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Fields of moduli of three-point G-covers with cyclic p-Sylow, I

Andrew Obus

We examine in detail the stable reduction of G-Galois covers of the projective line over a complete discrete valuation field of mixed characteristic $(0, p)$, where G has a *cyclic* p-Sylow subgroup of order p^n. If G is further assumed to be p-*solvable* (that is, G has no nonabelian simple composition factors with order divisible by p), we obtain the following consequence: Suppose $f : Y \to \mathbb{P}^1$ is a three-point G-Galois cover defined over $\mathbb{C}$. Then the n-th higher ramification groups above p for the upper numbering for the extension $K/\mathbb{Q}$ vanish, where K is the field of moduli of f. This extends work of Beckmann and Wewers. Additionally, we completely describe the stable model of a general three-point $\mathbb{Z}/p^n$-cover, where $p > 2$.

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1. Introduction

1A. ***Overview.*** This paper focuses on understanding how primes of $\mathbb{Q}$ ramify in the field of moduli of three-point Galois covers of the Riemann sphere. Our main result, Theorem 1.3, generalizes results of Beckmann and Wewers (Theorems 1.1 and 1.2) about ramification of primes p where p divides the order of the Galois group and the p-Sylow subgroup of the Galois group is cyclic.

Let X be the Riemann sphere $\mathbb{P}^1_{\mathbb{C}}$, and let $f : Y \to X$ be a finite branched cover of Riemann surfaces. By GAGA [Serre 1955–1956], Y is isomorphic to an algebraic variety, and f is the analytification of an algebraic, regular map. By a theorem of Weil, if the branch points of f are $\overline{\mathbb{Q}}$-rational (for example, if the cover is branched at three points, which we can always take to be 0, 1, and ∞ — such a cover is called a *three-point cover*), then the equations of the cover f can themselves be defined over $\overline{\mathbb{Q}}$ (in fact, over some number field). Let

$$\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) = G_{\mathbb{Q}}.$$

Since X is defined over $\mathbb{Q}$, we have that σ acts on the set of branched covers of X by acting on the coefficients of the defining equations. We write $f^\sigma : Y^\sigma \to X^\sigma$ for the cover thus obtained. If $f : Y \to X$ is a G-Galois cover, then so is f^σ. Let $\Gamma^{in} \subset G_{\mathbb{Q}}$ be the subgroup consisting of those σ that preserve the isomorphism class of f as well as the G-action. That is, Γ^{in} consists of those elements σ of $G_{\mathbb{Q}}$ such that there is an isomorphism $\phi : Y \to Y^\sigma$ commuting with the action of G that makes the following diagram commute:

$$\begin{tikzcd} Y \arrow[r, "\phi"] \arrow[d, "f"'] & Y^\sigma \arrow[d, "f^\sigma"] \\ X \arrow[r, equal] & X^\sigma \end{tikzcd} \tag{1-1}$$

The fixed field $\overline{\mathbb{Q}}^{\Gamma^{in}}$ is known as the *field of moduli* of f (as a G-cover). It is the intersection of all the fields of definition of f as a G-cover (that is, those fields of definition K of f such that the action of G can also be written in terms of polynomials with coefficients in K); see [Coombes and Harbater 1985, Proposition 2.7].

Now, since a branched G-Galois cover $f : Y \to X$ of the Riemann sphere is given entirely in terms of combinatorial data (the branch locus C, the Galois group G, and the monodromy action of $\pi_1(X \backslash C)$ on Y), it is reasonable to try to draw inferences about the field of moduli of f based on these data. However, not much is known about this, and this is the goal toward which we work.

The problem of determining the field of moduli of three-point covers has applications toward analyzing the *fundamental exact sequence*

$$1 \to \pi_1(\mathbb{P}^1_{\overline{\mathbb{Q}}} \setminus \{0, 1, \infty\}) \to \pi_1(\mathbb{P}^1_{\mathbb{Q}} \setminus \{0, 1, \infty\}) \to G_{\mathbb{Q}} \to 1,$$

where π_1 is the étale fundamental group functor. Our knowledge of this object is limited (note that a complete understanding would yield a complete understanding of $G_{\mathbb{Q}}$). The exact sequence gives rise to an outer action of $G_{\mathbb{Q}}$ on $\Pi = \pi_1(\mathbb{P}^1_{\overline{\mathbb{Q}}} \setminus \{0, 1, \infty\})$. This outer action would be particularly interesting to understand. Knowing about fields of moduli sheds light as follows: Say the G-Galois cover f corresponds to the normal subgroup $N \subset \Pi$ so that $\Pi/N \cong G$. Then the group Γ^{in} consists exactly of those elements of $G_{\mathbb{Q}}$ whose outer action on Π both preserves N and descends to an inner action on $\Pi/N \cong G$.

1B. *Main result.* One of the first major results in this direction is due to Beckmann:

Theorem 1.1 [Beckmann 1989]. *Let $f : Y \to X$ be a branched G-Galois cover of the Riemann sphere with branch points defined over $\overline{\mathbb{Q}}$. Then $p \in \mathbb{Q}$ can be ramified in the field of moduli of f as a G-cover only if p is ramified in the field of definition of a branch point, or $p \mid |G|$, or there is a collision of branch points modulo some prime dividing p. In particular, if f is a three-point cover and if $p \nmid |G|$, then p is unramified in the field of moduli of f.*

This result was partially generalized by Wewers:

Theorem 1.2 [Wewers 2003b]. *Let $f : Y \to X$ be a three-point G-Galois cover of the Riemann sphere, and suppose that p exactly divides $|G|$. Then p is tamely ramified in the field of moduli of f as a G-cover.*

In fact, Wewers shows somewhat more, in that he computes the index of tame ramification of p in the field of moduli in terms of some invariants of f.

To state our main theorem, which is a further generalization, we will need some group theory. We call a finite group G *p-solvable* if its only simple composition factors with order divisible by p are isomorphic to $\mathbb{Z}/p$. Clearly, any solvable group is p-solvable. Our main result is this:

Theorem 1.3. *Let $f : Y \to X$ be a three-point G-Galois cover of the Riemann sphere, and suppose that a p-Sylow subgroup $P \subset G$ is cyclic of order p^n. Let $K/\mathbb{Q}$ be the field of moduli of f. Then, if G is p-solvable, the n-th higher ramification groups for the upper numbering of (the Galois closure of) $K/\mathbb{Q}$ above p vanish.*

Remark 1.4. (i) Beckmann's and Wewers's theorems cover the cases $n = 0, 1$ in the notation above (and Wewers does not need the assumption of p-solvability).

(ii) The paper [Obus 2011b] will show that the result of Theorem 1.3 holds in many cases, even when G is not p-solvable, provided that the normalizer of P acts on P via a group of order 2.

(iii) If the normalizer of P in G is equal to the centralizer, then G is always p-solvable. This follows from [Zassenhaus 1958, Theorem 4, p. 169].

(iv) We will show (Proposition B.2) that if G has a cyclic p-Sylow subgroup and is *not* p-solvable, it must have a simple composition factor with order divisible by p^n. There seem to be limited examples of simple groups with cyclic p-Sylow subgroups of order greater than p. Furthermore, many of the examples that do exist are in the form discussed in part (ii) of this remark (for instance, $\mathrm{PSL}_2(q)$, where p^n exactly divides $q^2 - 1$).

Our main technique for proving Theorem 1.3 will be an analysis of the *stable reduction* of the cover f to characteristic p (Section 4). This is also the main technique used in [Wewers 2003b] to prove Theorem 1.2. The argument there relies on the fact that the stable reduction of a three-point G-Galois cover to characteristic p is relatively simple when p exactly divides $|G|$. When higher powers of p divide $|G|$, the stable reduction can be significantly more complicated. Many of the technical results needed for dealing with this situation are proven in [Obus 2012], and we will recall them as necessary. In particular, our proof depends on an analysis of *effective ramification invariants*, which are generalizations of the invariants σ_b of [Raynaud 1999; Wewers 2003b]

In proving Theorem 1.3, we will essentially be able to reduce to the case where $G \cong \mathbb{Z}/p^n \rtimes \mathbb{Z}/m$ at the cost of having to determine the minimal field of definition of the *stable model* of f, rather than just the field of moduli. In particular, if the normalizer and centralizer of P are equal, the proof of Theorem 1.3 boils down to understanding the stable model of an arbitrary three-point $\mathbb{Z}/p^n$-cover. A complete description of this stable model has been given when $p > 3$ and in certain cases when $p = 3$ in [Coleman and McCallum 1988]. We give a complete enough description for our purposes for arbitrary p in Lemma 7.8. Additionally, our description for $p = 2$ is used in [Obus 2011c] to complete the proof of a product formula due to Colmez for periods of CM-abelian varieties [Colmez 1993].

We should remark that when $G \cong \mathbb{Z}/p^n \rtimes \mathbb{Z}/m$, the cover f is very much like an *auxiliary cover*; see [Raynaud 1999; Wewers 2003b; Obus 2011b]. Our assumption of p-solvability allows us to avoid the auxiliary cover construction.

For other work on understanding stable models of mixed characteristic G-covers where the residue characteristic divides $|G|$, see for instance [Lehr and Matignon 2006; Matignon 2003; Raynaud 1990; Saïdi 2007; 1998a; 1998b]. These papers focus mostly on the case where G is a p-group, while allowing more than three branch points. For an application to computing the stable reduction of modular curves, see [Bouw and Wewers 2004].

1C. ***Section-by-section summary and walkthrough.*** In Sections 2A–2D, we recall well-known facts about group theory, fields of moduli, ramification, and models of $\mathbb{P}^1$. In Section 3, we give some explicit results on the reduction of $\mathbb{Z}/p^n$-torsors. In Section 4, we recall the relevant results about stable reduction from [Raynaud

1999; Obus 2012]. The most important of these is the *vanishing cycles formula*, which we then apply in the specific case of a p-solvable three-point cover. In Section 5, we recall the construction of deformation data given in [Obus 2012], which is a generalization of that given in [Henrio 2000a]. We also recall the *effective local vanishing cycles formula* from [Obus 2012]. In Section 6, we relate the field of moduli of a cover to that of its quotient covers. In Section 7, we prove our main result, Theorem 1.3. After reducing to a local problem, we first assume that $G \cong \mathbb{Z}/p^n \rtimes \mathbb{Z}/m$, with $p \nmid m$. We deal separately with the cases $m > 1$ (Section 7A) and $m = 1$ (Section 7B). Then, it is an easy application of the results of Section 6 to obtain the full statement of Theorem 1.3.

Appendix A gives a full description of the stable model of a general three-point $\mathbb{Z}/p^n$-cover when $p > 3$ and in certain cases when $p = 3$. It uses different techniques than [Coleman and McCallum 1988]. Furthermore, the techniques there can be adapted to give a full description whenever $p = 2$ or $p = 3$. Appendix B examines what kinds of groups with cyclic p-Sylow subgroups are not p-solvable, and thus are not covered by Theorem 1.3. Some technical calculations from Section 3 and Section 7B are postponed to Appendix C. Appendices Appendix A and Appendix B are not necessary for the proof of Theorem 1.3, and Appendix C is only necessary when $p \leq 3$.

1D. ***Notation and conventions.*** The letter k will always represent an algebraically closed field of characteristic $p > 0$.

If H is a subgroup of a finite group G, then $N_G(H)$ is the normalizer of H in G and $Z_G(H)$ is the centralizer of H in G. If G has a cyclic p-Sylow subgroup P, and p is understood, we write $m_G = |N_G(P)/Z_G(P)|$.

If K is a field, then $\overline{K}$ is its algebraic closure, and G_K is its absolute Galois group. If $H \leq G_K$, we write $\overline{K}^H$ for the fixed field of H in $\overline{K}$. Similarly, if Γ is a group of automorphisms of a ring A, we write A^Γ for the fixed ring under Γ. If K is discretely valued, then K^{ur} is the *completion* of the maximal unramified algebraic extension of K.

If K is any field and $a \in K$, then $K(\sqrt[n]{a})$ denotes a minimal field extension of K containing an n-th root of a (*not* necessarily the ring $K[x]/(x^n - a)$). For instance, $\mathbb{Q}(\sqrt{9}) \cong \mathbb{Q}$. In cases where K does not contain the n-th roots of unity, it will not matter which (conjugate) extension we take.

If R is any local ring, then $\hat{R}$ is the completion of R with respect to its maximal ideal. If R is any ring with a nonarchimedean absolute value $|\cdot|$, then $R\{T\}$ is the ring of power series $\sum_{i=0}^{\infty} c_i T^i$ such that $\lim_{i\to\infty} |c_i| = 0$. If R is a discrete valuation ring with fraction field K of characteristic 0 and residue field of characteristic p, we normalize the absolute value on K and on any subring of K so that $|p| = 1/p$. We normalize the valuation on R so that p has valuation 1.

A *branched cover* $f : Y \to X$ is a finite, surjective, generically étale morphism of geometrically connected, smooth, proper curves. If f is of degree d and G is a finite group of order d with $G \cong \mathrm{Aut}(Y/X)$, then f is called a *Galois cover with (Galois) group* G. If we choose an isomorphism $i : G \to \mathrm{Aut}(Y/X)$, then the datum (f, i) is called a G*-Galois cover* (or just a G*-cover*, for short). We will usually suppress the isomorphism i, and speak of f as a G-cover.

Suppose $f : Y \to X$ is a G-cover of smooth curves, and K is a field of definition for X. Then the *field of moduli of* f *relative to* K (*as a* G*-cover*) is the fixed field in $\overline{K}/K$ of $\Gamma^{in} \subset G_K$, where

$$\Gamma^{in} = \{\sigma \in G_K \mid f^\sigma \cong f \,(\text{as } G\text{-covers})\}$$

(see Section 1A). If X is $\mathbb{P}^1$, then the *field of moduli of* f means the field of moduli of f relative to $\mathbb{Q}$. Unless otherwise stated, a field of definition (or moduli) means a field of definition (or moduli) *as a* G*-cover* (see Section 1A). If we do not want to consider the G-action, we will always explicitly refer to the field of definition (or moduli) *as a mere cover*. For two covers to be isomorphic as mere covers, the isomorphism ϕ of Section 1A does not need to commute with the G-action.

2. Background material

2A. ***Finite, p-solvable groups with cyclic p-Sylow subgroups.*** The following proposition is a structure theorem on p-solvable groups that is integral to the paper (recall that a group G is p*-solvable* if its only simple composition factors with order divisible by p are isomorphic to $\mathbb{Z}/p$). Note that for any finite group G, there is a unique maximal prime-to-p normal subgroup (as the subgroup of G generated by two normal prime-to-p subgroups is also normal and prime to p).

Proposition 2.1. *Suppose G is a p-solvable finite group with cyclic p-Sylow subgroup of order p^n, $n \geq 1$. Let N be the maximal prime-to-p normal subgroup of G. Then $G/N \cong \mathbb{Z}/p^n \rtimes \mathbb{Z}/m_G$, where the conjugation action of $\mathbb{Z}/m_G$ on $\mathbb{Z}/p^n$ is faithful.*

Proof. Clearly, G/N has no nontrivial normal subgroups of prime-to-p order. Since G is p-solvable, so is G/N. Thus, a minimal normal subgroup of G/N, being the product of isomorphic simple groups [Aschbacher 2000, 8.2, 8.3], must be isomorphic to $\mathbb{Z}/p$. It is readily verified that $m_G = m_{G/N}$, so the proposition follows from [Obus 2012, Lemma 2.3]. □

2B. ***G-covers versus mere covers.*** Let $f : Y \to X$ be a G-cover of smooth, proper, geometrically connected curves. Let K be a field of definition for X, and let L/K be a field containing the field of moduli of f as a *mere cover* (which is equivalent to L being a field of definition of f as a mere cover; see [Coombes and Harbater

1985, Proposition 2.5]. This gives rise to a homomorphism $h : G_L \to \mathrm{Out}(G)$ as follows. For $\sigma \in G_L$, consider the diagram (1-1), which we reproduce here:

$$\begin{tikzcd} Y \arrow[r, "\phi"] \arrow[d, "f"'] & Y^\sigma \arrow[d, "f^\sigma"] \\ X \arrow[r, equal] & X^\sigma \end{tikzcd}$$

The isomorphism ϕ is well defined up to composition with an element of G acting on Y^σ. Thus, the map h_σ given by $h_\sigma(g) := \phi \circ g \circ \phi^{-1}$ is well defined as an element of $\mathrm{Out}(G)$ (the input is thought of as an automorphism of Y, and the output is thought of as an automorphism of Y^σ). Then L contains the field of moduli of f (as a G-cover) if and only if h_σ is inner (because then there will be a choice of ϕ making the diagram G-equivariant).

2C. *Wild ramification.* We state here some facts from [Serre 1979, IV] and derive some consequences. Let K be a complete discrete valuation field with residue field k. If L/K is a finite Galois extension of fields with Galois group G, then L is also a complete discrete valuation field with residue field k. Here G is of the form $P \rtimes \mathbb{Z}/m$, where P is a p-group and m is prime to p. The group G has a filtration $G = G_0 \supseteq G_i$ ($i \in \mathbb{R}_{\geq 0}$) for the lower numbering, and $G \supseteq G^i$ for the upper numbering ($i \in \mathbb{R}_{\geq 0}$). If $i \leq j$, then $G_i \supseteq G_j$ and $G^i \supseteq G^j$; see [Serre 1979, IV, Section 1, Section 3]. The subgroups G_i and G^i are known respectively as the *i-th higher ramification groups for the lower and upper numbering*. One knows that $G_0 = G^0 = G$, and that for sufficiently small $\epsilon > 0$, $G_\epsilon = G^\epsilon = P$. For sufficiently large i, $G_i = G^i = \{\mathrm{id}\}$. Any i such that $G^i \supsetneq G^{i+\epsilon}$ for all $\epsilon > 0$ is called an *upper jump* of the extension L/K. Likewise, if $G_i \supsetneq G_{i+\epsilon}$, then i is called a *lower jump* of L/K. The lower jumps are all prime-to-p integers. The greatest upper jump (that is, the greatest i such that $G^i \neq \{\mathrm{id}\}$) is called the *conductor* of higher ramification of L/K. The upper numbering is invariant under quotients [Serre 1979, IV, Proposition 14]. That is, if $H \leq G$ is normal, and $M = L^H$, then the i-th higher ramification group for the upper numbering for M/K is $G^i/(G^i \cap H) \subseteq G/H$.

Lemma 2.2. *Let $K \subseteq L \subseteq L'$ be a tower of field extensions such that L'/L is tame, L/K is ramified, and L'/K, L/K are finite Galois. Then the conductor of L'/K is equal to the conductor of L/K.*

Proof. This is an easy consequence of [Serre 1979, IV, Proposition 14]. □

Lemma 2.3. *Let $L_1, \dots, L_\ell$ be finite Galois extensions of K with compositum L in some algebraic closure of K. Denote by h_i the conductor of L_i/K and by h the conductor of L/K. Then $h = \max_i(h_i)$.*

Proof. Write $G = \mathrm{Gal}(L/K)$ and $N_i = \mathrm{Gal}(L/L_i)$. Suppose $g \in G^j \subseteq \mathrm{Gal}(L/K)$. Since L is the compositum of the L_i, the intersection of the N_i is trivial. So g is trivial if and only if its image in each G/N_i is trivial. Because the upper numbering is invariant under quotients, this shows that G^j is trivial if and only if the j-th higher ramification group for the upper numbering for L_i/K is trivial for all i. This means that $h = \max_i(h_i)$. $\square$

If A and B are the valuation rings of K and L, respectively, sometimes we will refer to the conductor and higher ramification groups of the extension B/A. If $f : Y \to X$ is a branched cover of curves and $f(y) = x$, then we refer to the higher ramification groups of $\hat{\mathcal{O}}_{Y,y}/\hat{\mathcal{O}}_{X,x}$ *as the higher ramification groups at* y (or, if f is Galois, and we only care about groups up to isomorphism, as the *higher ramification groups above* x).

We include two well-known lemmas. The first follows easily from the Hurwitz formula; see also [Stichtenoth 2009, Propositions 3.7.8, 6.4.1]. For the second, see [Pries 2002, Theorem 1.4.1 (i)].

Lemma 2.4. *Let $f : Y \to \mathbb{P}^1$ be a $\mathbb{Z}/p$-cover of curves over an algebraically closed field k of characteristic p, ramified at exactly one point of order p. If the conductor of higher ramification at this point is h, then the genus of Y is $(h-1)(p-1)/2$.*

Lemma 2.5. *Let $f : Y \to \mathbb{P}^1$ be a $\mathbb{Z}/p$-cover of k-curves, branched at one point. Then f can be given birationally by an equation $y^p - y = g(x)$, where the terms of $g(x) \in k[x]$ have prime-to-p degree (the branch point is $x = \infty$). If h is the conductor of higher ramification at ∞, then $h = \deg(g)$.*

2D. *Semistable models of* $\mathbb{P}^1$. Let R be a mixed characteristic $(0, p)$ complete discrete valuation ring with residue field k and fraction field K. If X is a smooth curve over K, then a *semistable* model for X is a relative flat curve $X_R \to \operatorname{Spec} R$ with $X_R \times_R K \cong X$ and semistable special fiber (that is, the special fiber is reduced with only ordinary double points for singularities). If X_R is smooth, it is called a *smooth model*.

Models. Let $X \cong \mathbb{P}^1_K$. Write v for the valuation on K. Let X_R be a smooth model of X over R. Then there is an element $T \in K(X)$ such that $K(T) \cong K(X)$ and the local ring at the generic point of the special fiber of X_R is the valuation ring of $K(T)$ corresponding to the Gauss valuation (which restricts to v on K). We say that our model corresponds to the Gauss valuation on $K(T)$, and we call T a *coordinate* of X_R. Conversely, if T is any rational function on X such that $K(T) \cong K(X)$, there is a smooth model X_R of X such that T is a coordinate of X_R. In simple terms, T is a coordinate of X_R if and only if, for all $a, b \in R$, the subvarieties of X_R cut out by $T - a$ and $T - b$ intersect exactly when $v(a - b) > 0$.

Now, let X'_R be a semistable model of X over R. The special fiber of X'_R is a tree-like configuration of copies of $\mathbb{P}^1_k$. Each irreducible component $\overline{W}$ of the special fiber $\overline{X}$ of X'_R yields a smooth model of X by blowing down all other irreducible components of $\overline{X}$. If T is a coordinate on the smooth model of X with $\overline{W}$ as special fiber, we will say that T corresponds to $\overline{W}$.

Disks and annuli. We give a brief overview here. For more details, see [Henrio 2000b].

Let X'_R be a semistable model for $X = \mathbb{P}^1_K$. Suppose x is a smooth point of the special fiber $\overline{X}$ of X'_R on the irreducible component $\overline{W}$. Let T be a coordinate corresponding to $\overline{W}$ such that $T = 0$ specializes to x. Then the set of points of $X(\overline{K})$ which specialize to x is the *open p-adic disk D* given by $v(T) > 0$. The ring of functions on the formal disk corresponding to D is $\hat{\mathcal{O}}_{X,x} \cong R\{T\}$.

Now, let x be an ordinary double point of $\overline{X}$ at the intersection of components $\overline{W}$ and $\overline{W}'$. Then the set of points of $X(\overline{K})$ which specialize to x is an *open annulus* A. If T is a coordinate corresponding to $\overline{W}$ such that $T = 0$ specializes to $\overline{W}' \setminus \overline{W}$, then A is given by $0 < v(T) < e$ for some $e \in v(K^\times)$. The ring of functions on the formal annulus corresponding to A is

$$\hat{\mathcal{O}}_{X,x} \cong \frac{R[[T, U]]}{(TU - p^e)}.$$

Observe that e is independent of the coordinate.

Suppose we have a preferred coordinate T on X and a semistable model X'_R of X whose special fiber $\overline{X}$ contains an irreducible component $\overline{X}_0$ corresponding to the coordinate T. If $\overline{W}$ is any irreducible component of $\overline{X}$ other than $\overline{X}_0$, then since $\overline{X}$ is a tree of copies of $\mathbb{P}^1$, there is a unique nonrepeating sequence of consecutive, intersecting components $\overline{X}_0, \dots, \overline{W}$. Let $\overline{W}'$ be the component in this sequence that intersects $\overline{W}$. Then the set of points in $X(\overline{K})$ that specialize to the connected component of $\overline{W}$ in $\overline{X} \setminus \overline{W}'$ is a closed p-adic disk D. If the established preferred coordinate (equivalently, the preferred component $\overline{X}_0$) is clear, we will abuse language and refer to the component $\overline{W}$ as *corresponding to the disk D*, and vice versa. If U is a coordinate corresponding to $\overline{W}$, and $U = \infty$ does not specialize to the connected component of $\overline{W}$ in $\overline{X} \setminus \overline{W}'$, then the ring of functions on the formal disk corresponding to D is $R\{U\}$.

3. Étale reduction of torsors

Let R be a mixed characteristic $(0, p)$ complete discrete valuation ring with residue field k and fraction field K. Let π be a uniformizer of R. Recall that we normalize the valuation of p (not π) to be 1. For any scheme or algebra S over R, write S_K and S_k for its base changes to K and k, respectively.

The following lemma will be used in the proof of Lemma 7.8 to analyze cyclic covers of closed p-adic disks given by explicit equations.

Lemma 3.1. *Assume that R contains the p^n-th roots of unity. Let $X = \mathrm{Spec}\, A$, where $A = R\{T\}$. Let $f : Y_K \to X_K$ be a μ_{p^n}-torsor given by the equation $y^{p^n} = g$, where $g = 1 + \sum_{i=1}^{\infty} c_i T^i$. Suppose one of the following two conditions holds*:

(i) $\min_i v(c_i) = n + 1/(p-1)$ *and* $v(c_i) > n + 1/(p-1)$ *for all i divisible by p.*

(ii) *p is odd*, $v(c_1) > n$, $v(c_p) > n$, *and* $\min_{i \neq 1, p} v(c_i) = n + 1/(p-1)$. *Also,* $v(c_i) > n + 1/(p-1)$ *for all $i > p$ divisible by p. Lastly,*

$$v\left(c_p - \frac{c_1^p}{p^{(p-1)n+1}}\right) > n + \frac{1}{p-1}.$$

Let h be the largest i ($\neq p$) such that $v(c_i) = n + (1)/p - 1$. Then $f : Y_K \to X_K$ splits into a union of p^{n-1} disjoint μ_p-torsors. Let Y be the normalization of X in the total ring of fractions of Y_K. Then the map $Y_k \to X_k$ is étale and is birationally equivalent to the union of p^{n-1} disjoint Artin–Schreier covers of $\mathbb{P}^1_k$, each with conductor h.

Proof. Suppose (i) holds. We claim that g has a p^{n-1}-st root $1 + au$ in A such that $a \in R$, $v(a) = (p)/p - 1$, and the reduction $\bar{u}$ of u in $A_k = k[T]$ is of degree h with only prime-to-p degree terms. By [Henrio 2000a, Chapter 5, Proposition 1.6] (the étale reduction case) and Lemma 2.5, this suffices to prove the lemma.

We prove the claim. Write $g = 1 + bw$ with $b \in R$ and $v(b) = n + 1/(p-1)$. Suppose $n > 1$. Then, using the binomial theorem, a p^{n-1}-st root of g is given by

$$\sqrt[p^{n-1}]{g} = 1 + \frac{1/p^{n-1}}{1!} bw + \frac{(1/p^{n-1})((1/p^{n-1}) - 1)}{2!} (bw)^2 + \cdots .$$

Since $v(b) = n + 1/(p-1)$, this series converges and is in A. Since the coefficients of all terms in this series of degree ≥ 2 have valuation greater than $p/(p-1)$, the series can be written as $\sqrt[p^{n-1}]{g} = 1 + au$, where $a = b/p^{n-1} \in R$, $v(a) = p/(p-1)$, and u congruent to $w \pmod{\pi}$. By assumption, the reduction $\bar{w}$ of w has degree h and only prime-to-p degree terms. Thus $\bar{u}$ does as well.

Now assume (ii) holds. It clearly suffices to show that there exists $a \in A$ such that $a^{p^n} g$ satisfies (i). Let $a = 1 + \eta T$, where $\eta = -c_1/p^n$. Now, by assumption, $v(c_1^p) - (p-1)n - 1 \geq \min(v(c_p), n + 1/(p-1))$. Since $v(c_p) > n$, we derive that $v(c_1^p) > pn + 1$. Thus $v(\eta) > 1/p$. Then there exists $\epsilon \in \mathbb{Q}^{>0}$ such that

$$(1 + \eta T)^{p^n} \equiv 1 - c_1 T - \binom{p^n}{p} \frac{c_1^p}{p^{pn}} T^p \pmod{p^{n+1/(p-1)+\epsilon}}.$$

It is easy to show that $\binom{p^n}{p} \equiv p^{n-1} \pmod{p^n}$ for all $n \geq 1$. Furthermore, the valuation of the T^i term ($1 \leq i \leq p^n$) in $(1 + \eta T)^{p^n}$ is greater than $i/p + n - v(i)$.

For any i other than 1 and p, this is greater than $n+1/(p-1)$ (here we use that p is odd). So

$$(1+\eta T)^{p^n} \equiv 1 - c_1 T - \frac{c_1^p}{p^{(p-1)n+1}} T^p \pmod{p^{n+1/(p-1)+\epsilon}}.$$

By the assumption that $v(c_p - c_1^p/p^{(p-1)n+1}) > n + (1)/p - 1$, we now see that $(1+\eta T)^{p^n} g$ satisfies (i). In particular, $(1+\eta T)^{p^n} g \equiv 1 \pmod{p^{n+1/(p-1)}}$, and, for any $i \neq 1$, p such that $v(c_i) = n + 1/(p-1)$, the valuation of the coefficient of T^i in $(1+\eta T)^{p^n} g$ is $n + 1/(p-1)$. □

An analogous result, which is necessary to prove our main theorem in the case $p = 2$, is in Appendix C.

4. Stable reduction of covers

In this section, R is a mixed characteristic $(0, p)$ complete discrete valuation ring with residue field k and fraction field K. We set $X \cong \mathbb{P}^1_K$, and we fix a *smooth* model X_R of X. Let $f : Y \to X$ be a G-Galois cover defined over K, with G any finite group, such that the branch points of f are defined over K and their specializations do not collide on the special fiber of X_R. Assume that f is branched at at least 3 points. By a theorem of Deligne and Mumford [1969, Corollary 2.7] combined with work of Raynaud [1990; 1999] and Liu [2006], there is a minimal finite extension K^{st}/K with ring of integers R^{st}, and a unique model $f^{st} : Y^{st} \to X^{st}$ of $f_{K^{st}} := f \times_K K^{st}$ (called the *stable model* of f) such that:

- The special fiber $\overline{Y}$ of Y^{st} is semistable.
- The ramification points of $f_{K^{st}}$ specialize to *distinct* smooth points of $\overline{Y}$.
- Any genus zero irreducible component of $\overline{Y}$ contains at least three marked points (that is, ramification points or points of intersection with the rest of $\overline{Y}$).
- G acts on Y^{st}, and $X^{st} = Y^{st}/G$.

The field K^{st} is called the minimal field of definition of the stable model of f. If we are working over a finite extension K'/K^{st} with ring of integers R', we will sometimes abuse language and call $f^{st} \times_{R^{st}} R'$ the stable model of f.

Remark 4.1. Our definition of the stable model is the definition used in [Wewers 2003b]. This differs from the definition in [Raynaud 1999], where ramification points are allowed to coalesce on the special fiber.

Remark 4.2. Note that X^{st} can be naturally identified with a blowup of $X \times_R R^{st}$ centered at closed points. Furthermore, the nodes of $\overline{Y}$ lie above nodes of the special fiber $\overline{X}$ of X^{st} [Raynaud 1994, Lemme 6.3.5], and Y^{st} is the normalization of X^{st} in $K^{st}(Y)$.

If $\overline{Y}$ is smooth, the cover $f : Y \to X$ is said to have *potentially good reduction*. If f does not have potentially good reduction, it is said to have *bad reduction*. In any case, the special fiber $\overline{f} : \overline{Y} \to \overline{X}$ of the stable model is called the *stable reduction* of f. The strict transform of the special fiber of $X_{R^{st}}$ in $\overline{X}$ is called the *original component* and will be denoted $\overline{X}_0$.

Each $\sigma \in G_K$ acts on $\overline{Y}$ (via its action on Y). This action commutes with that of G and is called the *monodromy action*. Then it is known that the extension K^{st}/K is the fixed field of the group $\Gamma^{st} \leq G_K$ consisting of those $\sigma \in G_K$ such that σ acts trivially on $\overline{Y}$; see, for instance, [Obus 2012, Proposition 2.9]. Thus K^{st} is clearly Galois over K. Since k is algebraically closed, the action of G_K fixes $\overline{X}_0$ pointwise.

Lemma 4.3. *Let $X_{R^{st}}$ be a smooth model for $X \times_K K^{st}$, and let $Y_{R^{st}}$ be its normalization in $K^{st}(Y)$. Suppose that the special fiber of $Y_{R^{st}}$ has irreducible components whose normalizations have genus greater than* 0. *Then X^{st} is a blow up of $X_{R^{st}}$* (*in other words, the stable reduction $\overline{X}$ contains a component corresponding to the special fiber of $X_{R^{st}}$*).

Proof. Consider a modification $(X^{st})' \to X^{st}$ centered on the special fiber such that $X_{R^{st}}$ is a blow down of $(X^{st})'$. Let $(Y^{st})'$ be the normalization of $(X^{st})'$ in $K^{st}(Y)$. By the minimality of the stable model, we know that X^{st} is obtained by blowing down components of $(X^{st})'$ such that the components of $(Y^{st})'$ lying above them are curves of genus zero. By our assumption, the component corresponding to the special fiber of $X_{R^{st}}$ is not blown down in the map $(X^{st})' \to X^{st}$. Thus $X_{R^{st}}$ is a blow down of X^{st}. □

4A. ***The graph of the stable reduction.*** As in [Wewers 2003b], we construct the (unordered) dual graph $\mathcal{G}$ of the stable reduction of $\overline{X}$. An *unordered graph* $\mathcal{G}$ consists of a set of *vertices* $V(\mathcal{G})$ and a set of *edges* $E(\mathcal{G})$. Each edge has a *source vertex* $s(e)$ and a *target vertex* $t(e)$. Each edge has an *opposite edge* $\bar{e}$ such that $s(e) = t(\bar{e})$ and $t(e) = s(\bar{e})$. Also, $\bar{\bar{e}} = e$.

Given f, $\overline{f}$, $\overline{Y}$, and $\overline{X}$ as above, we construct two unordered graphs $\mathcal{G}$ and $\mathcal{G}'$. In our construction, $\mathcal{G}$ has a vertex v for each irreducible component of $\overline{X}$ and an edge e for each ordered triple $(\bar{x}, \overline{W}', \overline{W}'')$, where $\overline{W}'$ and $\overline{W}''$ are irreducible components of $\overline{X}$ whose intersection is $\bar{x}$. If e corresponds to $(\bar{x}, \overline{W}', \overline{W}'')$, then $s(e)$ is the vertex corresponding to $\overline{W}'$ and $t(e)$ is the vertex corresponding to $\overline{W}''$. The opposite edge of e corresponds to $(\bar{x}, \overline{W}'', \overline{W}')$. We denote by $\mathcal{G}'$ the *augmented* graph of $\mathcal{G}$ constructed as follows: consider the set B_{wild} of branch points of f with branching index divisible by p. For each $x \in B_{\mathrm{wild}}$, we know that x specializes to a unique irreducible component $\overline{W}_x$ of $\overline{X}$ corresponding to a vertex A_x of $\mathcal{G}$. Then $V(\mathcal{G}')$ consists of the elements of $V(\mathcal{G})$ with an additional vertex V_x for each $x \in B_{\mathrm{wild}}$. Also, $E(\mathcal{G}')$ consists of the elements of $E(\mathcal{G})$ with two additional opposite edges for each $x \in B_{\mathrm{wild}}$: one with source V_x and target A_x, and one with source A_x and target V_x. We write v_0 for the vertex corresponding to the original component $\overline{X}_0$.

We partially order the vertices of $\mathcal{G}$ (and $\mathcal{G}'$) such that $v_1 \preceq v_2$ if and only if $v_1 = v_2$, $v_1 = v_0$, or v_0 and v_2 are in different connected components of $\mathcal{G}' \backslash v_1$. The set of irreducible components of $\overline{X}$ inherits the partial order $\preceq$. If $a \preceq b$, where a and b are vertices of $\mathcal{G}$ (or $\mathcal{G}'$) or irreducible components of $\overline{X}$, we say that b lies *outward* from a.

4B. ***Inertia groups of the stable reduction.*** Maintain the notation from the beginning of Section 4.

Proposition 4.4 [Raynaud 1999, Proposition 2.4.11]. *The following are the inertia groups of $\bar{f} : \overline{Y} \to \overline{X}$ at points of $\overline{Y}$ (note that points in the same G-orbit have conjugate inertia groups):*

(i) *At the generic points of irreducible components, the inertia groups are p-groups.*

(ii) *At each node, the inertia group is an extension of a cyclic, prime-to-p order group by a p-group generated by the inertia groups of the generic points of the crossing components.*

(iii) *If a point $y \in Y$ above a branch point $x \in X$ specializes to a smooth point $\bar{y}$ on a component $\overline{V}$ of $\overline{Y}$, then the inertia group at $\bar{y}$ is an extension of the prime-to-p part of the inertia group at y by the inertia group of the generic point of $\overline{V}$.*

(iv) *At all other points q (automatically smooth, closed), the inertia group is equal to the inertia group of the generic point of the irreducible component of $\overline{Y}$ containing q.*

If $\overline{V}$ is an irreducible component of $\overline{Y}$, we will always write $I_{\overline{V}} \leq G$ for the inertia group of the generic point of $\overline{V}$ and $D_{\overline{V}}$ for the decomposition group.

For the rest of this subsection, assume G has a *cyclic* p-Sylow subgroup. When G has a cyclic p-Sylow subgroup, the inertia groups above a generic point of an irreducible component $\overline{W} \subset \overline{X}$ are conjugate cyclic groups of p-power order. If they are of order p^i, we call $\overline{W}$ a *p^i-component*. If $i = 0$, we call $\overline{W}$ an *étale component*, and if $i > 0$, we call $\overline{W}$ an *inseparable component*. For an inseparable component $\overline{W}$, the morphism $Y \times_X \overline{W} \to \overline{W}$ induced from f corresponds to an inseparable extension of the function field $k(\overline{W})$.

As in [Raynaud 1999], we call an irreducible component $\overline{W} \subseteq \overline{X}$ a *tail* if it is not the original component and intersects exactly one other irreducible component of $\overline{X}$. Otherwise, it is called an *interior component*. A tail of $\overline{X}$ is called *primitive* if it contains a branch point other than the point at which it intersects the rest of $\overline{X}$.

Otherwise it is called *new*. This follows [Wewers 2003b]. An inseparable tail that is a p^i-component will also be called a *p^i-tail*. Thus one can speak of, for instance, "new p^i-tails" or "primitive étale tails."

We call the stable reduction $\overline{f}$ of f *monotonic* if for every $\overline{W} \preceq \overline{W}'$, the inertia group of $\overline{W}'$ is contained in the inertia group of $\overline{W}$. In other words, the stable reduction is monotonic if the generic inertia does not increase as we move outward from $\overline{X}_0$ along $\overline{X}$.

Proposition 4.5. *If G is p-solvable, then $\overline{f}$ is monotonic.*

Proof. By Proposition 2.1, we know that there is a prime-to-p group N such that $G/N \cong \mathbb{Z}/p^n \rtimes \mathbb{Z}/m$. Since taking the quotient of a G-cover by a prime-to-p group does not affect monotonicity, we may assume that $G \cong \mathbb{Z}/p^n \rtimes \mathbb{Z}/m_G$. By [Obus 2012, Remark 4.5], it follows that $\overline{f}$ is monotonic. $\square$

Proposition 4.6 [Obus 2012, Proposition 2.13]. *If $x \in X$ is branched of index $p^a s$, where $p \nmid s$, then x specializes to a p^a-component of $\overline{X}$.*

Lemma 4.7 [Raynaud 1999, Proposition 2.4.8]. *If $\overline{W}$ is an étale component of $\overline{X}$, then $\overline{W}$ is a tail.*

Lemma 4.8 [Obus 2012, Lemma 2.16]. *If $\overline{W}$ is a p^a-tail of $\overline{X}$, then the component $\overline{W}'$ that intersects $\overline{W}$ is a p^b-component with $b > a$.*

Proposition 4.9. *Suppose f has monotonic stable reduction. Let K'/K be a field extension such that the following hold for each tail $\overline{X}_b$ of $\overline{X}$:*

(i) *There exists a smooth point $\overline{x}_b$ of $\overline{X}$ on $\overline{X}_b$ such that $\overline{x}_b$ is fixed by $G_{K'}$.*

(ii) *There exists a smooth point $\overline{y}_b$ of $\overline{Y}$ on some component $\overline{Y}_b$ lying above $\overline{X}_b$ such that $\overline{y}_b$ is fixed by $G_{K'}$.*

Then the stable model of f can be defined over a tame extension of K'.

Proof. We claim that $G_{K'}$ acts on $\overline{Y}$ through a group of prime-to-p order. This will yield the proposition.

Suppose $\gamma \in G_{K'}$ is such that γ^p acts trivially on $\overline{Y}$. For each tail $\overline{X}_b$, we have that γ fixes $\overline{x}_b$. Since γ fixes the original component pointwise, it fixes the point of intersection of $\overline{X}_b$ with the rest of $\overline{X}$. Any action on $\mathbb{P}^1_k$ with order dividing p and two fixed points is trivial, so γ fixes each $\overline{X}_b$ pointwise. By inward induction, γ fixes $\overline{X}$ pointwise. So γ acts "vertically" on $\overline{Y}$.

Now, γ also fixes each $\overline{y}_b$. By Propositions 4.4 and 4.6, the inertia of f^{st} at $\overline{y}_b$ is an extension of a prime-to-p group by the generic inertia of f^{st} on $\overline{Y}_b$. So some prime-to-p power γ^i of γ fixes $\overline{Y}_b$ pointwise. Since $p \nmid i$ and the action of γ has order p, it follows that γ fixes $\overline{Y}_b$ pointwise. Since γ and G commute, γ fixes all components above $\overline{X}_b$ pointwise.

We proceed to show that γ acts trivially on $\overline{Y}$ by inward induction. Suppose $\overline{W}$ is a component of $\overline{X}$ such that if $\overline{W}' \succ \overline{W}$, then γ fixes all components above $\overline{W}'$ pointwise. Suppose $\overline{W}' \succ \overline{W}$ is a component such that $\overline{W}' \cap \overline{W} = \{\overline{w}\} \neq \varnothing$. Let $\overline{V}$ be a component of $\overline{Y}$ above $\overline{W}$, and let $\overline{v}$ be a point of $\overline{V}$ above $\overline{w}$. By the inductive hypothesis, γ fixes $\overline{v}$. Since $\overline{f}$ is monotonic, Proposition 4.4 shows that the p-part of the inertia group at $\overline{v}$ is the same as the generic inertia group of $\overline{V}$. Thus γ fixes $\overline{V}$ pointwise. Because γ commutes with G, it fixes all components above $\overline{W}$ pointwise. This completes the induction. $\square$

4C. ***Ramification invariants and the vanishing cycles formula.*** Maintain the notation from the beginning of Section 4, and assume additionally that G has a *cyclic* p-Sylow group P. Recall that $m_G = |N_G(P)/Z_G(P)|$. Below, we define the *effective ramification invariant* σ_b corresponding to each tail $\overline{X}_b$ of $\overline{X}$.

Definition 4.10. Consider a tail $\overline{X}_b$ of $\overline{X}$. Suppose $\overline{X}_b$ intersects the rest of $\overline{X}$ at x_b. Let $\overline{Y}_b$ be a component of $\overline{Y}$ lying above $\overline{X}_b$, and let y_b be a point lying above x_b. Then the *effective ramification invariant* σ_b is defined as follows: If $\overline{X}_b$ is an étale tail, then σ_b is the conductor of higher ramification for the extension $\hat{\mathbb{O}}_{\overline{Y}_b,y_b}/\hat{\mathbb{O}}_{\overline{X}_b,x_b}$ (see Section 2C). If $\overline{X}_b$ is a p^i-tail ($i > 0$), then the extension $\hat{\mathbb{O}}_{\overline{Y}_b,y_b}/\hat{\mathbb{O}}_{\overline{X}_b,x_b}$ can be factored as

$$\hat{\mathbb{O}}_{\overline{X}_b,x_b} \overset{\alpha}{\hookrightarrow} S \overset{\beta}{\hookrightarrow} \hat{\mathbb{O}}_{\overline{Y}_b,y_b},$$

where α is Galois and β is purely inseparable of degree p^i. Then σ_b is the conductor of higher ramification for the extension $S/\hat{\mathbb{O}}_{\overline{X}_b,x_b}$.

The vanishing cycles formula [Raynaud 1999, 3.4.2 (5)] is a key formula that helps us understand the structure of the stable reduction of a branched G-cover of curves in the case where p exactly divides the order of G. The following theorem, which is the most important ingredient in the proof of Theorem 1.3, generalizes the vanishing cycles formula to the case where G has a cyclic p-Sylow group of arbitrary order.

Theorem 4.11 (vanishing cycles formula [Obus 2012, Corollary 3.15]). *Let $f : Y \to X \cong \mathbb{P}^1$ be a G-Galois cover over K with bad reduction, branched only above $\{0, 1, \infty\}$, where G has a cyclic p-Sylow subgroup. Let $\overline{f} : \overline{Y} \to \overline{X}$ be the stable reduction of f. Let B_{new} be an indexing set for the new étale tails and let B_{prim} be an indexing set for the primitive étale tails. Then we have the formula*

$$\sum_{b\in B_{\mathrm{new}}} (\sigma_b - 1) + \sum_{b\in B_{\mathrm{prim}}} \sigma_b = 1. \tag{4-1}$$

Lemma 4.12 [Obus 2012, Proposition 4.1]. *If b indexes an inseparable tail $\overline{X}_b$, then σ_b is an integer.*

Lemma 4.13 [Obus 2012, Lemma 4.2(i)]. *A new tail $\overline{X}_b$ (étale or inseparable) has $\sigma_b \geq 1 + 1/m$.*

Lemma 4.14. *Suppose $\overline{X}_b$ is a new inseparable p^i-tail with effective ramification invariant σ_b. Suppose further that the inertia group $I \cong \mathbb{Z}/p^i$ of some component $\overline{Y}_b$ above $\overline{X}_b$ is normal in G. Then $\overline{X}_b$ is a new (étale) tail of the stable reduction of the quotient cover $f' : Y/I \to X$ with effective ramification invariant σ_b.*

Proof. Let $(f')^{st} : (Y')^{st} \to (X')^{st}$ be the stable model of f'. Then, since $(Y^{st})/I$ is a semistable model of Y/I, we have that $(Y')^{st}$ is a contraction of $(Y^{st})/I$. Thus $(X')^{st}$ is a contraction of X^{st}. To prove the lemma, it suffices to prove that $\overline{X}_b$ is not contracted in the map $\alpha : X^{st} \to (X')^{st}$.

By Lemmas 4.12 and 4.13, we know $\sigma_b \geq 2$. A calculation using the Hurwitz formula (cf. [Raynaud 1999, Lemme 1.1.6]) shows that the genus of $\overline{Y}_b$ is greater than zero. Since the quotient morphism $Y \to Y/I$ is radicial on $\overline{Y}_b$, the normalization of X^{st} in $K^{st}(Y/I)$ has irreducible components of genus greater than zero lying above $\overline{X}_b$. By Lemma 4.3, $\overline{X}_b$ is a component of the special fiber of $(X^{st})'$, thus it is not contracted by α. □

Proposition 4.15. *Let $f : Y \to X = \mathbb{P}^1_K$ be a three-point G-cover with bad reduction, where G is p-solvable, G has cyclic p-Sylow subgroup, and $m_G > 1$. Then $\overline{X}$ has no inseparable tails or new tails.*

Proof. Since taking the quotient of a G-cover by a prime-to-p group affects neither ramification invariants (Lemma 2.2) nor inseparability, we may assume by Proposition 2.1 that $G \cong \mathbb{Z}/p^n \rtimes \mathbb{Z}/m_G$. Then all elements of G have either p-power order or prime-to-p order. The resulting cover is branched at three points (otherwise it would be cyclic), and at least two of these points have prime-to-p branching index.

We first show there are no inseparable tails. Say there is an inseparable p^i-tail $\overline{X}_b$ with effective ramification invariant σ_b. By Lemma 4.12, σ_b is an integer. By Lemma 4.13, $\sigma_b > 1$ if $\overline{X}_b$ does not contain the specialization of any branch point. Assume for the moment that this is the case. Then $\sigma_b \geq 2$. Let I be the common inertia group of all components of $\overline{Y}$ above $\overline{X}_b$. If $f' : Y/I \to X$ is the quotient cover, then we know f' is branched at three points, with at least two having prime-to-p ramification index. Thus the stable reduction $\overline{f}'$ has at least two primitive tails. By Lemma 4.14, it also has a new tail corresponding to the image of $\overline{X}_b$, which has effective ramification invariant $\sigma_b \geq 2$. Then the left-hand side of (4-1) for the cover f' is greater than 1, so we have a contradiction.

We now prove that no branch point of f specializes to $\overline{X}_b$. By Proposition 4.6, such a branch point x would have ramification index $p^i s$, where $p \nmid s$. Since $i \geq 1$, the only possible branching index for x is p^i (as it must be the order of an element of G). So in $f' : Y/I \to X$, x has ramification index 1. Thus $Z \to X$ is branched in at most two points, which contradicts the fact that f' is not cyclic.

Now we show there are no new tails. Suppose there is a new tail $\overline{X}_b$ with ramification invariant σ_b. If $\sigma_b \in \mathbb{Z}$, we get the same contradiction as in the inseparable case. If $\sigma_b \notin \mathbb{Z}$, and if $\overline{Y}_b \subseteq \overline{Y}$ is an irreducible component above $\overline{X}_b$, then $\overline{Y}_b \to \overline{X}_b$ is a $\mathbb{Z}/p^i \rtimes \mathbb{Z}/m_b$-cover branched at only one point, where $i \geq 1$ and $m_b > 1$. This violates the easy direction of Abhyankar's conjecture, as this group is not quasi-p; see, for instance, [SGA 1 1971, XIII, Corollaire 2.12]. □

5. Deformation data

Deformation data arise naturally from the stable reduction of covers. Much information is lost when we pass from the stable model of a cover to its stable reduction, and deformation data provide a way to retain some of this information. This process is described in detail in [Obus 2012, Section 3.2], and we recall some facts here.

5A. ***Generalities.*** Let $\overline{W}$ be any connected smooth proper curve over k. Let H be a finite group and χ a 1-dimensional character $H \to \mathbb{F}_p^\times$. A *deformation datum* over $\overline{W}$ of type (H, χ) is an ordered pair $(\overline{V}, \omega)$ such that $\overline{V} \to \overline{W}$ is an H-cover, ω is a meromorphic differential form on $\overline{V}$ that is either logarithmic or exact (that is, $\omega = du/u$ or du for $u \in k(\overline{V})$), and $\eta^*\omega = \chi(\eta)\omega$ for all $\eta \in H$. If ω is logarithmic or exact, the deformation datum is called multiplicative or additive, respectively. When $\overline{V}$ is understood, we will sometimes speak of the deformation datum ω.

If $(\overline{V}, \omega)$ is a deformation datum and $w \in \overline{W}$ is a closed point, we define m_w to be the order of the prime-to-p part of the ramification index of $\overline{V} \to \overline{W}$ at w. Define h_w to be $\mathrm{ord}_v(\omega) + 1$, where $v \in \overline{V}$ is any point which maps to $w \in \overline{W}$. This is well defined because $\eta^*\omega$ is a nonzero scalar multiple of ω for $\eta \in H$.

Lastly, define $\sigma_x = h_w/m_w$. We call w a *critical point* of the deformation datum $(\overline{V}, \omega)$ if $(h_w, m_w) \neq (1, 1)$. Note that every deformation datum contains only a finite number of critical points. The ordered pair (h_w, m_w) is called the *signature* of $(\overline{V}, \omega)$ (or of ω, if $\overline{V}$ is understood) at w, and σ_w is called the *invariant* of the deformation datum at w.

5B. ***Deformation data arising from stable reduction.*** Maintain the notation of Section 4. In particular, $X \cong \mathbb{P}^1_K$, we have a G-cover $f : Y \to X$ defined over K with bad reduction and at least three branch points, there is a smooth model of X where the reductions of the branch points do not coalesce, and f has stable model $f^{st} : Y^{st} \to X^{st}$ and stable reduction $\bar{f} : \overline{Y} \to \overline{X}$. We assume further that G has a cyclic p-Sylow subgroup. For each irreducible component of $\overline{Y}$ lying above a p^r-component of $\overline{X}$ with $r > 0$, we obtain r different deformation data. The details of this construction are given in [Obus 2012, Construction 3.4], and we only give a sketch here.

Suppose $\overline{V}$ is an irreducible component of $\overline{Y}$ with generic point η and nontrivial generic inertia group $I \cong \mathbb{Z}/p^r \subset G$. We write $B = \hat{\mathbb{O}}_{Y^{st},\eta}$, and $C = B^I$. The map Spec $B \to$ Spec C is given by a tower of r maps, each of degree p. We can write these maps as Spec $C_{i+1} \to$ Spec C_i for $1 \le i \le r$ such that $B = C_{r+1}$ and $C = C_1$. After a possible finite extension K'/K^{st}, each of these maps is given by an equation $y^p = z$ on the generic fiber, where z is well defined up to raising to a prime-to-p power. The morphism on the special fiber is purely inseparable. To such a degree p map, [Henrio 2000a, chapitre 5, définition 1.9] associates a meromorphic differential form ω_i, well defined up to multiplication by a scalar in $\mathbb{F}_p^\times$, on the special fiber Spec $C_i \times_{R^{st}} k =$ Spec C_i/π, where π is a uniformizer of R^{st}. This differential form is either logarithmic or exact. Since $C/\pi \cong k(\overline{V})^{p^r} \cong k(\overline{V})^{p^{r-i+1}} \cong C_i/\pi$ for any i, each ω_i can be thought of as a differential form on $\overline{V}' =$ Spec $C \times_{R^{st}} k$, where $k(\overline{V}') = k(\overline{V})^{p^r}$.

Let $H = D_{\overline{V}}/I_{\overline{V}} \cong D_{\overline{V}'}$. If $\overline{W}$ is the component of $\overline{X}$ lying below $\overline{V}$, we have that $\overline{W} = \overline{V}'/H$. In fact, each $(\overline{V}', \omega_i)$, for $1 \le i \le r$, is a deformation datum of type (H, χ) over $\overline{W}$, where χ is given by the conjugation action of H on $I_{\overline{V}}$. The invariant of σ_i at a point $w \in W$ will be denoted $\sigma_{i,w}$. We will sometimes call the deformation datum $(\overline{V}', \omega_1)$ the *bottom deformation datum* for $\overline{V}$.

For $1 \le i \le r$, denote the valuation of the different of $C_i \hookrightarrow C_{i+1}$ by δ_{ω_i}. If ω_i is multiplicative, then $\delta_{\omega_i} = 1$. Otherwise, $0 < \delta_{\omega_i} < 1$.

For the rest of this section, we will only concern ourselves with deformation data that arise from stable reduction in the manner described above. We will use the notation of Section 4 throughout.

Lemma 5.1 ([Obus 2012, Lemma 3.5], cf. [Wewers 2003b, Proposition 1.7]). *Say $(\overline{V}', \omega)$ is a deformation datum arising from the stable reduction of a cover, and let $\overline{W}$ be the component of $\overline{X}$ lying under $\overline{V}'$. Then a critical point x of the deformation datum on $\overline{W}$ is either a singular point of $\overline{X}$ or the specialization of a branch point of $Y \to X$ with ramification index divisible by p. In the first case, $\sigma_x \neq 0$, and in the second case, $\sigma_x = 0$ and ω is logarithmic.*

Proposition 5.2. *Let $(\overline{V}', \omega_1)$ be the bottom deformation datum for some irreducible component $\overline{V}$ of $\overline{Y}$. If ω_1 is multiplicative, then $\omega_i = \omega_1$ for $2 \le i \le r$. In particular, all ω_i are multiplicative.*

Proof. As is mentioned at the beginning of [Obus 2012, Section 3.2.2], we may work over a finite extension K'/K^{st} containing the p^r-th roots of unity. Let B and C be as in our construction of deformation data. Let R' be the ring of integers of K'. By Kummer theory, we can write $B \otimes_{R'} K' = (C \otimes_{R'} K')[\theta]/(\theta^{p^r} - \theta_1)$. After a further extension of K', we can assume $v(\theta_1) = 0$.

By [Henrio 2000a, chapitre 5, définition 1.9], if ω_1 is logarithmic, then the reduction $\overline{\theta}_1$ of θ_1 to k is not a p-th power in $C \otimes_{R'} k$. Again, by [Henrio 2000a, chapitre 5, définition 1.9], we thus know that $\omega_1 = d\bar{\theta}_1/\bar{\theta}_1$. It easy to see that ω_i arises from the equation $y^p = \theta_i$ where $\theta_i = \sqrt[p^{i-1}]{\theta_1}$. Under the p^{i-1}-st power isomorphism $\iota : C_i \otimes_{R'} k \to C \otimes_{R'} k$, $\iota(\theta_i) = \theta_1$. So, again by [Henrio 2000a, chapitre 5, définition 1.9], ω_i is logarithmic and is equal to $d\theta_1/\theta_1$, which is equal to ω_1. □

Lemma 5.3. *If f is a three-point cover, then the original component of $\overline{X}$ is a p^n-component, and all deformation data above the original component are multiplicative.*

Proof. Since G is p-solvable, we know by Proposition 2.1 that $f : Y \to X$ has a quotient cover $f' : Y' \to X$ with Galois group $\mathbb{Z}/p^n \rtimes \mathbb{Z}/m_G$. Since $Y \to Y'$ is of prime-to-p degree, we may assume that $Y = Y'$ and $G \cong \mathbb{Z}/p^n \rtimes \mathbb{Z}/m_G$. Let $J < G$ be the unique subgroup of order p^{n-1}. Then the quotient cover $\eta : Z = Y/J \to X$ has Galois group $\mathbb{Z}/p \rtimes \mathbb{Z}/m_G$. If all branch points of η have prime-to-p branching index, then [Wewers 2003a, Section 1.4] shows that, in the language of that paper, η is of *multiplicative type*. Then η has bad reduction by [ibid., Corollary 1.5], and the original component for the stable reduction $\overline{Z} \to \overline{X}$ is a p-component. Furthermore, the deformation datum on the irreducible component of $\overline{Z}$ above the original component of $\overline{X}$ is multiplicative (also due to the same corollary).

If η has a branch point x with ramification index divisible by p, then η has bad reduction. By Proposition 4.6, x specializes to a p-component. By [Wewers 2003b, Theorem 2, p. 992], this is the original component $\overline{X}_0$, which is the only p-component. The deformation datum above $\overline{X}_0$ must be multiplicative here, as $\overline{X}_0$ contains the specialization of a branch point with p dividing the branching index (see Lemma 5.1).

So in all cases, the original component is a p-component for η with multiplicative deformation datum. Thus the bottom deformation datum above $\overline{X}_0$ for f is multiplicative. Now, we claim that $\overline{X}_0$ is a p^n-component for f. Let I be the inertia group of a component of $\overline{Y}$ lying above $\overline{X}_0$. Since η is inseparable above $\overline{X}_0$, we must have that $I \supsetneq J$. Thus $|I| = p^n$, proving the claim. Finally, Proposition 5.2 shows that all the deformation data above $\overline{X}_0$ for f are multiplicative. □

5C. ***Effective invariants of deformation data.*** Maintain the Section 5B notation. Recall that $\mathscr{G}'$ is the augmented dual graph of $\overline{X}$. To each edge e of $\mathscr{G}'$ we will associate an invariant σ_e^{eff}, called the *effective invariant*.

Definition 5.4 (cf. [Obus 2012, Definition 3.10]).

- If $s(e)$ corresponds to a p^r-component $\overline{W}$ and $t(e)$ corresponds to a $p^{r'}$-component $\overline{W}'$ with $r \geq r'$, then $r \geq 1$ by Lemma 4.7. Let ω_i, $1 \leq i \leq r$, be the deformation data above $\overline{W}$. If $\{w\} = \overline{W} \cap \overline{W}'$, define $\sigma_{i,w}$ to be the invariant of ω_i at w. Then

$$\sigma_e^{\text{eff}} := \left(\sum_{i=1}^{r-1} \frac{p-1}{p^i}\sigma_{i,w}\right) + \frac{1}{p^{r-1}}\sigma_{r,w}.$$

Note that this is a weighted average of the $\sigma_{i,w}$.

- If $s(e)$ corresponds to a p^r-component and $t(e)$ corresponds to a $p^{r'}$-component with $r < r'$, then $\sigma_e^{\text{eff}} := -\sigma_{\bar{e}}^{\text{eff}}$.
- If either $s(e)$ or $t(e)$ is a vertex of $\mathcal{G}'$ but not $\mathcal{G}$, then $\sigma_e^{\text{eff}} := 0$.

Lemma 5.5 [Obus 2012, Lemma 3.11 (i), (iii)].

(i) *For any $e \in E(\mathcal{G}')$, we have $\sigma_e^{\text{eff}} = -\sigma_{\bar{e}}^{\text{eff}}$.*

(ii) *If $t(e)$ corresponds to an étale tail $\overline{X}_b$, then $\sigma_e^{\text{eff}} = \sigma_b$.*

Lemma 5.6 (effective local vanishing cycles formula [Obus 2012, Lemma 3.12]). *Let $v \in V(\mathcal{G}')$ correspond to a p^j-component $\overline{W}$ of $\overline{X}$ with genus g_v. Then*

$$\sum_{s(e)=v} (\sigma_e^{\text{eff}} - 1) = 2g_v - 2.$$

Lemma 5.7. *Let e be an edge of $\mathcal{G}$ such that $s(e) \prec t(e)$. Write $\overline{W}$ for the component corresponding to $t(e)$. Let Π_e be the set of branch points of f with branching index divisible by p that specialize to or outward from $\overline{W}$. Let B_e index the set of étale tails $\overline{X}_b$ such that $\overline{X}_b \succeq \overline{W}$. Then the following formula holds*:

$$\sigma_e^{\text{eff}} - 1 = \sum_{b \in B_e} (\sigma_b - 1) - |\Pi_e|.$$

Proof. For the context of this proof, call a set A of edges of $\mathcal{G}'$ *admissible* if:

- For each $a \in A$, we have $s(e) \preceq s(a) \prec t(a)$.
- For each $b \in B_e$, there is exactly one $a \in A$ such that $t(a) \preceq v_b$, where v_b is the vertex corresponding to $\overline{X}_b$.
- For each $c \in \Pi_e$, there is exactly one $a \in A$ such that $t(a) \preceq v_c$, where v_c is the vertex corresponding to c.

For an admissible set A, write $F(A) = \sum_{a \in A}(\sigma_a^{\text{eff}} - 1)$. We claim that $F(A) = \sum_{b \in B_e}(\sigma_b - 1) - |\Pi_e|$ for all admissible A. Since the set $\{e\}$ is clearly admissible, this claim proves the lemma.

Now, if A is an admissible set of edges, then we can form a new admissible set A' by eliminating an edge α such that $t(\alpha)$ is not a leaf of $\mathcal{G}'$, and replacing it with the set of all edges β such that $t(\alpha) = s(\beta)$. Since $t(\alpha)$ always corresponds to a vertex of genus 0, Lemmas 5.5(i) and 5.6 show that $F(A) = F(A')$. By repeating this process, we see that $F(A) = F(D)$, where D consists of all edges

d such that $t(d) = v_b$ or $t(d) = v_c$ with $b \in B_e$ or $c \in \Pi_e$. But by Lemma 5.5(ii), $F(D) = \sum_{b \in B_e}(\sigma_b - 1) + \sum_{c \in |\Pi_e|}(0-1)$, proving the claim. □

The remainder of this section will be used only in Appendix A, and may be skipped by a reader who does not wish to read that section.

Consider two intersecting components $\overline{W}$ and $\overline{W}'$ of $\overline{X}$ as in Definition 5.4. Suppose $\overline{W}$ is a p^r-component and $\overline{W}'$ is a $p^{r'}$-component, $r \geq r$. If $\overline{V}$ and $\overline{V}'$ are intersecting components lying above $\overline{W}$ and $\overline{W}'$, respectively, then for each i, $1 \leq i \leq r$, there is a deformation datum with differential form ω_i associated to $\overline{V}$. Likewise, for each i', $1 \leq i' \leq r'$, there is a deformation datum with differential form $\omega'_{i'}$ associated to $\overline{V}'$. Let $(h_{i,w}, m_w)$ be the invariants of ω_i at w, the intersection point of $\overline{W}$ and $\overline{W}'$. Suppose v is an intersection point of $\overline{V}$ and $\overline{V}'$. We have the following proposition relating the change in the differents of the deformation data (see just before Lemma 5.1) and the épaisseur of the annulus corresponding to w:

Proposition 5.8. *Let ϵ_w be the épaisseur of the formal annulus corresponding to w.*

- *If $i = i' + r - r'$, then $\delta_{\omega_i} - \delta'_{\omega'_{i'}} = \epsilon_w \sigma_{i,w}(p-1)/p^i$.*
- *If $i \leq r - r'$, then $\delta_{\omega_i} = \epsilon_w \sigma_{i,w}(p-1)/p^i$.*

Proof. Write I_i for the unique subgroup of order p^i of the inertia group of $\overline{f}$ at v in G. Let $\mathcal{A} = \operatorname{Spec} \hat{\mathcal{O}}_{Y^{st},v}$. Let ϵ be the épaisseur of $\mathcal{A}/(I_{r-i+1})$. Then, in the case $i = i' + r - r'$, [Henrio 2000a, chapitre 5, proposition 1.10] shows that

$$\delta_{\omega_i} - \delta'_{\omega'_{i'}} = \epsilon h_{i,w}(p-1).$$

In the case $i < r - r'$, the same proposition shows $\delta_{\omega_i} - 0 = \epsilon h_{i,w}(p-1)$. Also, [Raynaud 1999, Proposition 2.3.2 (a)] shows that $\epsilon_w = p^i m_w \epsilon$. The proposition follows. □

It will be useful to work with the *effective different*, which we define now.

Definition 5.9. Let $\overline{W}$ be a p^r-component of $\overline{X}$, and let ω_i, $1 \leq i \leq r$, be the deformation data above $\overline{W}$. Define the *effective different* $\delta^{\text{eff}}_{\overline{W}}$ by

$$\delta^{\text{eff}}_{\overline{W}} = \left(\sum_{i=1}^{r-1} \delta_{\omega_i}\right) + \frac{p}{p-1}\delta_{\omega_r}.$$

Lemma 5.10. *Assume the notation of Proposition 5.8. Let e be an edge of $\mathcal{G}$ such that $s(e)$ corresponds to $\overline{W}$ and $t(e)$ corresponds to $\overline{W}'$. Then*

$$\delta^{\text{eff}}_{\overline{W}} - \delta^{\text{eff}}_{\overline{W}'} = \sigma^{\text{eff}}_e \epsilon_w.$$

Proof. We sum the equations from Proposition 5.8 for $1 \leq i \leq r-1$. Then we add $p/(p-1)$ times the equation for $i = r$. This exactly gives $\delta^{\text{eff}}_{\overline{W}} - \delta^{\text{eff}}_{\overline{W}'} = \sigma^{\text{eff}}_e \epsilon_w$. □

6. Quotient covers

In this section, we relate the minimal field of definition of the stable model of a G-cover to that of its quotient G/N-covers when $p \nmid |N|$. This allows a significant simplification of the group theory in Section 7.

Lemma 6.1. *Let $f : Y \to X$ be any G-Galois cover of smooth, proper, geometrically connected curves over any field (we do not assume that a p-Sylow subgroup of G is cyclic). Suppose G has a normal subgroup N such that $p \nmid |N|$ and $Z := Y/N$. So f factorizes as*

$$Y \xrightarrow{q} Z \xrightarrow{\eta} X.$$

Suppose L is a field such that $\eta : Z \to X$ is defined over L, and let Z_L be a model for Z over L. Suppose further that $q : Y \to Z$ can be defined over L, with respect to the model Z_L. Then the field of moduli L' of f with respect to L satisfies $p \nmid [L' : L]$.

Proof. Clearly, f is defined as a *mere* cover over L. So let Y_L be a model for Y over L such that $Y_L/N = Z_L$ (and set $X_L = Z_L/(G/N)$). Then the cover $Y_L \to X_L$ gives rise to a homomorphism $h : G_L \to \mathrm{Out}(G)$, as in Section 2B, whose kernel is the subgroup of G_L fixing the field of moduli of f. Since q is defined over L, the image of h acts by inner automorphisms on N. Thus, there is a natural homomorphism $r : (\mathrm{im}\ h) \to \mathrm{Out}(G/N)$. Since η is defined over L, the image of $r \circ h$ acts by inner automorphisms on G/N. Take $\overline{\alpha} \in \mathrm{im}\ h$. It is easy to see that we can find a representative $\alpha \in \mathrm{Aut}(G)$ of $\overline{\alpha}$ that fixes N pointwise and whose image in $\mathrm{Aut}(G/N)$ fixes G/N pointwise. If $g \in G$, then $\alpha(g) = gs$ for some $s \in N$. Since α fixes N, we see that $\alpha^i(g) = gs^i$. Since $s \in N$, we know $s^{|N|}$ is trivial, so $\alpha^{|N|}$ is trivial. Thus $\overline{\alpha}$ has prime-to-p order, implying that $G_L/(\ker h)$ does as well. We conclude that the field of moduli L' of f relative to L is a prime-to-p extension of L. □

For the next proposition, K is a characteristic zero complete discrete valuation field with residue field k.

Proposition 6.2. *Let $f : Y \to X \cong \mathbb{P}^1_K$ be a G-cover with bad reduction and stable model f^{st} as in Section 4. Suppose G has a normal subgroup N such that $p \nmid |N|$, and let $Z = Y/N$. Let L/K be a finite extension such that the stable model $\eta^{st} : Z^{st} \to X^{st}$ of $\eta : Z \to X$ and each of the branch points of the canonical map $q : Y \to Z$ can be defined over L. Then the stable model f^{st} of f can be defined over a tame extension of L.*

Proof. By [Liu 2006, Remark 2.21], the minimal modification $(Z^{st})'$ of Z^{st} that separates the specializations of the branch points of q is defined over L. Note that q, being an N-cover, is tamely ramified. We claim that $q^{st} : Y^{st} \to (Z^{st})'$ is defined over a tame extension of L (along with the N-action).

The proof of the claim is almost completely contained in the proof of [Saïdi 1997, théorème 3.7], so we only give a sketch. Break up the formal completion $\mathcal{X}$ of $(Z^{st})'$ at its special fiber into three pieces: The piece $\mathcal{X}_1$ is the disjoint union of the formal annuli corresponding to the completion of each double point; the piece $\mathcal{X}_2$ is the disjoint union of the formal disks corresponding to the completion of the specialization of each branch point of q; and the piece $\mathcal{X}_3$ is $\mathcal{X}\backslash(\mathcal{X}_1 \cup \mathcal{X}_2)$. Let $\hat{Z}_1$, $\hat{Z}_2$, and $\hat{Z}_3$ be the respective special fibers of $\mathcal{X}_1$, $\mathcal{X}_2$, and $\mathcal{X}_3$. Saïdi's proof shows how to lift the covers $q^{st}|_{\hat{Z}_1}$ and $q^{st}|_{\hat{Z}_3}$ to covers of $\mathcal{X}_1$ and $\mathcal{X}_3$, étale on the generic fiber, after a possible tame extension of L. Now, each connected component $\mathcal{C}_i$ of $\mathcal{X}_2$ is isomorphic to Spf $S[[z_i]]$, where S is the ring of integers of L. The special fiber $\hat{C}_i$ of $\mathcal{C}_i$ is isomorphic to Spec $k[[z_i]]$. The cover $q^{st}|_{\hat{C}_i}$ is given by a disjoint union of identical covers $\hat{D}_i \to \hat{C}_i$, each $\hat{D}_i$ being given by extracting a m_i-th root of z_i, where m_i is the branching index of the branch point of q specializing to $\hat{C}_i$. Since each branch point of q is defined over L, there is a unique lift (over L) of $q^{st}|_{\hat{C}_i}$ to a cover of $\mathcal{C}_i$, étale on the generic fiber outside the appropriate point. Using the arguments of Saïdi's proof, the covers of $\mathcal{X}_1$, $\mathcal{X}_2$, and $\mathcal{X}_3$ patch together uniquely to give a cover of $\mathcal{X}$ defined over a tame extension of L. By Grothendieck's existence theorem, this cover is algebraic and it must be the base change of q^{st}. Thus q^{st} is defined over a tame extension of L, and the claim is proved.

Let M/L be a tame extension such that q^{st} is defined over M. By Lemma 6.1 applied to $q : Y \to Z$ and $\eta : Z \to X$, the field of moduli of f is contained in some tame extension M' of M. Since M' has cohomological dimension 1, it follows [Coombes and Harbater 1985, Proposition 2.5] that f can be defined (as a G-cover) over M'. Furthermore, $G_{M'} \leq G_M$ acts trivially on the special fiber $\overline{Y}$ of Y^{st}. Thus f^{st} is defined over M'. □

Remark 6.3. Suppose $f : Y \to X$ is a G-cover, $N \leq G$ is prime-to-p and normal, and the field of *moduli* of $f' : Y' := Y/N \to X$ is L. One can ask if this implies that the field of moduli of f is a tame extension of L (Proposition 6.2 is the analogous statement for the minimal field of definition of the stable model). If the answer to this question is yes, then some of the proofs in Section 7 would be much easier. Unfortunately, I believe the answer is no.

7. Proof of the main theorem

In this section, we will prove Theorem 1.3. Throughout Section 7, if $G \cong \mathbb{Z}/p^n \rtimes \mathbb{Z}/m$ and $p \nmid m$, then Q_i $(0 \leq i \leq n)$ is the unique subgroup of order p^i.

Let $f : Y \to X = \mathbb{P}^1$ be a three-point Galois cover defined over $\overline{\mathbb{Q}}$. Our first step is to reduce to a local problem, which is the content of Proposition 7.1. Let $\mathbb{Q}_p^{ur}$ be the *completion* of the maximal unramified extension of $\mathbb{Q}$. For an embedding $\iota : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}_p^{ur}}$, let f_ι be the base change of f to $\overline{\mathbb{Q}_p^{ur}}$ via ι. The following proposition shows that, for the purposes of Theorem 1.3, we need only consider covers defined over $\overline{\mathbb{Q}_p^{ur}}$.

Proposition 7.1. *Let K_{gl} be the field of moduli of f (with respect to $\mathbb{Q}$) and let $K_{\mathrm{loc},\iota}$ be the field of moduli of f_ι with respect to $\mathbb{Q}_p^{ur}$. Fix $n \geq 0$ and suppose that for all embeddings ι, the n-th higher ramification groups of the Galois closure $L_{\mathrm{loc},\iota}$ of $K_{\mathrm{loc},\iota}/\mathbb{Q}_p^{ur}$ for the upper numbering vanish. Then all the n-th higher ramification groups of the Galois closure L_{gl} of $K_{gl}/\mathbb{Q}$ above p for the upper numbering vanish.*

Proof. Pick a prime q of L_{gl} above p. We will show that the n-th higher ramification groups at q vanish. Choose a place r of $\overline{\mathbb{Q}}$ above q. Then r gives rise to an embedding $\iota_r : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}_p^{ur}}$ preserving the higher ramification filtrations at r for the upper numbering (and the lower numbering). Specifically, if $L/\mathbb{Q}_p^{ur}$ is a finite extension such that the n-th higher ramification group for the upper numbering vanishes, then the n-th higher ramification group for the upper numbering vanishes for $\iota_r^{-1}(L)/\mathbb{Q}$ at the unique prime of $\iota_r^{-1}(L)$ below r. By assumption, the n-th higher ramification group for the upper numbering vanishes for $L_{\mathrm{loc},\iota_r}/\mathbb{Q}_p^{ur}$. Also, the field $L' := \iota_r^{-1}(L_{\mathrm{loc},\iota_r})$ is Galois over $\mathbb{Q}$. So if $K_{gl} \subseteq L'$, then $L_{gl} \subseteq L'$. We know the n-th higher ramification groups for $L'/\mathbb{Q}$ vanish. We are thus reduced to showing that $K_{gl} \subseteq L'$.

Pick $\sigma \in G_{L'}$. Then σ extends by continuity to a unique automorphism τ in $G_{L_{\mathrm{loc},\iota_r}}$. By the definition of a field of moduli, $f_{\iota_r}^\tau \cong f_{\iota_r}$. But then $f^\sigma \cong f$. By the definition of a field of moduli, $K_{gl} \subseteq L'$. □

So, in order to prove Theorem 1.3, we can consider three-point covers defined over $\overline{\mathbb{Q}_p^{ur}}$. In fact, we generalize slightly, and consider three-point covers defined over algebraic closures of complete mixed characteristic discrete valuation fields with algebraically closed residue fields. In particular, throughout this section, K_0 is the fraction field of the ring R_0 of Witt vectors over k. On all extensions of K_0, we normalize the valuation v so that $v(p) = 1$. Also, write $K_n := K_0(\zeta_{p^n})$, with valuation ring R_n (here ζ_{p^n} is a primitive n-th root of unity). Let G be a finite, p-solvable group with a cyclic p-Sylow subgroup P of order p^n. We assume $f : Y \to X = \mathbb{P}^1$ is a three-point G-Galois cover of curves, branched at 0, 1, and ∞, *a priori* defined over some finite extension K/K_0. Since K has cohomological dimension 1, the field of moduli of f relative to K_0 is the same as the minimal field of definition of f that is an extension of K_0 [Coombes and Harbater 1985, Proposition 2.5]. We will therefore go back and forth between fields of moduli and fields of definition without further notice. Our default smooth model X_R of X is always the unique one such that the specializations of 0, 1, and ∞ do not collide on the special fiber. As in Section 4, the stable model of f is $f^{st} : Y^{st} \to X^{st}$ and the stable reduction is $\bar{f} : \overline{Y} \to \overline{X}$. The original component of $\overline{X}$ will be denoted $\overline{X}_0$.

We will first prove Theorem 1.3 in the case that $G \cong \mathbb{Z}/p^n \rtimes \mathbb{Z}/m_G$. The cases $m_G > 1$ and $m_G = 1$ have quite different flavors, and we deal with them separately. We in fact determine more than we need for Theorem 1.3; namely, we determine bounds on the higher ramification filtrations of the extension K^{st}/K_0, where K^{st} is the minimal field of definition of the stable model of f. In Section 7C, we will generalize to the p-solvable case.

7A. *The case where $G \cong \mathbb{Z}/p^n \rtimes \mathbb{Z}/m_G$, $m_G > 1$.* Let $G \cong \mathbb{Z}/p^n \rtimes \mathbb{Z}/m$ be such that the conjugation action of $\mathbb{Z}/m$ is faithful (note that this implies $m = m_G$). We will show that the field of moduli with respect to K_0 of f as a *mere* cover is in fact K_0. Then, we will show that its field of moduli with respect to K_0 as a G-cover is contained in K_n. Lastly, we will show that its stable model can be defined over a tame extension of K_n.

Let $\chi : \mathbb{Z}/m \to \mathbb{F}_p^\times$ correspond to the conjugation action of $\mathbb{Z}/m$ on any order p subquotient of $\mathbb{Z}/p^n$. Now, there is an intermediate $\mathbb{Z}/m$-cover $\eta : Z \to X$ where $Z = Y/Q_n$. If $q : Y \to Z$ is the quotient map, then $f = \eta \circ q$. Because it will be easier for our purposes here, let us assume that the three branch points of f are $x_1, x_2, x_3 \in R_0$ and that they have pairwise distinct reduction to k (in particular, none is ∞). Since the m-th roots of unity are contained in K_0, the cover η can be given birationally by the equation $z^m = (x - x_1)^{a_1}(x - x_2)^{a_2}(x - x_3)^{a_3}$ with $0 \le a_i < m$ for all $i \in \{1, 2, 3\}$, where $a_1 + a_2 + a_3 \equiv 0 \pmod{m}$ and not all $a_i \equiv 0 \pmod{m}$. Since g^*z/z is an m-th root of unity, we can and do choose z so that $g^*z = \chi(g)z$ for any $g \in \mathbb{Z}/m$. We know from Lemma 5.3 that the original component $\overline{X}_0$ is a p^n-component, and all of the deformation data above $\overline{X}_0$ are multiplicative.

Consider the $\mathbb{Z}/p \rtimes \mathbb{Z}/m$-cover $f' : Y' \to X$, where $Y' = Y/Q_{n-1}$. The stable reduction $\overline{f}' : \overline{Y}' \to \overline{X}'$ of this cover has a multiplicative deformation datum (ω, χ) over the original component $\overline{X}_0$. For all $x \in \overline{X}_0$, recall that (h_x, m_x) is the signature of the deformation datum at x, and $\sigma_x = h_x/m_x$ (see Section 5). Also, since there are no new tails (Proposition 4.15), it follows from [Wewers 2003b, Theorem 2, p. 992] that the stable reduction $\overline{X}'$ consists only of the original component $\overline{X}_0$ along with a primitive étale tail $\overline{X}_i$ for each branch point x_i of f (or f') with prime-to-p ramification index. The tail $\overline{X}_i$ intersects $\overline{X}_0$ at the specialization of x_i to $\overline{X}_0$.

Proposition 7.2. *For $i = 1, 2, 3$, let $\overline{x}_i$ be the specialization of x_i to $\overline{X}_0$. For short, write h_i, m_i, and σ_i for $h_{\overline{x}_i}$, $m_{\overline{x}_i}$, and $\sigma_{\overline{x}_i}$.*

(i) *For $i = 1, 2, 3$, $h_i \equiv a_i/\gcd(m, a_i) \pmod{m_i}$.*

(ii) *In fact, the h_i depend only on the $\mathbb{Z}/m$-cover $\eta : Z \to X$.*

Proof. (i) (cf. [Wewers 2003a, Proposition 2.5]): Let $\overline{Z}_0$ be the unique irreducible component lying above $\overline{X}_0$, and suppose that $\overline{z}_i \in \overline{Z}_0$ lies above $\overline{x}_i$. Let t_i be the formal parameter at $\bar{z}_i$ given by $z^{\alpha}(x-x_i)^{\beta}$, where $\alpha a_i + \beta m = \gcd(m, a_i)$. Then

$$\omega = \Big(c_0 t_i^{h_i - 1} + \sum_{j=1}^{\infty} c_j t_i^{h_i - 1 + j}\Big) dt_i$$

in a formal neighborhood of $\bar{z}_i$. Recall that, for $g \in \mathbb{Z}/m$, $g^* z = \chi(g) z$ and $g^* \omega = \chi(g)\omega$. Then

$$\chi(g) = \frac{g^*\omega}{\omega} = \Big(\frac{g^* t_i}{t_i}\Big)^{h_i} = \Big(\frac{g^* z}{z}\Big)^{\alpha h_i} = \chi(g^{\alpha h_i}).$$

So $\alpha h_i \equiv 1 \pmod m$. It follows that $h_i \gcd(m, a_i) \equiv h_i(\alpha a_i + \beta m) \equiv a_i \pmod m$. It is clear that the ramification index m_i at $\bar{x}_i$ is $m/\gcd(m, a_i)$. Dividing

$$h_i \gcd(m, a_i) \equiv a_i \pmod m$$

by $\gcd(m, a_i)$ yields (i).

(ii) Since we know the congruence class of h_i modulo m_i, it follows that the fractional part $\langle \sigma_i \rangle$ of σ_i is determined by $\eta : Z \to X$. But if x_i corresponds to a primitive tail, the vanishing cycles formula (4-1) shows that $0 < \sigma_i < 1$. If x_i corresponds to a wild branch point, then $\sigma_i = 0$. Thus σ_i is determined by $\langle \sigma_i \rangle$, so it is determined by $\eta : Z \to X$. Since $h_i = \sigma_i m_i$, we are done. □

Corollary 7.3. *The differential form ω corresponding to the cover $f' : Y' \to X$ is determined (up to multiplication by an element of $\mathbb{F}_p^{\times}$) by $\eta : Z \to X$.*

Proof. Proposition 7.2 determines the divisor corresponding to ω from $\eta : Z \to X$. Two meromorphic differential forms on a complete curve can have the same divisor only if they differ by a scalar multiple. Also, if ω is logarithmic and $c \in k$, then $c\omega$ is logarithmic if and only if $c \in \mathbb{F}_p$ by basic properties of the Cartier operator; see, for instance, [Wewers 2003a, p. 136]. □

We will now show that $\eta : Z \to X$ determines not only the differential form ω, but also the entire cover $f : Y \to X$ as a mere cover. This is the key lemma of this section. We will prove it in several stages using an induction.

Lemma 7.4. *Assume $m > 1$.*

(i) *If $f : Y \to X$ is a three-point $\mathbb{Z}/p^n \rtimes \mathbb{Z}/m$-cover (with faithful conjugation action of $\mathbb{Z}/m$ on $\mathbb{Z}/p^n$) defined over some finite extension K/K_0, then it is determined as a mere cover by the map $\eta : Z = Y/Q_n \to X$.*

(ii) *If $f : Y \to X$ is a three-point $\mathbb{Z}/p^n \rtimes \mathbb{Z}/m$-cover (with faithful conjugation action of $\mathbb{Z}/m$ on $\mathbb{Z}/p^n$) defined over some finite extension K/K_0, its field of moduli (as a mere cover) with respect to K_0 is K_0, and f can be defined over K_0 (as a mere cover).*

(iii) *In the situation of part* (ii), *the field of moduli of f (as a $\mathbb{Z}/p^n \rtimes \mathbb{Z}/m$-cover) with respect to K_0 is contained in $K_n = K_0(\zeta_{p^n})$. Thus f can be defined over K_n (as a $\mathbb{Z}/p^n \rtimes \mathbb{Z}/m$-cover).*

Proof. (i) We first assume $n = 1$, so $G \cong \mathbb{Z}/p \rtimes \mathbb{Z}/m$. Write Z^{st} for Y^{st}/Q_1 and $\overline{Z}$ for the special fiber of Z^{st}. We know from Corollary 7.3 that η determines (up to a scalar multiple in $\mathbb{F}_p^\times$) the logarithmic differential form ω that is part of the deformation datum $(\overline{Z}_0, \omega)$ on the irreducible component $\overline{Z}_0$ above $\overline{X}_0$. Let ξ be the generic point of $\overline{Z}_0$. Then ω is of the form $d\bar{u}/\bar{u}$, where $\bar{u} \in k(\overline{Z}_0)$ is the reduction of some function $u \in \hat{\mathbb{O}}_{(Z')^{st},\xi}$. Moreover, by [Henrio 2000a, chapitre 5, définition 1.9], we can choose u such that the cover $Y \to Z$ is given birationally by extracting a p-th root of u (viewing $u \in K(Z) \cap \hat{\mathbb{O}}_{(Z')^{st},\xi}$). That is,

$$K(Y) = K(Z)[t]/(t^p - u).$$

We wish to show that knowledge of $d\bar{u}/\bar{u}$ up to a scalar multiple $c \in \mathbb{F}_p^\times$ determines u up to raising to the c-th power, and then possibly multiplication by a p-th power in $K(Z)$ (as this shows $Y' \to X$ is uniquely determined as a mere cover). This is equivalent to showing that knowledge of $d\bar{u}/\bar{u}$ determines u up to a p-th power (that is, that if $d\bar{u}/\bar{u} = d\bar{v}/\bar{v}$, then u/v is a p-th power in $K(Z)$).

Suppose that there exist $u, v \in K(Z) \cap \mathbb{O}_{(Z')^{st},\xi}$ such that $d\bar{u}/\bar{u} = d\bar{v}/\bar{v}$. Then $\bar{u} = \bar{\kappa}\,\bar{v}$, with $\bar{\kappa} \in k(\overline{Z}_0)^p$. Since $\bar{\kappa}$ is a p-th power, it lifts to some p-th power κ in K. Multiplying v by κ, we can assume that $\bar{u} = \bar{v}$. Consider the cover $Y' \to Z$ given birationally by the field extension $K(Y') = K(Z)[t]/(t^p - u/v)$. Since $\bar{u} = \bar{v}$, we have that u/v is congruent to 1 in the residue field of $\mathbb{O}_{(Z')^{st},\xi}$. This means that the cover $Y' \to Z$ cannot have multiplicative reduction; see [Henrio 2000a, chapitre 5, proposition 1.6]. But the cover $Y' \to Z \to X$ is a $\mathbb{Z}/p \rtimes \mathbb{Z}/m$-cover, branched at three points, so it must have multiplicative reduction if the $\mathbb{Z}/p$ part is nontrivial (Lemma 5.3). Thus it is trivial, which means that u/v is a p-th power in $K(Z)$, that is, $u = \phi^p v$ for some $\phi \in K(Z)$. This proves the case $n = 1$.

For $n > 1$, we proceed by induction. We assume that (i) is known for $\mathbb{Z}/p^{n-1} \rtimes \mathbb{Z}/m$-covers. Given $\eta : Z \to X$, we wish to determine $u \in K(Z)^\times/(K(Z)^\times)^{p^n}$ such that $K(Y)$ is given by $K(Z)[t]/(t^{p^n} - u)$. By the induction hypothesis, we know that u is well-determined up to multiplication by a p^{n-1}-st power. Suppose that extracting p^n-th roots of u and v both give $\mathbb{Z}/p^n \rtimes \mathbb{Z}/m$-covers branched at 0, 1, and ∞. Consider the cover $Y' \to Z \to X$ of smooth curves given birationally by $K(Y') = K(Z)[t]/(t^{p^n} - u/v)$. Since u/v is a p^{n-1}-st power in $K(Z)$, this cover splits into a disjoint union of p^{n-1} different $\mathbb{Z}/p \rtimes \mathbb{Z}/m$-covers. By our previous argument, each of these covers can be given by extracting a p-th root of some power of u itself! So $\sqrt[p^{n-1}]{u/v} = u^c w^p$, where $w \in K(Z)$ and $c \in \mathbb{Z}$. Thus $v = u^{1-p^{n-1}c} w^{-p^n}$, which means that extracting p^n-th roots of either u or v gives the same mere cover.

(ii) We know that the cyclic cover η of part (i) is defined over K_0 because we have written it down explicitly. Now, for $\sigma \in G_{K_0}$, f^σ is a $\mathbb{Z}/p^n \rtimes \mathbb{Z}/m$-cover with quotient cover η, branched at 0, 1, and ∞. By part (i), there is only one such (mere) cover, so $f^\sigma \cong f$ as mere covers. So the field of moduli of f as a mere cover with respect to K_0 is K_0. It is also a field of definition by [Coombes and Harbater 1985, Proposition 2.5].

(iii) Since f is defined over K_0 as a mere cover, it is certainly defined over K_n as a mere cover. We thus obtain a homomorphism $h : G_{K_n} \to \mathrm{Out}(G)$, as in Section 2B. By Kummer theory, we can write $K_n(Z) \hookrightarrow K_n(Y)$ as a Kummer extension with Galois action defined over K_n. This means that the image of h acts trivially on $\mathbb{Z}/p^n$. Furthermore, $\eta : Z \to X$ is defined over K_0 as a $\mathbb{Z}/m$-cover. Thus, if $r : \mathrm{Out}(G) \to \mathrm{Out}(\mathbb{Z}/m)$ is the natural map, the image of $r \circ h$ acts trivially on $\mathbb{Z}/m$. But the only automorphisms of G satisfying both of these properties are inner, so h is trivial. This shows that the field of moduli of f with respect to K_0 is K_n. Since K_0 has cohomological dimension 1, we see that $f : Y \to X$ is defined over K_n as a $\mathbb{Z}/p^n \rtimes \mathbb{Z}/m$-cover. □

We know from Lemma 7.4 that f is defined over K_0 as a mere cover and over K_n as a G-cover. Recall from Section 4 that the minimal field of definition of the stable model K^{st} is the fixed field of the subgroup $\Gamma^{st} \le G_{K_0}$ that acts trivially on the stable reduction $\bar{f} : \bar{Y} \to \bar{X}$. Recall also that the action of G_{K_n} centralizes the action of G.

Lemma 7.5. *If $g \in G_{K_n}$ acts on $\bar{Y}$ with order p, then g acts trivially on $\bar{Y}$.*

Proof. First, note that since each tail $\bar{X}_b$ of $\bar{X}$ is primitive (Proposition 4.15), each contains the specialization of a K_0-rational point (which must be fixed by g). As in the proof of Proposition 4.9, g fixes all of $\bar{X}$ pointwise.

There are at least two primitive tails, because, for $G \cong \mathbb{Z}/p^n \rtimes \mathbb{Z}/m$ with $m > 1$ and faithful conjugation action, a three-point G-cover must have at least two branch points with prime-to-p branching index. Since G has trivial center, [Obus 2012, Lemmas 5.4 and 5.8] shows that g acts trivially on $\bar{X}$. □

Proposition 7.6. *Assume $m > 1$. Let $f : Y \to X$ be a three-point G-cover, where $G \cong \mathbb{Z}/p^n \rtimes \mathbb{Z}/m$ (with faithful conjugation action of $\mathbb{Z}/m$ on $\mathbb{Z}/p^n$). Choose a model for f over K_n, as in Lemma 7.4*(iii)*. Then there is a tame extension K^{stab}/K_n such that the stable model $f^{st} : Y^{st} \to X^{st}$ is defined over K^{stab}. In particular, the n-th higher ramification groups for the upper numbering of K^{stab}/K_0 vanish.*

Proof. By Lemma 7.5, no element of G_{K_n} acts with order p on $\bar{Y}$. So the subgroup of G_{K_n} that acts trivially on $\bar{Y}$ has prime-to-p index, and its fixed field K^{stab} is a tame extension of K_n. By [Serre 1979, Corollary to IV, Proposition 18], the n-th higher ramification groups for the upper numbering of the extension K_n/K_0 vanish.

