$ is π'-semiample by Theorem 6.2. Thus, H is π-semiample. □

Remark 6.7. (i) It is obvious that $\mathrm{Supp}(-\mathbf{N}(X, B)_X) \subseteq \mathrm{Nklt}(X, B)$. In general, $\mathrm{Supp}(-\mathbf{N}(X, B)_X) \subsetneq \mathrm{Nklt}(X, B)$. In particular, $\mathrm{Nklt}(X, B)$ is not necessarily of pure codimension one in X.

(ii) If (X, B) is dlt, then $\mathrm{Nklt}(X, B) = \mathrm{Supp}(-\mathbf{N}(X, B)_X) = \lfloor B \rfloor$. Therefore, if (X, B) is dlt and S is a point, then Theorem 1.1 is nothing but Fukuda's result [Fukuda 2002, Proposition 3.3].

(iii) The reader can find applications of this corollary in [Fukuda 2002; Fujino 2010; Fujino and Gongyo 2011].

By combining Theorem 1.1 with [Gongyo 2010, Theorem 1.5], we obtain the following result.

Corollary 6.8. *Let (X, B) be a projective dlt pair such that $\nu(K_X+B)=\kappa(K_X+B)$ and that $(K_X + B)|_{\lfloor B \rfloor}$ is numerically trivial. Then $K_X + B$ is semiample.*

Remark 6.9. We can easily generalize Theorem 6.2 and Theorem 1.1 to varieties in class $\mathscr{C}$ by suitable modifications. For details, see [Fujino 2011d, Section 4].

7. Basepoint-free theorems for pseudo-klt pairs

In this section, we generalize the Kawamata–Shokurov base point free theorem and Kawamata's theorem: Theorem 4.2 for *klt pairs* to *pseudo-klt pairs*. We think that our formulation is useful when we study lc centers (see Proposition 7.8). First, we introduce the notion of *pseudo-klt pairs*.

Definition 7.1 (Pseudo-klt pair). Let W be a normal variety. Assume the following conditions:

(1) there exist a sub-klt pair (V, B) and a projective surjective morphism $f : V \to W$ with connected fibers.

(2) $f_*\mathscr{O}_V(\lceil \mathbf{A}(V, B) \rceil) \simeq \mathscr{O}_W$.

(3) There exists a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $\mathscr{H}$ on W such that $K_V + B \sim_{\mathbb{Q}} f^*\mathscr{H}$.

Then the pair $[W, \mathscr{H}]$ is called a *pseudo-klt pair*.

Although it is the first time that we use the name of *pseudo-klt pair*, the notion of pseudo-klt pair appeared in [Fujino 1999], where we proved the cone and contraction theorem for *pseudo-klt* pairs (cf. [Fujino 1999, Section 4]). We note that all the fundamental theorems for the log minimal model program for pseudo-klt pairs can be proved by the theory of quasilog varieties (cf. [Ambro 2003; Fujino 2009b; 2011b]).

Remark 7.2. In Definition 7.1, we assume that W is normal. However, the normality of W follows from condition (2) and the normality of V. Note that $\lceil \mathbf{A}(V, B) \rceil$ is effective.

Remark 7.3. In the definition of pseudo-klt pairs, if (V, B) is klt, the condition $f_*\mathscr{O}_V(\lceil \mathbf{A}(V, B) \rceil) \simeq \mathscr{O}_W$ is automatically satisfied. This is because $\lceil \mathbf{A}(V, B) \rceil$ is effective and exceptional over V.

We note that a pseudo-klt pair is a very special example of Ambro's quasilog varieties (see [Ambro 2003, Definition 4.1]). More precisely, if $[V, \mathcal{H}]$ is a pseudo-klt pair, then we can easily check that $[V, \mathcal{H}]$ is a *qlc pair*. See, for example, [Fujino 2011b, Definition 3.1]. For the details of the theory of quasilog varieties, see [Fujino 2009b].