By Lemma 2.2, the n-th higher ramification groups for the upper numbering of K^{stab}/K_0 vanish. □

7B. *The case where $G \cong \mathbb{Z}/p^n$.* Maintaining the notation of this section, we now set $G \cong \mathbb{Z}/p^n$. Finding the field of moduli is easy in this case, but understanding the stable model (which is needed to apply Proposition 6.2) is more difficult.

Proposition 7.7. *The field of moduli of $f : Y \to X$ relative to K_0 is $K_n = K_0(\zeta_{p^n})$.*

Proof. Since the field of moduli of f relative to K_0 is the intersection of all extensions of K_0 which are fields of definition of f, it suffices to show that K_n is the minimal such extension. By Kummer theory, f can be defined over $\overline{K}_0$ birationally by the equation $y^{p^n} = x^a(x-1)^b$ for some integral a and b. The Galois action is generated by $y \mapsto \zeta_{p^n} y$. This cover is clearly defined over K_n as a G-cover.

Since Y is connected, f is totally ramified above at least one of the branch points x_0 (that is, with index p^n). Let $y_0 \in Y$ be the unique point above x_0. Assume f is defined over some finite extension K/K_0 as a G-cover, where Y and X are considered as K-varieties. Then, by [Raynaud 1999, Proposition 4.2.11], the residue field $K(y_0)$ of y_0 contains the p^n-th roots of unity. Since y_0 is totally ramified, $K(y_0) = K(x_0) = K$, and thus $K \supseteq K_n$. So K_n is the minimal extension of K_0 which is a field of definition of f. Thus K_n is the field of moduli of f with respect to K_0. □

In the rest of this section, we analyze the stable model of three-point G-covers $f : Y \to X$ (a complete description, at least in the case $p > 3$, is given in Appendix A). By Kummer theory, f can be given (over $\overline{K}_0$) by an equation of the form $y^{p^n} = cx^a(x-1)^b$ for any $c \in \overline{K}_0$ (note that different values of c might give different models over subfields of $\overline{K}_0$). The ramification indices above 0, 1, and ∞ are $p^{n-v(a)}$, $p^{n-v(b)}$, and $p^{n-v(a+b)}$, respectively. Since Y is connected, we must have that at least two of a, b, and $a+b$ are prime to p. Note that if $p = 2$, then exactly two of a, b, and $a+b$ are prime to p. In all cases, we assume without loss of generality that f is totally ramified above 0 and ∞, and we set s to be such that p^s is the ramification index above 1. Then $v(b) = n - s$.

As in Section 4, write $f^{st} : Y^{st} \to X^{st}$ for the stable model of f, and $\overline{f} : \overline{Y} \to \overline{X}$ for the stable reduction.

Lemma 7.8 (cf. [Coleman and McCallum 1988, Section 3]). *The stable reduction $\overline{X}$ (over $\overline{K}_0$) contains exactly one étale tail $\overline{X}_b$, which is a new tail with effective invariant $\sigma_b = 2$.*

If $p > 3$, or $p = 3$ and either $s > 1$ or $s = n = 1$, set $d = a/(a+b)$. If $p = 3$ and $n > s = 1$, set

$$d = \frac{a}{a+b} + \frac{\sqrt[3]{3^{2n+1}\binom{b}{3}}}{a+b},$$

where we choose any cube root. If $p = 2$, *set*

$$d = \frac{a}{a+b} + \frac{\sqrt{2^n bi}}{(a+b)^2},$$

where $i^2 = -1$ *and the square root sign represents either square root.*

Then $\overline{X}_b$ *corresponds to the disk of radius* $|e|$ *centered at* d, *where* $e \in \overline{K}_0$ *has* $v(e) = \frac{1}{2}(2n - s + 1/(p-1))$.

Proof. By the Hasse–Arf theorem, the effective ramification invariant σ of any étale tail is an integer. Clearly there are no primitive tails, as there are no branch points with prime-to-p branching index. By Lemma 4.13, any new tail has $\sigma \geq 2$. By the vanishing cycles formula (4-1), there is exactly one new tail $\overline{X}_b$ and its invariant σ_b is equal to 2.

We know that f is given by an equation of the form $y^{p^n} = g(x) := cx^a(x-1)^b$, and that any value of c yields f over $\overline{K}_0$. Taking K sufficiently large, we may (and do) assume that $c = d^{-a}(d-1)^{-b}$. Note that, in all cases, $g(d) = 1$, $v(d) = v(a) = 0$, and $v(d-1) = v(b) = n - s$.

Let K be a subfield of $\overline{K}_0$ containing $K_0(\zeta_{p^n}, e, d)$. Let R be the valuation ring of K. Consider the smooth model X'_R of $\mathbb{P}^1_K$ corresponding to the coordinate t, where $x = d + et$. The formal disk D corresponding to the completion of $D_k := X'_k \backslash \{t = \infty\}$ in X'_R is the closed disk of radius 1 centered at $t = 0$, or, equivalently, the disk of radius $|e|$ centered at $x = d$; see Section 2D. Its ring of functions is $R\{t\}$.

In order to calculate the normalization of X'_R in $K(Y)$, we calculate the normalization E of D in the fraction field of

$$R\{t\}[y]/(y^{p^n} - g(x)) = R\{t\}[y]/(y^{p^n} - g(d+et)).$$

Now, $g(d+et) = \sum_{\ell=0}^{a+b} c_\ell t^\ell$, where

$$c_\ell = e^\ell \sum_{j=0}^{\ell} \binom{a}{\ell - j}\binom{b}{j} d^{j-\ell}(d-1)^{-j}. \tag{7-1}$$

If $s = n$ and $\ell \geq 3$, then clearly $v(c_\ell) \geq v(e^\ell) = \frac{\ell}{2}(n + 1/(p-1)) > n + 1/(p-1)$. If $s < n$ and $\ell \geq 3$, then the $j = \ell$ term is the term of least valuation in (7-1), and thus it has the same valuation as c_ℓ. We obtain

$$v(c_\ell) = \ell v(e) + v(b) - v(\ell) - \ell(n-s) = n + \frac{1}{p-1} + \frac{\ell - 2}{2}\left(s + \frac{1}{p-1}\right) - v(\ell) \tag{7-2}$$

(unless, of course, $c_\ell = 0$).

Now, assume either that $p > 3$, or if $p = 3$, then $s > 1$ or $s = n = 1$. Then $d = a/(a+b)$. It is easy to check, using (7-2), that $v(c_\ell) > n + 1/(p-1)$ for $\ell \geq 3$. Equation (7-1) shows that $c_0 = 1$, $c_1 = 0$, and $c_2 = (a+b)^3 e^2/(ab)$, which has valuation

$n+1/(p-1)$. Thus we are in the situation (i) of Lemma 3.1 (with $h=2$), and the special fiber E_k of E is a disjoint union of p^{n-1} étale covers of $D_k \cong \mathbb{A}^1_k$. Each of these extends to an Artin–Schreier cover of conductor 2 over $\mathbb{P}^1_k$. By Lemma 2.4, these have genus $(p-1)/2 > 0$. Therefore, by Lemma 4.3, the component $\overline{X}_b$ corresponding to D is included in the stable model. By Lemma 4.7, it is a tail. Since there is only one tail of $\overline{X}$, and it has effective ramification invariant 2, it must correspond to $\overline{X}_b$.

For the cases where either $p=2$, or $p=3$ and $n>s=1$, see Lemma C.2. □

Remark 7.9. The computation of Lemma 7.8 is similar to the relevant parts of [Coleman and McCallum 1988, Section 3], in particular Lemma 3.6 and Case 5 of Theorem 3.18. Our task is simplified because we know from the outset what we are looking for, that is, a new tail with $\sigma_b = 2$.

Corollary 7.10. (i) *If f is totally ramified above $\{0, 1, \infty\}$, then $\overline{X}$ has no inseparable tails.*

(ii) *If $p > 3$ and f is totally ramified above only $\{0, \infty\}$, then if $\overline{X}$ has an inseparable tail, the tail contains the specialization of $x = 1$.*

(iii) *If $p = 3$, suppose f is totally ramified above only $\{0, \infty\}$ and ramified of index 3^s above 1. Then any inseparable tail of $\overline{X}$ not containing the specialization of $x = 1$ is a p^{s-1}-tail (in particular, $s \geq 2$). Furthermore, such a tail corresponds to the disk of radius $|e'|$ centered at d', where $v(e') = n - s + \frac{2}{3}$ and*

$$d' = \frac{a}{a+b} + \frac{\sqrt[3]{3^{2(n-s+1)+1}\binom{b}{3}}}{a+b}.$$

(iv) *If $p = 2$, suppose f is totally ramified above only $\{0, \infty\}$ and ramified of index 2^s above 1. Then any inseparable tail of $\overline{X}$ not containing the specialization of $x = 1$ is a p^j-tail for some $j < s$. Furthermore, such a tail corresponds to the disk of radius $|e_j|$ centered at d_j, where $v(e_j) = \frac{1}{2}(2n - s - j + 1)$,*

$$d_j = \frac{a}{a+b} + \frac{\sqrt{2^{n-j}bi}}{(a+b)^2},$$

and $i^2 = -1$.

Proof. (i) Let $d = a/(a+b)$ as in Lemma 7.8. Suppose there is an inseparable p^j-tail $\overline{X}_c \subset \overline{X}$ (we know $j < n$ by Lemma 4.8). By Proposition 4.6, $\overline{X}_c$ is a new inseparable tail. By Lemma 4.14, $\overline{X}_c$ is a new (étale) tail of the stable reduction of $Y/Q_j \to X$. Its corresponding disk must contain d, by Lemma 7.8 (substituting $n-j$ for n in the statement). But this is absurd, because the disks corresponding to $\overline{X}_c$ and the étale tail $\overline{X}_b$ are disjoint.

(ii) Assume f is ramified above $x = 1$ of index p^s, $s < n$. Let $\overline{X}_c$ be a new inseparable p^j-tail of $\overline{X}$, and σ_c its ramification invariant. By Lemma 4.13, $\sigma_c > 1$.

Let $\overline{Y}_c$ be a component of $\overline{Y}$ lying above $\overline{X}_c$. If $j \geq s$, we see that $Y/Q_j \to X$ is branched at two points, and thus has genus zero. Since $Q_j \leq I_{\overline{Y}_c}$, the constancy of arithmetic genus in flat families shows that $\overline{Y}_c$ has genus zero. But any component $\overline{Y}_c$ above $\overline{X}_c$ must have genus greater than 1; see [Raynaud 1999, Lemme 1.1.6]. This is a contradiction.

Now suppose $j < s$. Then $Y/Q_j \to X$ is a three-point cover. So we obtain the same contradiction as in (i).

(iii) As in (ii), we see that any new inseparable p^j-tail $\overline{X}_c$ of $\overline{X}$ must satisfy $j < s$. In particular, $s \geq 2$. As in (i), $\overline{X}_c$ must correspond to the same disk as the new étale tail of the stable reduction of $f' : Y/Q_j \to X$, but the disk must not contain the specialization of $d = a/(a+b)$. By Lemma 7.8, this can only happen if f' has degree greater than 3, but is branched of index 3 above 1. Thus $j = s-1$. Thus f' is a $\mathbb{Z}/p^{n-s+1}$-cover. We conclude using Lemma 7.8, replacing n by $n-s+1$ and s by 1.

(iv) Let j and $\overline{X}_c$ be as in part (ii). As in (ii), we may assume $j < s$. As in (i), $\overline{X}_c$ is the new étale tail of the stable reduction of $f' : Y/Q_j \to X$. The cover f' is a $\mathbb{Z}/p^{n-j}$-cover totally ramified above 0 and ∞ and ramified of index 2^{s-j} above 1. We conclude using Lemma 7.8, replacing n by $n-j$ and s by $s-j$. □

Corollary 7.11. *In cases* (*ii*), (*iii*), *and* (*iv*) *above*, $x = 1$ *in fact specializes to an inseparable tail.*

Proof. If $x = 1$ specializes to a component $\overline{W}$ that is not a tail, then there exists a tail $\overline{X}_c$ lying outward from $\overline{W}$. If $\overline{X}_c$ is a p^i-tail, then Lemma 4.8 and Proposition 4.6 show that $i < s$. By Lemma 4.14, $\overline{X}_c$ is an étale tail of the stable model of $Y/Q_i \to X$. As $i < s$, this is still a three-point cover. So we may assume (still, for the sake of contradiction) that there is an étale tail lying outwards from the specialization of $x = 1$. By Lemma 7.8, we have $\sigma_c = 2$, and $\overline{X}_c$ is the only étale tail of f.

Let e_0 and e_1 be the edge of $\mathcal{G}'$ with source corresponding to $\overline{W}$ and target corresponding to the branch point $x = 1$ and, respectively, the immediately following component of $\overline{X}$ in the direction of $\overline{X}_c$. Then $\sigma_{e_1}^{\text{eff}} = 2$ by Lemma 5.7, and $\sigma_{e_0}^{\text{eff}} = 0$. The deformation data above $\overline{W}$ are multiplicative and identical, and σ^{eff} is given by a weighted average of invariants. So for any deformation datum ω above $\overline{W}$, we have $\sigma_{x_0} = 0$ and $\sigma_{x_1} = 2$, where the points x_0 and x_1 correspond to e_0 and e_1, respectively. Furthermore, $\sigma_x = 1$ for all x other than x_0, x_1, and the intersection point x_2 of $\overline{W}$ and the next most inward component.

Now, by a similar argument as in the first part of the proof of Corollary 7.10(ii), any component of $\overline{Y}$ above $\overline{W}$ must have genus zero. Thus ω has degree -2. Since ω has simple poles above x_0 and simple zeroes above x_1, it must have a double pole above x_2. But a logarithmic differential form cannot have a double pole. This is a contradiction. □

Remark 7.12. In Corollary 7.10(iv), there in fact does exist an inseparable p^j-tail for each $1 \leq j < s$. Each of these is the same as the unique new tail of the cover $Y/Q_j \to X$. We omit the details.

We give the major result of this section:

Proposition 7.13. *Assume $G = \mathbb{Z}/p^n$, $n \geq 1$, and $f : Y \to X$ is a three-point G-cover defined over $\overline{K}_0$, totally ramified above $\{0, \infty\}$ and ramified of index p^s above 1. Suppose f is given over $\overline{K}_0$ by $y^{p^n} = x^a(x-1)^b$.*

(i) *If $s = n$ (that is, f is totally ramified above 1), then there is a model for f defined over $K_n = K_0(\zeta_{p^n})$ whose stable model can be defined over a tame extension K^{stab}/K_n.*

(ii) *If $p > 3$ and $s < n$, then there is a model for f over K_n whose stable model can be defined over a tame extension $K^{\mathrm{stab}}/K_n(\sqrt[p^{n-s}]{a/(a+b)})$.*

(iii) *If $p = 3$ and $1 = s < n$, then there is a model for f over $K_n\left(\sqrt[3]{3^{2n+1}\binom{b}{3}}\right)$ whose stable model can be defined over a tame extension K^{stab} of*

$$K_n\left(\sqrt[3]{3^{2n+1}\binom{b}{3}},\ \sqrt[3^{n-1}]{\frac{a}{a+b}}\right).$$

(iv) *Assume $p = 3$ and $1 < s < n$. Let*

$$d' = \frac{a}{a+b} + \frac{\sqrt[3]{3^{(2(n-s+1)+1)}\binom{b}{3}}}{a+b}.$$

Then there is a model for f over K_n whose stable model can be defined over a tame extension K^{stab} of

$$K_n\left(d',\ \sqrt[3^{n-s}]{\frac{a}{a+b}},\ \sqrt[3^{n-s+1}]{\frac{(d')^a(d'-1)^b}{a^ab^b(a+b)^{-(a+b)}}}\right).$$

(v) *Assume $p = 2$. For $0 \leq j < s$, let*

$$d_j = \frac{a}{a+b} + \frac{\sqrt{2^{n-j}bi}}{(a+b)^2},$$

where $i^2 = -1$, and the square root sign represents either square root. Then there is a model for f over K_n whose stable model can defined over a tame extension K^{stab} of

$$K := K_n\left(\sqrt[2^{n-1}]{d_0},\ \sqrt[2^{s-1}]{d_0 - 1},\ \sqrt[2^{n-j}]{d_j},\ \sqrt[2^{s-j}]{d_j - 1}\right)_{1 \leq j < s}.$$

Proof. In each case of the proposition, let d be as in Lemma 7.8. Set

$$c = d^{-a}(d-1)^{-b}.$$

The model of f we will use will always be the one given by the equation $y^{p^n} = cx^a(x-1)^b$. In all cases, there is a unique étale tail $\overline{W}$ of $\overline{X}$ containing the specialization of $x = d$, which is a smooth point of $\overline{X}$. Furthermore, the points in the fiber of f above $x = d$ are all K_n-rational.

(i) Since $s = n$, we have $d = a/(a+b)$ and a, b, $a+b$ are prime to p. Our model for f is defined over K_n. By Corollary 7.10, the tail $\overline{W}$ is the unique tail of $\overline{X}$. Since the point $x = d$ and all points in the fiber of f above $x = d$ are K_n-rational, their specializations are fixed by G_{K_n}. By Proposition 4.9, the stable model of f is defined over a tame extension of K_n.

(ii) and (iii): By Corollary 7.10(ii, iii), there is a unique inseparable tail $\overline{W}'$ containing the specialization of $x = 1$ (to a smooth point of $\overline{X}$). Now, consider Y/Q_s (note that Q_s is the inertia group above $x = 1$). This is a cover of X given birationally by the equation $y^{p^{n-s}} = cx^a(x-1)^b$. Since p^{n-s} exactly divides b, we set $y' = y/(x-1)^{b/p^{(n-s)}}$. The new equation $(y')^{p^{n-s}} = cx^a$ shows that the points above $x = 1$ in Y/Q_s are defined over the field $K_{n-s}(c, \sqrt[p^{n-s}]{c}) = K_{n-s}(d, \sqrt[p^{n-s}]{d})$, and their specializations are thus fixed by its absolute Galois group. Since the map $Y^{st} \to Y^{st}/Q_s$ is radicial above $\overline{W}'$, all points of $\overline{Y}$ above the specialization of $x = 1$ are fixed by $G_{K_{n-s}(d, \sqrt[p^{n-s}]{d})}$. By Proposition 4.9, the stable model of f is defined over a tame extension of $K_n(d, \sqrt[p^{n-s}]{d})$.

If $p > 3$ and $s < n$, then $K_n(d, \sqrt[p^{n-s}]{d}) = K_n(\sqrt[p^{n-s}]{a/(a+b)})$, finishing the proof of (ii). If $p = 3$ and $s = 1$, then

$$d = \frac{a}{a+b}\Big(1 + \frac{B}{a}\Big),$$

where $B = \sqrt[3]{3^{2n+1}\binom{b}{3}}$. Since $v(B) = n - \frac{1}{3}$, the binomial theorem shows that $1 + B/a$ is a 3^{n-1}-st power in $K_n(B)$. Thus

$$K_n(d, \sqrt[3^{n-1}]{d}) = K_n\left(\sqrt[3]{3^{2n+1}\tbinom{b}{3}},\ \sqrt[3^{n-1}]{\frac{a}{a+b}}\right),$$

finishing the proof of (iii).

(iv) Here $d = a/(a+b)$, and our model of f is defined over K_n. There is an inseparable tail $\overline{W}'$ containing the specialization of $x = 1$ and a unique new inseparable tail containing the specialization of $x = d'$ by Corollary 7.10(iii). As in parts (ii) and (iii), the fiber of $\bar{f}$ above the specialization of $x = 1$ is pointwise fixed by the absolute Galois group of $K_n(\sqrt[3^{n-s}]{a/(a+b)})$. Likewise, the fiber of $\bar{f}$ above the specialization of $x = d'$ is fixed by the absolute Galois group of $K_n(\sqrt[3^{n-s+1}]{c(d')^a(d'-1)^b})$. By Proposition 4.9, the stable model of f is defined over a tame extension of the compositum of these two fields, which is exactly the field given in part (iv) of the proposition.

(v) In this case, $d = d_0$. Note that $n \geq 2$, as there are no three-point $\mathbb{Z}/2$-covers. One sees that $c = d_0^{-a}(d_0 - 1)^{-b} \in K_n$ (in fact, $c \in K_3$ always, and $c \in K_2$ for $n = 2$). So our model of f is defined over K_n.

By Corollary 7.10(iv) (and Remark 7.12), there is a unique inseparable p^j-tail $\overline{W}_j$ of $\overline{X}$ for each $1 \leq j < s$. Also, there is an inseparable tail containing the specialization of $x = 1$ (even if these inseparable tails did not exist, our proof would still carry through — only our K would overestimate the minimal field of definition of the stable model). Each tail $\overline{W}_j$ contains the specialization of $x = d_j$ to a smooth point of $\overline{X}$.

As in (iv), the fiber of $\bar{f}$ above the specialization of $x = d_j$, for $1 \leq j < s$, is pointwise fixed by G_{L_j}, where

$$L_j = K_n\left(\sqrt[2^{n-j}]{\frac{d_j^a(d_j-1)^b}{d_0^a(d_0-1)^b}}\right).$$

As in (ii) and (iii), the fiber above the specialization of $x = 1$ is pointwise fixed by $G_{L'}$, where

$$L' = K_n\left(\sqrt[2^{n-s}]{d_0^{-a}(d_0 - 1)^{-b}}\right).$$

Keeping in mind that $v(b) = n - s$, we see that K (as defined in the proposition) contains the compositum of L' and all the extensions L_j. We conclude using Proposition 4.9. □

Corollary 7.14. *In each case covered in Proposition 7.13, the n-th higher ramification group of K^{stab}/K_0 for the upper numbering vanishes.*

Proof. We first note that any tame extension of a Galois extension of K_0 is itself Galois over K_0. In case (i) of Proposition 7.13, K^{stab} is contained in a tame extension of K_n. The n-th higher ramification groups for the upper numbering for K_n/K_0 vanish by [Serre 1979, Corollary to IV, Proposition 18]. By Lemma 2.2, the n-th higher ramification groups vanish for K^{stab}/K_0 as well.

For case (ii) of Proposition 7.13, we note that $v(a/(a+b)) = 0$. So

$$K_n(\sqrt[p^{n-s}]{a/(a+b)})/K_0$$

has trivial n-th higher ramification groups for the upper numbering by [Viviani 2004, Theorem 5.8]. We again conclude using Lemma 2.2.

For cases (iii) and (iv) of Proposition 7.13, Lemma C.3 shows that K^{stab} is a tame extension of an extension of K_0 for which the n-th higher ramification groups for the upper numbering vanish. For case (v), this fact is shown by Proposition C.5. We again conclude using Lemma 2.2. □

7C. *The general p-solvable case.* We maintain the notation of earlier subsections.

Proposition 7.15. *Let G be a p-solvable finite group with a cyclic p-Sylow subgroup P of order p^n. If $f : Y \to X$ is a three-point G-cover of $\mathbb{P}^1$ defined over $\overline{K}_0$, then there exists a field extension K'/K_0 such that*:

(i) *The cover f has a model whose stable model is defined over K'.*

(ii) *The n-th higher ramification group of K'/K_0 for the upper numbering vanishes.*

In particular, *if K is the field of moduli of f relative to K_0, then $K \subseteq K'$, so the n-th higher ramification group of K/K_0 for the upper numbering vanishes.*

Proof. By Proposition 2.1, we know that there is a prime-to-p subgroup N such that G/N is of the form $\mathbb{Z}/p^n \rtimes \mathbb{Z}/m_G$. Let $f^{\dagger} : Y^{\dagger} \to X$ be the quotient G/N-cover.

Suppose first that $f^{\dagger}$ is a three-point cover. Then we know from Propositions 7.6 and 7.13, along with Corollary 7.14, that there exists a model of $f^{\dagger}$ whose stable model can be defined over a field K^{stab} such that the n-th higher ramification groups for the upper numbering for K^{stab}/K_0 vanish. Let $\overline{f}^{\dagger} : \overline{Y}^{\dagger} \to \overline{X}^{\dagger}$ be the stable reduction of $f^{\dagger}$. The branch points of $Y \to Y^{\dagger}$ are all ramification points of $f^{\dagger}$, because $f^{\dagger}$ is branched at three points. Thus, by definition, their specializations do not coalesce on $\overline{Y}^{\dagger}$. Since $G_{K^{\text{stab}}}$ acts trivially on $\overline{Y}^{\dagger}$, it permutes the ramification points of $f^{\dagger}$ trivially, and thus these points are defined over K^{stab}. By Proposition 6.2, the stable model f^{st} of f can be defined over a tame extension K'/K^{stab}. By Lemma 2.2, K' satisfies the properties of the proposition.

Now, suppose that $f^{\dagger}$ is branched at fewer than three points. Since $\text{char}(\overline{K}_0) = 0$, the cover $f^{\dagger}$ must be a $\mathbb{Z}/p^n$-cover branched at two points, say (without loss of generality) 0 and ∞. Then the branch points of $Y \to Y^{\dagger}$ include the points of $Y^{\dagger}$ lying over $x = 1$, as well as the ramification points of $f^{\dagger}$. We may assume that $f^{\dagger} : Y^{\dagger} \to X$ is given by the equation $y^{p^n} = x$, which is defined over K_n as a $\mathbb{Z}/p^n$-cover. Then, the points lying above $x = 1$ are also defined over K_n. By Proposition 6.2, we can take K' to be a tame extension of K_n. The n-th higher ramification group of K_n/K_0 for the upper numbering vanishes [Serre 1979, Corollary to IV, Proposition 18]. By Lemma 2.2, the n-th higher ramification group of K'/K_0 for the upper numbering vanishes. □

Theorem 1.3 now follows from Propositions 7.1 and 7.15.

Remark 7.16. The proofs of Propositions 7.6 and 7.13, and Corollary 7.14, which are the main ingredients in the proof of Theorem 1.3, depend on writing down explicit extensions and calculating their higher ramification groups. It would be interesting to find a method to place bounds on the conductor without writing down explicit extensions. Such a method might be more easily generalizable to the non-p-solvable case.

Appendix A. Explicit determination of the stable model of a three-point $\mathbb{Z}/p^n$-cover, $p > 2$

Throughout this appendix, we assume the notations of Section 7B (in particular, that $f : Y \to X$ is given by $y^{p^n} = cx^a(x-1)^b$ for some c, and that d is as in Lemma 7.8). So $G \cong \mathbb{Z}/p^n$, and Q_i is the unique subgroup of order p^i for $0 \le i \le n$. For a three-point G-cover f defined over $\overline{K}_0$, the methods of Section 7B are sufficient to bound the conductor of the field of moduli of f above p. But we can also completely determine the structure of the stable model of f (Propositions A.3, A.4, and A.5). Although this is essentially already done in [Coleman and McCallum 1988, Section 3], we include this appendix for three reasons. First, we compute the stable reduction of the cover f, as opposed to the curve Y. Second, we have fewer restrictions than Coleman in the case $p = 3$ (we allow not only covers with full ramification above all three branch points, but also covers with ramification index 3 above one of the branch points). Most importantly, our proof requires significantly less computation and guesswork, and takes advantage of the vanishing cycles formula as well as the effective different (Definition 5.9). Indeed, the majority of the computation required is already encapsulated in Lemma 7.8.

While it would be a somewhat tedious calculation, our proof can be adapted to the case of all cyclic three-point covers without using new techniques. However, for simplicity, we assume throughout this appendix that either $p > 3$, or that $p = 3$ and either f is totally ramified above three points, or f is totally ramified above two points and ramified of index 3 above the third.

Lemma A.1. *The stable reduction $\overline{X}$ cannot have a p^i-component intersecting a p^{i+j}-component for $j \ge 2$.*

Proof. Let $\overline{X}_c$ be such a p^i-component. Then, a calculation with the Hurwitz formula shows that the genus of any component $\overline{Y}_c$ above $\overline{X}_c$ is greater than 0. By Lemma 4.3, $\overline{X}_c$ is a component of the stable reduction of the cover $f' : Y/Q_i \to X$. It is étale, and thus a tail by Lemma 4.7. Let σ_c be its effective ramification invariant. By [Obus 2012, Lemma 4.2], $\sigma_c \ge p > 2$. But this contradicts the vanishing cycles formula (4-1). □

Lemma A.2. *Suppose $\overline{W}$ is a p^i-component of $\overline{X}$ that does not contain the specialization of a branch point of f and does not intersect a p^j-component with $j > i$. Then $\overline{W}$ intersects at least three other components.*

Proof. Let $\overline{V}$ be an irreducible component of $\overline{Y}$ lying above $\overline{W}$. Let $\overline{V}'$ be the smooth, proper curve with function field $k(\overline{V})^{p^i}$. Then f^{st} induces a natural map $\alpha : \overline{V}' \to \overline{W}$. By Proposition 4.4(i, ii), this map is tamely ramified and is branched only at points where $\overline{W}$ intersects another component. If there are only two such points, then α is totally ramified, and $\overline{V}'$ (and thus $\overline{V}$) has genus zero. This violates the three-point condition of the stable model. $\square$

We now give the structure of the stable reduction when f has three totally ramified points.

Proposition A.3. *Suppose that f is totally ramified above all three branch points. Then $\overline{X}$ is a chain, with one p^{n-i}-component $\overline{X}_i$ for each i for $0 \le i \le n$ ($\overline{X}_0$ is the original component). For each $i > 0$, the component $\overline{X}_{n-i}$ corresponds to the closed disk of radius $p^{-(1/2)(i+1/(p-1))}$ centered at $d = a/(a+b)$.*

Proof. We know from Lemma 7.8 and Corollary 7.10 that $\overline{X}$ has only one tail, so it must be a chain. The original component contains the specializations of the branch points, so it must be a p^n-component. By Lemma A.1, there must be a p^{n-i}-component for each i, $0 \le i \le n$. Also, by Lemma A.2, there cannot be two intersecting components $\overline{W} \prec \overline{W}'$ of $\overline{X}$ with the same size inertia groups. Since $\overline{f}$ is monotonic by Proposition 4.5, there must be exactly one p^i-component of $\overline{X}$ for each $0 \le i \le n$.

It remains to show that the disks are as claimed. For $i = n$, this follows from Lemma 7.8. For $i < n$, consider the cover $Y/Q_{n-i} \to X$. The stable model of this cover is a contraction of $Y^{st}/Q_{n-i} \to X^{st}$. By Lemma 7.8 (using i in place of n), the stable reduction of $Y/Q_{n-i} \to X$ has a new étale tail corresponding to a closed disk centered at d with radius $p^{-(1/2)(i+1/(p-1))}$. Thus $\overline{X}$ also contains such a component. This is true for every i, proving the proposition. $\square$

Things are more complicated when f has only two totally ramified points:

Proposition A.4. *Suppose that f is totally ramified above 0 and ∞, and ramified of index p^s above 1, for some $0 < s < n$. If $p = 3$, assume further that $s = 1$. Then the augmented dual graph $\mathcal{G}'$ of the stable reduction of $\overline{X}$ is as in Figure 1.*

In particular, the original component $\overline{X}_0$ is a p^n-component, as labeled in Figure 1. For $s+1 \le i < n$, $\overline{X}_{n-i}$ is a p^i-component corresponding to the disk of

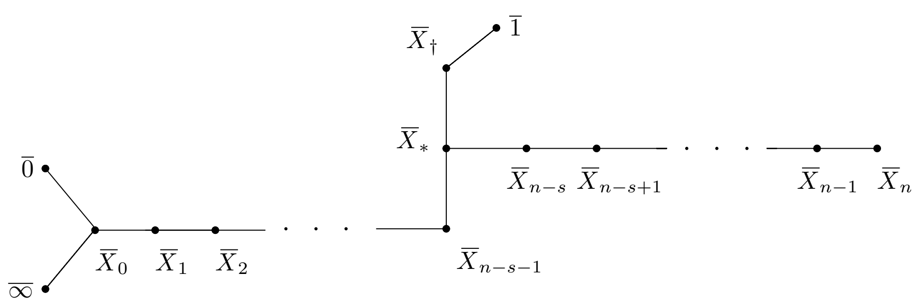

Figure 1. The augmented dual graph $\mathcal{G}'$ of the stable reduction of a three-point $\mathbb{Z}/p^n$-cover with two totally ramified points, $p \neq 2$.

radius $p^{-(1/2)(n-i+1/(p-1))}$ *centered at* d. *For* $0 \leq i \leq s$, $\overline{X}_{n-i}$ *is a* p^i*-component corresponding to the disk of radius* $p^{-(1/2)(2n-s-i+1/(p-1))}$ *centered at* d. *The component* $\overline{X}_*$ *is a* p^{s+1}*-component corresponding to the disk of radius* $p^{-(n-s)}$ *centered at* d. *The component* $\overline{X}_\dagger$ *is a* p^s*-component corresponding to the disk of radius* $p^{-(n-s+1/(p-1))}$ *centered at* 1. *The vertices corresponding to* 0, 1, *and* ∞ *are marked as* $\overline{0}$, $\overline{1}$, *and* $\overline{\infty}$.

Proof. Recall that $v(1-d) = v(b) = n-s$ so long as $p \geq 3$. By Corollary 7.11 and Lemma 7.8, $\overline{X}$ contains exactly two tails: an inseparable tail $\overline{X}_\dagger$ containing the specialization of $x = 1$, and an étale tail $\overline{X}_n$ containing the specialization of d. There must be a component of $\overline{X}$ "separating" 1 and d, that is, corresponding to the disk centered at d (equivalently, 1) of radius $|1-d| = p^{-(n-s)}$. Call this component $\overline{X}_*$. Then $\overline{X}$ looks like a chain from the original component $\overline{X}_0$ to $\overline{X}_*$ followed by two chains: one going out to $\overline{X}_\dagger$ and one going out to $\overline{X}_n$.

Let us first discuss the component $\overline{X}_*$. Consider the cover $f' : Y^{st}/Q_s \to X^{st}$. For any edge e of $\mathscr{G}'$ corresponding to a singular point on $\overline{X}$, we will take $(\sigma_e^{\mathrm{eff}})'$ to mean the effective invariant for the cover f'. Now, the generic fiber of f' is a cover branched at two points, so Y^{st}/Q_s has genus 0 fibers. By Lemma 2.4, any tail $\overline{X}_b$ for the blow-down of the special fiber $\overline{f}'$ of f' to a stable curve must have $\sigma_b = 1$. Lemma 5.7 shows that if $s(e) \prec t(e)$, then $(\sigma_e^{\mathrm{eff}})' = 1$. Since the deformation data above $\overline{X}_0$ are multiplicative, the effective different $(\delta^{\mathrm{eff}})'$ for f' above $\overline{X}_0$ is $n-s+1/(p-1)$. So above $\overline{X}_*$ it is

$$n - s + \frac{1}{p-1} - (n-s) = \frac{1}{p-1} > 0$$

by Lemma 5.10 applied to each of the singular points between $\overline{X}_0$ and $\overline{X}_*$ in succession. This means that $\overline{X}_*$ is an inseparable component for f', which means that it is at least a p^{s+1}-component for f.

Next, we examine the part of $\overline{X}$ between $\overline{X}_0$ and $\overline{X}_*$. By Lemma A.1, there must be a p^i-component of $\overline{X}$ for each i such that $s+1 \leq i \leq n$. Then if we take $f_i' : Y^{st}/Q_i \to X^{st}$, the effective different for f_i' above $\overline{X}_0$ is $n-i+1/(p-1)$. As in the previous paragraph, Lemma 5.10 shows that above a component corresponding to the closed disk of radius $p^{-(n-i+1/(p-1))}$ centered at d, the effective different for f_i' will be 0. This means that this component is the innermost p^i-component. In Figure 1, we label this component $\overline{X}_{n-i}$. In particular, the p^{s+1}-component $\overline{X}_{n-s-1}$ corresponds to the closed disk of radius $p^{-(n-s-1+1/(p-1))}$ around d. Note that $\overline{X}_*$ corresponds to the closed disk of radius $p^{-(n-s)}$ around d, and thus lies outward from $\overline{X}_{n-s-1}$. By monotonicity, $\overline{X}_*$ is a p^{s+1}-component. By Lemma A.2, $\overline{X}_*$ intersects $\overline{X}_{n-s-1}$, and for $s+1 < i \leq n$, there is exactly one p^i-component, namely $\overline{X}_{n-i}$. So the part of $\overline{X}$ between $\overline{X}_0$ and $\overline{X}_*$ is as in Figure 1, and radii of the corresponding disks are as in the proposition.

Now, let us examine the part of $\overline{X}$ between $\overline{X}_*$ and $\overline{X}_\dagger$. We have seen that $\overline{X}_*$ is a p^{s+1}-component, and $\overline{X}_\dagger$ is a p^s-component by Proposition 4.6. So, by Lemma A.2, this part of $\overline{X}$ consists only of these two components. Recall that if we quotient out Y^{st} by Q_s, the effective different above $\overline{X}_*$ is $1/(p-1)$. Also, recall that the effective invariant $(\sigma_e^{\mathrm{eff}})'$ for $s(e)$, $t(e)$ corresponding to $\overline{X}_*$, $\overline{X}_\dagger$ is 1. So by Lemma 5.10, the épaisseur of this annulus is $1/(p-1)$, and $\overline{X}_\dagger$ corresponds to the disk of radius $p^{-(n-s+1/(p-1))}$ centered at 1.

Lastly, let us examine the part of $\overline{X}$ between $\overline{X}_*$ and the new tail $\overline{X}_n$. We know there must be a p^i-component for each i, $0 \leq i \leq s+1$. This component must be unique, by Lemma A.2. These components are labeled $\overline{X}_{n-i}$ in Figure 1 (with the exception of $\overline{X}_*$, which corresponds to $i = s+1$). We calculate the radius of the closed disk corresponding to each $\overline{X}_{n-i}$. For $i = s$, the radius is $p^{-(n-s+1/(p-1))}$ for the same reasons as for $\overline{X}_\dagger$. For $i = 0$, we already know from Lemma 7.8 that the radius is $p^{-(1/2)(2n-s+1/(p-1))}$. For $1 \leq i \leq s-1$, we consider the cover $Y/Q_i \to X$. The stable model of this cover is a contraction of $Y^{st}/Q_i \to X^{st}$. Since $Y/Q_i \to X$ is still a three-point cover, we can use Lemma 7.8 (with $n-i$ and $s-i$ in place of n and s) to obtain that the stable reduction of $Y/Q_i \to X$ has a new tail corresponding to a closed disk centered at d with radius $p^{-(1/2)(2n-s-i+1/(p-1))}$. This is the component $\overline{X}_{n-i}$. □

Propositions A.3 and A.4 give us the entire structure of the stable reduction $\overline{X}$. The following proposition gives us the structure of $\overline{Y}$.

Proposition A.5. *Suppose we are in the situation of either Proposition A.3 or A.4. If $\overline{W}$ is a p^i-component of $\overline{X}$ which does not intersect a p^{i+1}-component, then $\overline{f}^{-1}(\overline{W})$ consists of p^{n-i} connected components, each of which is a genus zero radicial extension of $\overline{W}$. If $\overline{W}$ borders a p^{i+1}-component $\overline{W}'$, then $\overline{f}^{-1}(\overline{W})$ consists of p^{n-i-1} connected components, each a radicial extension of an Artin–Schreier cover of $\overline{W}$, branched of order p at the point of intersection w of $\overline{W}$ and $\overline{W}'$. The conductor of this cover at its unique ramification point is* 2, *unless we are in the situation of Proposition A.4 and $i \geq s$, in which case the conductor is* 1.

The rest of the structure of $\overline{Y}$ is determined by the fact that $\overline{Y}$ is tree-like (*that is, the dual graph of its irreducible components is a tree*).

Proof. That $\overline{Y}$ is tree-like follows from [Raynaud 1990, théorème 1]. This means that any two irreducible components of $\overline{Y}$ can intersect at at most one point. Everything else except the statement about the conductors follows from Proposition 4.4, Lemma A.1, and the fact that if H is a cyclic p-group, then an H-Galois cover of $\mathbb{P}^1$ branched at one point with inertia groups $\mathbb{Z}/p$ must, in fact, be a $\mathbb{Z}/p$-cover. We omit the details.

For the remainder of the proof, let $\overline{W}$ be a p^i-component intersecting a p^{i+1}-component $\overline{W}'$.

Suppose we are in the situation of Proposition A.4 and $i \geq s$. Then $Y/Q_i \to X$ is branched at two points, so Y has genus zero. So any component of the special fiber of Y^{st}/Q_i must also have genus zero. Since Q_i acts trivially above $\overline{W}$, every component above $\overline{W}$ must have genus zero. If such a component is a radicial extension of an Artin–Schreier cover, then Lemma 2.4 shows that the Artin–Schreier cover must have conductor 1.

Now, suppose that f has three totally ramified points or that we are in the situation of Proposition A.4 and $i < s$. Then $\overline{W}$ is the unique p^i-component of $\overline{X}$ (Propositions A.3 and A.4), and is thus the unique étale tail of the stable reduction of the three-point cover $f' : Y/Q_i \to X$. By Lemma 7.8, the irreducible components above $\overline{W}$ in the stable reduction of $f' : Y/Q_i \to X$ are Artin–Schreier covers with conductor 2. Since $\overline{W}$ is a p^i-component, the irreducible components of $\overline{Y}$ above $\overline{W}$ are radicial extensions of Artin–Schreier covers with conductor 2. □

Appendix B. Composition series of groups with cyclic p-Sylow subgroup

In this appendix, we prove Proposition B.2, which shows that a finite, non-p-solvable group with cyclic p-Sylow subgroup has a unique composition factor with order divisible by p. Before we prove Proposition B.2, we prove a lemma. Our proof depends on the classification of finite simple groups.

Lemma B.1. *Let S be a nonabelian finite simple group with a (nontrivial) cyclic p-Sylow subgroup. Then any element $\bar{x} \in \mathrm{Out}(S)$ with order p lifts to an automorphism $x \in \mathrm{Aut}(S)$ with order p.*

Proof. All facts about finite simple groups used in this proof that are not clear from the definitions or otherwise cited can be found in [Conway et al. 1985].

First, note that no nonabelian simple group has a cyclic 2-Sylow subgroup, so we assume $p \neq 2$. Note also that no primes other than 2 divide the order of the outer automorphism group of any alternating or sporadic group. So we may assume that S is of Lie type.

We first show that p does not divide the order g of the graph automorphism group or the order d of the diagonal automorphism group of S. The only simple groups S of Lie type for which an odd prime divides g are those of the form $O_8^+(q)$. In this case $3|g$. But $O_8^+(q)$ contains $(O_4^+(q))^2$ in block form, and the order of $O_4^+(q)$ is $(1/(4, q^2-1))(q^2(q^2-1)^2)$. This is divisible by 3, so $O_8^+(q)$ contains the group $\mathbb{Z}/3 \times \mathbb{Z}/3$, and does not have a cyclic 3-Sylow subgroup.

The simple groups S of Lie type for which an odd prime p divides d are the following:

(1) $\mathrm{PSL}_n(q)$, for $p|(n, q-1)$.

(2) $\mathrm{PSU}_n(q^2)$, for $p|(n, q+1)$.

(3) $E_6(q)$, for $p = 3$ and $3|(q-1)$.

(4) ${}^2E_6(q^2)$, $p = 3$ and $3|(q+1)$.

Now, $\mathrm{PSL}_n(q)$ contains a split maximal torus $((\mathbb{Z}/q)^\times)^{n-1}$. Since $p|(q-1)$, this group contains $(\mathbb{Z}/p)^{n-1}$, which is not cyclic, as $p|n$ and $p \neq 2$. So a p-Sylow subgroup of $\mathrm{PSL}_n(q)$ is not cyclic. The diagonal matrices in $\mathrm{PSU}_n(q^2)$ form the group $(\mathbb{Z}/(q+1))^{n-1}$, which also contains a noncyclic p-group (as $p > 2$ and $p|(n, q+1)$). The group $E_6(q)$ has a split maximal torus $((\mathbb{Z}/q)^\times)^6$ [Humphreys 1975, Section 35], and thus contains a noncyclic 3-group. Lastly, ${}^2E_6(q^2)$ is constructed as a subgroup of $E_6(q^2)$. When $q \equiv -1 \pmod 3$, the ratio $|E_6(q^2)|/|{}^2E_6(q^2)|$ is not divisible by 3, so a 3-Sylow subgroup of ${}^2E_6(q^2)$ is isomorphic to one of $E_6(q^2)$, which we already know is not cyclic.

So if there exists an element $\bar{x} \in \mathrm{Out}(S)$ of order p, then p divides f, the order of the group of field automorphisms. Also, since the group of field automorphisms is cyclic and p does not divide d or g, a p-Sylow subgroup of $\mathrm{Out}(S)$ is cyclic. This means that all elements of order p in $\mathrm{Out}(S)$ are conjugate in $\mathrm{Out}(S)$, up to a power with exponent prime to p. At the same time, there exists an automorphism α in $\mathrm{Aut}(S)$ which has order p and is not inner. Namely, we view S as the $\mathbb{F}_q$-points of some $\mathbb{Z}$-scheme, where $q = \wp^f$ for some prime $\wp$, and we act on these points by the (f/p)-th power of the Frobenius at $\wp$. Let $\bar{\alpha}$ be the image of α in $\mathrm{Out}(S)$. Since there exists c prime to p such that $\bar{\alpha}^c$ is conjugate to $\bar{x}$ in $\mathrm{Out}(S)$, there exists some x conjugate to α^c in $\mathrm{Aut}(S)$ such that $\bar{x}$ is the image of x in $\mathrm{Out}(S)$. Since α^c has order p, so does x. It is the automorphism we seek. □

The main theorem we wish to prove in this section states that a finite group with a cyclic p-Sylow subgroup is either p-solvable or "as far from p-solvable as possible."

Proposition B.2. *Let G be a finite group with a cyclic p-Sylow subgroup P of order p^n. Then at least one of the following two statements is true*:

- *G is p-solvable.*
- *G has a simple composition factor S with $p^n \mid |S|$.*

Proof. We may replace G by G/N, where N is the maximal prime-to-p normal subgroup of G. So assume that any nontrivial normal subgroup of G has order divisible by p. Let S be a minimal normal subgroup of G. Then S is a direct product of isomorphic simple groups [Aschbacher 2000, 8.2, 8.3]. Since G has cyclic p-Sylow subgroup and no nontrivial normal subgroups of prime-to-p order, we see that S is a simple group with $p \mid |S|$. If $S \cong \mathbb{Z}/p$, then [Obus 2012, Corollary 2.4 (i)] shows that G is p-solvable. So assume, for a contradiction, that $p^n \nmid |S|$ and $S \ncong \mathbb{Z}/p$. Then G/S contains a subgroup of order p. Let H be the inverse image of this subgroup in G. It follows that H is an extension of the form

$$1 \to S \to H \to H/S \cong \mathbb{Z}/p \to 1. \tag{B-1}$$

We claim that H cannot have a cyclic p-Sylow subgroup, thus obtaining the desired contradiction.

To prove our claim, we show that H is in fact a semidirect product $S \rtimes H/S$, that is, we can lift H/S to a subgroup of H. Let $\overline{x}$ be a generator of H/S. We need to find a lift x of $\overline{x}$ which has order p. It suffices to find x lifting $\overline{x}$ such that conjugation by x^p on S is the trivial isomorphism, as S is center-free. Since the possible choices of x correspond to the possible automorphisms of S which lift the outer automorphism $\phi_{\overline{x}}$ given by $\overline{x}$, we need only find an automorphism of S of order p which lifts $\phi_{\overline{x}}$. Since $\phi_{\overline{x}}$ has order p, our desired automorphism is provided by Lemma B.1, finishing the proof. □

Remark B.3. As was mentioned in the introduction, there are limited examples of simple groups with cyclic p-Sylow subgroups of order greater than p. For instance, there are no sporadic groups or alternating groups. There are some of the form $\mathrm{PSL}_r(\ell)$, including all groups of the form $\mathrm{PSL}_2(\ell)$ with $v_p(\ell^2-1) > 1$ and p, ℓ odd. There is also the Suzuki group $Sz(32)$. All other examples are too large to be included in [Conway et al. 1985].

Appendix C. Computations for $p = 2, 3$

We collect some technical computations involving small primes that would have disrupted the continuity of the main text.

For the following proposition, R is a mixed characteristic $(0, 2)$ complete discrete valuation ring with residue field k and fraction field K. For any scheme S over R, we write S_k and S_K for $S \times_R k$ and $S \times_R K$, respectively.

Proposition C.1. *Assume that R contains the 2^n-th roots of unity, where $n \geq 2$. Let $X = \mathrm{Spec}\, A$, where $A = R\{T\}$. Let $f : Y_K \to X_K$ be a μ_{2^n}-torsor given by the equation $y^{2^n} = s$, where $s \equiv 1 + c_1T + c_2T^2 \pmod{2^{n+1}}$, such that $v(c_2) = n$, c_2 is a square in R, and $c_1^2/c_2 \equiv 2^{n+1}i \pmod{2^{n+2}}$, where i is either square root of -1. Then $f : Y_K \to X_K$ splits into a union of 2^{n-2} disjoint μ_4-torsors. Let Y be the normalization of X in the total ring of fractions of Y_K. Then the map $Y_k \to X_k$ is étale, and is birationally equivalent to the union of 2^{n-2} disjoint $\mathbb{Z}/4$-covers of $\mathbb{P}^1_k$, each branched at one point, with first upper jump equal to* 1.

Proof. Using the binomial theorem, we see that $\sqrt[2^{n-2}]{s}$ exists in A and is congruent to $1 + b_1T + b_2T^2 \pmod 8$, with $b_1 = c_1/2^{n-2}$ and $b_2 = c_2/2^{n-2}$. Then $v(b_2) = 2$, b_2 is a square in R, and $b_1^2/b_2 \equiv 8i \pmod{16}$. Thus, we reduce to the case $n = 2$.

Let $Z_K \cong Y_K/\mu_2$. The natural maps $r : Z_K \to X_K$ and $q : Y_K \to Z_K$ are given by the equations

$$z^2 = g, \quad y^2 = z,$$

respectively. Let us write $g' = g(1+T\sqrt{-b_2})^2$ and $z' = z(1+T\sqrt{-b_2})$. Then r is also given by the equation

$$(z')^2 = g'.$$

Now, $g' = 1 + 2T\sqrt{-b_2} + \epsilon$, where ϵ is a power series whose coefficients all have valuation greater than 2 (note that, by assumption, $v(b_1) = \frac{5}{2}$). By [Henrio 2000a, chapitre 5, proposition 1.6] and Lemma 2.5, the torsor r has (nontrivial) étale reduction $Z_k \to X_k$, which is birationally equivalent to an Artin–Schreier cover with conductor 1. By Lemma 2.4, Z_k has genus zero. Let U be such that $1 - 2U = z'$. Then the cover $Z_k \to X_k$ is given by the equation

$$(\bar{u})^2 - \bar{u} = \overline{T}\sqrt{(\overline{-b_2/4})},$$

where an overline represents reduction modulo π. Then $\bar{u}$ is a parameter for Z_k, and the normalization of A in Z_K is $R\{U\}$.

It remains to show that q has étale reduction. The cover q is given by the equation

$$y^2 = z = z'(1+T\sqrt{-b_2})^{-1} = (1-2U)(1+T\sqrt{-b_2})^{-1}. \tag{C-1}$$

By [Henrio 2000a, chapitre 5, proposition 1.6], it will suffice to show that, up to multiplication by a square in $R\{U\}$, the right-hand side of (C-1) is congruent to 1 (mod 4) in $R\{U\}$. Equivalently, we must show that the right-hand side is congruent modulo 4 to a square in $R\{U\}$. Modulo 4, we can rewrite the right-hand side as

$$1 - 2U - T\sqrt{-b_2}. \tag{C-2}$$

We also have that

$$(1-2U)^2 = z^2(1+T\sqrt{-b_2})^2 = g(1+T\sqrt{-b_2})^2 \equiv 1 + T(b_1 + 2\sqrt{-b_2}) \pmod 8.$$

Rearranging, this yields that

$$\frac{-4U+4U^2}{(b_1/\sqrt{-b_2})+2} \equiv T\sqrt{-b_2} \pmod 4.$$

Since $b_1^2/b_2 \equiv 8i \pmod{16}$, it is clear that $b_1^2/(-b_2) \equiv 8i \pmod{16}$. One can then show that $b_1/\sqrt{-b_2} \equiv 2+2i \pmod 4$. We obtain $T\sqrt{-b_2} \equiv 2iU - 2iU^2 \pmod 4$. So (C-2) is congruent to $1-(2+2i)U + 2iU^2$ modulo 4. This is $(1-(1+i)U)^2$, so we are done. □

Lemma C.2. *Lemma 7.8 holds when $p = 2$, and also when $p = 3$ and $n > s = 1$.*

Proof. Use the notation of Lemma 7.8, and let $R/W(k)$ be a large enough finite extension. As in the proof of Lemma 7.8, we must show that if D is the formal disk with ring of functions $R\{t\}$, then the normalization E of D in the fraction field of $R\{t\}[y]/(y^{p^n} - g(d+et))$ has special fiber with irreducible components of positive genus. Here $g(d+et) = \sum_{\ell=0}^{\infty} c_\ell t^\ell$, and

$$c_\ell = e^\ell \sum_{j=0}^{\ell} \binom{a}{\ell-j}\binom{b}{j} d^{j-\ell}(d-1)^{-j}. \tag{C-3}$$

In (C-3), $v(a)=0$, $v(b)=n-s$, $v(d)=0$, $v(d-1)=n-s$, and

$$v(e) = \frac{1}{2}\Big(2n - s + \frac{1}{p-1}\Big).$$

First, assume that $p=3$ and $n > s = 1$. Then $d = \frac{a}{a+b} + \frac{\sqrt[3]{3^{2n+1}\binom{b}{3}}}{a+b}$. Using (C-3), one calculates $c_0 = 1$,

$$c_1 = e\Big(\frac{(a+b)d-a}{d(d-1)}\Big) = e\left(\frac{\sqrt[3]{3^{2n+1}\binom{b}{3}}}{d(d-1)}\right),$$

and $v(c_2) = n+\frac{1}{2}$. By (7-2), we have $v(c_\ell) \geq n+\frac{1}{2}$ except when $\ell = 3$. Furthermore, each term in (C-3) for $\ell = 3$, other than $j = 3$, has valuation greater than $n+\frac{1}{2}$. So

$$c_3 \equiv e^3\binom{b}{3}(d-1)^{-3} \pmod{3^{n+\frac{1}{2}+\epsilon}},$$

for some $\epsilon > 0$. Thus $v(c_1) = n + \frac{5}{12} > n$ and $v(c_3) = n+\frac{1}{4} > n$. Note also that $v(c_\ell) > n+\frac{1}{2}$ for $\ell \geq 4$. Now, $c_1^3/3^{2n+1} = e^3\binom{b}{3}(d-1)^{-3}d^{-3}$. Since $v(d-1) = n-s > \frac{1}{4}$, and $v(e^3\binom{b}{3}(d-1)^{-3}) = n+\frac{1}{4}$, we obtain that

$$\frac{c_1^3}{3^{2n+1}} \equiv e^3\binom{b}{3}(d-1)^{-3} \equiv c_3 \pmod{3^{n+\frac{1}{2}+\epsilon}}$$

for some $\epsilon > 0$. We are now in the situation (ii) of Lemma 3.1 (with $h=2$), and we conclude using Lemma 2.4.

Next, assume $p=2$. First, note that $n > s$ as there are no three-point $\mathbb{Z}/2^n$-covers of $\mathbb{P}^1$ that are totally ramified above all three branch points. Consider (C-3). Clearly $c_0 = 1$. We claim that $v(c_2) = n$, that $c_1^2/c_2 \equiv 2^{n+1}i \pmod{2^{n+2}}$, and that $v(c_\ell) \geq n+1$ for $\ell \geq 3$. We may assume that K contains $\sqrt{c_2}$. Given the claim, we can apply Proposition C.1 to see that the special fiber E_k of E is a disjoint union of 2^{n-2} étale $\mathbb{Z}/4$-covers of the special fiber D_k of D, each of which extends to a cover $\phi : E'_k \to \mathbb{P}^1_k$ branched at one point with first upper jump equal to 1. By [Pries 2006, Lemma 19], such a cover has conductor at least 2. A Hurwitz formula calculation shows that the each component of E'_k has positive genus, proving the lemma.

Now we prove the claim. The term in (C-3) for c_2 with lowest valuation corresponds to $j = 2$, and this term has valuation $2v(e) + v(b) - 1 - 2v(d-1)$, which is equal to n. For c_ℓ, $\ell \geq 3$, we have $v(c_\ell) = n + 1 + ((\ell - 2)/2)(s+1) - v(\ell)$ by (7-2). Since $s \geq 1$, we obtain $v(c_\ell) \geq n+1$ for $\ell \geq 3$.

Lastly, we must show that $c_1^2/c_2 \equiv 2^{n+1} i \pmod{2^{n+2}}$. Choose

$$d = \frac{a}{a+b} + \frac{\sqrt{2^n bi}}{(a+b)^2}$$

as in Lemma 7.8. Using (C-3), we compute

$$\frac{c_1^2}{c_2} = \frac{(a(d-1)+bd)^2}{\binom{a}{2}(d-1)^2 + abd(d-1) + \binom{b}{2}d^2}.$$

Then the congruence $\dfrac{c_1^2}{c_2} \equiv 2^{n+1} i \pmod{2^{n+2}}$ is equivalent to

$$\frac{2((a+b)d-a)^2}{-bd^2} \equiv 2^{n+1} i \pmod{2^{n+2}}$$

(as the other terms in the denominator become negligible). Plugging in d to

$$\frac{2((a+b)d-a)^2}{-bd^2},$$

we obtain

$$\frac{2^{n+1} bi}{-b(a^2 + (2a/(a+b))\sqrt{2^n bi} + 2^n bi/(a+b)^2)} \equiv 2^{n+1} i \pmod{2^{n+2}}.$$

This is equivalent to $-1/a^2 \equiv 1 \pmod 2$, as the terms in the denominator, other than $-ba^2$, are negligible. This is certainly true, as a is odd. This completes the proof of the claim, and thus the lemma. $\square$

Lemma C.3. *Let $p = 3$, let $n > s$ be positive integers, and let a and b be integers with $v_3(a) = 0$ and $v_3(b) = n - s$. Write $K_0 = \mathrm{Frac}(W(k))$ and, for all $i > 0$, write $K_i = K_0(\zeta_{3^i})$, where ζ_{3^i} is a primitive 3^i-th root of unity. If $s = 1$, then the n-th higher ramification groups for the upper numbering of*

$$K_n\left(\sqrt[3]{3^{2n+1}\binom{b}{3}},\ \sqrt[3^{n-1}]{\frac{a}{a+b}}\right)\Big/K_0$$

vanish. If $s > 1$, let

$$d' = \frac{a}{a+b} + \frac{\sqrt[3]{3^{2(n-s+1)+1}\binom{b}{3}}}{a+b}.$$

Then the n-th higher ramification groups for the upper numbering of

$$K_n\left(d',\ \sqrt[3^{n-s}]{\frac{a}{a+b}},\ \sqrt[3^{n-s+1}]{\frac{(d')^a(d'-1)^b}{a^ab^b(a+b)^{-(a+b)}}}\right)\Big/K_0$$

vanish.

Proof. Assume $s=1$. Then $3^{2n+1}\binom{b}{3}$ has valuation $3n-1$. Since $n\geq 2$, the n-th higher ramification groups for the upper numbering of

$$L=K_1\left(\sqrt[3]{3^{2n+1}\binom{b}{3}}\right)\Big/K_0$$

vanish by [Viviani 2004, Theorem 6.5]. Also, the n-th higher ramification groups for the upper numbering of

$$L=K_n\left(\sqrt[3^{n-1}]{\frac{a}{a+b}}\right)\Big/K_0$$

vanish by [Viviani 2004, Theorem 5.8]. By Lemma 2.3,

$$K_n\left(\sqrt[3]{3^{2n+1}\binom{b}{3}},\ \sqrt[3^{n-1}]{\frac{a}{a+b}}\right)\Big/K_0$$

has trivial n-th higher ramification groups for the upper numbering.

Now, assume $s>1$. We use case (ii) of Corollary 7.14 and Lemma 2.3 to reduce to showing that the conductor of K/K_0 is less than n, where

$$K:=K_n\left(d',\ \sqrt[3^{n-s+1}]{\frac{(d')^a(d'-1)^b}{a^ab^b(a+b)^{-(a+b)}}}\right)\Big/K_0.$$

Since $v(b)=n-s$ and $v(a)=0$, one calculates that $d'':=d'/(a/(a+b))$ can be written as $1+r$, where $v(r)=n-s+\frac{2}{3}$. The same is true for $(d'')^a$. By the binomial expansion, $(d'')^a$ is a 3^{n-s}-th power in $K_n(d')$. Thus so is $(d'')^a((d'-1)^b/b^b(a+b)^{-b})$. So we can write $K=K_n(d',\ \sqrt[3]{d'''})$, for some $d'''\in K_n(d')$. Using Lemma 2.3 again, we need only show that the conductor h of $K_1(d',\ \sqrt[3]{d'''})/K_0$ is less than n. Note that $n\geq 3$.

Let $L=K_1(d')$ and $M=K_1(d',\ \sqrt[3]{d'''})$. By [Obus 2011a, Lemma 3.2], the conductor of L/K_1 is 3. Since the lower numbering is invariant under subgroups, the greatest lower jump for the higher ramification filtration of L/K_0 is 3. Thus the conductor of L/K_0 is $\frac{3}{2}$. By [Obus 2011a, Lemma 3.2], the conductor of M/L is ≤ 9. Applying [Obus 2011a, Lemma 2.1] to $K_0\subseteq L\subseteq M$ yields that h is either $\frac{3}{2}$ or satisfies $h\leq\frac{3}{2}+\frac{1}{6}(9-3)<3\leq n$. $\square$

For the rest of the appendix, K_0 is the fraction field of $W(k)$, where k is algebraically closed of characteristic 2. We set $K_r:=K_0(\zeta_{2^r})$.

We state an easy lemma from elementary number theory without proof:

Lemma C.4. *Choose $d \in \mathbb{Q}$, and a square root i of -1 in K_2. Let v be the standard 2-adic valuation. If $v(d)$ is even, then di is a square in K_3, but not in K_2. Also, d is a square in K_2. If $v(d)$ is odd, then di is a square in K_2, and d is a square in K_3.*

We turn to the field extension in Proposition 7.13(v). Recall that $n \geq 2$ is a positive integer, and s is an integer satisfying $0 < s < n$. Also, a is an odd integer and b is an integer exactly divisible by 2^{n-s}. For each $0 \leq j < s$, set

$$d_j = \frac{a}{a+b} + \frac{\sqrt{2^{n-j}bi}}{(a+b)^2},$$

where $i^2 = -1$ and the square root symbol represents either square root. Lastly, as in Proposition 7.13 (iii), set

$$K := K_n\big(\sqrt[2^{n-1}]{d_0},\ \sqrt[2^{s-1}]{d_0 - 1},\ \sqrt[2^{n-j}]{d_j},\ \sqrt[2^{s-j}]{d_j - 1}\big)_{1 \leq j < s}.$$

For the purpose of the proof below, we let v_ℓ be the valuation on K_ℓ normalized so that a uniformizer has valuation 1 (in contrast to the convention for the rest of this paper).

Proposition C.5. *Let L be the Galois closure of K over K_0. Then the conductor h_{L/K_0} is less than n.*

Proof. Note that $h_{K_n/K_0} = n - 1$. Thus, by Lemma 2.3, we need only consider the extensions $K_{n-1}(\sqrt[2^{n-1}]{d_0})$, $K_{s-1}(\sqrt[2^{s-1}]{d_0 - 1})$, $K_{n-j}(\sqrt[2^{n-j}]{d_j})$ $(1 \leq j < s)$, and $K_{s-j}(\sqrt[2^{s-j}]{d_j - 1})$ $(1 \leq j < s)$ of K_0, and we consider them separately. Write $\ell(j)$ for the smallest ℓ such that $d_j \in K_\ell$. By Lemma C.4, we have that $\ell(j) = 2$ for $s + j$ odd and $\ell(j) = 3$ for $s + j$ even. By [Obus 2011a, Corollary 4.4], we need only consider those fields $K_c(\sqrt[2^c]{d_j})$ and $K_c(\sqrt[2^c]{d_j - 1})$ such that $c + \ell(j) > n$. Since $\ell(j) \leq 3$ for all j, we are reduced to bounding the conductors of the (Galois closures of the) following fields over K_0:

$$K_{n-1}(\sqrt[2^{n-1}]{d_j})_{j\in\{0,1\}},\ K_{n-2}(\sqrt[2^{n-2}]{d_2}),\ K_{n-2}(\sqrt[2^{n-2}]{d_j - 1})_{j\in\{0,1\}}.$$

For any of the above field extensions involving d_j, we may assume that $s > j$.

- Let M be the Galois closure of $K_{n-1}(\sqrt[2^{n-1}]{d_j})$ over K_0, where $j \in \{0, 1\}$. First, assume $\ell(j) = 2$. Then $v_2(d_j - 1) = v_2(b) = 2(n - s) > 1$. We conclude using [Obus 2011a, Corollary 4.4] (with $c = n - 1$, $\ell = 2$, and $t_\alpha = d_j - 1$) that $h_{M/K_0} < n$.

  Now, assume $\ell(j) = 3$. Suppose $d_j \in (K_3^\times)^2$. We know that $v_3(d_j - 1) = 4(n - s) \geq 4$. By [Obus 2011a, Lemma 3.1], if $(d'_j)^2 = d_j$, then $v_3(d'_j - 1) > 1$. Then M is the Galois closure of $K_{n-1}(\sqrt[2^{n-2}]{d_j'})$ over K_0. We conclude using

[Obus 2011a, Corollary 4.4] (with $c = n-2$, $\ell = 3$, and $t_\alpha = d'_j - 1$) that $h_{M/K_0} < n$.

Lastly, suppose $d_j \notin (K_3^\times)^2$. Write $d_j = \alpha_j \beta^2$, where $\beta^2 = a/(a+b)$, and $\beta \in K_2$ (Lemma C.4). Write $\alpha_j = 1 + t_j$. One can find $\gamma_j \in K_3$ such that $\alpha_j = \alpha'_j \gamma_j^2$, where $\alpha'_j = 1 + t'_j$ and $v_3(t'_j)$ is odd; this is the main content of [Obus 2011a, Lemma 3.2(i)]. Then $d_j = \alpha'_j (\beta\gamma_j)^2$. By [Obus 2011a, Remark 3.4], we have

$$v_3(t'_j) \geq v_3(t_j) = 4\Big(n - \frac{s+j}{2}\Big) > 5,$$

the last inequality holding because $n > s > j$. This means that

$$v_3(\gamma_j^2 - 1) = v_3\Big(\frac{\alpha_j - \alpha'_j}{\alpha'_j}\Big) \geq v_3(t_j) > 5.$$

Also, $v_3(\beta^2 - 1) = 4(n-s) \geq 4$. So $v_3((\beta\gamma_j)^2 - 1) \geq 4$. By [Obus 2011a, Lemma 3.1], we obtain $v_3(\beta\gamma_j - 1) > 1$. We conclude using [Obus 2011a, Corollary 4.3] (with $c = n-1$, $\ell = 3$, $t_{\alpha'} = t'_j$, and $t_\beta = \beta\gamma_j - 1$) that $h_{M/K_0} < n$.

- Let M be the Galois closure of $K_{n-2}(\sqrt[2^{n-2}]{d_2})$ over K_0. If $\ell(2) = 2$, then $h_{M/K_0} < n$ by [Obus 2011a, Corollary 4.4] (with $c = n-2$ and $\ell = 2$). So assume $\ell(2) = 3$. Since $v_3(d_2 - 1) = 4(n-s) > 1$, we obtain that $h_{M/K_0} < n$ by [Obus 2011a, Corollary 4.4] (with $c = n-2$, $\ell = 3$, and $t_\alpha = d_2 - 1$).
- Let M be the Galois closure of $K_{n-2}(\sqrt[2^{n-2}]{d_j - 1})$ over K_0, where $j \in \{0, 1\}$. As in the previous case, we may assume that $\ell(2) = 3$. By Lemma C.4, there exists $\gamma \in K_3$ such that $\gamma^2 = -b/(a+b)$. Write $d_j - 1 = \alpha'_j \gamma^2$. Then M is contained in the compositum of the Galois closures M' of $K_{n-2}\big(\sqrt[2^{n-2}]{\alpha'_j}\big)$ and M'' of $K_{n-2}(\sqrt[2^{n-3}]{\gamma})$ over K_0.

  Now, $v_3(\alpha'_j - 1) = 4(n - (s + j/2) - (n-s)) = 4((s - j/2)) > 1$. By [Obus 2011a, Corollary 4.4] (with $c = n-2$, $\ell = 3$, and $t_\alpha = \alpha'_j - 1$), we have $h_{M'/K_0} < n$. Also by [Obus 2011a, Corollary 4.4] (with $c = n-3$, $\ell = 3$, and $t_\alpha = \gamma - 1$), we have $h_{M''/K_0} < n$. By Lemma 2.3, we have $h_{M/K_0} < n$. □

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Toroidal compactifications of PEL-type Kuga families

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We explain how compactifications of Kuga families of abelian varieties over PEL-type Shimura varieties, including for example all those products of universal abelian schemes, can be constructed (up to good isogenies not affecting the relative cohomology) by a uniform method. We also calculate the relative cohomology and explain its various properties crucial for applications to the cohomology of automorphic bundles.

Introduction

To study the relations between automorphic forms and Galois representations, it is desirable to understand the cohomology of Shimura varieties with coefficients in algebraic representations of the associated reductive groups (i.e., the so-called *automorphic bundles*).

In the case of PEL-type Shimura varieties, the associated reductive groups are (up to center) twists of products of symplectic, orthogonal, or general linear groups. According to Weyl's construction [1997] (see also [Fulton and Harris 1991;

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Goodman and Wallach 2009]), all algebraic representations of a classical group can be realized as summands in the tensor powers of the *standard representation* of the group. In geometry, one is led to consider the cohomology of fiber products of the universal families of abelian varieties over the PEL-type Shimura varieties. Such fiber products are special cases of what we will call *PEL-type Kuga families*, or simply Kuga families. When the PEL-type Shimura variety in question is not compact, the total spaces of such Kuga families are not compact either.

To study cohomology properly, one is often led to the question of the existence of projective smooth compactifications with good properties, such as allowing the Hecke operators to act on their cohomology spaces (but not necessarily the geometric spaces). In what follows, let us simply call such compactifications *good* compactifications. In characteristic zero, such questions can often be handled by the embedded resolution of singularities due to Hironaka [1964a; 1964b]. However, more explicit theories exist in our context. The work of Mumford and his collaborators in [Ash et al. 1975] provides a systematic collection of good compactifications of Shimura varieties with explicit descriptions of local structures, while the work of Pink [1990] provides a systematic construction of good compactifications of the Kuga families as well. These compactifications are called *toroidal compactifications*. Their methods are analytic in nature and cannot be truly generalized in mixed characteristics.

Based on the theory of degeneration of polarized abelian varieties initiated by Mumford [1972], Faltings and Chai [Faltings 1985; Chai 1985; Faltings and Chai 1990] constructed good compactifications over the integers for Siegel moduli spaces defined by the moduli space of principally polarized abelian varieties. In [Faltings and Chai 1990], they also constructed good compactifications of fiber products of the universal families by gluing weak relatively complete models along the boundary. We ought to point out that, although most works on compactifications spend most of their pages on the construction of boundary charts, it is only the *gluing* argument that validates the whole construction. (This is not necessarily the case for works using the moduli-theoretic approach, such as [Alexeev and Nakamura 1999; Alexeev 2002; Olsson 2008]. However, the questions there are not less challenging: What can one say about the boundary structures? Are they equally useful for applications to cohomology?) Thus, even if the construction of toroidal compactifications of Siegel moduli spaces in [Faltings and Chai 1990, Chapter IV] has been generalized for all PEL-type Shimura varieties in [Lan 2008], the gluing of weak relatively complete models has to be carried out separately when one works along the original idea of [Faltings and Chai 1990, Chapter VI]. (This is the case in for example [Rozensztajn 2006], in which the assumption that the boundary divisors are regular, i.e., have no crossings, unfortunately rules out all cases where choices of cone decompositions are needed for the Shimura varieties.)

Note that gluing is not just about techniques of descent. Any theory of descent requires an input of some *descent data*. Since a naive generalization of the constructions in [Faltings and Chai 1990, Chapter IV] introduces unwanted boundary components, which have to be studied and removed carefully by imposing liftability and pairing conditions as in [Lan 2008], we have reason to believe that a naive generalization of the construction in [Faltings and Chai 1990, Chapter VI, §1] requires delicate modifications, without which even the strongest descent theory cannot be applied.

The aim of this article is to avoid any further argument of gluing, and to treat all PEL-type cases on an equal footing. We shall reduce the construction of toroidal compactifications of PEL-type Kuga families to the construction of toroidal compactifications of Shimura varieties in [Lan 2008], by systematically realizing the Kuga families as locally closed boundary strata in the toroidal compactifications of (larger) PEL-type Shimura varieties. Partly inspired by Kato's theory of log abelian schemes, we can show that, up to refinements of cone decompositions, the structural morphisms from the Kuga families to the Shimura varieties extend (up to good isogenies not affecting the relative cohomology) to log smooth morphisms with nice properties between the toroidal compactifications. This approach differs fundamentally from the one in [Faltings and Chai 1990, Chapter VI]. As Chai pointed out, although no technique can be truly shared between analytic and algebraic constructions, our idea is close in spirit to that of [Pink 1990]. (See Remark 3.10 below.)

Since we replace Faltings and Chai's construction with a different one, we need to explain that our simpler (but perhaps cruder) construction is not less useful. Thus our second task is to calculate the relative (log) de Rham cohomology of the compactified families. We show that such relative cohomology not only enjoys the same expected properties as in [Faltings and Chai 1990, Chapter VI, §1], but also admits natural Hecke actions defined by parabolic subgroups of larger reductive algebraic groups, because our construction uses toroidal boundaries of larger Shimura varieties. This exhibits a large class of endomorphisms on our cohomology spaces, including ones needed in the geometric realization of Weyl's construction (i.e., the realization of automorphic bundles as summands in the relative cohomology of Kuga families).

The outline of this article is as follows. In Section 1, we review some of the results we need from [Lan 2008]. We consider the investment of this summary worthwhile because, although we do not need to carry out another gluing argument, we do need the full strength of the long work [Lan 2008]. In Section 2, we define what we mean by PEL-type Kuga families, state our main theorem, and give an outline of the proof. In Section 3, we carry out the construction of toroidal compactifications for these Kuga families that admit log smooth morphisms to the

Shimura varieties in question. (This section serves roughly the same purpose as [Faltings and Chai 1990, Chapter VI, §1].) In Sections 4 and 5, we show that these toroidal compactifications are indeed *good* by justifying what we mentioned in the previous paragraph. (These two sections serve roughly the same purpose as [Faltings and Chai 1990, Chapter VI, §2].) We would like to mention that the use of nerve spectral sequences in Section 4 imitates immediate analogues in [Harris and Zucker 1994; 2001] (based on techniques that can be traced back to [Kempf et al. 1973, Chapter I, §3]), while the use of log extensions of polarizations is inspired by Kato's idea of (relative) log Picard groups [Illusie 1994, 3.3]. (See Remark 5.7.) The article ends with Section 6, in which we explain how to define canonical extensions of the so-called principal bundles.

Although used as the main motivation for our construction, applications to cohomology of automorphic bundles will be deferred to some forthcoming papers. There the readers will find the construction of proper smooth integral models useful for studying cohomology with not only rational coefficients, but also integral and torsion coefficients.

We shall follow [Lan 2008, Notations and Conventions] unless otherwise specified. (Although our references to [Lan 2008] use the numbering in the original version, the reader is advised to consult the errata and revision (available online) for corrections of typos and minor mistakes, and for improved exposition.)

1. PEL-type moduli problems and their compactifications

In this section, we summarize definitions and main results in [Lan 2008] that will be needed in this article. We will emphasize definitions such as the ones involved in the description of boundary structures, but will have to be less comprehensive on some fundamental definitions including the ones of level structures.

1A. ***Linear algebraic data.*** Let $\mathcal{O}$ be an order in a finite-dimensional semisimple algebra over $\mathbb{Q}$ with a positive involution $\star$. Here an *involution* means an antiautomorphism of order two, and *positivity* of $\star$ means $\mathrm{Tr}_{\mathcal{O}\otimes_{\mathbb{Z}}\mathbb{R}/\mathbb{R}}(xx^\star) > 0$ for any $x \neq 0$ in $\mathcal{O} \otimes_{\mathbb{Z}} \mathbb{R}$. We assume that $\mathcal{O}$ is mapped to itself under $\star$. We shall denote the center of $\mathcal{O} \otimes_{\mathbb{Z}} \mathbb{Q}$ by F.

Let $\mathbb{Z}(1) := \ker(\exp : \mathbb{C} \to \mathbb{C}^\times)$, which is a free $\mathbb{Z}$-module of rank one. Any choice $\sqrt{-1}$ of a square-root of -1 in $\mathbb{C}$ determines an isomorphism $(2\pi\sqrt{-1})^{-1} : \mathbb{Z}(1) \xrightarrow{\sim} \mathbb{Z}$, but there is no canonical isomorphism between $\mathbb{Z}(1)$ and $\mathbb{Z}$. For any commutative $\mathbb{Z}$-algebra R, we denote by $R(1)$ the module $R \otimes_{\mathbb{Z}} \mathbb{Z}(1)$.

By a PEL-type $\mathcal{O}$-lattice $(L, \langle \cdot , \cdot \rangle, h)$ (as in [Lan 2008, Definition 1.2.1.3]), we mean the following data:

(1) An $\mathcal{O}$-lattice, namely a $\mathbb{Z}$-lattice L with the structure of an $\mathcal{O}$-module.

(2) An alternating pairing $\langle\,\cdot\,,\,\cdot\,\rangle : L \times L \to \mathbb{Z}(1)$ satisfying $\langle bx, y\rangle = \langle x, b^\star y\rangle$ for any $x, y \in L$ and $b \in \mathcal{O}$, together with an $\mathbb{R}$-algebra homomorphism $h : \mathbb{C} \to \mathrm{End}_{\mathcal{O}\otimes_{\mathbb{Z}}\mathbb{R}}(L \otimes_{\mathbb{Z}} \mathbb{R})$ satisfying:

   (a) For any $z \in \mathbb{C}$ and $x, y \in L \otimes_{\mathbb{Z}} \mathbb{R}$, we have $\langle h(z)x, y\rangle = \langle x, h(z^c)y\rangle$, where $\mathbb{C} \to \mathbb{C} : z \mapsto z^c$ is the complex conjugation.

   (b) For any choice of $\sqrt{-1}$ in $\mathbb{C}$, the $\mathbb{R}$-bilinear pairing
$$(2\pi\sqrt{-1})^{-1}\langle\,\cdot\,, h(\sqrt{-1})\,\cdot\,\rangle : (L \otimes_{\mathbb{Z}} \mathbb{R}) \times (L \otimes_{\mathbb{Z}} \mathbb{R}) \to \mathbb{R}$$
is symmetric and positive definite. (This last condition forces $\langle\,\cdot\,,\,\cdot\,\rangle$ to be nondegenerate.)

The tuple $(\mathcal{O}, ^\star, L, \langle\,\cdot\,,\,\cdot\,\rangle, h)$ (over $\mathbb{Z}$) then gives us an integral version of the tuple $(B, ^\star, V, \langle\,\cdot\,,\,\cdot\,\rangle, h)$ (over $\mathbb{Q}$) in [Kottwitz 1992] and related works. (We favor lattices over $\mathbb{Z}$ rather than their analogues over $\mathbb{Q}$ (or over $\mathbb{Z}_{(p)}$ for some p) because we will work with isomorphism classes rather than isogeny classes; cf. Remark 1.7.)

Definition 1.1 [Lan 2008, Definition 1.2.1.5]. Let a PEL-type $\mathcal{O}$-lattice $(L, \langle\,\cdot\,,\cdot\,\rangle, h)$ be given as above. For any $\mathbb{Z}$-algebra R, set
$$\mathrm{G}(R) := \big\{(g, r) \in \mathrm{GL}_{\mathcal{O}\otimes_{\mathbb{Z}}R}(L \otimes_{\mathbb{Z}} R) \times \mathbf{G}_{\mathrm{m}}(R) : \langle gx, gy\rangle = r\langle x, y\rangle, \forall x, y \in L\big\}.$$
In other words, $\mathrm{G}(R)$ is the group of symplectic automorphisms of $L \otimes_{\mathbb{Z}} R$ (respecting the pairing $\langle\,\cdot\,,\,\cdot\,\rangle$ up to a scalar multiple; cf. [Lan 2008, Definition 1.1.4.11]). For any $\mathbb{Z}$-algebra homomorphism $R \to R'$, we have by definition a natural homomorphism $\mathrm{G}(R) \to \mathrm{G}(R')$, making G a group functor (or in fact an affine group scheme) over $\mathbb{Z}$.

The projection to the second factor $(g, r) \mapsto r$ defines a morphism $\nu : \mathrm{G} \to \mathbf{G}_{\mathrm{m}}$, which we call the *similitude character*. For simplicity, we shall often denote elements (g, r) in G by simply g, and denote by $\nu(g)$ the value of r when we need it. (If $L \neq \{0\}$, then the value of r is uniquely determined by g. Hence there is little that we lose when suppressing r from the notation. However, this is indeed an abuse of notation when $L = \{0\}$, in which case we have $\mathrm{G} = \mathbf{G}_{\mathrm{m}}$.)

Let $\Box$ be any set of rational primes. (It can be either an empty set, a finite set, or an infinite set.) We denote by $\mathbb{Z}_{(\Box)}$ the unique localization of $\mathbb{Z}$ (at the multiplicative subset of $\mathbb{Z}$ generated by nonzero integers prime to $\Box$) having $\Box$ as its set of height one primes, and denote by $\hat{\mathbb{Z}}^\Box$ (resp. $\mathbb{A}^{\infty,\Box}$, resp. $\mathbb{A}^\Box$) the integral adeles (resp. finite adeles, resp. adeles) away from $\Box$. Then we have definitions for $\mathrm{G}(\mathbb{Q})$, $\mathrm{G}(\mathbb{A}^{\infty,\Box})$, $\mathrm{G}(\mathbb{A}^\infty)$, $\mathrm{G}(\mathbb{R})$, $\mathrm{G}(\mathbb{A}^\Box)$, $\mathrm{G}(\mathbb{A})$, $\mathrm{G}(\mathbb{Z})$, $\mathrm{G}(\mathbb{Z}/n\mathbb{Z})$, $\mathrm{G}(\hat{\mathbb{Z}}^\Box)$, $\mathrm{G}(\hat{\mathbb{Z}})$, $\mathcal{U}^\Box(n) := \ker(\mathrm{G}(\hat{\mathbb{Z}}^\Box) \to \mathrm{G}(\hat{\mathbb{Z}}^\Box/n\hat{\mathbb{Z}}^\Box) = \mathrm{G}(\mathbb{Z}/n\mathbb{Z}))$ for any n prime to $\Box$, and $\mathcal{U}(n) := \ker(\mathrm{G}(\hat{\mathbb{Z}}) \to \mathrm{G}(\hat{\mathbb{Z}}/n\hat{\mathbb{Z}}) = \mathrm{G}(\mathbb{Z}/n\mathbb{Z}))$.

Following Pink [1990, 0.6], we define the neatness of open compact subgroups $\mathcal{H}$ of $\mathrm{G}(\hat{\mathbb{Z}}^\Box)$ as follows: View $\mathrm{G}(\hat{\mathbb{Z}}^\Box)$ as a *subgroup* of $\mathrm{GL}_{\mathcal{O}\otimes_{\mathbb{Z}}\hat{\mathbb{Z}}^\Box}(L \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}^\Box) \times \mathbf{G}_{\mathrm{m}}(\hat{\mathbb{Z}}^\Box)$.

(Or we may use any faithful linear algebraic representation of G.) Then, for each rational prime $p > 0$ not in $\Box$, it makes sense to talk about *eigenvalues* of elements g_p in $\mathrm{G}(\mathbb{Z}_p)$, which are elements in $\bar{\mathbb{Q}}_p^\times$. Let $g = (g_p) \in \mathrm{G}(\hat{\mathbb{Z}}^\Box)$, with p running through rational primes such that $\Box \nmid p$. For each such p, let Γ_{g_p} be the subgroup of $\bar{\mathbb{Q}}_p^\times$ generated by eigenvalues of g_p. For any embedding $\bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$, consider the subgroup $(\bar{\mathbb{Q}}^\times \cap \Gamma_{g_p})_{\mathrm{tors}}$ of torsion elements of $\bar{\mathbb{Q}}^\times \cap \Gamma_{g_p}$, which is independent of the choice of the embedding $\bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$.

Definition 1.2 [Lan 2008, Definition 1.4.1.8]. We say that $g = (g_p)$ is *neat* if $\bigcap_{p \notin \Box} (\bar{\mathbb{Q}}^\times \cap \Gamma_{g_p})_{\mathrm{tors}} = \{1\}$. We say that an open compact subgroup $\mathcal{H}$ of $\mathrm{G}(\hat{\mathbb{Z}}^\Box)$ is *neat* if all its elements are neat.

Remark 1.3. The usual Serre's lemma that no nontrivial root of unity can be congruent to 1 modulo n if $n \geq 3$ shows that $\mathcal{H}$ is neat if $\mathcal{H} \subset \mathcal{U}^\Box(n)$ for some $n \geq 3$ such that $\Box \nmid n$.

Remark 1.4. Definition 1.2 makes no reference to the group $\mathrm{G}(\mathbb{Q})$ of rational elements. For the related notion of neatness for arithmetic groups, see [Borel 1969, 17.1].

1B. *Definition of moduli problems.* Let us fix a PEL-type $\mathcal{O}$-lattice $(L, \langle \cdot, \cdot \rangle, h)$ as in the previous section. Let F_0 be the *reflex field* of $(L \otimes_{\mathbb{Z}} \mathbb{R}, \langle \cdot, \cdot \rangle, h)$ defined as in [Kottwitz 1992, page 389] or [Lan 2008, Definition 1.2.5.4]. We shall denote the ring of integers in F_0 by $\mathcal{O}_{F_0}$, and use similar notations for other number fields. (This is in conflict with the notation of the order $\mathcal{O}$, but the precise interpretation will be clear from the context.)

Let $\mathrm{Disc} = \mathrm{Disc}_{\mathcal{O}/\mathbb{Z}}$ be the discriminant of $\mathcal{O}$ over $\mathbb{Z}$ (as in [Lan 2008, Definition 1.1.1.6]; see also [Lan 2008, Proposition 1.1.1.12]). Closely related to Disc is the invariant $\mathrm{I}_{\mathrm{bad}}$ for $\mathcal{O}$ defined in [Lan 2008, Definition 1.2.1.17], which is either 2 or 1, depending on whether type D factors are involved. Let $L^\# := \{x \in L \otimes_{\mathbb{Z}} \mathbb{Q} : \langle x, y \rangle \in \mathbb{Z}(1), \forall y \in L\}$ denote the dual lattice of L with respect to the pairing $\langle \cdot, \cdot \rangle$.

Definition 1.5. We say that a prime number p is *bad* if $p \mid \mathrm{I}_{\mathrm{bad}} \mathrm{Disc}[L^\# : L]$. We say a prime number p is *good* if it is not bad. We say that $\Box$ is a set of good primes if it does not contain any bad primes.

Let us fix a choice of a set $\Box$ of *good primes*. By abuse of notation, let $\mathcal{O}_{F_0,(\Box)}$ be the localization of $\mathcal{O}_{F_0}$ at the multiplicative set generated by rational prime numbers not in $\Box$. Let $\mathsf{S}_0 := \mathrm{Spec}(\mathcal{O}_{F_0,(\Box)})$ and let $(\mathrm{Sch}/\mathsf{S}_0)$ be the category of schemes over S_0. For any open compact subgroup $\mathcal{H}$ of $\mathrm{G}(\hat{\mathbb{Z}}^\Box)$, there is an associated moduli problem $\mathsf{M}_{\mathcal{H}}$ defined as follows:

Definition 1.6 [Lan 2008, Definition 1.4.1.4]. The moduli problem $\mathsf{M}_{\mathcal{H}}$ is defined as the category fibered in groupoids over $(\mathrm{Sch}/\mathsf{S}_0)$ whose fiber over each

S is the groupoid $\mathsf{M}_{\mathcal{H}}(S)$ described as follows: The objects of $\mathsf{M}_{\mathcal{H}}(S)$ are tuples $(G, \lambda, i, \alpha_{\mathcal{H}})$, where:

(1) G is an abelian scheme over S.

(2) $\lambda : G \to G^{\vee}$ is a polarization of degree prime to $\Box$.

(3) $i : \mathcal{O} \to \mathrm{End}_S(G)$ defines an $\mathcal{O}$-structure of (G, λ) (satisfying the Rosati condition $i(b)^{\vee} \circ \lambda = \lambda \circ i(b^{\star})$ for any $b \in \mathcal{O}$).

(4) $\underline{\mathrm{Lie}}_{G/S}$ with its $\mathcal{O} \otimes_{\mathbb{Z}} \mathbb{Z}_{(\Box)}$-module structure given naturally by i satisfies the determinantal condition in [Lan 2008, Definition 1.3.4.2] given by

$$(L \otimes_{\mathbb{Z}} \mathbb{R}, \langle \cdot , \cdot \rangle, h).$$

(5) $\alpha_{\mathcal{H}}$ is an (integral) level-$\mathcal{H}$ structure of (G, λ, i) of type $(L \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}^{\Box}, \langle \cdot , \cdot \rangle)$ as in [Lan 2008, Definition 1.3.7.8].

The isomorphisms $(G, \lambda, i, \alpha_{\mathcal{H}}) \sim_{\text{isom.}} (G', \lambda', i', \alpha'_{\mathcal{H}})$ of $\mathsf{M}_{\mathcal{H}}(S)$ are given by (naive) isomorphisms $f : G \xrightarrow{\sim} G'$ such that $\lambda = f^{\vee} \circ \lambda' \circ f$, $f \circ i(b) = i'(b) \circ f$ for all $b \in \mathcal{O}$, and $f \circ \alpha_{\mathcal{H}} = \alpha'_{\mathcal{H}}$ (symbolically).

Remark 1.7. The definition here using isomorphism classes is not as canonical as the ones proposed by Grothendieck and Deligne using quasiisogeny classes (as in [Kottwitz 1992]). For the relation between their definitions and ours, see [Lan 2008, §1.4]. We introduce the definition (using isomorphisms) here mainly because this is the definition most concrete for the study of compactifications.

Theorem 1.8 [Lan 2008, Theorem 1.4.1.12 and Corollary 7.2.3.10]. *The moduli problem* $\mathsf{M}_{\mathcal{H}}$ *is a smooth separated algebraic stack of finite type over* S_0. *It is representable by a quasiprojective scheme if the objects it parametrizes have no nontrivial automorphism*, *which is in particular the case when* $\mathcal{H}$ *is* ***neat*** (as in Definition 1.2).

We shall insist *from now on* the following technical condition on PEL-type $\mathcal{O}$-lattices:

Condition 1.9 [Lan 2008, Condition 1.4.3.9]. *The PEL-type* $\mathcal{O}$*-lattice* $(L, \langle \cdot , \cdot \rangle, h)$ *is chosen such that the action of* $\mathcal{O}$ *on* L *extends to an action of some maximal order* $\mathcal{O}'$ *in* B *containing* $\mathcal{O}$.

1C. ***Cusp labels.*** Although there is no *rational boundary components* in the theory of arithmetic compactifications (in mixed characteristics), we have developed in [Lan 2008, §5.4] the notion of *cusp labels* that serves a similar purpose. (While $\mathrm{G}(\mathbb{Q})$ plays an important role in the analytic theory over $\mathbb{C}$, it does not play any obvious role in the algebraic theory over $\mathcal{O}_{F_0,(\Box)}$. This is partly due to the so-called failure of Hasse's principle; see for example [Kottwitz 1992, §8] and [Lan 2008, Remark 1.4.3.11].)

Unlike in the analytic theory over $\mathbb{C}$, where boundary components are naturally parametrized by group-theoretic objects, the only algebraic machinery we have is the theory of semiabelian degenerations of abelian varieties with PEL structures. The cusp labels are (by their very design) part of the parameters (which we call the *degeneration data*) for such (semiabelian) degenerations.

Definition 1.10 [Lan 2008, §1.2.6]. Let R be any noetherian $\mathbb{Z}$-algebra. Suppose we have an increasing filtration $\mathtt{F} = \{\mathtt{F}_{-i}\}$ on $L \otimes_{\mathbb{Z}} R$, indexed by nonpositive integers $-i$, such that $\mathtt{F}_0 = L \otimes_{\mathbb{Z}} R$.

(1) We say that $\mathtt{F}$ is *integrable* if, for any i, $\mathrm{Gr}^{\mathtt{F}}_{-i} := \mathtt{F}_{-i}/\mathtt{F}_{-i-1}$ is integrable in the sense that $\mathrm{Gr}^{\mathtt{F}}_{-i} \cong M_i \otimes_{\mathbb{Z}} R$ (as R-modules) for some $\mathcal{O}$-lattice M_i.

(2) We say that $\mathtt{F}$ is *split* if there exists (noncanonically) some isomorphism $\mathrm{Gr}^{\mathtt{F}} := \oplus_i \mathrm{Gr}^{\mathtt{F}}_{-i} \xrightarrow{\sim} \mathtt{F}_0$ of R-modules.

(3) We say that $\mathtt{F}$ is *admissible* if it is both integrable and split.

(4) Let m be an integer. We say that $\mathtt{F}$ is *m-symplectic with respect to* $(L, \langle\,\cdot\,,\,\cdot\,\rangle)$ if, for any i, $\mathtt{F}_{-m+i}$ and $\mathtt{F}_{-i}$ are annihilators of each other under the pairing $\langle\,\cdot\,,\,\cdot\,\rangle$ on $\mathtt{F}_0$.

We shall only work with $m = 3$, and we shall suppress m in what follows. The fact that $\hat{\mathbb{Z}}^{\Box}$ involves *bad primes* (cf. Definition 1.5) is the main reason that we may have to allow nonprojective filtrations.

Definition 1.11 [Lan 2008, Definition 5.2.7.1]. We say that a symplectic admissible filtration $\mathtt{Z}$ on $L \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}^{\Box}$ is *fully symplectic with respect to* $(L, \langle\,\cdot\,,\,\cdot\,\rangle)$ if there is a symplectic admissible filtration $\mathtt{Z}_{\mathbb{A}^{\Box}} = \{\mathtt{Z}_{-i,\mathbb{A}^{\Box}}\}$ on $L \otimes_{\mathbb{Z}} \mathbb{A}^{\Box}$ that *extends* $\mathtt{Z}$ in the sense that $\mathtt{Z}_{-i,\mathbb{A}^{\Box}} \cap (L \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}^{\Box}) = \mathtt{Z}_{-i}$ in $L \otimes_{\mathbb{Z}} \mathbb{A}^{\Box}$ for all i.

Definition 1.12 [Lan 2008, Definition 5.2.7.3]. A symplectic-liftable admissible filtration $\mathtt{Z}_n$ on L/nL is called *fully symplectic-liftable* with respect to $(L, \langle\,\cdot\,,\,\cdot\,\rangle)$ if it is the reduction modulo n of some admissible filtration $\mathtt{Z}$ on $L \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}^{\Box}$ that is fully symplectic with respect to $(L, \langle\,\cdot\,,\,\cdot\,\rangle)$ as in Definition 1.11.

Degenerations into semiabelian schemes induce filtrations on Tate modules and on Lie algebras of the generic fibers. While the symplectic-liftable admissible filtrations represent (certain orbits of) filtrations on $L \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}^{\Box}$ induced by filtrations on Tate modules via the level structures, the fully symplectic-liftable ones are equipped with (certain orbits of) filtrations on $L \otimes_{\mathbb{Z}} \mathbb{R}$ induced by the filtrations on Lie algebras via the Lie algebra condition (4) in Definition 1.6. (One may interpret the Lie algebra condition as the "de Rham" (or rather "Hodge") component of a certain "complete level structure", the direct product of whose "ℓ-adic" components being a level structure in the usual sense.) Such (orbits of) filtrations are the crudest invariants of degenerations we consider.

Definition 1.13 [Lan 2008, Definition 5.4.1.3]. Given a fully symplectic admissible filtration Z on $L \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}^{\Box}$ with respect to $(L, \langle \cdot , \cdot \rangle)$ as in Definition 1.11, a *torus argument* Φ for Z is a tuple $\Phi := (X, Y, \phi, \varphi_{-2}, \varphi_0)$, where:

(1) X and Y are $\mathcal{O}$-lattices of the same $\mathcal{O}$-multirank (see [Lan 2008, Definition 5.2.2.5]), and $\phi : Y \hookrightarrow X$ is an $\mathcal{O}$-equivariant embedding.

(2) $\varphi_{-2} : \mathrm{Gr}^{\mathrm{Z}}_{-2} \xrightarrow{\sim} \mathrm{Hom}_{\hat{\mathbb{Z}}^{\Box}}(X \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}^{\Box}, \hat{\mathbb{Z}}^{\Box}(1))$ and $\varphi_0 : \mathrm{Gr}^{\mathrm{Z}}_0 \xrightarrow{\sim} Y \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}^{\Box}$ are isomorphisms (of $\hat{\mathbb{Z}}^{\Box}$-modules) such that the pairing $\langle \cdot , \cdot \rangle_{20} : \mathrm{Gr}^{\mathrm{Z}}_{-2} \times \mathrm{Gr}^{\mathrm{Z}}_0 \to \hat{\mathbb{Z}}^{\Box}(1)$ defined by Z is the pullback of the pairing

$$\langle \cdot , \cdot \rangle_{\phi} : \mathrm{Hom}_{\hat{\mathbb{Z}}^{\Box}}(X \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}^{\Box}, \hat{\mathbb{Z}}^{\Box}(1)) \times (Y \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}^{\Box}) \to \hat{\mathbb{Z}}^{\Box}(1)$$

defined by the composition

$$\begin{aligned}&\mathrm{Hom}_{\hat{\mathbb{Z}}^{\Box}}(X \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}^{\Box}, \hat{\mathbb{Z}}^{\Box}(1)) \times (Y \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}^{\Box}) \\ &\qquad \xrightarrow{\mathrm{Id} \times \phi} \mathrm{Hom}_{\hat{\mathbb{Z}}^{\Box}}(X \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}^{\Box}, \hat{\mathbb{Z}}^{\Box}(1)) \times (X \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}^{\Box}) \to \hat{\mathbb{Z}}^{\Box}(1),\end{aligned}$$

with the sign convention that $\langle \cdot , \cdot \rangle_{\phi}(x, y) = x(\phi(y)) = (\phi(y))(x)$ for any $x \in \mathrm{Hom}_{\hat{\mathbb{Z}}^{\Box}}(X \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}^{\Box}, \hat{\mathbb{Z}}^{\Box}(1))$ and any $y \in Y \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}^{\Box}$.

Definition 1.14 [Lan 2008, Definitions 5.4.1.4 and 5.4.1.5]. Given a fully symplectic-liftable admissible filtration Z_n on L/nL with respect to $(L, \langle \cdot , \cdot \rangle)$ as in Definition 1.12, a *torus argument* Φ_n *at level* n for Z_n is a tuple $\Phi_n := (X, Y, \phi, \varphi_{-2,n}, \varphi_{0,n})$, where:

(1) X and Y are $\mathcal{O}$-lattices of the same $\mathcal{O}$-multirank, and $\phi : Y \hookrightarrow X$ is an $\mathcal{O}$-equivariant embedding.

(2) $\varphi_{-2,n} : \mathrm{Gr}^{\mathrm{Z}}_{-2,n} \xrightarrow{\sim} \mathrm{Hom}(X/nX, (\mathbb{Z}/n\mathbb{Z})(1))$ (resp. $\varphi_{0,n} : \mathrm{Gr}^{\mathrm{Z}}_{0,n} \xrightarrow{\sim} Y/nY$) is an isomorphism that is the reduction modulo n of some isomorphism $\varphi_{-2} : \mathrm{Gr}^{\mathrm{Z}}_{-2} \xrightarrow{\sim} \mathrm{Hom}_{\hat{\mathbb{Z}}^{\Box}}(X \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}^{\Box}, \hat{\mathbb{Z}}^{\Box}(1))$ (resp. $\varphi_0 : \mathrm{Gr}^{\mathrm{Z}}_0 \xrightarrow{\sim} (Y \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}^{\Box})$), such that $\Phi = (X, Y, \phi, \varphi_{-2}, \varphi_0)$ form a torus argument as in Definition 1.13.

We say in this case that Φ_n is the reduction modulo n of Φ.

Two torus arguments $\Phi_n = (X, Y, \phi, \varphi_{-2,n}, \varphi_{0,n})$ and $\Phi'_n = (X', Y', \phi', \varphi'_{-2,n}, \varphi'_{0,n})$ at level n are *equivalent* if and only if there exists a pair of isomorphisms

$$(\gamma_X : X' \xrightarrow{\sim} X, \gamma_Y : Y \xrightarrow{\sim} Y')$$

(of $\mathcal{O}$-lattices) such that $\phi = \gamma_X \phi' \gamma_Y$, $\varphi'_{-2,n} = {}^t\gamma_X \varphi_{-2,n}$, and $\varphi'_{0,n} = \gamma_Y \varphi_{0,n}$. In this case, we say that Φ_n and Φ'_n are equivalent under the pair of isomorphisms $\gamma = (\gamma_X, \gamma_Y)$, which we denote as $\gamma = (\gamma_X, \gamma_Y) : \Phi_n \xrightarrow{\sim} \Phi'_n$.

The torus arguments record the isomorphism classes of the torus parts of degenerations of abelian schemes with PEL structures. These are the second crudest invariants of degenerations we consider.

Definition 1.15 [Lan 2008, Definition 5.4.1.9]. A (*principal*) *cusp label at level n* for a PEL-type $\mathcal{O}$-lattice $(L, \langle\cdot,\cdot\rangle, h)$, or a cusp label of the moduli problem M_n, is an equivalence class $[(\mathrm{Z}_n, \Phi_n, \delta_n)]$ of triples $(\mathrm{Z}_n, \Phi_n, \delta_n)$, where:

(1) Z_n is an admissible filtration on L/nL that is fully symplectic-liftable in the sense of Definition 1.12.

(2) Φ_n is a torus argument at level n for Z_n.

(3) $\delta_n : \mathrm{Gr}_n^{\mathrm{Z}} \xrightarrow{\sim} L/nL$ is a liftable splitting.

Two triples $(\mathrm{Z}_n, \Phi_n, \delta_n)$ and $(\mathrm{Z}'_n, \Phi'_n, \delta'_n)$ are equivalent if Z_n and Z'_n are identical, and if Φ_n and Φ'_n are equivalent as in Definition 1.14.

The liftable splitting δ_n in any triple $(\mathrm{Z}_n, \Phi_n, \delta_n)$ is noncanonical and auxiliary in nature. Such splittings are needed for analyzing the "degeneration of pairings" in general PEL cases (unlike in the special case in [Faltings and Chai 1990, Chapter IV, §6]).

To proceed from principal cusp labels at level n to general cusp labels at level $\mathcal{H}$, where $\mathcal{H}$ is an open compact subgroup of $\mathrm{G}(\hat{\mathbb{Z}}^{\Box})$, we form *étale orbits* of the objects we have thus defined. The precise definitions are complicated (see [Lan 2008, Definitions 5.4.2.1, 5.4.2.2, and 5.4.2.4]) but the idea is simple: For any $\mathcal{H}$ as above, consider those $n \geq 1$ sufficiently divisible such that $\Box \nmid n$ and $\mathcal{U}^{\Box}(n) \subset \mathcal{H}$. Then we have a compatible system of finite groups $\mathcal{H}_n = \mathcal{H}/\mathcal{U}^{\Box}(n)$, and an object at level $\mathcal{H}$ is simply defined to be a compatible system of étale $\mathcal{H}_n$-orbits of objects at running levels n as above. Then we arrive at the notions of *torus arguments* $\Phi_{\mathcal{H}} = (X, Y, \phi, \varphi_{-2,n}, \varphi_{0,n})$ *at level* $\mathcal{H}$, and of *representatives* $(\mathrm{Z}_{\mathcal{H}}, \Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ *of cusp labels* $[(\mathrm{Z}_{\mathcal{H}}, \Phi_{\mathcal{H}}, \delta_{\mathcal{H}})]$ *at level* $\mathcal{H}$. (The liftability condition is implicit in such a definition, as in the definition of level structures we omitted.) By abuse of language, we call these $\mathcal{H}$-orbits of $\Phi = (X, Y, \phi, \varphi_{-2}, \varphi_0)$, $(\mathrm{Z}, \Phi, \delta)$, and $[(\mathrm{Z}, \Phi, \delta)]$, respectively.

For simplicity, we shall often omit $\mathrm{Z}_{\mathcal{H}}$ from the notation.

Lemma 1.16 [Lan 2008, Lemma 5.2.7.5 in the revision]. *Let Z_n be an admissible filtration on L/nL that is fully symplectic-liftable with respect to $(L, \langle\cdot,\cdot\rangle)$. Let $(\mathrm{Gr}_{-1}^{\mathrm{Z}}, \langle\cdot,\cdot\rangle_{11})$ be induced by some fully symplectic lifting Z of Z_n, and let $(\mathrm{Gr}_{-1,\mathbb{R}}^{\mathrm{Z}}, \langle\cdot,\cdot\rangle_{11,\mathbb{R}}, h_{-1})$ be determined by* [Lan 2008, Proposition 5.1.2.2 in the revision] *by any extension $\mathrm{Z}_{\mathbb{A}^{\Box}}$ in Definition 1.11* (*which has the same reflex field F_0 as $(L \otimes_{\mathbb{Z}} \mathbb{R}, \langle\cdot,\cdot\rangle, h)$ does*)*. Then there is associated* (*noncanonically*) *a PEL-type $\mathcal{O}$-lattice $(L_n^{\mathrm{Z}}, \langle\cdot,\cdot\rangle_n^{\mathrm{Z}}, h_n^{\mathrm{Z}})$ satisfying Condition 1.9 such that*:

(1) $[(L_n^{\mathrm{Z}})^{\#} : L_n^{\mathrm{Z}}]$ *is prime to* $\Box$.

(2) *There exist* (*noncanonical*) *$\mathcal{O}$-equivariant isomorphisms*

$$(\mathrm{Gr}_{-1}^{\mathrm{Z}}, \langle\cdot,\cdot\rangle_{11}) \xrightarrow{\sim} (L^{\mathrm{Z}_n} \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}^{\Box}, \langle\cdot,\cdot\rangle^{\mathrm{Z}_n})$$

and

$$(\mathrm{Gr}^{\mathsf{Z}}_{-1,\mathbb{R}}, \langle\,\cdot\,,\,\cdot\,\rangle_{11,\mathbb{R}}, h_{-1}) \xrightarrow{\sim} (L^{\mathsf{Z}_n} \otimes_{\mathbb{Z}} \mathbb{R}, \langle\,\cdot\,,\,\cdot\,\rangle^{\mathsf{Z}_n}, h^{\mathsf{Z}_n}).$$

(3) *The moduli problem $\mathsf{M}_n^{\mathsf{Z}_n}$ defined by the noncanonical $(L^{\mathsf{Z}_n}, \langle\,\cdot\,,\,\cdot\,\rangle^{\mathsf{Z}_n}, h^{\mathsf{Z}_n})$ as in Definition 1.6 is canonical in the sense that it depends* (*up to isomorphism*) *only on Z_n, but not on the choice of $(L^{\mathsf{Z}_n}, \langle\,\cdot\,,\,\cdot\,\rangle^{\mathsf{Z}_n}, h^{\mathsf{Z}_n})$.*

Definition 1.17 [Lan 2008, Definition 5.4.2.6]. The PEL-type $\mathcal{O}$-lattice

$$(L^{\mathsf{Z}_{\mathcal{H}}}, \langle\,\cdot\,,\,\cdot\,\rangle^{\mathsf{Z}_{\mathcal{H}}}, h^{\mathsf{Z}_{\mathcal{H}}})$$

is a fixed (noncanonical) choice of any of the PEL-type $\mathcal{O}$-lattice $(L^{\mathsf{Z}_n}, \langle\,\cdot\,,\,\cdot\,\rangle^{\mathsf{Z}_n}, h^{\mathsf{Z}_n})$ in Lemma 1.16 for any element Z_n in any $\mathsf{Z}_{\mathcal{H}_n}$ (in $\mathsf{Z}_{\mathcal{H}} = \{\mathsf{Z}_{\mathcal{H}_n}\}$, a compatible collection of étale orbits $\mathsf{Z}_{\mathcal{H}_n}$ at various levels n such that $\Box \nmid n$ and $\mathcal{U}^{\Box}(n) \subset \mathcal{H}$). The elements of $\mathcal{H}_n$ leaving Z_n invariant induce a subgroup of $\mathrm{G}_{(L^{\mathsf{Z}_n}, \langle\cdot,\cdot\rangle^{\mathsf{Z}_n}, h^{\mathsf{Z}_n})}(\mathbb{Z}/n\mathbb{Z})$. Let $\mathcal{H}_h$ be the preimage of this subgroup under

$$\mathrm{G}_{(L^{\mathsf{Z}_n}, \langle\cdot,\cdot\rangle^{\mathsf{Z}_n}, h^{\mathsf{Z}_n})}(\hat{\mathbb{Z}}^{\Box}) \twoheadrightarrow \mathrm{G}_{(L^{\mathsf{Z}_n}, \langle\cdot,\cdot\rangle^{\mathsf{Z}_n}, h^{\mathsf{Z}_n})}(\mathbb{Z}/n\mathbb{Z}).$$

Then we define $\mathsf{M}_{\mathcal{H}}^{\mathsf{Z}_{\mathcal{H}}}$ to be the moduli problem defined by $(L^{\mathsf{Z}_n}, \langle\,\cdot\,,\,\cdot\,\rangle^{\mathsf{Z}_n}, h^{\mathsf{Z}_n})$ with level-$\mathcal{H}_h$ structures as in Lemma 1.16. (The isomorphism class of this final moduli problem is independent of the choice of $(L^{\mathsf{Z}_{\mathcal{H}}}, \langle\,\cdot\,,\,\cdot\,\rangle^{\mathsf{Z}_{\mathcal{H}}}, h^{\mathsf{Z}_{\mathcal{H}}}) = (L^{\mathsf{Z}_n}, \langle\,\cdot\,,\,\cdot\,\rangle^{\mathsf{Z}_n}, h^{\mathsf{Z}_n})$.)

Such *boundary moduli problems* $\mathsf{M}_{\mathcal{H}}^{\mathsf{Z}_{\mathcal{H}}}$ are the fundamental building blocks in the construction of toroidal boundary charts for $\mathsf{M}_{\mathcal{H}}$. (They actually appear in the boundary of the minimal compactification of $\mathsf{M}_{\mathcal{H}}$, which we call *cusps*. They are parametrized by the cusp labels of $\mathsf{M}_{\mathcal{H}}$.)

It is important to study the relations among cusp labels of different multiranks.

Definition 1.18 [Lan 2008, Definition 5.4.1.15]. A *surjection*

$$(\mathsf{Z}_n, \Phi_n, \delta_n) \twoheadrightarrow (\mathsf{Z}'_n, \Phi'_n, \delta'_n)$$

between representatives of cusp labels at level n, where $\Phi_n = (X, Y, \phi, \varphi_{-2,n}, \varphi_{0,n})$ and $\Phi'_n = (X', Y', \phi', \varphi'_{-2,n}, \varphi'_{0,n})$, is a pair (of surjections) $(s_X : X \twoheadrightarrow X', s_Y : Y \twoheadrightarrow Y')$ (of $\mathcal{O}$-lattices) such that:

(1) Both s_X and s_Y are admissible surjections (i.e., with kernels defining filtrations that are admissible in the sense of Definition 1.10), and they are compatible with ϕ and ϕ' in the sense that $s_X \phi = \phi' s_Y$.

(2) $\mathsf{Z}'_{-2,n}$ is an admissible submodule of $\mathsf{Z}_{-2,n}$, and the natural embedding $\mathrm{Gr}^{\mathsf{Z}'}_{-2,n} \hookrightarrow \mathrm{Gr}^{\mathsf{Z}_{-2,n}}$ satisfies $\varphi_{-2,n} \circ (\mathrm{Gr}^{\mathsf{Z}'}_{-2,n} \hookrightarrow \mathrm{Gr}^{\mathsf{Z}_{-2,n}}) = s_X^* \circ \varphi'_{-2,n}$.

(3) $\mathsf{Z}_{-1,n}$ is an admissible submodule of $\mathsf{Z}'_{-1,n}$, and the natural surjection $\mathrm{Gr}^{\mathsf{Z}_{0,n}} \twoheadrightarrow \mathrm{Gr}^{\mathsf{Z}'}_{0,n}$ satisfies $s_Y \circ \varphi_{0,n} = \varphi'_{0,n} \circ (\mathrm{Gr}^{\mathsf{Z}_{0,n}} \twoheadrightarrow \mathrm{Gr}^{\mathsf{Z}'}_{0,n})$.

In this case, we write $s = (s_X, s_Y) : (\mathrm{Z}_n, \Phi_n, \delta_n) \twoheadrightarrow (\mathrm{Z}'_n, \Phi'_n, \delta'_n)$

By taking orbits as before, there is a corresponding notion for general cusp labels:

Definition 1.19 [Lan 2008, Definition 5.4.2.12]. A *surjection* $(\mathrm{Z}_{\mathcal{H}}, \Phi_{\mathcal{H}}, \delta_{\mathcal{H}}) \twoheadrightarrow (\mathrm{Z}'_{\mathcal{H}}, \Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$ between representatives of cusp labels at level $\mathcal{H}$, where $\Phi_{\mathcal{H}} = (X, Y, \phi, \varphi_{-2,\mathcal{H}}, \varphi_{0,\mathcal{H}})$ and $\Phi'_{\mathcal{H}} = (X', Y', \phi', \varphi'_{-2,\mathcal{H}}, \varphi'_{0,\mathcal{H}})$, is a pair (of surjections) $s = (s_X : X \twoheadrightarrow X', s_Y : Y \twoheadrightarrow Y')$ (of $\mathcal{O}$-lattices) such that:

(1) Both s_X and s_Y are admissible surjections, and they are compatible with ϕ and ϕ' in the sense that $s_X \phi = \phi' s_Y$.

(2) $\mathrm{Z}'_{\mathcal{H}}$ and $(\varphi'_{-2,\mathcal{H}}, \varphi'_{0,\mathcal{H}})$ are assigned to $\mathrm{Z}_{\mathcal{H}}$ and $(\varphi_{-2,\mathcal{H}}, \varphi_{0,\mathcal{H}})$ respectively under $s = (s_X, s_Y)$ as in [Lan 2008, Lemma 5.4.2.11].

In this case, we write $s = (s_X, s_Y) : (\mathrm{Z}_{\mathcal{H}}, \Phi_{\mathcal{H}}, \delta_{\mathcal{H}}) \twoheadrightarrow (\mathrm{Z}'_{\mathcal{H}}, \Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$.

Definition 1.20 [Lan 2008, Definition 5.4.2.13]. We say that there is a *surjection* from a cusp label at level $\mathcal{H}$ represented by some $(\mathrm{Z}_{\mathcal{H}}, \Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ to a cusp label at level $\mathcal{H}$ represented by some $(\mathrm{Z}'_{\mathcal{H}}, \Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$ if there is a surjection (s_X, s_Y) from $(\mathrm{Z}_{\mathcal{H}}, \Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ to $(\mathrm{Z}'_{\mathcal{H}}, \Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$.

This is well defined by [Lan 2008, Lemma 5.4.1.16].

The surjection among cusp labels can be naturally seen when we have the so-called *two-step degenerations* (see [Faltings and Chai 1990, Chapter III, §10] and [Lan 2008, §4.5.6 in the revision]). This notion will be further developed in Definitions 1.32, 1.37, and 1.38 below.

1D. ***Cone decompositions.*** For any torus argument $\Phi_n = (X, Y, \phi, \varphi_{-2,n}, \varphi_{0,n})$ at level n, consider the finitely generated commutative group (i.e., $\mathbb{Z}$-module)

$$\dddot{\mathbf{S}}_{\Phi_n} := ((\tfrac{1}{n}Y) \otimes_{\mathbb{Z}} X) \Big/ \begin{pmatrix} y \otimes \phi(y') - y' \otimes \phi(y) \\ (b\tfrac{1}{n}y) \otimes \chi - (\tfrac{1}{n}y) \otimes (b^\star \chi) \end{pmatrix}_{\substack{y, y' \in Y \\ \chi \in X,\ b \in \mathcal{O}}} \tag{1.21}$$

and set $\mathbf{S}_{\Phi_n} := \dddot{\mathbf{S}}_{\Phi_n,\mathrm{free}}$, the free quotient of $\dddot{\mathbf{S}}_{\Phi_n}$. (See [Lan 2008, (6.2.3.5) and Convention 6.2.3.26].) Then, for a general torus argument $\Phi_{\mathcal{H}} = (X, Y, \phi, \varphi_{-2,\mathcal{H}}, \varphi_{0,\mathcal{H}})$ at level $\mathcal{H}$, there is a recipe [Lan 2008, Lemma 6.2.4.4] that gives a corresponding free commutative group $\mathbf{S}_{\Phi_{\mathcal{H}}}$ (which can be identified with a finite index subgroup of some $\mathbf{S}_{\Phi_n}$).

The group $\mathbf{S}_{\Phi_{\mathcal{H}}}$ provides indices for certain "Laurent series expansions" near the boundary strata. In the modular curve case, it is canonically isomorphic to $\mathbb{Z}$, which means there is a canonical parameter q near the boundary — i.e., the *cusps*. The expansion of modular forms with respect to this parameter then gives the familiar q-expansion along the cusps. The compactification of the modular curves can be described locally near each of the cusps by $\mathrm{Spec}(R[q^i]_{i \in \mathbb{Z}}) \hookrightarrow \mathrm{Spec}(R[q^i]_{i \in \mathbb{Z}_{\geq 0}})$

for some suitable base ring R. For $\mathsf{M}_{\mathcal{H}}$, we would like to have an analogous theory in which the torus with the character group $\mathbf{S}_{\Phi_{\mathcal{H}}}$ can be partially compactified by adding normal crossings divisors in a smooth scheme. This is best achieved by the theory of *toroidal embeddings* developed in [Kempf et al. 1973]. Many terminologies in such a theory will naturally show up in our description of the toroidal boundary charts, and we will review them in what follows.

Let $\mathbf{S}_{\Phi_{\mathcal{H}}}^{\vee} := \mathrm{Hom}_{\mathbb{Z}}(\mathbf{S}_{\Phi_{\mathcal{H}}}, \mathbb{Z})$ be the $\mathbb{Z}$-dual of $\mathbf{S}_{\Phi_{\mathcal{H}}}$, and let $(\mathbf{S}_{\Phi_{\mathcal{H}}})_{\mathbb{R}}^{\vee} := \mathbf{S}_{\Phi_{\mathcal{H}}}^{\vee} \otimes_{\mathbb{Z}} \mathbb{R} = \mathrm{Hom}_{\mathbb{Z}}(\mathbf{S}_{\Phi_{\mathcal{H}}}, \mathbb{R})$. By construction of $\mathbf{S}_{\Phi_{\mathcal{H}}}$, the $\mathbb{R}$-vector space $(\mathbf{S}_{\Phi_{\mathcal{H}}})_{\mathbb{R}}^{\vee}$ is isomorphic to the space of Hermitian pairings $(\!|\cdot,\cdot|\!) : (Y \otimes_{\mathbb{Z}} \mathbb{R}) \times (Y \otimes_{\mathbb{Z}} \mathbb{R}) \to \mathcal{O} \otimes_{\mathbb{Z}} \mathbb{R} = B \otimes_{\mathbb{Q}} \mathbb{R}$, by sending a Hermitian pairing $(\!|\cdot,\cdot|\!)$ to the function $y \otimes \phi(y') \mapsto \mathrm{Tr}_{B/\mathbb{Q}}(\!|y, y'|\!)$ in $\mathrm{Hom}_{\mathbb{R}}((Y \otimes_{\mathbb{Z}} \mathbb{R}) \times (Y \otimes_{\mathbb{Z}} \mathbb{R}), \mathbb{R}) \cong (\mathbf{S}_{\Phi_{\mathcal{H}}})_{\mathbb{R}}^{\vee}$. (See [Lan 2008, Lemma 1.1.4.6].)

Definition 1.22 [Lan 2008, beginning of §6.1.1]. (1) A subset of $(\mathbf{S}_{\Phi_{\mathcal{H}}})_{\mathbb{R}}^{\vee}$ is called a *cone* if it is invariant under the natural multiplication action of $\mathbb{R}_{>0}^{\times}$ on the $\mathbb{R}$-vector space $(\mathbf{S}_{\Phi_{\mathcal{H}}})_{\mathbb{R}}^{\vee}$.

(2) A cone in $(\mathbf{S}_{\Phi_{\mathcal{H}}})_{\mathbb{R}}^{\vee}$ is *nondegenerate* if its closure does not contain any nonzero $\mathbb{R}$-vector subspace of $(\mathbf{S}_{\Phi_{\mathcal{H}}})_{\mathbb{R}}^{\vee}$.

(3) A *rational polyhedral cone* in $(\mathbf{S}_{\Phi_{\mathcal{H}}})_{\mathbb{R}}^{\vee}$ is a cone in $(\mathbf{S}_{\Phi_{\mathcal{H}}})_{\mathbb{R}}^{\vee}$ of the form $\sigma = \mathbb{R}_{>0}v_1 + \ldots + \mathbb{R}_{>0}v_n$ with $v_1, \ldots, v_n \in (\mathbf{S}_{\Phi_{\mathcal{H}}})_{\mathbb{Q}}^{\vee} = \mathbf{S}_{\Phi_{\mathcal{H}}}^{\vee} \otimes_{\mathbb{Z}} \mathbb{Q}$.

(4) A *supporting hyperplane* of σ is a hyperplane P in $(\mathbf{S}_{\Phi_{\mathcal{H}}})_{\mathbb{R}}^{\vee}$ such that σ does not overlap with both sides of P.

(5) A *face* of σ is a rational polyhedral cone τ such that $\overline{\tau} = \overline{\sigma} \cap P$ for some supporting hyperplane P of σ. (Here an overline on a cone means its closure in the ambient space $(\mathbf{S}_{\Phi_{\mathcal{H}}})_{\mathbb{R}}^{\vee}$.)

Let $\mathbf{P}_{\Phi_{\mathcal{H}}}$ be the subset of $(\mathbf{S}_{\Phi_{\mathcal{H}}})_{\mathbb{R}}^{\vee}$ corresponding to positive semidefinite Hermitian pairings $(\!|\cdot,\cdot|\!) : (Y \otimes_{\mathbb{Z}} \mathbb{R}) \times (Y \otimes_{\mathbb{Z}} \mathbb{R}) \to B \otimes_{\mathbb{Q}} \mathbb{R}$, with radical (namely the annihilator of the whole space) *admissible* in the sense that it is the $\mathbb{R}$-span of some *admissible submodule* Y' of Y. (We say a submodule Y' of Y is admissible if $Y' \subset Y$ defines an admissible filtration on Y; cf. Definition 1.10. In particular, the quotient Y/Y' is also an $\mathcal{O}$-lattice.)

Definition 1.23 [Lan 2008, Definitions 6.2.4.1 and 5.4.1.6]. The group $\Gamma_{\Phi_{\mathcal{H}}}$ is the subgroup of elements $\gamma = (\gamma_X, \gamma_Y)$ in $\mathrm{GL}_{\mathcal{O}}(X) \times \mathrm{GL}_{\mathcal{O}}(Y)$ satisfying $\phi = \gamma_X \phi \gamma_Y$, $\varphi_{-2,\mathcal{H}} = {}^t\gamma_X \varphi_{-2,\mathcal{H}}$, and $\varphi_{0,\mathcal{H}} = \gamma_Y \varphi_{0,\mathcal{H}}$ (if we view the latter two as collections of orbits).

The group $\Gamma_{\Phi_{\mathcal{H}}}$ acts on $\mathbf{S}_{\Phi_{\mathcal{H}}}$, and its induced action preserves the subset $\mathbf{P}_{\Phi_{\mathcal{H}}}$ of $(\mathbf{S}_{\Phi_{\mathcal{H}}})_{\mathbb{R}}^{\vee}$. (The group $\Gamma_{\Phi_{\mathcal{H}}}$ is the automorphism group of the torus argument $\Phi_{\mathcal{H}}$. Such automorphism groups show up naturally because torus arguments are only determined up to isomorphism.)

Definition 1.24 [Lan 2008, Definition 6.1.1.12]. A $\Gamma_{\Phi_\mathcal{H}}$*-admissible rational polyhedral cone decomposition* of $\mathbf{P}_{\Phi_\mathcal{H}}$ is a collection $\Sigma = \{\sigma_j\}_{j\in J}$ with some indexing set J such that:

(1) Every σ_j is a nondegenerate rational polyhedral cone.

(2) $\mathbf{P}_{\Phi_\mathcal{H}}$ is the disjoint union of all the σ_j's in Σ. For each $j \in J$, the closure of σ_j in $\mathbf{P}_{\Phi_\mathcal{H}}$ is a disjoint union of σ_k's with $k \in J$. In other words, $\mathbf{P}_{\Phi_\mathcal{H}} = \coprod_{j\in J} \sigma_j$ is a stratification of $\mathbf{P}_{\Phi_\mathcal{H}}$.

(3) Σ is invariant under the action of $\Gamma_{\Phi_\mathcal{H}}$ on $(\mathbf{S}_{\Phi_\mathcal{H}})^\vee_\mathbb{R}$, in the sense that $\Gamma_{\Phi_\mathcal{H}}$ permutes the cones in Σ. Under this action, the set $\Sigma/\Gamma_{\Phi_\mathcal{H}}$ of $\Gamma_{\Phi_\mathcal{H}}$-orbits is finite.

Definition 1.25 [Lan 2008, Definition 6.1.1.13]. A rational polyhedral cone σ in $(\mathbf{S}_{\Phi_\mathcal{H}})^\vee_\mathbb{R}$ is *smooth* with respect to the integral structure given by $\mathbf{S}^\vee_{\Phi_\mathcal{H}}$ if we have $\sigma = \mathbb{R}_{>0}v_1 + \ldots + \mathbb{R}_{>0}v_n$ with $v_1, \ldots, v_n$ part of a $\mathbb{Z}$-basis of $\mathbf{S}^\vee_{\Phi_\mathcal{H}}$.

Definition 1.26 [Lan 2008, Definition 6.1.1.14]. A $\Gamma_{\Phi_\mathcal{H}}$*-admissible smooth rational polyhedral cone decomposition* of $\mathbf{P}_{\Phi_\mathcal{H}}$ is a $\Gamma_{\Phi_\mathcal{H}}$-admissible rational polyhedral cone decomposition $\{\sigma_j\}_{j\in J}$ of $\mathbf{P}_{\Phi_\mathcal{H}}$ in which every σ_j is *smooth*.

Definition 1.27 [Lan 2008, Definition 7.3.1.1]. Let

$$\Sigma_{\Phi_\mathcal{H}} = \{\sigma_j\}_{j\in J}$$

be any $\Gamma_{\Phi_\mathcal{H}}$-admissible rational polyhedral cone decomposition of $\mathbf{P}_{\Phi_\mathcal{H}}$. An (*invariant*) *polarization function* on $\mathbf{P}_{\Phi_\mathcal{H}}$ for the cone decomposition $\Sigma_{\Phi_\mathcal{H}}$ is a $\Gamma_{\Phi_\mathcal{H}}$-invariant continuous piecewise linear function $\mathrm{pol}_{\Phi_\mathcal{H}} : \mathbf{P}_{\Phi_\mathcal{H}} \to \mathbb{R}_{\geq 0}$ such that:

(1) $\mathrm{pol}_{\Phi_\mathcal{H}}$ is linear (i.e., coincides with a linear function) on each cone σ_j in $\Sigma_{\Phi_\mathcal{H}}$. (In particular, $\mathrm{pol}_{\Phi_\mathcal{H}}(tx) = t\,\mathrm{pol}_{\Phi_\mathcal{H}}(x)$ for any $x \in \mathbf{P}_{\Phi_\mathcal{H}}$ and $t \in \mathbb{R}_{\geq 0}$.)

(2) $\mathrm{pol}_{\Phi_\mathcal{H}}((\mathbf{P}_{\Phi_\mathcal{H}} \cap \mathbf{S}^\vee_{\Phi_\mathcal{H}}) - \{0\}) \subset \mathbb{Z}_{>0}$. (In particular, $\mathrm{pol}_{\Phi_\mathcal{H}}(x) > 0$ for any nonzero x in $\mathbf{P}_{\Phi_\mathcal{H}}$.)

(3) $\mathrm{pol}_{\Phi_\mathcal{H}}$ is linear (in the above sense) on a rational polyhedral cone σ in $\mathbf{P}_{\Phi_\mathcal{H}}$ if and only if σ is contained in some cone σ_j in $\Sigma_{\Phi_\mathcal{H}}$.

(4) For any $x, y \in \mathbf{S}_{\Phi_\mathcal{H}}$, we have $\mathrm{pol}_{\Phi_\mathcal{H}}(x + y) \geq \mathrm{pol}_{\Phi_\mathcal{H}}(x) + \mathrm{pol}_{\Phi_\mathcal{H}}(y)$. This is called the *convexity* of $\mathrm{pol}_{\Phi_\mathcal{H}}$.

If such a polarization function exists, then we say that the $\Gamma_{\Phi_\mathcal{H}}$-admissible rational polyhedral cone decomposition $\Sigma_{\Phi_\mathcal{H}}$ is *projective*.

Definition 1.28. An *admissible boundary component* of $\mathbf{P}_{\Phi_\mathcal{H}}$ is the image of $\mathbf{P}_{\Phi'_\mathcal{H}}$ under the embedding $(\mathbf{S}_{\Phi'_\mathcal{H}})^\vee_\mathbb{R} \hookrightarrow (\mathbf{S}_{\Phi_\mathcal{H}})^\vee_\mathbb{R}$ defined by some surjection $(\Phi_\mathcal{H}, \delta_\mathcal{H}) \twoheadrightarrow (\Phi'_\mathcal{H}, \delta'_\mathcal{H})$. (See Definition 1.19.)

We shall always assume that the following technical condition is satisfied:

Condition 1.29 (cf. [Faltings and Chai 1990, Chapter IV, Remark 5.8(a)]; see also [Lan 2008, Condition 6.2.5.25 in the revision]). *The cone decomposition $\Sigma_{\Phi_{\mathcal{H}}} = \{\sigma_j\}_{j\in J}$ of $\mathbf{P}_{\Phi_{\mathcal{H}}}$ is chosen such that, for any $j \in J$, if $\gamma\bar{\sigma}_j \cap \bar{\sigma}_j \neq \{0\}$ for some $\gamma \in \Gamma_{\Phi_{\mathcal{H}}}$, then γ acts as the identity on the smallest admissible boundary component of $\mathbf{P}_{\Phi_{\mathcal{H}}}$ containing σ_j.*

This condition is used to ensure that there are no self-intersections of toroidal boundary strata when the level $\mathcal{H}$ is neat.

To describe the toroidal boundary of $\mathsf{M}_{\mathcal{H}}$, we will need not only cusp labels but also the cones:

Definition 1.30 [Lan 2008, Definition 6.2.6.1]. Let $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ and $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$ be two representatives of cusp labels at level $\mathcal{H}$, let $\sigma \subset (\mathbf{S}_{\Phi_{\mathcal{H}}})^\vee_{\mathbb{R}}$, and let $\sigma' \subset (\mathbf{S}_{\Phi'_{\mathcal{H}}})^\vee_{\mathbb{R}}$. We say that the two triples $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)$ and $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}}, \sigma')$ are *equivalent* if there exists a pair of isomorphisms $\gamma = (\gamma_X : X' \xrightarrow{\sim} X, \gamma_Y : Y \xrightarrow{\sim} Y')$ (of $\mathcal{O}$-lattices) such that:

(1) The two representatives $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ and $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$ are equivalent under γ (as in [Lan 2008, Definition 5.4.2.4], the general level analogue of Definition 1.15).

(2) The isomorphism $(\mathbf{S}_{\Phi'_{\mathcal{H}}})^\vee_{\mathbb{R}} \xrightarrow{\sim} (\mathbf{S}_{\Phi_{\mathcal{H}}})^\vee_{\mathbb{R}}$ induced by γ sends σ' to σ.

In this case, we say that the two triples $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)$ and $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}}, \sigma')$ are equivalent under the pair of isomorphisms $\gamma = (\gamma_X, \gamma_Y)$.

Definition 1.31 [Lan 2008, Definition 6.2.6.2]. Let $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ and $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$ be two representatives of cusp labels at level $\mathcal{H}$, and let $\Sigma_{\Phi_{\mathcal{H}}}$ (resp. $\Sigma_{\Phi'_{\mathcal{H}}}$) be a $\Gamma_{\Phi_{\mathcal{H}}}$-admissible (resp. $\Gamma_{\Phi'_{\mathcal{H}}}$-admissible) smooth rational polyhedral cone decomposition of $\mathbf{P}_{\Phi_{\mathcal{H}}}$ (resp. $\mathbf{P}_{\Phi'_{\mathcal{H}}}$). We say that the two triples $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \Sigma_{\Phi_{\mathcal{H}}})$ and $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}}, \Sigma_{\Phi'_{\mathcal{H}}})$ are *equivalent* if $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ and $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$ are equivalent under some pair of isomorphisms $\gamma = (\gamma_X : X' \xrightarrow{\sim} X, \gamma_Y : Y \xrightarrow{\sim} Y')$, and if under one (and hence every) such γ the cone decomposition $\Sigma_{\Phi_{\mathcal{H}}}$ of $\mathbf{P}_{\Phi_{\mathcal{H}}}$ is identified with the cone decomposition $\Sigma_{\Phi'_{\mathcal{H}}}$ of $\mathbf{P}_{\Phi'_{\mathcal{H}}}$. In this case we say that the two triples $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \Sigma_{\Phi_{\mathcal{H}}})$ and $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}}, \Sigma_{\Phi'_{\mathcal{H}}})$ are equivalent under the pair of isomorphisms $\gamma = (\gamma_X, \gamma_Y)$.

The compatibility among cone decompositions over different cusp labels is described as follows:

Definition 1.32 [Lan 2008, Definition 6.2.6.4]. Let $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ and $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$ be two representatives of cusp labels at level $\mathcal{H}$, and let $\Sigma_{\Phi_{\mathcal{H}}}$ (resp. $\Sigma_{\Phi'_{\mathcal{H}}}$) be a $\Gamma_{\Phi_{\mathcal{H}}}$-admissible (resp. $\Gamma_{\Phi'_{\mathcal{H}}}$-admissible) smooth rational polyhedral cone decomposition of $\mathbf{P}_{\Phi_{\mathcal{H}}}$ (resp. $\mathbf{P}_{\Phi'_{\mathcal{H}}}$). A *surjection* $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \Sigma_{\Phi_{\mathcal{H}}}) \twoheadrightarrow (\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}}, \Sigma_{\Phi'_{\mathcal{H}}})$ is given by a surjection $s = (s_X : X \twoheadrightarrow X', s_Y : Y \twoheadrightarrow Y') : (\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}) \twoheadrightarrow (\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$ (see Definition 1.19) that induces an embedding $\mathbf{P}_{\Phi'_{\mathcal{H}}} \hookrightarrow \mathbf{P}_{\Phi_{\mathcal{H}}}$ such that the restriction $\Sigma_{\Phi_{\mathcal{H}}}|_{\mathbf{P}_{\Phi'_{\mathcal{H}}}}$ of the cone decomposition $\Sigma_{\Phi_{\mathcal{H}}}$ of $\mathbf{P}_{\Phi_{\mathcal{H}}}$ to $\mathbf{P}_{\Phi'_{\mathcal{H}}}$ is the cone decomposition $\Sigma_{\Phi'_{\mathcal{H}}}$ of $\mathbf{P}_{\Phi'_{\mathcal{H}}}$.

This allows us to define:

Definition 1.33 [Lan 2008, Condition 6.3.3.1 and Definition 6.3.3.2]. A *compatible choice of admissible smooth rational polyhedral cone decomposition data* for $\mathsf{M}_{\mathcal{H}}$ is a complete set $\Sigma = \{\Sigma_{\Phi_{\mathcal{H}}}\}$ of *compatible choices* of $\Sigma_{\Phi_{\mathcal{H}}}$ (satisfying Condition 1.29) such that, for every surjection $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}) \twoheadrightarrow (\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$ of representatives of cusp labels, the cone decompositions $\Sigma_{\Phi_{\mathcal{H}}}$ and $\Sigma_{\Phi'_{\mathcal{H}}}$ define a surjection $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \Sigma_{\Phi_{\mathcal{H}}}) \twoheadrightarrow (\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}}, \Sigma_{\Phi'_{\mathcal{H}}})$ as in Definition 1.32.

Definition 1.34 [Lan 2008, Definition 7.3.1.3]. We say that a compatible choice $\Sigma = \{\Sigma_{\Phi_{\mathcal{H}}}\}$ of admissible smooth rational polyhedral cone decomposition data for $\mathsf{M}_{\mathcal{H}}$ (see Definition 1.33) is *projective* if it satisfies the following condition: There is a collection $\mathsf{pol} = \{\mathsf{pol}_{\Phi_{\mathcal{H}}} : \mathbf{P}_{\Phi_{\mathcal{H}}} \to \mathbb{R}_{\geq 0}\}$ of polarization functions labeled by representatives $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ of cusp labels, each $\mathsf{pol}_{\Phi_{\mathcal{H}}}$ being a polarization function of the cone decomposition $\Sigma_{\Phi_{\mathcal{H}}}$ in Σ (see Definition 1.27), which are *compatible* in the following sense: For any surjection $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}) \twoheadrightarrow (\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$ of representatives of cusp labels (see Definition 1.19) inducing an embedding $\mathbf{P}_{\Phi'_{\mathcal{H}}} \hookrightarrow \mathbf{P}_{\Phi_{\mathcal{H}}}$, we have $\mathsf{pol}_{\Phi_{\mathcal{H}}}|_{\mathbf{P}_{\Phi'_{\mathcal{H}}}} = \mathsf{pol}_{\Phi'_{\mathcal{H}}}$.

The most important relations among cone decompositions and among compatible choices of them are the so-called *refinements*:

Definition 1.35 [Lan 2008, Definition 6.2.6.3]. Let $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ and $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$ be two representatives of cusp labels at level $\mathcal{H}$, and let $\Sigma_{\Phi_{\mathcal{H}}}$ (resp. $\Sigma_{\Phi'_{\mathcal{H}}}$) be a $\Gamma_{\Phi_{\mathcal{H}}}$-admissible (resp. $\Gamma_{\Phi'_{\mathcal{H}}}$-admissible) smooth rational polyhedral cone decomposition of $\mathbf{P}_{\Phi_{\mathcal{H}}}$ (resp. $\mathbf{P}_{\Phi'_{\mathcal{H}}}$). We say that the triple $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \Sigma_{\Phi_{\mathcal{H}}})$ is a *refinement* of the triple $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}}, \Sigma_{\Phi'_{\mathcal{H}}})$ if $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ and $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$ are equivalent under some pair of isomorphisms $\gamma = (\gamma_X, \gamma_Y)$, and if under one (and hence every) such γ the cone decomposition $\Sigma_{\Phi_{\mathcal{H}}}$ of $\mathbf{P}_{\Phi_{\mathcal{H}}}$ is identified with a refinement of the cone decomposition $\Sigma_{\Phi'_{\mathcal{H}}}$ of $\mathbf{P}_{\Phi'_{\mathcal{H}}}$. In this case we say that the triple $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \Sigma_{\Phi_{\mathcal{H}}})$ is a refinement of the triple $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}}, \Sigma_{\Phi'_{\mathcal{H}}})$ under the pair of isomorphisms $\gamma = (\gamma_X, \gamma_Y)$.

Definition 1.36 [Lan 2008, Definition 6.4.2.2]. Let $\Sigma = \{\Sigma_{\Phi_{\mathcal{H}}}\}$ and $\Sigma' = \{\Sigma'_{\Phi_{\mathcal{H}}}\}$ be two compatible choices of admissible smooth rational polyhedral cone decomposition data for $\mathsf{M}_{\mathcal{H}}$. We say that Σ *refines* Σ' if the triple $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \Sigma_{\Phi_{\mathcal{H}}})$ is a refinement of the triple $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \Sigma'_{\Phi_{\mathcal{H}}})$, as in Definition 1.35, for $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ running through all representatives of cusp labels.

Finally, we would like to describe the relations among the equivalence classes $[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)]$, which will describe the "incidence relations" among (closures of) the toroidal boundary strata.

Definition 1.37 [Lan 2008, Definition 6.3.2.14]. Let $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ be a representative of a cusp label at level $\mathcal{H}$, and let $\sigma \subset \mathbf{P}^{+}_{\Phi_{\mathcal{H}}}$ be a nondegenerate smooth rational polyhedral cone. We say that a triple $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}}, \sigma')$ is a *face* of $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)$ if:

(1) $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$ is the representative of some cusp label at level $\mathcal{H}$, such that there exists a surjection $s = (s_X, s_Y) : (\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}) \twoheadrightarrow (\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$ as in Definition 1.19.

(2) $\sigma' \subset \mathbf{P}^+_{\Phi'_{\mathcal{H}}}$ is a nondegenerate smooth rational polyhedral cone, such that for one (and hence every) surjection $s = (s_X, s_Y)$ as above, the image of σ' under the induced embedding $\mathbf{P}_{\Phi'_{\mathcal{H}}} \hookrightarrow \mathbf{P}_{\Phi_{\mathcal{H}}}$ is contained in the $\Gamma_{\Phi_{\mathcal{H}}}$-orbit of a face of σ.

Note that this definition is insensitive to the choices of representatives in the classes $[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)]$ and $[(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}}, \sigma')]$. This justifies the following:

Definition 1.38 [Lan 2008, Definition 6.3.2.15]. We say that the equivalence class $[(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}}, \sigma')]$ of $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}}, \sigma')$ is a *face* of the equivalence class $[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)]$ of $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)$ if some triple equivalent to $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}}, \sigma')$ is a face of some triple equivalent to $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)$.

1E. *Arithmetic toroidal compactifications.*

Definition 1.39 [Lan 2008, Definition 5.3.2.1]. Let S be a normal locally noetherian algebraic stack. A tuple $(G, \lambda, i, \alpha_{\mathcal{H}})$ over S is called a *degenerating family of type* $\mathsf{M}_{\mathcal{H}}$, or simply a *degenerating family* when the context is clear, if there exists a dense subalgebraic stack S_1 of S, such that S_1 is defined over $\operatorname{Spec}(\mathcal{O}_{F_0,(\Box)})$, and such that:

(1) By viewing group schemes as relative schemes (cf. [Hakim 1972]), G is a semiabelian scheme over S whose restriction G_{S_1} to S_1 is an abelian scheme. In this case, the dual semiabelian scheme $G^\vee$ exists (up to unique isomorphism), whose restriction $G^\vee_{S_1}$ to S_1 is the dual abelian scheme of G_{S_1}.

(2) $\lambda : G \to G^\vee$ is a group homomorphism that induces by restriction a prime-to-$\Box$ polarization λ_{S_1} of G_{S_1}.

(3) $i : \mathcal{O} \to \operatorname{End}_S(G)$ is a homomorphism that defines by restriction an $\mathcal{O}$-structure $i_{S_1} : \mathcal{O} \to \operatorname{End}_{S_1}(G_{S_1})$ of (G_{S_1}, λ_{S_1}).

(4) $(G_{S_1}, \lambda_{S_1}, i_{S_1}, \alpha_{\mathcal{H}}) \to S_1$ defines a tuple parametrized by the moduli problem $\mathsf{M}_{\mathcal{H}}$.

We will only talk about (semiabelian) degenerations (of abelian varieties with PEL structures) of this form.

Definition 1.40 [Lan 2008, Definition 6.3.1]. Let $(G, \lambda, i, \alpha_{\mathcal{H}})$ be a degenerating family of type $\mathsf{M}_{\mathcal{H}}$ over S (as in Definition 1.39) over $\mathsf{S}_0 = \operatorname{Spec}(\mathcal{O}_{F_0,(\Box)})$. Let $\underline{\mathrm{Lie}}^\vee_{G/S} := e^*_G\Omega^1_{G/S}$ be the dual of $\underline{\mathrm{Lie}}_{G/S}$, and let $\underline{\mathrm{Lie}}^\vee_{G^\vee/S} := e^*_G\Omega^1_{G^\vee/S}$ be the dual of $\underline{\mathrm{Lie}}_{G^\vee/S}$. Note that $\lambda : G \to G^\vee$ induces an $\mathcal{O}$-equivariant morphism $\lambda^* : \underline{\mathrm{Lie}}^\vee_{G^\vee/S} \to \underline{\mathrm{Lie}}^\vee_{G/S}$. (Here the $\mathcal{O}$-action on $\underline{\mathrm{Lie}}^\vee_{G/S}$ is a left action after twisting by the involution $\star$.) Then we define the sheaf $\underline{\mathrm{KS}} = \underline{\mathrm{KS}}_{(G,\lambda)/S} = \underline{\mathrm{KS}}_{(G,\lambda,i,\alpha_{\mathcal{H}})/S}$ by setting

$$\underline{\mathrm{KS}} := (\underline{\mathrm{Lie}}^{\vee}_{G/S} \otimes_{\mathcal{O}_S} \underline{\mathrm{Lie}}^{\vee}_{G^{\vee}/S}) / \begin{pmatrix} \lambda^*(y) \otimes z - \lambda^*(z) \otimes y \\ (b^{\star}x) \otimes y - x \otimes (by) \end{pmatrix}_{\substack{x \in \underline{\mathrm{Lie}}^{\vee}_{G/S} \\ y,z \in \underline{\mathrm{Lie}}^{\vee}_{G^{\vee}/S} \\ b \in \mathcal{O}}} .$$

Analogues of the sheaf $\underline{\mathrm{KS}}$ appear naturally in the deformation theory of abelian varieties with PEL structures (without degenerations). The point of Definition 1.40 is that it extends the conventional definition (for abelian schemes with PEL structures) to the context of (semiabelian) degenerating families (see Definition 1.39).

Theorem 1.41 [Lan 2008, Theorems 6.4.1.1 and 7.3.3.4]. *To each compatible choice $\Sigma = \{\Sigma_{\Phi_{\mathcal{H}}}\}$ of admissible smooth rational polyhedral cone decomposition data as in Definition 1.33, there is associated a proper smooth algebraic stack $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ over $\mathsf{S}_0 = \mathrm{Spec}(\mathcal{O}_{F_0,(\Box)})$, which is an algebraic space when $\mathcal{H}$ is neat* (as in Definition 1.2), *containing $\mathsf{M}_{\mathcal{H}}$ as an open dense subalgebraic stack, together with a degenerating family $(G, \lambda, i, \alpha_{\mathcal{H}})$ over $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$* (as in Definition 1.39) *such that*:

(1) *The restriction $(G_{\mathsf{M}_{\mathcal{H}}}, \lambda_{\mathsf{M}_{\mathcal{H}}}, i_{\mathsf{M}_{\mathcal{H}}}, \alpha_{\mathcal{H}})$ of the degenerating family $(G, \lambda, i, \alpha_{\mathcal{H}})$ to $\mathsf{M}_{\mathcal{H}}$ is the tautological (i.e., universal) tuple over $\mathsf{M}_{\mathcal{H}}$.*

(2) *$\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$ has a stratification by locally closed subalgebraic stacks*

$$\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma} = \coprod_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]} \mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]},$$

with $[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)]$ running through a complete set of equivalence classes of $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)$ (as in Definition 1.30) *with $\sigma \subset \mathbf{P}^+_{\Phi_{\mathcal{H}}}$ and $\sigma \in \Sigma_{\Phi_{\mathcal{H}}} \in \Sigma$.* (Here $\mathsf{Z}_{\mathcal{H}}$ is suppressed in the notation by our convention.)

In this stratification, the $[(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}}, \sigma')]$-stratum $\mathsf{Z}_{[(\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}},\sigma')]}$ lies in the closure of the $[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)]$-stratum $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}$ if and only if $[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)]$ is a face of $[(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}}, \sigma')]$ as in Definition 1.38.

The $[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)]$-stratum $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}$ is smooth and isomorphic to the support of the formal algebraic stack $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma} / \Gamma_{\Phi_{\mathcal{H}},\sigma}$ for any representative $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)$ of $[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)]$, where the formal algebraic stack $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}$ (before quotient by $\Gamma_{\Phi_{\mathcal{H}},\sigma}$, the subgroup of $\Gamma_{\Phi_{\mathcal{H}}}$ formed by elements mapping σ to itself) admits a canonically defined structure of a torus-torsor over an abelian scheme over the smooth algebraic stack $\mathsf{M}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}$ in Definition 1.17. (Note that $\mathsf{Z}_{\mathcal{H}}$ and the isomorphism class of $\mathsf{M}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}$ depend only on the class $[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)]$, but not on the choice of the representative $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)$.)

In particular, $\mathsf{M}_{\mathcal{H}}$ is an open dense stratum in this stratification.

(3) *The complement of $\mathsf{M}_{\mathcal{H}}$ in $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ (with its reduced structure) is a relative Cartier divisor $\mathsf{D}_{\infty,\mathcal{H}}$ with normal crossings, such that each connected component of a stratum of $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}} - \mathsf{M}_{\mathcal{H}}$ is open dense in an intersection of irreducible components*

of $\mathsf{D}_{\infty,\mathcal{H}}$ (*including possible self-intersections*). *When* $\mathcal{H}$ *is neat*, *the irreducible components of* $\mathsf{D}_{\infty,\mathcal{H}}$ *have no self-intersections* (cf. Condition 1.29, [Lan 2008, Remark 6.2.5.26 in the revision], and [Faltings and Chai 1990, Chapter IV, Remark 5.8(a)]).

(4) *The extended Kodaira–Spencer morphism* [Lan 2008, Definition 4.6.3.32] *for* $G \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$ *induces an isomorphism*

$$\mathrm{KS}_{G/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}/\mathsf{S}_0} : \underline{\mathrm{KS}}_{G/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}} \xrightarrow{\sim} \Omega^1_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}/\mathsf{S}_0}[d \log \infty]$$

(see Definition 1.40). *Here the sheaf* $\Omega^1_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}/\mathsf{S}_0}[d \log \infty]$ *is the sheaf of modules of log* 1*-differentials on* $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$ *over* S_0, *with respect to the relative Cartier divisor* $\mathsf{D}_{\infty,\mathcal{H}}$ *with normal crossings.*

(5) *The formal completion*

$$(\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}})^{\wedge}_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}}$$

of $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$ *along the* $[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)]$*-stratum* $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}$ *is canonically isomorphic to the formal algebraic stack* $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma} / \Gamma_{\Phi_{\mathcal{H}},\sigma}$ *for any representative* $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)$ *of* $[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)]$. (*To form the formal completion along a given locally closed stratum*, *we first remove the other strata appearing in the closure of this stratum from the total space*, *and then form the formal completion of the remaining space along this stratum.*)

This isomorphism respects stratifications in the sense that, *given any formally étale morphism* $\mathrm{Spf}(R, I) \to \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma} / \Gamma_{\Phi_{\mathcal{H}},\sigma}$ *inducing a morphism* $\mathrm{Spec}(R) \to \Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma) / \Gamma_{\Phi_{\mathcal{H}},\sigma}$, *the stratification of* $\mathrm{Spec}(R)$ (inherited from $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma) / \Gamma_{\Phi_{\mathcal{H}},\sigma}$; see [Lan 2008, Proposition 6.3.1.6 and Definition 6.3.2.16 in the revision]) *makes the induced morphism* $\mathrm{Spec}(R) \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$ *a strata-preserving morphism.*

The pullback to $(\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}})^{\wedge}_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}}$ *of the degenerating family* $(G, \lambda, i, \alpha_{\mathcal{H}})$ *over* $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$ *is the Mumford family*

$$({}^{\heartsuit}G,\ {}^{\heartsuit}\lambda,\ {}^{\heartsuit}i,\ {}^{\heartsuit}\alpha_{\mathcal{H}})$$

over $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma} / \Gamma_{\Phi_{\mathcal{H}},\sigma}$ (see [Lan 2008, §6.2.5]) *after we identify the bases using the isomorphism.* (Here both the pullback of $(G, \lambda, i, \alpha_{\mathcal{H}})$ and the Mumford family $({}^{\heartsuit}G,\ {}^{\heartsuit}\lambda,\ {}^{\heartsuit}i,\ {}^{\heartsuit}\alpha_{\mathcal{H}})$ are considered as relative schemes with additional structures; cf. [Hakim 1972].)

(6) *Let* S *be an irreducible noetherian normal scheme over* S_0. *Suppose we have a degenerating family* $(G^{\dagger}, \lambda^{\dagger}, i^{\dagger}, \alpha^{\dagger}_{\mathcal{H}})$ *of type* $\mathsf{M}_{\mathcal{H}}$ *over* S *as in Definition 1.39. Then* $(G^{\dagger}, \lambda^{\dagger}, i^{\dagger}, \alpha^{\dagger}_{\mathcal{H}}) \to S$ *is the pullback of* $(G, \lambda, i, \alpha_{\mathcal{H}}) \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$ *via a* (*necessarily unique*) *morphism* $S \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$ (*over* S_0) *if and only if the following condition is satisfied*:

Consider any dominant morphism $\mathrm{Spec}(V) \to S$ *centered at a geometric point* $\bar{s}$ *of* S, *where* V *is a complete discrete valuation ring with quotient field* K, *algebraically closed residue field* k, *and discrete valuation* υ. *Let*

$$(G^{\ddagger}, \lambda^{\ddagger}, i^{\ddagger}, \alpha^{\ddagger}_{\mathcal{H}}) \to \mathrm{Spec}(V)$$

be the pullback of $(G^{\dagger}, \lambda^{\dagger}, i^{\dagger}, \alpha^{\dagger}_{\mathcal{H}}) \to S$. *This pullback family defines an object of* $\mathrm{DEG}_{\mathrm{PEL},\mathsf{M}_{\mathcal{H}}}$ *over* $\mathrm{Spec}(V)$, *which corresponds to a tuple*

$$(A^{\ddagger}, \lambda^{\ddagger}_A, i^{\ddagger}_A, \underline{X}^{\ddagger}, \underline{Y}^{\ddagger}, \phi^{\ddagger}, c^{\ddagger}, (c^{\vee})^{\ddagger}, \tau^{\ddagger}, [(\alpha^{\natural}_{\mathcal{H}})^{\ddagger}])$$

in $\mathrm{DD}_{\mathrm{PEL},\mathsf{M}_{\mathcal{H}}}$ (under [Lan 2008, Theorem 5.3.1.17]). *Then we have a fully symplectic-liftable admissible filtration* $\mathrm{Z}^{\ddagger}_{\mathcal{H}}$ *determined by* $[(\alpha^{\natural}_{\mathcal{H}})^{\ddagger}]$. *Moreover, the étale sheaves* $\underline{X}^{\ddagger}$ *and* $\underline{Y}^{\ddagger}$ *are necessarily constant, because the base ring* V *is strict local. Hence it makes sense to say we also have a uniquely determined torus argument* $\Phi^{\ddagger}_{\mathcal{H}}$ *at level* $\mathcal{H}$ *for* $\mathrm{Z}^{\ddagger}_{\mathcal{H}}$.

On the other hand, we have objects $\underline{\Phi}_{\mathcal{H}}(G^{\ddagger})$, $\underline{\mathbf{S}}_{\Phi_{\mathcal{H}}(G^{\ddagger})}$, *and* $\underline{B}(G^{\ddagger})$ (see [Lan 2008, Construction 6.3.1.1]), *which define objects* $\Phi^{\ddagger}_{\mathcal{H}}$, $\mathbf{S}_{\Phi^{\ddagger}_{\mathcal{H}}}$ *and in particular* $B^{\ddagger} : \mathbf{S}_{\Phi^{\ddagger}_{\mathcal{H}}} \to \mathrm{Inv}(V)$ *over the special fiber. Then*

$$\upsilon \circ B^{\ddagger} : \mathbf{S}_{\Phi^{\ddagger}_{\mathcal{H}}} \to \mathbb{Z}$$

defines an element of $\mathbf{S}^{\vee}_{\Phi^{\ddagger}_{\mathcal{H}}}$, *where* $\upsilon : \mathrm{Inv}(V) \to \mathbb{Z}$ *is the homomorphism induced by the discrete valuation of* V.

Then the condition is that, for any $\mathrm{Spec}(V) \to S$ *as above, and for any choice of* $\delta^{\ddagger}_{\mathcal{H}}$ (*which is immaterial, because this choice will not be used*), *there is a cone* $\sigma^{\ddagger}$ *in the cone decomposition* $\Sigma_{\Phi^{\ddagger}_{\mathcal{H}}}$ *of* $\mathbf{P}_{\Phi^{\ddagger}_{\mathcal{H}}}$ (given by the choice of Σ; cf. Definition 1.33) *such that* $\overline{\sigma}^{\ddagger}$ *contains all the* $\upsilon \circ B^{\ddagger}$ *obtained in this way.*

(7) *If* $\mathcal{H}$ *is neat and* Σ *is projective* (see Definition 1.34), *then* $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ *is projective* (*and hence a scheme*) *over* S_0.

Statement (1) means the tautological tuple over $\mathsf{M}_{\mathcal{H}}$ extends to a degenerating family $(G, \lambda, i, \alpha_{\mathcal{H}})$ over $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$. (Since $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$ is normal, this extension is unique by a result of Raynaud; see [Raynaud 1970, IX, 1.4] or [Faltings and Chai 1990, Chapter I, Proposition 2.7].) Statements (2)–(5) and (7) are self-explanatory. Statement (6) can be interpreted as a "universal property" for the degenerating family $(G, \lambda, i, \alpha_{\mathcal{H}}) \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$ among degenerating families over normal locally noetherian bases, as in Definition 1.39, satisfying moreover some conditions describing the "degenerating patterns" over pullbacks to complete discrete valuation rings with algebraically closed residue fields. This "universal property" will be crucial in the main construction of this article (in Section 3A below).

2. Kuga families and their compactifications

Let $\mathcal{O}$, $\star$, $(L, \langle\,\cdot\,,\,\cdot\,\rangle)$, h, and $\Box$ be as in the previous section. Then we have a moduli problem $\mathsf{M}_{\mathcal{H}}$ over $\mathsf{S}_0 = \mathrm{Spec}(\mathcal{O}_{F_0,(\Box)})$ for each open compact $\mathcal{H}$ of $\mathrm{G}(\hat{\mathbb{Z}}^{\Box})$, with a toroidal compactification $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ for each choice of Σ.

For simplicity, let us maintain the following:

Convention 2.1. All morphisms between schemes or algebraic stacks over $\mathsf{S}_0 = \mathrm{Spec}(\mathcal{O}_{F_0,(\Box)})$ will be defined over S_0, unless otherwise specified.

2A. *PEL-type Kuga families.* Let Q be any $\mathcal{O}$-lattice. Consider the abelian scheme $G_{\mathsf{M}_{\mathcal{H}}}$ over $\mathsf{M}_{\mathcal{H}}$ in (1) of Theorem 1.41. By [Lan 2008, Proposition 5.2.3.8], the group functor $\underline{\mathrm{Hom}}_{\mathcal{O}}(Q, G_{\mathsf{M}_{\mathcal{H}}})$ over $\mathsf{M}_{\mathcal{H}}$ is representable by a proper smooth group scheme which is an extension of a finite étale group scheme, whose rank has no prime factors other than those of Disc, by an abelian scheme $\underline{\mathrm{Hom}}_{\mathcal{O}}(Q, G_{\mathsf{M}_{\mathcal{H}}})^\circ$, which we call the *fiberwise geometric identity component* of $\underline{\mathrm{Hom}}_{\mathcal{O}}(Q, G_{\mathsf{M}_{\mathcal{H}}})$.

Example 2.2. If $Q \cong \mathcal{O}^{\oplus s}$ for some integer $s \geq 0$, then $\underline{\mathrm{Hom}}_{\mathcal{O}}(Q, G_{\mathsf{M}_{\mathcal{H}}})^\circ = \underline{\mathrm{Hom}}_{\mathcal{O}}(Q, G_{\mathsf{M}_{\mathcal{H}}}) \cong G^{\times s}_{\mathsf{M}_{\mathcal{H}}}$ is the s-fold fiber product of $G_{\mathsf{M}_{\mathcal{H}}}$ over $\mathsf{M}_{\mathcal{H}}$.

Example 2.3. If $\mathcal{O} \cong \mathrm{M}_k(\mathcal{O}_F)$ and Q is of finite index in $\mathcal{O}_F^{\oplus k}$ for some integer $k \geq 1$, then the relative dimension of $\underline{\mathrm{Hom}}_{\mathcal{O}}(Q, G_{\mathsf{M}_{\mathcal{H}}})^\circ$ over $\mathsf{M}_{\mathcal{H}}$ is $1/k$ of the relative dimension of $G_{\mathsf{M}_{\mathcal{H}}}$ over $\mathsf{M}_{\mathcal{H}}$.

Definition 2.4. A *PEL-type Kuga family* over $\mathsf{M}_{\mathcal{H}}$ is an abelian scheme $\mathsf{N} \to \mathsf{M}_{\mathcal{H}}$ that is $\mathbb{Z}^\times_{(\Box)}$-isogenous to $\underline{\mathrm{Hom}}_{\mathcal{O}}(Q, G_{\mathsf{M}_{\mathcal{H}}})^\circ$ for some $\mathcal{O}$-lattice Q.

Consider $\mathrm{Diff}^{-1} = \mathrm{Diff}^{-1}_{\mathcal{O}/\mathbb{Z}}$, the inverse different of $\mathcal{O}$ over $\mathbb{Z}$ [Lan 2008, Definition 1.1.1.11] with its canonical left $\mathcal{O}$-module structure. Since the trace pairing $\mathrm{Diff}^{-1} \times \mathcal{O} \to \mathbb{Z} : (y, x) \mapsto \mathrm{Tr}_{\mathcal{O}/\mathbb{Z}}(yx)$ is perfect by definition, for each $\mathcal{O}$-lattice Q, we may identify $Q^\vee := \mathrm{Hom}_{\mathbb{Z}}(Q, \mathbb{Z})$ with $\mathrm{Hom}_{\mathcal{O}}(Q, \mathrm{Diff}^{-1})$. By composition with the involution $\star : \mathcal{O} \xrightarrow{\sim} \mathcal{O}^{\mathrm{op}}$, the natural right action of $\mathcal{O}$ on Diff^{-1} induced a left action of $\mathcal{O}$ on Diff^{-1}, which commutes with the natural left action of $\mathcal{O}$ on Diff^{-1}. Accordingly, the $\mathbb{Z}$-module $Q^\vee$ is torsion-free and has a canonical left $\mathcal{O}$-structure induced by the right action of $\mathcal{O}^{\mathrm{op}}$ on Diff^{-1} (and $\star : \mathcal{O} \xrightarrow{\sim} \mathcal{O}^{\mathrm{op}}$). In other words, $Q^\vee$ is an $\mathcal{O}$-lattice. Then the trace pairing induces a perfect pairing

$$\langle\,\cdot\,,\,\cdot\,\rangle_Q : Q^\vee \times Q \to \mathbb{Z} : (f, x) \mapsto \mathrm{Tr}_{\mathcal{O}/\mathbb{Z}}(f(x)).$$

For any $b \in \mathcal{O}$, $f \in Q^\vee$, and $x \in Q$, we have

$$\langle bf, x\rangle_Q = \mathrm{Tr}_{\mathcal{O}/\mathbb{Z}}(f(x)b^\star) = \mathrm{Tr}_{\mathcal{O}/\mathbb{Z}}(b^\star f(x)) = \mathrm{Tr}_{\mathcal{O}/\mathbb{Z}}(f(b^\star x)) = \langle f, b^\star x\rangle.$$

Lemma 2.5. *There exists an embedding $j_Q : Q^\vee \hookrightarrow Q$ of $\mathcal{O}$-lattices inducing an isomorphism $j_Q : Q^\vee \otimes_{\mathbb{Z}} \mathbb{Z}_{(\Box)} \xrightarrow{\sim} Q \otimes_{\mathbb{Z}} \mathbb{Z}_{(\Box)}$ of $\mathcal{O} \otimes_{\mathbb{Z}} \mathbb{Z}_{(\Box)}$-modules such that the* *pairing*

$$\langle j_Q^{-1}(\cdot),\ \cdot\rangle_Q : (Q \otimes_{\mathbb{Z}} \mathbb{R}) \times (Q \otimes_{\mathbb{Z}} \mathbb{R}) \to \mathbb{R}$$

is positive definite.

Proof. By the explicit classification [Lan 2008, (1.2.1.10), Proposition 1.2.1.13, and Lemma 1.2.1.23], there exists an isomorphism $j_{Q,0} : Q^\vee \otimes_{\mathbb{Z}} \mathbb{R} \xrightarrow{\sim} Q \otimes_{\mathbb{Z}} \mathbb{R}$ of $\mathcal{O} \otimes_{\mathbb{Z}} \mathbb{R}$-modules such that the induced pairing $\langle j_{Q,0}^{-1}(\cdot),\ \cdot\rangle_Q : (Q\otimes_{\mathbb{Z}}\mathbb{R})\times(Q\otimes_{\mathbb{Z}}\mathbb{R}) \to \mathbb{R}$ is positive definite. If $\Box$ is the set of all rational prime numbers, then necessarily $\mathcal{O} = \mathbb{Z}$, and the lemma is clear. Otherwise, we know that $\mathrm{Isom}_{\mathcal{O}\otimes_{\mathbb{Z}}\mathbb{Z}_{(\Box)}}(Q^\vee \otimes_{\mathbb{Z}} \mathbb{Z}_{(\Box)}, Q \otimes_{\mathbb{Z}} \mathbb{Z}_{(\Box)})$ is dense in $\mathrm{Isom}_{\mathcal{O}\otimes_{\mathbb{Z}}\mathbb{R}}(Q^\vee \otimes_{\mathbb{Z}} \mathbb{R}, Q \otimes_{\mathbb{Z}} \mathbb{R})$ (with the topology induced by $\mathbb{R}$). Hence there exists an element $j_{Q,1} : Q^\vee \otimes_{\mathbb{Z}} \mathbb{Z}_{(\Box)} \xrightarrow{\sim} Q \otimes_{\mathbb{Z}} \mathbb{Z}_{(\Box)}$ close to $j_{Q,0}$ in $\mathrm{Isom}_{\mathcal{O}\otimes_{\mathbb{Z}}\mathbb{R}}(Q^\vee \otimes_{\mathbb{Z}} \mathbb{R}, Q \otimes_{\mathbb{Z}} \mathbb{R})$ such that the induced pairing $\langle j_{Q,1}^{-1}(\cdot),\ \cdot\rangle_Q : (Q\otimes_{\mathbb{Z}}\mathbb{R})\times(Q\otimes_{\mathbb{Z}}\mathbb{R}) \to \mathbb{R}$ is still positive definite. By multiplying $j_{Q,1}$ by a positive element in $\mathbb{Z}_{(\Box)}^\times$, we may assume that it maps $Q^\vee$ to a submodule of Q. Then the induced morphism $j_Q : Q^\vee \to Q$ satisfies the requirement of the lemma. $\Box$

Lemma 2.6. *The abelian scheme* $\underline{\mathrm{Hom}}_{\mathbb{Z}}(Q^\vee, G^\vee_{\mathsf{M}_{\mathcal{H}}})$ *is isomorphic to the dual abelian scheme of* $\underline{\mathrm{Hom}}_{\mathbb{Z}}(Q, G_{\mathsf{M}_{\mathcal{H}}})$.

Proof. Let s be the common rank of Q and $Q^\vee$ as free $\mathbb{Z}$-modules. Let $\{e_1, \dots, e_s\}$ be a $\mathbb{Z}$-basis of Q, and let $\{e_1^\vee, \dots, e_s^\vee\}$ be the dual $\mathbb{Z}$-basis of $Q^\vee$, such that $e_i^\vee(e_j) = \delta_{ij}$ for any $1 \le i, j \le s$. Then the choices of bases define canonical isomorphisms

$$\underline{\mathrm{Hom}}_{\mathbb{Z}}(Q, G_{\mathsf{M}_{\mathcal{H}}}) \cong G_{\mathsf{M}_{\mathcal{H}}}^{\times s} \tag{2.7}$$

and

$$\underline{\mathrm{Hom}}_{\mathbb{Z}}(Q^\vee, G^\vee_{\mathsf{M}_{\mathcal{H}}}) \cong (G^\vee_{\mathsf{M}_{\mathcal{H}}})^{\times s}. \tag{2.8}$$

As a result, $\underline{\mathrm{Hom}}_{\mathbb{Z}}(Q^\vee, G^\vee_{\mathsf{M}_{\mathcal{H}}}) \cong G_{\mathsf{M}_{\mathcal{H}}}^{\times s}$ is isomorphic to the dual abelian scheme of $\underline{\mathrm{Hom}}_{\mathbb{Z}}(Q, G_{\mathsf{M}_{\mathcal{H}}}) \cong (G^\vee_{\mathsf{M}_{\mathcal{H}}})^{\times s}$. $\Box$

Lemma 2.9. *Let* $j_Q : Q^\vee \hookrightarrow Q$ *be as in Lemma 2.5. Then the isogeny*

$$\lambda_{\mathsf{M}_{\mathcal{H}}, j_Q, \mathbb{Z}} : \underline{\mathrm{Hom}}_{\mathbb{Z}}(Q, G_{\mathsf{M}_{\mathcal{H}}}) \to \underline{\mathrm{Hom}}_{\mathbb{Z}}(Q^\vee, G^\vee_{\mathsf{M}_{\mathcal{H}}})$$

induced canonically by j_Q *and* $\lambda_{\mathsf{M}_{\mathcal{H}}} : G_{\mathsf{M}_{\mathcal{H}}} \to G^\vee_{\mathsf{M}_{\mathcal{H}}}$, *which is of degree prime to* $\Box$ *because both* $[Q : j_Q(Q^\vee)]$ *and* $\deg(\lambda_{\mathsf{M}_{\mathcal{H}}})$ *are prime to* $\Box$, *is a polarization.*

Proof. We need to show that the invertible sheaf

$$(\mathrm{Id}_{\underline{\mathrm{Hom}}_{\mathbb{Z}}(Q, G_{\mathsf{M}_{\mathcal{H}}})}, \lambda_{\mathsf{M}_{\mathcal{H}}, j_Q, \mathbb{Z}})^* \mathcal{P}_{\underline{\mathrm{Hom}}_{\mathbb{Z}}(Q, G_{\mathsf{M}_{\mathcal{H}}})}$$

is relative ample over $\mathsf{M}_{\mathcal{H}}$. Using the choice of basis $\{e_1, \dots, e_s\}$ (resp. $\{e_1^\vee, \dots, e_s^\vee\}$) of Q (resp. $Q^\vee$) as in the proof of Lemma 2.6, the morphism j_Q can be represented by $e_i^\vee \mapsto \sum_{1\le j\le s} a_{ij} e_j$ for some integers a_{ij}, for each $1 \le i \le s$. These integers form a positive definite matrix $a = (a_{ij})$, because the induced pairing $\langle j_Q^{-1}(\,\cdot\,),\,\cdot\,\rangle_Q : (Q \otimes_{\mathbb{Z}} \mathbb{R}) \times (Q \otimes_{\mathbb{Z}} \mathbb{R}) \to \mathbb{R}$ is positive definite. By completion of squares for quadratic forms, we know that there exist an integer $m \geq 1$ such that $ma = ud\,{}^t u$ for some matrices d and u with integral coefficients, where $d = \operatorname{diag}(d_1, \ldots, d_s)$ is diagonal with positive entries. As a result, the morphism $m\lambda_{\mathsf{M}_{\mathcal{H}}, j_Q, \mathbb{Z}}$ factors as a composition

$$m\lambda_{\mathsf{M}_{\mathcal{H}}, j_Q, \mathbb{Z}} = [{}^t u]^* \circ \lambda_{\mathsf{M}_{\mathcal{H}}, d, \mathbb{Z}} \circ [u]^*$$

of morphisms

$$\begin{aligned} [u]^* &: \underline{\operatorname{Hom}}_{\mathbb{Z}}(Q, G_{\mathsf{M}_{\mathcal{H}}}) \to \underline{\operatorname{Hom}}_{\mathbb{Z}}(Q, G_{\mathsf{M}_{\mathcal{H}}}), \\ \lambda_{\mathsf{M}_{\mathcal{H}}, d, \mathbb{Z}} &: \underline{\operatorname{Hom}}_{\mathbb{Z}}(Q, G_{\mathsf{M}_{\mathcal{H}}}) \to \underline{\operatorname{Hom}}_{\mathbb{Z}}(Q^\vee, G^\vee_{\mathsf{M}_{\mathcal{H}}}), \\ [{}^t u]^* &: \underline{\operatorname{Hom}}_{\mathbb{Z}}(Q^\vee, G^\vee_{\mathsf{M}_{\mathcal{H}}}) \to \underline{\operatorname{Hom}}_{\mathbb{Z}}(Q^\vee, G^\vee_{\mathsf{M}_{\mathcal{H}}}). \end{aligned}$$

If we identify $\underline{\operatorname{Hom}}_{\mathbb{Z}}(Q, G_{\mathsf{M}_{\mathcal{H}}})$ and $\underline{\operatorname{Hom}}_{\mathbb{Z}}(Q^\vee, G^\vee_{\mathsf{M}_{\mathcal{H}}})$ as dual abelian schemes of each other using the canonical isomorphisms (2.7) and (2.8) defined by the dual bases $\{e_1, \ldots, e_s\}$ and $\{e_1^\vee, \ldots, e_s^\vee\}$, then $[{}^t u]^* = ([u]^*)^\vee$, and $\lambda_{\mathsf{M}_{\mathcal{H}}, d, \mathbb{Z}} = (d_1\lambda_{\mathsf{M}_{\mathcal{H}}}) \times (d_2\lambda_{\mathsf{M}_{\mathcal{H}}}) \times \ldots \times (d_s\lambda_{\mathsf{M}_{\mathcal{H}}}) : G^{\times s}_{\mathsf{M}_{\mathcal{H}}} \to (G^\vee_{\mathsf{M}_{\mathcal{H}}})^{\times s}$ is a polarization. Since $[u]^*$ is finite, this implies that $\lambda_{\mathsf{M}_{\mathcal{H}}, j_Q, \mathbb{Z}}$ is also a polarization, as desired. $\square$

Proposition 2.10. *The abelian scheme* $\underline{\operatorname{Hom}}_{\mathcal{O}}(Q^\vee, G^\vee_{\mathsf{M}_{\mathcal{H}}})^\circ$ *is* $\mathbb{Z}^\times_{(\Box)}$*-isogenous to the dual abelian scheme of* $\underline{\operatorname{Hom}}_{\mathcal{O}}(Q, G_{\mathsf{M}_{\mathcal{H}}})^\circ$.

Proof. Since $\lambda_{\mathsf{M}_{\mathcal{H}}, j_Q, \mathbb{Z}}$ is a polarization by Lemma 2.9, the induced morphism

$$\begin{aligned} \lambda_{\mathsf{M}_{\mathcal{H}}, j_Q} :\ & \underline{\operatorname{Hom}}_{\mathcal{O}}(Q, G_{\mathsf{M}_{\mathcal{H}}})^\circ \hookrightarrow \underline{\operatorname{Hom}}_{\mathbb{Z}}(Q, G_{\mathsf{M}_{\mathcal{H}}}) \\ & \overset{\lambda_{\mathsf{M}_{\mathcal{H}}, j_Q, \mathbb{Z}}}{\to} \underline{\operatorname{Hom}}_{\mathbb{Z}}(Q^\vee, G^\vee_{\mathsf{M}_{\mathcal{H}}}) \twoheadrightarrow (\underline{\operatorname{Hom}}_{\mathcal{O}}(Q, G_{\mathsf{M}_{\mathcal{H}}})^\circ)^\vee \end{aligned} \tag{2.11}$$

is also a polarization. (Since the condition of being a polarization can be checked fiber by fiber [Deligne and Pappas 1994, 1.2–1.4], it suffices to note that the restriction of an ample invertible sheaf to a closed subscheme is again ample.) Since $\lambda_{\mathsf{M}_{\mathcal{H}}, j_Q, \mathbb{Z}}$ maps $\underline{\operatorname{Hom}}_{\mathcal{O}}(Q, G_{\mathsf{M}_{\mathcal{H}}})^\circ$ onto the subscheme $\underline{\operatorname{Hom}}_{\mathcal{O}}(Q^\vee, G^\vee_{\mathsf{M}_{\mathcal{H}}})^\circ$ of $\underline{\operatorname{Hom}}_{\mathbb{Z}}(Q^\vee, G^\vee_{\mathsf{M}_{\mathcal{H}}})$, we obtain an isogeny

$$\underline{\operatorname{Hom}}_{\mathcal{O}}(Q^\vee, G^\vee_{\mathsf{M}_{\mathcal{H}}})^\circ \to (\underline{\operatorname{Hom}}_{\mathcal{O}}(Q, G_{\mathsf{M}_{\mathcal{H}}})^\circ)^\vee.$$

The degree of this isogeny is prime to $\Box$ because $\lambda_{\mathsf{M}_{\mathcal{H}}, j_Q, \mathbb{Z}}$ is. $\square$

Corollary 2.12 (of the proof of Proposition 2.10). *Let* $j_Q : Q^\vee \hookrightarrow Q$ *be as in Lemma 2.5. Then the canonical morphism*

$$\lambda_{\mathsf{M}_{\mathcal{H}}, j_Q} : \underline{\operatorname{Hom}}_{\mathcal{O}}(Q, G_{\mathsf{M}_{\mathcal{H}}})^\circ \to (\underline{\operatorname{Hom}}_{\mathcal{O}}(Q, G_{\mathsf{M}_{\mathcal{H}}})^\circ)^\vee$$

induced by j_Q *and* $\lambda_{\mathsf{M}_{\mathcal{H}}} : G_{\mathsf{M}_{\mathcal{H}}} \to G^\vee_{\mathsf{M}_{\mathcal{H}}}$ (as in (2.11)) *is a polarization of degree prime to* $\Box$.

Corollary 2.13. *If a Kuga family $\mathsf{N} \to \mathsf{M}_{\mathcal{H}}$ is $\mathbb{Z}^{\times}_{(\Box)}$-isogenous to $\underline{\mathrm{Hom}}_{\mathcal{O}}(Q, G_{\mathsf{M}_{\mathcal{H}}})^{\circ}$ for some $\mathcal{O}$-lattice Q, then we have canonical isomorphisms over $\mathsf{M}_{\mathcal{H}}$:*

$$\underline{\mathrm{Lie}}_{\mathsf{N}/\mathsf{M}_{\mathcal{H}}} \cong \underline{\mathrm{Hom}}_{\mathcal{O}}(Q, \underline{\mathrm{Lie}}_{G/\mathsf{M}_{\mathcal{H}}}), \qquad \underline{\mathrm{Lie}}_{\mathsf{N}^{\vee}/\mathsf{M}_{\mathcal{H}}} \cong \underline{\mathrm{Hom}}_{\mathcal{O}}(Q^{\vee}, \underline{\mathrm{Lie}}_{G^{\vee}/\mathsf{M}_{\mathcal{H}}}),$$
$$\underline{\mathrm{Lie}}^{\vee}_{\mathsf{N}/\mathsf{M}_{\mathcal{H}}} \cong \underline{\mathrm{Hom}}_{\mathcal{O}}(Q^{\vee}, \underline{\mathrm{Lie}}^{\vee}_{G/\mathsf{M}_{\mathcal{H}}}), \qquad \underline{\mathrm{Lie}}^{\vee}_{\mathsf{N}^{\vee}/\mathsf{M}_{\mathcal{H}}} \cong \underline{\mathrm{Hom}}_{\mathcal{O}}(Q, \underline{\mathrm{Lie}}^{\vee}_{G^{\vee}/\mathsf{M}_{\mathcal{H}}}).$$

Remark 2.14. We do not need to choose a polarization $\mathsf{N} \to \mathsf{N}^{\vee}$ in the isomorphisms in Corollary 2.13. The sheaves on the right-hand sides of the isomorphisms are locally free because the order $\mathcal{O}$ is maximal at any good prime (see Definition 1.5 and [Lan 2008, Proposition 1.1.1.17]), and because lattices over maximal orders are projective modules (see [Lan 2008, Proposition 1.1.1.20]).

2B. *Main theorem.* (Convention 2.1 will persist until the end of this article.)

Theorem 2.15. *Let Q be any $\mathcal{O}$-lattice. Suppose that $\mathcal{H}$ is **neat*** (as in Definition 1.2), *so that the moduli problem $\mathsf{M}_{\mathcal{H}}$ it defines is representable by a quasiprojective scheme, and so that $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}} = \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ is a proper smooth algebraic space over S_0. Then there is a directed partially ordered set $\mathbf{K}_{Q,\mathcal{H},\Sigma}$ parametrizing the following data:*

(1) *For each $\kappa \in \mathbf{K}_{Q,\mathcal{H},\Sigma}$, there is a $\mathbb{Z}^{\times}_{(\Box)}$-isogeny $\kappa^{\mathrm{isog}} : \underline{\mathrm{Hom}}_{\mathcal{O}}(Q, G_{\mathsf{M}_{\mathcal{H}}})^{\circ} \to \mathsf{N}_{\kappa}$ over $\mathsf{M}_{\mathcal{H}}$, together with an open immersion $\kappa^{\mathrm{tor}} : \mathsf{N}_{\kappa} \hookrightarrow \mathsf{N}^{\mathrm{tor}}_{\kappa}$ of schemes over S_0, such that the scheme $\mathsf{N}^{\mathrm{tor}}_{\kappa}$ is **projective** and **smooth** over S_0, and that the complement of N_{κ} in $\mathsf{N}^{\mathrm{tor}}_{\kappa}$ (with its reduced structure) is a relative Cartier divisor $\mathsf{E}_{\infty,\kappa}$ with simple normal crossings.*

For each relation $\kappa' \succ \kappa$ in $\mathbf{K}_{Q,\mathcal{H},\Sigma}$, there is a proper log étale morphism $f^{\mathrm{tor}}_{\kappa',\kappa} : \mathsf{N}^{\mathrm{tor}}_{\kappa'} \to \mathsf{N}^{\mathrm{tor}}_{\kappa}$ extending the canonical $\mathbb{Z}^{\times}_{(\Box)}$-isogeny

$$f_{\kappa',\kappa} := \kappa^{\mathrm{isog}} \circ ((\kappa')^{\mathrm{isog}})^{-1} : \mathsf{N}_{\kappa'} \to \mathsf{N}_{\kappa}$$

such that $R^i (f^{\mathrm{tor}}_{\kappa',\kappa})_* \mathcal{O}_{\mathsf{N}^{\mathrm{tor}}_{\kappa'}} = 0$ for $i > 0$.

(2) *For each $\kappa \in \mathbf{K}_{Q,\mathcal{H},\Sigma}$, the structural morphism $f_{\kappa} : \mathsf{N}_{\kappa} \to \mathsf{M}_{\mathcal{H}}$ extends (necessarily uniquely) to a morphism $f^{\mathrm{tor}}_{\kappa} : \mathsf{N}^{\mathrm{tor}}_{\kappa} \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$, which is **proper** and **log smooth*** (as in [Kato 1989, 3.3] and [Illusie 1994, 1.6]) *if we equip $\mathsf{N}^{\mathrm{tor}}_{\kappa}$ and $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$ with the canonical (fine) log structures given respectively by the relative Cartier divisors with (simple) normal crossings $\mathsf{E}_{\infty,\kappa}$ and $\mathsf{D}_{\infty,\mathcal{H}}$* (see (1) above and (3) of Theorem 1.41). *Then we have the following commutative diagram:*

$$\begin{tikzcd}
\mathsf{N}_{\kappa} \arrow[r, hook, "+\text{NCD}"] \arrow[d, "f_{\kappa}", "\substack{\text{proper}\\ \text{smooth}}"'] & \mathsf{N}^{\mathrm{tor}}_{\kappa} \arrow[d, "f^{\mathrm{tor}}_{\kappa}", "\substack{\text{proper}\\ \text{log smooth}}"'] \arrow[dr, "\substack{\text{projective}\\ \text{smooth}}"] & \\
\mathsf{M}_{\mathcal{H}} \arrow[r, hook, "+\text{NCD}"'] & \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}} \arrow[r, "\substack{\text{proper}\\ \text{smooth}}"'] & \mathsf{S}_0
\end{tikzcd}$$

If $\kappa' \succ \kappa$, then we have the compatibility $f^{\mathrm{tor}}_{\kappa'} = f^{\mathrm{tor}}_{\kappa} \circ f^{\mathrm{tor}}_{\kappa',\kappa}$.

(3) *Let us fix a choice of $\kappa \in \mathbf{K}_{Q,\mathcal{H},\Sigma}$ and suppress the subscript κ from the notation.* (*All canonical isomorphisms will be required to be compatible with the canonical isomorphisms defined by pullback under $f^{\mathrm{tor}}_{\kappa',\kappa}$ for each relation $\kappa' \succ \kappa$.*) *Then the following are true*:

(3a) *Let $\Omega^1_{\mathsf{N}^{\mathrm{tor}}/\mathsf{S}_0}[d\log\infty]$ and $\Omega^1_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}/\mathsf{S}_0}[d\log\infty]$ denote the sheaves of modules of log 1-differentials over S_0 given by the* (*respective*) *canonical log structures defined in* (2). *Let*

$$\overline{\Omega}^1_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}} := (\Omega^1_{\mathsf{N}^{\mathrm{tor}}/\mathsf{S}_0}[d\log\infty])/((f^{\mathrm{tor}})^*(\Omega^1_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}/\mathsf{S}_0}[d\log\infty])).$$

Then there is a canonical isomorphism

$$(f^{\mathrm{tor}})^*(\underline{\mathrm{Hom}}_{\mathcal{O}}(Q^\vee, \underline{\mathrm{Lie}}^\vee_{G/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}})) \cong \overline{\Omega}^1_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}} \tag{2.16}$$

between locally free sheaves over $\mathsf{N}^{\mathrm{tor}}$, extending the composition of canonical isomorphisms

$$f^*(\underline{\mathrm{Hom}}_{\mathcal{O}}(Q^\vee, \underline{\mathrm{Lie}}^\vee_{G_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}}})) \cong f^*\underline{\mathrm{Lie}}^\vee_{\mathsf{N}/\mathsf{M}_{\mathcal{H}}} \cong \Omega^1_{\mathsf{N}/\mathsf{M}_{\mathcal{H}}} \tag{2.17}$$

over N.