Theorem 7.4. *Let $[W, \mathcal{H}]$ be a pseudo-klt pair. Assume that (V, B) is klt and W is projective or that W is affine. Then we can find an effective $\mathbb{Q}$-divisor B_W on W such that (W, B_W) is klt and that $\mathcal{H} \sim_{\mathbb{Q}} K_W + B_W$.*

Proof. When (X, B) is klt and W is projective, we can find B_W by [Ambro 2005a, Theorem 4.1]. When W is affine, this theorem follows from [Fujino 1999, Theorem 1.2]. □

Remark 7.5. It is conjectured that one can always find an effective $\mathbb{Q}$-divisor B_W on W such that (W, B_W) is klt and $\mathcal{H} \sim_{\mathbb{Q}} K_W + B_W$.

We now collect basic examples of pseudo-klt pairs.

Example 7.6. A klt pair is a pseudo-klt pair.

Example 7.7. Let $f : X \to W$ be a Mori fiber space. Then we can find a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $\mathcal{H}$ on W such that $[W, \mathcal{H}]$ is a pseudo-klt pair. It is because we can find an effective $\mathbb{Q}$-divisor B on X such that $K_X + B \sim_{\mathbb{Q}, f} 0$ and (X, B) is klt.

Proposition 7.8. *An exceptional lc center W of an lc pair (X, B) is a pseudo-klt pair for some $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $\mathcal{H}$ on W.*

Proof. We take a resolution $g : Y \to X$ such that $\mathrm{Exc}(g) \cup g_*^{-1}B$ has a simple normal crossing support. We put $K_Y + B_Y = g^*(K_X + B)$. Then $-B_Y = \mathbf{A}(X, B)_Y = \mathbf{A}_Y = \mathbf{A}^*_Y + \mathbf{N}_Y$, where $\mathbf{N}_Y = -\sum_{i=0}^{k} E_i$. Without loss of generality, we can assume that $f(E) = W$ and $E = E_0$. By shrinking X around W, we can assume that $\mathbf{N}_Y = -E$. Note that $R^1 g_*\mathcal{O}_Y(\lceil \mathbf{A}^*_Y \rceil - E) = 0$ by the Kawamata–Viehweg vanishing theorem since $\lceil \mathbf{A}^*_Y \rceil - E = K_Y + \{-\mathbf{A}^*_Y\} - g^*(K_X + B)$. Therefore, $g_*\mathcal{O}_Y(\lceil \mathbf{A}^*_Y \rceil) \simeq \mathcal{O}_X \to g_*\mathcal{O}_E(\lceil \mathbf{A}^*_Y|_E \rceil)$ is surjective. This implies that $g_*\mathcal{O}_E(\lceil \mathbf{A}^*_Y|_E \rceil) \simeq \mathcal{O}_W$. In particular, W is normal. If we put $K_E + B_E = (K_Y + B_Y)|_E$, then (E, B_E) is sub-klt and $\mathbf{A}^*_Y|_E = \mathbf{A}(E, B_E)_E = -B_E$. So, $g_*\mathcal{O}_E(\lceil \mathbf{A}(E, B_E) \rceil) = g_*\mathcal{O}_E(\lceil -B_E \rceil) \simeq \mathcal{O}_W$. Since $K_E + B_E = (K_Y + B_Y)|_E$ and $K_Y + B_Y = g^*(K_X + B)$, we can find a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $\mathcal{H}$ on W such that $K_E + B_E \sim_{\mathbb{Q}} g^*\mathcal{H}$. Therefore, W is a pseudo-klt pair. □

We make an important remark on *minimal* lc centers.

Remark 7.9 (Subadjunction for minimal lc center). Let (X, B) be a projective or affine lc pair and let W be a minimal lc center of the pair (X, B). Then we can find an effective $\mathbb{Q}$-divisor B_W on W such that (W, B_W) is klt and $K_W + B_W \sim_{\mathbb{Q}} (K_X + B)|_W$. For the details, see [Fujino and Gongyo 2012, Theorems 4.1, 7.1].