(3b) *For any integer $b \geq 0$, there exists a canonical isomorphism*

$$R^b f^{\mathrm{tor}}_*(\overline{\Omega}^a_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}) \cong \big(\textstyle\bigwedge^b(\underline{\mathrm{Hom}}_{\mathcal{O}}(Q^\vee, \underline{\mathrm{Lie}}_{G^\vee/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}))\big) \otimes_{\mathcal{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}} \big(\textstyle\bigwedge^a(\underline{\mathrm{Hom}}_{\mathcal{O}}(Q^\vee, \underline{\mathrm{Lie}}^\vee_{G/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}))\big). \tag{2.18}$$

of locally free sheaves over $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$, compatible with cup products and exterior products, extending the canonical isomorphism over $\mathsf{M}_{\mathcal{H}}$ induced by the composition of canonical isomorphisms

$$R^b f_*(\mathcal{O}_{\mathsf{N}}) \cong \textstyle\bigwedge^b \underline{\mathrm{Lie}}_{\mathsf{N}^\vee/\mathsf{M}_{\mathcal{H}}} \cong \bigwedge^b(\underline{\mathrm{Hom}}_{\mathcal{O}}(Q^\vee, \underline{\mathrm{Lie}}_{G^\vee_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}}})). \tag{2.19}$$

(3c) *Let $\overline{\Omega}^\bullet_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}} := \bigwedge^\bullet \overline{\Omega}^1_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}$ be the log de Rham complex associated with $f^{\mathrm{tor}} : \mathsf{N}^{\mathrm{tor}} \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$* (*with differentials inherited from $\Omega^\bullet_{\mathsf{N}/\mathsf{M}_{\mathcal{H}}}$*). *Let the* (*relative*) ***log de Rham cohomology*** *be defined by*

$$\underline{H}^i_{\text{log-dR}}(\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}) := R^i f^{\mathrm{tor}}_*(\overline{\Omega}^\bullet_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}).$$

Then the (*relative*) ***Hodge spectral sequence***

$$E_1^{a,b} := R^b f^{\mathrm{tor}}_*(\overline{\Omega}^a_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}) \Rightarrow \underline{H}^{a+b}_{\text{log-dR}}(\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}) \tag{2.20}$$

*degenerates at E_1 terms, and defines a **Hodge filtration** on $\underline{H}^i_{\text{log-dR}}(\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}})$ with locally free graded pieces given by $R^b f^{\mathrm{tor}}_*(\overline{\Omega}^a_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}})$ for integers $a+b=i$, extending the canonical Hodge filtration on $\underline{H}^i_{\mathrm{dR}}(\mathsf{N}/\mathsf{M}_{\mathcal{H}})$.*

As a result, for any integer $i \geq 0$, there is a canonical isomorphism

$$\textstyle\bigwedge^i \underline{H}^1_{\text{log-dR}}(\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}) \xrightarrow{\sim} \underline{H}^i_{\text{log-dR}}(\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}),$$

compatible with the Hodge filtrations defined by (2.20), *extending the canonical isomorphism* $\bigwedge^i \underline{H}^1_{\mathrm{dR}}(\mathsf{N}/\mathsf{M}_{\mathcal{H}}) \xrightarrow{\sim} \underline{H}^i_{\mathrm{dR}}(\mathsf{N}/\mathsf{M}_{\mathcal{H}})$ *over* $\mathsf{M}_{\mathcal{H}}$ (*defined by cup product*).

(3d) *For any* $j_Q : Q^\vee \hookrightarrow Q$ *as in Lemma 2.5, the* $\mathbb{Z}^\times_{(\Box)}$*-polarization* $\lambda_{\mathsf{M}_{\mathcal{H}}, j_Q}$: $\underline{\mathrm{Hom}}_{\mathcal{O}}(Q, G_{\mathsf{M}_{\mathcal{H}}})^\circ \to (\underline{\mathrm{Hom}}_{\mathcal{O}}(Q, G_{\mathsf{M}_{\mathcal{H}}})^\circ)^\vee$ *in Corollary 2.12 defines canonically* (as in [Deligne and Pappas 1994, 1.5]) *a perfect pairing*

$$\langle \cdot , \cdot \rangle_{\lambda_{\mathsf{M}_{\mathcal{H}}, j_Q}} : \underline{H}^1_{\mathrm{dR}}(\mathsf{N}/\mathsf{M}_{\mathcal{H}}) \times \underline{H}^1_{\mathrm{dR}}(\mathsf{N}/\mathsf{M}_{\mathcal{H}}) \to \mathcal{O}_{\mathsf{M}_{\mathcal{H}}}(1).$$

Then $\underline{H}^1_{\text{log-dR}}(\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}})$ *is the unique subsheaf of* $(\mathsf{M}_{\mathcal{H}} \hookrightarrow \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}})_*(\underline{H}^1_{\mathrm{dR}}(\mathsf{N}/\mathsf{M}_{\mathcal{H}}))$ *satisfying the following conditions*:

(i) $\underline{H}^1_{\text{log-dR}}(\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}})$ *is locally free of finite rank over* $\mathcal{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}$.

(ii) *The sheaf* $f^{\mathrm{tor}}_*(\overline{\Omega}^1_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}})$ *can be identified as the subsheaf of*

$$(\mathsf{M}_{\mathcal{H}} \hookrightarrow \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}})_*(f_*(\Omega^1_{\mathsf{N}/\mathsf{M}_{\mathcal{H}}}))$$

formed (*locally*) *by sections that are also sections of* $\underline{H}^1_{\text{log-dR}}(\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}})$. (*Here we are viewing all these sheaves canonically as subsheaves of* $(\mathsf{M}_{\mathcal{H}} \hookrightarrow \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}})_*(\underline{H}^1_{\mathrm{dR}}(\mathsf{N}/\mathsf{M}_{\mathcal{H}}))$.)

(iii) $\underline{H}^1_{\text{log-dR}}(\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}})$ *is self-dual under the push-forward*

$$(\mathsf{M}_{\mathcal{H}} \hookrightarrow \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}})_*\langle \cdot , \cdot \rangle_{\lambda_{\mathsf{M}_{\mathcal{H}}, j_Q}}.$$

(3e) *The Gauss–Manin connection*

$$\nabla : \underline{H}^\bullet_{\mathrm{dR}}(\mathsf{N}/\mathsf{M}_{\mathcal{H}}) \to \underline{H}^\bullet_{\mathrm{dR}}(\mathsf{N}/\mathsf{M}_{\mathcal{H}}) \otimes_{\mathcal{O}_{\mathsf{M}_{\mathcal{H}}}} \Omega^1_{\mathsf{M}_{\mathcal{H}}/\mathsf{S}_0} \tag{2.21}$$

extends to an integrable connection

$$\nabla : \underline{H}^\bullet_{\text{log-dR}}(\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}) \to \underline{H}^\bullet_{\text{log-dR}}(\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}) \otimes_{\mathcal{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}} \overline{\Omega}^1_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}/\mathsf{S}_0} \tag{2.22}$$

with log poles along $\mathsf{D}_{\infty,\mathcal{H}}$, *called the* ***extended Gauss–Manin connection***, *satisfying the usual Griffith transversality with the Hodge filtration defined by* (2.20).

(4) (Hecke actions.) *Suppose we have an element* $g_h \in \mathrm{G}(\mathbb{A}^{\infty,\Box})$, *and suppose we have a* (*neat*) *open compact subgroup* $\mathcal{H}'$ *of* $\mathrm{G}(\hat{\mathbb{Z}}^\Box)$ *such that* $g_h^{-1}\mathcal{H}'g_h \subset \mathcal{H}$. *Suppose* $\Sigma' = \{\Sigma'_{\Phi'_{\mathcal{H}'}}\}$ *is a compatible choice of admissible smooth rational polyhedral cone decomposition data for* $\mathsf{M}_{\mathcal{H}'}$, *which* g_h*-refines* Σ (as in [Lan 2008, Definition 6.4.3.3]; the notion was called "dominance" in the original version, but changed to the more common "refinement" in the revision). *Then there is also a directed partially ordered set* $\mathbf{K}_{Q,\mathcal{H}',\Sigma'}$ *parametrizing* (*for* $\kappa' \in \mathbf{K}_{Q,\mathcal{H}',\Sigma'}$) $\mathbb{Z}^\times_{(\Box)}$*-isogenies* $\underline{\mathrm{Hom}}_{\mathcal{O}}(Q, G_{\mathsf{M}_{\mathcal{H}'}})^\circ \to \mathsf{N}'_{\kappa'}$ *over* $\mathsf{M}_{\mathcal{H}'}$, *together with open immersions* $\mathsf{N}'_{\kappa'} \hookrightarrow (\mathsf{N}'_{\kappa'})^{\mathrm{tor}}$ *of schemes over* S_0, *satisfying analogues of properties* (1)–(3) *above. The constructions of* $\mathbf{K}_{Q,\mathcal{H},\Sigma}$ *and* $\mathbf{K}_{Q,\mathcal{H}',\Sigma'}$ (*and the*

objects they parametrize) satisfy the compatibility with g_h in the sense that, for each $\kappa \in \mathbf{K}_{Q,\mathcal{H},\Sigma}$, there is an element $\kappa' \in \mathbf{K}_{Q,\mathcal{H}',\Sigma'}$ such that the following are true:

(4a) *There exists a (necessarily unique) finite étale morphism $[g_h]_{\kappa',\kappa} : \mathsf{N}'_{\kappa'} \to \mathsf{N}_\kappa$ covering the morphism $[g_h] : \mathsf{M}_{\mathcal{H}'} \to \mathsf{M}_{\mathcal{H}}$ given by* [Lan 2008, Proposition 6.4.3.4], *inducing a prime-to-$\Box$ isogeny $\mathsf{N}'_{\kappa'} \to \mathsf{N}_\kappa \times_{\mathsf{M}_{\mathcal{H}}} \mathsf{M}_{\mathcal{H}'}$, which agrees with the $\mathbb{Z}^\times_{(\Box)}$-isogeny induced by $(\kappa')^{\mathrm{isog}}$, κ^{isog}, and the $\mathbb{Z}^\times_{(\Box)}$-isogeny $G_{\mathsf{M}_{\mathcal{H}'}} \to G_{\mathsf{M}_{\mathcal{H}}} \times_{\mathsf{M}_{\mathcal{H}}} \mathsf{M}_{\mathcal{H}'}$ realizing $G_{\mathsf{M}_{\mathcal{H}}} \times_{\mathsf{M}_{\mathcal{H}}} \mathsf{M}_{\mathcal{H}'}$ as a Hecke twist of $G_{\mathsf{M}_{\mathcal{H}'}}$ by g_h. (Here all the base changes from $\mathsf{M}_{\mathcal{H}}$ to $\mathsf{M}_{\mathcal{H}'}$ use the morphism $[g_h]$.)*

(4b) *There exists a (necessarily unique) proper log étale morphism*

$$[g_h]^{\mathrm{tor}}_{\kappa',\kappa} : (\mathsf{N}'_{\kappa'})^{\mathrm{tor}} \to \mathsf{N}^{\mathrm{tor}}_\kappa \tag{2.23}$$

extending the morphism $[g_h]_{\kappa',\kappa}$ and covering the morphism $[g_h]^{\mathrm{tor}} : \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}',\Sigma'} \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ given by [Lan 2008, Proposition 6.4.3.4], *such that*

$$R^i([g_h]^{\mathrm{tor}}_{\kappa',\kappa})_* \mathcal{O}_{(\mathsf{N}'_{\kappa'})^{\mathrm{tor}}} = 0 \tag{2.24}$$

for any $i > 0$.

(4c) *There is a canonical isomorphism*

$$([g_h]^{\mathrm{tor}})^* \underline{H}^{a+b}_{\log\text{-dR}}(\mathsf{N}^{\mathrm{tor}}_\kappa / \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}) \xrightarrow{\sim} \underline{H}^{a+b}_{\log\text{-dR}}((\mathsf{N}'_{\kappa'})^{\mathrm{tor}} / \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}',\Sigma'})$$

respecting the Hodge filtrations and compatible with the canonical isomorphisms

$$\begin{aligned} ([g_h]^{\mathrm{tor}}_{\kappa',\kappa})^* \overline{\Omega}^1_{\mathsf{N}^{\mathrm{tor}}_\kappa / \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}} &\xrightarrow{\sim} \overline{\Omega}^1_{(\mathsf{N}'_{\kappa'})^{\mathrm{tor}} / \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}',\Sigma'}}, \\ ([g_h]^{\mathrm{tor}})^* \underline{\mathrm{Lie}}_{G^\vee / \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}} &\xrightarrow{\sim} \underline{\mathrm{Lie}}_{G^\vee / \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}',\Sigma'}}, \\ ([g_h]^{\mathrm{tor}})^* \underline{\mathrm{Lie}}^\vee_{G / \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}} &\xrightarrow{\sim} \underline{\mathrm{Lie}}^\vee_{G / \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}',\Sigma'}}, \end{aligned}$$

and the canonical isomorphisms in (3) *for $\mathsf{N}^{\mathrm{tor}}_\kappa$ and $(\mathsf{N}'_{\kappa'})^{\mathrm{tor}}$.*

(5) ($\mathbb{Z}^\times_{(\Box)}$-isogenies.) *Let g_l be an element of $\mathrm{GL}_{\mathcal{O} \otimes_{\mathbb{Z}} \mathbb{A}^{\infty,\Box}}(Q \otimes_{\mathbb{Z}} \mathbb{A}^{\infty,\Box})$. Then the submodule $g_l(Q \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}^\Box)$ in $Q \otimes_{\mathbb{Z}} \mathbb{A}^{\infty,\Box}$ determines a unique $\mathcal{O}$-lattice Q' (up to isomorphism), together with a unique choice of an isomorphism $[g_l]_Q : Q \otimes_{\mathbb{Z}} \mathbb{Z}_{(\Box)} \xrightarrow{\sim} Q' \otimes_{\mathbb{Z}} \mathbb{Z}_{(\Box)}$, inducing an isomorphism $Q \otimes_{\mathbb{Z}} \mathbb{A}^{\infty,\Box} \xrightarrow{\sim} Q' \otimes_{\mathbb{Z}} \mathbb{A}^{\infty,\Box}$ matching $g_l(Q \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}^\Box)$ with $Q' \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}^\Box$, and inducing a canonical $\mathbb{Z}^\times_{(\Box)}$-isogeny*

$$[g_l]^*_Q : \underline{\mathrm{Hom}}_{\mathcal{O}}(Q', G_{\mathsf{M}_{\mathcal{H}}})^\circ \to \underline{\mathrm{Hom}}_{\mathcal{O}}(Q, G_{\mathsf{M}_{\mathcal{H}}})^\circ$$

defined by $[g_l]_Q$. *For* $\underline{\mathrm{Hom}}_{\mathcal{O}}(Q', G_{\mathsf{M}_{\mathcal{H}}})^\circ$, *there is also a directed partially ordered set* $\mathbf{K}_{Q',\mathcal{H},\Sigma}$ *parametrizing* (*for* $\kappa' \in \mathbf{K}_{Q',\mathcal{H},\Sigma}$) $\mathbb{Z}^\times_{(\Box)}$*-isogenies*

$$\underline{\mathrm{Hom}}_{\mathcal{O}}(Q', G_{\mathsf{M}_{\mathcal{H}}})^\circ \to \mathsf{N}'_{\kappa'}$$

over $\mathsf{M}_{\mathcal{H}}$, *together with open immersions* $\mathsf{N}'_{\kappa'} \hookrightarrow (\mathsf{N}'_{\kappa'})^{\mathrm{tor}}$ *of schemes over* S_0, *satisfying analogues of properties* (1)–(3) *above. The constructions of* $\mathbf{K}_{Q,\mathcal{H},\Sigma}$ *and* $\mathbf{K}_{Q',\mathcal{H},\Sigma}$ (*and the objects they parametrize*) *satisfy the compatibility with* g_l *in the sense that, for each* $\kappa \in \mathbf{K}_{Q,\mathcal{H},\Sigma}$, *there is an element* $\kappa' \in \mathbf{K}_{Q',\mathcal{H},\Sigma}$ *such that the following are true*:

(5a) *The* $\mathbb{Z}^\times_{(\Box)}$*-isogeny* $[g_l]^*_{\kappa',\kappa} := \kappa^{\mathrm{isog}} \circ [g_l]^*_Q \circ ((\kappa')^{\mathrm{isog}})^{-1} : \mathsf{N}'_{\kappa'} \to \mathsf{N}_\kappa$ *is an* ***isogeny*** (*not just a quasiisogeny*), *and hence defines a finite étale morphism.*

(5b) *There exists a* (*necessarily unique*) *proper log étale morphism*

$$([g_l]^*_{\kappa',\kappa})^{\mathrm{tor}} : (\mathsf{N}'_{\kappa'})^{\mathrm{tor}} \to \mathsf{N}^{\mathrm{tor}}_\kappa \tag{2.25}$$

extending the morphism $[g_l]^*_{\kappa',\kappa}$ *over* $\mathsf{M}_{\mathcal{H}}$, *such that*

$$R^i([g_l]^*_{\kappa',\kappa})^{\mathrm{tor}}_* \mathcal{O}_{(\mathsf{N}'_{\kappa'})^{\mathrm{tor}}} = 0 \tag{2.26}$$

for any $i > 0$.

(5c) *For any integer* $i \geq 0$, *there is a canonical isomorphism*

$$(([g_l]^*_{\kappa',\kappa})^{\mathrm{tor}})^* : \underline{H}^i_{\mathrm{log\text{-}dR}}(\mathsf{N}^{\mathrm{tor}}_\kappa/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}) \xrightarrow{\sim} \underline{H}^i_{\mathrm{log\text{-}dR}}((\mathsf{N}'_{\kappa'})^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma})$$

extending the canonical isomorphism

$$([g_l]^*_{\kappa',\kappa})^* : \underline{H}^i_{\mathrm{dR}}(\mathsf{N}_\kappa/\mathsf{M}_{\mathcal{H}}) \xrightarrow{\sim} \underline{H}^i_{\mathrm{dR}}(\mathsf{N}'_{\kappa'}/\mathsf{M}_{\mathcal{H}})$$

induced by $[g_l]_Q$, *respecting the Hodge filtrations and inducing canonical isomorphisms*

$$(([g_l]^*_{\kappa',\kappa})^{\mathrm{tor}})^* : R^b f^{\mathrm{tor}}_*(\overline{\Omega}^a_{\mathsf{N}^{\mathrm{tor}}_\kappa/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}) \xrightarrow{\sim} R^b f^{\mathrm{tor}}_*(\overline{\Omega}^a_{(\mathsf{N}'_{\kappa'})^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}})$$

(*for integers* $a + b = i$) *compatible* (*under the canonical isomorphisms in* (3) *for* $\mathsf{N}^{\mathrm{tor}}_\kappa$ *and* $(\mathsf{N}'_{\kappa'})^{\mathrm{tor}}$) *with the canonical isomorphisms*

$$([g_l]^*_Q)^* : \underline{\mathrm{Hom}}_{\mathcal{O}}(Q^\vee, \underline{\mathrm{Lie}}_{G^\vee/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}) \xrightarrow{\sim} \underline{\mathrm{Hom}}_{\mathcal{O}}((Q')^\vee, \underline{\mathrm{Lie}}_{G^\vee/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}})$$

and

$$([g_l]^*_Q)^* : \underline{\mathrm{Hom}}_{\mathcal{O}}(Q^\vee, \underline{\mathrm{Lie}}^\vee_{G/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}) \xrightarrow{\sim} \underline{\mathrm{Hom}}_{\mathcal{O}}((Q')^\vee, \underline{\mathrm{Lie}}^\vee_{G/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}).$$

2C. ***Outline of the proof.*** The proof of Theorem 2.15 consists of seven steps:

(1) Find a PEL-type $\mathcal{O}$-lattice $(\widetilde{L}, \langle\,\cdot\,,\,\cdot\,\rangle^{\sim}, \tilde{h})$, a fully symplectic admissible filtration $\widetilde{\mathrm{Z}}$ on $\widetilde{L} \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}^{\Box}$, a torus argument $\widetilde{\Phi}$, and a splitting $\tilde{\delta}$ for $\widetilde{\mathrm{Z}}$, such that, for some choices of $\widetilde{\mathcal{H}}$, $\widetilde{\Sigma}$, and $\widetilde{\sigma}$, the $[(\widetilde{\Phi}_{\widetilde{\mathcal{H}}}, \tilde{\delta}_{\widetilde{\mathcal{H}}}, \widetilde{\sigma})]$-stratum $\widetilde{\mathsf{Z}}_{[(\widetilde{\Phi}_{\widetilde{\mathcal{H}}}, \tilde{\delta}_{\widetilde{\mathcal{H}}}, \widetilde{\sigma})]}$ of the toroidal compactification $\widetilde{\mathsf{M}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}}} = \widetilde{\mathsf{M}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}}, \widetilde{\Sigma}}$ has a canonical structure of an abelian scheme over $\mathsf{M}_{\mathcal{H}}$, and such that there exists a canonical $\mathbb{Z}^{\times}_{(\Box)}$-isogeny

$$\kappa^{\mathrm{isog}} : \underline{\mathrm{Hom}}_{\mathcal{O}}(Q, G_{\mathsf{M}_{\mathcal{H}}})^{\circ} \to \widetilde{\mathsf{Z}}_{[(\widetilde{\Phi}_{\widetilde{\mathcal{H}}}, \tilde{\delta}_{\widetilde{\mathcal{H}}}, \widetilde{\sigma})]}.$$

Then we take N_{κ} to be this $\widetilde{\mathsf{Z}}_{[(\widetilde{\Phi}_{\widetilde{\mathcal{H}}}, \tilde{\delta}_{\widetilde{\mathcal{H}}}, \widetilde{\sigma})]}$.

Take $\mathbf{K}^{\mathrm{pre}}_{Q,\mathcal{H},\Sigma}$ to be the set of all such triples $\kappa = (\widetilde{\mathcal{H}}, \widetilde{\Sigma}, \widetilde{\sigma})$, with directed partial order defined by the relation

$$\kappa' = (\widetilde{\mathcal{H}}', \widetilde{\Sigma}', \widetilde{\sigma}') \succ \kappa = (\widetilde{\mathcal{H}}, \widetilde{\Sigma}, \widetilde{\sigma})$$

when $\widetilde{\mathcal{H}}' \subset \widetilde{\mathcal{H}}$ and $\widetilde{\Sigma}'$ refines $\widetilde{\Sigma}$ as in [Lan 2008, Definition 6.4.2.8], and when the $[(\widetilde{\Phi}_{\widetilde{\mathcal{H}}'}, \tilde{\delta}_{\widetilde{\mathcal{H}}'}, \widetilde{\sigma}')]$-stratum of $\widetilde{\mathsf{M}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}}', \widetilde{\Sigma}'}$ is mapped (surjectively) to the $[(\widetilde{\Phi}_{\widetilde{\mathcal{H}}}, \tilde{\delta}_{\widetilde{\mathcal{H}}}, \widetilde{\sigma})]$-stratum of $\widetilde{\mathsf{M}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}}} = \widetilde{\mathsf{M}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}}, \widetilde{\Sigma}}$ under the canonical morphism $\widetilde{\mathsf{M}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}}', \widetilde{\Sigma}'} \to \widetilde{\mathsf{M}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}}, \widetilde{\Sigma}}$ given by [Lan 2008, Proposition 6.4.2.9].

For $\kappa = (\widetilde{\mathcal{H}}, \widetilde{\Sigma}, \widetilde{\sigma})$, take $\mathsf{N}^{\mathrm{tor}}_{\kappa}$ to be the closure of the $[(\widetilde{\Phi}_{\widetilde{\mathcal{H}}}, \tilde{\delta}_{\widetilde{\mathcal{H}}}, \widetilde{\sigma})]$-stratum in $\widetilde{\mathsf{M}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}}, \widetilde{\Sigma}}$. For $\kappa' = (\widetilde{\mathcal{H}}', \widetilde{\Sigma}', \widetilde{\sigma}') \succ \kappa = (\widetilde{\mathcal{H}}, \widetilde{\Sigma}, \widetilde{\sigma})$, the morphism $f^{\mathrm{tor}}_{\kappa',\kappa} : \mathsf{N}^{\mathrm{tor}}_{\kappa'} \to \mathsf{N}^{\mathrm{tor}}_{\kappa}$ is just the morphism induced by the canonical proper morphism $\widetilde{\mathsf{M}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}}', \widetilde{\Sigma}'} \to \widetilde{\mathsf{M}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}}, \widetilde{\Sigma}}$ given by [Lan 2008, Proposition 6.4.2.9].

(2) Show that $\mathsf{N}^{\mathrm{tor}}_{\kappa}$ is projective and smooth over S_0 for $\kappa \in \mathbf{K}^{\mathrm{pre}}_{Q,\mathcal{H},\Sigma}$.

(3) Find a condition on κ that guarantees the existence of a morphism $f^{\mathrm{tor}}_{\kappa} : \mathsf{N}^{\mathrm{tor}}_{\kappa} \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$ extending the structural morphism $f_{\kappa} : \mathsf{N}_{\kappa} \to \mathsf{M}_{\mathcal{H}}$.

(4) Take $\mathbf{K}_{Q,\mathcal{H},\Sigma}$ to be the subset of $\mathbf{K}^{\mathrm{pre}}_{Q,\mathcal{H},\Sigma}$ consisting of elements κ satisfying the condition we have found. Show that this subset is nonempty and has an induced directed partial order by showing that the conditions we need can be achieved after suitable refinements of cone decompositions. This verifies (1) and (2) of Theorem 2.15.

(5) For each $\kappa \in \mathbf{K}_{Q,\mathcal{H},\Sigma}$, verify that the morphism $f^{\mathrm{tor}}_{\kappa} : \mathsf{N}^{\mathrm{tor}}_{\kappa} \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$ extending $\mathsf{N}_{\kappa} \to \mathsf{M}_{\mathcal{H}}$ is log smooth, and verify (3a) of Theorem 2.15.

(6) Assuming (3b) and (3c), verify (4) and (5) of Theorem 2.15 using the Hecke actions on the double tower $\{\widetilde{\mathsf{M}}_{\widetilde{\mathcal{H}}, \widetilde{\Sigma}}\}_{\widetilde{\mathcal{H}}, \widetilde{\Sigma}}$.

(7) Verify (3b), (3c), and (3d) of Theorem 2.15 using explicit descriptions of the formal fibers of $f^{\mathrm{tor}}_{\kappa}$ along (locally closed) strata of $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$. (A crucial step for (3b) requires the notion of log extensions of polarizations we mentioned in the introduction.)

We will carry out these steps in Sections 3–5. We will make frequent references to results cited in Section 1, and also to the original statements in [Lan 2008].

2D. ***System of notation.*** Although the underlying ideas are simple, the notation can be quite heavy. (This seems unavoidable in general works on compactifications.) We decided to keep the notation informative (and hence complicated), because we believe it is more difficult to keep track of three sets of cusp labels and cone decompositions with simplified notation. We understand that the heaviness of notation will inevitably be an enormous burden on the readers, and hence we would like to provide some guidance by explaining the key features in the system of notation, as follows:

- The superscript $^{\mathrm{tor}}$ stands for *toroidal compactifications* (or objects related to them). For morphisms this typically means extensions to morphisms between toroidal compactifications.
- Depending on the context, the overlines can have different meanings:
  - For geometric objects they almost always mean *closures*.
  - For sheaves of differentials (or related objects) they mean the *log versions*.
  - Notable exceptions (to the above two) are in Sections 3B–3C below, where overlines can also stand for *quotients* of group schemes or sheaves.
- Objects for the "given" moduli problem $\mathsf{M}_{\mathcal{H}}$ and its compactifications are denoted as in Section 1.
- Objects for the "larger" moduli problem $\widetilde{\mathsf{M}}_{\widetilde{\mathcal{H}}}$ (mentioned in step (1) above) will be denoted with either $\tilde{}$ (tilde) or $\breve{}$ (breve) on top of the symbols in Section 1. The difference is the following:
  - Symbols with $\tilde{}$ will be used for defining $\widetilde{\mathsf{M}}_{\widetilde{\mathcal{H}}}$ and its compactifications $\widetilde{\mathsf{M}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}},\widetilde{\Sigma}}$, and for realizing the Kuga families we would like to compactify as boundary strata $\widetilde{\mathsf{Z}}_{[(\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\tilde{\delta}_{\widetilde{\mathcal{H}}},\tilde{\sigma})]}$ of $\widetilde{\mathsf{M}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}},\widetilde{\Sigma}}$.
  - Symbols with $\breve{}$ will be used for the boundary strata of $\widetilde{\mathsf{M}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}},\widetilde{\Sigma}}$ appearing in the closure of the realizations

$$\widetilde{\mathsf{Z}}_{[(\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\tilde{\delta}_{\widetilde{\mathcal{H}}},\tilde{\sigma})]}.$$

    (These strata are parametrized by faces $[(\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}},\breve{\tau})]$ of $[(\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\tilde{\delta}_{\widetilde{\mathcal{H}}},\tilde{\sigma})]$.) In other words, they parametrize the boundary strata of the toroidal compactification of the Kuga families we consider.
- In the local descriptions of toroidal boundary structures, we will encounter notations of the forms $(\,\cdot\,)(\sigma)$ and $(\,\cdot\,)_\sigma$.
  - When the object $(\,\cdot\,)$ being modified is a scheme with action by some torus, $(\,\cdot\,)(\sigma)$ will stand for the affine toroidal embedding adding the σ-stratum

(which then also adds all the strata for nontrivial faces of σ), while $(\,\cdot\,)_\sigma$ will stand for the closed σ-stratum (without the nontrivial face strata).

- The formal version of $(\,\cdot\,)_\sigma$ (often denoted in Fraktur) will mean the formal completion of $(\,\cdot\,)(\sigma)$ along $(\,\cdot\,)_\sigma$.

The notation will be most heavy in Sections 4–5, where the calculation of relative cohomology is carried out in detail. For readers only interested in applications to cohomology of Shimura varieties, the statements of Theorem 2.15, the two propositions in Section 3D, and the applications in Section 6 are all they need.

3. Constructions of compactifications and morphisms

3A. ***Kuga families as toroidal boundary strata.*** The goal of this subsection is to carry out steps (1) and (2) of Section 2C.

Let Q be an $\mathcal{O}$-lattice as in Theorem 2.15. Identify $Q^\vee$ with $\operatorname{Hom}_{\mathcal{O}}(Q, \operatorname{Diff}^{-1})$ and give it an $\mathcal{O}$-lattice structure as in Section 2A. The (surjective) trace map $\operatorname{Tr}_{\mathcal{O}/\mathbb{Z}} : \operatorname{Diff}^{-1} \to \mathbb{Z}$ induces a *perfect* pairing

$$\langle\,\cdot\,,\,\cdot\,\rangle_Q : Q^\vee \times Q \to \mathbb{Z} : (f, x) \mapsto \operatorname{Tr}_{\mathcal{O}/\mathbb{Z}}(f(x)).$$

By extension of scalars, the pairing $\langle\,\cdot\,,\,\cdot\,\rangle_Q$ induces a perfect pairing between $Q^\vee \otimes_{\mathbb{Z}} \mathbb{Q}$ and $Q \otimes_{\mathbb{Z}} \mathbb{Q}$. By Condition 1.9, the action of $\mathcal{O}$ on L extends to an action of some maximal order $\mathcal{O}'$ in B containing $\mathcal{O}$. Let us fix the choice of such a maximal order $\mathcal{O}'$. By [Lan 2008, Proposition 1.1.1.17], $\mathcal{O} \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)} \neq \mathcal{O}' \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$ for a prime number $p > 0$ only when $p \mid \operatorname{Disc}$. Let $Q_0 := \mathcal{O}' \cdot Q \subset Q \otimes_{\mathbb{Z}} \mathbb{Q}$ and $Q_{-2} := \operatorname{Hom}_{\mathcal{O}}(Q, \operatorname{Diff}^{-1}_{\mathcal{O}'/\mathbb{Z}})(1) \subset Q^\vee \otimes_{\mathbb{Z}} \mathbb{Q}(1)$. Then the induced pairing

$$\langle\,\cdot\,,\,\cdot\,\rangle_Q : Q_{-2} \times Q_0 \to \mathbb{Q}(1)$$

has values in $\mathbb{Z}(1)$. The localizations of this pairing at primes of $\mathbb{Z}$ are perfect except at those dividing Disc.

Let $(\widetilde{L}, \langle\,\cdot\,,\,\cdot\,\rangle^{\sim}, \tilde{h})$ be the symplectic $\mathcal{O}$-lattice given by the following data:

(1) An $\mathcal{O}$-lattice $\widetilde{L} := Q_{-2} \oplus L \oplus Q_0$, where Q_{-2} and Q_0 are defined as above. (Note that $\widetilde{L}$ satisfies Condition 1.9 by construction.)

(2) A symplectic $\mathcal{O}$-pairing $\langle\,\cdot\,,\,\cdot\,\rangle^{\sim} : \widetilde{L} \times \widetilde{L} \to \mathbb{Z}(1)$ defined (symbolically) by the matrix

$$\langle x, y\rangle^{\sim} := {}^t\begin{pmatrix} x_{-2} \\ x_{-1} \\ x_0 \end{pmatrix} \begin{pmatrix} & & \langle\,\cdot\,,\,\cdot\,\rangle_Q \\ & \langle\,\cdot\,,\,\cdot\,\rangle & \\ -{}^t\langle\,\cdot\,,\,\cdot\,\rangle_Q & & \end{pmatrix} \begin{pmatrix} y_{-2} \\ y_{-1} \\ y_0 \end{pmatrix},$$

namely by

$$\langle x, y\rangle^{\sim} := \langle x_{-2}, y_0\rangle_Q + \langle x_{-1}, y_{-1}\rangle - \langle y_{-2}, x_0\rangle_Q,$$

where $x = \begin{pmatrix} x_{-2} \\ x_{-1} \\ x_0 \end{pmatrix}$ and $y = \begin{pmatrix} y_{-2} \\ y_{-1} \\ y_0 \end{pmatrix}$ are elements of $\widetilde{L} = Q_{-2} \oplus L \oplus Q_0$ expressed (vertically) in terms of components in the direct summands.

Let $j_Q : Q^\vee \hookrightarrow Q$ be an embedding of $\mathcal{O}$-lattices given by Lemma 2.5, so that the pairing $\langle j_Q^{-1}(\,\cdot\,),\,\cdot\,\rangle_Q : (Q \otimes_{\mathbb{Z}} \mathbb{R}) \times (Q \otimes_{\mathbb{Z}} \mathbb{R}) \to \mathbb{R}$ is positive definite. Consider the $\mathbb{R}$-algebra homomorphism $\tilde{h} : \mathbb{C} \to \mathrm{End}_{\mathcal{O}\otimes_{\mathbb{Z}}\mathbb{R}}(\widetilde{L} \otimes_{\mathbb{Z}} \mathbb{R})$ defined by

$$\begin{aligned} z &= z_1 + \sqrt{-1}\, z_2 \\ &\mapsto \tilde{h}(z) := \begin{pmatrix} z_1 \,\mathrm{Id}_{Q_{-2}\otimes_{\mathbb{Z}}\mathbb{R}} & & -z_2((2\pi\sqrt{-1}) \circ j_Q^{-1}) \\ & h(z) & \\ z_2(j_Q \circ (2\pi\sqrt{-1})^{-1}) & & z_1 \,\mathrm{Id}_{Q_0\otimes_{\mathbb{Z}}\mathbb{R}} \end{pmatrix}, \end{aligned}$$

where $2\pi\sqrt{-1} : \mathbb{Z} \xrightarrow{\sim} \mathbb{Z}(1)$ and $(2\pi\sqrt{-1})^{-1} : \mathbb{Z}(1) \xrightarrow{\sim} \mathbb{Z}$ stand for the isomorphisms defined by the choice of $\sqrt{-1}$ in $\mathbb{C}$, and where the matrix acts (symbolically) on elements of $\widetilde{L} \otimes_{\mathbb{Z}} \mathbb{R}$ by left multiplication. In other words,

$$\tilde{h}(z) \begin{pmatrix} x_{-2} \\ x_{-1} \\ x_0 \end{pmatrix} = \begin{pmatrix} z_1 x_{-2} - z_2((2\pi\sqrt{-1}) \circ j_Q^{-1})(x_0) \\ h(z) x_{-1} \\ z_2(j_Q \circ (2\pi\sqrt{-1})^{-1})(x_{-2}) + z_1 x_0 \end{pmatrix}.$$

Then $\tilde{h}$ is a polarization of $(\widetilde{L}, \langle\,\cdot\,,\,\cdot\,\rangle^{\sim})$ making $(\widetilde{L}, \langle\,\cdot\,,\,\cdot\,\rangle^{\sim}, \tilde{h})$ a PEL-type $\mathcal{O}$-lattice. Note that the reflex field of $(\widetilde{L} \otimes_{\mathbb{Z}} \mathbb{R}, \langle\,\cdot\,,\,\cdot\,\rangle^{\sim}, \tilde{h})$ is also F_0.

By construction of $(\widetilde{L}, \langle\,\cdot\,,\,\cdot\,\rangle^{\sim})$, there is a fully symplectic admissible filtration on $\widetilde{L} \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}^\square$ induced by

$$0 \subset Q_{-2} \subset Q_{-2} \oplus L \subset Q_{-2} \oplus L \oplus Q_0 = \widetilde{L}.$$

More precisely, we have

$$\begin{aligned} \widetilde{\mathrm{Z}}_{-3} &:= 0, \\ \widetilde{\mathrm{Z}}_{-2} &:= Q_{-2} \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}^\square, \\ \widetilde{\mathrm{Z}}_{-1} &:= (Q_{-2} \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}^\square) \oplus (L \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}^\square), \\ \widetilde{\mathrm{Z}}_{0} &:= (Q_{-2} \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}^\square) \oplus (L \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}^\square) \oplus (Q_0 \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}^\square) = \widetilde{L} \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}^\square, \end{aligned}$$

so that there are canonical isomorphisms

$$\mathrm{Gr}_{-2}^{\widetilde{\mathrm{Z}}} \cong Q_{-2} \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}^\square, \quad \mathrm{Gr}_{-1}^{\widetilde{\mathrm{Z}}} \cong L \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}^\square, \qquad \mathrm{Gr}_{0}^{\widetilde{\mathrm{Z}}} \cong Q_0 \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}^\square$$

matching the pairings $\mathrm{Gr}_{-2}^{\widetilde{\mathrm{Z}}} \times \mathrm{Gr}_{0}^{\widetilde{\mathrm{Z}}} \to \hat{\mathbb{Z}}^\square(1)$ and $\mathrm{Gr}_{-1}^{\widetilde{\mathrm{Z}}} \times \mathrm{Gr}_{-1}^{\widetilde{\mathrm{Z}}} \to \hat{\mathbb{Z}}^\square(1)$ induced by $\langle\,\cdot\,,\,\cdot\,\rangle^{\sim}$ with $\langle\,\cdot\,,\,\cdot\,\rangle_Q$ and $\langle\,\cdot\,,\,\cdot\,\rangle$, respectively.

Let $\widetilde{X} := \mathrm{Hom}_{\mathcal{O}}(Q_{-2}, \mathrm{Diff}^{-1}(1))$ and $\widetilde{Y} := Q_0$. The pairing

$$\langle \cdot , \cdot \rangle_Q : Q_{-2} \times Q_0 \to \mathbb{Z}(1)$$

induces a canonical embedding $\widetilde{\phi} : \widetilde{Y} \hookrightarrow \widetilde{X}$ and there are canonical isomorphisms $\widetilde{\varphi}_{-2} : \mathrm{Gr}_{-2}^{\widetilde{\mathrm{Z}}} \xrightarrow{\sim} \mathrm{Hom}_{\hat{\mathbb{Z}}^\square}(\widetilde{X} \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}^\square, \hat{\mathbb{Z}}^\square(1))$ and $\widetilde{\varphi}_0 : \mathrm{Gr}_0^{\widetilde{\mathrm{Z}}} \xrightarrow{\sim} \widetilde{Y} \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}^\square$ (of $\hat{\mathbb{Z}}^\square$-modules). These data define a torus argument $\widetilde{\Phi} := (\widetilde{X}, \widetilde{Y}, \widetilde{\phi}, \widetilde{\varphi}_{-2}, \widetilde{\varphi}_0)$ for $\widetilde{\mathrm{Z}}$ as in Definition 1.13.

Let $\tilde{\delta}$ be the obvious splitting of $\widetilde{\mathrm{Z}}$ induced by the equality $Q_{-2} \oplus L \oplus Q_0 = \widetilde{L}$.

Let $\widetilde{\mathrm{G}}$ be the group functor defined by $(\widetilde{L}, \langle \cdot , \cdot \rangle^{\sim})$ as in Definition 1.1. For any $\hat{\mathbb{Z}}^\square$-algebra R, let $\widetilde{\mathrm{P}}_{\widetilde{\mathrm{Z}}}(R)$ denote the subgroup of $\widetilde{\mathrm{G}}(R)$ consisting of elements g such that $g(\widetilde{\mathrm{Z}}_{-2} \otimes_{\hat{\mathbb{Z}}^\square} R) = \widetilde{\mathrm{Z}}_{-2} \otimes_{\hat{\mathbb{Z}}^\square} R$ and $g(\widetilde{\mathrm{Z}}_{-1} \otimes_{\hat{\mathbb{Z}}^\square} R) = \widetilde{\mathrm{Z}}_{-1} \otimes_{\hat{\mathbb{Z}}^\square} R$. Any element g in $\widetilde{\mathrm{P}}_{\widetilde{\mathrm{Z}}}(R)$ defines an isomorphism

$$\mathrm{Gr}_{-1}^{\widetilde{\mathrm{Z}}}(g) : \mathrm{Gr}_{-1}^{\widetilde{\mathrm{Z}}} \otimes_{\hat{\mathbb{Z}}^\square} R \xrightarrow{\sim} \mathrm{Gr}_{-1}^{\widetilde{\mathrm{Z}}} \otimes_{\hat{\mathbb{Z}}^\square} R,$$

which corresponds under the canonical isomorphism $\mathrm{Gr}_{-1}^{\widetilde{\mathrm{Z}}} \otimes_{\hat{\mathbb{Z}}^\square} R \cong L \otimes_{\mathbb{Z}} R$ above to an element of $\mathrm{G}(R)$. This defines in particular a homomorphism

$$\mathrm{Gr}_{-1}^{\widetilde{\mathrm{Z}}} : \widetilde{\mathrm{P}}_{\widetilde{\mathrm{Z}}}(\hat{\mathbb{Z}}^\square) \to \mathrm{G}(\hat{\mathbb{Z}}^\square).$$

Let $\widetilde{\mathcal{H}}$ be any *neat* open compact subgroup of $\widetilde{\mathrm{G}}(\hat{\mathbb{Z}}^\square)$ such that the image $\mathrm{Gr}_{-1}^{\widetilde{\mathrm{Z}}}(\widetilde{\mathcal{H}} \cap \widetilde{\mathrm{P}}_{\widetilde{\mathrm{Z}}}(\hat{\mathbb{Z}}^\square))$ is exactly $\mathcal{H}$. (Such an $\widetilde{\mathcal{H}}$ exists because the pairing $\langle \cdot , \cdot \rangle^{\sim}$ is the direct sum of the pairings on $Q_{-2} \oplus Q_0$ and on L.) The data of $\mathcal{O}$, $(\widetilde{L}, \langle \cdot , \cdot \rangle^{\sim}, \tilde{h})$, $\square$, and $\widetilde{\mathcal{H}} \subset \widetilde{\mathrm{G}}(\hat{\mathbb{Z}}^\square)$ define a moduli problem $\widetilde{\mathsf{M}}_{\widetilde{\mathcal{H}}}$ as in Definition 1.6.

Take any compatible choice $\widetilde{\Sigma}$ of admissible smooth rational polyhedral cone decomposition data for $\widetilde{\mathsf{M}}_{\widetilde{\mathcal{H}}}$ that is *projective* (see Definitions 1.33 and 1.34). Since $\widetilde{\mathcal{H}}$ is neat, any such $\widetilde{\Sigma}$ defines a toroidal compactification $\widetilde{\mathsf{M}}_{\widetilde{\mathcal{H}}}^{\mathrm{tor}} = \widetilde{\mathsf{M}}_{\widetilde{\mathcal{H}}, \widetilde{\Sigma}}^{\mathrm{tor}}$ which is projective and smooth over S_0 by (7) of Theorem 1.41.

Let $(\widetilde{\mathrm{Z}}, \widetilde{\Phi}, \tilde{\delta})$ be as above, and let $(\widetilde{\mathrm{Z}}_{\widetilde{\mathcal{H}}}, \widetilde{\Phi}_{\widetilde{\mathcal{H}}} = (\widetilde{X}, \widetilde{Y}, \widetilde{\phi}, \widetilde{\varphi}_{-2,\widetilde{\mathcal{H}}}, \widetilde{\varphi}_{0,\widetilde{\mathcal{H}}}), \tilde{\delta}_{\widetilde{\mathcal{H}}})$ be the induced triple at level $\widetilde{\mathcal{H}}$, inducing a cusp label $[(\widetilde{\mathrm{Z}}_{\widetilde{\mathcal{H}}}, \widetilde{\Phi}_{\widetilde{\mathcal{H}}}, \tilde{\delta}_{\widetilde{\mathcal{H}}})]$ at level $\widetilde{\mathcal{H}}$.

Let $\tilde{\sigma} \subset \mathbf{P}_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}}}^{+}$ be any *top-dimensional* nondegenerate rational polyhedral cone in the cone decomposition $\widetilde{\Sigma}_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}}}$ in $\widetilde{\Sigma}$. Then, by (2) of Theorem 1.41, we have a stratum $\widetilde{\mathsf{Z}}_{[(\widetilde{\Phi}_{\widetilde{\mathcal{H}}}, \tilde{\delta}_{\widetilde{\mathcal{H}}}, \tilde{\sigma})]}$ of $\widetilde{\mathsf{M}}_{\widetilde{\mathcal{H}}}^{\mathrm{tor}}$.

Since $\tilde{\sigma}$ is a *top-dimensional* cone in $\widetilde{\Sigma}_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}}}$, the locally closed stratum $\widetilde{\mathsf{Z}}_{[(\widetilde{\Phi}_{\widetilde{\mathcal{H}}}, \tilde{\delta}_{\widetilde{\mathcal{H}}}, \tilde{\sigma})]}$ (not its closure) is a zero-dimensional torus bundle over the abelian scheme $\widetilde{C}_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}}, \tilde{\delta}_{\widetilde{\mathcal{H}}}}$ over $\mathsf{M}_{\mathcal{H}}$. In other words, $\widetilde{\mathsf{Z}}_{[(\widetilde{\Phi}_{\widetilde{\mathcal{H}}}, \tilde{\delta}_{\widetilde{\mathcal{H}}}, \tilde{\sigma})]}$ is canonically *isomorphic* to $\widetilde{C}_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}}, \tilde{\delta}_{\widetilde{\mathcal{H}}}}$. By the construction of $\widetilde{C}_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}}, \tilde{\delta}_{\widetilde{\mathcal{H}}}}$ in [Lan 2008, §§6.2.3–6.2.4], it is canonically $\mathbb{Z}_{(\square)}^{\times}$-isogenous to the abelian scheme $\underline{\mathrm{Hom}}_{\mathcal{O}}(Q, G_{\mathsf{M}_{\mathcal{H}}})^{\circ}$. Let us define N_{κ} to be this stratum $\widetilde{\mathsf{Z}}_{[(\widetilde{\Phi}_{\widetilde{\mathcal{H}}}, \tilde{\delta}_{\widetilde{\mathcal{H}}}, \tilde{\sigma})]}$, and denote the canonical morphism $\mathsf{N}_{\kappa} \to \mathsf{M}_{\mathcal{H}}$ by f_{κ}. This gives the $\mathbb{Z}_{(\square)}^{\times}$-isogeny $\kappa^{\mathrm{isog}} : \underline{\mathrm{Hom}}_{\mathcal{O}}(Q, G_{\mathsf{M}_{\mathcal{H}}})^{\circ} \to \mathsf{N}_{\kappa}$. Note that $\mathsf{N}_{\kappa} = \widetilde{\mathsf{Z}}_{[(\widetilde{\Phi}_{\widetilde{\mathcal{H}}}, \tilde{\delta}_{\widetilde{\mathcal{H}}}, \tilde{\sigma})]}$ is canonically isomorphic to $\widetilde{C}_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}}, \tilde{\delta}_{\widetilde{\mathcal{H}}}}$ for every $\widetilde{\Sigma}$ and every top-dimensional cone $\tilde{\sigma}$ in $\widetilde{\Sigma}_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}}}$.

As planned in step (1) of Section 2C, let us take $\mathbf{K}^{\mathrm{pre}}_{Q,\mathcal{H},\Sigma}$ to be the set of all possible such triples $\kappa = (\widetilde{\mathcal{H}}, \widetilde{\Sigma}, \widetilde{\sigma})$, with directed partial order defined by the relation $\kappa' = (\widetilde{\mathcal{H}}', \widetilde{\Sigma}', \widetilde{\sigma}') \succ \kappa = (\widetilde{\mathcal{H}}, \widetilde{\Sigma}, \widetilde{\sigma})$ when $\widetilde{\mathcal{H}}' \subset \widetilde{\mathcal{H}}$, when $\widetilde{\Sigma}'$ refines $\widetilde{\Sigma}$ as in [Lan 2008, Definition 6.4.2.8], and when $(\widetilde{\Phi}_{\widetilde{\mathcal{H}}'}, \widetilde{\delta}_{\widetilde{\mathcal{H}}'}, \widetilde{\sigma}')$ refines $(\widetilde{\Phi}_{\widetilde{\mathcal{H}}}, \widetilde{\delta}_{\widetilde{\mathcal{H}}}, \widetilde{\sigma})$ as in [Lan 2008, Definition 6.4.2.6]. In this case, the $[(\widetilde{\Phi}_{\widetilde{\mathcal{H}}'}, \widetilde{\delta}_{\widetilde{\mathcal{H}}'}, \widetilde{\sigma}')]$-stratum of $\widetilde{\mathsf{M}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}}',\widetilde{\Sigma}'}$ is mapped to the $[(\widetilde{\Phi}_{\widetilde{\mathcal{H}}}, \widetilde{\delta}_{\widetilde{\mathcal{H}}}, \widetilde{\sigma})]$-stratum of $\widetilde{\mathsf{M}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}},\widetilde{\Sigma}}$ by the canonical morphism $\widetilde{\mathsf{M}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}}',\widetilde{\Sigma}'} \to \widetilde{\mathsf{M}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}},\widetilde{\Sigma}}$ given by [Lan 2008, Proposition 6.4.2.9]. Note that the induced morphism $f_{\kappa',\kappa} : \mathsf{N}_{\kappa'} \to \mathsf{N}_{\kappa}$, which is $\kappa^{\mathrm{isog}} \circ ((\kappa')^{\mathrm{isog}})^{-1}$ by definition, can be identified with the canonical $\mathbb{Z}^{\times}_{(\Box)}$-isogeny $\widetilde{C}_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}'},\widetilde{\delta}_{\widetilde{\mathcal{H}}'}} \to \widetilde{C}_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\widetilde{\delta}_{\widetilde{\mathcal{H}}}}$. In particular, it is surjective and is an isogeny of degree prime to $\Box$.

For $\kappa = (\widetilde{\mathcal{H}}, \widetilde{\Sigma}, \widetilde{\sigma})$, take $\mathsf{N}^{\mathrm{tor}}_{\kappa}$ to be the closure of $\widetilde{\mathsf{Z}}_{[(\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\widetilde{\delta}_{\widetilde{\mathcal{H}}},\widetilde{\sigma})]}$ in $\widetilde{\mathsf{M}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}},\widetilde{\Sigma}}$. Then we obtain the canonical immersion $\kappa^{\mathrm{tor}} : \mathsf{N}_{\kappa} \hookrightarrow \mathsf{N}^{\mathrm{tor}}_{\kappa}$.

When $\kappa' = (\widetilde{\mathcal{H}}', \widetilde{\Sigma}', \widetilde{\sigma}') \succ \kappa = (\widetilde{\mathcal{H}}, \widetilde{\Sigma}, \widetilde{\sigma})$, the morphism $f^{\mathrm{tor}}_{\kappa',\kappa} : \mathsf{N}^{\mathrm{tor}}_{\kappa'} \to \mathsf{N}^{\mathrm{tor}}_{\kappa}$ is simply the morphism induced by the canonical proper morphism $\widetilde{\mathsf{M}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}}',\widetilde{\Sigma}'} \to \widetilde{\mathsf{M}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}},\widetilde{\Sigma}}$ given by [Lan 2008, Proposition 6.4.2.9]. Note that the latter morphism is étale locally given by equivariant morphisms between toric schemes, and the same is true for the induced morphism $f^{\mathrm{tor}}_{\kappa',\kappa} : \mathsf{N}^{\mathrm{tor}}_{\kappa'} \to \mathsf{N}^{\mathrm{tor}}_{\kappa}$. Therefore, both the morphism $\widetilde{\mathsf{M}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}}',\widetilde{\Sigma}'} \to \widetilde{\mathsf{M}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}},\widetilde{\Sigma}}$ and the induced morphism $f^{\mathrm{tor}}_{\kappa',\kappa} : \mathsf{N}^{\mathrm{tor}}_{\kappa'} \to \mathsf{N}^{\mathrm{tor}}_{\kappa}$ are log étale essentially by definition (see [Kato 1989, Theorem 3.5]). Moreover, as in [Faltings and Chai 1990, Chapter V, Remark 1.2(b)] and in the proof of [Lan 2008, Lemma 7.1.1.3], we have $R^i(f^{\mathrm{tor}}_{\kappa',\kappa})_*\mathcal{O}_{\mathsf{N}^{\mathrm{tor}}_{\kappa'}} = 0$ for $i > 0$ by [Kempf et al. 1973, Chapter I, §3].

Lemma 3.1. *Under the assumption that $\widetilde{\mathcal{H}}$ is neat, the closure of every stratum in $\widetilde{\mathsf{M}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}},\widetilde{\Sigma}}$ has no self-intersection.*

Proof. According to Definitions 1.33 and 1.34, the collection $\widetilde{\Sigma}$ of cone decompositions for $\widetilde{\mathsf{M}}_{\widetilde{\mathcal{H}}}$ satisfies Condition 1.29. Hence [Lan 2008, Lemma 6.2.5.27 in the revision] implies that the closure of any stratum does not intersect itself. (See also [Faltings and Chai 1990, Chapter IV, Remark 5.8a].) $\Box$

Corollary 3.2. *For any $\kappa = (\widetilde{\mathcal{H}}, \widetilde{\Sigma}, \widetilde{\sigma}) \in \mathbf{K}^{\mathrm{pre}}_{Q,\mathcal{H},\Sigma}$, the closure $\mathsf{N}^{\mathrm{tor}}_{\kappa}$ of $\mathsf{N}_{\kappa} = \widetilde{\mathsf{Z}}_{[(\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\widetilde{\delta}_{\widetilde{\mathcal{H}}},\widetilde{\sigma})]}$ in $\widetilde{\mathsf{M}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}},\widetilde{\Sigma}}$ is projective and smooth over S_0, and the complement of N_{κ} in $\mathsf{N}^{\mathrm{tor}}_{\kappa}$ (with its reduced structure) is a relative Cartier divisor with simple normal crossings. Thus the collection of open embeddings $\kappa^{\mathrm{tor}} : \mathsf{N}_{\kappa} \hookrightarrow \mathsf{N}^{\mathrm{tor}}_{\kappa}$, with $\kappa \in \mathbf{K}^{\mathrm{pre}}_{Q,\mathcal{H},\Sigma}$, satisfies* (1) *of Theorem 2.15.*

Proof. Combine Lemma 3.1 with (3) and (7) of Theorem 1.41. $\Box$

From now on, let us fix a choice of $\kappa = (\widetilde{\mathcal{H}}, \widetilde{\Sigma}, \widetilde{\sigma}) \in \mathbf{K}^{\mathrm{pre}}_{Q,\mathcal{H},\Sigma}$, and suppress κ and $\widetilde{\Sigma}$ from the notation. The compatibility of various objects under compositions with or pullbacks by $f^{\mathrm{tor}}_{\kappa',\kappa} : \mathsf{N}^{\mathrm{tor}}_{\kappa'} \to \mathsf{N}^{\mathrm{tor}}_{\kappa}$ (for $\kappa' \succ \kappa$ in $\mathbf{K}^{\mathrm{pre}}_{Q,\mathcal{H},\Sigma}$) will be obvious from the constructions.

3B. ***Extendability of structural morphisms.*** The goal of this subsection is to carry out steps (3) and (4) of Section 2C.

Let $(\widetilde{G}, \widetilde{\lambda}, \tilde{i}, \widetilde{\alpha}_{\widetilde{\mathcal{H}}})$ be the degenerating family of type $\widetilde{\mathsf{M}}_{\widetilde{\mathcal{H}}}$ over $\widetilde{\mathsf{M}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}}}$. By construction of N as a boundary stratum of $\widetilde{\mathsf{M}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}}}$, the restriction $\widetilde{G}_{\mathsf{N}}$ of $\widetilde{G}$ to N is an extension of the pullback of the abelian scheme $G_{\mathsf{M}_{\mathcal{H}}}$ over $\mathsf{M}_{\mathcal{H}}$ to N by $f : \mathsf{N} \to \mathsf{M}_{\mathcal{H}}$, by the split torus $\widetilde{T}_{\mathsf{N}}$ over N with character group $\widetilde{X}$. The data of $\widetilde{\lambda}$, $\tilde{i}$, and $\widetilde{\alpha}_{\widetilde{\mathcal{H}}}$ induce respectively a polarization, an $\mathcal{O}$-endomorphism structure, and a level $\mathcal{H}$-structure on the abelian part of $\widetilde{G}_{\mathsf{N}}$, which agree with the pullbacks of the data λ, i, and $\alpha_{\mathcal{H}}$ over $\mathsf{M}_{\mathcal{H}}$ to N by $f : \mathsf{N} \to \mathsf{M}_{\mathcal{H}}$. By normality of (the closure) $\mathsf{N}^{\mathrm{tor}}$ (of N in $\widetilde{\mathsf{M}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}}}$), and by a result of Raynaud (see [Raynaud 1970, IX 2.4] or [Faltings and Chai 1990, Chapter I, Proposition 2.9]), the embedding $\widetilde{T}_{\mathsf{N}} \hookrightarrow \widetilde{G}_{\mathsf{N}}$ of group schemes extends (uniquely) to an embedding $\widetilde{T}_{\mathsf{N}^{\mathrm{tor}}} \hookrightarrow \widetilde{G}_{\mathsf{N}^{\mathrm{tor}}}$ of group schemes, and the quotient

$$\overline{G} := \widetilde{G}_{\mathsf{N}^{\mathrm{tor}}} / \widetilde{T}_{\mathsf{N}^{\mathrm{tor}}}$$

is a semiabelian scheme whose restriction to N can be identified with the pullback of G from $\mathsf{M}_{\mathcal{H}}$ to N. Similarly, we obtain $\overline{G}^{\vee} := \widetilde{G}^{\vee}_{\mathsf{N}^{\mathrm{tor}}} / \widetilde{T}^{\vee}_{\mathsf{N}^{\mathrm{tor}}}$. By another result of Raynaud (see [Raynaud 1970, IX 1.4] or [Faltings and Chai 1990, Chapter I, Proposition 2.7]), the semiabelian $\overline{G}$ carries (unique) additional structures $\overline{\lambda} : \overline{G} \to \overline{G}^{\vee}$, $\overline{i}$, and $\overline{\alpha}_{\mathcal{H}}$ such that the restriction of $(\overline{G}, \overline{\lambda}, \overline{i}, \overline{\alpha}_{\mathcal{H}})$ to N is the pullback of the tautological tuple over $\mathsf{M}_{\mathcal{H}}$ by $f : \mathsf{N} \to \mathsf{M}_{\mathcal{H}}$, and so that $(\overline{G}, \overline{\lambda}, \overline{i}, \overline{\alpha}_{\mathcal{H}})$ defines a degenerating family of type $\mathsf{M}_{\mathcal{H}}$ over $\mathsf{N}^{\mathrm{tor}}$.

Now the question is whether the structural morphism $f : \mathsf{N} \to \mathsf{M}_{\mathcal{H}}$ extends (necessarily uniquely) to a (proper) morphism $f^{\mathrm{tor}} : \mathsf{N}^{\mathrm{tor}} \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$ between the compactifications. By (6) of Theorem 1.41, this extendability can be verified after pullback to complete discrete valuation rings (with algebraically closed residue fields).

The stratification of $\widetilde{\mathsf{M}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}}}$ induces one on $\mathsf{N}^{\mathrm{tor}}$. By (2) of Theorem 1.41, the strata of $\mathsf{N}^{\mathrm{tor}}$ are parametrized by the faces of $[(\widetilde{\Phi}_{\widetilde{\mathcal{H}}}, \tilde{\delta}_{\widetilde{\mathcal{H}}}, \widetilde{\sigma})]$ (as in Definition 1.38). Concretely, the faces of $[(\widetilde{\Phi}_{\widetilde{\mathcal{H}}}, \tilde{\delta}_{\widetilde{\mathcal{H}}}, \widetilde{\sigma})]$ are equivalence classes $[(\breve{\Phi}_{\widetilde{\mathcal{H}}}, \breve{\delta}_{\widetilde{\mathcal{H}}}, \breve{\tau})]$ of $\widetilde{\mathcal{H}}$-orbits of data of the following form:

(1) A fully symplectic admissible filtration $\breve{\mathrm{Z}} = \{\breve{\mathrm{Z}}_{-i}\}$ on $\widetilde{L} \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}^{\square}$ satisfying

$$\widetilde{\mathrm{Z}}_{-2} \subset \breve{\mathrm{Z}}_{-2} \subset \breve{\mathrm{Z}}_{-1} \subset \widetilde{\mathrm{Z}}_{-1}. \tag{3.3}$$

Any such filtration $\breve{\mathrm{Z}}$ induces a fully symplectic admissible filtration $\mathrm{Z} = \{\mathrm{Z}_{-i}\}$ on $L \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}^{\square}$ by $\mathrm{Z}_{-2} := \breve{\mathrm{Z}}_{-2} / \widetilde{\mathrm{Z}}_{-2}$ and $\mathrm{Z}_{-1} := \breve{\mathrm{Z}}_{-1} / \widetilde{\mathrm{Z}}_{-2}$, so that there is a canonical isomorphism

$$\mathrm{Z}_0 / \mathrm{Z}_{-1} \cong \widetilde{\mathrm{Z}}_{-1} / \breve{\mathrm{Z}}_{-1}. \tag{3.4}$$

Conversely, any fully symplectic admissible filtration Z on $L \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}^{\square}$ induces a fully symplectic admissible filtration $\breve{\mathrm{Z}}$ on $\widetilde{L} \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}^{\square}$ satisfying (3.3) and (3.4).

(2) A torus argument $\breve{\Phi} = (\breve{X}, \breve{Y}, \breve{\phi}, \breve{\varphi}_{-2}, \breve{\varphi}_0)$ for $\breve{\mathrm{Z}}$ (as in Definition 1.13), together with admissible surjections $s_{\breve{X}} : \breve{X} \twoheadrightarrow \widetilde{X}$ and $s_{\breve{Y}} : \breve{Y} \twoheadrightarrow \widetilde{Y}$ satisfying $s_{\breve{X}}\breve{\phi} = \widetilde{\phi} s_{\breve{Y}}$ and other natural compatibilities with $\breve{\varphi}_{-2}$, $\breve{\varphi}_0$, $\widetilde{\varphi}_{-2}$, and $\widetilde{\varphi}_0$. (See Definitions 1.18–1.20.)

Any $\breve{\Phi}$, $s_{\breve{X}}$, and $s_{\breve{Y}}$ determine a torus argument $\Phi = (X, Y, \phi, \varphi_{-2}, \varphi_0)$ for Z by $X := \ker(s_{\breve{X}})$, $Y := \ker(s_{\breve{Y}})$, and $\phi := \breve{\phi}|_Y$, so that there is a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & Y \arrow[r] \arrow[d, "\phi"'] & \breve{Y} \arrow[r, "s_{\breve{Y}}"] \arrow[d, "\breve{\phi}"'] & \widetilde{Y} \arrow[r] \arrow[d, "\widetilde{\phi}"'] & 0 \\ 0 \arrow[r] & X \arrow[r] & \breve{X} \arrow[r, "s_{\breve{X}}"'] & \widetilde{X} \arrow[r] & 0 \end{tikzcd} \tag{3.5}$$

whose horizontal rows are exact sequences.

(3) The existence of some splitting of $\breve{\mathrm{Z}}$, inducing some liftable splitting $\breve{\delta}_{\widetilde{\mathcal{H}}}$ defining the cusp label $(\breve{\mathrm{Z}}_{\widetilde{\mathcal{H}}}, \breve{\Phi}_{\widetilde{\mathcal{H}}}, \breve{\delta}_{\widetilde{\mathcal{H}}})$ at level $\widetilde{\mathcal{H}}$.

Given the liftable splitting $\tilde{\delta}_{\widetilde{\mathcal{H}}}$, the existence of the liftable splitting $\breve{\delta}_{\widetilde{\mathcal{H}}}$ is equivalent to the existence of some liftable splitting $\delta_{\mathcal{H}}$ of $\mathrm{Z}_{\mathcal{H}}$. Then we see that there is a canonical bijection between cusp labels $[(\mathrm{Z}_{\mathcal{H}}, \Phi_{\mathcal{H}}, \delta_{\mathcal{H}})]$ at level $\mathcal{H}$ and cusp labels $[(\breve{\mathrm{Z}}_{\widetilde{\mathcal{H}}}, \breve{\Phi}_{\widetilde{\mathcal{H}}}, \breve{\delta}_{\widetilde{\mathcal{H}}})]$ at level $\widetilde{\mathcal{H}}$ admitting a surjection to $[(\widetilde{\mathrm{Z}}_{\widetilde{\mathcal{H}}}, \widetilde{\Phi}_{\widetilde{\mathcal{H}}}, \tilde{\delta}_{\widetilde{\mathcal{H}}})]$.

(4) Let $\Phi_{\mathcal{H}}$ (resp. $\breve{\Phi}_{\widetilde{\mathcal{H}}}$) be the torus argument for $\mathrm{Z}_{\mathcal{H}}$ (resp. $\widetilde{\mathrm{Z}}_{\widetilde{\mathcal{H}}}$) at level $\mathcal{H}$ (resp. $\widetilde{\mathcal{H}}$) induced by Φ (resp. $\breve{\Phi}$). Then (3.5) induces morphisms

$$\mathbf{S}_{\Phi_{\mathcal{H}}} \hookrightarrow \mathbf{S}_{\breve{\Phi}_{\widetilde{\mathcal{H}}}} \twoheadrightarrow \mathbf{S}_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}}}, \tag{3.6}$$

where the first morphism is canonical, and where the second morphism is defined by $s_{\breve{X}}$ and $s_{\breve{Y}}$, whose composition is zero. (In general, the morphisms in (3.6) *do not* form an exact sequence.)

The dual of (3.6) defines morphisms

$$\mathbf{P}^{+}_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}}} \hookrightarrow \mathbf{P}_{\breve{\Phi}_{\widetilde{\mathcal{H}}}} \twoheadrightarrow \mathbf{P}_{\Phi_{\mathcal{H}}}, \tag{3.7}$$

where the first morphism is defined by $s_{\breve{X}}$ and $s_{\breve{Y}}$, and where the second morphism is canonical, whose composition is zero.

Then $\breve{\tau} \subset \mathbf{P}^{+}_{\breve{\Phi}_{\widetilde{\mathcal{H}}}}$ is a cone in the cone decomposition $\widetilde{\Sigma}_{\breve{\Phi}_{\widetilde{\mathcal{H}}}}$ having a face $\breve{\sigma}$ that is a $\Gamma_{\breve{\Phi}_{\widetilde{\mathcal{H}}}}$-translation (see Definition 1.23) of the image of $\widetilde{\sigma} \subset \mathbf{P}^{+}_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}}}$ under the first morphism in (3.7).

By (5) of Theorem 1.41, the formal completion

$$(\widetilde{\mathsf{M}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}}})^{\wedge}_{\widetilde{\mathsf{Z}}_{[(\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}},\check{\tau})]}}$$

is isomorphic to the formal scheme $\widetilde{\mathfrak{X}}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}},\check{\tau}} = \widetilde{\mathfrak{X}}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}},\check{\tau}}/\Gamma_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\tau}}$ for any representative $(\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}},\check{\tau})$ of $[(\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}},\check{\tau})]$. Here $\Gamma_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\tau}}$ is trivial by [Lan 2008, Lemma 6.2.5.27 in the revision], and $\widetilde{\mathfrak{X}}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}},\check{\tau}}$ is the formal completion of $\widetilde{\Xi}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}(\check{\tau})$ along its $\check{\tau}$-stratum $(\widetilde{\Xi}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}})_{\check{\tau}}$.

Let us describe the structure of the scheme $\widetilde{\Xi}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}(\check{\tau})$ in more detail:

(1) By construction, $\widetilde{\Xi}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}(\check{\tau})$ is a scheme over $\widetilde{\mathsf{M}}^{\check{\mathsf{Z}}_{\widetilde{\mathcal{H}}}}_{\widetilde{\mathcal{H}}}$, the latter of which is isomorphic to $\mathsf{M}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}$ because of (3.3) and (3.4). (Note that $\widetilde{\mathsf{M}}^{\check{\mathsf{Z}}_{\widetilde{\mathcal{H}}}}_{\widetilde{\mathcal{H}}} \cong \mathsf{M}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}$ is a scheme by [Lan 2008, Corollary 7.2.3.10].)

By abuse of notation, we shall simply denote the push-forward

$$(\widetilde{\Xi}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}(\check{\tau}) \twoheadrightarrow \widetilde{C}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}})_{*}\mathcal{O}_{\widetilde{\Xi}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}(\check{\tau})}$$

by $\mathcal{O}_{\widetilde{\Xi}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}(\check{\tau})}$, and view $\mathcal{O}_{\widetilde{\Xi}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}(\check{\tau})}$ as an $\mathcal{O}_{\widetilde{C}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}}$-algebra when there is no confusion. We shall adopt a similar convention for other affine morphisms.

(2) Let $(A, \lambda_A, i_A, \varphi_{-1,\mathcal{H}})$ be the tautological object over $\mathsf{M}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}$. Then $\widetilde{C}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}$ is the abelian scheme over $\mathsf{M}_{\mathcal{H}}$ parametrizing liftings (to level $\widetilde{\mathcal{H}}$) of data of the form $(\check{c} : \check{X} \to A^{\vee},\ \check{c}^{\vee} : \check{Y} \to A)$, compatible with $\check{\phi} : \check{Y} \hookrightarrow \check{X}$ and satisfying certain liftability and pairing conditions (coming from the so-called symplectic-liftability on the level structures). By construction, $\widetilde{C}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}$ is $\mathbb{Z}^{\times}_{(\Box)}$-isogenous to $\underline{\mathrm{Hom}}_{\mathcal{O}}(\check{Y}, A)^{\circ}$.

(3) The scheme $\widetilde{\Xi}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}$ is a torsor over $\widetilde{C}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}$ under (the pullback of) the split torus $E_{\check{\Phi}_{\widetilde{\mathcal{H}}}} = \underline{\mathrm{Hom}}(\mathbf{S}_{\check{\Phi}_{\widetilde{\mathcal{H}}}}, \mathbf{G}_{\mathrm{m}})$, which can be identified with the relative spectrum

$$\underline{\mathrm{Spec}}_{\mathcal{O}_{\widetilde{C}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}}}\Big(\bigoplus_{\check{\ell}\in\mathbf{S}_{\check{\Phi}_{\widetilde{\mathcal{H}}}}} \widetilde{\Psi}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}(\check{\ell})\Big),$$

where $\widetilde{\Psi}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}(\check{\ell})$ is the subsheaf of $\mathcal{O}_{\widetilde{\Xi}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}}$ (considered as an $\mathcal{O}_{\widetilde{C}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}}$-algebra by our convention) on which $E_{\check{\Phi}_{\widetilde{\mathcal{H}}}}$ acts by the character $\check{\ell}$. In the case when $\check{\ell} = [\check{y} \otimes \check{\chi}]$, where $\check{y} \in \check{Y}$ and $\check{\chi} \in \check{X}$, there is a canonical identification $\widetilde{\Psi}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}(\check{\ell})$ and the pullback of $(\check{c}^{\vee}(\check{y}), \check{c}(\check{\chi}))^{*}\mathcal{P}_A$ over $\widetilde{C}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}$. (See [Lan 2008, Convention 6.2.3.26 and end of §6.2.4].)

(4) Consider the subsemigroups of $\mathbf{S}_{\check{\Phi}_{\widetilde{\mathcal{H}}}}$ (see [Lan 2008, Definitions 6.1.1.9 and 6.1.2.5]) given by

$$\check{\tau}^{\vee} = \{\check{\ell} \in \mathbf{S}_{\check{\Phi}_{\widetilde{\mathcal{H}}}} : \langle \check{\ell}, y\rangle \geq 0, \forall y \in \check{\tau}\},$$
$$\check{\tau}^{\vee}_0 = \{\check{\ell} \in \mathbf{S}_{\check{\Phi}_{\widetilde{\mathcal{H}}}} : \langle \check{\ell}, y\rangle > 0, \forall y \in \check{\tau}\},$$
$$\check{\tau}^{\perp} = \{\check{\ell} \in \mathbf{S}_{\check{\Phi}_{\widetilde{\mathcal{H}}}} : \langle \check{\ell}, y\rangle = 0, \forall y \in \check{\tau}\} \cong \check{\tau}^{\vee}/\check{\tau}^{\vee}_0.$$

The scheme $\widetilde{\Xi}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}(\breve{\tau})$ is constructed as an *affine toroidal embedding*

$$\widetilde{\Xi}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}} \hookrightarrow \widetilde{\Xi}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}(\breve{\tau})$$

along $\breve{\tau}$ over the abelian scheme $\widetilde{C}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}$, which can be identified with the relative spectrum

$$\underline{\mathrm{Spec}}_{\mathbb{O}_{\widetilde{C}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}}}\Big(\bigoplus_{\breve{\ell}\in\breve{\tau}^\vee}\widetilde{\Psi}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}(\breve{\ell})\Big).$$

(5) Finally, the sheaf of ideals

$$\widetilde{\mathcal{I}}_{\breve{\tau}} = \bigoplus_{\breve{\ell}\in\breve{\tau}_0^\vee}\widetilde{\Psi}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}(\breve{\ell})$$

(see [Lan 2008, Lemma 6.1.2.6]) defines the $\breve{\tau}$-stratum $(\widetilde{\Xi}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}})_{\breve{\tau}}$, which can be identified with the relative spectrum

$$\underline{\mathrm{Spec}}_{\mathbb{O}_{\widetilde{C}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}}}\Big(\bigoplus_{\breve{\ell}\in\breve{\tau}^\perp}\widetilde{\Psi}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}(\breve{\ell})\Big).$$

Here $\widetilde{\mathcal{I}}_{\breve{\tau}}$ is an $\mathbb{O}_{\widetilde{\Xi}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}(\breve{\tau})}$-ideal represented as an $\mathbb{O}_{\widetilde{C}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}}$-submodule of $\mathbb{O}_{\widetilde{\Xi}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}(\breve{\tau})}$ (the latter being viewed as an $\mathbb{O}_{\widetilde{C}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}}$-algebra by our convention).

Suppose $\breve{\sigma}$ is the face of $\breve{\tau}$ that is a $\Gamma_{\breve{\Phi}_{\widetilde{\mathcal{H}}}}$-translation of the image of $\widetilde{\sigma}\subset\mathbf{P}^+_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}}}$ under the first morphism in (3.7). Similar to the definition of $\breve{\tau}^\vee$, $\breve{\tau}_0^\vee$, and $\breve{\tau}^\perp$ above, consider the following subsemigroups of $\mathbf{S}_{\breve{\Phi}_{\widetilde{\mathcal{H}}}}$:

$$\begin{aligned}
\breve{\sigma}^\vee &= \{\breve{\ell}\in\mathbf{S}_{\breve{\Phi}_{\widetilde{\mathcal{H}}}} : \langle\breve{\ell},y\rangle\ge 0, \forall y\in\breve{\sigma}\},\\
\breve{\sigma}_0^\vee &= \{\breve{\ell}\in\mathbf{S}_{\breve{\Phi}_{\widetilde{\mathcal{H}}}} : \langle\breve{\ell},y\rangle> 0, \forall y\in\breve{\sigma}\},\\
\breve{\sigma}^\perp &= \{\breve{\ell}\in\mathbf{S}_{\breve{\Phi}_{\widetilde{\mathcal{H}}}} : \langle\breve{\ell},y\rangle= 0, \forall y\in\breve{\sigma}\}\cong\breve{\sigma}^\vee/\breve{\sigma}_0^\vee.
\end{aligned}$$

Note that $\breve{\tau}^\vee\subset\breve{\sigma}^\vee$ and $\breve{\tau}^\perp\subset\breve{\sigma}^\perp$, but $\breve{\tau}_0^\vee\not\subset\breve{\sigma}_0^\vee$ in general. The closure $(\widetilde{\Xi}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}})_{\breve{\sigma}}(\breve{\tau})$ of the $\breve{\sigma}$-stratum on $\widetilde{\Xi}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}(\breve{\tau})\cong\underline{\mathrm{Spec}}_{\mathbb{O}_{\widetilde{C}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}}}\big(\bigoplus_{\breve{\ell}\in\breve{\tau}^\vee}\widetilde{\Psi}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}(\breve{\ell})\big)$ is defined by the sheaf of ideals $\bigoplus_{\breve{\ell}\in\breve{\sigma}_0^\vee\cap\breve{\tau}^\vee}\widetilde{\Psi}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}(\breve{\ell})$. Then we have a canonical isomorphism

$$(\widetilde{\Xi}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}})_{\breve{\sigma}}(\breve{\tau})\cong\underline{\mathrm{Spec}}_{\mathbb{O}_{\widetilde{C}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}}}\Big(\bigoplus_{\breve{\ell}\in\breve{\sigma}^\perp\cap\breve{\tau}^\vee}\widetilde{\Psi}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}(\breve{\ell})\Big),$$

with the $\breve{\tau}$-stratum

$$(\widetilde{\Xi}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}})_{\breve{\tau}}\cong\underline{\mathrm{Spec}}_{\mathbb{O}_{\widetilde{C}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}}}\Big(\bigoplus_{\breve{\ell}\in\breve{\tau}^\perp}\widetilde{\Psi}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}(\breve{\ell})\Big)$$

(as a closed subscheme of $(\widetilde{\Xi}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}})_{\breve{\sigma}}(\breve{\tau})$) defined by the sheaf of ideals

$$\widetilde{\mathcal{I}}_{\breve{\sigma},\breve{\tau}} := \bigoplus_{\breve{\ell}\in\breve{\sigma}^\perp\cap\breve{\tau}_0^\vee}\widetilde{\Psi}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}(\breve{\ell}).$$

Let $\mathfrak{X}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}},\breve{\sigma},\breve{\tau}}$ denote the formal completion of $(\widetilde{\Xi}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}})_{\breve{\sigma}}(\breve{\tau})$ along $(\widetilde{\Xi}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}})_{\breve{\tau}}$, which can be canonically identified as a closed formal subscheme of $\mathfrak{X}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}},\breve{\tau}}$, inducing the closures of the $[(\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\tilde{\delta}_{\widetilde{\mathcal{H}}},\widetilde{\sigma})]$-strata on any good formal $(\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}},\breve{\tau})$-model. (See [Lan 2008, Definition 6.3.1.11] for the definition of good formal models, and see [Lan 2008, Definition 6.3.2.16 in the revision] for the labeling of the strata by equivalence classes of triples of the form $[(\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\tilde{\delta}_{\widetilde{\mathcal{H}}},\widetilde{\sigma})]$.) By (5) of Theorem 1.41, the *strata-preserving* canonical isomorphism $(\widetilde{\mathsf{M}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}},\widetilde{\Sigma}})^{\wedge}_{\mathsf{Z}_{[(\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}},\breve{\tau})]}} \cong \mathfrak{X}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}},\breve{\tau}}$ then induces a canonical isomorphism

$$(\mathsf{N}^{\mathrm{tor}})^{\wedge}_{\widetilde{\mathsf{Z}}_{[(\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}},\breve{\tau})]}} \cong \mathfrak{X}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}},\breve{\sigma},\breve{\tau}}.$$

(Alternatively, one may refer directly to the gluing construction of $\widetilde{\mathsf{M}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}}}$ in [Lan 2008, §6.3.3], based on the crucial [Lan 2008, Proposition 6.3.2.13].)

By the theory of two-step constructions (see [Faltings and Chai 1990, Chapter III Theorem 10.2] and [Lan 2008, §4.5.6 in the revision]), the degeneration data of the pullback of $(\overline{G},\overline{\lambda},\overline{i},\overline{\alpha}_{\mathcal{H}})$ to affine open formal subschemes of $\mathfrak{X}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}},\breve{\sigma},\breve{\tau}}$ can be obtained from the degeneration data of pullback of $(\widetilde{G},\widetilde{\lambda},\tilde{i},\widetilde{\alpha}_{\widetilde{\mathcal{H}}})$ to affine open formal subschemes of $\widetilde{\mathfrak{X}}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}},\breve{\tau}}$ by restricting objects defined on $\breve{X}$ and $\breve{Y}$ to the subgroups X and Y. Therefore, in order to verify (6) of Theorem 1.41, it suffices to verify the following:

Condition 3.8 (cf. [Faltings and Chai 1990, Chapter VI, Definition 1.3]). *For each $(\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}},\breve{\tau})$ as above, the image of $\breve{\tau}$ in $\mathbf{P}_{\Phi_{\mathcal{H}}}$ under the (canonical) second morphism in* (3.7) *is contained in some cone $\tau \subset \mathbf{P}^{+}_{\Phi_{\mathcal{H}}}$ in the cone decomposition $\Sigma_{\Phi_{\mathcal{H}}}$.*

If Condition 3.8 is satisfied (for $\kappa = (\widetilde{\mathcal{H}},\widetilde{\Sigma},\widetilde{\sigma})$), then the structural morphism $f : \mathsf{N} \to \mathsf{M}_{\mathcal{H}}$ extends to a (unique) morphism $f^{\mathrm{tor}} : \mathsf{N}^{\mathrm{tor}} \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$, which is étale locally given by morphisms between toric schemes equivariant under (surjective) morphisms between tori. By construction, we have a commutative diagram

$$\begin{tikzcd}
\mathsf{N}^{\mathrm{tor}} \arrow[d, "f^{\mathrm{tor}}"'] & \mathfrak{X}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}},\breve{\sigma},\breve{\tau}} \arrow[l] \arrow[r] \arrow[d] & \widetilde{C}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}} \arrow[d] \\
\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}} & \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau} \arrow[l] \arrow[r] & C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}
\end{tikzcd} \tag{3.9}$$

of canonical morphisms whenever the image of $\breve{\tau}$ under the (canonical) second morphism in (3.7) is contained in τ.

Remark 3.10. Condition 3.8 is analogous to the condition in [Pink 1990, 6.25(b)], used in for example [Harris and Zucker 1994, Lemma 1.6.5] and related works based on [Ash et al. 1975]. Unfortunately, we must point out that, apart from some pleasant (and often suggestive) analogies, there is no logical implication between the analytic theory in [Ash et al. 1975; Pink 1990], and the algebraic theory in [Faltings and Chai 1990; Lan 2008]. (One cannot even use $\mathrm{G}(\mathbb{Q})$ in the algebraic theory.) The applicability of Condition 3.8 in our work cannot be proved using [Pink 1990, 6.25(b)].

As planned in step (4) of Section 2C, let us take $\mathbf{K}_{Q,\mathcal{H},\Sigma}$ to be the subset of $\mathbf{K}^{\mathrm{pre}}_{Q,\mathcal{H},\Sigma}$ consisting of elements κ satisfying Condition 3.8. Since Condition 3.8 can be achieved by replacing any given $\widetilde{\Sigma}$ with a refinement, we see that $\mathbf{K}_{Q,\mathcal{H},\Sigma}$ is nonempty and has an induced directed partial order.

From now on, assume that our fixed choice $\kappa = (\widetilde{\mathcal{H}}, \widetilde{\Sigma}, \widetilde{\sigma})$ lies in $\mathbf{K}_{Q,\mathcal{H},\Sigma}$.

3C. *Logarithmic smoothness of f^{tor}*. The aim of this subsection is to carry out step (5) of Section 2C.

We need to show that the morphism f^{tor} is log smooth (as in [Kato 1989, 3.3] and [Illusie 1994, 1.6]) if we equip $\mathsf{N}^{\mathrm{tor}}$ and $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$ with the canonical fine log structures given respectively by the relative Cartier divisors with simple normal crossings given by the complements $\mathsf{N}^{\mathrm{tor}} - \mathsf{N}$ and $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}} - \mathsf{M}_{\mathcal{H}}$ with their reduced structures. According to [Kato 1989, 3.12], we have the following:

Lemma 3.11. *To show that the morphism f^{tor} is log smooth, it suffices to show that the first morphism in the canonical exact sequence*

$$(f^{\mathrm{tor}})^*(\Omega^1_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}/\mathsf{S}_0}[d\log\infty]) \to \Omega^1_{\mathsf{N}^{\mathrm{tor}}/\mathsf{S}_0}[d\log\infty] \to \overline{\Omega}^1_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}} \to 0 \tag{3.12}$$

*is **injective**, and that $\overline{\Omega}^1_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}$ is **locally free** of finite rank.*

By (4) of Theorem 1.41, the extended Kodaira–Spencer morphism [Lan 2008, Definition 4.6.3.32] for $G \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$ induces an isomorphism

$$\mathrm{KS}_{G/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}/\mathsf{S}_0} : \underline{\mathrm{KS}}_{G/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}} \xrightarrow{\sim} \Omega^1_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}/\mathsf{S}_0}[d\log\infty]$$

over $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$, while the extended Kodaira–Spencer morphism for $\widetilde{G} \to \widetilde{\mathsf{M}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}}}$ induces an isomorphism

$$\mathrm{KS}_{\widetilde{G}/\widetilde{\mathsf{M}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}}}/\mathsf{S}_0} : \underline{\mathrm{KS}}_{\widetilde{G}/\widetilde{\mathsf{M}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}}}} \xrightarrow{\sim} \Omega^1_{\widetilde{\mathsf{M}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}}}/\mathsf{S}_0}[d\log\infty]$$

over $\widetilde{\mathsf{M}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}}}$. Over $\mathsf{N}^{\mathrm{tor}}$, we have canonical extensions

$$0 \to \widetilde{T}_{\mathsf{N}^{\mathrm{tor}}} \to \widetilde{G}_{\mathsf{N}^{\mathrm{tor}}} \to \overline{G} \to 0 \quad \text{and} \quad 0 \to \widetilde{T}^{\vee}_{\mathsf{N}^{\mathrm{tor}}} \to \widetilde{G}^{\vee}_{\mathsf{N}^{\mathrm{tor}}} \to \overline{G}^{\vee} \to 0$$

of group schemes, inducing exact sequences

$$0 \to \mathrm{Lie}^{\vee}_{\overline{G}/\mathsf{N}^{\mathrm{tor}}} \to \mathrm{Lie}^{\vee}_{\widetilde{G}_{\mathsf{N}^{\mathrm{tor}}}/\mathsf{N}^{\mathrm{tor}}} \to \mathrm{Lie}^{\vee}_{\widetilde{T}_{\mathsf{N}^{\mathrm{tor}}}/\mathsf{N}^{\mathrm{tor}}} \to 0$$

and

$$0 \to \mathrm{Lie}^{\vee}_{\overline{G}^{\vee}/\mathsf{N}^{\mathrm{tor}}} \to \mathrm{Lie}^{\vee}_{\widetilde{G}^{\vee}_{\mathsf{N}^{\mathrm{tor}}}/\mathsf{N}^{\mathrm{tor}}} \to \mathrm{Lie}^{\vee}_{\widetilde{T}^{\vee}_{\mathsf{N}^{\mathrm{tor}}}/\mathsf{N}^{\mathrm{tor}}} \to 0.$$

Therefore, there is a canonical surjection

$$\underline{\mathrm{KS}}_{\widetilde{G}_{\mathsf{N}^{\mathrm{tor}}}/\mathsf{N}^{\mathrm{tor}}} \twoheadrightarrow \underline{\mathrm{KS}}_{\widetilde{T}_{\mathsf{N}^{\mathrm{tor}}}/\mathsf{N}^{\mathrm{tor}}}, \tag{3.13}$$

where $\underline{\mathrm{KS}}_{\widetilde{T}_{\mathsf{N}^{\mathrm{tor}}}/\mathsf{N}^{\mathrm{tor}}}$ is the pullback of the sheaf

$$\underline{\mathrm{KS}}_{\widetilde{T}_{\mathsf{S}_0}/\mathsf{S}_0} := (\underline{\mathrm{Lie}}^{\vee}_{\widetilde{T}_{\mathsf{S}_0}/\mathsf{S}_0} \otimes_{\mathcal{O}_{\mathsf{S}_0}} \underline{\mathrm{Lie}}^{\vee}_{\widetilde{T}^{\vee}_{\mathsf{S}_0}/\mathsf{S}_0}) \Big/ \begin{pmatrix} \lambda^*(y) \otimes z - \lambda^*(z) \otimes y \\ (b^\star x) \otimes y - x \otimes (by) \end{pmatrix}_{\substack{x \in \underline{\mathrm{Lie}}^{\vee}_{\widetilde{T}_{\mathsf{S}_0}/\mathsf{S}_0} \\ y,z \in \underline{\mathrm{Lie}}^{\vee}_{\widetilde{T}^{\vee}_{\mathsf{S}_0}/\mathsf{S}_0} \\ b \in \mathcal{O}}}$$

defined (as for degenerating families in Definition 1.40) by the split tori $\widetilde{T}$ and $\widetilde{T}^{\vee}$ over S_0 with respective character groups $\widetilde{X}$ and $\widetilde{Y}$. The kernel

$$\underline{\mathrm{K}} := \ker(\underline{\mathrm{KS}}_{\widetilde{G}_{\mathsf{N}^{\mathrm{tor}}}/\mathsf{N}^{\mathrm{tor}}} \twoheadrightarrow \underline{\mathrm{KS}}_{\widetilde{T}_{\mathsf{N}^{\mathrm{tor}}}/\mathsf{N}^{\mathrm{tor}}})$$

contains $\underline{\mathrm{KS}}_{\overline{G}/\mathsf{N}^{\mathrm{tor}}}$ as a natural subsheaf, and the quotient of $\underline{\mathrm{K}}$ by $\underline{\mathrm{KS}}_{\overline{G}/\mathsf{N}^{\mathrm{tor}}}$ is isomorphic to

$$\begin{aligned}
&(\underline{\mathrm{Lie}}^{\vee}_{\overline{G}/\mathsf{N}^{\mathrm{tor}}} \otimes_{\mathcal{O}_{\mathsf{N}^{\mathrm{tor}}}} \underline{\mathrm{Lie}}^{\vee}_{\widetilde{T}^{\vee}_{\mathsf{N}^{\mathrm{tor}}}/\mathsf{N}^{\mathrm{tor}}}) / \big((b^\star x) \otimes y - x \otimes (by)\big)_{\substack{x \in \underline{\mathrm{Lie}}^{\vee}_{\overline{G}/\mathsf{N}^{\mathrm{tor}}} \\ y \in \underline{\mathrm{Lie}}^{\vee}_{\widetilde{T}^{\vee}_{\mathsf{N}^{\mathrm{tor}}}/\mathsf{N}^{\mathrm{tor}}} \\ b \in \mathcal{O}}} \\
&\cong \underline{\mathrm{Hom}}_{\mathcal{O} \otimes_{\mathbb{Z}} \mathcal{O}_{\mathsf{N}^{\mathrm{tor}}}}(\underline{\mathrm{Lie}}_{\widetilde{T}^{\vee}_{\mathsf{N}^{\mathrm{tor}}}/\mathsf{N}^{\mathrm{tor}}}, \underline{\mathrm{Lie}}^{\vee}_{\overline{G}/\mathsf{N}^{\mathrm{tor}}}) \\
&\cong \underline{\mathrm{Hom}}_{\mathcal{O} \otimes_{\mathbb{Z}} \mathcal{O}_{\mathsf{N}^{\mathrm{tor}}}}(\underline{\mathrm{Hom}}_{\mathbb{Z}}(\widetilde{Y}, \mathcal{O}_{\mathsf{N}^{\mathrm{tor}}}), \underline{\mathrm{Lie}}^{\vee}_{\overline{G}/\mathsf{N}^{\mathrm{tor}}}) \\
&\cong \underline{\mathrm{Hom}}_{\mathcal{O}}(\widetilde{Y}^{\vee}, \underline{\mathrm{Lie}}^{\vee}_{\overline{G}/\mathsf{N}^{\mathrm{tor}}}) \\
&\cong \underline{\mathrm{Hom}}_{\mathcal{O}}(Q^{\vee}, \underline{\mathrm{Lie}}^{\vee}_{\overline{G}/\mathsf{N}^{\mathrm{tor}}}).
\end{aligned}$$

Since the pullback of $(G, \lambda, i, \alpha_{\mathcal{H}})$ under $\mathsf{N}^{\mathrm{tor}} \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$ is isomorphic to $(\overline{G}, \overline{\lambda}, \overline{i}, \overline{\alpha}_{\mathcal{H}})$, we have canonical isomorphisms

$$(f^{\mathrm{tor}})^* \underline{\mathrm{KS}}_{G/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}} \cong \underline{\mathrm{KS}}_{\overline{G}/\mathsf{N}^{\mathrm{tor}}}$$

and

$$(f^{\mathrm{tor}})^*(\underline{\mathrm{Hom}}_{\mathcal{O}}(Q^{\vee}, \underline{\mathrm{Lie}}^{\vee}_{G/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}})) \cong \underline{\mathrm{Hom}}_{\mathcal{O}}(Q^{\vee}, \underline{\mathrm{Lie}}^{\vee}_{\overline{G}/\mathsf{N}^{\mathrm{tor}}}).$$

Since the étale local structure of $\widetilde{\mathsf{M}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}}}$ along the $[(\breve{\Phi}_{\widetilde{\mathcal{H}}}, \breve{\delta}_{\widetilde{\mathcal{H}}}, \breve{\tau})]$-stratum is the same as $\widetilde{\Xi}_{\breve{\Phi}_{\widetilde{\mathcal{H}}}, \breve{\delta}_{\widetilde{\mathcal{H}}}}(\breve{\tau})$, the calculation in the proof of [Lan 2008, Proposition 6.2.5.14] shows that the isomorphism $\mathrm{KS}_{\widetilde{G}/\widetilde{\mathsf{M}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}}}/\mathsf{S}_0}$ induces by restriction (to the closure $\mathsf{N}^{\mathrm{tor}}$ of the $[(\widetilde{\Phi}_{\widetilde{\mathcal{H}}}, \widetilde{\delta}_{\widetilde{\mathcal{H}}}, \widetilde{\sigma})]$-stratum) an isomorphism

$$\underline{\mathrm{K}} \xrightarrow{\sim} \Omega^1_{\mathsf{N}^{\mathrm{tor}}/\mathsf{S}_0}[d \log \infty] \tag{3.14}$$

making the diagram

$$\begin{array}{ccc}
(f^{\mathrm{tor}})^*\underline{\mathrm{KS}}_{G/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}} & \hookrightarrow & \underline{\mathrm{K}} \\
{\scriptstyle \mathrm{KS}_{G/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}/\mathsf{S}_0}}\downarrow\wr & & \wr\downarrow{\scriptstyle (3.14)} \\
(f^{\mathrm{tor}})^*(\Omega^1_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}/\mathsf{S}_0}[d\log\infty]) & \longrightarrow & \Omega^1_{\mathsf{N}^{\mathrm{tor}}/\mathsf{S}_0}[d\log\infty]
\end{array}$$

commutative. In particular, the bottom arrow (which is the first morphism in (3.12)) is *injective*, and the isomorphism (3.14) induces a canonical isomorphism

$$(f^{\mathrm{tor}})^*(\underline{\mathrm{Hom}}_{\mathcal{O}}(Q^\vee, \underline{\mathrm{Lie}}^\vee_{G/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}})) \xrightarrow{\sim} \overline{\Omega}^1_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}} \tag{3.15}$$

of coherent sheaves over $\mathsf{N}^{\mathrm{tor}}$. (The restriction of (3.15) to N is compatible with the composition of isomorphisms (2.17) because of the same calculation in the proof of [Lan 2008, Proposition 6.2.5.14].)

Thus the desired isomorphism (2.16) is a consequence of (3.15). Moreover, since $\underline{\mathrm{Hom}}_{\mathcal{O}}(Q^\vee, \underline{\mathrm{Lie}}^\vee_{G/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}})$ (see Remark 2.14) is locally free of finite rank over $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$, the isomorphism (3.15) shows that the sheaf $\overline{\Omega}^1_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}$ is also locally free of finite rank over $\mathsf{N}^{\mathrm{tor}}$. By Lemma 3.11, this shows that f^{tor} is log smooth, and completes the proof of (2) and (3a) of Theorem 2.15.

3D. ***Equidimensionality of f^{tor}.*** Let us take a closer look at the diagram (3.9). By construction of f^{tor}, given any stratum $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}$ of $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$, the preimage

$$\widetilde{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]} := (f^{\mathrm{tor}})^{-1}(\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]})$$

has a stratification formed by $\widetilde{\mathsf{Z}}_{[(\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}},\breve{\tau})]}$, where $\breve{\tau}$ runs through cones in $\widetilde{\Sigma}_{\breve{\Phi}_{\widetilde{\mathcal{H}}}}$ satisfying the following conditions:

(1) $\breve{\tau} \subset \mathbf{P}^+_{\breve{\Phi}_{\widetilde{\mathcal{H}}}}$.

(2) $\breve{\tau}$ has a face $\breve{\sigma}$ that is a $\Gamma_{\breve{\Phi}_{\widetilde{\mathcal{H}}}}$-translation of the image of $\widetilde{\sigma} \subset \mathbf{P}^+_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}}}$ under the first morphism in (3.7).

(3) The image of $\breve{\tau}$ under the (canonical) second morphism in (3.7) is contained in $\tau \subset \mathbf{P}^+_{\Phi_{\mathcal{H}}}$.

The formal completion $(\mathsf{N}^{\mathrm{tor}})^\wedge_{\widetilde{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}}$ admits a canonical morphism

$$(\mathsf{N}^{\mathrm{tor}})^\wedge_{\widetilde{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}} \to C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}},$$

whose precomposition with the canonical morphism

$$(\mathsf{N}^{\mathrm{tor}})^\wedge_{\widetilde{\mathsf{Z}}_{[(\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}},\breve{\tau})]}} \to (\mathsf{N}^{\mathrm{tor}})^\wedge_{\widetilde{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}},$$

for any stratum $\widetilde{\mathsf{Z}}_{[(\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}},\check{\tau})]}$ of $\widetilde{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}$, coincides with the composition of canonical morphisms $\mathfrak{X}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}},\check{\sigma},\check{\tau}} \to \widetilde{C}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}} \to C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ by its very construction.

Since f^{tor} is étale locally given by morphisms between toric schemes equivariant under (surjective) morphisms between tori, to determine if f^{tor} is equidimensional (cf. [Faltings and Chai 1990, Chapter VI, Definition 1.3 and Remark 1.4]), it suffices to determine if the relative dimension of each of the induced (smooth) morphism $\widetilde{\mathsf{Z}}_{[(\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}},\check{\tau})]} \to \mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}$ between strata is at most $\dim_{\mathsf{M}_{\mathcal{H}}}(\mathsf{N})$, the relative dimension of $f : \mathsf{N} \to \mathsf{M}_{\mathcal{H}}$.

By abuse of language, we define the $\mathbb{R}$-dimension of a cone to be the $\mathbb{R}$-dimension of its $\mathbb{R}$-span. Then the codimension of $\mathsf{N} = \widetilde{\mathsf{Z}}_{[(\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\widetilde{\delta}_{\widetilde{\mathcal{H}}},\widetilde{\sigma})]}$ in $\widetilde{\mathsf{M}}^{\text{tor}}_{\widetilde{\mathcal{H}}}$ is $\dim_{\mathbb{R}}(\widetilde{\sigma}) = \dim_{\mathbb{R}}((\mathbf{S}_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}}})^{\vee}_{\mathbb{R}})$ because $\widetilde{\sigma}$ is top-dimensional. The codimension of

$$\widetilde{\mathsf{Z}}_{[(\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}},\check{\tau})]} \cong (\widetilde{\Xi}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}})_{\check{\tau}}$$

in $\widetilde{\mathsf{M}}^{\text{tor}}_{\widetilde{\mathcal{H}}}$ is equal to $\dim_{\mathbb{R}}(\check{\tau})$. Therefore, the codimension of $\widetilde{\mathsf{Z}}_{[(\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}},\check{\tau})]}$ in N^{tor} is equal to $\dim_{\mathbb{R}}(\check{\tau}) - \dim_{\mathbb{R}}(\widetilde{\sigma}) = \dim_{\mathbb{R}}(\check{\tau}) - \dim_{\mathbb{R}}((\mathbf{S}_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}}})^{\vee}_{\mathbb{R}})$. On the other hand, the codimension of $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]} \cong (\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}})_{\tau}$ in $\mathsf{M}^{\text{tor}}_{\mathcal{H}}$ is $\dim_{\mathbb{R}}(\tau)$. Hence we have

$$\begin{aligned}\dim_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}}&(\widetilde{\mathsf{Z}}_{[(\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}},\check{\tau})]})\\ &= \dim_{\mathsf{M}_{\mathcal{H}}}(\mathsf{N}) - (\dim_{\mathbb{R}}(\check{\tau}) - \dim_{\mathbb{R}}((\mathbf{S}_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}}})^{\vee}_{\mathbb{R}})) + \dim_{\mathbb{R}}(\tau). \quad (3.16)\end{aligned}$$

Let τ' denote the image of $\check{\tau}$ in $(\mathbf{S}_{\Phi_{\mathcal{H}}})^{\vee}_{\mathbb{R}}$. By assumption on $\check{\tau}$, we have $\tau' \subset \tau$. If $\tau' = \tau$, then

$$\dim_{\mathbb{R}}(\tau) = \dim_{\mathbb{R}}(\tau') \leq \dim_{\mathbb{R}}(\check{\tau}) - \dim_{\mathbb{R}}((\mathbf{S}_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}}})^{\vee}_{\mathbb{R}}),$$

and hence (3.16) implies

$$\dim_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}}(\widetilde{\mathsf{Z}}_{[(\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}},\check{\tau})]}) \leq \dim_{\mathsf{M}_{\mathcal{H}}}(\mathsf{N}).$$

(If this is true for all $\widetilde{\mathsf{Z}}_{[(\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}},\check{\tau})]}$, then f^{tor} is equidimensional.) On the other hand, suppose $\tau' \subsetneq \tau$. Then there exists a face of τ'' of τ' such that $\tau'' \subset \tau$ and $\dim_{\mathbb{R}}(\tau'') < \dim_{\mathbb{R}}(\tau)$. Note that τ'' is the image of at least one face of $\check{\tau}$ satisfying the three conditions in the first paragraph of this section. By dropping redundant basis vectors, we may assume moreover that this face $\check{\tau}''$ of $\check{\tau}$ satisfies $\dim_{\mathbb{R}}(\tau'') = \dim_{\mathbb{R}}(\check{\tau}'') - \dim_{\mathbb{R}}((\mathbf{S}_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}}})^{\vee}_{\mathbb{R}})$. Then we have

$$\dim_{\mathbb{R}}(\tau) > \dim_{\mathbb{R}}(\tau'') = \dim_{\mathbb{R}}(\check{\tau}'') - \dim_{\mathbb{R}}((\mathbf{S}_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}}})^{\vee}_{\mathbb{R}}),$$

and hence (3.16) implies

$$\dim_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}}(\widetilde{\mathsf{Z}}_{[(\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}},\check{\tau}'')]}) > \dim_{\mathsf{M}_{\mathcal{H}}}(\mathsf{N}),$$

which means f^{tor} cannot be equidimensional.

This motivates the following strengthening of Condition 3.8:

Condition 3.17 (cf. [Faltings and Chai 1990, Chapter VI, Definition 1.3]). *For each* $(\breve{\Phi}_{\widetilde{\mathcal{H}}}, \breve{\delta}_{\widetilde{\mathcal{H}}}, \breve{\tau})$ *such that* $\mathsf{Z}_{[(\breve{\Phi}_{\widetilde{\mathcal{H}}}, \breve{\delta}_{\widetilde{\mathcal{H}}}, \breve{\tau})]}$ *is* (*a stratum*) *in* $\mathsf{N}^{\mathrm{tor}}$, *the image of* $\breve{\tau} \subset \mathbf{P}^{+}_{\Phi_{\mathcal{H}}}$ *under the* (*canonical*) *second morphism in* (3.7) *is* ***exactly*** *some cone* $\tau \subset \mathbf{P}^{+}_{\Phi_{\mathcal{H}}}$ *in the cone decomposition* $\Sigma_{\Phi_{\mathcal{H}}}$.

Proposition 3.18. *The morphism* $f^{\mathrm{tor}} : \mathsf{N}^{\mathrm{tor}} \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$ *is equidimensional* (*with relative dimension equal to the one of* $f : \mathsf{N} \to \mathsf{M}_{\mathcal{H}}$), *and hence flat*, *if and only if Condition 3.17 is satisfied*, *if and only if* f^{tor} *is log integral* (see [Kato 1989, Definition 4.3]).

Proof. The equivalence between Condition 3.17 and equidimensionality has been explained above. Since both $\mathsf{N}^{\mathrm{tor}}$ and $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$ are regular (because they are smooth over $\mathsf{S}_0 = \mathrm{Spec}(\mathcal{O}_{F_0,(\Box)})$), the equidimensionality and flatness of f^{tor} are equivalent by [EGA IV$_3$ 1966, 15.4.2 b) $\Leftrightarrow$ e$'$)]. By [Kato 1989, Proposition 4.1(2)], the log integrality of f^{tor} is equivalent to the flatness of each of the canonical morphisms $\mathbb{Z}[\tau^{\vee}] \hookrightarrow \mathbb{Z}[\breve{\tau}^{\vee}]$ (defined when $\mathsf{Z}_{[(\breve{\Phi}_{\widetilde{\mathcal{H}}}, \breve{\delta}_{\widetilde{\mathcal{H}}}, \breve{\tau})]}$ is mapped to $\mathsf{Z}_{[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \tau)]}$), which is equivalent to the equidimensionality of any such morphism (by the smoothness of $\mathbb{Z}[\tau^{\vee}]$ and $\mathbb{Z}[\breve{\tau}^{\vee}]$ over $\mathbb{Z}$, and by [EGA IV$_3$ 1966, 15.4.2 b) $\Leftrightarrow$ e$'$)] again), which is equivalent to Condition 3.17 by the same (dimension comparison) argument. $\Box$

Proposition 3.19 (cf. [Faltings and Chai 1990, Chapter VI, Remark 1.4]). *Condition 3.17 can be achieved by replacing both the cone decompositions* $\widetilde{\Sigma}$ *and* Σ *with some refinements.*

Proof. Instead of taking refinements of $\widetilde{\Sigma}$ and Σ separately, we consider the morphism $\mathbf{P}_{\breve{\Phi}_{\widetilde{\mathcal{H}}}} \twoheadrightarrow \mathbf{P}_{\Phi_{\mathcal{H}}}$ in (3.7) and consider the *graph* of $\widetilde{\Sigma}$. More precisely, using the canonical morphisms $X \hookrightarrow \breve{X}$ and $Y \hookrightarrow \breve{Y}$ compatible with ϕ and $\breve{\phi}$, we obtain canonical morphisms $X' := \breve{X} \oplus X \to \breve{X}$ and $Y' := \breve{Y} \oplus Y \to \breve{Y}$ compatible with $\phi' := \breve{\phi} \oplus \phi$ and $\breve{\phi}$, inducing morphisms $\mathbf{S}_{\breve{\Phi}_{\widetilde{\mathcal{H}}}} \oplus \mathbf{S}_{\Phi_{\mathcal{H}}} \twoheadrightarrow \mathbf{S}_{\breve{\Phi}_{\widetilde{\mathcal{H}}}}$ and $\mathbf{P}_{\breve{\Phi}_{\widetilde{\mathcal{H}}}} \hookrightarrow \mathbf{P}_{\breve{\Phi}_{\widetilde{\mathcal{H}}}} \oplus \mathbf{P}_{\Phi_{\mathcal{H}}}$. The image of this latter morphism is the *graph* of $\mathbf{P}_{\breve{\Phi}_{\widetilde{\mathcal{H}}}} \twoheadrightarrow \mathbf{P}_{\Phi_{\mathcal{H}}}$. Let us define $\dddot{\mathbf{S}}'$ by X', Y', and ϕ' as in (1.21), and let $\mathbf{S}'$ be its free quotient. Define $\mathbf{P}'$ accordingly as the subset of $(\mathbf{S}')^{\vee}_{\mathbb{R}}$ consisting of positive semidefinite pairings with admissible radicals, containing the graph of $\mathbf{P}_{\breve{\Phi}_{\widetilde{\mathcal{H}}}} \twoheadrightarrow \mathbf{P}_{\Phi_{\mathcal{H}}}$ canonically as an admissible boundary component (cf. Definition 1.28). The cone decomposition $\widetilde{\Sigma}_{\breve{\Phi}_{\widetilde{\mathcal{H}}}}$ defines a cone decomposition on this graph, which might fail to be projective or smooth with respect to the structure of the ambient space. But we can find a projective smooth cone decomposition of $\mathbf{P}'$, admissible with respect to the actions of all elements in $\mathrm{GL}_{\mathcal{O}}(X') \times \mathrm{GL}_{\mathcal{O}}(Y')$ respecting ϕ', such that its restriction to the graph refine the cone decomposition defined by $\widetilde{\Sigma}_{\breve{\Phi}_{\widetilde{\mathcal{H}}}}$. Thus we obtain a simultaneous smooth projective refinement of $\widetilde{\Sigma}_{\breve{\Phi}_{\widetilde{\mathcal{H}}}}$ and $\Sigma_{\Phi_{\mathcal{H}}}$, such that image of cones in $\widetilde{\Sigma}_{\breve{\Phi}_{\widetilde{\mathcal{H}}}}$ under $\mathbf{P}_{\breve{\Phi}_{\widetilde{\mathcal{H}}}} \twoheadrightarrow \mathbf{P}_{\Phi_{\mathcal{H}}}$ are cones in $\Sigma_{\Phi_{\mathcal{H}}}$. Since this construction is compatible with surjections between different choices of $\check{\Phi}_{\widetilde{\mathcal{H}}}$ and $\Phi_{\mathcal{H}}$, we can conclude by induction on magnitude of cusp labels $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ as in the proofs of [Lan 2008, Propositions 6.3.3.3 and 7.3.1.5]. $\square$

Remark 3.20. We will not need Propositions 3.18 and 3.19 in what follows. We supply them here because knowing flatness or log integrality of f^{tor} is useful in many applications.

3E. ***Hecke actions.*** The aim of this subsection is to explain the proof of statements (4) and (5) of Theorem 2.15, with (4c) and (5c) conditional on (3b) and (3c) of Theorem 2.15. These statements might seem elaborate, but they are self-explanatory and based on the following simple idea: Since N and $\mathsf{N}^{\mathrm{tor}}$ are constructed using the toroidal compactifications of $\widetilde{\mathsf{M}}_{\widetilde{\mathcal{H}}}$, we can use the Hecke actions on $\widetilde{\mathsf{M}}_{\widetilde{\mathcal{H}}}$ and their (compatible) extensions to toroidal compactifications provided by [Lan 2008, Proposition 6.4.3.4 in the revision].

Let g_h, $\mathcal{H}'$, Σ', g_l, and Q' be as in (4) and (5) of Theorem 2.15. (For proving (4) and (5) of Theorem 2.15, we may assume in what follows either $g_h = 1$ or $g_l = 1$, although the theory works in a more general context.) Using the splitting $\tilde{\delta}$ of $\widetilde{\mathrm{Z}}$, we obtain an element $\tilde{g}$ in $\widetilde{\mathrm{P}}_{\widetilde{\mathrm{Z}}}(\mathbb{A}^{\infty,\square})$ such that $\mathrm{Gr}^{\widetilde{\mathrm{Z}}}_{-1}(\tilde{g}) = g_h$, and such that $\mathrm{Gr}^{\widetilde{\mathrm{Z}}}_{0}(\tilde{g})$ is identified with g_l^{-1} under $\widetilde{\varphi}_0 : \mathrm{Gr}^{\widetilde{\mathrm{Z}}}_0 \xrightarrow{\sim} Q_0 \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}^{\square} \cong Q \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}^{\square}$. (See Section 3A.) Let $\widetilde{\mathcal{H}}'$ be a (necessarily neat) subgroup of $\widetilde{\mathrm{G}}(\hat{\mathbb{Z}}^{\square})$ such that $\tilde{g}^{-1}\widetilde{\mathcal{H}}'\tilde{g} \subset \widetilde{\mathcal{H}}$, and such that $\mathcal{H}' = \mathrm{Gr}^{\widetilde{\mathrm{Z}}}_{-1}(\widetilde{\mathcal{H}}' \cap \mathrm{P}_{\widetilde{\mathrm{Z}}}(\hat{\mathbb{Z}}^{\square}))$. By [Lan 2008, Proposition 6.4.3.4 in the revision], there exist some choices of $\widetilde{\Sigma}'$ such that the canonical morphism $[\tilde{g}] : \widetilde{\mathsf{M}}_{\widetilde{\mathcal{H}}'} \to \widetilde{\mathsf{M}}_{\widetilde{\mathcal{H}}}$ extends canonically to $[\tilde{g}]^{\mathrm{tor}} : \widetilde{\mathsf{M}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}}',\widetilde{\Sigma}'} \to \widetilde{\mathsf{M}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}},\widetilde{\Sigma}}$. By replacing $\widetilde{\Sigma}'$ with a refinement such that it satisfies Condition 3.8 (with Σ' and) with some choice of $\widetilde{\sigma}'$, and such that the morphism $[\tilde{g}]^{\mathrm{tor}}$ sends the stratum $\widetilde{\mathsf{Z}}_{[(\check{\Phi}_{\widetilde{\mathcal{H}}'},\check{\delta}_{\widetilde{\mathcal{H}}'},\widetilde{\sigma}')]}$ to $\widetilde{\mathsf{Z}}_{[(\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}},\widetilde{\sigma})]}$, we see that the induced morphism from the closure of $\widetilde{\mathsf{Z}}_{[(\check{\Phi}_{\widetilde{\mathcal{H}}'},\check{\delta}_{\widetilde{\mathcal{H}}'},\widetilde{\sigma}')]}$ to the closure of $\widetilde{\mathsf{Z}}_{[(\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}},\widetilde{\sigma})]}$ gives the existences of the morphisms $[g_h]_{\kappa',\kappa}$, $[g_h]^{\mathrm{tor}}_{\kappa',\kappa}$, $[g_l]^{*}_{\kappa',\kappa}$, and $([g_l]^{*}_{\kappa',\kappa})^{\mathrm{tor}}$ as in (4a), (4b), (5a), and (5b) of Theorem 2.15, where $\kappa' = (\widetilde{\mathcal{H}}', \widetilde{\Sigma}', \widetilde{\sigma}')$ lies in $\mathbf{K}_{Q',\mathcal{H}',\Sigma'}$, except that (2.24) and (2.26) still have to be explained.

As in the case of showing $R^i(f^{\mathrm{tor}}_{\kappa',\kappa})_*\mathcal{O}_{\mathsf{N}^{\mathrm{tor}}_{\kappa'}} = 0$ for $i > 0$ in Section 3A, since the morphisms $[g_h]^{\mathrm{tor}}_{\kappa',\kappa}$ and $([g_l]^{*}_{\kappa',\kappa})^{\mathrm{tor}}$ are étale locally given by equivariant morphisms between toric schemes, we have (by [Kempf et al. 1973, Chapter I, §3]) $R^i([g_h]^{\mathrm{tor}}_{\kappa',\kappa})_*(\mathcal{O}_{(\mathsf{N}'_{\kappa'})^{\mathrm{tor}}}) = 0$ and $R^i([g_l]^{*}_{\kappa',\kappa})^{\mathrm{tor}}_*(\mathcal{O}_{(\mathsf{N}'_{\kappa'})^{\mathrm{tor}}}) = 0$ for $i > 0$, which are (2.24) and (2.26) of Theorem 2.15.

The remaining statements in (4c) and (5c) of Theorem 2.15 now follow if we assume statements (3b) and (3c) of Theorem 2.15. (See the end of Section 5, p. 957.)

4. Calculation of formal cohomology

Throughout this section, unless otherwise specified, we fix the choice of an arbitrary (locally closed) stratum $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}$ of $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$. The aim of this section is to calculate the relative cohomology of the pullback of the structure morphism f^{tor} to the formal completion $(\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}})^{\wedge}_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}}$. (See (5) of Theorem 1.41 for a description of this formal completion. See also the first paragraph of Section 3D for a description of the formal completion $(\mathsf{N}^{\mathrm{tor}})^{\wedge}_{\widetilde{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}}$ of $\mathsf{N}^{\mathrm{tor}}$ along $\widetilde{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]} = (f^{\mathrm{tor}})^{-1}(\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]})$.)

4A. *Formal fibers of f^{tor}.* Let $\Gamma_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\tau}$ be the subgroup of elements in $\Gamma_{\breve{\Phi}_{\widetilde{\mathcal{H}}}}$ stabilizing (both) X and Y and inducing an element in $\Gamma_{\Phi_{\mathcal{H}},\tau}$ (the subgroup of $\Gamma_{\Phi_{\mathcal{H}}}$ formed by elements mapping τ to itself). Since we have tacitly assumed that $\Gamma_{\Phi_{\mathcal{H}},\tau}$ is trivial by Condition 1.29 and [Lan 2008, Lemma 6.2.5.27 in the revision], $\Gamma_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\tau}$ is also the subgroup of elements in $\Gamma_{\breve{\Phi}_{\widetilde{\mathcal{H}}}}$ fixing (both) X and Y. Let $\Gamma_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}}$ be the subgroup of elements in $\mathrm{Hom}_{\mathcal{O}}(\widetilde{X}, X)$ sending $\widetilde{\phi}(\widetilde{Y})$ to $\phi(Y)$ that are compatible with $\widetilde{\varphi}_{-2,\widetilde{\mathcal{H}}}$, $\widetilde{\varphi}_{0,\widetilde{\mathcal{H}}}$, $\varphi_{-2,\mathcal{H}}$, and $\varphi_{0,\mathcal{H}}$. Note that these compatibility conditions imply that the subgroup $\Gamma_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}}$ has index prime to $\Box$ in $\mathrm{Hom}_{\mathcal{O}}(\widetilde{X}, X)$. The two surjections

$$s_{\breve{X}} : \breve{X} \twoheadrightarrow \widetilde{X} \quad \text{and} \quad s_{\breve{Y}} : \breve{Y} \twoheadrightarrow \widetilde{Y}$$

identify $\Gamma_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}}$ as a subgroup of $\Gamma_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\tau}$. (More precisely, any $t \in \Gamma_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}}$ defines a translation action $x \mapsto x + t(s_{\breve{X}}(x))$ on $\breve{X}$, inducing compatibly a translation action on $\breve{Y}$, and hence defining an element in $\Gamma_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\tau}$ fixing both X and Y.)

Since $\Gamma_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}}$ does not modify $s_{\breve{X}}$ and $s_{\breve{Y}}$, it does not modify the first morphism in (3.7). Therefore, if we denote the image of $\widetilde{\sigma}$ in $\mathbf{P}_{\breve{\Phi}_{\widetilde{\mathcal{H}}}}$ by $\breve{\sigma}$, then $\Gamma_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}}$ maps $\breve{\sigma}$ to itself. On the other hand, by Condition 1.29 (and Lemma 3.1), if a cone $\breve{\tau} \subset \mathbf{P}^{+}_{\breve{\Phi}_{\widetilde{\mathcal{H}}}}$ in $\Sigma_{\breve{\Phi}_{\widetilde{\mathcal{H}}}}$ has a face that is a $\Gamma_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\tau}$-translation of $\breve{\sigma}$, then it cannot have a different face that is also a $\Gamma_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\tau}$-translation of $\breve{\sigma}$. Let us denote by $\Sigma_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\sigma},\tau}$ the subset of $\Sigma_{\breve{\Phi}_{\widetilde{\mathcal{H}}}}$ consisting of cones $\breve{\tau}$ satisfying the following conditions (cf. similar conditions in the first paragraph of Section 3D):

(1) $\breve{\tau} \subset \mathbf{P}^{+}_{\breve{\Phi}_{\widetilde{\mathcal{H}}}}$.

(2) $\breve{\tau}$ has $\breve{\sigma}$ as a face.

(3) The image of $\breve{\tau}$ under the (canonical) second morphism in (3.7) is contained in $\tau \subset \mathbf{P}^{+}_{\Phi_{\mathcal{H}}}$.

Then, to obtain a complete list of representatives of the cusp labels $[(\breve{\Phi}_{\widetilde{\mathcal{H}}}, \breve{\delta}_{\widetilde{\mathcal{H}}}, \breve{\tau})]$ parametrizing the strata of $\widetilde{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}$, it suffices to take representatives of $\Sigma_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\sigma},\tau}$ modulo the action of $\Gamma_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}}$. (That is, we do not have to consider $\Gamma_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}}$-translates of $\breve{\sigma}$.)

Let $\widetilde{\Xi}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}(\tau)$ denote the toroidal embedding of $\widetilde{\Xi}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}$ formed by gluing the affine toroidal embeddings $\widetilde{\Xi}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}(\breve{\tau})$ over $\widetilde{C}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}$, where $\breve{\tau}$ runs through cones in $\Sigma_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\sigma},\tau}$. To minimize confusion, we shall distinguish $\widetilde{\Xi}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}(\breve{\tau}_1)$ and $\widetilde{\Xi}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}(\breve{\tau}_2)$ even when $[(\breve{\Phi}_{\widetilde{\mathcal{H}}}, \breve{\delta}_{\widetilde{\mathcal{H}}}, \breve{\tau}_1)] = [(\breve{\Phi}_{\widetilde{\mathcal{H}}}, \breve{\delta}_{\widetilde{\mathcal{H}}}, \breve{\tau}_2)]$. For each $\breve{\tau}$ as above (having $\breve{\sigma}$ as a face), recall that we have denoted the closure of the $\breve{\sigma}$-stratum of $\widetilde{\Xi}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}(\breve{\tau})$ by $(\widetilde{\Xi}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}})_{\breve{\sigma}}(\breve{\tau})$. Let $(\widetilde{\Xi}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}})_{\breve{\sigma}}(\tau)$ denote the union of all such $(\widetilde{\Xi}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}})_{\breve{\sigma}}(\breve{\tau})$, let

$(\widetilde{\Xi}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}})_\tau$ denote the union of all such $(\widetilde{\Xi}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}})_{\breve{\tau}}$, and let $\mathfrak{X}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}},\breve{\sigma},\tau}$ denote the formal completion of $(\widetilde{\Xi}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}})_{\breve{\sigma}}(\tau)$ along $(\widetilde{\Xi}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}})_\tau$.

For each $\breve{\tau} \in \Sigma_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\sigma},\tau}$, consider the open subscheme $U_{\breve{\tau}}$ of $(\widetilde{\Xi}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}})_\tau$ formed by the union of all (locally closed) strata of $(\widetilde{\Xi}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}})_\tau$ that contains the stratum $(\widetilde{\Xi}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}})_{\breve{\tau}}$ in its closure, and consider the open formal subscheme $\mathfrak{U}_{\breve{\tau}}$ of $\mathfrak{X}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}},\breve{\sigma},\tau}$ supported on $U_{\breve{\tau}}$. The subscheme $U_{\breve{\tau}}$ of $(\widetilde{\Xi}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}})_\tau$ is the closed subscheme of $\widetilde{\Xi}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}(\breve{\tau})$ given by the intersection of $\widetilde{\Xi}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}(\breve{\tau})$ and $(\widetilde{\Xi}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}})_\tau$ in $\widetilde{\Xi}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}(\tau)$. The formal subscheme $\mathfrak{U}_{\breve{\tau}}$ of $\mathfrak{X}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}},\breve{\sigma},\tau}$ is the formal completion of $(\widetilde{\Xi}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}})_{\breve{\sigma}}(\breve{\tau})$ along $U_{\breve{\tau}}$. The collection $\{U_{\breve{\tau}}\}_{\breve{\tau}\in\Sigma_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\sigma},\tau}}$ forms an open covering of $(\widetilde{\Xi}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}})_\tau$. We can interpret $\mathfrak{X}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}},\breve{\sigma},\tau}$ as constructed by gluing the collection $\{\mathfrak{U}_{\breve{\tau}}\}_{\breve{\tau}\in\Sigma_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\sigma},\tau}}$ of formal schemes along their intersections (of supports).

Explicitly, let us denote by $\breve{\tau}_{\breve{\sigma}}^\vee$ the intersection of $(\breve{\tau}')_0^\vee$ for $\breve{\tau}'$ running through faces of $\breve{\tau}$ in $\Sigma_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\sigma},\tau}$ (including $\breve{\tau}$ itself). Then we have the canonical isomorphism

$$U_{\breve{\tau}} \cong \underline{\operatorname{Spec}}_{\mathcal{O}_{\widetilde{C}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}}}\Big(\Big(\bigoplus_{\breve{\ell}\in\breve{\tau}^\vee}\Psi_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}(\breve{\ell})\Big)/\Big(\bigoplus_{\breve{\ell}\in\breve{\tau}_{\breve{\sigma}}^\vee}\Psi_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}(\breve{\ell})\Big)\Big)$$

of schemes affine over $\widetilde{C}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}$. As $\mathcal{O}_{\widetilde{C}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}}$-modules, we have a canonical isomorphism

$$\Big(\bigoplus_{\breve{\ell}\in\breve{\tau}^\vee}\Psi_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}(\breve{\ell})\Big)/\Big(\bigoplus_{\breve{\ell}\in\breve{\tau}_{\breve{\sigma}}^\vee}\Psi_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}(\breve{\ell})\Big) \cong \bigoplus_{\breve{\ell}\in\breve{\tau}^\vee-\breve{\tau}_{\breve{\sigma}}^\vee}\Psi_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}(\breve{\ell}).$$

If we equip $\breve{\tau}^\vee - \breve{\tau}_{\breve{\sigma}}^\vee$ with the semigroup structure induced by the canonical bijection $(\breve{\tau}^\vee - \breve{\tau}_{\breve{\sigma}}^\vee) \to \breve{\tau}^\vee/\breve{\tau}_{\breve{\sigma}}^\vee$, then we may interpret $\bigoplus_{\breve{\ell}\in\breve{\tau}^\vee-\breve{\tau}_{\breve{\sigma}}^\vee}\Psi_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}(\breve{\ell})$ as an $\mathcal{O}_{\widetilde{C}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}}$-algebra, with algebra structure given by canonical isomorphisms

$$\Psi_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}(\breve{\ell}) \otimes_{\mathcal{O}_{\widetilde{C}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}}} \Psi_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}(\breve{\ell}') \xrightarrow{\sim} \Psi_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}(\breve{\ell}+\breve{\ell}')$$

(inherited from those of $\mathcal{O}_{\widetilde{\Xi}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}} \cong \bigoplus_{\breve{\ell}\in\mathbf{S}_{\breve{\Phi}_{\widetilde{\mathcal{H}}}}}\widetilde{\Psi}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}(\breve{\ell})$) if $\breve{\ell}+\breve{\ell}' \in \breve{\tau}^\vee - \breve{\tau}_{\breve{\sigma}}^\vee$ and by

$$\Psi_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}(\breve{\ell}) \otimes_{\mathcal{O}_{\widetilde{C}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}}} \Psi_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}(\breve{\ell}') \to 0$$

otherwise. Then we have a canonical isomorphism

$$U_{\breve{\tau}} \cong \underline{\operatorname{Spec}}_{\mathcal{O}_{\widetilde{C}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}}}\Big(\bigoplus_{\breve{\ell}\in\breve{\tau}^\vee-\breve{\tau}_{\breve{\sigma}}^\vee}\Psi_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}(\breve{\ell})\Big).$$

By definition, we have

$$\breve{\tau}^\vee - \breve{\tau}_{\breve{\sigma}}^\vee = \Big(\bigcup_{\substack{\breve{\tau}' \text{ face of } \breve{\tau} \\ \text{in } \Sigma_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\sigma},\tau}}}\big((\breve{\tau}')^\perp \cap \breve{\tau}^\vee\big)\Big) \subset \breve{\sigma}^\perp \cap \breve{\tau}^\vee.$$

The formal scheme $\mathfrak{U}_{\check{\tau}}$, being the formal completion of

$$(\widetilde{\Xi}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}})_{\check{\sigma}}(\check{\tau}) \cong \underline{\mathrm{Spec}}_{\mathcal{O}_{\widetilde{C}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}}}\Big(\bigoplus_{\check{\ell}\in\check{\sigma}^{\perp}\cap\check{\tau}^{\vee}} \widetilde{\Psi}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}(\check{\ell})\Big)$$

along $U_{\check{\tau}}$, can be canonically identified with the relative formal spectrum of the $\mathcal{O}_{\widetilde{C}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}}$-algebra $\widehat{\bigoplus}_{\check{\ell}\in\check{\sigma}^{\perp}\cap\check{\tau}^{\vee}} \Psi_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}(\check{\ell})$ over $\widetilde{C}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}$, where $\widehat{\bigoplus}$ denotes the completion of the sum with respect to the $\mathcal{O}_{\widetilde{C}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}}$-ideal $\bigoplus_{\check{\ell}\in\check{\sigma}^{\perp}\cap\check{\tau}^{\vee}_{\check{\sigma}}} \Psi_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}(\check{\ell})$. Note that all the above canonical isomorphisms correspond to canonical morphisms of $\mathcal{O}_{\widetilde{C}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}}$-algebras formed by sums of sheaves of the form $\widetilde{\Psi}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}(\check{\ell})$ (with $\mathcal{O}_{\widetilde{C}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}}$-algebra structures inherited from that of $\mathcal{O}_{\widetilde{\Xi}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}}$).

The descriptions above imply the following simple but important facts:

Lemma 4.1. *Suppose $\check{\tau}$ and $\check{\tau}'$ are two cones in $\Sigma_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\sigma},\tau}$ such that $\check{\tau}'$ is a face of $\check{\tau}$.*

(1) *We have a canonical open immersion $\mathfrak{U}_{\check{\tau}'} \hookrightarrow \mathfrak{U}_{\check{\tau}}$ (resp. $U_{\check{\tau}'} \hookrightarrow U_{\check{\tau}}$) of formal subschemes of $\mathfrak{X}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}},\check{\sigma},\tau}$.*

(2) *The canonical restriction morphism from $\mathfrak{U}_{\check{\tau}}$ to $\mathfrak{U}_{\check{\tau}'}$ corresponds to the canonical morphism*

$$\widehat{\bigoplus_{\check{\ell}\in\check{\sigma}^{\perp}\cap\check{\tau}^{\vee}}} \Psi_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}(\check{\ell}) \to \widehat{\bigoplus_{\check{\ell}\in\check{\sigma}^{\perp}\cap(\check{\tau}')^{\vee}}} \Psi_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}(\check{\ell})$$

of $\mathcal{O}_{\widetilde{C}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}}$-algebras, where the two symbols $\widehat{\bigoplus}$ denote completions of the sums with respect to the sheaves of ideals $\bigoplus\limits_{\check{\ell}\in\check{\sigma}^{\perp}\cap\check{\tau}^{\vee}_{\check{\sigma}}} \Psi_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}(\check{\ell})$ and $\bigoplus\limits_{\check{\ell}\in\check{\sigma}^{\perp}\cap(\check{\tau}')^{\vee}_{\check{\sigma}}} \Psi_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}(\check{\ell})$, respectively.

(3) *The canonical restriction morphism from $U_{\check{\tau}}$ to $U_{\check{\tau}'}$ corresponds to the canonical morphism*

$$\bigoplus_{\check{\ell}\in\check{\tau}^{\vee}-\check{\tau}^{\vee}_{\check{\sigma}}} \Psi_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}(\check{\ell}) \to \bigoplus_{\check{\ell}\in(\check{\tau}')^{\vee}-(\check{\tau}')^{\vee}_{\check{\sigma}}} \Psi_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}(\check{\ell})$$

of $\mathcal{O}_{\widetilde{C}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}}$-algebras, which maps $\Psi_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}(\check{\ell})$ to $\Psi_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}(\check{\ell})$ when

$$\check{\ell} \in (\check{\tau}^{\vee} - (\check{\tau}')^{\vee}_{\check{\sigma}}) = (\check{\tau}^{\vee} - \check{\tau}^{\vee}_{\check{\sigma}}) \cap ((\check{\tau}')^{\vee} - (\check{\tau}')^{\vee}_{\check{\sigma}}),$$

and to zero otherwise.

(4) *The correspondences in* (2) *and* (3) *above are canonically compatible with each other.*

By Condition 1.29 (and Lemma 3.1), the action of $\Gamma_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}}$ induces only the trivial action on each stratum it stabilizes. Therefore, the quotient morphism

$$\mathfrak{X}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}},\check{\sigma},\tau} \to \mathfrak{X}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}},\check{\sigma},\tau}/\,\Gamma_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}} \tag{4.2}$$

of formal schemes over S_0 is a *local isomorphism*. The morphism (4.2) is not defined over $\widetilde{C}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}$ when the action of $\Gamma_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}}$ on $\widetilde{C}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}$ is nontrivial. Nevertheless, since $\Gamma_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}}$ acts trivially on $\Phi_{\mathcal{H}}$, it acts trivially on $C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$, and hence (4.2) is defined over $C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$.

Proposition 4.3. *There is a canonical isomorphism*

$$(\mathsf{N}^{\mathrm{tor}})^{\wedge}_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}} \cong \mathfrak{X}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}},\check{\sigma},\tau} / \Gamma_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}} \tag{4.4}$$

of formal schemes over $C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$, characterized by the identifications

$$(\mathsf{N}^{\mathrm{tor}})^{\wedge}_{\mathsf{Z}_{[(\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}},\check{\tau})]}} \cong \mathfrak{X}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}},\check{\sigma},\check{\tau}}$$

of formal schemes over $\widetilde{C}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}$ (compatible with the canonical morphisms

$$(\mathsf{N}^{\mathrm{tor}})^{\wedge}_{\mathsf{Z}_{[(\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}},\check{\tau})]}} \to (\mathsf{N}^{\mathrm{tor}})^{\wedge}_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}}$$

and $\widetilde{C}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}} \to C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$). (The formation of the formal completion here is similar to the one in (5) of Theorem 1.41.)

Proof. Let $\check{\tau} \in \Sigma_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\sigma},\tau}$. Let $\widetilde{\mathfrak{U}}_{\check{\tau}}$ denote the completion of $\widetilde{\Xi}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}(\check{\tau})$ along $U_{\check{\tau}}$, which contains $\mathfrak{U}_{\check{\tau}}$ as a closed formal subscheme (with the same support $U_{\check{\tau}}$).

Since $U_{\check{\tau}}$ is the union of $(\widetilde{\Xi}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}})_{\check{\tau}'}$ with $\check{\tau}'$ running through faces of $\check{\tau}$ in $\Sigma_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\sigma},\tau}$, which are cones in $\mathbf{P}^{+}_{\check{\Phi}_{\widetilde{\mathcal{H}}}}$, the tautological degeneration data over $\widetilde{\mathfrak{U}}_{\check{\tau}}$ satisfies the positivity condition (with respect to the ideal defining $U_{\check{\tau}}$), and we obtain by Mumford's construction a degenerating family $({}^{\heartsuit}\widetilde{G}, {}^{\heartsuit}\widetilde{\lambda}, {}^{\heartsuit}\widetilde{i}, {}^{\heartsuit}\widetilde{\alpha}_{\widetilde{\mathcal{H}}}) \to \widetilde{\mathfrak{U}}_{\check{\tau}}$ as in [Lan 2008, §6.2.5; especially the paragraph preceding Definition 6.2.5.17], called a *Mumford family*. Note that a Mumford family is defined in the sense of relative schemes, namely as a functorial assignment to each affine open formal subscheme $\mathrm{Spf}(R)$ of $\widetilde{\mathfrak{U}}_{\check{\tau}}$ a degenerating family over $\mathrm{Spec}(R)$. Therefore (6) of Theorem 1.41 applies, and implies the existence of a canonical (strata-preserving) morphism $\widetilde{\mathfrak{U}}_{\check{\tau}} \to \widetilde{\mathsf{M}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}}}$ under which $({}^{\heartsuit}\widetilde{G}, {}^{\heartsuit}\widetilde{\lambda}, {}^{\heartsuit}\widetilde{i}, {}^{\heartsuit}\widetilde{\alpha}_{\widetilde{\mathcal{H}}}) \to \widetilde{\mathfrak{U}}_{\check{\tau}}$ is the pullback of $(\widetilde{G}, \widetilde{\lambda}, \widetilde{i}, \widetilde{\alpha}_{\widetilde{\mathcal{H}}}) \to \widetilde{\mathsf{M}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}}}$. Moreover, if $\check{\tau}' \in \Sigma_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\sigma},\tau}$, then the morphisms from $\widetilde{\mathfrak{U}}_{\check{\tau}}$ and from $\widetilde{\mathfrak{U}}_{\check{\tau}'}$ to $\widetilde{\mathsf{M}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}}}$ agree over the intersection $\widetilde{\mathfrak{U}}_{\check{\tau}} \cap \widetilde{\mathfrak{U}}_{\check{\tau}'}$.

By taking the closures of the $[(\widetilde{\Phi}_{\widetilde{\mathcal{H}}}, \widetilde{\delta}_{\widetilde{\mathcal{H}}}, \widetilde{\sigma})]$-strata (not as closed subschemes of the supports, but as closed formal subschemes, as in the second last paragraph preceding Condition 3.8), we obtain canonical morphisms $\mathfrak{U}_{\check{\tau}} \to \mathsf{N}^{\mathrm{tor}}$ for all $\check{\tau}$ in $\Sigma_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\sigma},\tau}$, which patch together, cover all strata above $[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \tau)]$, and define (4.4) as desired. □

By (5) of Theorem 1.41, we have a canonical isomorphism

$$(\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}})^{\wedge}_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}} \cong \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau}. \tag{4.5}$$

By the very constructions, we may and we shall identify the pullback of f^{tor} to $(\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}})^{\wedge}_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}}$ with the canonical morphism $\mathfrak{X}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}},\breve{\sigma},\tau}/\Gamma_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}} \to \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau}$. By abuse of notation, we shall also denote this pullback by

$$f^{\mathrm{tor}} : \mathfrak{X}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}},\breve{\sigma},\tau}/\Gamma_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}} \to \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau}.$$

For each $\breve{\tau} \in \Sigma_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\sigma},\tau}$, let $\mathfrak{U}_{[\breve{\tau}]}$ denote the image of $\mathfrak{U}_{\breve{\tau}}$ under (4.2), which is isomorphic to $\mathfrak{U}_{\breve{\tau}}$ as a formal scheme over $C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$. By admissibility of $\Sigma_{\breve{\Phi}_{\widetilde{\mathcal{H}}}}$, we know that the set $\Sigma_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\sigma},\tau}/\Gamma_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}}$ is *finite*. Then $\mathfrak{X}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}},\breve{\sigma},\tau}/\Gamma_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}}$ can be constructed by gluing the finite collection $\{\mathfrak{U}_{[\breve{\tau}]}\}_{[\breve{\tau}]\in\Sigma_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\sigma},\tau}/\Gamma_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}}}$ of formal schemes over their intersections. Let us denote by

$$f^{\mathrm{tor}}_{[\breve{\tau}]} : \mathfrak{U}_{[\breve{\tau}]} \to \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau}$$

the restriction of f^{tor} to $\mathfrak{U}_{[\breve{\tau}]}$. If we choose a representative $\breve{\tau}$ of $[\breve{\tau}]$, then we can identify $f^{\mathrm{tor}}_{[\breve{\tau}]} : \mathfrak{U}_{[\breve{\tau}]} \to \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau}$ with the canonical morphism $f^{\mathrm{tor}}_{\breve{\tau}} : \mathfrak{U}_{\breve{\tau}} \to \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau}$ induced by the canonical morphism $\mathfrak{X}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}},\breve{\sigma},\tau} \to \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau}$. Let us denote by

$$g_{\breve{\tau}} : \mathfrak{U}_{\breve{\tau}} \to \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau} \times_{C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}} \widetilde{C}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}, \quad h : \widetilde{C}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}} \to C_{\Phi_{\mathcal{H}},\mathsf{Z}_{\mathcal{H}}},$$

and

$$h_{\tau} : \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau} \times_{C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}} \widetilde{C}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}} \to \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau}$$

the canonical morphisms. Then we have a canonical identification $f^{\mathrm{tor}}_{\breve{\tau}} = h_{\tau} \circ g_{\breve{\tau}}$. (Note that $g_{\breve{\tau}}$ is a morphism between affine formal schemes over $\widetilde{C}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}$, and that h_{τ} is the pullback of h to the affine formal scheme $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau}$ over $C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$.)

For simplicity, let us view $\mathcal{O}_{\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau}}$ and $\mathcal{O}_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}}$ as sheaves over $C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$, and suppress $(\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau} \to C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}})_*$ and $(\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]} \to C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}})_*$ from the notation. For push-forwards (to $C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$) of sheaves over $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau}$, we shall use the notation $\widehat{\bigoplus}$ to denote the completion with respect to (the push-forward of) the ideal of definition of $\mathcal{O}_{\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau}}$.

Based on Lemma 4.1, we have the following important facts:

Lemma 4.6. (1) *For any $\breve{\tau} \in \Sigma_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\sigma},\tau}$, and any integer $d \geq 0$, we have the canonical isomorphisms*

$$R^d(f^{\mathrm{tor}}_{\breve{\tau}})_*(\mathcal{O}_{\mathfrak{U}_{\breve{\tau}}}) \cong \widehat{\bigoplus_{\breve{\ell}\in\breve{\sigma}^{\perp}\cap\breve{\tau}^{\vee}}} R^d h_*(\Psi_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}(\breve{\ell})) \tag{4.7}$$

and

$$R^d(f^{\mathrm{tor}}_{\breve{\tau}})_*(\mathcal{O}_{U_{\breve{\tau}}}) \cong \bigoplus_{\breve{\ell}\in\breve{\tau}^{\vee}-\breve{\tau}^{\vee}_{\breve{\sigma}}} R^d h_*(\Psi_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}(\breve{\ell})) \tag{4.8}$$

over $C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$.

(2) *For any $\gamma \in \Gamma_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}}, \Phi_{\mathcal{H}}}$, we have a commutative diagram*

$$\begin{tikzcd}
\mathfrak{U}_{\check{\tau}} \arrow[r, "\gamma"] \arrow[d, "g_{\check{\tau}}"'] & \mathfrak{U}_{\gamma\check{\tau}} \arrow[d, "g_{\gamma\check{\tau}}"] \\
\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau} \times_{C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}} \widetilde{C}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}} \arrow[r, "\gamma"] \arrow[d, "h_\tau"'] & \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau} \times_{C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}} \widetilde{C}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}} \arrow[d, "h_\tau"] \\
\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau} \arrow[r, equal] & \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau}
\end{tikzcd}$$

of formal schemes, (*naturally*) *compatible with the commutative diagram*

$$\begin{tikzcd}
U_{\check{\tau}} \arrow[r, "\gamma"] \arrow[d, "g_{\check{\tau}}"'] & U_{\gamma\check{\tau}} \arrow[d, "g_{\gamma\check{\tau}}"] \\
(\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}})_\tau \times_{C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}} \widetilde{C}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}} \arrow[r, "\gamma"] \arrow[d, "h_\tau"'] & (\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}})_\tau \times_{C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}} \widetilde{C}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}} \arrow[d, "h_\tau"] \\
(\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}})_\tau \arrow[r, equal] & (\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}})_\tau
\end{tikzcd}$$

of their supports. Then (4.7) *and* (4.8) *are compatible with the canonical isomorphisms $\gamma^*\mathcal{O}_{\mathfrak{U}_{\gamma\check{\tau}}} \to \mathcal{O}_{\mathfrak{U}_{\check{\tau}}}$ induced by the canonical isomorphisms $\gamma^*\Psi_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}(\gamma\check{\ell}) \xrightarrow{\sim} \Psi_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}(\check{\ell})$ over $\widetilde{C}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}$.*

(3) *For any integer $d \geq 0$, if $\check{\tau}'$ is a face of $\check{\tau}$, then the canonical morphism $R^d(f^{\mathrm{tor}}_{\check{\tau}})_*\mathcal{O}_{\mathfrak{U}_{\check{\tau}}} \to R^d(f^{\mathrm{tor}}_{\check{\tau}'})_*\mathcal{O}_{\mathfrak{U}_{\check{\tau}'}}$ induced by restriction from $\mathfrak{U}_{\check{\tau}}$ to $\mathfrak{U}_{\check{\tau}'}$ corresponds to the morphism*

$$\widehat{\bigoplus_{\check{\ell}\in\check{\sigma}^\perp\cap\check{\tau}^\vee}} R^d h_*(\Psi_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}(\check{\ell})) \to \widehat{\bigoplus_{\check{\ell}\in\check{\sigma}^\perp\cap(\check{\tau}')^\vee}} R^d h_*(\Psi_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}(\check{\ell}))$$

over $C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$, and the canonical morphism $R^d(f^{\mathrm{tor}}_{\check{\tau}})_*\mathcal{O}_{U_{\check{\tau}}} \to R^d(f^{\mathrm{tor}}_{\check{\tau}'})_*\mathcal{O}_{U_{\check{\tau}'}}$ induced by restriction from $U_{\check{\tau}}$ to $U_{\check{\tau}'}$ corresponds to the morphism

$$\bigoplus_{\check{\ell}\in\check{\tau}^\vee-\check{\tau}^\vee_{\check{\sigma}}} R^d h_*(\Psi_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}(\check{\ell})) \to \bigoplus_{\check{\ell}\in(\check{\tau}')^\vee-(\check{\tau}')^\vee_{\check{\sigma}}} R^d h_*(\Psi_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}(\check{\ell}))$$

over $C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$. Both of these morphisms send $R^d h_*(\Psi_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}(\check{\ell}))$ (*identically*) *to $R^d h_*(\Psi_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}(\check{\ell}))$ when it is defined on both sides, and to zero otherwise.*

4B. *Relative cohomology of structural sheaves.* Using (4.5), we shall identify $(\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}})^\wedge_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}}$ with $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau}$, and identify $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}$ with $(\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}})_\tau$. For simplicity of notation, we shall use $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau}$ and $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}$ more often than their counterparts.

Recall that $C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ is an abelian scheme over the moduli problem $\mathsf{M}_{\mathcal{H}}^{\mathsf{Z}_{\mathcal{H}}}$ (see Definition 1.17). Let $(A, \lambda_A, i_A, \alpha_{\mathcal{H}_h})$ be the tautological tuple over $\mathsf{M}_{\mathcal{H}}^{\mathsf{Z}_{\mathcal{H}}}$. Let T (resp. $T^\vee$) be the split torus with character group X (resp. Y). For simplicity of notation, we shall denote the pullbacks of A, $A^\vee$, T, and $T^\vee$, respectively, by the same symbols. The pullback of G (resp. $G^\vee$) to $\mathfrak{X}_{\Phi_{\mathcal{H}},\mathsf{Z}_{\mathcal{H}},\tau}$ is an extension of A (resp. $A^\vee$) by T (resp. $T^\vee$), and this extension is a pullback of the tautological extension $G^\natural$ (resp. $G^{\vee,\natural}$) over $C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$. For simplicity, we shall also denote the pullbacks of $G^\natural$ and $G^{\vee,\natural}$, respectively, by the same symbols.

Lemma 4.9. *The morphism $h : \widetilde{C}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}} \to C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ is proper and smooth, and is a torsor under the pullback to $C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ of an abelian scheme $\mathbb{Z}^\times_{(\Box)}$-isogenous to $\underline{\mathrm{Hom}}_{\mathcal{O}}(\widetilde{X}, A)^\circ \to \mathsf{M}_{\mathcal{H}}^{\mathsf{Z}_{\mathcal{H}}}$.*

Proof. For simplicity, let us denote the kernel of $\widetilde{C}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}} \to C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ by C, viewed as a scheme over $\mathsf{M}_{\mathcal{H}}^{\mathsf{Z}_{\mathcal{H}}}$.

While the abelian scheme $\widetilde{C}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}} \to \mathsf{M}_{\mathcal{H}}^{\mathsf{Z}_{\mathcal{H}}}$ parametrizes liftings of pairs of the form $(\check{c} : \check{X} \to A^\vee, \check{c}^\vee : \check{Y} \to A) \to \mathsf{M}_{\mathcal{H}}^{\mathsf{Z}_{\mathcal{H}}}$ satisfying the compatibility $\check{c}\check{\phi} = \lambda_A \check{c}^\vee$ and the liftability and pairing conditions, and while the abelian scheme $C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}} \to \mathsf{M}_{\mathcal{H}}^{\mathsf{Z}_{\mathcal{H}}}$ parametrizes liftings of pairs of the form $(c : X \to A^\vee, c^\vee : Y \to A)$ satisfying the compatibility $c\phi = \lambda_A c^\vee$ and the liftability and pairing conditions, the scheme $C \to \mathsf{M}_{\mathcal{H}}^{\mathsf{Z}_{\mathcal{H}}}$ parametrizes lifts of pairs of the form $(\tilde{c} : \widetilde{X} \to A^\vee, \tilde{c}^\vee : \widetilde{Y} \to A)$ satisfying the compatibility $\tilde{c}\widetilde{\phi} = \lambda_A \tilde{c}^\vee$ and the liftability and pairing conditions induced by the ones of the pairs over $\widetilde{C}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}} \to \mathsf{M}_{\mathcal{H}}^{\mathsf{Z}_{\mathcal{H}}}$. Therefore, the same (component annihilating) argument in [Lan 2008, §6.2.3–6.2.4] shows that the kernel C of h is an abelian scheme $\mathbb{Z}^\times_{(\Box)}$-isogenous to $\underline{\mathrm{Hom}}_{\mathcal{O}}(\widetilde{X}, A)^\circ$.

Consequently, all geometric fibers of h are smooth and have the same dimension (as the relative dimension of $C \to \mathsf{M}_{\mathcal{H}}^{\mathsf{Z}_{\mathcal{H}}}$). Since both $\widetilde{C}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}$ and $C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ are smooth over S_0, the morphism h is smooth by [EGA IV$_3$ 1966, 15.4.2 e′) ⇒ b)] and [EGA IV$_4$ 1967, 17.5.1 b) ⇒ a)]. By [Bosch et al. 1990, §2.2, Proposition 14], smooth morphisms between schemes have sections étale locally. This shows that h is a torsor under the pullback of C to $C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$. (Regardless of this argument, the morphism h is proper because the morphism $\widetilde{C}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}} \to \mathsf{M}_{\mathcal{H}}^{\mathsf{Z}_{\mathcal{H}}}$ is.) $\Box$

The nerve of the open covering $\{\mathfrak{U}_{\check{\tau}}\}_{\check{\tau} \in \Sigma_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\sigma},\tau}}$ of $\mathfrak{X}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}},\check{\sigma},\tau}$, or equivalently the open covering $\{U_{\check{\tau}}\}_{\check{\tau} \in \Sigma_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\sigma},\tau}}$ of $(\widetilde{\Xi}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}})_{\check{\sigma}}(\tau)$ (by the supports of the formal schemes $\{\mathfrak{U}_{\check{\tau}}\}_{\check{\tau} \in \Sigma_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\sigma},\tau}}$), defines a simplicial complex $\widetilde{\mathfrak{N}}_{\check{\sigma},\tau}$ formed (up to scaling by the multiplicative action of $\mathbb{R}_{>0}$, inducing homotopy equivalences harmless for our purpose) by the union of the cones $\check{\tau}$ in $\Sigma_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\sigma},\tau}$ (with natural incidence relations among their closures inherited from their realizations as locally closed subsets of $(\mathbf{S}_{\check{\Phi}_{\widetilde{\mathcal{H}}}})^\vee_{\mathbb{R}}$). Then the nerve of the open covering

$$\{\mathfrak{U}_{[\check{\tau}]}\}_{[\check{\tau}] \in \Sigma_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\sigma},\tau} / \Gamma_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}}} \quad \text{of} \quad (\mathsf{N}^{\mathrm{tor}})^\wedge_{\check{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}} \cong \mathfrak{X}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}},\check{\sigma},\tau} / \Gamma_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}},$$

or equivalently the open covering

$$\{U_{\check{\tau}}\}_{\check{\tau}\in\Sigma_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\sigma},\tau}/\Gamma_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}}} \quad \text{of} \quad \widetilde{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]} \cong (\widetilde{\Xi}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}})_\tau/\Gamma_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}}$$

of the supports of formal schemes, is naturally identified with $\mathfrak{N}_{\check{\sigma},\tau} := \widetilde{\mathfrak{N}}_{\check{\sigma},\tau}/\Gamma_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}}$.

The simplicial complex $\widetilde{\mathfrak{N}}_{\check{\sigma},\tau}$ has a *closed covering* by the closures $\check{\tau}^{\mathrm{cl}}$ (in $\widetilde{\mathfrak{N}}_{\check{\sigma},\tau}$) of the cones $\check{\tau}$ in $\Sigma_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\sigma},\tau}$, which induces a closed covering of $\mathfrak{N}_{\check{\sigma},\tau}$ by the closures $[\check{\tau}]^{\mathrm{cl}}$ (in $\mathfrak{N}_{\check{\sigma},\tau}$) of the subsets $[\check{\tau}]$ of $\Sigma_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\sigma},\tau}/\Gamma_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}}$. For any sheaf $\mathcal{M}$ on $(\widetilde{\Xi}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}})_\tau/\Gamma_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}}$ (such as $\mathcal{O}_{(\mathsf{N}^{\mathrm{tor}})^{\wedge}_{\widetilde{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}}} \cong \mathcal{O}_{\mathfrak{X}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}},\check{\sigma},\tau}/\Gamma_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}}}$), define for any integer $d \geq 0$ the local system $\mathcal{H}^d(\mathcal{M})$ on $\mathfrak{N}_{\check{\sigma},\tau}$ which associates with each $[\check{\tau}]$ in $\Sigma_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\sigma},\tau}/\Gamma_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}}$ the coefficients

$$\mathcal{H}^d(\mathcal{M})([\check{\tau}]^{\mathrm{cl}}) := H^d(U_{[\check{\tau}]}, \mathcal{M}|_{U_{[\check{\tau}]}}).$$

Then, by [Godement 1958, II, 5.4.1], there is a spectral sequence

$$E_2^{c,d} := H^c(\mathfrak{N}_{\check{\sigma},\tau}, \mathcal{H}^d(\mathcal{M})) \Rightarrow H^{c+d}((\widetilde{\Xi}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}})_\tau/\Gamma_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}}, \mathcal{M}). \tag{4.10}$$

The construction of $\mathfrak{N}_{\check{\sigma},\tau}$ depends only on the cone decomposition $\Sigma_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\sigma},\tau}$, while the constructions of both $\mathcal{H}^d(\mathcal{M})$ and the spectral sequence (4.10) are compatible with restrictions to affine open subschemes of $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}$. Therefore, we can define the sheaves $\underline{\mathcal{H}}^d(\mathcal{M})$ (of local systems on $\mathfrak{N}_{\check{\sigma},\tau}$) over $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}$, and obtain a spectral sequence

$$E_2^{c,d} := H^c(\mathfrak{N}_{\check{\sigma},\tau}, \underline{\mathcal{H}}^d(\mathcal{M})) \Rightarrow R^{c+d} f_*^{\mathrm{tor}}(\mathcal{M}). \tag{4.11}$$

Here $H^c(\mathfrak{N}_{\check{\sigma},\tau}, \underline{\mathcal{H}}^d(\mathcal{M}))$ is interpreted as a sheaf on $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}$, and the formation of (4.11) is compatible with morphisms in $\mathcal{M}$. In particular, we have compatible spectral sequences

$$E_2^{c,d} := H^c(\mathfrak{N}_{\check{\sigma},\tau}, \underline{\mathcal{H}}^d(\mathcal{O}_{(\mathsf{N}^{\mathrm{tor}})^{\wedge}_{\widetilde{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}}})) \Rightarrow R^{c+d} f_*^{\mathrm{tor}}(\mathcal{O}_{(\mathsf{N}^{\mathrm{tor}})^{\wedge}_{\widetilde{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}}}) \tag{4.12}$$

and

$$E_2^{c,d} := H^c(\mathfrak{N}_{\check{\sigma},\tau}, \underline{\mathcal{H}}^d(\mathcal{O}_{\widetilde{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}})) \Rightarrow R^{c+d} f_*^{\mathrm{tor}}(\mathcal{O}_{\widetilde{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}}). \tag{4.13}$$

To calculate the left-hand sides of (4.12) and (4.13), we define the sheaves $\underline{\mathcal{H}}^d(\mathcal{O}_{\mathfrak{X}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}},\check{\sigma},\tau}})$ and $\underline{\mathcal{H}}^d(\mathcal{O}_{(\widetilde{\Xi}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}})_\tau})$ of local systems on $\widetilde{\mathfrak{N}}_{\check{\sigma},\tau}$ (in the obvious way), which, by Lemma 4.6, carry canonical equivariant actions of the group $\Gamma_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}}$, and descend to the sheaves $\underline{\mathcal{H}}^d(\mathcal{O}_{(\mathsf{N}^{\mathrm{tor}})^{\wedge}_{\widetilde{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}}})$ and $\underline{\mathcal{H}}^d(\mathcal{O}_{\widetilde{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}})$ on $\mathfrak{N}_{\check{\sigma},\tau}$, respectively. Hence we obtain compatible spectral sequences

$$\begin{aligned} E_2^{c-e,e} := H^{c-e}(\Gamma_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}}, H^e(\widetilde{\mathfrak{N}}_{\check{\sigma},\tau}, \underline{\mathcal{H}}^d(\mathcal{O}_{\mathfrak{X}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}},\check{\sigma},\tau}}))) \\ \Rightarrow H^c(\mathfrak{N}_{\check{\sigma},\tau}, \underline{\mathcal{H}}^d(\mathcal{O}_{(\mathsf{N}^{\mathrm{tor}})^{\wedge}_{\widetilde{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}}})) \end{aligned} \tag{4.14}$$

and

$$E_2^{c-e,e} := H^{c-e}(\Gamma_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}}, H^e(\widetilde{\mathfrak{N}}_{\breve{\sigma},\tau}, \underline{\mathscr{H}}^d(\mathcal{O}_{(\breve{\Xi}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}})_\tau})))$$
$$\Rightarrow H^c(\mathfrak{N}_{\breve{\sigma},\tau}, \underline{\mathscr{H}}^d(\mathcal{O}_{\widetilde{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}})). \quad (4.15)$$

Lemma 4.16. *For any $d \geq 0$, the canonical morphisms*

$$R^d h_*(\mathcal{O}_{\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau} \times_{C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}} \widetilde{C}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}}) \to H^0(\widetilde{\mathfrak{N}}_{\breve{\sigma},\tau}, \underline{\mathscr{H}}^d(\mathcal{O}_{\mathfrak{X}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}},\breve{\sigma},\tau}})) \quad (4.17)$$

and

$$R^d h_*(\mathcal{O}_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]} \times_{C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}} \widetilde{C}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}}) \to H^0(\widetilde{\mathfrak{N}}_{\breve{\sigma},\tau}, \underline{\mathscr{H}}^d(\mathcal{O}_{(\breve{\Xi}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}})_\tau})) \quad (4.18)$$

are isomorphisms compatible with each other. Moreover, for any integer $e > 0$, we have

$$H^e(\widetilde{\mathfrak{N}}_{\breve{\sigma},\tau}, \underline{\mathscr{H}}^d(\mathcal{O}_{\mathfrak{X}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}},\breve{\sigma},\tau}})) = 0 \quad (4.19)$$

and

$$H^e(\widetilde{\mathfrak{N}}_{\breve{\sigma},\tau}, \underline{\mathscr{H}}^d(\mathcal{O}_{(\breve{\Xi}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}})_\tau})) = 0. \quad (4.20)$$

Proof. By (4.7), we have

$$\underline{\mathscr{H}}^d(\mathcal{O}_{\mathfrak{X}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}},\breve{\sigma},\tau}})(\breve{\tau}^{\mathrm{cl}}) \cong R^d(f_{\breve{\tau}}^{\mathrm{tor}})_*(\mathcal{O}_{\mathfrak{U}_{\breve{\tau}}}) \cong \widehat{\bigoplus_{\breve{\ell}\in\breve{\sigma}^\perp\cap\breve{\tau}^\vee}} R^d h_*(\Psi_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}(\breve{\ell})),$$

and for any face $\breve{\tau}'$ of $\breve{\tau}$, the canonical morphism

$$\underline{\mathscr{H}}^d(\mathcal{O}_{\mathfrak{X}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}},\breve{\sigma},\tau}})(\breve{\tau}^{\mathrm{cl}}) \to \underline{\mathscr{H}}^d(\mathcal{O}_{\mathfrak{X}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}},\breve{\sigma},\tau}})((\breve{\tau}')^{\mathrm{cl}})$$

sends the subsheaf $R^d h_*(\Psi_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}(\breve{\ell}))$ either (identically) to $R^d h_*(\Psi_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}(\breve{\ell}))$ when $\breve{\ell} \in \breve{\sigma}^\perp \cap (\breve{\tau}')^\vee$, or to zero otherwise. Since $\bigcup_{\breve{\ell}\in\breve{\tau}^\vee} \breve{\tau}^{\mathrm{cl}} = \bigcup_{\breve{\ell}\in\breve{\tau}^\vee} \breve{\tau}$ is a convex subset of $\widetilde{\mathfrak{N}}_{\breve{\sigma},\tau}$ for any given $\breve{\ell} \in \breve{\sigma}^\perp$, this shows (4.19) as usual (by the argument in [Kempf et al. 1973, Chapter I, §3]). On the other hand, since $\bigcap_{\breve{\tau}\in\Sigma_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\sigma},\tau}} (\breve{\sigma}^\perp \cap \breve{\tau}^\vee) = \tau^\vee$, we see that (4.17) is an isomorphism. The proofs for (4.20) and (4.18) are similar. □

Lemma 4.21. *The topological space $\mathfrak{N}_{\breve{\sigma},\tau}$ is homotopic to the real torus*

$$\mathbf{T}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}} := (\Gamma_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}})^\vee_{\mathbb{R}} / \Gamma_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}},$$

whose cohomology groups (by contractibility of $(\Gamma_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}})^\vee_{\mathbb{R}}$) are

$$H^j(\mathbf{T}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}}, \mathbb{Z}) \cong H^j(\Gamma_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}}, \mathbb{Z}) \cong \textstyle\bigwedge^j(\mathrm{Hom}_{\mathbb{Z}}(\Gamma_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}}, \mathbb{Z}))$$

for any $j \geq 0$. Over $C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$, we have a canonical isomorphism

$$H^j(\Gamma_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}}, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathcal{O}_{C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}} \cong \textstyle\bigwedge^j(\mathrm{Hom}_{\mathcal{O}}(Q^\vee, \underline{\mathrm{Lie}}_{T^\vee/C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}})). \quad (4.22)$$

Proof. Since $\widetilde{\sigma}$ is a top-dimensional cone in $\mathbf{P}^+_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}}}$, any $\check{\tau} \in \Sigma_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\sigma},\tau}$ (which has $\check{\sigma}$ as a face) is generated by $\check{\sigma}$ and some rational basis vectors not contained in the image of the first morphism in (3.7). Moreover, the image of $\check{\tau}$ under the second morphism in (3.7) is contained in $\tau \subset \mathbf{P}^+_{\Phi_{\mathcal{H}}}$. By choosing some (noncanonical) splitting of $s_{\check{X}} \otimes_{\mathbb{Z}} \mathbb{Q} : \check{X} \otimes_{\mathbb{Z}} \mathbb{Q} \twoheadrightarrow \widetilde{X} \otimes_{\mathbb{Z}} \mathbb{Q}$, we can decompose the real vector space $(\mathbf{S}_{\check{\Phi}_{\widetilde{\mathcal{H}}}})^\vee_{\mathbb{R}}$ (noncanonically) as a direct sum $(\mathbf{S}_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}}})^\vee_{\mathbb{R}} \oplus (\Gamma_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}})^\vee_{\mathbb{R}} \oplus (\mathbf{S}_{\Phi_{\mathcal{H}}})^\vee_{\mathbb{R}}$, on which the action of $\Gamma_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}}$ is realized by its canonical translation action on the second factor. Along the directions of $(\mathbf{S}_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}}})^\vee_{\mathbb{R}}$ and $(\mathbf{S}_{\Phi_{\mathcal{H}}})^\vee_{\mathbb{R}}$, we can contract the simplicial complex $\widetilde{\mathfrak{N}}_{\check{\sigma},\tau}$ (say, towards some arbitrarily chosen points in the convex sets $\widetilde{\sigma}$ and τ) in a way compatible with the actions of $\Gamma_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}}$. Therefore, $\mathfrak{N}_{\check{\sigma},\tau} = \widetilde{\mathfrak{N}}_{\check{\sigma},\tau} / \Gamma_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}}$ is homotopic to the real torus $\mathbf{T}_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}} = (\Gamma_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}})^\vee_{\mathbb{R}} / \Gamma_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}}$.

The canonical isomorphism (4.22) then follows from the composition of the following canonical isomorphisms:

$$\begin{aligned} H^j(\Gamma_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}}, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{O}_{C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}} &\cong \big(\textstyle\bigwedge^j (\mathrm{Hom}_{\mathbb{Z}}(\Gamma_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}}, \mathbb{Z}))\big) \otimes_{\mathbb{Z}} \mathbb{O}_{C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}} \\ &\cong \big(\textstyle\bigwedge^j (\mathrm{Hom}_{\mathbb{Z}}(\mathrm{Hom}_{\mathcal{O}}(\widetilde{X}, X), \mathbb{Z}_{(\Box)}))\big) \otimes_{\mathbb{Z}_{(\Box)}} \mathbb{O}_{C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}} \\ &\cong \textstyle\bigwedge^j (\mathrm{Hom}_{\mathcal{O}}(Q^\vee, \mathrm{Hom}_{\mathbb{Z}}(Y, \mathbb{O}_{C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}))) \\ &\cong \textstyle\bigwedge^j (\mathrm{Hom}_{\mathcal{O}}(Q^\vee, \underline{\mathrm{Lie}}_{T^\vee/C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}})). \end{aligned}$$

□

Lemma 4.23. *There are compatible canonical isomorphisms*

$$R^d h_*(\mathbb{O}_{\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau} \times_{C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}} \widetilde{C}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}}) \cong \textstyle\bigwedge^d (\underline{\mathrm{Hom}}_{\mathcal{O}}(Q^\vee, \underline{\mathrm{Lie}}_{A^\vee/\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau}})$$

and

$$R^d h_*(\mathbb{O}_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]} \times_{C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}} \widetilde{C}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}}) \cong \textstyle\bigwedge^d (\underline{\mathrm{Hom}}_{\mathcal{O}}(Q^\vee, \underline{\mathrm{Lie}}_{A^\vee/\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}}))$$

for any integer $d \geq 0$.

Proof. By Lemma 4.9, the morphism $h : \widetilde{C}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}} \to C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ is a torsor under an abelian scheme $\mathbb{Z}^\times_{(\Box)}$-isogenous to $\underline{\mathrm{Hom}}_{\mathcal{O}}(Q, A)^\circ$ (and hence has a section étale locally). Since the cohomology of abelian schemes (with coefficients in the structural sheaves) are free and are compatible with arbitrary base changes (see [Berthelot et al. 1982, Proposition 2.5.2; Mumford 1970, §5]), we obtain compatible canonical isomorphisms

$$\begin{aligned} R^d h_*(\mathbb{O}_{\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau} \times_{C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}} \widetilde{C}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}}) &\cong \textstyle\bigwedge^d (\underline{\mathrm{Lie}}_{(\underline{\mathrm{Hom}}_{\mathcal{O}}(Q,A)^\circ)^\vee/\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau}}) \\ &\cong \textstyle\bigwedge^d (\underline{\mathrm{Hom}}_{\mathcal{O}}(Q^\vee, \underline{\mathrm{Lie}}_{A^\vee/\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau}})), \\ R^d h_*(\mathbb{O}_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]} \times_{C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}} \widetilde{C}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}}) &\cong \textstyle\bigwedge^d (\underline{\mathrm{Lie}}_{(\underline{\mathrm{Hom}}_{\mathcal{O}}(Q,A)^\circ)^\vee/\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}}) \\ &\cong \textstyle\bigwedge^d (\underline{\mathrm{Hom}}_{\mathcal{O}}(Q^\vee, \underline{\mathrm{Lie}}_{A^\vee/\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}})) \end{aligned}$$

for any integer $d \geq 0$. □

Proposition 4.24. *There are compatible canonical isomorphisms*

$$H^c(\mathfrak{N}_{\breve{\sigma},\tau}, \underline{\mathscr{H}}^d(\mathcal{O}_{(\mathsf{N}^{\mathrm{tor}})^{\wedge}_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}}})) \cong \big(\textstyle\bigwedge^c(\underline{\mathrm{Hom}}_{\mathcal{O}}(Q^\vee, \underline{\mathrm{Lie}}_{T^\vee/\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau}}))\big) \otimes_{\mathcal{O}_{\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau}}} \big(\textstyle\bigwedge^d(\underline{\mathrm{Hom}}_{\mathcal{O}}(Q^\vee, \underline{\mathrm{Lie}}_{A^\vee/\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau}}))\big) \quad (4.25)$$

and

$$H^c(\mathfrak{N}_{\breve{\sigma},\tau}, \underline{\mathscr{H}}^d(\mathcal{O}_{\widetilde{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}})) \cong \big(\textstyle\bigwedge^c(\underline{\mathrm{Hom}}_{\mathcal{O}}(Q^\vee, \underline{\mathrm{Lie}}_{T^\vee/\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}}))\big) \otimes_{\mathcal{O}_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}}} \big(\textstyle\bigwedge^d(\underline{\mathrm{Hom}}_{\mathcal{O}}(Q^\vee, \underline{\mathrm{Lie}}_{A^\vee/\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}}))\big) \quad (4.26)$$

for any integers $c, d \geq 0$.

Proof. By Lemma 4.16, the spectral sequences (4.14) and (4.15) degenerate and show that for any integers c and d we have compatible canonical isomorphisms

$$\begin{aligned} H^c(\mathfrak{N}_{\breve{\sigma},\tau}, \underline{\mathscr{H}}^d(\mathcal{O}_{(\mathsf{N}^{\mathrm{tor}})^{\wedge}_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}}})) &\cong H^c(\Gamma_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}}, H^0(\widetilde{\mathfrak{N}}_{\breve{\sigma},\tau}, \underline{\mathscr{H}}^d(\mathcal{O}_{\mathfrak{X}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}},\breve{\sigma},\tau}}))) \\ &\cong H^c(\Gamma_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}}, \mathbb{Z}) \otimes_{\mathbb{Z}} R^d h_*(\mathcal{O}_{\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau} \times_{C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}} \widetilde{C}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}}) \end{aligned} \quad (4.27)$$

and

$$\begin{aligned} H^c(\mathfrak{N}_{\breve{\sigma},\tau}, \underline{\mathscr{H}}^d(\mathcal{O}_{\widetilde{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}})) &\cong H^c(\Gamma_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}}, H^0(\widetilde{\mathfrak{N}}_{\breve{\sigma},\tau}, \underline{\mathscr{H}}^d(\mathcal{O}_{(\widetilde{\Xi}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}})_\tau}))) \\ &\cong H^c(\Gamma_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}}, \mathbb{Z}) \otimes_{\mathbb{Z}} R^d h_*(\mathcal{O}_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]} \times_{C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}} \widetilde{C}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}}). \end{aligned} \quad (4.28)$$

Now combine (4.27) and (4.28) with Lemmas 4.21 and 4.23. □

Lemma 4.29. *The spectral sequence* (4.12) *degenerates at E_2 terms. Consequently, since the choice of the stratum $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}$ is arbitrary, by Grothendieck's fundamental theorem* [EGA III_1 1961, 4.1.5] (and by fpqc descent for the property of local freeness [SGA 1 1971, VIII, 1.11]), *the sheaf $R^b f_*^{\mathrm{tor}}(\mathcal{O}_{\mathsf{N}^{\mathrm{tor}}})$ is locally free of the same rank as $\bigwedge^b(\underline{\mathrm{Hom}}_{\mathcal{O}}(Q^\vee, \underline{\mathrm{Lie}}_{G^\vee/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}))$ over $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$.*

*If, for every **maximal point** s of $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}$* (see [Grothendieck 1971, 0 2.1.2]), *we have*

$$\begin{aligned} &\dim_{k(s)}((R^b f_*^{\mathrm{tor}}(\mathcal{O}_{\widetilde{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}})) \otimes_{\mathcal{O}_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}}} k(s)) \\ &\qquad \geq \dim_{k(s)}((R^b f_*^{\mathrm{tor}}(\mathcal{O}_{(\mathsf{N}^{\mathrm{tor}})^{\wedge}_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}}})) \otimes_{\mathcal{O}_{\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau}}} k(s)), \end{aligned} \quad (4.30)$$

then the spectral sequence (4.13) *degenerates at E_2 terms as well*, *and there is a canonical isomorphism*

$$R^b f_*^{\mathrm{tor}}(\mathcal{O}_{\mathsf{N}^{\mathrm{tor}}}) \otimes_{\mathcal{O}_{\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}}} \mathcal{O}_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}} \xrightarrow{\sim} R^b f_*^{\mathrm{tor}}(\mathcal{O}_{\tilde{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}}). \tag{4.31}$$

Proof. Let $\mathrm{Spf}(R, I)$ be any connected affine open formal subscheme of $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau}$, with the ideal of definition I satisfying $\mathrm{rad}(I) = I$ for simplicity. Since $\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$ is smooth and of finite type over $\mathsf{S}_0 = \mathrm{Spec}(\mathcal{O}_{F_0,(\Box)})$, the ring R is a noetherian domain. (See [Matsumura 1980, 33.I and 34.A].) Since $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}$ is a smooth subscheme of $\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$, the quotient R/I is also a noetherian domain. Let $K := \mathrm{Frac}(R)$ and $k := \mathrm{Frac}(R/I)$ be the fraction fields. By abuse of notation, we shall denote pullbacks of schemes to $\mathrm{Spec}(K)$ (resp. $\mathrm{Spec}(k)$) by the subscript K (resp. k).

Since we have an exact sequence

$$0 \to \underline{\mathrm{Lie}}_{T^\vee/\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau}} \to \underline{\mathrm{Lie}}_{G^{\vee,\natural}/\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau}} \to \underline{\mathrm{Lie}}_{A^\vee/\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau}} \to 0$$

of locally free sheaves, we have an equality

$$\begin{aligned}\sum_{c+d=b} \dim_K \big(\textstyle\bigwedge^c (\mathrm{Hom}_{\mathcal{O}}(Q^\vee, \mathrm{Lie}_{A_K^\vee}))\big) \otimes_K \big(\textstyle\bigwedge^d (\mathrm{Hom}_{\mathcal{O}}(Q^\vee, \mathrm{Lie}_{T_K^\vee}))\big)&\\ = \dim_K \big(\textstyle\bigwedge^b (\mathrm{Hom}_{\mathcal{O}}(Q^\vee, \mathrm{Lie}_{G_K^{\vee,\natural}}))\big)&\\ = \dim_K \big(\textstyle\bigwedge^b (\mathrm{Hom}_{\mathcal{O}}(Q^\vee, \mathrm{Lie}_{G_K^\vee}))\big)&, \end{aligned}\tag{4.32}$$

and an analogous equality with K replaced with k.

By construction of the spectral sequences (4.12) and (4.13), by the canonical isomorphisms (4.25) and (4.26), and by the equality (4.32), we have

$$\begin{aligned}\sum_{c+d=b} \dim_K (H^c(\mathfrak{N}_{\breve{\sigma},\tau}, \underline{\mathscr{H}}^d(\mathcal{O}_{(\mathsf{N}^{\mathrm{tor}})^\wedge_{\tilde{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}}})) \otimes_{\mathcal{O}_{\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau}}} K)&\\ = \dim_K \big(\textstyle\bigwedge^b (\mathrm{Hom}_{\mathcal{O}}(Q^\vee, \mathrm{Lie}_{G_K^\vee}))\big)&\\ \geq \dim_K ((R^b f_*^{\mathrm{tor}}(\mathcal{O}_{(\mathsf{N}^{\mathrm{tor}})^\wedge_{\tilde{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}}})) \otimes_{\mathcal{O}_{\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau}}} K)&\end{aligned}\tag{4.33}$$

and

$$\begin{aligned}\sum_{c+d=b} \dim_k (H^c(\mathfrak{N}_{\breve{\sigma},\tau}, \underline{\mathscr{H}}^d(\mathcal{O}_{\tilde{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}})) \otimes_{\mathcal{O}_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}}} k)&\\ = \dim_k \big(\textstyle\bigwedge^b (\mathrm{Hom}_{\mathcal{O}}(Q^\vee, \mathrm{Lie}_{G_k^\vee}))\big)&\\ \geq \dim_k (R^b f_*^{\mathrm{tor}}(\mathcal{O}_{\tilde{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}}) \otimes_{\mathcal{O}_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}}} k)&. \end{aligned}\tag{4.34}$$

Since the pullback of f^{tor} to the open dense subscheme $\mathsf{M}_{\mathcal{H}}$ of $\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$ is simply the abelian scheme $f : \mathsf{N}' \to \mathsf{M}_{\mathcal{H}}$, we have

$$\begin{aligned}(R^b f^{\mathrm{tor}}_*(\mathcal{O}_{\mathsf{N}^{\mathrm{tor}}})) \otimes_{\mathcal{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}} \mathcal{O}_{\mathsf{M}_{\mathcal{H}}} &\cong R^b f_*(\mathcal{O}_{\mathsf{N}}) \\ &\cong \textstyle\bigwedge^b \underline{\mathrm{Lie}}_{\mathsf{N}^\vee/\mathsf{M}_{\mathcal{H}}} \cong \bigwedge^b (\underline{\mathrm{Hom}}_{\mathcal{O}}(Q^\vee, \underline{\mathrm{Lie}}_{G^\vee_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}}})).\end{aligned}$$

Since the canonical morphism $\mathrm{Spec}(K) \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$ factors through some maximal point of $\mathsf{M}_{\mathcal{H}}$, this implies that the inequality in (4.33) is an equality, and hence that the spectral sequence (4.12) degenerates at E_2 terms after pullback to K. Since all E_2 terms of this spectral sequence are locally free sheaves, this shows that (4.12) degenerates at E_2 terms after pullback to R. Since the choice of R is arbitrary, this shows that (4.12) degenerates over the whole $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau}$, and hence $R^b f^{\mathrm{tor}}_*(\mathcal{O}_{\mathsf{N}^{\mathrm{tor}}})$ is locally free of the same rank as $\bigwedge^b(\underline{\mathrm{Hom}}_{\mathcal{O}}(Q^\vee, \underline{\mathrm{Lie}}_{G^\vee/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}))$ over $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$. (Nevertheless, since f^{tor} is not necessarily flat, this does not imply that the formation of $R^b f^{\mathrm{tor}}_*(\mathcal{O}_{\mathsf{N}^{\mathrm{tor}}})$ is compatible with arbitrary base change.)

Since the canonical morphism $\mathrm{Spec}(k) \to \mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}$ factors through some maximal point of $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}$, the inequality (4.30) implies that

$$\begin{aligned}&\dim_k(R^b f^{\mathrm{tor}}_*(\mathcal{O}_{\tilde{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}}) \otimes_{\mathcal{O}_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}}} k) \\ &\qquad \geq \dim_k((R^b f^{\mathrm{tor}}_*(\mathcal{O}_{(\mathsf{N}^{\mathrm{tor}})^\wedge_{\tilde{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}}})) \otimes_{\mathcal{O}_{\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau}}} k) \\ &\qquad = \dim_K((R^b f^{\mathrm{tor}}_*(\mathcal{O}_{(\mathsf{N}^{\mathrm{tor}})^\wedge_{\tilde{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}}})) \otimes_{\mathcal{O}_{\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau}}} K),\end{aligned}$$

and hence the *equality* in (4.33) implies the equality in (4.34), because

$$\dim_k\bigl(\textstyle\bigwedge^b(\mathrm{Hom}_{\mathcal{O}}(Q^\vee, \mathrm{Lie}_{G^\vee_k}))\bigr) = \dim_K\bigl(\bigwedge^b(\mathrm{Hom}_{\mathcal{O}}(Q^\vee, \mathrm{Lie}_{G^\vee_K}))\bigr).$$

Therefore, by the same reasoning as in the case of (4.12) above, the spectral sequence (4.13) also degenerates at E_2 terms. Since the spectral sequences (4.12) and (4.13) are compatible with each other (by their very construction), their degeneracy implies that the canonical morphism

$$R^b f^{\mathrm{tor}}_*(\mathcal{O}_{(\mathsf{N}^{\mathrm{tor}})^\wedge_{\tilde{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}}}) \otimes_{\mathcal{O}_{\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau}}} \mathcal{O}_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}} \to R^b f^{\mathrm{tor}}_*(\mathcal{O}_{\tilde{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}})$$

is an isomorphism (by comparing graded pieces) and induces (4.31). □

Remark 4.35. By upper semicontinuity for proper flat morphisms (see [Mumford 1970, §5 Corollary (a)]), the assumption (4.30) is satisfied when f^{tor} is flat, or equivalently when Condition 3.17 is satisfied (by Proposition 3.18), which can be achieved by refining both $\tilde{\Sigma}$ and Σ (by Proposition 3.19).

Corollary 4.36. *For any integer $b \geq 0$, the canonical* (*cup product*) *morphism $\bigwedge^b(R^1 f^{\mathrm{tor}}_*(\mathcal{O}_{\mathsf{N}^{\mathrm{tor}}})) \to R^b f^{\mathrm{tor}}_*(\mathcal{O}_{\mathsf{N}^{\mathrm{tor}}})$ is an isomorphism.*

Proof. As in Lemma 4.29, by properness of f^{tor}, this is true if and only if it is true over the formal completion along each stratum $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}$, which is the case because the canonical morphism induces isomorphisms on all graded pieces defined by spectral sequences such as (4.12), which are compatible with cup products by the very construction (see [Godement 1958, II, §5–6]). □

4C. *Degeneracy of the (relative) Hodge spectral sequence.* As in (3c) of Theorem 2.15, let $\underline{H}^i_{\text{log-dR}}(\mathsf{N}^{\text{tor}}/\mathsf{M}^{\text{tor}}_{\mathcal{H}}) := R^i f^{\text{tor}}_* \overline{\Omega}^\bullet_{\mathsf{N}^{\text{tor}}/\mathsf{M}^{\text{tor}}_{\mathcal{H}}}$ be the (relative) log de Rham cohomology. By the definition of $\underline{H}^i_{\text{log-dR}}(\mathsf{N}^{\text{tor}}/\mathsf{M}^{\text{tor}}_{\mathcal{H}})$ as the "relative hypercohomology", the natural (Hodge) filtration on the complex $\overline{\Omega}^\bullet_{\mathsf{N}^{\text{tor}}/\mathsf{M}^{\text{tor}}_{\mathcal{H}}}$ defines the (relative) Hodge spectral sequence (2.20):

$$E_1^{a,b} := R^b f^{\text{tor}}_* (\overline{\Omega}^a_{\mathsf{N}^{\text{tor}}/\mathsf{M}^{\text{tor}}_{\mathcal{H}}}) \Rightarrow \underline{H}^{a+b}_{\text{log-dR}}(\mathsf{N}^{\text{tor}}/\mathsf{M}^{\text{tor}}_{\mathcal{H}}).$$

By (3a) of Theorem 2.15 (which we have proved in Section 3C), there is a canonical isomorphism

$$\begin{aligned}\overline{\Omega}^a_{\mathsf{N}^{\text{tor}}/\mathsf{M}^{\text{tor}}_{\mathcal{H}}} &\cong \textstyle\bigwedge^a\big[(f^{\text{tor}})^*(\underline{\mathrm{Hom}}_{\mathcal{O}}(Q^\vee, \underline{\mathrm{Lie}}^\vee_{G/\mathsf{M}^{\text{tor}}_{\mathcal{H}}}))\big] \\ &\cong (f^{\text{tor}})^*\big[\textstyle\bigwedge^a(\underline{\mathrm{Hom}}_{\mathcal{O}}(Q^\vee, \underline{\mathrm{Lie}}^\vee_{G/\mathsf{M}^{\text{tor}}_{\mathcal{H}}}))\big]\end{aligned}$$

of locally free sheaves over N^{tor}. Then (by the projection formula [EGA I 1960, chapitre 0, 5.4.10.1]) we have canonical isomorphisms

$$R^b f^{\text{tor}}_*(\overline{\Omega}^a_{\mathsf{N}^{\text{tor}}/\mathsf{M}^{\text{tor}}_{\mathcal{H}}}) \cong (R^b f^{\text{tor}}_*(\mathcal{O}_{\mathsf{N}^{\text{tor}}})) \otimes_{\mathcal{O}_{\mathsf{M}^{\text{tor}}_{\mathcal{H}}}} \big(\textstyle\bigwedge^a(\underline{\mathrm{Hom}}_{\mathcal{O}}(Q^\vee, \underline{\mathrm{Lie}}^\vee_{G/\mathsf{M}^{\text{tor}}_{\mathcal{H}}}))\big). \quad (4.37)$$

Lemma 4.38. *If $R^b f^{\text{tor}}_*(\mathcal{O}_{\mathsf{N}^{\text{tor}}})$ is locally free for every integer $b \geq 0$, then the spectral sequence* (2.20) *degenerates at the E_1 terms.*

Proof. By (4.37), if $R^b f^{\text{tor}}_*(\mathcal{O}_{\mathsf{N}^{\text{tor}}})$ is locally free for every integer $b \geq 0$, then all the E_1 terms $R^b f^{\text{tor}}_*(\overline{\Omega}^a_{\mathsf{N}^{\text{tor}}/\mathsf{M}^{\text{tor}}_{\mathcal{H}}})$ of the spectral sequence (2.20) are locally free. Therefore, to show that (2.20) degenerates at E_1 terms, it suffices to show that it degenerates at E_1 terms over the open dense subscheme $\mathsf{M}_{\mathcal{H}}$ of $\mathsf{M}^{\text{tor}}_{\mathcal{H}}$, which is true because $f^{\text{tor}}|_{\mathsf{N}} = f : \mathsf{N} \to \mathsf{M}_{\mathcal{H}}$ is an abelian scheme. (See for example [Berthelot et al. 1982, Proposition 2.5.2].) □

This proves (3c) of Theorem 2.15, because the local freeness of $R^b f^{\text{tor}}_*(\mathcal{O}_{\mathsf{N}^{\text{tor}}})$ has been established in Section 4B for every integer $b \geq 0$.