The following theorem is the Kawamata–Shokurov basepoint-free theorem for pseudo-klt pairs. We give a simple proof depending on Kawamata's positivity theorem. Although Theorem 7.10 seems to be contained in [Ambro 2003, Theorem 7.2], no proof is given there.

Theorem 7.10. *Let $[W, \mathcal{K}]$ be a pseudo-klt pair, let $\pi : W \to S$ be a proper morphism onto a variety S and let D be a π-nef Cartier divisor on W. Assume that $rD - \mathcal{K}$ is π-nef and π-big for some positive integer r. Then mD is π-generated for every $m \gg 0$.*

Proof. Without loss of generality, we can assume that S is affine. By the usual technique (see [Kawamata 1998, Theorem 1] and [Fujino 1999, Theorem 1.2]), we have

$$\mathcal{K} + \varepsilon(rD - \mathcal{K}) \sim_{\mathbb{Q}} K_W + \Delta_W$$

such that (W, Δ_W) is klt for some sufficiently small rational number $0 < \varepsilon \ll 1$ (see also [Kollár 2007, Theorem 8.6.1]). Then $rD - (K_W + \Delta_W) \sim_{\mathbb{Q}} (1 - \varepsilon)(rD - \mathcal{K})$, which is π-nef and π-big. Therefore, mD is π-generated for every $m \gg 0$ by the usual Kawamata–Shokurov basepoint-free theorem. □

The next theorem is the main theorem of this section. It is a generalization of Kawamata's theorem in [Kawamata 1985] (cf. Theorem 4.2) for pseudo-klt pairs.

Theorem 7.11. *Let $[W, \mathcal{K}]$ be a pseudo-klt pair and let $\pi : W \to S$ be a proper morphism onto a variety S. Assume the following conditions*:

(i) *H is a π-nef $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on W.*

(ii) *$H - \mathcal{K}$ is π-nef and π-abundant.*

(iii) *$\kappa(W_\eta, (aH - \mathcal{K})_\eta) \geq 0$ and $\nu(W_\eta, (aH - \mathcal{K})_\eta) = \nu(W_\eta, (H - \mathcal{K})_\eta)$ for some $a \in \mathbb{Q}$ with $a > 1$, where η is the generic point of S.*

Then H is π-semiample.

Proof. By definition, there exists a proper surjective morphism $f : V \to W$ from a sub-klt pair (V, B). Without loss of generality, we can assume that V is nonsingular and Supp B is a simple normal crossing divisor. By definition, $f_*\mathcal{O}_V(\lceil -B \rceil) \simeq \mathcal{O}_W$. From now on, we assume that H is Cartier by replacing it with a multiple. Then $f_*\mathcal{O}_V(\lceil -B \rceil + jH_V) \simeq \mathcal{O}_W(jH)$ by the projection formula for every integer j, where $H_V = f^*H$. Pushing forward by π, we have

$$\begin{aligned} p_*\mathcal{O}_V(\lceil \mathbf{A}(V, B) \rceil + j\overline{H_V}) &= p_*\mathcal{O}_V(\lceil -B \rceil + jH_V) \\ &\simeq \pi_*\mathcal{O}_W(jH) \\ &\simeq p_*\mathcal{O}_V(jH_V) \end{aligned}$$

for every integer j, where $p = \pi \circ f$. This is nothing but the saturation condition Theorem 4.4(e). We put $L = H - \mathcal{K}$. We consider the Iitaka fibration with respect to L over S as in [Fujino 2011d, Proof of Theorem 1.1]. Then we obtain the following commutative diagram:

$$\begin{array}{ccc} V & = & V \\ {\scriptstyle f}\downarrow & & \downarrow \\ W & \xleftarrow{\ \mu\ } & U \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle g} \\ S & \xleftarrow[\ \varphi\ ]{} & Z \end{array}$$

where $g : U \to Z$ is the Iitaka fibration over S and $\mu : U \to W$ is a birational morphism. Note that we can assume that $f : V \to W$ factors through U by blowing up V.

Lemma 7.12. $\operatorname{rank} h_*\mathcal{O}_V(\lceil \mathbf{A}(V, B)\rceil) = 1$, *where* $h : V \to U \to Z$.