4D. *Gauss–Manin connections with log poles.* In Section 3C, we proved the log smoothness of $f^{\text{tor}} : \mathsf{N}^{\text{tor}} \to \mathsf{M}^{\text{tor}}_{\mathcal{H}}$ by verifying Lemma 3.11. For simplicity, let us set

$$\overline{\Omega}^1_{\mathsf{M}^{\text{tor}}_{\mathcal{H}}/\mathsf{S}_0} := \Omega^1_{\mathsf{M}^{\text{tor}}_{\mathcal{H}}/\mathsf{S}_0}[d\log\infty] \quad \text{and} \quad \overline{\Omega}^1_{\mathsf{N}^{\text{tor}}/\mathsf{S}_0} := \Omega^1_{\mathsf{N}^{\text{tor}}/\mathsf{S}_0}[d\log\infty].$$

Then (3.12) can be rewritten as the exact sequence

$$0 \to (f^{\text{tor}})^*(\overline{\Omega}^1_{\mathsf{M}^{\text{tor}}_{\mathcal{H}}/\mathsf{S}_0}) \to \overline{\Omega}^1_{\mathsf{N}^{\text{tor}}/\mathsf{S}_0} \to \overline{\Omega}^1_{\mathsf{N}^{\text{tor}}/\mathsf{M}^{\text{tor}}_{\mathcal{H}}} \to 0, \quad (4.39)$$

which induces the *Koszul filtration* [Katz 1972, 1.2, 1.3]

$$\mathtt{K}^a(\overline{\Omega}^\bullet_{\mathsf{N}^{\mathrm{tor}}/\mathsf{S}_0}) := \mathrm{image}(\overline{\Omega}^{\bullet-a}_{\mathsf{N}^{\mathrm{tor}}/\mathsf{S}_0} \otimes_{\mathcal{O}_\mathsf{N}} (f^{\mathrm{tor}})^*(\overline{\Omega}^a_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}/\mathsf{S}_0}) \to \overline{\Omega}^\bullet_{\mathsf{N}^{\mathrm{tor}}/\mathsf{S}_0})$$

on $\overline{\Omega}^\bullet_{\mathsf{N}^{\mathrm{tor}}/\mathsf{S}_0}$, with graded pieces $\mathrm{Gr}^a_{\mathtt{K}}(\overline{\Omega}^\bullet_{\mathsf{N}^{\mathrm{tor}}/\mathsf{S}_0}) \cong \overline{\Omega}^{\bullet-a}_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}} \otimes_{\mathcal{O}_\mathsf{N}} (f^{\mathrm{tor}})^*(\overline{\Omega}^a_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}/\mathsf{S}_0})$.

On the other hand, we have the Hodge filtration

$$\mathtt{F}^a(\overline{\Omega}^\bullet_{\mathsf{N}^{\mathrm{tor}}/\mathsf{S}_0}) := \overline{\Omega}^{\bullet\geq a}_{\mathsf{N}^{\mathrm{tor}}/\mathsf{S}_0}$$

on $\overline{\Omega}^\bullet_{\mathsf{N}^{\mathrm{tor}}/\mathsf{S}_0}$, giving the Hodge filtration

$$\mathtt{F}^a(\underline{H}^i_{\text{log-dR}}(\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}})) := \mathrm{image}(R^i f^{\mathrm{tor}}_*(\mathtt{F}^a(\overline{\Omega}^\bullet_{\mathsf{N}^{\mathrm{tor}}/\mathsf{S}_0})) \to R^i f^{\mathrm{tor}}_*(\overline{\Omega}^\bullet_{\mathsf{N}^{\mathrm{tor}}/\mathsf{S}_0}))$$

on $\underline{H}^i_{\text{log-dR}}(\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}})$. By applying $R^\bullet f^{\mathrm{tor}}_*$ to the short exact sequence

$$0 \to \overline{\Omega}^{\bullet-1}_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}} \otimes_{\mathcal{O}_\mathsf{N}} (f^{\mathrm{tor}})^*(\overline{\Omega}^1_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}/\mathsf{S}_0}) \to \mathtt{K}^2/\mathtt{K}^0 \to \overline{\Omega}^\bullet_{\mathsf{N}^{\mathrm{tor}}/\mathsf{S}_0} \to 0, \tag{4.40}$$

we obtain in the long exact sequence the connecting homomorphisms

$$\begin{aligned}&\underline{H}^i_{\text{log-dR}}(\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}) = R^i f^{\mathrm{tor}}_*(\overline{\Omega}^\bullet_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}})\\ &\xrightarrow{\nabla} R^{i+1} f^{\mathrm{tor}}_*(\overline{\Omega}^{\bullet-1}_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}} \otimes_{\mathcal{O}_\mathsf{N}} \overline{\Omega}^1_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}/\mathsf{S}_0}) \cong \underline{H}^i_{\text{log-dR}}(\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}) \otimes_{\mathcal{O}_{\mathsf{M}_{\mathcal{H}}}} \overline{\Omega}^1_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}/\mathsf{S}_0}.\end{aligned} \tag{4.41}$$

As explained in [Katz 1972, 1.4], the pullback of ∇ in (4.41) to $\mathsf{M}_{\mathcal{H}}$ is nothing but the usual Gauss–Manin connection on $\underline{H}^i_{\mathrm{dR}}(\mathsf{N}/\mathsf{M}_{\mathcal{H}})$. Since the sheaves involved in (4.41) are all locally free,

$$\nabla : \underline{H}^i_{\text{log-dR}}(\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}) \to \underline{H}^i_{\text{log-dR}}(\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}) \otimes_{\mathcal{O}_{\mathsf{M}_{\mathcal{H}}}} \overline{\Omega}^1_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}/\mathsf{S}_0}$$

satisfies the necessary conditions for being an integrable connection with log poles (because its restriction to the dense subscheme $\mathsf{M}_{\mathcal{H}}$ does). If we take the $\mathtt{F}$-filtration on (4.40), we obtain

$$0 \to (\mathtt{F}^{a-1}(\overline{\Omega}^\bullet_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}) \otimes_{\mathcal{O}_\mathsf{N}} (f^{\mathrm{tor}})^*(\overline{\Omega}^1_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}/\mathsf{S}_0}))[-1] \to \mathtt{F}^a(\mathtt{K}^2/\mathtt{K}^0) \to \mathtt{F}^a(\overline{\Omega}^\bullet_{\mathsf{N}^{\mathrm{tor}}/\mathsf{S}_0}) \to 0$$

and hence the *Griffith transversality*

$$\nabla(\mathtt{F}^a(\underline{H}^i_{\text{log-dR}}(\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}))) \subset \mathtt{F}^{a-1}(\underline{H}^i_{\text{log-dR}}(\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}})) \otimes_{\mathcal{O}_{\mathsf{M}_{\mathcal{H}}}} \overline{\Omega}^1_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}/\mathsf{S}_0}$$

(as in [Katz 1972, Proposition 1.4.1.6]). This proves (3e) of Theorem 2.15.

Remark 4.42. By (3c) of Theorem 2.15, the (relative) Hodge spectral sequence

$$E_1^{a,i-a} := R^{i-a} f^{\mathrm{tor}}_*(\overline{\Omega}^a_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}) \Rightarrow \underline{H}^i_{\text{log-dR}}(\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}})$$

degenerates. Then we have $\mathrm{Gr}^a_{\mathtt{F}}(\underline{H}^i_{\text{log-dR}}(\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}})) \cong R^{i-a} f^{\mathrm{tor}}_*(\overline{\Omega}^a_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}})$, and we can conclude (as in [Katz 1972, Proposition 1.4.1.7]) that the induced morphism

$\nabla : \mathrm{Gr}^a_{\mathsf{F}}\, \underline{H}^i_{\text{log-dR}}(\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}) \to \mathrm{Gr}^{a-1}_{\mathsf{F}}\, \underline{H}^i_{\text{log-dR}}(\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}) \otimes_{\mathcal{O}_{\mathsf{M}_{\mathcal{H}}}} \overline{\Omega}^1_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}/\mathsf{S}_0}$ agrees with the morphism

$$R^{i-a} f^{\mathrm{tor}}_*(\overline{\Omega}^a_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}) \to R^{i-a+1} f^{\mathrm{tor}}_*(\overline{\Omega}^{a-1}_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}) \otimes_{\mathcal{O}_{\mathsf{M}_{\mathcal{H}}}} \overline{\Omega}^1_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}/\mathsf{S}_0}$$

defined by cup product with the Kodaira–Spencer class defined by the extension class of (4.39). We will revisit a special case of this in Section 6B.

5. Polarizations

The aim of this section is to prove (3b) and (3d) of Theorem 2.15, by studying the log extension of polarizations on the relative de Rham cohomology.

5A. *Identification of $R^b f^{\mathrm{tor}}_*(\mathcal{O}_{\mathsf{N}^{\mathrm{tor}}})$*. By Corollary 2.12, any morphism $j_Q : Q^\vee \hookrightarrow Q$ in Lemma 2.5 (together with the tautological polarization $\lambda_{\mathsf{M}_{\mathcal{H}}} : G_{\mathsf{M}_{\mathcal{H}}} \to G^\vee_{\mathsf{M}_{\mathcal{H}}}$ over $\mathsf{M}_{\mathcal{H}}$) induces canonically a polarization

$$\lambda_{\mathsf{M}_{\mathcal{H}}, j_Q} : \underline{\mathrm{Hom}}_{\mathcal{O}}(Q, G_{\mathsf{M}_{\mathcal{H}}})^\circ \to (\underline{\mathrm{Hom}}_{\mathcal{O}}(Q, G_{\mathsf{M}_{\mathcal{H}}})^\circ)^\vee$$

of degree prime to $\Box$, and hence an isomorphism

$$d\lambda_{\mathsf{M}_{\mathcal{H}}, j_Q} : \underline{\mathrm{Hom}}_{\mathcal{O}}(Q, \underline{\mathrm{Lie}}_{G_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}}}) \xrightarrow{\sim} \underline{\mathrm{Hom}}_{\mathcal{O}}(Q^\vee, \underline{\mathrm{Lie}}_{G^\vee_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}}}).$$

Therefore, it induces canonically a $\mathbb{Z}^\times_{(\Box)}$-polarization $\lambda_{\mathsf{M}_{\mathcal{H}}, j_Q} : \mathsf{N} \to \mathsf{N}^\vee$, and hence an isomorphism $d\lambda_{\mathsf{M}_{\mathcal{H}}, j_Q} : \underline{\mathrm{Lie}}_{\mathsf{N}/\mathsf{M}_{\mathcal{H}}} \to \underline{\mathrm{Lie}}_{\mathsf{N}^\vee/\mathsf{M}_{\mathcal{H}}}$. Over $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$, the morphisms $j_Q : Q^\vee \hookrightarrow Q$ and $d\lambda : \underline{\mathrm{Lie}}_{G/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}} \to \underline{\mathrm{Lie}}_{G^\vee/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}$ induce canonically an isomorphism $d\lambda_{j_Q} : \underline{\mathrm{Hom}}_{\mathcal{O}}(Q, \underline{\mathrm{Lie}}_{G/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}) \xrightarrow{\sim} \underline{\mathrm{Hom}}_{\mathcal{O}}(Q^\vee, \underline{\mathrm{Lie}}_{G^\vee/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}})$ extending $d\lambda_{\mathsf{M}_{\mathcal{H}}, j_Q} : \underline{\mathrm{Hom}}_{\mathcal{O}}(Q, \underline{\mathrm{Lie}}_{G_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}}}) \xrightarrow{\sim} \underline{\mathrm{Hom}}_{\mathcal{O}}(Q^\vee, \underline{\mathrm{Lie}}_{G^\vee_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}}})$.

Let us define $\overline{\underline{\mathrm{Der}}}_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}} := \underline{\mathrm{Hom}}_{\mathcal{O}_{\mathsf{N}^{\mathrm{tor}}}}(\overline{\Omega}^1_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}, \mathcal{O}_{\mathsf{N}^{\mathrm{tor}}})$. Its restriction to $\mathsf{M}_{\mathcal{H}}$ can be canonically identified with $\underline{\mathrm{Der}}_{\mathsf{N}/\mathsf{M}_{\mathcal{H}}} := \underline{\mathrm{Hom}}_{\mathcal{O}_{\mathsf{N}}}(\Omega^1_{\mathsf{N}/\mathsf{M}_{\mathcal{H}}}, \mathcal{O}_{\mathsf{N}})$.

Let us denote by $\jmath : \mathsf{M}_{\mathcal{H}} \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$ the canonical open immersion. Then we have the commutative diagram

$$\begin{tikzcd}
f^{\mathrm{tor}}_*(\overline{\underline{\mathrm{Der}}}_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}) \arrow[r, equal, "\sim", "\text{can.}"'] \arrow[d, hook, "\text{res.}"'] & \underline{\mathrm{Hom}}_{\mathcal{O}}(Q, \underline{\mathrm{Lie}}_{G_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}}}) \arrow[d, hook, "\text{res.}"] \arrow[ddd, bend left=60, "d\lambda_{j_Q}"] \\
\jmath_*(f_*(\underline{\mathrm{Der}}_{\mathsf{N}/\mathsf{M}_{\mathcal{H}}})) \arrow[r, equal, "\sim", "\text{can.}"'] \arrow[d, dotted] & \jmath_*(\underline{\mathrm{Hom}}_{\mathcal{O}}(Q, \underline{\mathrm{Lie}}_{G_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}}})) \arrow[d, "\jmath_*(d\lambda_{\mathsf{M}_{\mathcal{H}}, j_Q})"] \\
\jmath_*(R^1 f_*(\mathcal{O}_{\mathsf{N}})) \arrow[r, equal, "\sim", "\text{can.}"'] & \jmath_*(\underline{\mathrm{Hom}}_{\mathcal{O}}(Q, \underline{\mathrm{Lie}}_{G^\vee_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}}})) \\
R^1 f^{\mathrm{tor}}_*(\mathcal{O}_{\mathsf{N}^{\mathrm{tor}}}) \arrow[u, hook, "\text{res.}"] & \underline{\mathrm{Hom}}_{\mathcal{O}}(Q, \underline{\mathrm{Lie}}_{G^\vee/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}) \arrow[u, hook, "\text{res.}"']
\end{tikzcd} \tag{5.1}$$

of sheaves over $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$, with the dotted arrow induced by $\jmath_*(d\lambda_{\mathsf{M}_{\mathcal{H}},j_Q})$. By abuse of notation, let us denote the dotted arrow also by $\jmath_*(d\lambda_{\mathsf{M}_{\mathcal{H}},j_Q})$. We have the following simple observation:

Lemma 5.2. *If $\jmath_*(d\lambda_{\mathsf{M}_{\mathcal{H}},j_Q})$ maps the image of the canonical injection*

$$f^{\mathrm{tor}}_*(\overline{\underline{\mathrm{Der}}}_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}) \hookrightarrow \jmath_*(f_*(\underline{\mathrm{Der}}_{\mathsf{N}/\mathsf{M}_{\mathcal{H}}}))$$

***isomorphically** to the image of the canonical injection*

$$R^1 f^{\mathrm{tor}}_*(\mathcal{O}_{\mathsf{N}^{\mathrm{tor}}}) \hookrightarrow \jmath_*(R^1 f_*(\mathcal{O}_{\mathsf{N}})),$$

then (5.1) *induces the desired canonical isomorphism*

$$R^1 f^{\mathrm{tor}}_*(\mathcal{O}_{\mathsf{N}^{\mathrm{tor}}}) \cong \underline{\mathrm{Hom}}_{\mathcal{O}}(Q, \underline{\mathrm{Lie}}_{G^\vee/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}) \tag{5.3}$$

extending the canonical isomorphism $R^1 f_*(\mathcal{O}_{\mathsf{N}}) \cong \underline{\mathrm{Hom}}_{\mathcal{O}}(Q, \underline{\mathrm{Lie}}_{G^\vee_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}}})$ over $\mathsf{M}_{\mathcal{H}}$.

Remark 5.4. The question is whether the assumption of Lemma 5.2 can be satisfied. Since this is a question about morphisms between locally free sheaves over the normal base scheme $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$, it suffices to verify the statement after localizations at points of codimension one. Therefore, since the statement is tautologically true over $\mathsf{M}_{\mathcal{H}}$, it suffices to verify it over $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}} \otimes_{\mathbb{Z}} \mathbb{Q}$.

5B. *Logarithmic extension of polarizations.* By construction (see Section 3A), $\widetilde{X}^\vee(1) \cong \mathrm{Hom}_{\mathcal{O}}(\widetilde{X}, \mathrm{Diff}^{-1}(1))$ is the submodule Q_{-2} of $Q^\vee \otimes_{\mathbb{Z}} \mathbb{Z}_{(\Box)}(1)$, and $\widetilde{Y}$ is the submodule Q_0 of $Q \otimes_{\mathbb{Z}} \mathbb{Z}_{(\Box)}$. Therefore, the embedding $j_Q : Q^\vee \hookrightarrow Q$ corresponds to an element $\tilde{\ell}_{j_Q}$ of $\mathbf{S}_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}}} \otimes_{\mathbb{Z}} \mathbb{Z}_{(\Box)}$. The positive definiteness of the induced pairing $\langle j_Q^{-1}(\,\cdot\,),\,\cdot\,\rangle_Q$ then translates to the strong positivity condition that $\langle \tilde{\ell}_{j_Q}, y\rangle > 0$ for any $y \in \mathbf{P}_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}}} - \{0\}$. By replacing j_Q with a multiple by a positive integer prime to $\Box$, we may and we shall assume that $\tilde{\ell}_{j_Q} \in \mathbf{S}_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}}}$ (without altering the above strong positivity condition). Then we obtain an invertible sheaf $\Psi_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\tilde{\delta}_{\widetilde{\mathcal{H}}}}(\tilde{\ell}_{j_Q})$ over the abelian scheme $\mathsf{N} \to \mathsf{M}_{\mathcal{H}}$. Note that $\tilde{\ell}_{j_Q} \in \tilde{\sigma}_0^\vee$.

Lemma 5.5. *The invertible sheaf $\Psi_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\tilde{\delta}_{\widetilde{\mathcal{H}}}}(\tilde{\ell}_{j_Q})$ is relatively ample over $\mathsf{M}_{\mathcal{H}}$, and induces twice of a $\mathbb{Z}^\times_{(\Box)}$-polarization $\lambda_{\Psi_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\tilde{\delta}_{\widetilde{\mathcal{H}}}}(\tilde{\ell}_{j_Q})} : \mathsf{N} \to \mathsf{N}^\vee$ (namely a $\mathbb{Z}^\times_{(\Box)}$-isogeny whose sufficiently divisible positive multiple is a polarization). Under the canonical isomorphisms in Corollary 2.13, the induced morphism*

$$d\lambda_{\Psi_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\tilde{\delta}_{\widetilde{\mathcal{H}}}}(\tilde{\ell}_{j_Q})} : \underline{\mathrm{Lie}}_{\mathsf{N}/\mathsf{M}_{\mathcal{H}}} \to \underline{\mathrm{Lie}}_{\mathsf{N}^\vee/\mathsf{M}_{\mathcal{H}}}$$

is twice a positive $\mathbb{Z}^\times_{(\Box)}$-multiple of

$$d\lambda_{\mathsf{M}_{\mathcal{H}},j_Q} : \underline{\mathrm{Hom}}_{\mathcal{O}}(Q, \underline{\mathrm{Lie}}_{G_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}}}) \xrightarrow{\sim} \underline{\mathrm{Hom}}_{\mathcal{O}}(Q^\vee, \underline{\mathrm{Lie}}_{G^\vee_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}}}).$$

In particular, $d\lambda_{\Psi_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\tilde{\delta}_{\widetilde{\mathcal{H}}}}(\tilde{\ell}_{j_Q})}$ is an isomorphism over $\mathsf{M}_{\mathcal{H}} \otimes_{\mathbb{Z}} \mathbb{Q}$.

Proof. Just note that the morphism $\lambda_{\Psi_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\widetilde{\delta}_{\widetilde{\mathcal{H}}}}(\tilde{\ell}_{jQ})}$ is twice a positive $\mathbb{Z}^{\times}_{(\Box)}$-multiple of the $\mathbb{Z}^{\times}_{(\Box)}$-polarization $\lambda_{\mathsf{M}_{\mathcal{H}},jQ}$ in Corollary 2.12. $\Box$

The invertible sheaf $\Psi_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\widetilde{\delta}_{\widetilde{\mathcal{H}}}}(\tilde{\ell}_{jQ})$ over N defines a global section of $R^1 f_*(\mathcal{O}^{\times}_{\mathsf{N}})$, and the morphism

$$d\log : \mathcal{O}^{\times}_{\mathsf{N}} \to \Omega^1_{\mathsf{N}/\mathsf{M}_{\mathcal{H}}}, \quad a \mapsto a^{-1}da$$

induces a global section $D_{\tilde{\ell}_{jQ}} = d\log(\Psi_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\widetilde{\delta}_{\widetilde{\mathcal{H}}}}(\tilde{\ell}_{jQ}))$ of $R^1 f_*(\Omega^1_{\mathsf{N}/\mathsf{M}_{\mathcal{H}}})$. Then it is standard (cf. [Lan 2008, Proposition 2.1.5.14]) that the cup product with $D_{\tilde{\ell}_{jQ}}$ induces a composition of morphisms

$$f_*(\underline{\mathrm{Der}}_{\mathsf{N}/\mathsf{M}_{\mathcal{H}}}) \overset{\cup D_{\tilde{\ell}_{jQ}}}{\to} R^1 f_*(\underline{\mathrm{Der}}_{\mathsf{N}/\mathsf{M}_{\mathcal{H}}} \otimes_{\mathcal{O}_{\mathsf{N}}} \Omega^1_{\mathsf{N}/\mathsf{M}_{\mathcal{H}}}) \overset{\text{can.}}{\to} R^1 f_*(\mathcal{O}_{\mathsf{N}}),$$

and that this morphism $f_*(\underline{\mathrm{Der}}_{\mathsf{N}/\mathsf{M}_{\mathcal{H}}}) \to R^1 f_*(\mathcal{O}_{\mathsf{N}})$ can be identified with the morphism $d\lambda_{\Psi_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\widetilde{\delta}_{\widetilde{\mathcal{H}}}}(\tilde{\ell}_{jQ})}$ under the canonical isomorphisms

$$f_*(\underline{\mathrm{Der}}_{\mathsf{N}/\mathsf{M}_{\mathcal{H}}}) \cong \underline{\mathrm{Lie}}_{\mathsf{N}/\mathsf{M}_{\mathcal{H}}} \quad \text{and} \quad R^1 f_*(\mathcal{O}_{\mathsf{N}}) \cong \underline{\mathrm{Lie}}_{\mathsf{N}^{\vee}/\mathsf{M}_{\mathcal{H}}}.$$

The first question is whether we can extend the morphism $f_*(\underline{\mathrm{Der}}_{\mathsf{N}/\mathsf{M}_{\mathcal{H}}}) \to R^1 f_*(\mathcal{O}_{\mathsf{N}})$ to $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$; and the second question is whether the extended morphism is an isomorphism, at least in codimension one.

A naive approach is to extend the invertible sheaf $\Psi_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\widetilde{\delta}_{\widetilde{\mathcal{H}}}}(\tilde{\ell}_{jQ})$ to $\mathsf{N}^{\mathrm{tor}}$. Since $\mathsf{N}^{\mathrm{tor}}$ is projective and smooth over $\mathsf{S}_0 = \mathrm{Spec}(\mathcal{O}_{F_0,(\Box)})$, it is locally noetherian and locally factorial. Then [EGA IV_4 1967, 21.6.11] implies that the canonical restriction morphism $\mathrm{Pic}(\mathsf{N}^{\mathrm{tor}}) \to \mathrm{Pic}(\mathsf{N})$ is *surjective*.

However, since $f^{\mathrm{tor}} : \mathsf{N}^{\mathrm{tor}} \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$ is not smooth, we have little control on the canonical restriction morphism $R^1 f^{\mathrm{tor}}_*(\Omega^1_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}) \overset{\text{res.}}{\to} \jmath_*(R^1 f_*(\Omega^1_{\mathsf{N}/\mathsf{M}_{\mathcal{H}}}))$, and there is no obvious reason that the image of the class defined by any extension of $\Psi_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\widetilde{\delta}_{\widetilde{\mathcal{H}}}}(\tilde{\ell}_{jQ})$ should induce an *isomorphism* extending $d\lambda_{\Psi_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\widetilde{\delta}_{\widetilde{\mathcal{H}}}}(\tilde{\ell}_{jQ})}$ (at least) in codimension one. (This is mentioned in [Faltings and Chai 1990, Chapter VI, end of §2], but with no details.)

An alternative approach is to consider the canonical restriction morphism

$$R^1 f^{\mathrm{tor}}_*(\overline{\Omega}^1_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}) \overset{\text{res.}}{\to} \jmath_*(R^1 f_*(\Omega^1_{\mathsf{N}/\mathsf{M}_{\mathcal{H}}})). \tag{5.6}$$

By Lemma 4.29, and by (3a) of Theorem 2.15, $R^1 f^{\mathrm{tor}}_*(\overline{\Omega}^1_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}})$ is locally free over $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$. Therefore, the morphism (5.6) is injective.

Remark 5.7. The use of $R^1 f^{\mathrm{tor}}_*(\overline{\Omega}^1_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}})$ is inspired by Kato's idea of (relative) *log Picard groups* mentioned in [Illusie 1994, 3.3]. An application of this idea has been carried out in [Olsson 2004].

So far we have refrained from introducing the log structures (because they had not been necessary), but they are needed (at least formally) here. We shall adopt a notation slightly different from those of [Kato 1989; Illusie 1994]. Let $\breve{\jmath} : \mathsf{N} \to \mathsf{N}^{\mathrm{tor}}$ denote the canonical open immersion. Then the canonical (fine) log structure on $\mathsf{N}^{\mathrm{tor}}$ (which we have been using so far) given by $\mathsf{N}^{\mathrm{tor}} - \mathsf{N}$ (with its reduced structure) can be defined explicitly as the sheaf of monoids $\overline{\mathbb{O}}^{\times}_{\mathsf{N}^{\mathrm{tor}}} := \mathbb{O}_{\mathsf{N}^{\mathrm{tor}}} \cap \breve{\jmath}_* \mathbb{O}^{\times}_{\mathsf{N}}$ (sheafification of the obvious presheaf), with associated sheaf of groups $\overline{\mathbb{O}}^{\times,\mathrm{gp}}_{\mathsf{N}^{\mathrm{tor}}}$. Clearly, the restriction of $\overline{\mathbb{O}}^{\times,\mathrm{gp}}_{\mathsf{N}^{\mathrm{tor}}}$ to N is canonically isomorphic to $\mathbb{O}^{\times}_{\mathsf{N}}$.

Definition 5.8. A relative log invertible sheaf over $f^{\mathrm{tor}} : \mathsf{N}^{\mathrm{tor}} \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$ is a global section of $R^1 f^{\mathrm{tor}}_* (\overline{\mathbb{O}}^{\times,\mathrm{gp}}_{\mathsf{N}^{\mathrm{tor}}})$.

Since we do not assume that f^{tor} is flat (or log integral), the appropriate interpretation of relative log invertible sheaves can be quite delicate (and beyond this article).

Lemma 5.9. *To define a global section of $R^1 f^{\mathrm{tor}}_* (\overline{\mathbb{O}}^{\times,\mathrm{gp}}_{\mathsf{N}^{\mathrm{tor}}})$, it suffices to have the following data*:

(1) *A collection of schemes $\bar{U}_\alpha$ over $\mathsf{N}^{\mathrm{tor}}$ forming an étale covering. We shall denote the fiber product $\bar{U}_\alpha \times_{\mathsf{N}^{\mathrm{tor}}} \bar{U}_\beta$ (i.e., "intersection" in the étale topology) by $\bar{U}_{\alpha\beta}$, denote $\bar{U}_{\alpha\beta}|_{\mathsf{N}} := \bar{U}_{\alpha\beta} \times_{\mathsf{N}^{\mathrm{tor}}} \mathsf{N}$ by $U_{\alpha\beta}$, and use similar notations for higher fiber products.*

(2) *A usual invertible sheaf $\mathcal{L}_\alpha$ over each $\bar{U}_\alpha$.*

(3) *A comparison isomorphism $\mathcal{L}_\alpha|_{U_{\alpha\beta}} \cong \mathcal{L}_\beta|_{U_{\alpha\beta}}$ over each $U_{\alpha\beta}$, satisfying the usual cocycle condition over triple fiber products $U_{\alpha\beta\gamma}$.*

Proof. Since the restriction morphism $\overline{\mathbb{O}}^{\times,\mathrm{gp}}_{\mathsf{N}^{\mathrm{tor}}}(\bar{U}_{\alpha\beta}) \to \overline{\mathbb{O}}^{\times,\mathrm{gp}}_{\mathsf{N}^{\mathrm{tor}}}(U_{\alpha\beta}) \cong \mathbb{O}^{\times}_{\mathsf{N}}(U_{\alpha\beta})$ is a bijection when the image of $\bar{U}_{\alpha\beta}$ in $\mathsf{N}^{\mathrm{tor}}$ is sufficiently small, the data above define a section of $H^1(\mathsf{N}^{\mathrm{tor}}, \overline{\mathbb{O}}^{\times,\mathrm{gp}}_{\mathsf{N}^{\mathrm{tor}}})$, which then defines a section of $H^0(\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}, R^1 f^{\mathrm{tor}}_* (\overline{\mathbb{O}}^{\times,\mathrm{gp}}_{\mathsf{N}^{\mathrm{tor}}}))$ by the Leray spectral sequence in low degrees. (See [Godement 1958, I 4.5.1].) □

In the construction of toroidal compactifications in [Lan 2008, §6.3.3] (following [Faltings and Chai 1990, Chapter IV, §5]), there is a strata-preserving étale covering $\widetilde{\mathsf{U}} \to \widetilde{\mathsf{M}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}}}$ (serving as an étale presentation for the algebraic stack $\widetilde{\mathsf{M}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}}}$), where $\widetilde{\mathsf{U}}$ is a finite union of the so-called *good algebraic models* of $\widetilde{\mathsf{M}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}}}$. (See [Lan 2008, Definition 6.3.2.5].) By taking the closures of the $[(\widetilde{\Phi}_{\widetilde{\mathcal{H}}}, \tilde{\delta}_{\widetilde{\mathcal{H}}}, \widetilde{\sigma})]$-strata, we obtain a strata-preserving étale covering $\breve{\mathsf{U}} \to \mathsf{N}^{\mathrm{tor}}$, with strata labeled by triples $[(\breve{\Phi}_{\widetilde{\mathcal{H}}}, \breve{\delta}_{\widetilde{\mathcal{H}}}, \breve{\tau})]$ having $[(\widetilde{\Phi}_{\widetilde{\mathcal{H}}}, \tilde{\delta}_{\widetilde{\mathcal{H}}}, \widetilde{\sigma})]$ as a face.

Each connected component $\bar{U}_\alpha$ of $\breve{\mathsf{U}}$ is given by the closure of the $[(\widetilde{\Phi}_{\widetilde{\mathcal{H}}}, \tilde{\delta}_{\widetilde{\mathcal{H}}}, \widetilde{\sigma})]$-stratum in a so-called good algebraic $(\breve{\Phi}_{\widetilde{\mathcal{H}}}, \breve{\delta}_{\widetilde{\mathcal{H}}}, \breve{\tau})$-model $\widetilde{U}_\alpha = \mathrm{Spec}(\widetilde{R}_\alpha) \to \widetilde{\mathsf{M}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}}}$, where $(\breve{\Phi}_{\widetilde{\mathcal{H}}}, \breve{\delta}_{\widetilde{\mathcal{H}}}, \breve{\tau})$ is a representative of some $[(\breve{\Phi}_{\widetilde{\mathcal{H}}}, \breve{\delta}_{\widetilde{\mathcal{H}}}, \breve{\tau})]$ having $[(\widetilde{\Phi}_{\widetilde{\mathcal{H}}}, \tilde{\delta}_{\widetilde{\mathcal{H}}}, \widetilde{\sigma})]$

as a face (cf. second property in [Lan 2008, Definition 6.3.2.5]), which we may assume to satisfy $\breve{\tau} \in \Sigma_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\sigma},\tau}$. (See Section 4A. There are usually many α for each $[(\breve{\Phi}_{\widetilde{\mathcal{H}}}, \breve{\delta}_{\widetilde{\mathcal{H}}}, \breve{\tau})]$.) Then we also have a strata-preserving étale morphism $\bar{U}_\alpha \to (\widetilde{\Xi}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}})_{\breve{\sigma}}(\breve{\tau})$, which we shall call a *good algebraic* $(\breve{\Phi}_{\widetilde{\mathcal{H}}}, \breve{\delta}_{\widetilde{\mathcal{H}}}, \breve{\tau})$*-model* of $\mathsf{N}^{\mathrm{tor}}$. The (open) $[(\widetilde{\Phi}_{\widetilde{\mathcal{H}}}, \tilde{\delta}_{\widetilde{\mathcal{H}}}, \widetilde{\sigma})]$-stratum in $\bar{U}_\alpha$ is exactly the open subscheme $U_\alpha := \bar{U}_\alpha \times_{\mathsf{N}^{\mathrm{tor}}} \mathsf{N}$ of $\bar{U}_\alpha$.

Lemma 5.10. *Suppose that, for each $\breve{\tau} \in \Sigma_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\sigma},\tau}$, we have chosen an element $\breve{\ell}_{jQ,\breve{\tau}}$ in $\breve{\tau}_0^\vee$ that is mapped to $\tilde{\ell}_{jQ}$ in $\widetilde{\sigma}_0^\vee$ under the second morphism in* (3.6), *and that $\breve{\ell}_{jQ,\gamma\breve{\tau}} = \gamma\breve{\ell}_{jQ,\breve{\tau}}$ for any $\gamma \in \Gamma_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}}$. (Note that the choice of $\breve{\ell}_{jQ,\breve{\tau}}$ is unique only up to translation by $\breve{\sigma}^\perp$.) Let $\breve{\mathsf{U}} \to \mathsf{N}^{\mathrm{tor}}$ be any strata-preserving étale covering formed by a finite union of good algebraic models. Then the choices of $\{\breve{\ell}_{jQ,\breve{\tau}}\}_{\breve{\tau}\in\Sigma_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\sigma},\tau}}$ and $\breve{\mathsf{U}}$ determine a relative log invertible sheaf $\overline{\mathcal{L}}$ over $\mathsf{N}^{\mathrm{tor}} \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$ extending the rigidified invertible sheaf $\Psi_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\tilde{\delta}_{\widetilde{\mathcal{H}}}}(\tilde{\ell}_{jQ})$ over N, in the following sense: For each good algebraic $(\breve{\Phi}_{\widetilde{\mathcal{H}}}, \breve{\delta}_{\widetilde{\mathcal{H}}}, \breve{\tau})$-model $\bar{U}_\alpha$ of $\mathsf{N}^{\mathrm{tor}}$, with $\breve{\tau} \in \Sigma_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\sigma},\tau}$, let $\mathcal{L}_\alpha$ denote the pullback of $\Psi_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}(\breve{\ell}_{jQ,\breve{\tau}})$ under the composition $\bar{U}_\alpha \to (\widetilde{\Xi}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}})_{\breve{\sigma}}(\breve{\tau}) \to \widetilde{C}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}$. Then $\mathcal{L}_\alpha|_{U_\alpha}$ is canonically isomorphic to the pullback of $\Psi_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\tilde{\delta}_{\widetilde{\mathcal{H}}}}(\tilde{\ell}_{jQ})$ (from $\mathsf{N} \cong \widetilde{C}_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\tilde{\delta}_{\widetilde{\mathcal{H}}}}$) to U_α. Furthermore, the collection $\{(U_\alpha, \mathcal{L}_\alpha)\}$ satisfies the requirements in Lemma 5.9, and defines a log invertible sheaf as in Definition 5.8.*

Proof. Let $(\widetilde{G}, \widetilde{\lambda}, \tilde{i}, \widetilde{\alpha}_{\widetilde{\mathcal{H}}})$ be the degenerating family of type $\widetilde{\mathsf{M}}_{\widetilde{\mathcal{H}}}$ over $\widetilde{\mathsf{M}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}}}$. Let $\underline{B}(\widetilde{G}) : \underline{\mathbf{S}}_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}}(\widetilde{G})} \to \underline{\mathrm{Inv}}(\widetilde{\mathsf{M}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}}})$ be constructed as in [Lan 2008, Construction 6.3.1.1]. If $\widetilde{U}_\alpha$ is a good algebraic $(\breve{\Phi}_{\widetilde{\mathcal{H}}}, \breve{\delta}_{\widetilde{\mathcal{H}}}, \breve{\tau})$-model, then for any $\breve{\ell} \in \mathbf{S}_{\breve{\Phi}_{\widetilde{\mathcal{H}}}}$, the invertible sheaf $\underline{B}(\widetilde{G})(\widetilde{U}_\alpha)(\breve{\ell})$ over $\widetilde{U}_\alpha$ is canonically isomorphic to the pullback of $\Psi_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}(\breve{\ell})$ under the composition $\widetilde{U}_\alpha \to (\widetilde{\Xi}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}})_{\breve{\sigma}}(\breve{\tau}) \to \widetilde{C}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}$ (cf. third property in [Lan 2008, Definition 6.3.2.5]).

Given that $\underline{B}(\widetilde{G})$ is defined over $\widetilde{\mathsf{M}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}}}$ and functorial with respect to pullback morphisms $\widetilde{U}_{\alpha\beta} \to \widetilde{U}_\alpha$, the restriction of the pullback of $\Psi_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}(\breve{\ell}_{jQ,\breve{\tau}})$ to the $[(\widetilde{\Phi}_{\widetilde{\mathcal{H}}}, \tilde{\delta}_{\widetilde{\mathcal{H}}}, \widetilde{\sigma})]$-stratum of $\widetilde{U}_\alpha$ is isomorphic to the pullback of $\Psi_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\tilde{\delta}_{\widetilde{\mathcal{H}}}}(\tilde{\ell}_{jQ})$ when $(\widetilde{\Phi}_{\widetilde{\mathcal{H}}}, \tilde{\delta}_{\widetilde{\mathcal{H}}}, \widetilde{\sigma})$ is a face of $[(\breve{\Phi}_{\widetilde{\mathcal{H}}}, \breve{\delta}_{\widetilde{\mathcal{H}}}, \breve{\tau})]$. In other words, $\mathcal{L}_\alpha|_{U_\alpha}$ is isomorphic to the pullback of $\Psi_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\tilde{\delta}_{\widetilde{\mathcal{H}}}}(\tilde{\ell}_{jQ})$ over each U_α. Since the isomorphisms $\mathcal{L}_\alpha|_{U_{\alpha\beta}} \cong \mathcal{L}_\beta|_{U_{\alpha\beta}}$ induced by such identifications satisfy the cocycle condition (because $\Psi_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\tilde{\delta}_{\widetilde{\mathcal{H}}}}(\tilde{\ell}_{jQ})$ is defined on N), the claim follows, as desired. □

Remark 5.11. Any (usual) invertible sheaf over $\mathsf{N}^{\mathrm{tor}}$ extending $\Psi_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\tilde{\delta}_{\widetilde{\mathcal{H}}}}(\tilde{\ell}_{jQ})$ satisfies the requirements in Lemma 5.9 trivially. The point of Lemma 5.10 is that it provides an explicit extension of $\Psi_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\tilde{\delta}_{\widetilde{\mathcal{H}}}}(\tilde{\ell}_{jQ})$ (useful for our later argument) over an étale covering of $\mathsf{N}^{\mathrm{tor}}$. (We do not have such an explicit description of a global invertible sheaf extension over $\mathsf{N}^{\mathrm{tor}}$.)

Definition 5.12. To any relative log invertible sheaf $\overline{\mathcal{L}}$ over $\mathsf{N}^{\mathrm{tor}} \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$ defined by a global section of $R^1 f^{\mathrm{tor}}_*(\overline{\mathbb{O}}^{\times,\mathrm{gp}}_{\mathsf{N}^{\mathrm{tor}}})$, we define $d\log(\overline{\mathcal{L}})$ to be the image of $\overline{\mathcal{L}}$ under the canonical morphism $R^1 f^{\mathrm{tor}}_*(\overline{\mathbb{O}}^{\times,\mathrm{gp}}_{\mathsf{N}^{\mathrm{tor}}}) \to R^1 f^{\mathrm{tor}}_*(\overline{\Omega}^1_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}})$ induced by the canonical morphism $d\log : \overline{\mathbb{O}}^{\times,\mathrm{gp}}_{\mathsf{N}^{\mathrm{tor}}} \to \overline{\Omega}^1_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}$.

Corollary 5.13. *There exists a* (*unique*) *global section $D^{\mathrm{tor}}_{\tilde{\ell}_{j_Q}}$ of $R^1 f^{\mathrm{tor}}_*(\overline{\Omega}^1_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}})$ whose image under the canonical injection* (5.6) *is $\jmath_*(D_{\tilde{\ell}_{j_Q}})$, and which satisfies $D^{\mathrm{tor}}_{\tilde{\ell}_{j_Q}} = d\log(\overline{\mathcal{L}})$ for any $\overline{\mathcal{L}}$ constructed in Lemma 5.10* (*with any choices of $\check{\ell}_{j_Q,\check{\tau}}$'s*).

Proof. Existence is clear because there is always some (usual) invertible sheaf over $\mathsf{N}^{\mathrm{tor}}$ extending $\Psi_{\tilde{\Phi}_{\tilde{\mathcal{H}}},\tilde{\delta}_{\tilde{\mathcal{H}}}}(\tilde{\ell}_{j_Q})$ (by [EGA IV$_4$ 1967, 21.6.11], since $\mathsf{N}^{\mathrm{tor}}$ is locally noetherian and locally factorial, as mentioned above). Uniqueness is clear because (5.6) is injective. Once we know the unique existence of $D^{\mathrm{tor}}_{\tilde{\ell}_{j_Q}}$, it has to agree with $d\log(\overline{\mathcal{L}})$ for any $\overline{\mathcal{L}}$ constructed in Lemma 5.10. □

Thus we are led to state the following:

Proposition 5.14. *Cup product with the global section $D^{\mathrm{tor}}_{\tilde{\ell}_{j_Q}}$ of $R^1 f^{\mathrm{tor}}_*(\overline{\Omega}^1_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}})$ in Corollary 5.13 induces a composition of morphisms*

$$f^{\mathrm{tor}}_*(\underline{\overline{\mathrm{Der}}}_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}) \xrightarrow{\cup D^{\mathrm{tor}}_{\tilde{\ell}_{j_Q}}} R^1 f^{\mathrm{tor}}_*(\underline{\overline{\mathrm{Der}}}_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}} \otimes_{\mathbb{O}_{\mathsf{N}^{\mathrm{tor}}}} \overline{\Omega}^1_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}) \xrightarrow{\text{can.}} R^1 f^{\mathrm{tor}}_*(\mathbb{O}_{\mathsf{N}^{\mathrm{tor}}}). \quad (5.15)$$

This composition is an isomorphism over $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}} \otimes_{\mathbb{Z}} \mathbb{Q}$. (*By Lemma 5.2 and Remark 5.4, this implies the existence of the canonical isomorphism* (5.3).)

We will carry out the proof of Proposition 5.14 in the next subsection.

5C. *Induced morphisms over formal fibers.* We fix the choices of $\{\check{\ell}_{j_Q,\check{\tau}}\}_{\check{\tau}\in\Sigma_{\check{\Phi}_{\tilde{\mathcal{H}}},\check{\sigma},\tau}}$ and $\breve{\mathsf{U}}$, so that $\overline{\mathcal{L}}$ is constructed as in Lemma 5.10, and so that $D^{\mathrm{tor}}_{\tilde{\ell}_{j_Q}} = d\log(\overline{\mathcal{L}})$ as in Corollary 5.13.

Since f^{tor} is proper and the sheaves involved are all coherent, by Grothendieck's fundamental theorem [EGA III$_1$ 1961, 4.1.5], Proposition 5.14 can be verified by pulling back to formal completions along strata of $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$. Let us fix the choice of a cusp label $[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)]$ of $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$, and consider the canonical morphism

$$\imath : \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau} \cong (\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}})^{\wedge}_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}} \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}.$$

By abuse of notation, we shall also denote by $\imath^*(\,\cdot\,)$ the pullbacks of objects under pullbacks of the morphism $\imath$. We would like to show that the morphism $\imath^* f^{\mathrm{tor}}_*(\underline{\overline{\mathrm{Der}}}_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}) \to \imath^* R^1 f^{\mathrm{tor}}_*(\mathbb{O}_{\mathsf{N}^{\mathrm{tor}}})$ defined by cup product with $\imath^*(D^{\mathrm{tor}}_{\tilde{\ell}_{j_Q}})$ is an *isomorphism* over $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau} \otimes_{\mathbb{Z}} \mathbb{Q}$.

As said in Section 4A, the pullback of f^{tor} to $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau}$ can be identified with the canonical morphism $\mathfrak{X}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}},\check{\sigma},\tau}/\Gamma_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}} \to \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau}$, and $\mathfrak{X}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}},\check{\sigma},\tau}/\Gamma_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}}$ has a finite open covering by the collection $\{\mathfrak{U}_{[\check{\tau}]}\}_{[\check{\tau}]\in\Sigma_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\sigma},\tau}/\Gamma_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}}}$ of open formal subschemes. Let $\check{\tau}\in\Sigma_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\sigma},\tau}$ be a representative of $[\check{\tau}]\in\Sigma_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\sigma},\tau}/\Gamma_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}}$. For each such $\check{\tau}$, recall that the formal scheme $\mathfrak{U}_{\check{\tau}}$ is the completion of $(\widetilde{\Xi}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}})_{\check{\sigma}}(\check{\tau})$ along $U_{\check{\tau}}$. By abuse of notation, let us denote the pullback of $\Psi_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}(\check{\ell}_{jQ,\check{\tau}})$ over $\widetilde{C}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}$ to $\mathfrak{U}_{\check{\tau}}$ by the same notation. For any $\gamma\in\Gamma_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}}$, since $\check{\ell}_{jQ,\gamma\check{\tau}}=\gamma\check{\ell}_{jQ,\check{\tau}}$ (see Lemma 5.10), we have a canonical isomorphism $\gamma^*\Psi_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}(\check{\ell}_{jQ,\gamma\check{\tau}}) \xrightarrow{\sim} \Psi_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}(\check{\ell}_{jQ,\check{\tau}})$, where $\gamma:\mathfrak{U}_{\check{\tau}}\xrightarrow{\sim}\mathfrak{U}_{\gamma\check{\tau}}$ is the canonical isomorphism (see Lemma 4.6). Hence $\Psi_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}(\check{\ell}_{jQ,\check{\tau}})$ descends to an unambiguous invertible sheaf $\Psi_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}(\check{\ell}_{jQ,[\check{\tau}]})$ on $\mathfrak{U}_{[\check{\tau}]}$.

The étale covering $\check{\mathsf{U}}\to\mathsf{N}^{\mathrm{tor}}$ induces (by taking formal completion along the pullback of $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}$) a formally étale covering of $(\mathsf{N}^{\mathrm{tor}})^{\wedge}_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}}$. If $\bar{U}_\alpha$ is a good algebraic $(\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}},\check{\tau})$-model of $\mathsf{N}^{\mathrm{tor}}$, then the formal completion $(\bar{U}_\alpha)^{\wedge}_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}}$ of $\bar{U}_\alpha$ along the pullback of $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}$ is formally étale over $\mathfrak{U}_{\check{\tau}}$.

Lemma 5.16. *The pullback of $\mathcal{L}_\alpha$ to $(\bar{U}_\alpha)^{\wedge}_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}}$ is isomorphic to the pullback of $\Psi_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}(\check{\ell}_{jQ,\check{\tau}})$ from $\mathfrak{U}_{\check{\tau}}$.*

Proof. The canonical morphisms

$$(\bar{U}_\alpha)^{\wedge}_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}}\to\bar{U}_\alpha\to\mathsf{N}^{\mathrm{tor}}\quad\text{and}\quad(\bar{U}_\alpha)^{\wedge}_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}}\to\mathfrak{U}_{\check{\tau}}\to\mathsf{N}^{\mathrm{tor}}$$

are induced respectively by morphisms

$$(\widetilde{U}_\alpha)^{\wedge}_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}}\to\widetilde{U}_\alpha\to\widetilde{\mathsf{M}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}}}\quad\text{and}\quad(\widetilde{U}_\alpha)^{\wedge}_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}}\to\widetilde{\mathfrak{U}}_{\check{\tau}}\to\widetilde{\mathsf{M}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}}}$$

over $\widetilde{\mathsf{M}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}}}$. Under both these morphisms, the pullback of $(\widetilde{G},\widetilde{\lambda},\widetilde{i},\widetilde{\alpha}_{\widetilde{\mathcal{H}}})\to\widetilde{\mathsf{M}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}}}$ is canonically isomorphic to the pullback of the Mumford family (as in the proof of Proposition 4.3). Since the isomorphism class of the pullback of $\mathcal{L}_\alpha$ to $(\bar{U}_\alpha)^{\wedge}_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}}$ is determined by the pullback of $\underline{B}(\widetilde{G}):\underline{\mathbf{S}}_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}}(\widetilde{G})}\to\underline{\mathrm{Inv}}(\widetilde{\mathsf{M}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}}})$ (as in the proof of Lemma 5.10), we can pullback along $(\bar{U}_\alpha)^{\wedge}_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}}\to\mathfrak{U}_{\check{\tau}}\to\mathsf{N}^{\mathrm{tor}}$ and conclude that $\mathcal{L}_\alpha$ is isomorphic to the pullback of $\Psi_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}(\check{\ell}_{jQ,\check{\tau}})$ from $\mathfrak{U}_{\check{\tau}}$. □

By Lemma 4.29, we have

$$\imath^*f^{\mathrm{tor}}_*(\mathcal{O}_{\mathsf{N}^{\mathrm{tor}}})\cong f^{\mathrm{tor}}_*(\mathcal{O}_{(\mathsf{N}^{\mathrm{tor}})^{\wedge}_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}}})\cong H^0(\mathfrak{N}_{\check{\sigma},\tau},\underline{\mathscr{H}}^0(\mathcal{O}_{(\mathsf{N}^{\mathrm{tor}})^{\wedge}_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}}})),$$

and $\imath^*R^1f^{\mathrm{tor}}_*(\mathcal{O}_{\mathsf{N}^{\mathrm{tor}}})\cong R^1f^{\mathrm{tor}}_*(\mathcal{O}_{(\mathsf{N}^{\mathrm{tor}})^{\wedge}_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}}})$ is equipped with a decreasing filtration with (locally free) graded pieces

$$\mathrm{Gr}^0(\imath^*R^1f^{\mathrm{tor}}_*(\mathcal{O}_{\mathsf{N}^{\mathrm{tor}}}))\cong H^0(\mathfrak{N}_{\check{\sigma},\tau},\underline{\mathscr{H}}^1(\mathcal{O}_{(\mathsf{N}^{\mathrm{tor}})^{\wedge}_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}}}))$$

and

$$\mathrm{Gr}^1(\imath^*R^1f^{\mathrm{tor}}_*(\mathcal{O}_{\mathsf{N}^{\mathrm{tor}}}))\cong H^1(\mathfrak{N}_{\check{\sigma},\tau},\underline{\mathscr{H}}^0(\mathcal{O}_{(\mathsf{N}^{\mathrm{tor}})^{\wedge}_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}}})).$$

Thus, to show that (5.15) is an isomorphism over $\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}} \otimes_{\mathbb{Z}} \mathbb{Q}$, it suffices (by comparison of ranks of locally free sheaves) to show that it induces surjections from subquotients of $\imath^* f_*^{\mathrm{tor}}(\overline{\underline{\mathrm{Der}}}_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}})$ to these graded pieces over $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau} \otimes_{\mathbb{Z}} \mathbb{Q}$.

By tensoring the above filtration with $\imath^*\overline{\Omega}^1_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}}$ (and by (3.15)), we obtain a decreasing filtration on $\imath^* R^1 f_*^{\mathrm{tor}}(\overline{\Omega}^1_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}})$ with

$$\mathrm{Gr}^0(\imath^* R^1 f_*^{\mathrm{tor}}(\overline{\Omega}^1_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}})) \cong H^0(\mathfrak{N}_{\check{\sigma},\tau}, \underline{\mathscr{H}}^1(\imath^*\overline{\Omega}^1_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}}))$$

and

$$\mathrm{Gr}^1(\imath^* R^1 f_*^{\mathrm{tor}}(\overline{\Omega}^1_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}})) \cong H^1(\mathfrak{N}_{\check{\sigma},\tau}, \underline{\mathscr{H}}^0(\imath^*\overline{\Omega}^1_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}})).$$

Since $\overline{\underline{\mathrm{Der}}}_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}} \cong (f^{\mathrm{tor}})^*(\underline{\mathrm{Hom}}_{\mathcal{O}}(Q, \underline{\mathrm{Lie}}_{G_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}}}))$, we have

$$\imath^* f_*^{\mathrm{tor}}(\overline{\underline{\mathrm{Der}}}_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}}) \cong H^0(\mathfrak{N}_{\check{\sigma},\tau}, \underline{\mathscr{H}}^0(\imath^*\overline{\underline{\mathrm{Der}}}_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}})),$$

and the morphism

$$\imath^* f_*^{\mathrm{tor}}(\overline{\underline{\mathrm{Der}}}_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}}) \to H^0(\mathfrak{N}_{\check{\sigma},\tau}, \underline{\mathscr{H}}^1(\mathcal{O}_{(\mathsf{N}^{\mathrm{tor}})^{\wedge}_{\widetilde{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}}}))$$

induced by (5.15) can be identified with the morphism

$$H^0(\mathfrak{N}_{\check{\sigma},\tau}, \underline{\mathscr{H}}^0(\imath^*\overline{\underline{\mathrm{Der}}}_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}})) \to H^0(\mathfrak{N}_{\check{\sigma},\tau}, \underline{\mathscr{H}}^1(\mathcal{O}_{(\mathsf{N}^{\mathrm{tor}})^{\wedge}_{\widetilde{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}}})) \tag{5.17}$$

given by cup product with the image of $\imath^*(D^{\mathrm{tor}}_{\tilde{\ell}_{j_Q}})$ in $\mathrm{Gr}^0(\imath^* R^1 f_*^{\mathrm{tor}}(\overline{\Omega}^1_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}})) \cong H^0(\mathfrak{N}_{\check{\sigma},\tau}, \underline{\mathscr{H}}^1(\imath^*\overline{\Omega}^1_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}}))$.

For simplicity, let us define $\widetilde{\mathfrak{X}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau} := \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau} \times_{C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}} \widetilde{C}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}$. Then the structural morphism $\mathfrak{X}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}},\check{\sigma},\tau} \to \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau}$ factors as $\mathfrak{X}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}},\check{\sigma},\tau} \to \widetilde{\mathfrak{X}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau} \to \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau}$. Over $\mathfrak{X}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}},\check{\sigma},\tau}$, there is an exact sequence

$$\begin{aligned} 0 \to (\mathfrak{X}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}},\check{\sigma},\tau} \to \widetilde{C}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}})^*(\Omega^1_{\widetilde{C}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}/C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}) \\ \to \imath^*\overline{\Omega}^1_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}} \to \overline{\Omega}^1_{\mathfrak{X}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}},\check{\sigma},\tau}/\widetilde{\mathfrak{X}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau}} \to 0 \end{aligned}$$

of locally free sheaves, where $\imath^*\overline{\Omega}^1_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}} \cong \overline{\Omega}^1_{\mathfrak{X}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}},\check{\sigma},\tau}/\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau}}$. By taking duals, we obtain an exact sequence

$$\begin{aligned} 0 \to \overline{\underline{\mathrm{Der}}}_{\mathfrak{X}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}},\check{\sigma},\tau}/\widetilde{\mathfrak{X}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau}} \to \imath^*\overline{\underline{\mathrm{Der}}}_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}} \\ \to (\mathfrak{X}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}},\check{\sigma},\tau} \to \widetilde{C}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}})^*(\underline{\mathrm{Der}}_{\widetilde{C}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}/C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}) \to 0. \end{aligned}$$

We have similar sequences with $\mathfrak{X}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}},\check{\sigma},\tau}$ replaced with the locally isomorphic quotient $\mathfrak{X}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}},\check{\sigma},\tau}/\Gamma_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}}$. (For simplicity, in the notation of such differentials, we shall suppress the locally isomorphic quotients by $\Gamma_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}}$.)

Since $\Psi_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}(\check{\ell}_{j_Q,\check{\tau}})$ is the pullback of an invertible sheaf on $\widetilde{C}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}$, the image of $\imath^*(D^{\mathrm{tor}}_{\tilde{\ell}_{j_Q}})$ in $H^0(\mathfrak{N}_{\check{\sigma},\tau}, \underline{\mathscr{H}}^1(\imath^*\overline{\Omega}^1_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}))$ lies locally over each $\mathfrak{U}_{\check{\tau}}$ in the image of

$$\begin{aligned}(\mathfrak{U}_{\check{\tau}} \to C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}})^* R^1 h_*(\Omega^1_{\widetilde{C}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}/C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}})&\\ \xrightarrow{\sim} \underline{\mathscr{H}}^1((\mathfrak{U}_{\check{\tau}} \to \widetilde{C}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}})^*(\Omega^1_{\widetilde{C}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}/C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}})) &\to \underline{\mathscr{H}}^1(\imath^*\overline{\Omega}^1_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}).\end{aligned}$$

Hence (5.17) factors as

$$\begin{aligned}&H^0(\mathfrak{N}_{\check{\sigma},\tau}, \underline{\mathscr{H}}^0(\imath^*\overline{\underline{\mathrm{Der}}}_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}))\\ &\twoheadrightarrow H^0(\mathfrak{N}_{\check{\sigma},\tau}, \underline{\mathscr{H}}^0((\mathfrak{X}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}},\check{\sigma},\tau} \to \widetilde{C}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}})^*(\underline{\mathrm{Der}}_{\widetilde{C}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}/C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}})))\\ &\xrightarrow{\sim} (\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau} \to C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}})^* R^0 h_*(\underline{\mathrm{Der}}_{\widetilde{C}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}/C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}})\\ &\to (\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau} \to C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}})^* R^1 h_*(\mathcal{O}_{\widetilde{C}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}})\\ &\xrightarrow{\sim} H^0(\mathfrak{N}_{\check{\sigma},\tau}, \underline{\mathscr{H}}^1(\mathcal{O}_{(\mathsf{N}^{\mathrm{tor}})^{\wedge}_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}}})).\end{aligned}$$

Lemma 5.18. *The morphism*

$$R^0 h_*(\underline{\mathrm{Der}}_{\widetilde{C}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}/C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}) \to R^1 h_*(\mathcal{O}_{\widetilde{C}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}})$$

defined by cup product with $d\log(\Psi_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}(\check{\ell}_{j_Q,\check{\tau}}))$ *depends only on the image* $\tilde{\ell}_{j_Q}$ *of* $\check{\ell}_{j_Q,\check{\tau}}$ *in* $\mathbf{S}_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}}}$ *under the second morphism in* (3.6) (*and hence is independent of the choice of* $\check{\ell}_{j_Q,\check{\tau}}$). *Moreover, this morphism is surjective over* $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau} \otimes_{\mathbb{Z}} \mathbb{Q}$.

Proof. By Lemma 4.9, the morphism $h : \widetilde{C}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}} \to C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ is a torsor under its kernel C, which is an abelian scheme $\mathbb{Z}^{\times}_{(\Box)}$-isogenous to $\underline{\mathrm{Hom}}_{\mathcal{O}}(Q, A)^{\circ} \to \mathsf{M}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}$. The restriction of $\Psi_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}(\check{\ell}_{j_Q,\check{\tau}})$ to C depends only on the image $\tilde{\ell}_{j_Q}$ of $\check{\ell}_{j_Q,\check{\tau}}$ in $\widetilde{\sigma}_0^{\vee}$, and is *relatively ample* by the same proofs of Corollary 2.12 and Lemma 5.5 (with $G_{\mathsf{M}_{\mathcal{H}}} \to \mathsf{M}_{\mathcal{H}}$ replaced with $A \to \mathsf{M}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}$). Hence the lemma follows. $\Box$

Corollary 5.19. *The morphism* (5.17) *is surjective over* $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}} \otimes_{\mathbb{Z}} \mathbb{Q}$. *Its kernel is the subsheaf* $H^0(\mathfrak{N}_{\check{\sigma},\tau}, \underline{\mathscr{H}}^0(\overline{\underline{\mathrm{Der}}}_{\mathfrak{X}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}},\check{\sigma},\tau}/\widetilde{\mathfrak{X}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau}}))$ *of* $H^0(\mathfrak{N}_{\check{\sigma},\tau}, \underline{\mathscr{H}}^0(\imath^*\overline{\underline{\mathrm{Der}}}_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}))$.

Now consider the induced morphism

$$\begin{aligned}H^0(\mathfrak{N}_{\check{\sigma},\tau}, \underline{\mathscr{H}}^0(\overline{\underline{\mathrm{Der}}}_{\mathfrak{X}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}},\check{\sigma},\tau}/\widetilde{\mathfrak{X}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau}})) &\hookrightarrow H^0(\mathfrak{N}_{\check{\sigma},\tau}, \underline{\mathscr{H}}^0(\imath^*\overline{\underline{\mathrm{Der}}}_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}))\\ \xrightarrow{\sim} R^0 f^{\mathrm{tor}}_*(\imath^*\overline{\underline{\mathrm{Der}}}_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}) &\to R^1 f^{\mathrm{tor}}_*(\mathcal{O}_{(\mathsf{N}^{\mathrm{tor}})^{\wedge}_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}}})\end{aligned}$$

defined by cup product with $\imath^*(D^{\mathrm{tor}}_{\tilde{\ell}_{j_Q}})$. This composition has image in

$$H^1(\mathfrak{N}_{\check{\sigma},\tau}, \underline{\mathscr{H}}^0(\mathcal{O}_{(\mathsf{N}^{\mathrm{tor}})^{\wedge}_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}}})),$$

because its further composition with

$$R^1 f_*^{\mathrm{tor}}(\mathcal{O}_{(\mathsf{N}^{\mathrm{tor}})^{\wedge}_{\widetilde{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}}}) \twoheadrightarrow H^0(\mathfrak{N}_{\check{\sigma},\tau}, \underline{\mathscr{H}}^1(\mathcal{O}_{(\mathsf{N}^{\mathrm{tor}})^{\wedge}_{\widetilde{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}}}))$$

is zero (by Corollary 5.19). Thus the question is whether cup product with $\iota^*(D^{\mathrm{tor}}_{\tilde{\ell}_{j_Q}})$ induces a morphism

$$H^0(\mathfrak{N}_{\check{\sigma},\tau}, \underline{\mathscr{H}}^0(\overline{\mathrm{Der}}_{\mathfrak{X}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}},\check{\sigma},\tau}/\widetilde{\mathfrak{X}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau}})) \to H^1(\mathfrak{N}_{\check{\sigma},\tau}, \underline{\mathscr{H}}^0(\mathcal{O}_{(\mathsf{N}^{\mathrm{tor}})^{\wedge}_{\widetilde{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}}})) \quad (5.20)$$

surjective over $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau} \otimes_{\mathbb{Z}} \mathbb{Q}$.

Lemma 5.21. *Suppose $\check{\tau} \in \Sigma_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\sigma},\tau}$, and $\check{\ell} \in \check{\sigma}^{\perp}$. Suppose $\mathfrak{V}$ is an affine open formal subscheme of $\widetilde{\mathfrak{X}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau}$ over which the pullback of $\Psi_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}(\check{\ell})$ is a principal ideal of $\mathcal{O}_{\mathfrak{V}}$ generated by some section x. Let $\mathfrak{U} := \mathfrak{U}_{\check{\tau}} \times_{\widetilde{\mathfrak{X}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau}} \mathfrak{V}$ and let $\overline{\mathcal{O}}_{\mathfrak{U}}^{\times,\mathrm{gp}}$ be the pullback of $\overline{\mathcal{O}}_{\mathsf{N}^{\mathrm{tor}}}^{\times,\mathrm{gp}}$ to $\mathfrak{U}$. Let*

$$\overline{\mathcal{O}}_{\mathfrak{V}}^{\times,\mathrm{gp}} := (\mathfrak{U} \to \mathfrak{V})_*(\overline{\mathcal{O}}_{\mathfrak{U}}^{\times,\mathrm{gp}}).$$

Then there exists a canonical injection $\Psi_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}(\check{\ell}) \hookrightarrow \overline{\mathcal{O}}_{\mathfrak{V}}^{\times,\mathrm{gp}}$ over $\mathfrak{V}$, and the value of the section $d\log(x)$ of $(\mathfrak{U} \to \mathfrak{V})_*\overline{\Omega}^1_{\mathfrak{U}/\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau}}$ determines a canonical section of $\overline{\Omega}^1_{\mathfrak{U}/\widetilde{\mathfrak{X}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau}}$ (which is independent of the choice of the generator x).

Proof. If we replace x with ax, for some $a \in \mathcal{O}_{\mathfrak{V}}^{\times}$, then $d\log(ax) = d\log(a) + d\log(x) = d\log(x)$ because $d\log(a) = 0$ in $(\mathfrak{U} \to \mathfrak{V})_*\overline{\Omega}^1_{\mathfrak{U}/\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau}}$. □

Corollary 5.22. *Suppose $\check{\tau} \in \Sigma_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\sigma},\tau}$, and $\check{\ell} \in \check{\sigma}^{\perp}$. Then the local generators of $\Psi_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}(\check{\ell})$ in Lemma 5.21 determine a well-defined section of $\overline{\Omega}^1_{\mathfrak{U}_{\check{\tau}}/\widetilde{\mathfrak{X}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau}}$, which we denote by $d\log(\Psi_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}(\check{\ell}))$.*

Proof. Since $\Psi_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}(\check{\ell})$ is defined over $\widetilde{\mathfrak{X}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau}$ (or rather $\widetilde{C}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}$), we can always cover $\mathfrak{U}_{\check{\tau}}$ by open formal subschemes $\mathfrak{U}$ as in Lemma 5.21. □

Lemma 5.23. *For any $\check{\tau}, \check{\tau}' \in \Sigma_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\sigma},\tau}$ such that $\check{\tau}$ and $\check{\tau}'$ are adjacent to each other, let us define the section $u_{[\check{\tau}],[\check{\tau}']}$ of $\underline{\mathscr{H}}^0(\overline{\Omega}^1_{\mathfrak{X}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}},\check{\sigma},\tau}/\widetilde{\mathfrak{X}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau}})([\check{\tau}]^{\mathrm{cl}} \cap [\check{\tau}']^{\mathrm{cl}})$ to be*

$$d\log(\Psi_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}(\check{\ell}_{j_Q,\check{\tau}} - \check{\ell}_{j_Q,\check{\tau}'}))$$

(as in Corollary 5.22). *Then this is well defined and determines a section u of $H^1(\mathfrak{N}_{\check{\sigma},\tau}, \underline{\mathscr{H}}^0(\iota^*\overline{\Omega}^1_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}))$ that induces by cup product the same morphism as* (5.20).

Proof. If $\check{\tau}$ and $\check{\tau}'$ are adjacent, then $\gamma\check{\tau}$ and $\gamma'\check{\tau}'$ are adjacent for $\gamma, \gamma' \in \Gamma_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}}$ only when $\gamma = \gamma'$ (by Condition 1.29; cf. Lemma 3.1), in which case

$$\check{\ell}_{j_Q,\gamma\check{\tau}} - \check{\ell}_{j_Q,\check{\tau}} = \gamma\check{\ell}_{j_Q,\check{\tau}} - \check{\ell}_{j_Q,\check{\tau}} = \gamma\check{\ell}_{j_Q,\check{\tau}'} - \check{\ell}_{j_Q,\check{\tau}'} = \check{\ell}_{j_Q,\gamma\check{\tau}'} - \check{\ell}_{j_Q,\check{\tau}'}$$

(because $\Gamma_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}}$ acts by the same translation on $\breve{\ell}_{j_Q,\breve{\tau}}$ and $\breve{\ell}_{j_Q,\breve{\tau}'}$). This shows that the assignment of $u_{[\breve{\tau}],[\breve{\tau}']}$ is independent of the choices of the respective representatives $\breve{\tau}$ and $\breve{\tau}'$ of $[\breve{\tau}]$ and $[\breve{\tau}']$, and that u is well defined.

Cup product with u induces the same morphism as (5.20) because the canonical morphism

$$\underline{\overline{\mathrm{Der}}}_{\mathfrak{X}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}},\breve{\sigma},\tau}/\widetilde{\mathfrak{X}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau}} \otimes \iota^*\overline{\Omega}^1_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}} \to \mathcal{O}_{(\mathsf{N}^{\mathrm{tor}})^{\wedge}_{\widetilde{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}}}$$

factors through

$$\underline{\overline{\mathrm{Der}}}_{\mathfrak{X}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}},\breve{\sigma},\tau}/\widetilde{\mathfrak{X}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau}} \otimes \overline{\Omega}^1_{\mathfrak{X}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}},\breve{\sigma},\tau}/\widetilde{\mathfrak{X}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau}} \to \mathcal{O}_{(\mathsf{N}^{\mathrm{tor}})^{\wedge}_{\widetilde{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}}},$$

and because cup product with the image of $\iota^*(D^{\mathrm{tor}}_{\ell_{j_Q}})$ in $H^0(\mathfrak{N}_{\breve{\sigma},\tau},\underline{\mathcal{H}}^1(\iota^*\overline{\Omega}^1_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}))$ induces the zero morphism (cf. the paragraph preceding Lemma 5.18). □

Consider any sequence $\breve{\tau}_1, \breve{\tau}_2, \ldots, \breve{\tau}_k$ of adjacent cones in $\Sigma_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\sigma},\tau}$, such that $\breve{\tau}_k = \gamma\breve{\tau}_1$ for some $\gamma \in \Gamma_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}}$. The union of the cones in any such sequence form a subset of $\widetilde{\mathfrak{N}}_{\breve{\sigma},\tau}$ contractible to a path joining a point in $\breve{\tau}$ with its translation by γ in $\gamma\breve{\tau}$, whose image in $\mathfrak{N}_{\breve{\sigma},\tau}$ defines a loop. Suppose we have a class s in $H^1(\mathfrak{N}_{\breve{\sigma},\tau},\underline{\mathcal{H}}^0(\mathcal{O}_{(\mathsf{N}^{\mathrm{tor}})^{\wedge}_{\widetilde{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}}}))$ represented by a collection of sections

$$s_{[\breve{\tau}],[\breve{\tau}']} \in \underline{\mathcal{H}}^0(\mathcal{O}_{(\mathsf{N}^{\mathrm{tor}})^{\wedge}_{\widetilde{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}}})([\breve{\tau}]^{\mathrm{cl}} \cap [\breve{\tau}']^{\mathrm{cl}})$$

for $[\breve{\tau}], [\breve{\tau}'] \in \Sigma_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\sigma},\tau}/\Gamma_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}}$, and suppose we define formally $s_{\breve{\tau},\breve{\tau}'} = s_{[\breve{\tau}],[\breve{\tau}']}$ for any $\breve{\tau}, \breve{\tau}' \in \Sigma_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\sigma},\tau}$. Then we can define the *path integral* of s along the sequence $\breve{\tau}_1, \breve{\tau}_2, \ldots, \breve{\tau}_k$ to be the sum

$$\sum_{i=1}^{k-1} s_{\breve{\tau}_i,\breve{\tau}_{i+1}}.$$

This defines a morphism

$$H^1(\mathfrak{N}_{\breve{\sigma},\tau},\underline{\mathcal{H}}^0(\mathcal{O}_{(\mathsf{N}^{\mathrm{tor}})^{\wedge}_{\widetilde{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}}})) \to \mathcal{O}_{\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau}}. \tag{5.24}$$

Note that this is a realization of the cap product

$$H_1(\mathfrak{N}_{\breve{\sigma},\tau},\mathbb{Z}) \times H^1(\mathfrak{N}_{\breve{\sigma},\tau},\underline{\mathcal{H}}^0(\mathcal{O}_{(\mathsf{N}^{\mathrm{tor}})^{\wedge}_{\widetilde{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}}})) \\ \to H_0(\mathfrak{N}_{\breve{\sigma},\tau},\underline{\mathcal{H}}^0(\mathcal{O}_{(\mathsf{N}^{\mathrm{tor}})^{\wedge}_{\widetilde{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}}})) \cong \mathcal{O}_{\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau}}.$$

Lemma 5.25. *For any $\breve{\ell} \in \mathbf{S}_{\breve{\Phi}_{\widetilde{\mathcal{H}}}}$ that is mapped to $\tilde{\ell}_{j_Q}$ in $\widetilde{\sigma}_0^{\vee}$ under the second morphism in* (3.6), *the assignment $\gamma \mapsto d\log(\Psi_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}(\gamma\breve{\ell}-\breve{\ell}))$ for $\gamma \in \Gamma_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}}$ induces a morphism*

$$\Gamma_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}} \otimes_{\mathbb{Z}} \mathcal{O}_{\mathfrak{X}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}},\breve{\sigma},\tau}} \to \overline{\Omega}^1_{\mathfrak{X}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}},\breve{\sigma},\tau}/\widetilde{\mathfrak{X}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau}},$$

which is an isomorphism over $\mathfrak{X}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}},\breve{\sigma},\tau} \otimes_{\mathbb{Z}} \mathbb{Q}$.

Proof. Since $\gamma\breve{\ell}$ and $\breve{\ell}$ have the same image $\tilde{\ell}_{j_Q}$ in $\tilde{\sigma}_0^\vee$ under the second morphism in (3.6), the difference $\gamma\breve{\ell} - \breve{\ell}$ lands in $\breve{\sigma}^\perp$. For any $\breve{\ell}' \in \breve{\sigma}^\perp$, an elementary matrix calculation (using any splitting of $s_{\breve{X}} \otimes_{\mathbb{Z}} \mathbb{Q} : \breve{X} \otimes_{\mathbb{Z}} \mathbb{Q} \twoheadrightarrow \widetilde{X} \otimes_{\mathbb{Z}} \mathbb{Q}$) shows that $\gamma\breve{\ell}' - \breve{\ell}'$ lies in $\mathbf{S}_{\Phi_{\mathcal{H}}} = (\mathbf{S}_{\Phi_{\mathcal{H}}} \otimes_{\mathbb{Z}} \mathbb{Q}) \cap \mathbf{S}_{\breve{\Phi}_{\widetilde{\mathcal{H}}}}$ (identified as the image of the first morphism in (3.6)). Therefore, we have $(\gamma_1\gamma_2\breve{\ell} - \breve{\ell}) - (\gamma_1\breve{\ell} - \breve{\ell}) - (\gamma_2\breve{\ell} - \breve{\ell}) = \gamma_1(\gamma_2\breve{\ell} - \breve{\ell}) - (\gamma_2\breve{\ell} - \breve{\ell}) \in \mathbf{S}_{\Phi_{\mathcal{H}}}$, which shows that the assignment $\gamma \mapsto \gamma\breve{\ell} - \breve{\ell}$ defines a group homomorphism $\Gamma_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}} \to (\breve{\sigma}^\perp/\mathbf{S}_{\Phi_{\mathcal{H}}})$. By the choice of j_Q, the element $\tilde{\ell}_{j_Q}$ is represented by a positive definite matrix with respect to any choice of basis, and hence the homomorphism $\Gamma_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}} \to (\breve{\sigma}^\perp/\mathbf{S}_{\Phi_{\mathcal{H}}})$ induced by $\gamma \mapsto \gamma\breve{\ell} - \breve{\ell}$ is injective (by another elementary matrix calculation over $\mathbb{Q}$). By comparison of dimensions, this shows that the induced injective homomorphism

$$\Gamma_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}} \otimes_{\mathbb{Z}} \mathbb{Q} \to (\breve{\sigma}^\perp/\mathbf{S}_{\Phi_{\mathcal{H}}}) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is bijective. Since $\overline{\Omega}^1_{\mathfrak{X}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}},\breve{\sigma},\tau}/\widetilde{\mathfrak{X}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau}}$ is generated over $\mathcal{O}_{\mathfrak{X}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}},\breve{\sigma},\tau}}$ by

$$\{d\log(\Psi_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}(\breve{\ell}')) : \breve{\ell}' \text{ representatives of } \breve{\sigma}^\perp/\mathbf{S}_{\Phi_{\mathcal{H}}}\},$$

the lemma follows. □

Lemma 5.26. *Let $\breve{\tau}_1, \breve{\tau}_2, \ldots, \breve{\tau}_k$ be a sequence of adjacent cones in $\Sigma_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\sigma},\tau}$, such that $\breve{\tau}_k = \gamma\breve{\tau}_1 \neq \breve{\tau}_1$ for some $\gamma \in \Gamma_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}}$. Then the composition of* (5.20) *and* (5.24) *is surjective over $\mathfrak{X}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}},\breve{\sigma},\tau} \otimes_{\mathbb{Z}} \mathbb{Q}$.*

Proof. If $\gamma\breve{\tau}_1 \neq \breve{\tau}_1$, then $\breve{\ell}_{j_Q,\gamma\breve{\tau}_1} = \gamma\breve{\ell}_{j_Q,\breve{\tau}_1} \neq \breve{\ell}_{j_Q,\breve{\tau}_1}$ by the proof of Lemma 5.25. By Lemma 5.25, this implies that $d\log(\Psi_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}(\breve{\ell}_{j_Q,\breve{\tau}_1} - \breve{\ell}_{j_Q,\breve{\tau}_k}))$ defines a nonzero section of $\overline{\Omega}^1_{\mathfrak{X}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}},\breve{\sigma},\tau}/\widetilde{\mathfrak{X}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau}}$ over every $\mathfrak{U}_{[\breve{\tau}]} \otimes_{\mathbb{Z}} \mathbb{Q}$. Let t be any section of $H^0(\mathfrak{N}_{\breve{\sigma},\tau}, \underline{\mathscr{H}}^0(\iota^*\underline{\mathrm{Der}}_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}))$. Cup product with u (see Lemma 5.23) sends t to the class s in $H^1(\mathfrak{N}_{\breve{\sigma},\tau}, \underline{\mathscr{H}}^0(\mathcal{O}_{(\mathsf{N}^{\mathrm{tor}})^{\wedge}_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}}}))$ represented (up to a sign convention) by the collection of sections

$$s_{[\breve{\tau}],[\breve{\tau}']} \in \underline{\mathscr{H}}^0(\mathcal{O}_{(\mathsf{N}^{\mathrm{tor}})^{\wedge}_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}}})([\breve{\tau}]^{\mathrm{cl}} \cap [\breve{\tau}']^{\mathrm{cl}})$$

determined by $s_{\breve{\tau},\breve{\tau}'} = t \cup (d\log(\Psi_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}(\breve{\ell}_{j_Q,\breve{\tau}} - \breve{\ell}_{j_Q,\breve{\tau}'})))$ for any $\breve{\tau}, \breve{\tau}' \in \Sigma_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\sigma},\tau}$. Therefore, if locally there exists t such that $t \cup (d\log(\Psi_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}(\breve{\ell}_{j_Q,\breve{\tau}_1} - \breve{\ell}_{j_Q,\breve{\tau}_k})))$ is the pullback of (local) generators of $\mathcal{O}_{\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau} \otimes_{\mathbb{Z}} \mathbb{Q}}$, which is possible by Lemma 5.25, then the path integral

$$\begin{aligned}\sum_{i=1}^{k-1} s_{\breve{\tau}_i,\breve{\tau}_{i+1}} &= \sum_{i=1}^{k-1} t \cup (d\log(\Psi_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}(\breve{\ell}_{j_Q,\breve{\tau}_i} - \breve{\ell}_{j_Q,\breve{\tau}_{i+1}}))) \\ &= t \cup (d\log(\Psi_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}(\breve{\ell}_{j_Q,\breve{\tau}_1} - \breve{\ell}_{j_Q,\breve{\tau}_k})))\end{aligned}$$

is defined locally by generators of $\mathcal{O}_{\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau} \otimes_{\mathbb{Z}} \mathbb{Q}}$. This shows that the composition of (5.20) with (5.24) is surjective over $\mathfrak{X}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}},\breve{\sigma},\tau} \otimes_{\mathbb{Z}} \mathbb{Q}$, as desired. □

Corollary 5.27. *The morphism* (5.20) *is surjective over* $\mathfrak{X}_{\check{\Phi}_{\tilde{\mathcal{H}}},\check{\delta}_{\tilde{\mathcal{H}}},\check{\sigma},\tau} \otimes_{\mathbb{Z}} \mathbb{Q}$.

Proof. By Lemma 4.21, (4.25), and Lemma 5.25, the morphism (5.20) is surjective over $\mathfrak{X}_{\check{\Phi}_{\tilde{\mathcal{H}}},\check{\delta}_{\tilde{\mathcal{H}}},\check{\sigma},\tau} \otimes_{\mathbb{Z}} \mathbb{Q}$ if its composition with (5.24) is surjective over $\mathfrak{X}_{\check{\Phi}_{\tilde{\mathcal{H}}},\check{\delta}_{\tilde{\mathcal{H}}},\check{\sigma},\tau} \otimes_{\mathbb{Z}} \mathbb{Q}$ for some collection of sequences $\check{\tau}_1, \check{\tau}_2, \ldots, \check{\tau}_k$ defining loops in $\mathfrak{N}_{\check{\sigma},\tau}$ generating $H_1(\mathfrak{N}_{\check{\sigma},\tau}, \mathbb{Z})$. Hence the corollary follows from Lemma 5.26. □

Now Proposition 5.14 follows from the combination of Corollaries 5.19 and 5.27. By Lemma 5.2 and Remark 5.4, Proposition 5.14 implies the existence of the canonical isomorphism (5.3). Thus Corollary 4.36 implies:

Corollary 5.28. *For any integer* $b \geq 0$, *we have a canonical isomorphism*

$$R^b f_*^{\mathrm{tor}}(\mathcal{O}_{\mathsf{N}^{\mathrm{tor}}}) \cong \bigwedge^b(\underline{\mathrm{Hom}}_{\mathcal{O}}(Q^\vee, \underline{\mathrm{Lie}}_{G^\vee/\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}}))$$

of locally free sheaves over $\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$, *compatible with cup products and exterior products, extending the composition of canonical isomorphisms* (2.19) *over* $\mathsf{M}_{\mathcal{H}}$.

This completes the proof of (3b) and (3d) of Theorem 2.15, using respectively (3a) and (3c) of Theorem 2.15. As explained in Section 3E, this also makes (4c) and (5c) of Theorem 2.15 unconditional. The proof of Theorem 2.15 is now complete.

6. Canonical extensions of principal bundles

6A. ***Principal bundles.*** Consider $(G_{\mathsf{M}_{\mathcal{H}}}, \lambda_{\mathsf{M}_{\mathcal{H}}}, i_{\mathsf{M}_{\mathcal{H}}}, \alpha_{\mathcal{H}}) \to \mathsf{M}_{\mathcal{H}}$, the restriction of the degenerating family $(G, \lambda, i, \alpha_{\mathcal{H}}) \to \mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$, which is isomorphic to the tautological tuple over $\mathsf{M}_{\mathcal{H}}$; and consider the relative de Rham cohomology $\underline{H}^1_{\mathrm{dR}}(G_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}})$ and the relative de Rham homology $\underline{H}_1^{\mathrm{dR}}(G_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}}) := \underline{\mathrm{Hom}}_{\mathcal{O}_{\mathsf{M}_{\mathcal{H}}}}(\underline{H}^1_{\mathrm{dR}}(G_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}}), \mathcal{O}_{\mathsf{M}_{\mathcal{H}}})$. We have the canonical pairing $\langle\cdot,\cdot\rangle_\lambda : \underline{H}_1^{\mathrm{dR}}(G_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}}) \times \underline{H}_1^{\mathrm{dR}}(G_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}}) \to \mathcal{O}_{\mathsf{M}_{\mathcal{H}}}(1)$ defined as the composition of $(\mathrm{Id} \times \lambda_{\mathsf{M}_{\mathcal{H}}})_*$ followed by the perfect pairing $\underline{H}_1^{\mathrm{dR}}(G_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}}) \times \underline{H}_1^{\mathrm{dR}}(G^\vee_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}}) \to \mathcal{O}_{\mathsf{M}_{\mathcal{H}}}(1)$ defined by the first Chern class of the Poincaré invertible sheaf over $G_{\mathsf{M}_{\mathcal{H}}} \times_{\mathsf{M}_{\mathcal{H}}} G^\vee_{\mathsf{M}_{\mathcal{H}}}$. (See for example [Deligne and Pappas 1994, 1.5].) Under the assumption that $\lambda_{\mathsf{M}_{\mathcal{H}}}$ has degree prime to $\Box$, we know that $\lambda_{\mathsf{M}_{\mathcal{H}}}$ is separable, that $(\lambda_{\mathsf{M}_{\mathcal{H}}})_*$ is an isomorphism, and hence that the pairing $\langle\cdot,\cdot\rangle_\lambda$ above is *perfect*. Let $\langle\cdot,\cdot\rangle_\lambda$ also denote the induced pairing on $\underline{H}^1_{\mathrm{dR}}(G_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}}) \times \underline{H}^1_{\mathrm{dR}}(G_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}})$ by duality. By [Berthelot et al. 1982, Lemma 2.5.3], we have canonical short exact sequences

$$0 \to \underline{\mathrm{Lie}}^\vee_{G^\vee_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}}} \to \underline{H}_1^{\mathrm{dR}}(G_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}}) \to \underline{\mathrm{Lie}}_{G_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}}} \to 0$$

and

$$0 \to \underline{\mathrm{Lie}}^\vee_{G_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}}} \to \underline{H}^1_{\mathrm{dR}}(G_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}}) \to \underline{\mathrm{Lie}}_{G^\vee_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}}} \to 0.$$

The submodules $\underline{\mathrm{Lie}}^\vee_{G^\vee_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}}}$ and $\underline{\mathrm{Lie}}^\vee_{G_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}}}$ are maximal totally isotropic with respect to $\langle\cdot,\cdot\rangle_\lambda$.

Consider the $\mathcal{O} \otimes_{\mathbb{Z}} \mathbb{C}$-module

$$L \otimes_{\mathbb{Z}} \mathbb{C} \to (L \otimes_{\mathbb{Z}} \mathbb{C})/\mathsf{P}_h, \tag{6.1}$$

where $\mathsf{P}_h := \{\sqrt{-1}x - h(\sqrt{-1})x : x \in L \otimes_{\mathbb{Z}} \mathbb{R}\} \subset L \otimes_{\mathbb{Z}} \mathbb{C}$.

Now suppose there exists a finite extension F_0' of F_0 in $\mathbb{C}$, and a subset $\Box'$ of $\Box$, such that F_0' is unramified at all primes in $\Box'$, and such that, by setting $R_0 := \mathcal{O}_{F_0', (\Box')}$, there exists an $\mathcal{O} \otimes_{\mathbb{Z}} R_0$-module L_0 such that $L_0 \otimes_{R_0} \mathbb{C} \cong (L \otimes_{\mathbb{Z}} \mathbb{C})/\mathsf{P}_h$. Once the choice of F_0' is fixed, the choice of L_0 is unique up to isomorphism because $\mathcal{O} \otimes_{\mathbb{Z}} R_0$-modules are uniquely determined by their multiranks. (See [Lan 2008, Lemma 1.1.3.4 and Definition 1.1.3.5] for the notion of multiranks.) Let

$$\langle \cdot , \cdot \rangle_{\mathrm{can.}} : (L_0 \oplus L_0^\vee(1)) \times (L_0 \oplus L_0^\vee(1)) \to R_0(1)$$

be the alternating pairing defined by $\langle (x_1, f_1), (x_2, f_2) \rangle_{\mathrm{can.}} := f_2(x_1) - f_1(x_2)$ (cf. [Lan 2008, Lemma 1.1.4.16]).

Definition 6.2. For any R_0-algebra R, set

$$\begin{aligned} \mathrm{G}_0(R) &:= \left\{ \begin{array}{c} (g, r) \in \mathrm{GL}_{\mathcal{O} \otimes_{\mathbb{Z}} R}((L_0 \oplus L_0^\vee(1)) \otimes_{R_0} R) \times \mathbf{G}_{\mathrm{m}}(R) : \\ \langle gx, gy \rangle = r \langle x, y \rangle, \forall x, y \in (L_0 \oplus L_0^\vee(1)) \otimes_{R_0} R \end{array} \right\}, \\ \mathrm{P}_0(R) &:= \{(g, r) \in \mathrm{G}_0(R) : g(L_0^\vee(1) \otimes_{R_0} R) = L_0^\vee(1) \otimes_{R_0} R\}, \\ \mathrm{M}_0(R) &:= \mathrm{GL}_{\mathcal{O} \otimes_{\mathbb{Z}} R}(L_0^\vee(1) \otimes_{R_0} R) \times \mathbf{G}_{\mathrm{m}}(R), \end{aligned}$$

where we view $\mathrm{M}_0(R)$ canonically as a quotient of $\mathrm{P}_0(R)$ by

$$\mathrm{P}_0(R) \to \mathrm{M}_0(R) : (g, r) \mapsto (g|_{L_0^\vee(1) \otimes_{R_0} R}, r).$$

The assignments are functorial in R and define group functors G_0, P_0, and M_0 over R_0.

Lemma 6.3. *For any complete local ring R over R_0 with separably closed residue field, there is an isomorphism*

$$(L \otimes_{\mathbb{Z}} R, \langle \cdot , \cdot \rangle) \cong (L_0 \oplus L_0^\vee(1), \langle \cdot , \cdot \rangle_{\mathrm{can.}}) \otimes_{R_0} R,$$

and hence an isomorphism $\mathrm{G}(R) \cong \mathrm{G}_0(R)$. (*Consequently*, $\mathrm{P}_0(R)$ *can be identified with a "parabolic" subgroup of* $\mathrm{G}(R)$.)

(In practice, it is not necessary to take R to be complete local. Much smaller rings would suffice for the existence of isomorphisms as in Lemma 6.3.)

In what follows, by abuse of notation, we shall replace $\mathsf{M}_{\mathcal{H}}$ etc. with their base extensions from $\mathrm{Spec}(\mathcal{O}_{F_0, (\Box)})$ to $\mathrm{Spec}(R_0)$, and replace $\mathsf{S}_0 = \mathrm{Spec}(\mathcal{O}_{F_0, (\Box)})$ with $\mathrm{Spec}(R_0)$.

Definition 6.4. The *principal* P_0-*bundle* over $\mathsf{M}_{\mathcal{H}}$ is the P_0-torsor

$$\mathcal{E}_{\mathrm{P}_0} := \underline{\mathrm{Isom}}_{\mathcal{O}\otimes_{\mathbb{Z}}\mathbb{O}_{\mathsf{M}_{\mathcal{H}}}}((\underline{H}_1^{\mathrm{dR}}(G_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}}), \langle\,\cdot\,,\,\cdot\,\rangle_\lambda, \mathbb{O}_{\mathsf{M}_{\mathcal{H}}}(1), \underline{\mathrm{Lie}}^\vee_{G^\vee_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}}}), \\ ((L_0 \oplus L_0^\vee(1)) \otimes_{R_0} \mathbb{O}_{\mathsf{M}_{\mathcal{H}}}, \langle\,\cdot\,,\,\cdot\,\rangle_{\mathrm{can.}}, \mathbb{O}_{\mathsf{M}_{\mathcal{H}}}(1), L_0^\vee(1) \otimes_{R_0} \mathbb{O}_{\mathsf{M}_{\mathcal{H}}})),$$

the sheaf of isomorphisms of $\mathbb{O}_{\mathsf{M}_{\mathcal{H}}}$-sheaves of symplectic $\mathcal{O}$-modules with maximal totally isotropic $\mathcal{O} \otimes_{\mathbb{Z}} R_0$-submodules. (The group P_0 acts as automorphisms on $(L \otimes_{\mathbb{Z}} \mathbb{O}_{\mathsf{M}_{\mathcal{H}}}, \langle\,\cdot\,,\,\cdot\,\rangle_\lambda, \mathbb{O}_{\mathsf{M}_{\mathcal{H}}}(1), L_0^\vee(1) \otimes_{R_0} \mathbb{O}_{\mathsf{M}_{\mathcal{H}}})$ by definition. The third entries in the tuples represent the values of the pairings.)

Definition 6.5. The *principal* M_0-*bundle* over $\mathsf{M}_{\mathcal{H}}$ is the M_0-torsor

$$\mathcal{E}_{\mathrm{M}_0} := \underline{\mathrm{Isom}}_{\mathcal{O}\otimes_{\mathbb{Z}}\mathbb{O}_{\mathsf{M}_{\mathcal{H}}}}((\underline{\mathrm{Lie}}^\vee_{G^\vee_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}}}, \mathbb{O}_{\mathsf{M}_{\mathcal{H}}}(1)), (L_0^\vee(1) \otimes_{R_0} \mathbb{O}_{\mathsf{M}_{\mathcal{H}}}, \mathbb{O}_{\mathsf{M}_{\mathcal{H}}}(1))),$$

the sheaf of isomorphisms of $\mathbb{O}_{\mathsf{M}_{\mathcal{H}}}$-sheaves of $\mathcal{O} \otimes_{\mathbb{Z}} R_0$-modules. (We view the second entries in the pairs as an additional structure, inherited from the corresponding objects for P_0. The group M_0 acts obviously on $(L_0^\vee(1) \otimes_{R_0} \mathbb{O}_{\mathsf{M}_{\mathcal{H}}}, \mathbb{O}_{\mathsf{M}_{\mathcal{H}}}(1))$ as automorphisms, by definition.)

These define étale torsors because, by the theory of infinitesimal deformations (cf. for example [Lan 2008, Chapter 2]) and the theory of Artin's approximations (cf. [Artin 1969, Theorem 1.10 and Corollary 2.5]),

$$(\underline{H}_1^{\mathrm{dR}}(G_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}}), \langle\,\cdot\,,\,\cdot\,\rangle_\lambda, \mathbb{O}_{\mathsf{M}_{\mathcal{H}}}(1), \underline{\mathrm{Lie}}^\vee_{G^\vee_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}}})$$

and

$$((L_0 \oplus L_0^\vee(1)) \otimes_{R_0} \mathbb{O}_{\mathsf{M}_{\mathcal{H}}}, \langle\,\cdot\,,\,\cdot\,\rangle_{\mathrm{can.}}, \mathbb{O}_{\mathsf{M}_{\mathcal{H}}}(1), L_0^\vee(1) \otimes_{R_0} \mathbb{O}_{\mathsf{M}_{\mathcal{H}}})$$

are étale locally isomorphic.

Definition 6.6. For any R_0-algebra E, we denote by $\mathrm{Rep}_E(\mathrm{P}_0)$ (resp. $\mathrm{Rep}_E(\mathrm{M}_0)$) the category of E-modules with algebraic actions of $\mathrm{P}_0 \otimes_{R_0} E$ (resp. $\mathrm{M}_0 \otimes_{R_0} E$).

Definition 6.7. Let E be any R_0-algebra. For any $W \in \mathrm{Rep}_E(\mathrm{P}_0)$, we define

$$\mathcal{E}_{\mathrm{P}_0,E}(W) := (\mathcal{E}_{\mathrm{P}_0} \otimes_{R_0} E) \times^{\mathrm{P}_0 \otimes_{R_0} E} W,$$

called the *automorphic sheaf* over $\mathsf{M}_{\mathcal{H}} \otimes_{R_0} E$ associated with W. It is called an *automorphic bundle* if W is locally free of finite rank over E. We define similarly $\mathcal{E}_{\mathrm{M}_0,E}(W)$ for $W \in \mathrm{Rep}_E(\mathrm{M}_0)$ by replacing P_0 with M_0 in the above expression.

Lemma 6.8. *Let E be any R_0-algebra. If we view an element $W \in \mathrm{Rep}_E(\mathrm{M}_0)$ as an element in $\mathrm{Rep}_E(\mathrm{P}_0)$ via the canonical surjection $\mathrm{P}_0 \twoheadrightarrow \mathrm{M}_0$, then we have a canonical isomorphism $\mathcal{E}_{\mathrm{P}_0,E}(W) \cong \mathcal{E}_{\mathrm{M}_0,E}(W)$.*

6B. ***Canonical extensions.*** By taking $Q = \mathcal{O}$, so that $\underline{\mathrm{Hom}}_{\mathcal{O}}(Q, G_{\mathsf{M}_{\mathcal{H}}})^{\circ} \cong G_{\mathsf{M}_{\mathcal{H}}}$ and so that there exists some $\mathbb{Z}^{\times}_{(\Box)}$-isogeny $\kappa^{\mathrm{isog}} : G_{\mathsf{M}_{\mathcal{H}}} \to \mathsf{N}$ as in Theorem 2.15, the locally free sheaf $\underline{H}^1_{\mathrm{dR}}(\mathsf{N}/\mathsf{M}_{\mathcal{H}}) \cong \underline{H}^1_{\mathrm{dR}}(G_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}})$ extends to the locally free sheaf $\underline{H}^1_{\text{log-dR}}(\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}})$ over $\mathbb{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}$. Let

$$\underline{H}_1^{\text{log-dR}}(\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}) := \underline{\mathrm{Hom}}_{\mathbb{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}}(\underline{H}^1_{\text{log-dR}}(\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}), \mathbb{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}).$$

Proposition 6.9. *There exists a* ***unique*** *locally free sheaf* $\underline{H}_1^{\mathrm{dR}}(G_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}})^{\mathrm{can}}$ *over* $\mathbb{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}$ *satisfying the following properties*:

(1) *The sheaf* $\underline{H}_1^{\mathrm{dR}}(G_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}})^{\mathrm{can}}$, *canonically identified as a subsheaf of the quasi-coherent sheaf* $(\mathsf{M}_{\mathcal{H}} \hookrightarrow \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}})_*(\underline{H}_1^{\mathrm{dR}}(G_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}}))$, *is self-dual under the pairing* $(\mathsf{M}_{\mathcal{H}} \hookrightarrow \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}})_*\langle\,\cdot\,,\,\cdot\,\rangle_{\lambda}$. *We shall denote the induced pairing by* $\langle\,\cdot\,,\,\cdot\,\rangle^{\mathrm{can}}_{\lambda}$.

(2) $\underline{H}_1^{\mathrm{dR}}(G_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}})^{\mathrm{can}}$ *contains* $\underline{\mathrm{Lie}}^{\vee}_{G^{\vee}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}$ *as a subsheaf totally isotropic under* $\langle\,\cdot\,,\,\cdot\,\rangle^{\mathrm{can}}_{\lambda}$.

(3) *The quotient sheaf* $\underline{H}_1^{\mathrm{dR}}(G_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}})^{\mathrm{can}}/\underline{\mathrm{Lie}}^{\vee}_{G^{\vee}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}$ *can be canonically identified with the subsheaf* $\underline{\mathrm{Lie}}_{G/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}$ *of* $(\mathsf{M}_{\mathcal{H}} \hookrightarrow \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}})_*\underline{\mathrm{Lie}}_{G_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}}}$.

(4) *The pairing* $\langle\,\cdot\,,\,\cdot\,\rangle^{\mathrm{can}}_{\lambda}$ *induces an isomorphism* $\underline{\mathrm{Lie}}_{G/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}} \xrightarrow{\sim} \underline{\mathrm{Lie}}_{G^{\vee}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}$ *which coincides with* $d\lambda$.

(5) *Let* $\underline{H}^1_{\mathrm{dR}}(G_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}})^{\mathrm{can}} := \underline{\mathrm{Hom}}_{\mathbb{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}}(\underline{H}_1^{\mathrm{dR}}(G_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}})^{\mathrm{can}}, \mathbb{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}})$. *The Gauss–Manin connection*

$$\nabla : \underline{H}^1_{\mathrm{dR}}(G_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}}) \to \underline{H}^1_{\mathrm{dR}}(G_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}}) \otimes_{\mathbb{O}_{\mathsf{M}_{\mathcal{H}}}} \Omega^1_{\mathsf{M}_{\mathcal{H}}/\mathsf{S}_0}$$

extends to an integrable connection

$$\nabla : \underline{H}^1_{\mathrm{dR}}(G_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}})^{\mathrm{can}} \to \underline{H}^1_{\mathrm{dR}}(G_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}})^{\mathrm{can}} \otimes_{\mathbb{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}} \overline{\Omega}^1_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}/\mathsf{S}_0} \tag{6.10}$$

with log poles along $\mathsf{D}_{\infty,\mathcal{H}}$, *called the extended Gauss–Manin connection*, *such that the composition*

$$\begin{aligned}\underline{\mathrm{Lie}}^{\vee}_{G/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}} &\hookrightarrow \underline{H}^1_{\mathrm{dR}}(G_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}})^{\mathrm{can}} \\ &\xrightarrow{\nabla} \underline{H}^1_{\mathrm{dR}}(G_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}})^{\mathrm{can}} \otimes_{\mathbb{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}} \overline{\Omega}^1_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}/\mathsf{S}_0} \twoheadrightarrow \underline{\mathrm{Lie}}_{G^{\vee}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}} \otimes_{\mathbb{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}} \overline{\Omega}^1_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}/\mathsf{S}_0}\end{aligned} \tag{6.11}$$

induces by duality the ***extended Kodaira–Spencer morphism***

$$\underline{\mathrm{Lie}}^{\vee}_{G/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}} \otimes_{\mathbb{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}} \underline{\mathrm{Lie}}^{\vee}_{G^{\vee}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}} \to \overline{\Omega}^1_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}/\mathsf{S}_0}$$

in [Lan 2008, Theorem 4.6.3.32], *which factors through* $\underline{\mathrm{KS}}$ (in Definition 1.40) *and induces the extended Kodaira–Spencer isomorphism* $\mathrm{KS}_{G/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}/\mathsf{S}_0}$ *in* (4) *of Theorem 1.41.*

With these characterizing properties, we say that $(\underline{H}_1^{\mathrm{dR}}(G_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}})^{\mathrm{can}}, \nabla)$ *is the* ***canonical extension*** *of* $(\underline{H}_1^{\mathrm{dR}}(G_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}}), \nabla)$.

Proof. The uniqueness of $\underline{H}_1^{\mathrm{dR}}(G_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}})^{\mathrm{can}}$ is clear by the first four properties. To show the existence, let us take $\underline{H}_1^{\mathrm{dR}}(G_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}})^{\mathrm{can}}$ to be the sheaf $\underline{H}_1^{\log\text{-}\mathrm{dR}}(\mathsf{N}^{\mathrm{tor}}/\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}})$ (for $Q = \mathcal{O}$, as mentioned before this proposition). It is locally free with a Hodge filtration by (3c) of Theorem 2.15. Moreover, by taking some integer $N > 0$ prime to $\Box$ such that $N\,\mathrm{Diff}^{-1} \subset \mathcal{O}$, we obtain by multiplication by N a morphism $j_Q : Q^\vee \cong \mathrm{Diff}^{-1} \hookrightarrow Q = \mathcal{O}$ as in Lemma 2.5 such that pullback by κ^{isog} identifies $\langle\cdot,\cdot\rangle_{\lambda_{\mathsf{M}_{\mathcal{H}}, j_Q}} : \underline{H}^1_{\mathrm{dR}}(\mathsf{N}/\mathsf{M}_{\mathcal{H}}) \times \underline{H}^1_{\mathrm{dR}}(\mathsf{N}/\mathsf{M}_{\mathcal{H}}) \to \mathbb{O}_{\mathsf{M}_{\mathcal{H}}}(1)$ canonically with $\langle\cdot,\cdot\rangle_{\lambda_{\mathsf{M}_{\mathcal{H}}}} : \underline{H}^1_{\mathrm{dR}}(G_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}}) \times \underline{H}^1_{\mathrm{dR}}(G_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}}) \to \mathbb{O}_{\mathsf{M}_{\mathcal{H}}}(1)$. Then (1)–(3) follow from (3d) of Theorem 2.15, and (4) follows from Proposition 5.14 (which is used to prove (3b) of Theorem 2.15). It remains to verify (5). By definition, $\underline{H}^1_{\mathrm{dR}}(G_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}})^{\mathrm{can}} \cong \underline{H}^1_{\log\text{-}\mathrm{dR}}(\mathsf{N}^{\mathrm{tor}}/\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}})$. The existence of ∇ in (6.10) follows from (3e) of Theorem 2.15. By Remark 4.42, the pullback of (6.11) to $\mathsf{M}_{\mathcal{H}}$ is induced by the usual Kodaira–Spencer class. Since the extended Kodaira–Spencer morphism in [Lan 2008, Theorem 4.6.3.32] is defined exactly as a morphism induced by the usual Kodaira–Spencer morphism (by normality of $\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$ and local freeness of the sheaves involved), it is induced by duality by (6.11), as desired. $\Box$

Remark 6.12. The notion of *canonical extensions* is closely related to the notion of *regular singularities* of algebraic differential equations. (See [Deligne 1970] and [Katz 1971] for the notion of regular singularities. See [Mumford 1977; Faltings and Chai 1990, Chapter VI; Harris 1989; 1990; Milne 1990] for the notion of canonical extensions over $\mathbb{C}$, and see [Mokrane and Tilouine 2002] for an earlier treatment of canonical extensions in mixed characteristics. See in particular [Harris 1989, Theorem 4.2] for the explanation of why and how the two notions are related.)

Then the principal bundle $\mathcal{E}_{\mathrm{P}_0}$ extends canonically to a principal bundle $\mathcal{E}_{\mathrm{P}_0}^{\mathrm{can}}$ over $\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$ by setting

$$\begin{aligned}\mathcal{E}_{\mathrm{P}_0}^{\mathrm{can}} := \underline{\mathrm{Isom}}_{\mathcal{O}\otimes_{\mathbb{Z}}\mathbb{O}_{\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}}}((&\underline{H}_1^{\mathrm{dR}}(G_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}})^{\mathrm{can}}, \langle\cdot,\cdot\rangle_\lambda^{\mathrm{can}}, \mathbb{O}_{\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}}(1), \underline{\mathrm{Lie}}^\vee_{G^\vee/\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}}),\\ &((L_0 \oplus L_0^\vee(1)) \otimes_{R_0} \mathbb{O}_{\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}}, \langle\cdot,\cdot\rangle_{\mathrm{can.}}, \mathbb{O}_{\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}}(1), L_0^\vee(1) \otimes_{R_0} \mathbb{O}_{\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}})),\end{aligned}$$

and the principal bundle $\mathcal{E}_{\mathrm{M}_0}$ extends canonically to a principal bundle $\mathcal{E}_{\mathrm{M}_0}^{\mathrm{can}}$ over $\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$ by setting

$$\mathcal{E}_{\mathrm{M}_0}^{\mathrm{can}} := \underline{\mathrm{Isom}}_{\mathcal{O}\otimes_{\mathbb{Z}}\mathbb{O}_{\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}}}((\underline{\mathrm{Lie}}^\vee_{G^\vee/\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}}, \mathbb{O}_{\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}}(1)), (L_0^\vee(1) \otimes_{R_0} \mathbb{O}_{\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}}, \mathbb{O}_{\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}}(1))).$$

Definition 6.13. Let E be any R_0-algebra. For any $W \in \mathrm{Rep}_E(\mathrm{P}_0)$, we define

$$\mathcal{E}_{\mathrm{P}_0,E}^{\mathrm{can}}(W) := (\mathcal{E}_{\mathrm{P}_0}^{\mathrm{can}} \otimes_{R_0} E) \times^{\mathrm{P}_0 \otimes_{R_0} E} W,$$

called the *canonical extension* of $\mathcal{E}_{\mathrm{P}_0,E}(W)$, and define

$$\mathcal{E}^{\mathrm{sub}}_{\mathrm{P}_0,E}(W) := \mathcal{E}^{\mathrm{can}}_{\mathrm{P}_0,E}(W) \otimes_{\mathcal{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}} \mathcal{I}_{\mathsf{D}_{\infty,\mathcal{H}}},$$

called the *subcanonical extension* of $\mathcal{E}_{\mathrm{P}_0,E}(W)$, where $\mathcal{I}_{\mathsf{D}_{\infty,\mathcal{H}}}$ is the $\mathcal{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}$-ideal defining the relative Cartier divisor $\mathsf{D}_{\infty,\mathcal{H}}$ (with its reduced structure) in (3) of Theorem 1.41. We define similarly $\mathcal{E}^{\mathrm{can}}_{\mathrm{M}_0,E}(W)$ and $\mathcal{E}^{\mathrm{sub}}_{\mathrm{M}_0,E}(W)$ with P_0 (and its principal bundle) replaced accordingly with M_0 (and its principal bundle).

Lemma 6.14. *Let E be any R_0-algebra. If we view an element in $W \in \mathrm{Rep}_E(\mathrm{M}_0)$ as an element in $\mathrm{Rep}_E(\mathrm{P}_0)$ in the canonical way, then we have canonical isomorphisms $\mathcal{E}^{\mathrm{can}}_{\mathrm{P}_0,E}(W) \cong \mathcal{E}^{\mathrm{can}}_{\mathrm{M}_0,E}(W)$ and $\mathcal{E}^{\mathrm{sub}}_{\mathrm{P}_0,E}(W) \cong \mathcal{E}^{\mathrm{sub}}_{\mathrm{M}_0,E}(W)$.*

6C. *Fourier–Jacobi expansions.* Let us fix a representative $(\mathsf{Z}_n, \Phi_n, \delta_n)$ of a cusp label $[(\mathsf{Z}_n, \Phi_n, \delta_n)]$ for $\mathsf{M}_{\mathcal{H}}$ (as in Section 1C). As usual, we shall omit $\mathsf{Z}_{\mathcal{H}}$ from the notation.

Definition 6.15. The *principal* M_0*-bundle* over $C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ is the M_0-torsor

$$\mathcal{E}^{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}_{\mathrm{M}_0} := \underline{\mathrm{Isom}}_{\mathcal{O}\otimes_{\mathbb{Z}}\mathcal{O}_{C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}}((\underline{\mathrm{Lie}}^{\vee}_{G^{\vee,\natural}/C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}, \mathcal{O}_{C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}(1)), \\ (L_0^{\vee}(1) \otimes_{R_0} \mathcal{O}_{C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}, \mathcal{O}_{C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}(1))),$$

with conventions as in Definition 6.5.

Then we define $\mathcal{E}^{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}_{\mathrm{M}_0,E}(W)$ for any R_0-algebra E and any $W \in \mathrm{Rep}_E(\mathrm{M}_0)$ as in Definition 6.7.

Lemma 6.16. *Let E be any R_0-algebra. For any $W \in \mathrm{Rep}_E(\mathrm{M}_0)$, there is a canonical isomorphism*

$$(\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma} \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}})^*\mathcal{E}^{\mathrm{can}}_{\mathrm{M}_0}(W) \cong (\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma} \to C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}})^*\mathcal{E}^{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}_{\mathrm{M}_0}(W).$$

Proof. This is because of the canonical isomorphism

$$(\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma} \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}})^*\underline{\mathrm{Lie}}^{\vee}_{G^{\vee}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}} \cong (\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma} \to C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}})^*\underline{\mathrm{Lie}}^{\vee}_{G^{\vee,\natural}/C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}. \quad \square$$

By the construction of $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma} \to C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ as a formal completion, we have a natural morphism

$$(\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma} \to C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}})_*\mathcal{O}_{\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}} \to \prod_{\ell\in\mathbf{S}_{\Phi_{\mathcal{H}}}} \Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell)$$

of $\mathcal{O}_{C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}$-modules. By Lemma 6.16, we have the composition of canonical morphisms

$$\begin{aligned}\Gamma(\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}, \mathcal{E}^{\mathrm{can}}_{\mathrm{M}_0}(W)) &\to \Gamma(\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}, (\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma} \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}})^* \mathcal{E}^{\mathrm{can}}_{\mathrm{M}_0}(W)) \\ &\to \Gamma(\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}, (\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma} \to C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}})^* \mathcal{E}^{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}_{\mathrm{M}_0}(W)) \\ &\to \prod_{\ell \in \mathbf{S}_{\Phi_{\mathcal{H}}}} \Gamma(C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}, \Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell) \otimes_{\mathcal{O}_{C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}} \mathcal{E}^{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}_{\mathrm{M}_0}(W)),\end{aligned}$$

which we call the morphism of *algebraic Fourier–Jacobi expansions*.

Definition 6.17. The *ℓ-th algebraic Fourier–Jacobi morphism*

$$\Gamma(\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}, \mathcal{E}^{\mathrm{can}}_{\mathrm{M}_0}(W)) \to \Gamma(C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}, \Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell) \otimes_{\mathcal{O}_{C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}} \mathcal{E}^{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}_{\mathrm{M}_0}(W))$$

is the ℓ-th factor of the morphism of algebraic Fourier–Jacobi expansions.

Remark 6.18. If $\mathrm{Gr}^{\mathsf{Z}}_{-1} = \{0\}$, then the abelian scheme $C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}} \to \mathsf{M}^{\mathsf{Z}_n}_n$ is trivial (i.e., the structural morphism is an isomorphism), and $\mathsf{M}^{\mathsf{Z}_n}_n$ is finite over $\mathsf{S}_0 = \mathrm{Spec}(R_0)$. Hence $\Gamma(C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}, \Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell) \otimes_{\mathcal{O}_{C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}} \mathcal{E}^{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}_{\mathrm{M}_0}(W)) \cong \Gamma(\mathsf{M}^{\mathsf{Z}_n}_n, \mathcal{O}_{\mathsf{M}^{\mathsf{Z}_n}_n} \otimes_{R_0} W)$. In this case, the Fourier–Jacobi expansions are often called q-expansions (because no genuine "Jacobi theta functions" are involved).

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Idempotents in representation rings of quivers

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For an acyclic quiver Q, we solve the Clebsch–Gordan problem for the projective representations by computing the multiplicity of a given indecomposable projective in the tensor product of two indecomposable projectives. Motivated by this problem for arbitrary representations, we study idempotents in the representation ring of Q (the free abelian group on the indecomposable representations, with multiplication given by tensor product). We give a general technique for constructing such idempotents and for decomposing the representation ring into a direct product of ideals, utilizing morphisms between quivers and categorical Möbius inversion.

1. Introduction

The problem of describing a tensor product of two representations of some algebraic object has appeared in many contexts. When the category of representations in question has the Krull–Schmidt property (unique decomposition into indecomposables), the problem can be stated for representations X, Y, Z as "What is the multiplicity of Z as a direct summand in $X \otimes Y$?" This is sometimes referred to as the *Clebsch–Gordan problem*, in honor of A. Clebsch and P. Gordan, who studied the problem for certain Lie groups in the language of invariant theory.

These multiplicities for representations of the groups $\mathrm{SU}(2)$ and $\mathrm{SO}(3, \mathbb{R})$ give rise to the Clebsch–Gordan coefficients used in quantum mechanics. In the case of representations of $\mathrm{GL}(n, \mathbb{C})$, these multiplicities are the Littlewood–Richardson coefficients, which play an important role in algebraic combinatorics and Schubert calculus [Fulton 1997].

Tensor products of quiver representations have been studied by Strassen [2000] in relation to orbit-closure degenerations, and Herschend [2008b] studied the relation to bialgebra structures on the path algebra. The Clebsch–Gordan problem for

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quiver representations is solved explicitly in various situations where a classification of indecomposables is known [Herschend 2009; 2008a; 2010], whereas other results on tensor product multiplicities without a classification of indecomposables have appeared in [Kinser 2008; 2010].

In this paper, we study the tensor products of representations of a quiver Q in terms of the representation ring $R(Q)$ of the quiver. This ring has a $\mathbb{Z}$-basis consisting of indecomposable representations of Q, with sum corresponding to direct sum and product to tensor product. The same construction has been used in modular representation theory of finite groups, where it is sometimes called the Green ring [Benson 1986]. Besides the actual representations, $R(Q)$ also contains formal additive inverses of representations, and thus “differences” of representations. Understanding the multiplication in this ring can be easier than directly working with the tensor product of representations. We recall the definition and basic properties of $R(Q)$ in Section 2.

In Section 3, we solve the Clebsch–Gordan problem for projective representations of an acyclic quiver Q with an explicit formula as follows. Let x, y, w be vertices in Q and $P(x)$, $P(y)$, $P(w)$ be the corresponding indecomposable projective representations.

Theorem 1. *The multiplicity of $P(w)$ in $P(x) \otimes P(y)$ equals*

$$n_{xw}n_{yw} - \sum_{z \to w} n_{xz}n_{yz},$$

where the sum is over all arrows with terminal vertex w, and n_{ij} denotes the number of paths from i to j in the quiver.

The proof technique is to give an integral change of basis in the subring of $R(Q)$ spanned by projectives to a new basis consisting of orthogonal idempotents. These are trivial to multiply, and then changing back to the original basis gives a multiplication formula for projective representations. This motivates the construction of other sets of orthogonal idempotents in $R(Q)$.

The projective representations of Q can be concretely presented in terms of discrete data from Q, namely, the set of paths in Q. In Section 4.1, we review a general method for constructing a representation which is not necessarily projective from discrete data, using a morphism of quivers $f : Q' \to Q$, also called a coloring of Q' by Q, or a quiver over Q. We describe how such a morphism gives rise to a representation of Q via linearization, which generalizes the process of passing from a permutation representation of a finite group to the associated linear representation. This can be thought of as the opposite course of action to taking a coefficient quiver of a representation [Crawley-Boevey 1990].

Linearization allows us to study certain representations combinatorially from the discrete data in a quiver over Q. A result of Herschend states that, under some mild technical hypotheses, linearization takes the fiber product of two quivers over Q to the tensor product of their linearizations [Herschend 2010]. Thus we expect to be able to analyze the tensor product of certain representations via quivers over Q.

The first main result of the paper, presented in Section 5, is a sufficient condition for a collection of quivers over Q to give rise to a set of orthogonal idempotents in $R(Q)$ (Theorem 9). The basic idea is to form an acyclic category (a generalization of a poset) from a collection of quivers over Q, then use a categorical form of Möbius inversion to orthogonalize the linearizations of these quivers in $R(Q)$.

The motivating application for Theorem 9 is covered in Section 6. For any acyclic quiver Q, we define a category PIE of quivers over Q such that the objects in PIE are in bijection with those indecomposable representations of Q which, after restriction to some subquiver of Q, are either projective, injective, or of dimension 1 at each vertex. We describe morphisms and fiber products in PIE and show that PIE satisfies the hypotheses of Theorem 9. This allows us to associate an idempotent $e_x \in R(Q)$ to every object $x \in \mathrm{PIE}$, and to prove our second main result:

Theorem 2. *Let Q be an acyclic quiver. Then $R(Q)$ has a direct product structure*

$$R(Q) \cong \prod_{x \in \mathrm{PIE}_0} \langle e_x \rangle,$$

where $\langle e_x \rangle$ is the principal ideal generated by e_x.

Finally, we present some closed-form expressions for certain values of the Möbius function of PIE.

2. Background

A *quiver* (or directed graph) is given by $Q = (Q_0, Q_1, s, t)$, where Q_0 is a vertex set, Q_1 is an arrow set, and s, t are functions from Q_1 to Q_0 giving the start and terminal vertex of an arrow, respectively. We assume Q_0 and Q_1 are finite in this paper. For any quiver Q and field K, there is a category $\mathrm{rep}_K(Q)$ of representations of Q over K. An object $V = (V_x, \varphi_\alpha)$ of $\mathrm{rep}_K(Q)$ is an assignment of a finite dimensional K-vector space V_x to each vertex $x \in Q_0$, and an assignment of a K-linear map $\varphi_\alpha \colon V_{s\alpha} \to V_{t\alpha}$ to each arrow $\alpha \in Q_1$. For any path p in Q, we get a K-linear map φ_p by composition. Morphisms in $\mathrm{rep}_K(Q)$ are given by linear maps at each vertex which form commutative diagrams over each arrow; see the book by Assem, Simson, and Skowroński [Assem et al. 2006] for a precise definition of morphisms, and other fundamentals of quiver representations. We will fix some arbitrary field K throughout the paper and hence omit it from notation when possible.

There is a natural *tensor product* of quiver representations, induced by the tensor product in the category of vector spaces. More precisely, the tensor product of $V = (V_x, \varphi_\alpha)$ and $W = (W_x, \psi_\alpha)$ is defined pointwise: the representation $V \otimes W =$

(U_x, ρ_α) is given by

$$U_x := V_x \otimes W_x \quad \text{for } x \in Q_0,$$
$$\rho_\alpha := \varphi_\alpha \otimes \psi_\alpha \quad \text{for } \alpha \in Q_1.$$

It is not difficult to see that $\otimes$ is an additive bifunctor which is commutative and associative, and distributive over $\oplus$ (up to isomorphism). In other words, this gives the category $\operatorname{rep}(Q)$ the structure of a *tensor category* in the sense of [Deligne and Milne 1982].

The category $\operatorname{rep}(Q)$ has the *Krull-Schmidt property* [Assem et al. 2006, Theorem I.4.10], meaning that each $V \in \operatorname{rep}(Q)$ has an essentially unique expression

$$V \simeq \bigoplus_{i=1}^{n} V_i$$

as a direct sum of indecomposable representations V_i. That is, given any other expression $V \simeq \bigoplus \tilde{V}_i$ with each $\tilde{V}_i$ indecomposable, there is a permutation σ of $\{1, \dots, n\}$ such that $\tilde{V}_i \simeq V_{\sigma i}$ for all i. Thus the Clebsch–Gordan problem is well defined for $\operatorname{rep}(Q)$.

Since the tensor product distributes over direct sum, to study $V \otimes W$ we can assume without loss of generality that V and W are indecomposable. A good starting point would then be to have a description of indecomposable objects in $\operatorname{rep}(Q)$. But a description of all indecomposables is not available for most quivers, so we approach the problem by placing the representations of Q inside a ring $R(Q)$, in which addition corresponds to direct sum and multiplication corresponds to tensor product (the split Grothendieck ring of $\operatorname{rep}(Q)$). Analyzing the properties of $R(Q)$ (for example ideals, idempotents, nilpotents) gives a way of stating and approaching problems involving tensor products of quiver representations even in the absence of an explicit description of the isomorphism classes in $\operatorname{rep}(Q)$.

Let $[V]$ denote the isomorphism class of a representation V. Then define $R(Q)$ to be the free abelian group generated by isomorphism classes of representations of Q, modulo the subgroup generated by all $[V \oplus W] - [V] - [W]$. The operation

$$[V] \cdot [W] := [V \otimes W] \quad \text{for } V, W \in \operatorname{rep}(Q)$$

induces a well-defined multiplication on $R(Q)$, making $R(Q)$ into a commutative ring, called the *representation ring* of Q.

The Krull–Schmidt property of $\operatorname{rep}(Q)$ gives that $R(Q)$ is a free $\mathbb{Z}$-module with the indecomposable representations as a basis. The ring $R(Q)$ generally depends on the base field K, but we omit K from the notation since this is fixed in our case. Also we usually omit the brackets [ ] and just refer to representations of Q as elements of $R(Q)$.

Although we introduce "virtual representations" (those with some negative coefficient in the basis of indecomposables), every element $r \in R(Q)$ can be written as a formal difference

$$r = V - W, \quad \text{with } V, W \in \operatorname{rep}(Q).$$

Then any additive or multiplicative relation $z = x + y$ or $z = xy$, respectively, can be rewritten to give some isomorphism of actual representations of Q.

Remark 3. If one wishes to consider an ideal of relations I for a quiver Q, the pointwise tensor product will not generally preserve these relations and thus not be defined for representations of the bound quiver (Q, I). However, if I is generated by commutativity relations (that is, relations of the form $p - q$ for paths p, q) then the representations of (Q, I) do generate a subring of $R(Q)$. If I is generated by zero relations (relations of the form $p = 0$ for p a path), then representations of (Q, I) generate an ideal in $R(Q)$ since the tensor product of any map with a zero map is still zero. The identity element of $R(Q)$ will not satisfy the zero relations, so the ring of representations satisfying I will not generally have an identity element. Thus, if I consists of zero relations and commutativity relations, we can get a representation ring $R(Q, I)$ without identity. Throughout the paper, we will not assume that the rings of representations that we work with have identity elements, and thus the term "subring" is taken to mean a nonempty subset of a ring which is closed under subtraction and multiplication (and possibly with a different identity element).

3. Projective representations

Let Q be a quiver without oriented cycles. For every vertex $x \in Q_0$, let $P(x)$ denote the indecomposable projective representation at x. For any two vertices x, y, denote by n_{xy} the number of paths from x to y in Q. The vector space $P(x)_y$ of the representation $P(x)$ at a vertex y has a basis consisting of all paths from x to y; thus $\dim P(x)_y = n_{xy}$.

We will first show in this section that the tensor product of two projective representations is projective, and then we compute the multiplicities c_{xy}^z in the direct sum decompositions

$$P(x) \otimes P(y) = \bigoplus_{z \in Q_0} c_{xy}^z P(z).$$

Lemma 4. *The tensor product of two projectives is projective.*

Proof. Since the tensor product is distributive over the direct sum, it is enough to show the statement for indecomposable projectives. Let i, j be two vertices in Q. We need to show that $P(i) \otimes P(j)$ is projective.

We will proceed by induction on the number of vertices in Q. If this number is one, then $i = j$, and $P(i)$ is a representation of dimension one, since Q has no oriented cycles, and thus $P(i) \otimes P(i) = P(i)$ is projective.

Now suppose Q has more than one vertex, and let i_0 be a sink in Q. If $i = i_0$ then $P(i)$ is the simple representation $S(i)$, and $P(i) \otimes P(j)$ is equal to $P(i)^{\oplus n_{ji}}$; in particular, it is equal to zero if there is no path from j to i. This shows that the lemma holds if $i = i_0$, and a similar argument shows that the lemma holds if $j = i_0$.

Suppose now that i and j are different from i_0. Denote by Q' the quiver obtained from Q by deleting the vertex i_0 and all arrows incident to it. Let $P(i)|_{Q'}$ be the representation of Q' obtained by restricting to the subquiver Q'. Since i_0 is a sink in Q, we have that $P(i)|_{Q'}$ is a projective Q' representation and therefore the induction hypothesis implies that $P(i)|_{Q'} \otimes P(j)|_{Q'}$ is a projective Q' representation, thus there is an isomorphism

$$f \ : \ \bigoplus_k c_{ij}^k P_{Q'}(k) \ \longrightarrow \ P(i)|_{Q'} \otimes P(j)|_{Q'},$$

for some $c_{ij}^k \geq 0$ and $P_{Q'}(k)$ the indecomposable projective Q' representation at vertex k. Let $\tilde{P} = (\tilde{P}_x, \tilde{\varphi}_\alpha)_{i \in Q_0, \alpha \in Q_1}$ be the corresponding projective Q representation, more precisely,

$$\tilde{P} = \bigoplus_k c_{ij}^k P_Q(k).$$

Let us use the notation $P(i) \otimes P(j) = (M_x, \varphi_\alpha)_{x \in Q_0, \alpha \in Q_1}$. Then for every vertex x, the vector space M_x has a basis consisting of pairs (c^i, c^j), where c^i is any path from i to x and c^j any path from j to x. On the other hand, since i, j are both different from i_0, the vector space M_{i_0} has a basis consisting of pairs $(c^i\alpha, c^j\beta)$, where α, β are arrows with terminal point i_0, and c^i is a path from i to $s(\alpha)$ and c^j is a path from j to $s(\beta)$. The maps φ_α are given by $\varphi_\alpha(c^i, c^j) = (c^i\alpha, c^j\alpha)$, in particular,

$$\bigoplus_{\alpha : x \to i_0} \varphi_\alpha : \bigoplus_{\alpha : x \to i_0} M_x \to M_{i_0}$$

is injective.

The morphism f induces a morphism $\tilde{f} = (\tilde{f}_x)_{x \in Q_0} : \tilde{P} \to P(i) \otimes P(j)$, where $\tilde{f}_x = f_x$ if $x \neq i_0$, and $\tilde{f}_{i_0}$ is defined on any path $c\alpha$, with α an arrow with $t(\alpha) = i_0$, as $\tilde{f}_{i_0}(c\alpha) = \varphi_\alpha \tilde{f}_{s(\alpha)}(c)$. Clearly, $\tilde{f}_x$ is an isomorphism for every $x \neq i_0$, and we will show that $\tilde{f}_{i_0}$ is injective.

Now in the commutative diagram

$$\begin{tikzcd}
\bigoplus_{\alpha:t(\alpha)=i_0}\tilde{P}_{s(\alpha)} \arrow[r, "\bigoplus_{\alpha:t(\alpha)=i_0}\tilde{\varphi}_\alpha"] \arrow[d, "\bigoplus_{\alpha:t(\alpha)=i_0}\tilde{f}_x"'] & \tilde{P}_{i_0} \arrow[d, "\tilde{f}_{i_0}"] \\
\bigoplus_{\alpha:t(\alpha)=i_0}M_{s(\alpha)} \arrow[r, "\bigoplus_{\alpha:t(\alpha)=i_0}\varphi_\alpha"] & M_{i_0}
\end{tikzcd}$$

the left column and the top row are isomorphisms, and the bottom row is injective. Therefore the right column $\tilde{f}_{i_0}$ is injective too.

Thus $\tilde{f} : \tilde{P} \to P(i) \otimes P(j)$ is injective with semisimple projective cokernel $P(i_0)^{\oplus t}$ for some integer t, and we get a split short exact sequence

$$0 \to \tilde{P} \to P(i) \otimes P(j) \to P(i_0)^{\oplus t} \to 0,$$

since $P(i_0)^{\oplus t}$ is projective. This shows that $P(i) \otimes P(j)$ is projective. □

The lemma implies that the free abelian group generated by all indecomposable projectives $P(x)$, $x \in Q_0$ has a ring structure whose addition is given by the direct sum and multiplication by the tensor product (that is, the projectives span a subring of $R(Q)$). As an additive group, this is isomorphic to $\mathbb{Z}^{Q_0}$ and an isomorphism is given by the Cartan matrix

$$C = \big[n_{xy}\big]_{x,y\in Q_0} = \big[\underline{\dim}\, P(1) \ \cdots \ \underline{\dim}\, P(n)\big],$$

where $n = \#Q_0$ and $\big[\underline{\dim}\, P(1) \ \cdots \ \underline{\dim}\, P(n)\big]$ is the $n \times n$ integer matrix whose x-th column is equal to the dimension vector of $P(x)$. The Cartan matrix is invertible. Since the dimension vector is multiplicative with respect to the tensor product, this is a ring isomorphism.

We also have that the (x, y) entry of the transposed inverse matrix $(C^{-1})^t$ can be computed by the formula $\dim \mathrm{Hom}(S(x), S(y)) - \dim \mathrm{Ext}(S(x), S(y))$; see for example [Assem et al. 2006, III.3.13]. Therefore

$$(C^{-1})^t_{x,y} = \begin{cases} 1 & \text{if } x = y, \\ -(\text{number of arrows } x \to y) & \text{if } x \neq y. \end{cases}$$

Let ϵ_x denote the standard basis vector $[0, \dots, 0, 1, 0, \dots, 0]^t$ with 1 at position x, and define $e(x)$ to be the inverse image of ϵ_x under the above isomorphism. In other words

$$e(x) = [P(1) \cdots P(n)]C^{-1}\epsilon_x,$$

where $[P(1) \cdots P(n)]$ denotes the $1 \times n$ matrix whose entries are the indecomposable projective modules, and $C^{-1}\epsilon_x$ is the x-th column of C^{-1}.

It follows that

$$e(x) = P(x) - \sum_{x \to y} P(y), \tag{1}$$

where the sum is over all arrows starting at x, and

$$P(x) = \sum_{z} n_{xz} e(z). \tag{2}$$

We are now ready to prove the main result of this section.

Theorem 5. *Let $x, y \in Q_0$. Then*

$$P(x) \otimes P(y) = \bigoplus_{w \in Q_0} c_{xy}^{w} P(w),$$

with $c_{xy}^{w} = n_{xw} n_{yw} - \sum_{z \to w} n_{xz} n_{yz}$, where the sum is over all arrows with terminal vertex w.

Proof. The proof is a simple computation in the representations ring with the orthogonal idempotents $\{e(z) \mid z \in Q_0\}$. We have

$$\begin{aligned} P(x) \otimes P(y) &= \textstyle\sum_z n_{xz} e(z) \sum_z n_{yz} e(z) \\ &= \textstyle\sum_z n_{xz} n_{yz} e(z), \end{aligned}$$

since the $e(z)$ are orthogonal idempotents. Now using (1), we get

$$P(x) \otimes P(y) = \sum_{z} n_{xz} n_{yz} \Big(P(z) - \sum_{z \to u} P(u) \Big).$$

For a fixed vertex w, we can compute c_{xy}^{w} by collecting terms. We then obtain $c_{xy}^{w} = n_{xw} n_{yw} - \sum_{z \to w} n_{xz} n_{yz}$, where the sum is over all arrows with terminal vertex w. This completes the proof. □

4. Linearization and Möbius rings

4.1. *Quivers over Q and linearization.* A *morphism of quivers* $f' \colon Q' \to Q$ sends vertices to vertices and arrows to arrows, and satisfies $s(f'(\alpha)) = f'(s(\alpha))$ and $t(f'(\alpha)) = f'(t(\alpha))$ for each arrow $\alpha \in Q'_1$. A *quiver over Q* is a pair (Q', f') where Q' is a quiver, and $f' \colon Q' \to Q$ is a morphism of quivers called the *structure map* of (Q', f'). A *morphism g of quivers over Q* is a morphism of quivers which commutes with the structure maps to Q:

$$\begin{tikzcd} Q' \arrow[rr, "g"] \arrow[dr, "f'"'] & \arrow[d, phantom, "\circlearrowleft"] & Q'' \arrow[dl, "f''"] \\ & Q & \end{tikzcd} \tag{3}$$

So the collection of all quivers over a given Q forms a category denoted by $\downarrow Q$, and we write $g \in \mathrm{Hom}_{\downarrow Q}(Q', Q'')$.

To simplify the notation, we consider the maps φ_α of a representation V to be defined on the total vector space $\bigoplus_{x \in Q_0} V_x$ by taking $\varphi_\alpha(V_y) = 0$ when $y \neq s(\alpha)$. If $f' : Q' \to Q$ is a morphism of quivers then the *pushforward* $f'_* V = (U_x, \rho_\alpha) \in \mathrm{rep}(Q)$ of a representation $V = (V_x, \varphi_\alpha) \in \mathrm{rep}(Q')$ is given by

$$U_x := \bigoplus_{y \in f'^{-1}(x)} V_y \quad \text{for } x \in Q_0, \tag{4}$$

$$\rho_\alpha := \sum_{\beta \in f'^{-1}(\alpha)} \varphi_\beta \quad \text{for } \alpha \in Q_1. \tag{5}$$

Extending f'_* linearly to $R(Q')$, we get an induced homomorphism

$$f'_* \colon R(Q') \to R(Q)$$

between additive groups, which will not generally be a ring homomorphism.

For a quiver Q, we denote by $\mathbb{1}_Q \in \mathrm{rep}(Q)$ the *identity representation* of Q: it has a one-dimensional vector space K at each vertex, and the identity map over each arrow. (The name comes from the fact that this is the identity element of the representation ring $R(Q)$). When $S \subset Q$ is a subquiver, we can consider $\mathbb{1}_S$ to be a representation of Q via extension by zero: that is, we assign the zero map or vector space to each arrow or vertex outside of S. More generally, we can take any quiver over Q and get a representation of Q by pushing forward the identity representation. Thus we get a map on objects

$$\begin{aligned} L \colon \downarrow Q &\longrightarrow \mathrm{rep}(Q) \\ (Q', f') &\longmapsto f'_* \mathbb{1}_{Q'} \end{aligned}$$

which we call the *linearization* map. The representation $f'_* \mathbb{1}_{Q'}$ has a standard basis $\{e_x \mid x \in Q'_0\}$. For example, when (Q', f') is the inclusion of a single vertex in Q, then its linearization is the simple representation concentrated at that vertex. When Q' is a quiver of type A with some technical conditions on f', the linearization is a string module. Similarly, we get a band module or tree module when Q is of type $\tilde{A}$ or when it is a tree, respectively.

Remark 6. There is a natural way that one would try to make the linearization functorial: if g is a morphism in $\downarrow Q$ as illustrated in (3), one might try to send a standard basis vector e_x of $f'_*\mathbb{1}_{Q'}$ to the vector $e_{g(x)}$ in $f''_*\mathbb{1}_{Q''}$. However, this will *not* be a morphism of quiver representations, in general. To see this, one need only take $Q = \bullet \to \bullet$ and consider the map of quivers given by the inclusion of the left vertex. The corresponding map of vector spaces just described would be a nontrivial morphism from the simple representation of dimension vector $(1, 0)$ to the indecomposable of dimension vector $(1, 1)$, which is not possible. By working in some (not necessarily full) subcategory of $\downarrow Q$, one may have some success in making the linearization functorial; see for example [Crawley-Boevey 1989; Kinser 2010, Theorem 18].

The categorical product of two objects $(Q', f') \times_Q (Q'', f'')$, which we refer to as the *fiber product* of Q' and Q'' over Q, exists in $\downarrow Q$. It can be realized concretely as having vertex set

$$(Q' \times_Q Q'')_0 = \{(x', x'') \in Q'_0 \times Q''_0 \mid f'(x') = f''(x'')\}$$

consisting of pairs of vertices lying over the same vertex of Q, with an arrow

$$(x', x'') \xrightarrow{(\alpha', \alpha'')} (y', y'')$$

for each pair of arrows $(x' \xrightarrow{\alpha'} y', x'' \xrightarrow{\alpha''} y'') \in Q'_1 \times Q''_1$ such that $f'(\alpha') = f''(\alpha'')$. This common value should be taken as the value of the structure map on the arrow (α', α'').

4.2. ***Acyclic categories and the Möbius function.*** In order to use an inclusion/exclusion technique to orthogonalize elements of the representation ring, we need a categorial analogue of Möbius inversion. This is provided by the work of Haigh [1980], and one may also see the more recent works [Leinster 2008; Kozlov 2008, Chapter 10]. We summarize here the tools that we need from this construction.

Following the terminology of Kozlov's book, we call a small category *acyclic* if the only endomorphisms are identity morphisms and only identity morphisms are invertible. This terminology is justified by the observation that if we draw a directed graph whose vertices are the objects and arrows are the morphisms of an acyclic category, then this graph will be acyclic. For brevity, we denote by $[x, y]_{\mathscr{C}}$ the number of morphisms from an object x to an object y in $\mathscr{C}$. An acyclic category $\mathscr{C}$ with finitely many objects $\mathscr{C}_0$ and morphisms $\mathscr{C}_1$ admits a *Möbius function*

$$\mu_{\mathscr{C}} : \mathscr{C}_0 \times \mathscr{C}_0 \to \mathbb{Z}$$

with the following properties:

$$\mu_{\mathscr{C}}(x, x) = 1 \qquad \text{for all } x,$$

$$\sum_{z \in \mathscr{C}_0} [x, z]_{\mathscr{C}}\, \mu_{\mathscr{C}}(z, y) = \begin{cases} 0 & \text{for } x \neq y, \\ 1 & \text{for } x = y. \end{cases}$$

We drop the subscripts $\mathscr{C}$ when this can cause no confusion.

For example, when $\mathscr{C}$ is a poset (whose elements are taken to be the objects of $\mathscr{C}$, with a unique morphism from x to y if and only if $x \leq y$), we get exactly the classical Möbius function of the poset [Stanley 1997, Section 3.7].

For any acyclic category $\mathscr{C}$, let $H_{\mathscr{C}}$ be the *Hom matrix* associated to $\mathscr{C}$, whose rows and columns are indexed by the objects of $\mathscr{C}$ such that the entry H_{xy} in row x and column y is $[x, y]$. One can choose an ordering of the objects of $\mathscr{C}$ such that this matrix is upper triangular with ones on the diagonal since $\mathscr{C}$ is acyclic, and then one can see from the definition of matrix multiplication that $M \stackrel{\text{def}}{=} H^{-1}$ will have the value $\mu(x, y)$ in row x, column y.

A few facts which will be used frequently are noted here:

(a) From the matrix description we see that

$$\sum_{z \in \mathscr{C}_0} \mu(x, z)[z, y] = 0$$

for all $x \neq y$.

(b) If $[x, y] = 0$, then $\mu(x, y) = 0$.

(c) The value $\mu(x, y)$ can be recursively calculated as

$$\mu(x, y) = -\sum_{x < z \leq y} [x, z]\mu(z, y), \tag{6}$$

where we write $x \leq y$ if there exists a morphism from x to y.

4.3. ***The Möbius ring of a finite acyclic category.*** The *Möbius ring* $M(\mathscr{C})$ of an acyclic category $\mathscr{C}$ [Haigh 1980] generalizes an object of the same name associated to a poset [Greene 1973]. The additive group of $M(\mathscr{C})$ is free on the set of objects of $\mathscr{C}$. A direct (but somewhat opaque) definition of the product xy of two basis vectors can be given, but we will first give a more computationally useful formulation. For each object x of $\mathscr{C}$, define an element

$$\delta_x \stackrel{\text{def}}{=} \sum_{z \in \mathscr{C}_0} \mu(z, x) z \tag{7}$$

in $M(\mathscr{C})$. The additive group of $M(\mathscr{C})$ is freely generated by $\{\delta_x\}_{x \in \mathscr{C}_0}$ also, since the Hom matrix and its inverse (which have determinant 1) give the change of basis between this and the defining basis. Then we just declare these basis elements to be orthogonal idempotents in $M(\mathscr{C})$:

$$\delta_x \delta_y = \begin{cases} \delta_x & \text{if } x = y, \\ 0 & \text{if } x \neq y, \end{cases} \tag{8}$$

and extend by $\mathbb{Z}$-linearity (so $M(\mathscr{C})$ is commutative). We can recover the original basis elements as

$$x = \sum_{z \in \mathscr{C}_0} [z, x]\, \delta_z, \tag{9}$$

and by substitution the product of two such elements is then

$$xy = \sum_{z \in \mathscr{C}_0} \biggl(\sum_{w \in \mathscr{C}_0} \mu(z, w)[w, x][w, y] \biggr) z, \tag{10}$$

recovering the standard definition.

Lemma 7. *If x is a terminal object for $\mathscr{C}$ (that is, each object of $\mathscr{C}$ has a unique morphism to x), then x serves as the identity element of $M(\mathscr{C})$.*

Proof. If $[w, x] = 1$ for all $w \in \mathscr{C}_0$, the formula (10) simplifies to

$$xy = \sum_{z \in \mathscr{C}_0} \biggl(\sum_{w \in \mathscr{C}_0} \mu(z, w)[w, y] \biggr) z.$$

The second sum is always 0 unless $z = y$, by fact (a) of the previous subsection, and 1 when $z = y$; thus we have $xy = y$ for all $y \in \mathscr{C}_0$. □

Remark 8. The finiteness of $\mathscr{C}$ can be relaxed in various ways. For example, the definition (7) still makes sense if, for each object x, there are only finitely many objects z such that $[z, x] \neq 0$.

5. Main result on Möbius rings

Let $\mathscr{C}$ be a full, acyclic subcategory of $\downarrow Q$. From here on, we will always assume that each object of $\mathscr{C}$ is a connected quiver over Q. Let $L : \mathscr{C} \to \operatorname{rep} Q$ be the linearization, which we recall is defined only on the objects of $\mathscr{C}$. Then L extends by $\mathbb{Z}$-linearity to a map $M(\mathscr{C}) \to R(Q)$, which we also denote by L. In this section, we will show that L is a ring homomorphism when $\mathscr{C}$ satisfies suitable conditions, and study the image of L in $R(Q)$. We give sufficient conditions on the category $\mathscr{C}$ so that this subring is isomorphic to the Möbius ring $M(\mathscr{C})$ of the category $\mathscr{C}$ and construct a basis of idempotents in that case.

We say that the category $\mathscr{C}$ is *closed under fiber products* if the fiber product of quivers in $\mathscr{C}$ is a disjoint union of quivers in $\mathscr{C}$. We need one more technical condition for linearization to behave well with respect to tensor product. Following the terminology of [Herschend 2010], we say that a morphism of quivers $f'\colon Q' \to Q$ is a *wrapping* if, for every pair of vertices $i', j' \in Q'_0$, the induced map

$$\{\text{arrows from } i' \text{ to } j'\} \xrightarrow{f'} \{\text{arrows from } f'(i') \text{ to } f'(j')\}$$

is injective. Intuitively, this says that f' does not collapse parallel arrows. The fiber product of two wrappings is again a wrapping.

Theorem 9. *Let $\mathscr{C}$ be an acyclic subcategory of $\downarrow Q$ whose objects are connected and wrappings, which is closed under fiber products, and such that for all $x, y \in \mathscr{C}$,*

$$L(x) \textit{ is indecomposable in } \operatorname{rep} Q \textit{ and } L(x) \not\cong L(y) \textit{ if } x \neq y. \tag{11}$$

Then the subring of $R(Q)$ generated by $L(\mathscr{C})$ is isomorphic to the Möbius ring $M(\mathscr{C})$ of $\mathscr{C}$.

Proof. The Möbius ring $M(\mathscr{C})$ has the two $\mathbb{Z}$-bases

$$\{x \mid x \in \mathscr{C}\} \quad \text{and} \quad \Big\{\delta_x = \sum_{z \in \mathscr{C}_0} \mu(z, x) z \;\Big|\; x \in \mathscr{C}\Big\}.$$

Consider the linearization map

$$L : M(\mathscr{C}) \longrightarrow R(Q),\ x = (Q', f') \mapsto L(x) = f'_* \mathbb{1}_{Q'}.$$

We will show that L is an injective ring homomorphism.

The map L is additive by definition, and by condition (11), L is injective. In $M(\mathscr{C})$ the product is given by $xy = \sum_{z \in \mathscr{C}_0} [z, x][z, y]\delta_z$, for $x, y \in \mathscr{C}$, using the basis of orthogonal idempotents. Now let $x \times_Q y = \sqcup_i w_i$ be the decomposition into connected components, where each $w_i \in \mathscr{C}$. For a fixed z, the set of pairs of maps $\{(z \xrightarrow{f} x,\ z \xrightarrow{g} y)\}$ is in bijection with the set of maps $\bigcup_i \{z \xrightarrow{h} w_i\}$, by the universal property of fiber products and the assumption that elements of $\mathscr{C}$ are connected quivers. This implies that $[z, x][z, y] = \sum_i [z, w_i]$ and so after applying L we have that

$$L(xy) = \sum_{z \in \mathscr{C}_0} \sum_i [z, w_i] L(\delta_z).$$

On the other hand, $L(x) \otimes L(y)$ is isomorphic to the linearization of $x \times_Q y$, by [Herschend 2010, Corollary 1] (which requires that x, y be wrappings). In the representation ring $R(Q)$, this gives $L(x)L(y) = \sum_i Lt(w_i)$. Now since we already know L is a homomorphism of additive groups, we can use formula (9) to obtain

$$\sum_i L(w_i) = \sum_i \sum_{z \in \mathscr{C}_0} [z, w_i]\, L(\delta_z).$$

This shows that L is a ring homomorphism, and moreover, the image of L is the subring of $R(Q)$ generated by $L(\mathscr{C})$; thus it is isomorphic to $M(\mathscr{C})$. $\square$

Corollary 10. *Let the assumptions be as in Theorem 9.*

(1) *The subring of $R(Q)$ generated by $L(\mathscr{C})$ has a basis $B = \{L(\delta_x) \mid x \in \mathscr{C}\}$ of orthogonal idempotents.*

(2) *When $(Q, \mathrm{id}) \in \mathscr{C}$, this results in a direct product decomposition*

$$R(Q) \cong \prod_{x \in \mathscr{C}} \langle L(\delta_x) \rangle,$$

where $\langle L(\delta_x) \rangle$ is the principal ideal of $R(Q)$ generated by $L(\delta_x)$.

Proof. Statement (1) is immediate from the theorem. Then statement (2) follows because the identity element of $R(Q)$ is the linearization of the identity element (Q, id) of $M(\mathscr{C})$, so $1 = \sum_x L(\delta_x)$ is a decomposition as a sum of orthogonal idempotents in $R(Q)$. □

6. The PIE category

In Section 3 we have seen that the projective representations of an acyclic quiver Q span a subring of $R(Q)$, in which multiplication can more easily be carried out using a basis of orthogonal idempotents. The duality functor gives a ring isomorphism $R(Q) \cong R(Q^{\mathrm{op}})$, so the same can be said for the injective representations of Q. In [Kinser 2010, Section 4.1], a similar construction is carried out for the collection of idempotent representations of Q (those which are the identity representation of some subquiver).

So the natural question arises as to whether these three sets of idempotents in $R(Q)$ have a common refinement. That is, we would like to find a subring of $R(Q)$ containing a complete set of orthogonal idempotents which span the set of projective, injective, and idempotent representations. The first problem one encounters is that the tensor product of a projective with an idempotent representation (which results in the restriction of the projective to a subquiver) is not necessarily projective, injective, or idempotent. So we need to enlarge the scope of representations that we look at.

6.1. ***Subprojective and subinjective representations.*** Recall that the *support* of a representation V of Q, written $\operatorname{supp} V$, is the subquiver of Q consisting of the vertices to which V assigns a nonzero vector space, and the arrows to which V assigns a nonzero map. For an object $X = (Q', f')$ of $\downarrow Q$, we define $\operatorname{supp} X = f'(Q')$, so that $\operatorname{supp} X = \operatorname{supp} L(X) \subseteq Q$ when X is a wrapping.

Definition 11. A representation V of a quiver is *subprojective* or *subinjective* if it restricts to a projective or injective representation of its support, respectively.

To utilize Theorem 9 in the study of tensor products of these representations, we must first present them as linearizations of some quivers over Q.

Definition 12. A *structure quiver* for $V \in \operatorname{rep}(Q)$ is an object $X \in \downarrow Q_0$ such that $L(X) \simeq V$. A structure quiver $X = (Q', f')$ for V is said to be *minimal* if any other structure quiver $Y = (Q'', f'')$ for V has at least as many arrows as Q'.

In the language of [Ringel 1998], a structure quiver is a "coefficient quiver" in some basis. By dimension reasons, any two structure quivers for a given V have the same number of vertices over each vertex of Q. But the following example shows a basic way that a structure quiver can fail to be minimal.

Example 13. Take for our base quiver

$$Q = \begin{tikzcd} \underset{3}{\bullet} \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\beta"'] & \underset{2}{\bullet} \arrow[rr, "\gamma"] & & \underset{1}{\bullet} \end{tikzcd}$$

and consider $P(3)$, the projective representation associated to vertex 3. The "natural" structure quiver for $P(3)$ is

$$Q' = \begin{tikzcd} & \underset{2}{\bullet} \arrow[r, "\gamma"] & \underset{1}{\bullet} \\ \underset{3}{\bullet} \arrow[ur, "\alpha"] \arrow[dr, "\beta"'] & & \\ & \underset{2}{\bullet} \arrow[r, "\gamma"] & \underset{1}{\bullet} \end{tikzcd}$$

(where we mark the vertices and edges according to what they lie over in Q). But one can quickly see that the linearization of

$$Q'' = \begin{tikzcd} & \underset{2}{\bullet} \arrow[r, "\gamma"] \arrow[ddr, "\gamma"] & \underset{1}{\bullet} \\ \underset{3}{\bullet} \arrow[ur, "\alpha"] \arrow[dr, "\beta"'] & & \\ & \underset{2}{\bullet} \arrow[r, "\gamma"] & \underset{1}{\bullet} \end{tikzcd}$$

will also give a representation isomorphic to $P(3)$, and that we have an embedding $Q' \subseteq Q''$ as quivers over Q.

6.2. ***Definition of the* PIE *category.*** We now present the natural structure quivers for subprojective, subinjective, and idempotent representations of an acyclic quiver. Then we justify calling them "natural" by showing that these are the unique minimal structure quivers for these representations. For each subquiver $T \subseteq Q$, consider the following quivers over Q.

- When T has a unique source t, we define the vertex set of the quiver P_T as the set of all paths in T starting at t; the structure map as a quiver over

Q sends such a path to its endpoint in Q. We put an arrow from the vertex associated to a path p to the one for a path q in P_T exactly when q is obtained by concatenating a single arrow α onto the end of p; in this case, that arrow in P_T is sent to the arrow $\alpha \in Q_1$ by the structure map. So in Example 13, we have $Q' = P_Q$. In [Enochs et al. 2004, Section 2], this is called the component of the "(left) path space" of Q associated to t.

- When T has a unique sink, I_T is defined dually; its vertex set is the collection of all paths within T that end at the sink.
- For any subquiver $T \subseteq Q$, the inclusion of T into Q will be denoted by E_T when being considered as a quiver over Q.

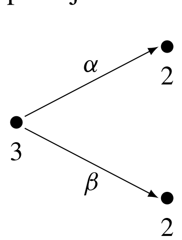
It will always be implicit that P_T or I_T is only defined when T has a unique source or sink, respectively.

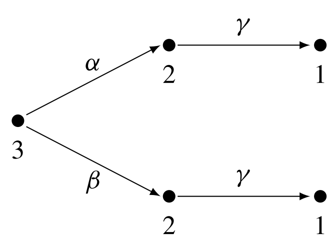
Remark 14. There are coincidences among the P-, I-, and E-type objects, which we record for reference later. Two distinct paths are said to be *parallel* if they start at the same vertex and end at the same vertex. Then $E_T = P_T$ if and only if T has a unique source and no parallel paths, while $E_T = I_T$ if and only if T has a unique sink and no parallel paths. We have $I_T = P_T$ exactly when T is just a single path, in which case we get that these both equal E_T as well.

Definition 15. Let PIE be the full subcategory of the category of quivers over Q whose objects are all the P_T, I_T, and E_T as T varies over all subquivers of Q.

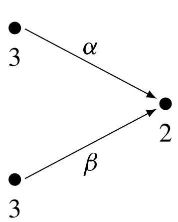
Example 16. With Q as in Example 13, the distinct objects of PIE are as follows:

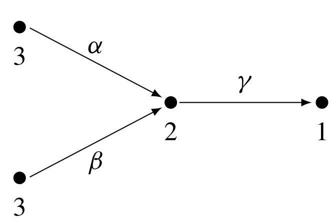
- The ten connected subquivers of Q.
- The P-type objects which are not subquivers:

$P_{\alpha\beta} =$ 

$P_Q =$ 

- The I-type objects not included above:

$I_{\alpha\beta} =$ 

$I_Q =$ 

Lemma 17. *The objects of* PIE *are the unique minimal structure quivers for the indecomposable subprojective*, *subinjective*, *and idempotent representations.*

Proof. It is easy to see that $L(P_T)$ is subprojective, $L(I_T)$ is subinjective, and $L(E_T)$ is idempotent, and that each of these is indecomposable; this is just the standard construction of projectives and injectives which can be found, for example, in [Assem et al. 2006, Lemma III.2.4]. Thus, we need to show that they are minimal and uniquely so.

If $X = (Q', f')$ is such that $L(X) = L(E_T)$ is an idempotent representation, there is exactly one vertex of Q' over each vertex of T. Consequently, all arrows of Q' over a given $\alpha \in T_1$ must be parallel; taking precisely one arrow over each $\alpha \in T_1$ is then the unique minimal choice, which is exactly the definition of E_T.

Now the P-type and I-type cases are dual (each follows from the other by working with quivers over Q^{op}), so it is enough to prove the statement for the P-type case. Suppose $X = (Q', f')$ is such that $L(X) = L(P_T)$, and fix an arrow $\alpha \in T_1$. Then the map $L(X)_\alpha$ is injective with rank equal to the number of paths in T from the source of T to $s(\alpha)$, by the description of projectives. Since a rank r map cannot be the sum of strictly less than r rank one maps, the pushforward construction (5) requires that Q' must have at least this many arrows over α. So P_T is minimal since it has precisely this many arrows.

To see that it is unique, we use induction on the number of arrows in T. When T has no arrows the uniqueness is clear. Now if T has arrows, let α be an arrow ending at some sink of T, and denote by $\tilde{T}$ the connected component of $T \setminus \alpha$ containing the source of T (that is, remove α, and if that isolates the vertex $t(\alpha)$, discard that vertex). Then working with representations over $\tilde{T}$ (which has a unique source), we define $\tilde{Q}' = f'^{-1}(\tilde{T})$ and see that the linearization of $\tilde{X} = (\tilde{Q}', f')$ is $L(P_{\tilde{T}})$.

Let $\{v'_1, \dots, v'_n\}$ be the vertices of Q' lying over $s(\alpha)$. Each v'_i must have at least one outgoing arrow α'_i in Q' lying over α, because otherwise the vector corresponding to v'_i in $L(X)$ would be in the kernel of the linear map over α, which is not possible since the maps in a projective representation are injective. By dimension count at the vertex $t(\alpha)$, each α'_i ends at a new vertex w'_i of Q' which is not in $\tilde{Q}'$. By the assumption that X is a minimal structure quiver for $L(P_T)$, we know that Q' has the same number of arrows as P_T. If some v'_i had more than one outgoing arrow over α, that would leave $\tilde{Q}'$ with fewer arrows than $P_{\tilde{T}}$, contradicting the fact that $P_{\tilde{T}}$ is minimal. So there are exactly n arrows over α in Q', and $\tilde{Q}'$ has the same number of arrows as $P_{\tilde{T}}$. By induction, we get that $\tilde{X} = P_{\tilde{T}}$, then the remaining arrows over α are configured exactly so that $X = P_T$. □

It is worth remarking that we have proven something slightly stronger, namely, that an object of the PIE category actually embeds in any quiver over Q giving the same linearization.

6.3. *Morphisms in* PIE. In order to see that the Theorem 9 can be applied to PIE, and eventually do some computations in its Möbius ring, we need to know the cardinalities of Hom sets. We first record some simple facts, continuing to use the notation $[X, Y]$ for the cardinality of $\mathrm{Hom}_{\downarrow Q}(X, Y)$.

Lemma 18. *Let X, Y be quivers over Q.*

(a) $[X, Y] = 0$ *unless* $\operatorname{supp} X \subseteq \operatorname{supp} Y$.

(b) *For $T \subseteq Q$ we have*

$$[X, E_T] = \begin{cases} 1 & \text{if } \operatorname{supp}(X) \subseteq T, \\ 0 & \text{otherwise.} \end{cases} \tag{12}$$

Proof. We can see (a) immediately from the diagram (3) in the definition of morphisms in $\downarrow Q$. Then specializing this diagram to the situation of (b), we see that the dotted line in

$$\begin{tikzcd} Q' \arrow[rr, dashed] \arrow[dr, "f'"'] \arrow[rr, phantom, shift right=6ex, "\circlearrowleft"] & & E_T \arrow[dl, hook', "\subseteq"] \\ & Q & \end{tikzcd}$$

can only be filled in when $\operatorname{supp}(X) = f'(Q') \subseteq T$, and only by the morphism f'. □

Describing maps to P-type objects is slightly more complicated, but we can get enough of a description to count morphism sets in PIE.

Proposition 19. *Let $T \subseteq Q$ be a subquiver, and $X = (Q', f')$ a quiver over Q with* $\operatorname{supp} X \subseteq T$.

(a) *Given a map of vertex sets $g_0 \colon Q'_0 \to (P_T)_0$ that respects the structure maps to Q, there is a unique map of arrow sets $g_1 \colon Q'_1 \to (P_T)_1$ which respects the structure maps to Q and also the start vertex function s.*

(b) *The maps in (a) give a morphism $g = (g_0, g_1) \colon Q' \to P_T$ in $\downarrow Q$ if and only if, when regarding the vertices of P_T as paths in T, the equation*

$$g_0(t(\alpha')) = g_0(s(\alpha'))f'(\alpha') \tag{13}$$

holds for each arrow $\alpha' \in Q'_1$. (The operation on the right hand side is concatenation.)

Proof. Given a map between vertex sets as in the hypotheses of (a), we explicitly describe the resulting map of arrows. For each $\alpha' \in Q'_1$, the arrow $g_1(\alpha')$ in P_T must start at $g_0(s(\alpha'))$ to respect the s function. To respect the structure maps to Q, this arrow must be labeled with $f'(\alpha')$. But in P_T, each vertex has at most one outgoing arrow labeled by a given arrow in Q, and the assumption that $\operatorname{supp} X \subseteq T$

guarantees that there is such an arrow for this vertex. So we can define $g_1(\alpha')$ as the unique arrow of P_T lying over $f'(\alpha')$ in Q and satisfying $s(g_1(\alpha')) = g_0(s(\alpha'))$. This shows (a).

Now suppose that the resulting map is a morphism in $\downarrow Q$. Then it must respect both the start and terminal vertex functions s, t, and so an arrow $s(\alpha') \xrightarrow{\alpha'} t(\alpha')$ is sent to

$$g_0(s(\alpha')) \xrightarrow{g_1(\alpha')} g_0(t(\alpha'))$$

in P_T, with $g_1(\alpha')$ lying over $f'(\alpha')$. But the construction of P_T is such that this is equivalent to (13). Conversely, we need to see that the function t is respected when this equation holds for all arrows. Since at least $s(g_1(\alpha')) = g_0(s(\alpha'))$, any arrow $s(\alpha') \xrightarrow{\alpha'} t(\alpha')$ is sent to an arrow

$$g_0(s(\alpha')) \xrightarrow{g_1(\alpha')} t(g_1(\alpha'))$$

in P_T. But then $g_1(\alpha')$ lying over $f'(\alpha')$ gives the equation of paths

$$t(g_1(\alpha')) = g_0(s(\alpha'))f'(\alpha')$$

by the construction of P_T again, which is exactly equal to $g_0(t(\alpha'))$ by assumption. So t is respected by these maps of vertices and arrows, and thus g is a morphism in $\downarrow Q$. □

Corollary 20. *If Q' has a unique source i', then any morphism $g\colon Q' \to P_T$ in $\downarrow Q$ is uniquely determined by $g(i')$. Consequently, $[P_S, P_T]$ is equal to the number of paths in T from the source of T to the source of S if $S \subseteq T$, and* 0 *otherwise.*

Proof. Part (a) of Proposition 19 tells us that the images of arrows under g are determined by the images of the vertices. Repeated use of (13) shows that $g(i')$ determines $g(j')$ for any vertex j' lying on a path starting at i'. Since i' is the unique source, this determines g completely.

To show the second statement of the corollary, observe first that if $S \nsubseteq T$ then Lemma 18 (a) implies that $[P_S, P_T] = 0$. Suppose now that $S \subseteq T$. Compatibility with structure maps requires that any morphism in $\downarrow Q$ sends the source of P_S to a vertex of P_T associated to a path q in T ending at the source of S. Any such choice extends to a morphism $P_S \to P_T$ in the obvious way, by sending a path in S to its concatenation with q, which is a path in T. Similarly, there is one obvious way to define the map on arrows of P_S. Now the previous paragraph implies that this extension to the rest of P_S is unique. □

Corollary 21. *If there exists a morphism $g\colon Q' \to P_T$ in $\downarrow Q$, then any two arrows with the same terminal vertex in Q' must lie over the same arrow in Q. That is, for $\alpha', \beta' \in Q'_1$ with $t(\alpha') = t(\beta')$, we have $f'(\alpha') = f'(\beta')$. Consequently, we get that $[E_S, P_T] = 0$ unless $E_S = P_S$, and $[I_S, P_T] = 0$ unless $I_S = P_S$.*

from\to	E_T	P_T	I_T
E_S	1	0 unless $E_S = P_S$	0 unless $E_S = I_S$
P_S	1	# paths in T from source T to source S	0 unless $P_S = I_S = E_S$
I_S	1	0 unless $I_S = P_S = E_S$	# paths in T from sink S to sink T

Table 1. Summary of morphisms in PIE if $S \subseteq T$.

Proof. If there exists such a morphism g, we apply (13) to both α' and β' and then use the assumption that $t(\alpha') = t(\beta')$ to get

$$g(s(\alpha'))f'(\alpha') = g(t(\alpha')) = g(t(\beta')) = g(s(\beta'))f'(\beta')$$

as paths in Q. Since a path can only end with one arrow, it must be that $f'(\alpha') = f'(\beta')$. Now if E_S is distinct from P_S, then the subquiver S must either have parallel paths or more than one source. In either case, there will be two arrows in E_S with the same terminal vertex but different labels, preventing any morphism from E_S to P_T. Similarly, if I_S is distinct from P_S, then there are distinct arrows in I_S with the same terminal vertex.

Thus there can be no morphism from I_S to P_T. □

The results of this subsection are summarized Table 1, keeping in mind that by Lemma 18(a) we need $S \subseteq T$ for any corresponding entry to be nonzero, though we don't write this in each entry of the table.

6.4. ***Fiber products in* PIE.**

Lemma 22. *For $T \subseteq Q$ and $X = (Q', f')$, we have $E_T \times_Q X \simeq f'^{-1}(T)$. In other words, fiber product with E_T restricts X to T.*

Proof. The universal property of the fiber product can be quickly verified: suppose we have a commutative diagram of quiver morphisms given by the solid lines in

$$\begin{tikzcd}
Z \arrow[drr, bend left, "g"] \arrow[dr, dashed] \arrow[ddr, bend right, "h"'] & & \\
& f'^{-1}(T) \arrow[r, hook] \arrow[d, "f'"] & Q' \arrow[d, "f'"] \\
& E_T \arrow[r, hook] & Q
\end{tikzcd}$$

where Z is an arbitrary quiver over Q. We need to see that there is a unique map along the dashed arrow making the diagram commutative everywhere. The outer square shows that $g(Z) \subseteq f'^{-1}(T)$, so filling in the dashed arrow with g gives a map from Z to $f'^{-1}(T)$ over Q making the two triangles commute. The upper triangle shows that g is unique. $\square$

We now show that PIE is closed under products with E-type objects. For a vertex i in a quiver Q, denote by $\vec{i}$ the *successor closure* of i in Q, that is, the full subquiver of Q containing the vertices which can be reached by a path starting at i.

Proposition 23. *For any $S, T \subseteq Q$, the fiber product $P_S \times_Q E_T$ is a disjoint union of P-type quivers over Q. More specifically, for each source i of $S \cap T$, the quiver $P_{\vec{i}}$ appears as a component of $P_S \times_Q E_T$ with multiplicity equal to the number of paths from the source of S to i in S, where the successor closure is taken inside $S \cap T$.*

Proof. We know from the previous lemma that $P_S \times_Q E_T$ can be identified with a subquiver of P_S lying over $S \cap T$. So, the vertices of $P_S \times_Q E_T$ can be identified with paths starting at the source of S and ending in $S \cap T$, with the arrows between them exactly the ones in P_S that lie over $S \cap T$; in particular, the arrows still fit the description of those in a P-type quiver over Q. Now each path ending in $S \cap T$ passes through precisely one source of $S \cap T$, naturally partitioning the vertices as described in the proposition. $\square$

As one would expect, describing the fiber product of an arbitrary $X = (Q', f')$ with P-type objects is more complicated. Roughly, we can think of $X \times_Q P_S$ as a path space for Q' that records only the labels from Q which are traversed to get to a vertex, rather than the exact path.

Proposition 24. *The fiber product of a P-type and an I-type quiver over Q is a disjoint union of paths in Q (that is, E-type quivers).*

Proof. Let $S, T \subseteq Q$ be subquivers, so that we want to describe $P_S \times_Q I_T$. By the definition of fiber products, we know that $P_S \times_Q I_T$ has support $S \cap T$, over which P_S and I_T decompose as disjoint unions of P-type and I-type quivers, respectively. So if $S \neq T$, we can distribute the product over these disjoint unions and then compute $P_S \times_Q I_T$ from the product of smaller P-type and I-type quivers. For each of these products, we can repeat the process until we are left with products over the same subquiver of Q in the base.

Hence we can assume without loss of generality that $S = T = Q$ for the remainder of the proof. Since, by assumption, P_Q and I_Q are defined, it follows that Q has a unique source i and a unique sink j. Then the vertices of $P_Q \times_Q I_Q$ lying over $k \in Q_0$ are pairs (p, q) consisting of a path p from i to k, and a path q from

k to j; in other words, each vertex corresponds to a maximal path pq in Q with a distinguished vertex k. Unraveling the definitions, we see that an arrow

$$(p_1, q_1) \xrightarrow{(a,b)} (p_2, q_2)$$

in $P_Q \times_Q I_Q$ occurs exactly when p_1q_1 and p_2q_2 are the same maximal path in Q and $a = b$ is an arrow between adjacent distinguished vertices on this path. Thus each connected component of $P_Q \times_Q I_Q$ is a maximal path in Q. □

Example 25. Continuing with the setup of Examples 13 and 16, we get that

$$P_Q \times_Q I_Q \simeq \quad \begin{array}{c} \underset{3}{\bullet} \xrightarrow{\;a\;} \underset{2}{\bullet} \xrightarrow{\;c\;} \underset{1}{\bullet} \\ \underset{3}{\bullet} \xrightarrow{\;b\;} \underset{2}{\bullet} \xrightarrow{\;c\;} \underset{1}{\bullet} \end{array}$$

can be identified with the two maximal paths in Q.

Proposition 26. *The fiber product of two P-type quivers over Q is a disjoint union of P-type quivers.*

Proof. The same argument as in Proposition 24 allows us to reduce to the case $P_Q \times_Q P_Q$, where Q has unique source i. Then the vertices of $P_Q \times_Q P_Q$ can be identified with pairs of paths (p, q) that start at i and end at the same vertex of Q, and since each vertex of P_Q has at most one incoming arrow, so must each vertex of $P_Q \times_Q P_Q$.

More precisely, an arrow

$$(p_1, q_1) \xrightarrow{(a,b)} (p_2, q_2)$$

in $P_Q \times_Q P_Q$ occurs exactly when a and b lie over the same arrow c of Q, and both $p_1c = p_2$ and $q_1c = q_2$ as paths in Q; in particular p_2 and q_2 are parallel paths starting at i that end with the same arrow. So any pair of paths $(p, q) \in (P_Q \times_Q P_Q)_0$ that do not end with the same arrow give a source of $P_Q \times_Q P_Q$, and, for each vertex of the form (pr, qr), where r varies over the paths starting at the common endpoint j of p and q, there is a unique path in $P_Q \times_Q P_Q$ starting at (p, q) and ending at (pr, qr). So in fact (p, q) is the unique source of a connected component of $P_Q \times_Q P_Q$ which is isomorphic to $P_{\vec{j}}$. Since all vertices fall into some connected component of this form (not forgetting the case where both p and q are the trivial path at i), we see that $P_Q \times_Q P_Q$ is a disjoint union of P-type quivers. □

6.5. *Main result on* PIE. We now apply Theorem 9 to the category PIE.

Lemma 27. *For any acyclic quiver Q, the corresponding* PIE *category satisfies the hypotheses of Theorem 9.*

Proof. The category PIE was defined so that the objects are connected, wrappings, and linearize to distinct indecomposables.

To see that PIE is acyclic, we demonstrate an ordering of its objects making the Hom matrix upper triangular unipotent. First, we "block" the objects together into sets $\mathcal{B}_S = \{P_S, I_S, E_S\}$ for each $S \subseteq Q$, keeping in mind our convention of omitting P_S or I_S when the object is undefined, and the possibility of coincidences among P_S, I_S and E_S. If these blocks are ordered so that $\mathcal{B}_S$ comes before $\mathcal{B}_T$ whenever $S \subseteq T$, the Hom matrix will be block lower triangular by Lemma 18(a). On the diagonal are then the blocks where $S = T$, which we see from Table 1 are always lower triangular: to get a nonzero entry above the main diagonal, we need a coincidence $E_S = P_S$ or $E_S = I_S$, but in this case the corresponding row and column would be omitted as redundant since $S = T$.

The fact that PIE is closed under fiber products follows from applying Lemma 22 and Propositions 23, 24 and 26 to Q and Q^{op}. $\square$

As in Section 5, each object x of PIE, defines an idempotent

$$\delta_x \overset{\text{def}}{=} \sum_{z \in \mathrm{PIE}_0} \mu(z, x) z \tag{14}$$

in $M(\mathrm{PIE})$. Let $e_x = L(\delta_x)$ be its image in $R(Q)$. (Note that e_x is different than the $e(x)$ of Section 3.)

We are ready for the main result of this section.

Theorem 28. *Let Q be a quiver without oriented cycles. Then $R(Q)$ has a direct product structure*

$$R(Q) \cong \prod_{x \in \mathrm{PIE}_0} \langle e_x \rangle,$$

where $\langle e_x \rangle$ is the principal ideal generated by e_x.

Proof. According to Lemma 27, Theorem 9 and its corollary apply in this situation. The result now follows. $\square$

Example 29. Continuing with the setup of Example 16, we can roughly visualize the PIE category as in Figure 1 (though we cannot count morphisms from this visualization). To get the idempotent associated to $x = E_{\alpha\beta}$, for example, we start by writing

$$\begin{aligned} e_x = {} & E_{\alpha\beta} + \mu(P_{\alpha\beta}, E_{\alpha\beta}) P_{\alpha\beta} + \mu(I_{\alpha\beta}, E_{\alpha\beta}) I_{\alpha\beta} + \mu(E_\alpha, E_{\alpha\beta}) E_\alpha \\ & + \mu(E_\beta, E_{\alpha\beta}) E_\beta + \mu(E_3, E_{\alpha\beta}) E_3 + \mu(E_2, E_{\alpha\beta}) E_2, \end{aligned}$$

where we have used the definition of e_x, that $\mu(x, x) = 1$, and that $\mu(z, x) = 0$ when $[z, x] = 0$. Then (6) can be used to calculate these coefficients, starting with

$$
\begin{tikzcd}
& E_Q & \\
E_{\alpha\beta} \arrow[ur] & E_{\alpha\gamma} \arrow[u] & E_{\beta\gamma} \arrow[ul] \\
P_{\alpha\beta} \arrow[u] & I_{\alpha\beta} \arrow[ul] & \\
E_\alpha \arrow[u] \arrow[ur] \arrow[uur] & E_\beta \arrow[ul] \arrow[u] \arrow[uur] & E_\gamma \arrow[uul] \arrow[uu] \\
E_3 \arrow[u] \arrow[ur] & E_2 \arrow[ul] \arrow[u] \arrow[ur] & E_1 \arrow[u]
\end{tikzcd}
$$

Figure 1. Visualization of the category PIE. The nodes are objects of PIE, and there is a path from x to y in the diagram if and only if there exists a morphism from x to y in PIE.

the ones closest to $E_{\alpha\beta}$. For example, we first get

$$\mu(P_{\alpha\beta}, E_{\alpha\beta}) = \mu(I_{\alpha\beta}, E_{\alpha\beta}) = -1$$

from the fact that $[x, x] = 1$. Similarly, we can find $\mu(E_\alpha, E_{\alpha\beta}) = \mu(E_\beta, E_{\alpha\beta}) = 1$. Then to get $\mu(E_2, E_{\alpha\beta})$, there is a unique morphism from E_2 to each object in the interval between E_2 and $E_{\alpha\beta}$ except $P_{\alpha\beta}$, for which we have $[E_2, P_{\alpha\beta}] = 2$. So here we find

$$\mu(E_2, E_{\alpha\beta}) = -1 - 2(-1) - (-1) - 1 - 1 = 0.$$

A similar computation shows $\mu(E_3, E_{\alpha\beta}) = 0$, so that finally

$$e_x = E_{\alpha\beta} - P_{\alpha\beta} - I_{\alpha\beta} + E_\alpha + E_\beta.$$

The entire basis of orthogonal idempotents for $M(\mathrm{PIE})$ is:

$$
\begin{aligned}
\{&E_1,\ E_2,\ E_3,\ E_\alpha - E_2 - E_3,\ E_\beta - E_2 - E_3,\ E_\gamma - E_2 - E_1,\\
&\qquad P_{\alpha\beta} - E_\alpha - E_\beta - E_3,\ I_{\alpha\beta} - E_\alpha - E_\beta - E_2,\\
&\qquad E_{\alpha\beta} - P_{\alpha\beta} - I_{\alpha\beta} + E_\alpha + E_\beta,\ E_{\alpha\gamma} - E_\alpha - E_\gamma - E_2,\\
&\qquad E_{\beta\gamma} - E_\beta - E_\gamma - E_2,\ E_Q - E_{\alpha\beta} - E_{\alpha\gamma} - E_{\beta\gamma} + E_\alpha + E_\beta + E_\gamma - E_2\}.
\end{aligned}
$$

6.6. ***Computation of specific Möbius functions.*** Although one generally cannot expect closed formulas for values of the Möbius function μ, even in the poset case, we can calculate them for some pairs of objects in the PIE category. Given two subquivers $S, T \subseteq Q$, we say that they have the same *skeleton* if, for every pair of vertices $v, w \in Q_0$, there is at least one edge between v and w in S exactly when there is at least one edge between v and w in T. When S and T have the same skeleton, P_S exists if and only if P_T exists, and similarly for I-type objects.

Proposition 30. *Let $S \subseteq T$ be subquivers of an acyclic quiver Q which have the same skeleton, and write $\mathcal{A} = T_1 \setminus S_1$ for the set of arrows of T which are not in S. Then the following hold in case $P_S \neq E_S \neq I_S$:*

$$\mu(E_S, P_T) = 0, \tag{15}$$

$$\mu(E_S, E_T) = (-1)^{\#\mathcal{A}}, \tag{16}$$

$$\mu(P_S, P_T) = (-1)^{\#\mathcal{A}}, \tag{17}$$

$$\mu(P_S, E_T) = (-1)^{\#\mathcal{A}+1}, \tag{18}$$

$$\mu(P_S, I_T) = 0. \tag{19}$$

When $X = P_S = E_S \neq I_S$, we have the following formulas:

$$\mu(X, E_T) = 0, \tag{20}$$

$$\mu(X, P_T) = (-1)^{\#\mathcal{A}}, \tag{21}$$

$$\mu(X, I_T) = 0. \tag{22}$$

In the case that $Y = P_S = E_S = I_S$, we have

$$\mu(Y, E_T) = (-1)^{\#\mathcal{A}+1}, \tag{23}$$

$$\mu(Y, P_T) = (-1)^{\#\mathcal{A}}. \tag{24}$$

Dual formulas also hold (that is, when P- and I-type objects are interchanged).

Proof. The key is that when S and T have the same skeleton, all the Hom sets involved in finding the formulas of the proposition have at most one element. In other words, we are computing values of the Möbius function of some poset in each case. For a given $T \subseteq Q$, there is a unique minimal subquiver of Q with the same skeleton as T. Remark 14 implies that this is the only possible subquiver with the same skeleton as T which may be simultaneously P- and E-type.

Equations (15) and (19) follow from fact (b) of Section 4.2. The top row of Table 1 shows that the full subcategory of PIE consisting of objects between E_S and E_T (in the Hom order) is isomorphic to the poset of subsets of $\mathcal{A}$. The Möbius function of this poset is well known [Stanley 1997, 3.8.3], giving (16). The same argument gives (17), since the only objects Z for which there exist morphisms $P_S \to Z \to P_T$ are P-type. To see (18), we use (6) to compute

$$-\mu(P_S, E_T) = \sum_{P_S < Z \leq E_T} [P_S, Z]\, \mu(Z, E_T)$$
$$= \mu(E_S, E_T) + \sum_{S \subsetneq Q' \subseteq T} \bigl(\mu(E_{Q'}, E_T) + \mu(P_{Q'}, E_T)\bigr) = \mu(E_S, E_T), \quad (25)$$

where the rightmost equality follows from induction by canceling out pairwise each term of the sum.

Now when $X = P_S = E_S$, the (25) still holds except that the term $\mu(E_S, E_T)$ is absent, so we get (20). Again, morphisms $X \to P_T$ can only factor through P-type objects, so the same argument for (17) applies to give (21). In this case there are still no morphisms from P_S to I_T, so (22) follows.

Finally, when $Y = P_S = E_S = I_S$ is just a path in Q, it has morphisms to objects of all types in PIE. So we get

$$-\mu(Y, E_T) = \sum_{P_S < Z \leq E_T} [Y, Z]\mu(Z, E_T)$$
$$= \sum_{S \subsetneq Q' \subseteq T} \bigl(\mu(E_{Q'}, E_T) + \mu(P_{Q'}, E_T) + \mu(I_{Q'}, E_T)\bigr) = \sum_{S \subsetneq Q' \subseteq T} \mu(P_{Q'}, E_T)$$
$$= -\sum_{S \subsetneq Q' \subseteq T} \mu(E_{Q'}, E_T) = -(-1)^{\#\mathcal{A}} = (-1)^{\#\mathcal{A}+1} \quad (26)$$

by applying formulas from the first group and canceling some terms. The same argument for (17) and (21) will give (24). By applying the formulas to Q^{op}, we get similar formulas on Q with P- and I-type objects interchanged. $\square$

The hypothesis that S and T have the same skeleton can be relaxed for several of the formulas; for example, the same proof shows that (15) and (19) hold for all subquivers S and T when $P_S \neq E_S$.

7. Future directions

Here we suggest a few directions for future work.

(1) What are other examples of categories of quivers over Q satisfying the hypotheses of the Theorem 9? For example, when Q is any quiver, Section 4 of [Herschend 2010] gives such a category (with infinitely many objects, but see Remark 8) in the course of studying string and band modules. Or when Q is a rooted tree quiver, there is a collection of "reduced quivers over Q" given in [Kinser 2010] which satisfies these hypotheses.

A result of Ringel states that if V is an exceptional representation of a quiver (that is, $\mathrm{Ext}^i(V, V) = 0$, for all $i \geq 1$), then V has a structure quiver which is a tree [Ringel 1998]. This structure quiver is not unique, but one may try to give “good” choices of structure quivers for some class of exceptional modules so that Theorem 9 can be applied.

(2) Can we get more closed formulas for values of μ, in addition to Proposition 30 (for the PIE category, or any other example)?

(3) When does Theorem 9 give *all* of the idempotents of $R(Q)$ (or how can it be improved to give all idempotents)? That is, under what conditions on $\mathscr{C}$ is it impossible to write each $L(\delta_x)$ as a nontrivial sum of idempotents? The PIE category will not generally give all idempotents, but the rooted tree case mentioned above does.

(4) Is there a representation theoretic interpretation for the idempotents obtained from the PIE category? For example, given $x \in \mathrm{PIE}_0$, what properties of $V \in \mathrm{rep}(Q)$ are necessary or sufficient for $e_x V = 0$? (See Propositions 32 and 35 of [Kinser 2010].)

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Cox rings and pseudoeffective cones of projectivized toric vector bundles

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We study projectivizations of a special class of toric vector bundles that includes cotangent bundles whose associated Klyachko filtrations are particularly simple. For these projectivized bundles, we give generators for the cone of effective divisors and a presentation of the Cox ring as a polynomial algebra over the Cox ring of a blowup of a projective space along a sequence of linear subspaces. As applications, we show that the projectivized cotangent bundles of some toric varieties are not Mori dream spaces and give examples of projectivized toric vector bundles whose Cox rings are isomorphic to that of $\overline{M}_{0,n}$.

1. Introduction

Projectivizations of toric vector bundles over complete toric varieties are a large class of rational varieties that have interesting moduli and share some of the pleasant properties of toric varieties and other Mori dream spaces. Hering, Mustaţă, and Payne [Hering et al. 2010] showed that their cones of effective curves are polyhedral and asked whether their Cox rings are indeed finitely generated. For rank-two bundles an affirmative answer is given in [Hausen and Süß 2010; González 2010] or can be derived from the results of [Knop 1993].

Here we apply general results of Hausen and Süß on Cox rings for varieties with torus actions to give a presentation of the Cox ring for certain projectivized toric vector bundles as a polynomial algebra over the Cox ring of the blowup of projective space along a collection of linear subspaces. The question of finite generation for the Cox rings of these blowups is completely understood when the collection of linear subspaces consists of finitely many points in very general position, through work of Mukai [2004], and Castravet and Tevelev [2006] in connection with Hilbert's fourteenth problem.

Let $\Bbbk$ be an algebraically closed field, and let X be a smooth projective toric

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variety of dimension d over $\Bbbk$, corresponding to a fan Σ with n rays. Throughout, we use r to denote the rank of a vector bundle on $X(\Sigma)$. By a *toric vector bundle* on X we mean a vector bundle admitting an action of the dense torus T in X that is linear on fibers and compatible with the action on the base. By the *projectivization* of a toric vector bundle we mean the bundle of rank-one quotients.

Theorem 1.1. *Suppose $\Bbbk$ is uncountable, $n > r \geq d$, and $\frac{1}{r} + \frac{1}{n-r} \leq \frac{1}{2}$. Then there is a nonsplit toric vector bundle $\mathcal{F}$ of rank r on $X(\Sigma)$ such that the Cox ring of the projectivization $\mathbb{P}(\mathcal{F})$ is not finitely generated.*

In particular, on any smooth projective toric surface corresponding to a fan with at least nine rays, there is a rank-three toric vector bundle whose projectivization is not a Mori dream space. The bundles that we construct in the proof of Theorem 1.1 are of a special form: in Klyachko's classification, they correspond to collections of filtrations each of which contains at most one nontrivial subspace; moreover this subspace has codimension one, and this arrangement of hyperplanes is in very general position. The inequality in the theorem is sharp; if $\frac{1}{r} + \frac{1}{n-r} > \frac{1}{2}$ and the hyperplanes are in general position, then the projectivization of any such bundle is a Mori dream space. See Corollary 3.7.

Remark 1.2. The techniques used to prove Theorem 1.1 give more information than just whether or not a Cox ring is finitely generated. In Section 3 we give presentations for the Cox rings of certain projectivized toric vector bundles as algebras over Cox rings of blowups of projective spaces along linear subspace arrangements. As one special case, we produce an example of a vector bundle on a toric surface whose projectivization has the same Cox ring as $\overline{M}_{0,n}$. See Example 3.9.

Remark 1.3. If $\mathbb{P}(\mathcal{F})$ is a projectivized bundle whose Cox ring is not finitely generated, it may still happen that the section ring of the tautological quotient line bundle $\mathcal{O}(1)$ on $\mathbb{P}(\mathcal{F})$ is finitely generated. However, Theorem 1.1 implies that there also exist toric vector bundles $\mathcal{F}'$ such that the section ring of $\mathcal{O}(1)$ on $\mathbb{P}(\mathcal{F}')$ is not finitely generated.

Suppose $\mathbb{P}(\mathcal{F})$ is a projectivized toric vector bundle on $X(\Sigma)$ whose Cox ring is not finitely generated, and let $\mathcal{L}_1, \dots, \mathcal{L}_k$ be line bundles that positively generate the Picard group of $X(\Sigma)$. Then the section ring of $\mathcal{O}(1)$ on the projectivization of $\mathcal{F}' = \mathcal{F} \oplus \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_k$ is not finitely generated. So Theorem 1.1 gives negative answers to Questions 7.1 and 7.2 of [Hering et al. 2010].

A necessary, but not sufficient, condition for a projective variety to be a Mori dream space is that its pseudoeffective cone be polyhedral. In many of the examples covered by Theorem 1.1, it is unclear whether this condition holds. However, by choosing the toric variety carefully, with an even larger number of rays, we produce examples of projectivized toric vector bundles whose pseudoeffective cones are not polyhedral.

Theorem 1.4. *Suppose $\Bbbk$ is uncountable, $n-d > r \geq d$ and $\frac{1}{r}+\frac{1}{n-d-r} \leq \frac{1}{2}$, and assume there is some cone $\sigma \in \Sigma$ such that every ray of Σ is contained in either σ or $-\sigma$. Then there is a nonsplit toric vector bundle $\mathscr{F}$ of rank r on $X(\Sigma)$ such that the pseudoeffective cone of $\mathbb{P}(\mathscr{F})$ is not polyhedral.*

Examples of toric varieties satisfying the hypotheses of Theorem 1.4 can be constructed through sequences of iterated blowups of $(\mathbb{P}^1)^d$, as in Example 1.7, below.

The constructions used to prove Theorems 1.1 and 1.4 involve choosing bundles that are very general in their moduli spaces. However, by choosing the fan sufficiently carefully, one gets examples of smooth projective toric varieties in characteristic zero whose projectivized cotangent bundles are not Mori dream spaces. For these examples, the bundle is determined by the combinatorial data in the fan.

Theorem 1.5. *Suppose $d \geq 3$ and the characteristic of $\Bbbk$ is not two or three. Then there exists a smooth projective toric variety $X(\Sigma')$ of dimension d over $\Bbbk$ such that the Cox ring of the projectivized cotangent bundle on $X(\Sigma')$ is not finitely generated.*

In this respect, cotangent bundles behave quite differently from tangent bundles, since the Cox ring of the projectivization of the tangent bundle on any smooth toric variety is finitely generated [Hausen and Süß 2010, Theorem 5.9]. So, Theorem 1.5 shows that there are toric vector bundles $\mathscr{F}$ such that $\mathbb{P}(\mathscr{F})$ is a Mori dream space, but the projectivized dual bundle $\mathbb{P}(\mathscr{F}^\vee)$ is not.