Proof. This proof is essentially the same as that of [Fujino 2011d, Lemma 2.3]. First, we can assume that S is affine. Let A be an ample divisor on Z such that $h_*\mathcal{O}_V(\lceil \mathbf{A}(V, B)\rceil) \otimes \mathcal{O}_Z(A)$ is φ-generated. We note that we can assume that $\mu^*L \sim_{\mathbb{Q}} g^*M$ since L is π-nef and π-abundant, where M is a φ-nef and φ-big $\mathbb{Q}$-divisor on Z. If we choose a large and divisible integer m, then $\mathcal{O}_Z(A) \subset \mathcal{O}_Z(mM)$. Thus

$$\begin{aligned} &\varphi_*(h_*\mathcal{O}_V(\lceil \mathbf{A}(V, B)\rceil) \otimes \mathcal{O}_Z(A)) \\ &\subseteq \varphi_*(h_*\mathcal{O}_V(\lceil \mathbf{A}(V, B)\rceil) \otimes \mathcal{O}_Z(mM)) \\ &\simeq p_*\mathcal{O}_V(\lceil \mathbf{A}(V, B)\rceil + m\overline{f^*L}) \\ &\simeq \pi_*\mathcal{O}_W(mL) \\ &\simeq \varphi_*\mathcal{O}_Z(mM). \end{aligned}$$

Therefore, $\operatorname{rank} h_*\mathcal{O}_V(\lceil \mathbf{A}(V, B)\rceil) \le 1$. Since $\mathcal{O}_Z \subset h_*\mathcal{O}_V \subset h_*\mathcal{O}_V(\lceil \mathbf{A}(V, B)\rceil)$, we obtain $\operatorname{rank} h_*\mathcal{O}_V(\lceil \mathbf{A}(V, B)\rceil) = 1$ □

Note that $h : V \to Z$ is the Iitaka fibration with respect to f^*L over S. Assumption (c) in Theorem 4.4 easily follows from (iii). Thus, by Theorem 4.4, we have that H_V is p-semiample. Equivalently, H is π-semiample. □

The final theorem of this paper is a basepoint-free theorem for minimal lc centers.

Theorem 7.13. *Let* (X, B) *be a quasi-projective lc pair and let* W *be a minimal lc center of* (X, B). *Let* $\pi : W \to S$ *be a proper morphism onto a variety* S. *Assume the following conditions*:

(i) *H is a π-nef $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on W.*

(ii) *$H-(K_X+B)|_W$ is π-nef and π-abundant.*

(iii) *$\kappa(W_\eta, (aH-(K_X+B))|_{W_\eta}) \geq 0$ and*

$$\nu(W_\eta, (aH-(K_X+B))|_{W_\eta}) = \nu(W_\eta, (H-(K_X+B))|_{W_\eta})$$

for some $a \in \mathbb{Q}$ with $a > 1$, where η is the generic point of S.

Then H is π-semiample.

Proof. Let $f : Y \to X$ be a dlt blow-up such that $K_Y + B_Y = f^*(K_X + B)$ (see, for example, [Fujino 2011a, Theorem 10.4] or [Fujino 2011e, Section 4]). Then we can take a minimal lc center Z of (Y, B_Y) such that $f(Z) = W$. Note that $K_Z + B_Z = (K_Y + B_Y)|_Z$ is klt. We also note that W is normal (see, for example, [Fujino 2011c, Theorem 2.4 (4)] or [Fujino 2011a, Theorem 9.1 (4)]). Let

$$f : Z \xrightarrow{\ g\ } V \xrightarrow{\ h\ } W$$

be the Stein factorization of $f : Z \to W$. Then $[V, h^*((K_X+B)|_W)]$ is a pseudo-klt pair by $g : (Z, B_Z) \to V$. We note that H is π-semiample if and only if h^*H is $\pi \circ h$-semiample. By Theorem 7.11, h^*H is semiample over S. This concludes the proof. □

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