Remark 1.6. In Theorems 1.1 and 1.4, we assume the field is uncountable in order to choose a configuration of points in very general position in the projective space $\mathbb{P}^{r-1}$. Examples constructed by Totaro in his work on Hilbert's 14th Problem over finite fields [2008] show that this restriction on the cardinality of the field is not necessary in some cases. For instance, to prove these theorems in the special case where r is three, it is enough to find a configuration S of nine points in $\mathbb{P}^2(\Bbbk)$ such that $\mathrm{Bl}_S\, \mathbb{P}^2$ contains infinitely many -1-curves, and Totaro constructed such configurations over $\mathbb{Q}$ and over $\mathbb{F}_p$, for $p > 3$.

We conclude the introduction with an example of a projectivized rank three bundle on an iterated blowup of $\mathbb{P}^1 \times \mathbb{P}^1$ at seven points whose effective cone agrees with the effective cone of $\mathbb{P}^2$ blown up at nine very general points, and hence is not polyhedral.

Example 1.7. Let $X(\Sigma)$ be the toric variety obtained by first blowing up one of the toric fixed points on $\mathbb{P}^1 \times \mathbb{P}^1$, then blowing up both of the toric fixed points in the exceptional divisor, and then blowing up all four of the torus fixed points in the new exceptional divisors. The corresponding fan is as shown in Figure 1.

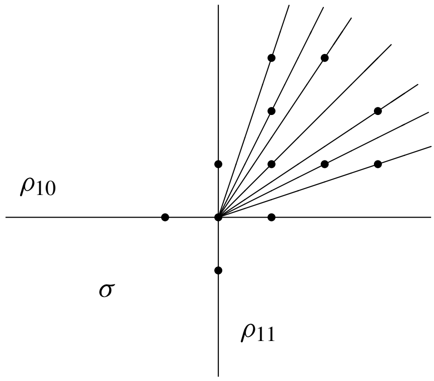

Figure 1. The fan of Example 1.7.

Note that every ray of the fan is contained in either the cone σ spanned by ρ_{10} and ρ_{11}, or in $-\sigma$, and $\frac{1}{3}+\frac{1}{11-2-3}=\frac{1}{2}$. So $X(\Sigma)$ satisfies the hypotheses of Theorems 1.1 and 1.4.

Let F be a three-dimensional vector space, and define filtrations

$$F^{\rho_i}(j)=\begin{cases} F & \text{for } j\le 0,\\ F_i & \text{for } j=1,\\ 0 & \text{for j > 1,}\end{cases}$$

where $F_1,\ldots,F_9$ are two-dimensional subspaces in very general position, and F_{10} and F_{11} are zero. By [Klyachko 1989] these filtrations give rise to a toric vector bundle $\mathscr{F}$ on $X(\Sigma)$, see also Section 2. The subspaces $F_1,\ldots,F_9$ correspond to a set $S=\{p_1,\ldots,p_9\}$ of nine points in very general position in the projective plane $\mathbb{P}_F$ of one-dimensional quotients of F. Our first main construction, in Section 3, shows that the Cox ring of $\mathbb{P}(\mathscr{F})$ is canonically isomorphic to a polynomial ring in two variables over the Cox ring of the blowup $\mathrm{Bl}_S\,\mathbb{P}_F$ of the plane at this set of points. Furthermore, in Section 5 we give an isomorphism of class groups $\mathrm{Cl}(\mathbb{P}(\mathscr{F}))\xrightarrow{\sim}\mathrm{Cl}(\mathrm{Bl}_S\,\mathbb{P}_F)$ that takes $\mathbb{O}(1)$ to the pullback of the hyperplane class of $\mathbb{P}_F$, and the class of $\mathbb{P}(\mathscr{F}|_{D_{\rho_i}})$ to the class of the exceptional divisor E_i, for $i=1,\ldots,9$, and show that this isomorphism induces an identification of the effective cones of the two spaces. Therefore, the pseudoeffective cone of $\mathbb{P}(\mathscr{F})$, like the pseudoeffective cone of $\mathrm{Bl}_S\,\mathbb{P}_F$, is not polyhedral, and $\mathbb{P}(\mathscr{F})$ is not a Mori dream space.

2. Preliminaries

We work over an uncountable field $\Bbbk$ of arbitrary characteristic with the exception of the proof of Theorem 1.5, where we restrict to characteristic not two or three.

Let T be a torus of dimension d, with character lattice M. Let $X(\Sigma)$ be a toric variety with dense torus T, and let $\rho_1, \dots, \rho_n$ be the rays of Σ. We write v_j for the primitive generator in $N = \mathrm{Hom}(M, \mathbb{Z})$ of the ray ρ_j, and D_{ρ_j} for the corresponding prime T-invariant divisor in $X(\Sigma)$.

Suppose $\mathcal{F}$ is a toric vector bundle of rank r on $X(\Sigma)$. The Klyachko filtrations associated to $\mathcal{F}$ are decreasing filtrations of the fiber F over the identity 1_T, indexed by the rays of Σ,

$$\cdots \supset F^{\rho_j}(k-1) \supset F^{\rho_j}(k) \supset F^{\rho_j}(k+1) \supset \cdots,$$

and characterized by the following property. If U_σ is the torus-invariant affine open subvariety of $X(\Sigma)$ corresponding to a cone σ in Σ, then the torus T acts on $H^0(U_\sigma, \mathcal{F})$ by $(ts)(x) = t(s(t^{-1}x))$. If u is a character of the torus, then the space of isotypical sections

$$H^0(U_\sigma, \mathcal{F})_u = \{s \in H^0(U_\sigma, \mathcal{F}) \mid ts = \chi^u(t)s \text{for all } t \in T\}$$

injects into F, by evaluation at 1_T, and the image is

$$F_u^\sigma = \bigcap_{\rho_j \preceq \sigma} F^{\rho_j}(\langle u, v_j \rangle).$$

In particular, if $F^{\rho_j}(0) = F$ for all j then the space of T-invariant sections of $\mathcal{F}$ is canonically isomorphic to F.

The Klyachko filtrations satisfy the following compatibility condition.

Klyachko's compatibility condition. For each maximal cone $\sigma \in \Sigma$, there are lattice points $u_1, \dots, u_r \in M$ and a decomposition into one-dimensional subspaces $F = L_1 \oplus \cdots \oplus L_r$ such that

$$F^{\rho_j}(k) = \bigoplus_{\langle u_i, v_j \rangle \geq k} L_i,$$

for each $\rho_j \preceq \sigma$ and all $k \in \mathbb{Z}$.

The bundle $\mathcal{F}$ can be recovered from the family of filtrations $\{F^{\rho_j}(k)\}$, and the induced correspondence between toric vector bundles and finite dimensional vector spaces with compatible families of filtrations gives an equivalence of categories. See [Klyachko 1989] or the summary in [Payne 2008, Section 2] for details.

We write $\mathbb{P}(\mathcal{F})$ for the projective bundle $\mathrm{Proj}(\mathrm{Sym}(\mathcal{F}))$ of rank one quotients of $\mathcal{F}$, and

$$\pi : \mathbb{P}(\mathcal{F}) \to X(\Sigma)$$

for its structure map. The fiber of $\mathbb{P}(\mathcal{F})$ over 1_T is the projective space $\mathbb{P}_F$ of one-dimensional quotients of F. If F' is a linear subspace of F then

$$\mathbb{P}_{F/F'} \subset \mathbb{P}_F$$

is a projective linear subspace of codimension equal to the dimension of F'.

Following the usual convention, we write $\mathcal{O}(1)$ for the tautological quotient bundle on $\mathbb{P}(\mathcal{F})$, which is relatively ample with respect to π, and $\mathcal{O}(m)$ for its mth tensor power.

For our primary examples in this paper, we will focus on bundles whose filtrations are especially simple, and in particular those satisfying

$$F^{\rho_j}(k) = \begin{cases} F & \text{for } k \leq 0, \\ F_j & \text{for } k = 1, \\ 0 & \text{for } k > 1, \end{cases} \tag{$*$}$$

where F_j is either 0 or a subspace of F of dimension at least two, and all of the nonzero F_j are distinct.

One reason for working with a bundle given by filtrations satisfying $(*)$ is that the T-invariant global sections of $\mathcal{O}(m)$ on $\mathbb{P}(\mathcal{F})$, and their orders of vanishing along the divisors $\pi^{-1}(D_{\rho_j})$, are particularly easy to understand. See Lemmas 5.1 and 5.2.

Remark 2.1. Suppose $\{F^{\rho_j}(j)\}$ is a collection of filtrations satisfying $(*)$ in which all of the F_j are hyperplanes. Using the fact that $X(\Sigma)$ is smooth, one checks that Klyachko's compatibility condition for a cone σ is satisfied for some $u_1, \dots, u_r$ if and only if the hyperplanes F_j for $\rho_j \preceq \sigma$ intersect transversely. Since at most r hyperplanes can meet transversely in a vector space of rank r, the condition $r \geq d$ appearing in Theorems 1.1 and 1.4 is necessary for such a collection of filtrations to define a toric vector bundle. If the F_j are chosen in general position, then the condition $r \geq d$ is also sufficient.

3. Torus quotients and Cox rings

Let X be a smooth variety whose divisor class group is finitely generated and torsion free. Choose divisors $D_1, \dots, D_k$ whose classes form a basis for the class group $\mathrm{Cl}(X)$. Then the Cox total coordinate ring of X is

$$\mathcal{R}(X) = \bigoplus_{(m_1,\dots,m_k)\in\mathbb{Z}^k} H^0(X, \mathcal{O}(m_1D_1 + \cdots + m_kD_k)),$$

with the natural multiplication map of global sections. See [Hu and Keel 2000] for further details and a discussion of the special properties of Mori dream spaces, those varieties whose Cox rings are finitely generated. If $X_0 \subset X$ is an open subvariety whose complement has codimension at least two, then $\mathrm{Cl}(X_0)$ and $\mathcal{R}(X_0)$ are naturally identified with $\mathrm{Cl}(X)$ and $\mathcal{R}(X)$, respectively.

Remark 3.1. Cox rings can be defined in greater generality, for possibly singular and nonseparated prevarieties whose class groups are finitely generated, but may contain torsion [Hausen 2008]. Cox rings of smooth and separated varieties with torsion free class groups suffice for all of the purposes of this paper, although we do consider nonseparated quotients in some generalizations of Theorem 3.3 presented in Section 6.

Our main technical result is a description of the Cox ring of certain projectivized toric vector bundles as a polynomial ring over the Cox ring of a blowup of projective space. Let S be a finite set of projective linear subspaces of $\mathbb{P}_F$ and let S' be the set of intersections of subspaces in S. Say $L_1, \dots, L_s$ are the elements of S'. We write $\mathrm{Bl}_{S'} \mathbb{P}_F$ for the space obtained by blowing up first the points in S', then the strict transforms of the lines in S', then the strict transforms of the two-dimensional subspaces in S', and so on. We write E_i for the exceptional divisor in $\mathrm{Bl}_{S'} \mathbb{P}_F$ dominating L_i, and define

$$\mathrm{Bl}_S \mathbb{P}_F = \mathrm{Bl}_{S'} \mathbb{P}_F \smallsetminus \bigcup_{L_i \notin S} E_i.$$

Example 3.2. Let x_1, x_2, and x_3 be noncollinear points in $\mathbb{P}^3$, and let L_{ij} be the line through x_i and x_j, and set

$$S = \{x_1, L_{12}, L_{13}, L_{23}\}.$$

Then $S' = S \cup \{x_2, x_3\}$ and $\mathrm{Bl}_S \mathbb{P}^3$ is the space obtained by blowing up first the points x_1, x_2, and x_3, and then the strict transforms of the lines L_{12}, L_{13}, and L_{23}, and then removing the exceptional divisors over x_2 and x_3.

Our main technical result can now be stated as follows. Let $\mathcal{F}$ be a toric vector bundle on a complete toric variety X given by filtrations satisfying the condition $(*)$ discussed in Section 2. After renumbering, say the F_i are distinct linear subspaces for $i \le s$, and F_j is zero for $s < j \le n$. Let

$$S = \{\mathbb{P}_{F/F_1}, \dots, \mathbb{P}_{F/F_s}\}$$

be the set of projective linear subspaces in $\mathbb{P}_F$ corresponding to $F_1, \dots, F_s$.

Theorem 3.3. *The Cox ring $\mathcal{R}(\mathbb{P}(\mathcal{F}))$ is isomorphic to a polynomial ring in $n - s$ variables over $\mathcal{R}(\mathrm{Bl}_S \mathbb{P}_F)$.*

Our proof of Theorem 3.3 will be an application of the following presentation of Cox rings for certain varieties with torus actions.

Proposition 3.4. *Let X be a smooth variety such that $H^0(X, \mathcal{O}_X^*) = \Bbbk^*$ and $\mathrm{Cl}(X)$ is free, and let T be a torus acting on X. Suppose $D_1, \dots, D_h$ are irreducible divisors in X with positive dimensional generic stabilizers, T acts freely on $X \smallsetminus (D_1 \cup \cdots \cup D_h)$, and the geometric quotient is a smooth variety Y with free class group. Then $\mathcal{R}(X)$ is isomorphic to a polynomial ring in h variables over $\mathcal{R}(Y)$.*

Proof. This is the special case of [Hausen and Süß 2010, Theorem 1.1], where X is smooth, the T-action on the complement of $D_1 \cup \cdots \cup D_h$ is free, and the geometric quotient Y is separated, with torsion free class group. Although stated in the case where X is complete, the proof of that theorem is also given under the assumption that $H^0(X, \mathcal{O}_X^*) = \Bbbk^*$ and $\mathrm{Cl}(X)$ is free, which is what we use here. □

We prove Theorem 3.3 by constructing a dominant rational map

$$\varphi : \mathbb{P}(\mathcal{F}) \dashrightarrow \mathrm{Bl}_S \mathbb{P}_F,$$

and producing open sets $U \subset U'$ in $\mathbb{P}(\mathcal{F})$ with the following properties:

(1) The complement of U' has codimension 2 in $\mathbb{P}(\mathcal{F})$.

(2) There are $n - s$ irreducible divisors in U' with positive dimensional generic stabilizers, the complement of these divisors is U, and T acts freely on U.

(3) The restriction $\varphi|_U$ is regular and a geometric quotient.

(4) The complement of $\varphi(U)$ has codimension 2 in $\mathrm{Bl}_S \mathbb{P}_F$.

To see that Theorem 3.3 follows from the existence of such a map, first note that class groups, global invertible functions and Cox rings are all invariant under the removal of sets of codimension 2. Therefore, (1) implies that $H^0(U', \mathcal{O}_{U'}^*) = \Bbbk^*$, the class group $\mathrm{Cl}(U')$ is free, and $\mathcal{R}(U') \cong \mathcal{R}(\mathbb{P}(\mathcal{F}))$. Then, by Proposition 3.4, properties (2) and (3) imply that $\mathcal{R}(U')$ is isomorphic to a polynomial ring in $n - s$ variables over $\mathcal{R}(\varphi(U))$. Finally, (4) gives $\mathcal{R}(\varphi(U)) \cong \mathcal{R}(\mathrm{Bl}_S \mathbb{P}_F)$. Therefore, (1)–(4) together imply that $\mathcal{R}(\mathbb{P}(\mathcal{F}))$ is isomorphic to a polynomial ring in $n - s$ variables over $\mathcal{R}(\mathrm{Bl}_S \mathbb{P}_F)$.

We construct the birational map φ and open sets U and U' as follows. There is a unique dominant, T-invariant rational map

$$\psi : \mathbb{P}(\mathcal{F}) \dashrightarrow \mathbb{P}_F$$

that restricts to the identity on the fiber $\mathbb{P}_F$ over 1_T. Over the dense torus T, this map takes a point x in the fiber over t to $t^{-1} \cdot x$. One can also describe ψ as the rational map associated to the T-invariant linear series $H^0(\mathbb{P}(\mathcal{F}), \mathcal{O}(1))_0$; the sections of $\mathcal{O}(1)$ are canonically identified with sections of $\mathcal{F}$, and evaluation at 1_T maps $H^0(X, \mathcal{F})_0$ isomorphically onto F. Alternatively, ψ can be constructed directly from the T-invariant sections of $\mathcal{F}$, which generate all fibers over T, following the general construction in [Lazarsfeld 2004, Example 6.1.15].

Since ψ is dominant, it induces a T-invariant rational map φ to $\mathrm{Bl}_S \mathbb{P}_F$. To prove Theorem 3.3, we produce open sets $U \subset U'$ in $\mathbb{P}(\mathcal{F})$ satisfying (1)–(4) with respect to φ.

We write x_i for the distinguished point in the codimension one orbit corresponding to ρ_i; see [Fulton 1993, Section 2.1]. The fiber of $\mathcal{F}$ over x_i is canonically isomorphic to $F_i \oplus F/F_i$; this is the eigenspace decomposition for the action of the one parameter subgroup corresponding to the primitive generator of ρ_i, which is the stabilizer of x_i. Let Z_i be the projective linear subspace corresponding to $\mathbb{P}_{F_i}$ in the fiber of $\mathbb{P}(\mathscr{F})$ over x_i. Let W_i be the projective linear subspace corresponding to $\mathbb{P}_{F/F_i}$ in the fiber over 1_T.

We now define U' to be the complement in $\mathbb{P}(\mathscr{F})$ of the following closed subsets:

- The preimages of the T-invariant closed subsets of codimension 2 in X.
- The torus orbit closures $\overline{T \cdot Z_i}$, for $F_i \neq 0$.
- The torus orbit closures $\overline{T \cdot W_i}$, for $F_i \neq 0$.

Note that the condition $(*)$ says that F_i has dimension at least 2 and codimension at least 1 whenever it is nonzero. Therefore, every component of the complement of U' has codimension at least 2.

This choice of closed subsets is closely related to the indeterminacy locus of φ. On the fiber over 1_T, this map is the birational inverse of the blowup morphism from $\mathrm{Bl}_S \mathbb{P}_F$ to $\mathbb{P}_F$, so its indeterminacy locus is the discriminant, which is the union of the W_i. The closures $\overline{T \cdot W_j}$ may meet the fiber over x_i, and these intersections are also in the indeterminacy locus of φ because the indeterminacy locus is closed. In the special case where $i = j$, the intersection of $\overline{T \cdot W_i}$ with the fiber over x_i is the linear subspace $\mathbb{P}_{F/F_i}$. Now, φ maps the fiber over x_i into the exceptional divisor over $\mathbb{P}(F/F_i)$, via the canonical rational map

$$\mathbb{P}(F_i \oplus F/F_i) \dashrightarrow \mathbb{P}_{F/F_i} \times \mathbb{P}_{F_i},$$

which is regular away from the linear subspaces W_i and Z_i. In particular, Z_i is the only remaining indeterminacy locus of φ in the fiber over x_i. Therefore, after removing the preimage of the codimension 2 strata in X, the open set U' is simply the locus where φ is regular.

For $i = s+1, \dots, n$, the subspace F_i is zero. Then the one parameter subgroup corresponding to the primitive generator of ρ_i acts trivially on $\mathbb{P}(\mathscr{F}|_{O_{\rho_i}})$. Let U be the complement in U' of these $n - s$ irreducible divisors with positive dimensional stabilizers.

We claim that T acts freely on U. Over the dense torus, T acts freely on the base. Over a codimension one orbit O_{ρ_i}, the stabilizer on the base is the one-parameter subgroup corresponding to the primitive generator of ρ_i. Because the eigenspace decomposition of the fiber of $\mathscr{F}$ over x_i is $F_i \oplus F/F_i$, with the one parameter subgroup acting by scaling on F_i and trivially on F/F_i, this subgroup acts freely away from the T-orbits of the linear subspaces $\mathbb{P}_{F_i}$ and $\mathbb{P}_{F/F_i}$, both of which are in the complement of U' and hence of U. Therefore, T acts freely on U.

To prove the theorem, it remains to show that $\varphi|_U$ is a geometric quotient and the image $\varphi(U)$ has codimension 2 in $\mathrm{Bl}_S \mathbb{P}_F$. We first treat the special case where the fan Σ has only a single ray ρ. Let U_ρ be the toric variety corresponding to a single ray ρ, and let $\mathcal{F}$ be the toric vector bundle on U_ρ given by the filtration

$$F^\rho(k) = \begin{cases} F & \text{for } k \le 0, \\ F_\rho & \text{for } k = 1, \\ 0 & \text{for } k > 1, \end{cases}$$

where F_ρ is a proper subspace of dimension at least two. Then $\mathcal{F}$ splits canonically as a sum $\mathcal{F} = \mathcal{F}_\rho \oplus \mathcal{F}/\mathcal{F}_\rho$, where $\mathcal{F}_\rho$ is the toric subbundle with fiber F_ρ over 1_T. Let Z be the projective linear subspace $\mathbb{P}_{F_\rho}$ in the fiber over x_ρ, and let W be the projective linear subspace $\mathbb{P}_{F/F_\rho}$ in the fiber over 1_T.

Proposition 3.5. *The torus T acts freely on the open set*

$$\mathbb{P}(\mathcal{F}) \smallsetminus (\overline{T \cdot Z} \cup \overline{T \cdot W})$$

with geometric quotient $\mathrm{Bl}_W \mathbb{P}_F$, *and the preimage of O_ρ under the structure map surjects onto the exceptional divisor over W.*

Proof. The open set $\mathbb{P}(\mathcal{F}) \smallsetminus (\overline{T \cdot Z} \cup \overline{T \cdot W})$ is the set denoted U in the discussion above, and hence T acts freely. We use a toric computation to compute the geometric quotient.

The projectivization of any toric vector bundle $\mathcal{G}$ of rank r on U_ρ is isomorphic to a toric variety. The toric variety is canonical, but the isomorphism depends on the choice of a splitting of the fiber over 1_T,

$$G = L_1 \oplus \cdots \oplus L_r,$$

satisfying Klyachko's compatibility condition. Fix such a splitting. For $1 \le j \le r$, define the integer $n_j = \max\{k \mid G^\rho(k) \text{ contains } L_j\}$. Let σ be the cone in $N_\mathbb{R} \times \mathbb{R}^r$ spanned by the standard vectors $(0, e_1), \dots, (0, e_r)$ and

$$\widetilde{v}_\rho = (v_\rho, n_1 e_1 + \cdots + n_r e_r),$$

where v_ρ is the primitive generator of ρ, and let Δ in $N_\mathbb{R} \times (\mathbb{R}^r/(1, \dots, 1))$ be the fan whose maximal cones are projections of the facets of σ that contain $\widetilde{v}_\rho$. Then there is a natural isomorphism from $\mathbb{P}(\mathcal{G})$ to $X(\Delta)$ taking $\mathbb{P}(\mathcal{G}|_{O_\rho})$ to the torus invariant divisor corresponding to the image of $\widetilde{v}_\rho$ and the codimension one projectivized subbundle corresponding to L_j to the torus invariant divisor corresponding to the image of $(0, e_j)$. For further details on this construction, see [Oda 1988, pp. 58–59].

We now apply the preceding general construction to our particular bundle $\mathcal{F}$. The decomposition into line bundles induces a decomposition of $F = L_1 \oplus \cdots \oplus L_r$ into one-dimensional coordinate subspaces. After relabeling we may assume that $F_\rho = L_1 \oplus \cdots \oplus L_k$ and so $\widetilde{v}_\rho = (v_\rho, e_1 + \cdots + e_k)$. Then the complement

$\mathbb{P}(\mathscr{F}) \smallsetminus (\overline{T \cdot Z} \cup \overline{T \cdot W})$, with its induced toric structure, corresponds to the fan Δ' in $N_{\mathbb{R}} \times (\mathbb{R}^r/(1, \dots, 1))$ obtained by removing from the fan for $\mathbb{P}(\mathscr{F})$ the cones containing either $\widetilde{v}_\rho$ and $(0, e_{k+1}), \dots, (0, e_r)$ (corresponding to $\overline{T \cdot Z}$), or all of $(0, e_1), \dots, (0, e_k)$ (corresponding to $\overline{T \cdot W}$).

The projection of $N_{\mathbb{R}} \times \mathbb{R}^r/(1, \dots, 1)$ onto $\mathbb{R}^r/(1, \dots, 1)$ induces a map of fans from Δ' to the fan of the blow up of $\mathbb{P}_F$ along $\mathbb{P}_{F/F_\rho}$. This map of fans satisfies the conditions of [A'Campo-Neuen and Hausen 1999, Proposition 3.2], and hence the corresponding morphism of toric varieties is a geometric quotient. $\square$

We now apply the special case treated above, where the fan consists of a single ray ρ, to prove the general case.

Proof of Theorem 3.3. By the discussion following Proposition 3.4, it remains to show that $\varphi|_U$ is a geometric quotient and the complement of $\varphi(U)$ has codimension 2 in $\mathrm{Bl}_S \mathbb{P}_F$. The property of being a geometric quotient is local on the base. For each i, let U_i be the complement in $\mathrm{Bl}_S \mathbb{P}_F$ of the exceptional divisors over W_j for $j \neq i$.

We claim that the preimage of U_i under φ is the preimage in U of the T-invariant affine open set U_{ρ_i}, under the structure map π. Indeed, by Proposition 3.5, the rational map φ takes the generic point of $\mathbb{P}(\mathscr{F})|_{O_{\rho_j}}$ to the generic point of the exceptional divisor over W_j. The part of U that lives over T maps into every U_i, but for the parts of U over codimension one orbits of X, only the part over O_{ρ_i} maps into U_i. This proves the claim.

The union of the sets U_i cover all but a codimension 2 locus in $\mathrm{Bl}_S \mathbb{P}_{\mathscr{F}}$, so it only remains to show that the restriction of φ to the preimage of U_i is a geometric quotient. Again, this follows from the local computation in Proposition 3.5, because U_i is just the complement of the codimension 2 loci given by the strict transforms of the W_j for $j \neq i$ in $\mathrm{Bl}_{W_i} \mathbb{P}_F$, and the restriction of φ to $\varphi^{-1}(U_i)$ is the restriction of the geometric quotient onto $\mathrm{Bl}_{W_i} \mathbb{P}_F$ described in Proposition 3.5. $\square$

Proof of Theorem 1.1. Let S be a subset of s very general points of $\mathbb{P}_F \cong \mathbb{P}^{r-1}$ such that $s \geq r + 2 + \frac{4}{r-2}$. Then $\mathscr{R}(\mathrm{Bl}_S \mathbb{P}_F)$ is not finitely generated, see [Mukai 2004]. Let X be any smooth toric variety of dimension at most $\dim(\mathbb{P}_F)$ with at least s rays. Then by Remark 2.1 there exists a vector bundle $\mathscr{F}$ on X satisfying $(*)$ such that the nonzero F_i correspond to the points p_i in S. The conclusion then follows from Theorem 3.3. $\square$

Remark 3.6. The isomorphism of Theorem 3.3 is not an isomorphism of graded rings. However, the pull back by the quotient map φ constructed in the proof of Theorem 3.3 induces a group homomorphism $\varphi^*\colon \mathrm{Cl}(\mathrm{Bl}_S(\mathbb{P}_F) \to \mathrm{Cl}(\mathbb{P}(\mathscr{F}))$. Letting $\deg(x_i) = [\pi^{-1}(D_{\rho_i})]$ for $s + 1 \leq i \leq n$, we obtain a $\mathrm{Cl}(\mathbb{P}(\mathscr{F}))$-grading of the polynomial ring in $n - s$ variables over the Cox ring of $\mathrm{Bl}_S(\mathbb{P}_F)$ such that the isomorphism of Theorem 3.3 is graded.

Corollary 3.7. *Suppose $\mathcal{F}$ is given by filtrations satisfying* $(*)$ *with the F_i being hyperplanes in general position. If $\frac{1}{r}+\frac{1}{n-r}>\frac{1}{2}$ then $\mathbb{P}(\mathcal{F})$ is a Mori dream space.*

Proof. Suppose $\frac{1}{r}+\frac{1}{n-r}>\frac{1}{2}$. Then the blow up of $\mathbb{P}^{r-1}$ at n points in general position is a Mori dream space [Castravet and Tevelev 2006, Theorem 1.3], and then so is the blow up $\mathrm{Bl}_s \mathbb{P}^{r-1}$ of $\mathbb{P}^{r-1}$ at s points in general position, where s is the number of rays ρ_j such that F_j is nonzero. The corollary then follows immediately from Theorem 3.3, which says that $\mathcal{R}(\mathbb{P}(\mathcal{F}))$ is finitely generated over $\mathcal{R}(\mathrm{Bl}_s \mathbb{P}^{r-1})$. □

If the points $p_1, \dots, p_s$ are not in general position then $\mathbb{P}(\mathcal{F})$ can be a Mori dream space, even when $\frac{1}{r}+\frac{1}{n-r}\le\frac{1}{2}$. For instance, if $p_1, \dots, p_s$ are collinear then $\mathrm{Bl}_S \mathbb{P}_F$ is a rational variety with a torus action with orbits of codimension one, and hence is a Mori dream space [Elizondo et al. 2004; Hausen and Süß 2010; Ottem 2011]. Also, if $p_1, \dots, p_s$ lie on a rational normal curve, then $\mathrm{Bl}_S \mathbb{P}_F$ is a Mori dream space [Castravet and Tevelev 2006, Theorem 1.2].

We conclude this section with the observation that the Cox ring of the blowup of projective space along an arbitrary arrangement of linear subspaces can be realized as the Cox ring of a projectivized toric vector bundle.

Corollary 3.8. *Let S be an arbitrary arrangement of n linear subspaces of codimension at least 2 in $\mathbb{P}_F$ and let Σ be a fan with n rays that defines a smooth projective toric surface. Then there is a toric vector bundle $\mathcal{F}$ on $X(\Sigma)$ such that*

$$\mathcal{R}(\mathbb{P}(\mathcal{F})) \cong \mathcal{R}(\mathrm{Bl}_S \mathbb{P}_F).$$

Proof. An arbitrary collection of filtrations of F indexed by the rays of Σ satisfies Klyachko's compatibility condition, because $X(\Sigma)$ is a smooth surface [Klyachko 1989, Example 2.3.4]. Therefore, if $S=\{\mathbb{P}_{F/F_1}, \dots, \mathbb{P}_{F/F_n}\}$ then the filtrations

$$F^{\rho_j}(k)=\begin{cases} F & \text{for } k\le 0,\\ F_j & \text{for } k=1,\\ 0 & \text{for } k>1,\end{cases}$$

satisfy $(*)$ and determine a toric vector bundle on $X(\Sigma)$. By Theorem 3.3, the Cox ring $\mathcal{R}(\mathbb{P}(\mathcal{F}))$ is isomorphic to $\mathcal{R}(\mathrm{Bl}_S \mathbb{P}_F)$. □

Example 3.9. Let S be the arrangement of all linear subspaces of codimension at least 2 spanned by subsets of a set of $r+1$ points in general position in $\mathbb{k}^r$. Then Kapranov's construction [1993] shows that $\mathrm{Bl}_S \mathbb{P}^{r-1}$ is isomorphic to the Deligne–Mumford moduli space $\overline{M}_{0,r+2}$. Therefore, there is a toric vector bundle $\mathcal{F}$ on a smooth projective toric surface such that $\mathcal{R}(\mathbb{P}(\mathcal{F}))$ is isomorphic to the Cox ring of $\overline{M}_{0,n}$. It is not known whether this ring is finitely generated.

4. Cotangent bundles

In the previous section, we gave a presentation of Cox rings of some projectivized toric vector bundles as polynomial rings over Cox rings of certain blowups of projective space, and used this to give examples where Cox rings of projectivized toric vector bundles are finitely generated, where they are not finitely generated, and where they are isomorphic to the Cox ring of $\overline{M}_{0,n}$. We now apply the same methods and results to study Cox rings of projectivized cotangent bundles of smooth projective toric varieties.

By [Klyachko 1989] the filtrations of cotangent bundles have the form

$$\Omega^{\rho_j}(k) = \begin{cases} M \otimes \Bbbk & \text{for } k \leq -1, \\ v_j^{\perp} & \text{for } k = 0, \\ 0 & \text{for } k > 0. \end{cases}$$

If the fan does not contain any pair of opposite rays, then the filtrations for the twist of the cotangent bundle by the anticanonical line bundle satisfy $(*)$. Since twisting by a line bundle does not change the projectivization, Theorem 3.3 shows that the Cox ring of the projectivized cotangent bundle is isomorphic to the Cox ring of $\mathrm{Bl}_S(\mathbb{P}_{M\otimes\Bbbk})$, where S is the set of points p_j corresponding to $v_j^{\perp}$. The case where the fan does contain opposite rays is treated on page 1014.

Example 4.1. For the cotangent bundle on projective space $\mathbb{P}^r$, the corresponding set S consists of $r+1$ points in linearly general position in $\mathbb{P}^{r-1}$. Then $\mathcal{R}(\mathbb{P}(\Omega^1_{\mathbb{P}^r}))$ is identified with $\mathcal{R}(\mathrm{Bl}_S\,\mathbb{P}^{r-1})$, which is isomorphic to the coordinate ring of the Grassmannian $\mathrm{Grass}(2, r+2)$ in its Plücker embedding; see [Castravet and Tevelev 2006, Remark 3.9].

Example 4.1 is a special case of the Cox rings of wonderful varieties studied by Brion [2007].

We now give an example of a smooth projective toric threefold whose projectivized cotangent bundle is not a Mori dream space. The construction uses a particularly nice configuration of nine points in $\mathbb{Z}^3$, due to Totaro, such that, for any field $\Bbbk$ of characteristic not two or three, the blowup of $\mathbb{P}^2(\Bbbk)$ at the corresponding nine $\Bbbk$-points is not a Mori dream space. The proof of Theorem 1.5 will be by induction on dimension, starting from this example.

Example 4.2. In this example, we work over a field $\Bbbk$ of characteristic not two or three. The vectors

$$v_1 = (0, 0, 1), \quad v_2 = (0, 1, 0), \quad v_3 = (1, 1, 1), \quad v_4 = (-1, -2, -2)$$

span the four rays of a unique complete fan Σ_4 in $\mathbb{R}^3$. The corresponding toric variety $X(\Sigma_4)$ is isomorphic to $\mathbb{P}^3$. Consider the vectors

$$\begin{array}{llll} v_5 = (1,1,2), & v_8 = (1,-1,1), & v_{11} = (-1,-1,1), & v_{13} = (-1,1,1), \\ v_6 = (0,-1,1), & v_9 = (-1,-2,-1), & v_{12} = (-1,0,1), & v_{14} = (0,1,1), \\ v_7 = (1,0,1), & v_{10} = (-1,-1,0), & & \end{array}$$

and let Σ_i be the stellar subdivision of Σ_{i-1} along the ray spanned by v_i, for $5 \leq i \leq 14$. For each such i, the vector v_i is the sum of two or three of the v_j that span a cone in Σ_{i-1}. Therefore, the toric variety $X(\Sigma_i)$ is the blowup of $X(\Sigma_{i-1})$ at either a point or a torus invariant smooth rational curve. In particular, if we set $\Sigma = \Sigma_{14}$, then the corresponding toric variety $X(\Sigma)$ is smooth and projective. The twist $\mathscr{F}$ of the cotangent bundle on $X(\Sigma)$ by the anticanonical bundle $\mathbb{O}(D_{\rho_1} + \cdots + D_{\rho_{14}})$ is given by the vector space $F = \Bbbk^3$ with filtrations

$$F^{\rho_i}(j) = \begin{cases} \Bbbk^3 & \text{for } j \leq 0, \\ v_i^{\perp} & \text{for } j = 1, \\ 0 & \text{for } j > 1. \end{cases}$$

Since the characteristic of $\Bbbk$ is not two or three, the points $v_i^{\perp}$ are all distinct in $\mathbb{P}^2_{\Bbbk}$, and hence the filtrations satisfy $(*)$. Twisting by a line bundle does not change the projectivization, so Theorem 3.3 says that the Cox ring of the projectivized cotangent bundle of $X(\Sigma)$ is isomorphic to the Cox ring of $\mathrm{Bl}_S\,\mathbb{P}^2_{\Bbbk}$, where $S = \{v_1^{\perp}, \dots, v_{14}^{\perp}\}$. The subset

$$S' = \{v_1^{\perp}, v_3^{\perp}, v_6^{\perp}, v_7^{\perp}, v_8^{\perp}, v_{11}^{\perp}, v_{12}^{\perp}, v_{13}^{\perp}, v_{14}^{\perp}\}$$

is the complete intersection of two smooth cubics, and the Cox ring of $\mathrm{Bl}_{S'}\,\mathbb{P}^2_{\Bbbk}$ is not finitely generated [Totaro 2008, Theorem 2.1, Corollary 5.1 and Theorem 5.2]. It follows that $\mathrm{Bl}_S\,\mathbb{P}^2_{\Bbbk}$ is not a Mori dream space, and neither is the projectivized cotangent bundle of $X(\Sigma)$.

We use the following lemma on Cox rings of blowups of projective space at finitely many points contained in a hyperplane in the proof of Theorem 1.5. Instances of this basic fact have appeared, for instance in [Hassett and Tschinkel 2004, Example 1.8]. However, lacking a suitable reference, we give a proof.

Lemma 4.3. *Let S be a finite set of points contained in a hyperplane H in $\mathbb{P}^d$, and assume $d > 2$. Then the Cox ring of $\mathrm{Bl}_S\,\mathbb{P}^d$ is isomorphic to a polynomial ring in one variable over the Cox ring of $\mathrm{Bl}_S\,H$.*

Proof. Choose coordinates on $\mathbb{P}^d$ so that H is a coordinate hyperplane, and let $\mathbb{G}_m$ act by scaling on the coordinate that cuts out H. The action of $\mathbb{G}_m$ lifts to an action on $\mathrm{Bl}_S\,\mathbb{P}^d$, and we let Y be the locus of fixed points of this action. Then $\mathbb{G}_m$ acts freely on $\mathrm{Bl}_S\,\mathbb{P}^d \smallsetminus Y$, with quotient $\mathrm{Bl}_S\,H$. The strict transform of H is the only divisor contained in Y, so the lemma follows by applying Proposition 3.4. □

Proof of Theorem 1.5. Let $\Bbbk$ be a field of characteristic not two or three. We must show that, for each dimension $d \geq 3$, there is a fan Σ in $\mathbb{R}^d$ such that

(1) The toric variety $X(\Sigma)$ is smooth and projective.

(2) The hyperplanes in $\Bbbk^d$ perpendicular to the primitive generators of the rays of Σ are distinct.

(3) The Cox ring of $\mathrm{Bl}_S \mathbb{P}^{d-1}_{\Bbbk}$ is not finitely generated, where S is the set of points corresponding to these hyperplanes.

For $d = 3$, we have Example 4.2, and we proceed by induction.

Suppose Σ is a fan in $\mathbb{R}^d$ satisfying (1), (2), and (3). Embed $\mathbb{R}^d$ as the last coordinate hyperplane in $\mathbb{R}^{d+1}$, and let Σ' be the fan in $\mathbb{R}^{d+1}$ whose maximal cones are spanned by a maximal cone of Σ together with either $(1, \ldots, 1)$ or $(1, \ldots, 1, -1)$. The corresponding toric variety $X(\Sigma')$ is smooth and projective and, since the characteristic of $\Bbbk$ is not two, the hyperplanes in $\Bbbk^{d+1}$ perpendicular to the rays of Σ' are distinct. It remains to show that Σ' satisfies (3). Let S' be the corresponding set of points in $\mathbb{P}^d_{\Bbbk}$. Now S' contains the subset S of points corresponding to rays of Σ, and S is contained in a hyperplane H. By hypothesis, the Cox ring of $\mathrm{Bl}_S H$ is not finitely generated. By Lemma 4.3, it follows that $\mathrm{Bl}_S \mathbb{P}^d_{\Bbbk}$ is not a Mori dream space, and neither is $\mathrm{Bl}_{S'} \mathbb{P}^d$. The theorem follows, since the Cox ring of the projectivized cotangent bundle of $X(\Sigma)$ is isomorphic to the Cox ring of $\mathrm{Bl}_{S'} \mathbb{P}^d_{\Bbbk}$, by Theorem 3.3. $\square$

5. Pseudoeffective cones

In this section we prove Theorem 1.4. The techniques of the proof are independent from those of Section 3.

The pseudoeffective cone of a projective variety X is the closure of the cone spanned by the classes of all effective divisors in the space of numerical equivalence classes of divisors $N^1(X)_{\mathbb{R}} = N^1(X) \otimes_{\mathbb{Z}} \mathbb{R}$. For projectivized toric vector bundles and for blowups of projective spaces at finite sets of points, linear equivalence and numerical equivalence coincide and then we identify $N^1(X)_{\mathbb{R}}$ and $\mathrm{Cl}(X)_{\mathbb{R}} = \mathrm{Cl}(X) \otimes_{\mathbb{Z}} \mathbb{R}$.

Now we consider again a toric vector bundle $\mathcal{F}$ on a complete toric variety $X(\Sigma)$. Any effective divisor D on $\mathbb{P}(\mathcal{F})$ is linearly equivalent to a torus invariant effective divisor; this can be seen by applying the Borel fixed-point theorem to the torus orbit closure of the point $[D]$ in the Chow variety of effective codimension 1 cycles on $\mathbb{P}(\mathcal{F})$. So the pseudoeffective cone of $\mathbb{P}(\mathcal{F})$ is the closure of the cone generated by classes of torus invariant prime divisors. Note that every torus invariant prime divisor in $\mathbb{P}(\mathcal{F})$ is either the preimage of a torus invariant prime divisor in $X(\Sigma)$ or surjects onto $X(\Sigma)$. If a torus invariant prime divisor surjects onto $X(\Sigma)$ then it must be the closure of the torus orbit of its intersection with the fiber over the identity. We write $\mathcal{D}_H$ for the closure of the torus orbit of a hypersurface H in $\mathbb{P}_F$. One key step toward understanding the pseudoeffective cone of $\mathbb{P}(\mathcal{F})$ is to express the class of each such $\mathcal{D}_H$ as a linear combination of $\mathcal{O}(1)$ and the $\pi^{-1}(D_{\rho_i})$. Such expressions may be somewhat complicated in general, but are relatively simple for bundles given by filtrations of the special form discussed in Section 2.

Suppose the filtrations $\{F^{\rho_i}(j)\}$ associated to the vector bundle $\mathcal{F}$ satisfy condition $(*)$ of Section 2 and all proper subspaces $F^{\rho_i}(j) \subset F$ are distinct hyperplanes.

Lemma 5.1. *Restriction to the fiber $\mathbb{P}_F$ gives an isomorphism from the space of T-invariant global sections of $\mathcal{O}(m)$ on $\mathbb{P}(\mathcal{F})$ to* $\mathrm{Sym}^m(F)$.

Proof. For any bundle $\mathcal{F}$, global sections of $\mathcal{O}(m)$ on $\mathbb{P}(\mathcal{F})$ are naturally identified with global sections of $\mathrm{Sym}^m \mathcal{F}$. Now, $\mathrm{Sym}^m \mathcal{F}$ is a toric vector bundle, with fiber $\mathrm{Sym}^m F$ over 1_T, and since the filtrations defining $\mathcal{F}$ satisfy $(*)$, the filtrations defining $\mathrm{Sym}^m \mathcal{F}$ are given by

$$\mathrm{Sym}^m F^{\rho_i}(j) = \begin{cases} \mathrm{Sym}^m F & \text{for } j \leq 0, \\ \mathrm{Image}(\mathrm{Sym}^j F_i \otimes \mathrm{Sym}^{m-j} F \to \mathrm{Sym}^m F) & \text{for } 1 \leq j \leq m, \\ 0 & \text{for } j > m. \end{cases}$$

The space of T-invariant sections of $\mathrm{Sym}^m \mathcal{F}$ is the intersection of all of these filtrations evaluated at zero, and the lemma follows, because $\mathrm{Sym}^m F^{\rho_i}(0)$ is $\mathrm{Sym}^m F$ for every ray ρ_i. □

Let p_i be the point in $\mathbb{P}_F$ corresponding to the one-dimensional quotient F/F_i, whenever F_i is nonzero. We write D_j for the T-invariant prime divisor $\pi^{-1}(D_{\rho_j})$ in $\mathbb{P}(\mathcal{F})$.

Lemma 5.2. *Let H be a hypersurface of degree m in $\mathbb{P}_F$, and let m_i be the multiplicity of H at p_i. Then there is a linear equivalence*

$$\mathcal{D}_H \sim \mathcal{O}(m) - \sum_i m_i(\pi^{-1}(D_{\rho_i})),$$

where the sum is over those i such that F_i is nonzero.

Proof. Let $h \in \mathrm{Sym}^m F$ be a defining equation for H. Then h corresponds to a torus invariant section s of $\mathcal{O}(m)$ on $\mathbb{P}(\mathcal{F})$, by Lemma 5.1. If F_i is zero then s does not vanish along D_i and if F_i is nonzero then m_i is the largest integer such that h is contained in the image of $\mathrm{Sym}^{m_i} F_i \otimes \mathrm{Sym}^{m-m_i} F$ in $\mathrm{Sym}^m F$. The one parameter subgroup corresponding to v_i extends to an embedding of the affine line $\mathbf{A}^1$ in $X(\Sigma)$ meeting D_{ρ_i} transversely at the image of zero. After restricting the section s to the preimage of $\mathbf{A}^1$, we must show that its order of vanishing along the preimage of zero is m_i. The isotypical decomposition of the module of global sections of $\mathcal{O}(1)$ on the preimage of $\mathbf{A}^1$, for the action of the one-parameter subgroup corresponding to v_i, is exactly $\bigoplus_j F^{\rho_i}(j)$, and multiplication by the coordinate x on $\mathbf{A}^1$ decreases degree by one. The sections of $\mathcal{O}(m)$ are given by the mth symmetric power of this module, in which the image of $\mathrm{Sym}^k F_i \otimes \mathrm{Sym}^{m-k} F$ in $\mathrm{Sym}^m F$ appears in degree

k, for nonnegative integers k. It follows that the T-invariant section s is equal to x^{m_i} times a section that is nonvanishing along the preimage of zero, and hence vanishes to order m_i, as required. □

Now, we fix a maximal cone σ and, after renumbering, we may assume σ is spanned by $\rho_1, \ldots, \rho_d$. Moreover, for the remainder of the section we assume that

$$F_i = 0, \text{ for } 1 \le i \le d.$$

The class of $\mathscr{O}(1)$ and the classes of $D_{d+1}, \ldots, D_n$ form a basis for $\mathrm{Cl}(\mathbb{P}(\mathscr{F}))$.

Let $f : \mathrm{Bl}_S \mathbb{P}_F \to \mathbb{P}_F$ be the blowup of $\mathbb{P}_F$ at the finite set of distinct points $\{p_i\}$, corresponding to the nonzero F_i, for $i > d$. Let L be a hyperplane in $\mathbb{P}_F$, and let E_i be the exceptional divisor over p_i. Then f^*L and $\{E_i\}$ together form a basis for $\mathrm{Cl}(\mathrm{Bl}_S \mathbb{P}_F)$.

We consider the linear map $\varphi^* : \mathrm{Cl}(\mathrm{Bl}_S \mathbb{P}_F)_{\mathbb{R}} \to \mathrm{Cl}(\mathbb{P}(\mathscr{F}))_{\mathbb{R}}$, taking f^*L to $\mathscr{O}(1)$ and the class of E_i to the class of D_i, for $i > d$. If H is a hypersurface of degree m in $\mathbb{P}_F$ passing through p_i with multiplicity m_i, then the class of the strict transform of H in $\mathrm{Bl}_S \mathbb{P}_F$ is $f^*mL - \sum_i m_i E_i$. So Lemma 5.2 says that φ^* maps the class of the strict transform of H to the class of $\mathscr{D}_H$.

Remark 5.3. One can show that the map φ^* is the map on class groups induced by the map φ of the proof of Theorem 3.3; see [González 2011, Section 5]. However, note that Lemmas 5.1 and 5.2 give an independent proof of the fact that we get a morphism of class groups, without having to construct the morphism φ.

Proposition 5.4. *The pseudoeffective cone of* $\mathbb{P}(\mathscr{F})$ *is generated by the image under* φ^* *of the pseudoeffective cone of* $\mathrm{Bl}_S \mathbb{P}_F$ *together with the classes of those* D_i *such that* F_i *is zero.*

Proof. Every effective divisor on $\mathbb{P}(\mathscr{F})$ is in the cone generated by the classes $\mathscr{D}_H$, for hypersurfaces H in $\mathbb{P}_F$, and the classes D_i. On $\mathrm{Bl}_S \mathbb{P}_F$, every effective divisor is in the cone generated by the classes of the strict transforms of the hypersurfaces H in $\mathbb{P}_F$, and the classes E_i. Now, the classes D_i such that F_i is nonzero are the images under φ^* of the classes E_i, and Lemma 5.2 says that the class of $\mathscr{D}_H$ is the image under φ^* of the strict transform of the hypersurface H in $\mathbb{P}_F$. Therefore, the cone of effective classes on $\mathbb{P}(\mathscr{F})$ is equal to the cone generated by the image under φ^* of the cone of effective classes on $\mathrm{Bl}_S \mathbb{P}_F$ together with the classes of those D_i such that F_i is zero. The proposition follows by taking closures. □

Proof of Theorem 1.4. Let σ be the cone spanned by $\rho_1, \ldots, \rho_d$, and choose the toric variety $X(\Sigma)$ so that each of the other rays ρ_i is contained in $-\sigma$. This can be accomplished, as in Example 1.7, by taking a suitable sequence of blowups of $(\mathbb{P}^1)^d$. Choose the filtrations defining $\mathscr{F}$ so that $F_{d+1}, \ldots, F_n$ are distinct hyperplanes, and $F_i = 0$ for $i \le d$.

The choice of the filtrations ensures that φ^* is an isomorphism on class groups, since it maps the basis elements $f^*L, E_{d+1}, \dots, E_n$ for $\mathrm{Cl}(\mathrm{Bl}_S \mathbb{P}_F)$ to the basis elements $\mathcal{O}(1), D_{d+1}, \dots, D_n$ for $\mathrm{Cl}(\mathbb{P}(\mathcal{F}))$, respectively. Furthermore, the choice of the fan Σ ensures that, for $i \le d$, the divisor D_{ρ_i} is linearly equivalent to an effective combination of the D_{ρ_j}, for $j > d$. So the classes of $D_1, \dots, D_d$ are in the cone spanned by the classes of D_i for $i > d$, and hence are in the image under φ^* of the pseudoeffective cone of $\mathrm{Bl}_S \mathbb{P}_F$. Therefore, by Proposition 5.4, the linear isomorphism φ^* identifies the pseudoeffective cone of $\mathrm{Bl}_S \mathbb{P}_F$ with the pseudoeffective cone of $\mathbb{P}(\mathcal{F})$. If $F_{d+1}, \dots, F_n$ are in very general position, then the inequalities on r and n imply that the pseudoeffective cone of $\mathrm{Bl}_S \mathbb{P}_F$ is not polyhedral [Mukai 2004], and the theorem follows. □

Remark 5.5. As in Corollary 3.8, a similar construction produces toric vector bundles $\mathcal{F}$ such that the effective cone of $\mathbb{P}(\mathcal{F})$ is canonically isomorphic to the effective cone of $\mathrm{Bl}_S \mathbb{P}_F$, for an arbitrary arrangement S of linear subspaces in $\mathbb{P}_F$.

6. Some generalizations

The techniques developed here can also be applied more generally to describe Cox rings of toric vector bundles where the condition $(*)$ is weakened to allow F_i to appear for multiple steps in the Klyachko filtrations, where some of the F_i are allowed to be 1-dimensional, and where the subspaces are not necessarily distinct. The results are similar to those in Section 3, only the presentations of the Cox rings are slightly more complex.

Longer steps in the filtrations. Consider a toric vector bundle $\mathcal{F}$ given by Klyachko filtrations of the form

$$F^{\rho_j}(k) = \begin{cases} F & \text{for } k \le 0, \\ F_j & \text{for } 1 \le k \le a_j, \\ 0 & \text{for } k > a_j, \end{cases}$$

for some positive integers a_j, and distinct linear subspaces $F_j \subsetneq F$ of dimension at least 2, for $j = 1, \dots, s$. The bundles that satisfy the condition $(*)$ are exactly those where each a_j is equal to 1. The Cox ring of $\mathbb{P}(\mathcal{F})$ can be analyzed just as in Section 3, except that T does not act freely on U; if D_j denotes the preimage of O_{ρ_j} in U, then D_j has a stabilizer of order a_j. In this case, the Cox ring of $\mathbb{P}(\mathcal{F})$ is a finite extension of a polynomial ring over $\mathcal{R}(\mathrm{Bl}_S \mathbb{P}_F)$ with a presentation of the form

$$\mathcal{R}(\mathbb{P}(\mathcal{F})) \cong \mathcal{R}(\mathrm{Bl}_S \mathbb{P}_F)[x_1, \dots, x_n]/\langle 1_{E_j} - x_j^{a_j} \mid 1 \le j \le s \rangle,$$

by [Hausen and Süß 2010, Theorem 1.1]. Here, 1_{E_i} denotes the canonical section of the bundle $\mathcal{O}(E_i)$ associated to the exceptional divisor E_i over $\mathbb{P}_{F/F_i}$. It follows that $\mathcal{R}(\mathbb{P}(\mathcal{F}))$ is finitely generated if and only if $\mathcal{R}(\mathrm{Bl}_S \mathbb{P}_F)$ is finitely generated.

One-dimensional subspaces. We now discuss toric vector bundles given by Klyachko filtrations of the form $(*)$, but where some F_j are allowed to be 1-dimensional. Consider the special case where the fan Σ consists of a single ray ρ, and $\mathcal{F}$ is given by the filtration

$$F^\rho(k) = \begin{cases} F & \text{for } k \le 0, \\ L & \text{for } k = 1, \\ 0 & \text{for } k > 1, \end{cases}$$

where L is 1-dimensional. The analysis of such a bundle is similar to that in Proposition 3.5, except that $\overline{T \cdot W}$ is a divisor. Still, the torus T acts freely on the toric variety $\mathbb{P}(\mathcal{F}) \setminus \overline{T \cdot Z}$, and a toric computation shows that the geometric quotient exists as a nonseparated toric prevariety; it is $\mathbb{P}_F$ with the hyperplane $\mathbb{P}_{F/L}$ doubled.

Now, consider the general case, and let S be the set of linear subspaces of $\mathbb{P}_F$ corresponding to the F_j that have dimension at least 2. Suppose the rays are numbered so that $F_1, \ldots, F_\ell$ are 1-dimensional and the rest are not. Then the analysis in the proof of Theorem 3.3 produces open subsets U and U' satisfying (1)–(4), except that the target of φ is $\mathrm{Bl}_S \mathbb{P}_F$ doubled along the strict transforms of the hyperplanes $H_i = \mathbb{P}_{F/F_i}$ for $1 \le i \le \ell$. Then [Hausen and Süß 2010] gives a presentation of the Cox ring $\mathcal{R}(\mathbb{P}(\mathcal{F}))$ as a polynomial ring in $n - s$ variables over

$$\mathcal{R}(\mathrm{Bl}_S \mathbb{P}_F)[x_1, \ldots, x_\ell, y_1, \ldots, y_\ell]/\langle 1_{H_i} - x_i y_i \mid 1 \le i \le \ell \rangle, \tag{1}$$

where 1_{H_i} is the canonical section of $\mathcal{O}(H_i)$. Setting the $n - s$ free variables equal to zero and $y_1, \ldots, y_\ell$ equal to 1, one can obtain $\mathcal{R}(\mathrm{Bl}_S \mathbb{P}_F)$ as a quotient of $\mathcal{R}(\mathbb{P}(\mathcal{F}))$, and hence $\mathcal{R}(\mathbb{P}(\mathcal{F}))$ is finitely generated if and only if $\mathcal{R}(\mathrm{Bl}_S \mathbb{P}_F)$ is so.

Example 6.1. Using the above observations, in [Hausen and Süß 2010] the Cox ring of the projectivized tangent bundle of a toric variety was calculated as follows. Let X be a toric variety associated to fan Σ with rays $\rho_1, \ldots, \rho_n$ having $v_1, \ldots, v_n \in N$ as their primitive generators. By [Klyachko 1989] the tangent bundle T_X corresponds to the filtrations of the form $(*)$ with $F^{\rho_j} = \Bbbk \cdot v_j \subset N \otimes \Bbbk$. In particular, all the subspaces are one-dimensional. Hence, the set S is empty and $\mathcal{R}(\mathrm{Bl}_S \mathbb{P}_F) = \mathcal{R}(\mathbb{P}_F)$ is simply the polynomial ring $\mathrm{Sym}(F)$. The element 1_{H_j} can be identified with $v_j \in \mathrm{Sym}(F)$. If there are no opposite rays in Σ, by the formula (1) we obtain

$$\Bbbk[x_1, \ldots, x_n, y_1, \ldots, y_n]/\big\langle \textstyle\sum_i \lambda_i \cdot x_i y_i \;\big|\; \underline{\lambda} \in \Bbbk^n, \text{ s.t. } \sum_i \lambda_i v_i = 0 \big\rangle$$

as the Cox ring of $\mathbb{P}(T_X)$.

Repetitions of subspaces and combinations. If some subspace is repeated, so $F_i = F_j$ for some $i \neq j$, then the arguments in Section 3 again go through, but the geometric quotient is nonseparated, with one copy of the exceptional divisor over $\mathbb{P}_{F/F_i}$ for each time that F_i appears. Again, this construction leads to a presentation of $\mathcal{R}(\mathbb{P}(\mathcal{F}))$ as a finitely generated algebra over $\mathcal{R}(\mathrm{Bl}_S \mathbb{P}_F)$ that is finitely generated if and only if $\mathcal{R}(\mathrm{Bl}_S \mathbb{P}_F)$ is so.

These generalizations can be combined to give a presentation of the Cox ring of an arbitrary toric vector bundle for which the Klyachko filtrations contain at most one nontrivial subspace for each ray.

Proposition 6.2. *Let $\mathcal{F}$ be a toric vector bundle corresponding to Klyachko filtrations $\{F^{\rho}(j)\}$ such that at most one proper subspace of F appears in each filtration, and let S be the collection of linear subspaces of $\mathbb{P}_F$ corresponding to these proper subspaces. Then $\mathcal{R}(\mathbb{P}(\mathcal{F}))$ is finitely generated if and only if $\mathcal{R}(\mathrm{Bl}_S \mathbb{P}_F)$ is.*

Remark 6.3. It may be possible to carry through a similar analysis for more general toric vector bundles. However, even when the fan consists of a single ray, if multiple proper subspaces occur in a single filtration then the torus quotients that appear are weighted blowups of projective space instead of ordinary blowups. Since very little is known about Cox rings of weighted blowups of projective space, we have not considered such bundles in this work.

A bundle on the Losev–Manin moduli space. We conclude with an example of a bundle on the Losev–Manin moduli space of pointed stable curves.

Let $v_0, \dots, v_d$ be vectors that generate the rank-d lattice N and sum to zero. Then the fan Σ whose nonzero cones are spanned by proper subsets of $\{v_0, \dots, v_d\}$ corresponds to projective space $\mathbb{P}^d$, and the barycentric subdivision Σ' is the normal fan of a permutahedron. The corresponding toric variety is the Losev–Manin moduli space $\overline{L}_{d+1}$ of pointed stable curves studied in [Losev and Manin 2000].

Let $\mathcal{F}$ be the pullback of the cotangent bundle $\Omega_{\mathbb{P}^d}$ to the Losev–Manin moduli space $\overline{L}_{d+1} \cong X(\Sigma')$. The rays of Σ' are naturally indexed by the proper subsets of $\{0, \dots, d\}$, where the primitive generator of the ray ρ_I is

$$v_I = \sum_{i \in I} v_i .$$

The fiber of $\mathcal{F}$ over 1_T is canonically identified with $M \otimes_{\mathbb{Z}} \Bbbk$, and we write M_I for the linear subspace perpendicular to the linear span of the v_i for $i \in I$. The Klyachko filtrations corresponding to $\mathcal{F}$ are then

$$F^{\rho_I}(k) = \begin{cases} M \otimes \Bbbk & \text{for } k \leq -1, \\ M_I & \text{for } k = 0, \\ 0 & \text{for } k > 0. \end{cases}$$

These filtrations almost satisfy $(*)$, except that the subspaces M_I corresponding to sets I of size $d-1$ are 1-dimensional, and the last nonzero subspace appears in the wrong place in the filtration. Tensoring with an appropriate line bundle puts the last nonzero subspace in the correct place in the filtration and does not change the projectivization. Then, applying the computation for filtrations with 1-dimensional subspaces (page 1013), we find that $\mathcal{R}(\mathbb{P}(\mathcal{F}))$ is a polynomial ring in $d+1$ variables over

$$\mathcal{R}(\mathrm{Bl}_S\,\mathbb{P}_F)[x_1,\ldots,x_{\binom{d+1}{2}},y_1,\ldots,y_{\binom{d+1}{2}}]/\big\langle 1_{H_i}-x_iy_i \;\big|\; 1\le i\le \tbinom{d+1}{2}\big\rangle,$$

where H_i runs over runs over all hyperplanes $\mathbb{P}_{F/F_I}$ with index sets I of size $d-1$.

Now, S consists of all linear subspaces spanned by $d+1$ points in general position in $\mathbb{P}_F \cong \mathbb{P}^{d-1}$. As in Example 3.9, the blowup $\mathrm{Bl}_S\,\mathbb{P}_F$ is isomorphic to the Deligne–Mumford moduli space $\overline{M}_{0,d+2}$.

Corollary 6.4. *The projectivization of the pullback of the cotangent bundle on $\mathbb{P}^d$ to the Losev–Manin moduli space $\overline{L}_{d+1}$ is a Mori dream space if and only if $\overline{M}_{0,d+2}$ is a Mori dream space.*

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Squareful numbers in hyperplanes

Karl Van Valckenborgh

Let $n \geqslant 4$. In this article, we will determine the asymptotic behavior of the size of the set of integral points $(a_0 : \cdots : a_n)$ on the hyperplane $\sum_{i=0}^{n} X_i = 0$ in $\mathbb{P}^n$ such that a_i is squareful (an integer a is called squareful if the exponent of each prime divisor of a is at least two) and $|a_i| \leqslant B$ for each $i \in \{0, \ldots, n\}$, when B goes to infinity. For this, we will use the classical Hardy–Littlewood method. The result obtained supports a possible generalization of the Batyrev–Manin program to Fano orbifolds.

1. Introduction

The problem we consider can be related to a question Campana posed concerning rational points on orbifolds. A good overview is given for example in [Abramovich 2009; Poonen 2006; Campana 2005]. Examining the orbifold $(\mathbb{P}^1, \Delta)$ with $\mathbb{Q}$-divisor $\Delta = 1/2 \cdot [0] + 1/2 \cdot [1] + 1/2 \cdot [\infty]$, it is explained for example in [Poonen 2006] why it is reasonable to expect that the set

$$\{(a_1, a_2, a_3) \in \mathbb{Z}^3 : a_1 + a_2 = a_3,\ a_1, a_2, a_3 \text{ are squareful}, \\ \max\{|a_1|, |a_2|, |a_3|\} \leqslant B, \gcd(a_1, a_2, a_3) = 1\}$$

will asymptotically behave as $C \cdot B^{1/2}$ as B tends to infinity.

Since this question turns out to be too difficult at the moment, we generalize to a higher-dimensional analogue $(\mathbb{P}^{n-1}, \Delta)$, where now Δ is the $\mathbb{Q}$-divisor $\Delta = 1/2 \cdot [H_0] + \cdots + 1/2 \cdot [H_n]$ with H_i the hyperplane defined by $X_i = 0$ for $i \in \{0, \ldots, n-1\}$ and H_n defined by $X_0 + \cdots + X_{n-1} = 0$. In analogy with the one-dimensional case, a point $P = (a_0 : \cdots : a_{n-1}) \in \mathbb{P}^{n-1}(\mathbb{Q})$ (we assume $a_0, \ldots, a_{n-1} \in \mathbb{Z}$ and $\gcd(a_0, \ldots, a_{n-1}) = 1$) will be called a rational point in Campana's sense on $(\mathbb{P}^{n-1}, \Delta)$ if for every $i \in \{0, \ldots, n\}$ and every prime p for which the reduction of P is contained in the reduction of H_i modulo p, we have $i_p(P, H_i) \geqslant 2$, where $i_p(P, H_i)$ denotes the intersection number of P and H_i above the prime p. These conditions will be satisfied if a_i is squareful for every

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$i \in \{0, \ldots, n-1\}$ and if $\sum_{i=0}^{n-1} a_i$ is also squareful. We denote the set of all such rational points by $(\mathbb{P}^{n-1}, \Delta)(\mathbb{Q})$. Using the height function

$$H(x_0 : \cdots : x_{n-1}) = \max\left\{ |x_0|, \ldots, |x_{n-1}|, \left|\sum_{i=0}^{n-1} x_i\right| \right\},$$

the set of points $P \in (\mathbb{P}^{n-1}, \Delta)(\mathbb{Q})$ of bounded height is denoted $(\mathbb{P}^{n-1}, \Delta)(\mathbb{Q})_{\leqslant B}$.

Defining the canonical divisor of the orbifold $(\mathbb{P}^{n-1}, \Delta)$ as

$$K_{(\mathbb{P}^{n-1},\Delta)} = K_{\mathbb{P}^{n-1}} + \Delta,$$

we have $K_{(\mathbb{P}^{n-1},\Delta)} \sim (-(n-1)/2) \cdot H$ in $\mathrm{Pic}(\mathbb{P}^{n-1})_{\mathbb{Q}}$, where H is the hyperplane class of $\mathbb{P}^{n-1}$. Since the height function we use is associated to H, a very naïve generalization of Manin's conjecture would predict that $\#(\mathbb{P}^{n-1}, \Delta)(\mathbb{Q})_{\leqslant B} \sim C \cdot B^{(n-1)/2}$ for some constant $C > 0$, as B tends to infinity. Our main goal is to prove the following theorem.

Theorem 1.1. *For $n \geqslant 4$, there exists a $\delta > 0$ so that*

$$\#(\mathbb{P}^{n-1}, \Delta)(\mathbb{Q})_{\leqslant B} = C \cdot B^{(n-1)/2} + O(B^{(n-1)/2-\delta})$$

for some constant $C > 0$.

In Section 5 we will give an explicit description of the constant C and examine the distribution of rational points on the orbifold $(\mathbb{P}^{n-1}, \Delta)$.

2. Description of the proof

Throughout the article, we will use the following notation.

We will denote the $(n+1)$-tuple $(x_0, \ldots, x_n) \in A^{n+1}$ for any ring A by $\underline{x}$. For the nonzero integers we use the notation $\mathbb{Z}_0$, that is $\mathbb{Z}_0 = \mathbb{Z} \setminus \{0\}$. If there exists a constant $C > 0$ such that $|f(x)| \leqslant Cg(x)$ for real-valued functions f and g with g only taking positive values, we write $f(x) \ll g(x)$ or $f(x) = O(g(x))$. If C depends on other parameters, this will be denoted explicitly when this dependence is important for the computations. We will write $f(x) \sim g(x)$ if $f(x)/g(x)$ tends to one if x goes to infinity. Also, we allow the small positive constant ε to take different values at different points of the arguments. Finally, for any $\alpha \in \mathbb{R}$ we will write $e(\alpha) = \exp(2\pi i \alpha)$.

To prove Theorem 1.1, we first restrict ourselves to the set of points

$$(a_0 : \cdots : a_{n-1}) \in (\mathbb{P}^{n-1}, \Delta)(\mathbb{Q})$$

for which $a_i \neq 0$ for each $i \in \{0, \ldots, n-1\}$ and $\sum_{i=0}^{n-1} a_i \neq 0$. We denote this subset by $(\mathbb{P}^{n-1}, \Delta)(\mathbb{Q})^+$. Also, $(\mathbb{P}^{n-1}, \Delta)(\mathbb{Q})^+_{\leqslant B}$ indicates the intersection of $(\mathbb{P}^{n-1}, \Delta)(\mathbb{Q})^+$ with $(\mathbb{P}^{n-1}, \Delta)(\mathbb{Q})_{\leqslant B}$.

By the definition of $(\mathbb{P}^{n-1}, \Delta)(\mathbb{Q})$, we can identify $(\mathbb{P}^{n-1}, \Delta)(\mathbb{Q})^+_{\leqslant B}$ with the set

$$\left\{(a_0:\ldots:a_n)\in H(\mathbb{Q}):a_i\in\mathbb{Z}_0,\, a_i \text{ is squareful}, \gcd(a_0,\ldots,a_n)=1,\, \max_{0\leqslant i\leqslant n}|a_i|\leqslant B\right\},$$

where $H\subset\mathbb{P}^n$ is the hyperplane defined by $X_0+\cdots+X_n=0$.

Since a squareful integer can be written uniquely (up to the sign of x) as x^2y^3, where y is squarefree, the latter set in turn corresponds to

$$\begin{aligned}\{(x_0^2y_0^3:\cdots:x_n^2y_n^3)\in H(\mathbb{Q}):x_i,\,y_i\in\mathbb{Z}_0 \text{ and } y_i \text{ is squarefree},\\ \gcd(x_0y_0,\ldots,x_ny_n)=1,\, \max_{0\leqslant i\leqslant n}|x_i^2y_i^3|\leqslant B\}.\end{aligned}\tag{1}$$

Definition. We define $M(B)$ as the set

$$\left\{\begin{aligned}(\underline{x},\underline{y})\in\mathbb{Z}_0^{2n+2}:\sum_{i=0}^n x_i^2y_i^3=0,\ \gcd(x_0y_0,\ldots,x_ny_n)=1,\\ \max_{0\leqslant i\leqslant n}|x_i^2y_i^3|\leqslant B,\ \prod_{i=0}^n\mu^2(|y_i|)=1\end{aligned}\right\}.$$

(Note that for any integer $y\in\mathbb{Z}$, the condition $\mu^2(|y_i|)=1$ means that y_i is squarefree.) Also, we denote by $M_{\underline{a},t}(B)$ the set

$$\left\{(\underline{x},\underline{y})\in\mathbb{Z}_0^{2n+2}:\sum_{i=0}^n a_ix_i^2y_i^3=t,\ \max_{0\leqslant i\leqslant n}|a_ix_i^2y_i^3|\leqslant B,\ \prod_{i=0}^n\mu_i'(y_i)=1\right\},$$

where $a_0,\ldots,a_n,t\in\mathbb{Z}$ are fixed, $\gcd(a_0,\ldots,a_n)=1$ and $\prod_{i=0}^n a_i\neq 0$. Here, μ_i' denotes an arbitrary function $\mathbb{Z}_0\to\{0,1\}$, for each $i\in\{0,\ldots,n\}$.

As a first step in the proof, we will use the classical Hardy–Littlewood circle method to determine an expression for the cardinality of the set $M_{\underline{a},t}(B)$. Notice that in the definition of $M_{\underline{a},t}(B)$, we replaced the function $\mu^2(\cdot)$ in the definition of $M(B)$ with the more general function $\mu_i'(\cdot)$. We shall see that applying the circle method is independent of this condition, but nevertheless necessary to derive an asymptotic formula for $\#M(B)$ since squarefree conditions on multiples of the y_i will appear as we will explain below. We see that $M(B)$ is a subset of $M_{(1,\ldots,1),0}(B)$ (if we take $\mu_i'(\cdot)$ to be $\mu^2(\cdot)$ for each i), with the additional gcd condition $\gcd(x_0y_0,\ldots,x_ny_n)=1$ on the solutions. We will take this gcd condition into account using an adapted version of the Möbius inversion.

Identifying $(\mathbb{P}^{n-1},\Delta)(\mathbb{Q})^+_{\leqslant B}$ with (1), it readily follows that

$$\#(\mathbb{P}^{n-1},\Delta)(\mathbb{Q})^+_{\leqslant B}=\frac{1}{2^{n+2}}\#M(B),$$

which implies that an asymptotic formula for $\#M(B)$ induces an asymptotic formula for $\#(\mathbb{P}^{n-1},\Delta)(\mathbb{Q})^+_{\leqslant B}$.

Finally, we will explain why this result suffices to prove Theorem 1.1.

3. Calculating $\#M_{\underline{a},t}(B)$

Let us first fix the framework of the circle method.

Let T be $\mathbb{R}/\mathbb{Z}$. For $0 < \Delta \leqslant 1$ and $P \geqslant 1$ (we always suppose $B \geqslant 1$), we define $\mathfrak{M}(\Delta, q, a)$ as the image in T of $\{\alpha \in \mathbb{R} : |\alpha - a/q| < P^{\Delta-2}\}$ with $a, q \in \mathbb{Z}$ and

$$\mathfrak{M}(\Delta) = \bigcup_{\substack{1\leqslant a\leqslant q\leqslant P^{\Delta}\\ \gcd(a,q)=1}} \mathfrak{M}(\Delta, q, a).$$

We call $\mathfrak{M}(\Delta)$ the union of the *major arcs* and $T \setminus \mathfrak{M}(\Delta) = \mathfrak{m}(\Delta)$ the union of the *minor arcs*. We shall clarify the constraint on the constant Δ and the dependence of P on B in Proposition 3.7 and Theorem 3.8.

The circle method calculates $\#M_{\underline{a},t}(B)$ by integrating an exponential sum over T, namely

$$\#M_{\underline{a},t}(B) = \int_T \sum_{\substack{1\leqslant|a_i x_i^2 y_i^3|\leqslant B\\ i=0,\dots,n}} \biggl(\prod_{i=0}^{n} \mu_i'(y_i)\biggr) e(\alpha f(\underline{x}, \underline{y}))\, d\alpha, \tag{2}$$

where $f(\underline{x}, \underline{y}) = \sum_{i=0}^{n} a_i x_i^2 y_i^3 - t$. We will denote the integrand of (2) by $E(\alpha)$ and will set

$$S_i(\alpha) = \sum_{1\leqslant|a_i x^2 y^3|\leqslant B} \mu_i'(y) e(\alpha a_i x^2 y^3).$$

Therefore,

$$E(\alpha) = e(-\alpha t) \prod_{i=0}^{n} S_i(\alpha).$$

As usual, the integral over $\mathfrak{M}(\Delta)$ will provide the main term while the integral over $\mathfrak{m}(\Delta)$ will only contribute to the error term.

Major arcs. We refer to [Schmidt 1984, Section 5; Davenport 2005, Chapter 4] for avoid conflict with theorems. (Many authors improperly cite a detailed description of the circle method over the major arcs for the classical case of diagonal equations. In order to apply this to $\int_{\mathfrak{M}(\Delta)} E(\alpha)\, d\alpha$, we will first fix $\underline{y}$ and thus consider the diagonal equation $f(\underline{x}, \underline{y}) = f_{\underline{y}}(\underline{x}) = 0$; afterwards we will take the sum of the obtained expression over all admitted $\underline{y}$.

Since we fix $\underline{y}$, we only look at x_i satisfying $1/|a_i y_i^3|^{1/2} \leqslant |x_i| \leqslant (B/|a_i y_i^3|)^{1/2}$. Most of the time, it suffices to consider only positive x_i; we will denote the corresponding interval for positive x_i with D_i, that is,

$$D_i = \bigl[1/|a_i y_i^3|^{1/2}, B^{1/2}/|a_i y_i^3|^{1/2}\bigr]. \tag{3}$$

We will also use the notation

$$B_{a_i,y_i} = B^{1/2}/|a_i y_i^3|^{1/2}. \tag{4}$$

Note that since we consider only $\underline{y}$ with $1 \leqslant |y_i^3| \leqslant B$, we have $1 \leqslant B_{a_i,y_i} \leqslant B^{1/2}$ for each $i \in \{0, \ldots, n\}$.

Because we first wish to examine the exponential sum $E(\alpha)$ (for $\alpha \in \mathfrak{M}(\Delta)$) for some $\underline{y}$ fixed, we denote this part of $E(\alpha)$ by

$$E_{\underline{y}}(\alpha) = \sum_{\substack{1/|a_i y_i^3|^{1/2} \leqslant |x_i| \leqslant B_{a_i,y_i} \\ i=0,\ldots,n}} e(\alpha f_{\underline{y}}(\underline{x})).$$

Furthermore, for every positive integer q and every integer a relatively prime to q, we define

$$\sigma_{\underline{y}}\Big(\frac{a}{q}\Big) = q^{-(n+1)} \sum_{\underline{z} \in (\mathbb{Z}/q\mathbb{Z})^{n+1}} e\Big(\frac{a f_{\underline{y}}(\underline{z})}{q}\Big), \tag{5}$$

and for every $\beta \in \mathbb{R}$,

$$\tau_{\underline{y},B}(\beta) = \int_{D_0} \cdots \int_{D_n} e(\beta f_{\underline{y}}(\underline{x}))\, d\underline{x}. \tag{6}$$

Proposition 3.1. *For $\alpha = a/q + \beta \in \mathfrak{M}(\Delta; q, a)$, we have*

$$E_{\underline{y}}(\alpha) = 2^{n+1} \sigma_{\underline{y}}\Big(\frac{a}{q}\Big) \tau_{\underline{y},B}(\beta) + O\bigg(q \frac{\sum_{i=0}^n |a_i y_i^3|^{1/2}}{\prod_{i=0}^n |a_i y_i^3|^{1/2}} B^{(n+2)/2} P^{\Delta-2}\bigg)$$

under the condition $B P^{\Delta-2} \geqslant 1$ on P and Δ.

Proof. Combining positive and negative signs of x_i, we have

$$E_{\underline{y}}(\alpha) = 2^{n+1} e(-\alpha t) \prod_{i=0}^n \sum_{x_i \in D_i} e(\alpha a_i x_i^2 y_i^3). \tag{7}$$

For $\alpha = a/q + \beta$, the inner sum over x_i equals

$$\sum_{1 \leqslant z_i \leqslant q} e\Big(\frac{a a_i z_i^2 y_i^3}{q}\Big) \sum_{\substack{v_i \in \mathbb{Z} \\ q v_i + z_i \in D_i}} e(\beta a_i (q v_i + z_i)^2 y_i^3). \tag{8}$$

Euler's summation formula (in its simplest version) implies

$$\sum_{X \leqslant qv+z \leqslant Y} e(\zeta (qv+z)^2) = \frac{1}{q} \int_X^Y e(\zeta \eta^2)\, d\eta + O\Big(1 + \frac{Y}{q} |\zeta| q Y\Big)$$

for any real numbers $0 \leqslant X < Y$, $\zeta \in \mathbb{R}$, $q, z \in \mathbb{N}$. Taking $Y = B_{a_i,y_i}$, $\zeta = \beta a_i y_i^3$

and recalling the definition of D_i in (3), we can rewrite (8) as

$$\sum_{1\leqslant z_i\leqslant q} e\Big(\frac{aa_i z_i^2 y_i^3}{q}\Big)\Big(\frac{1}{q}\int_{D_i} e(\beta a_i x_i^2 y_i^3)\,dx_i + O(1+|\beta|B)\Big).$$

We substitute these expressions successively back into (7) and obtain the desired main term. Using the trivial upper bounds

$$\Big|\sum_{x_i\in D_i} e(\alpha a_i x_i^2 y_i^3)\Big| + \Big|\frac{1}{q}\sum_{1\leqslant z_i\leqslant q} e\Big(\frac{aa_i z_i^2 y_i^3}{q}\Big)\int_{D_i} e(\beta a_i x_i^2 y_i^3)\,dx_i\Big| \ll B_{a_i,y_i},$$

we get the total error term $O\big(q(1+|\beta|B)\max_{0\leqslant i\leqslant n}\prod_{j\neq i} B_{a_j,y_j}\big)$. Using (4) and $1+|\beta|B \ll P^{\Delta-2}B$, we complete the proof. □

From this result, we can now derive an expression for the integral of $E_{\underline{y}}(\alpha)$ over $\mathfrak{M}(\Delta)$ by first integrating the expression for $E_{\underline{y}}(\alpha)$ obtained in Proposition 3.1 over $\mathfrak{M}(\Delta; q, a)$ and then summing over all admitted a and q.

We first define

$$\mathfrak{I}_{\underline{\varepsilon},t,B}(L) = \int_{|\gamma|<L} e(-\gamma t/B)\,d\gamma \int_{[B^{-1/2},1]^{n+1}} e\Big(\gamma\sum_{i=0}^{n}\varepsilon_i x_i'^2\Big)\,d\underline{x}',$$

(where $\varepsilon_i = \operatorname{sgn}(a_i y_i)$) and

$$\mathfrak{S}_{\underline{y},\underline{a},t}(L) = \sum_{q\leqslant L}\sum_{\substack{0<\frac{a}{q}\leqslant 1\\ \gcd(a,q)=1}} \sigma_{\underline{y}}\Big(\frac{a}{q}\Big).$$

We have

$$\int_{|\beta|<P^{\Delta-2}} \tau_{\underline{y},B}(\beta)\,d\beta = \frac{B^{(n-1)/2}}{\prod_{i=0}^{n}|a_i y_i^3|^{1/2}}\,\mathfrak{I}_{\underline{\varepsilon},t,B}(BP^{\Delta-2}),$$

and therefore

$$\int_{\mathfrak{M}(\Delta)} E_{\underline{y}}(\alpha)\,d\alpha = \frac{2^{n+1}\mathfrak{S}_{\underline{y},\underline{a},t}(P^{\Delta})\mathfrak{I}_{\underline{\varepsilon},t,B}(BP^{\Delta-2})}{\prod_{i=0}^{n}|a_i y_i^3|^{1/2}}\cdot B^{(n-1)/2} + O\bigg(\frac{\sum_{i=0}^{n}|a_i y_i^3|^{1/2}}{\prod_{i=0}^{n}|a_i y_i^3|^{1/2}}\,B^{(n+2)/2}P^{5\Delta-4}\bigg). \quad (9)$$

Note that the integral $\mathfrak{I}_{\underline{\varepsilon},t,B}(L)$ only depends on the signs of $\underline{y}$ and $\underline{a}$ and no longer on their actual values.

Next, we make the coefficient of $B^{(n-1)/2}$ in this expression independent of B. We first focus on the factor $\mathfrak{S}_{\underline{y},\underline{a},t}(P^{\Delta})$.

The singular series.

Lemma 3.2. *We have*

$$\left|\sigma_{\underline{y}}\left(\frac{a}{q}\right)\right| \ll q^{-(n+1)/2} \cdot \prod_{i=0}^{n} \gcd(a_i y_i^3, q)^{1/2}.$$

Proof. Using elementary properties of generalized Gauss sums (see for example [Berndt et al. 1998, Chapter 1]), we obtain for positive integers a and c that

$$\left|\sum_{n=0}^{c-1} e\left(\frac{an^2}{c}\right)\right| \ll \gcd(a, c)^{1/2}\sqrt{c}.$$

Applying this to (5) implies the statement. □

Corollary 3.3. *For $n \geqslant 4$, the series*

$$\mathfrak{S}_{\underline{y},\underline{a},t} = \sum_{q=1}^{\infty} \sum_{\substack{0<a/q\leqslant 1 \\ \gcd(a,q)=1}} \sigma_{\underline{y}}\left(\frac{a}{q}\right), \tag{10}$$

called the singular series, converges absolutely. In particular, we have

$$\mathfrak{S}_{\underline{y},\underline{a},t} \ll \frac{\prod_{i=0}^{n} |a_i y_i^3|^{1/2+\varepsilon}}{\operatorname{lcm}(a_0 y_0^3, \ldots, a_n y_n^3)^{1/2}} \tag{11}$$

and

$$\mathfrak{S}_{\underline{y},\underline{a},t}(P^{\Delta}) = \mathfrak{S}_{\underline{y},\underline{a},t} + O\left(\frac{\prod_{i=0}^{n} |a_i y_i^3|^{1/2+\varepsilon}}{\operatorname{lcm}(a_0 y_0^3, \ldots, a_n y_n^3)^{1/2}} \cdot P^{\Delta(-n+3)/2}\right) \tag{12}$$

for any $\varepsilon > 0$.

Proof. From the previous lemma, we deduce that

$$\begin{aligned}
\mathfrak{S}_{\underline{y},\underline{a},t} &\ll \sum_{q=1}^{\infty} q^{-(n-1)/2} \prod_{i=0}^{n} \gcd(a_i y_i^3, q)^{1/2} \\
&\ll \sum_{\substack{d_i | a_i y_i^3 \\ i=0,\ldots,n}} (d_0 \cdots d_n)^{1/2} \sum_{\substack{q=1 \\ \operatorname{lcm}(d_0,\ldots,d_n)|q}}^{\infty} q^{-(n-1)/2} \\
&\ll \sum_{\substack{d_i | a_i y_i^3 \\ i=0,\ldots,n}} \frac{(d_0 \cdots d_n)^{1/2}}{\operatorname{lcm}(d_0, \ldots, d_n)^{(n-1)/2}} \sum_{q=1}^{\infty} q^{-(n-1)/2}.
\end{aligned}$$

Since $n \geqslant 4$, the latter expression converges and we get

$$\mathfrak{S}_{\underline{y},\underline{a},t} \ll \sum_{\substack{d_i \mid a_i y_i^3 \\ i=0,\dots,n}} \frac{(d_0 \cdots d_n)^{1/2}}{\operatorname{lcm}(d_0, \dots, d_n)^{(n-1)/2}} \ll \frac{\prod_{i=0}^n |a_i y_i^3|^{1/2+\varepsilon}}{\operatorname{lcm}(a_0 y_0^3, \dots, a_0 y_0^3)^{1/2}}$$

for any $\varepsilon > 0$. Moreover, we obtain in the same way that

$$\begin{aligned}\left|\mathfrak{S}_{\underline{y},\underline{a},t} - \mathfrak{S}_{\underline{y},\underline{a},t}(P^\Delta)\right| &\leqslant \sum_{q > P^\Delta} q^{-(n-1)/2} \prod_{i=0}^n \gcd(a_i y_i^3, q)^{1/2} \\ &\ll \sum_{\substack{d_i \mid a_i y_i^3 \\ 0 \leqslant i \leqslant n}} \frac{(d_0 \cdots d_n)^{1/2}}{\operatorname{lcm}(d_0, \dots, d_n)^{(n-1)/2}} \sum_{\substack{q > P^\Delta \\ \operatorname{lcm}(d_0,\dots,d_n) \mid q}}^{\infty} q^{-(n-1)/2} \\ &\ll \frac{\prod_{i=0}^n |a_i y_i^3|^{1/2+\varepsilon}}{\operatorname{lcm}(a_0 y_0^3, \dots, a_n y_n^3)^{1/2}} \cdot P^{\Delta(-n+3)/2}. \qquad \square\end{aligned}$$

Remark 3.4. One can prove (see for example [Davenport 2005, Lemmas 5.2-5.3]) for $n \geqslant 4$ that $\mathfrak{S}_{\underline{y},\underline{a},t}$ can be written as an Euler product of p-adic densities

$$\lim_{l \to \infty} \frac{\#\{(x_0, \dots, x_n) \in (\mathbb{Z}/p^l\mathbb{Z})^{n+1} : \sum_{i=0}^n a_i y_i^3 x_i^2 \equiv t \bmod p^l\}}{p^{ln}}.$$

The singular integral. Examining $\mathfrak{I}_{\underline{\varepsilon},t,B}(BP^{\Delta-2})$ in (9), we have the following proposition.

Proposition 3.5. *For $n \geqslant 3$, we have*

$$\mathfrak{I}_{\underline{\varepsilon},t,B}(BP^{\Delta-2}) = \mathfrak{I}_{\underline{\varepsilon},t,B} + O\big(B^{(1-n)/2} P^{(\Delta-2)(1-n)/2}\big) \tag{13}$$

with

$$\mathfrak{I}_{\underline{\varepsilon},t,B} = \int_{-\infty}^{+\infty} e(-\gamma t/B)\, d\gamma \int_{[B^{-1/2},1]^{n+1}} e\Big(\gamma \sum_{i=0}^n \varepsilon_i x_i^2\Big)\, d\underline{x}$$

under the condition $BP^{\Delta-2} \geqslant 1$.

Proof. As proved in [Davenport 2005, Proof of Theorem 4.1], we have

$$\left|\int_{B^{-1/2}}^1 e(\gamma \varepsilon_i x_i^2)\, dx_i\right| \ll \min\{1, |\gamma|^{-1/2}\},$$

and thus

$$\left|\int_{[B^{-1/2},1]^{n+1}} e\Big(\gamma \sum_{i=0}^n \varepsilon_i x_i^2\Big)\, d\underline{x}\right| \ll \min\{1, |\gamma|^{-1/2}\}^{n+1}. \tag{14}$$

This implies that the integral $\mathfrak{I}_{\underline{\varepsilon},t,B}$ converges, since

$$\left|\mathfrak{I}_{\underline{\varepsilon},t,B}\right| \ll \int_{-\infty}^{+\infty} \min\{1, |\gamma|^{-1/2}\}^{n+1} d\gamma < +\infty.$$

Also,

$$\begin{aligned}\left|\mathfrak{I}_{\underline{\varepsilon},t,B}(BP^{\Delta-2}) - \mathfrak{I}_{\underline{\varepsilon},t,B}\right| &\ll \int_{|\gamma|>BP^{\Delta-2}} |\gamma|^{-(n+1)/2} d\gamma \\ &\ll B^{(1-n)/2} P^{(\Delta-2)(1-n)/2}.\end{aligned}$$ □

Defining *the singular integral* as

$$\mathfrak{I}_{\underline{\varepsilon}} = \int_{-\infty}^{+\infty} d\gamma \int_{[0,1]^{n+1}} e\Big(\gamma \sum_{i=0}^{n} \varepsilon_i x_i^2\Big)\, d\underline{x}, \tag{15}$$

it follows from the last proof that this integral is also convergent.

Lemma 3.6. *It holds that* $\mathfrak{I}_{\underline{\varepsilon},t,B} \to \mathfrak{I}_{\underline{\varepsilon}}$ *as* B *goes to infinity.*

Proof. We have

$$\begin{aligned}\left|\mathfrak{I}_{\underline{\varepsilon},t,B} - \mathfrak{I}_{\underline{\varepsilon}}\right| \leqslant \int_{-\infty}^{+\infty} &\left|(e(-\gamma t/B) - 1)\right| d\gamma \left|\int_{[B^{-1/2},1]^{n+1}} e\Big(\gamma \sum_{i=0}^{n} \varepsilon_i x_i^2\Big)\, d\underline{x}\right| \\ &+ \int_{-\infty}^{+\infty} d\gamma \left|\int_{([B^{-1/2},1]^{n+1})^c} e\Big(\gamma \sum_{i=0}^{n} \varepsilon_i x_i^2\Big)\, d\underline{x}\right| \\ = I_1(B,t) + I_2(B),&\end{aligned}$$

where $\big([B^{-1/2}, 1]^{n+1}\big)^c$ denotes the complement of $[B^{-1/2}, 1]^{n+1}$ in the hypercube $[0, 1]^{n+1}$.

Since $|(e(-\gamma t/B) - 1)| = 2|\sin(\pi\gamma t B^{-1})| \leqslant \min\{2, 2\pi|\gamma||t|B^{-1}\}$, we obtain the following for $I_1(B, t)$, recalling (14):

$$I_1(B,t) \ll \int_{-\infty}^{+\infty} \min\{1, \pi|\gamma||t|B^{-1}\} \cdot \min\{1, |\gamma|^{-1/2}\}^{n+1} d\gamma.$$

Splitting up the latter integral into three parts according to the appropriate range of γ, we get $I_1(B, t) \ll |t|B^{-1}$ for B big enough.

For $I_2(B)$, one has

$$\left|\int_0^1 e(\gamma\varepsilon_i x_i^2)\, dx_i\right| \ll \min\{1, |\gamma|^{-1/2}\}$$

and

$$\left|\int_0^{B^{-1/2}} e(\gamma\varepsilon_i x_i^2)\, dx_i\right| \ll \min\{B^{-1/2}, |\gamma|^{-1/2}\}.$$

Applying the exclusion-inclusion principle to $I_2(B)$ and observing the symmetric form of the integrand, we get

$$I_2(B) \ll \sum_{i=1}^{n+1} \int_{-\infty}^{+\infty} \min\{B^{-1/2}, |\gamma|^{-1/2}\}^i \cdot \min\{1, |\gamma|^{-1/2}\}^{n+1-i} d\gamma.$$

It follows that $I_2(B) \ll B^{-1/2}$. Hence,

$$\left|\mathfrak{I}_{\underline{\varepsilon},t,B} - \mathfrak{I}_{\underline{\varepsilon}}\right| \ll_t B^{-1/2} \tag{16}$$

for B big enough, completing the proof. □

Note that from Proposition 3.5 and (16), one has

$$\mathfrak{I}_{\underline{\varepsilon},t,B}(BP^{\Delta-2}) = \mathfrak{I}_{\underline{\varepsilon}} + O\big(B^{-1/2} + B^{(1-n)/2}P^{(\Delta-2)(1-n)/2}\big). \tag{17}$$

We now return to the integral of $E_{\underline{y}}(\alpha)$ over the major arcs.

Proposition 3.7. *For $n \geqslant 4$ and for any Δ with $0 < \Delta < 1/5$, there exists a $\delta > 0$ so that*

$$\int_{\mathfrak{M}(\Delta)} E_{\underline{y}}(\alpha)\, d\alpha = \frac{2^{n+1}\mathfrak{S}_{\underline{y},\underline{a},t}\mathfrak{I}_{\underline{\varepsilon}}}{\prod_{i=0}^n |a_i y_i^3|^{1/2}} \cdot B^{(n-1)/2} + O_{\underline{y},\underline{a}}\big(B^{(n-1)/2-\delta}\big). \tag{18}$$

Proof. Substituting (12) and (17) into formula (9), we get

$$\begin{aligned}\int_{\mathfrak{M}(\Delta)} E_{\underline{y}}(\alpha)\, d\alpha &= \frac{2^{n+1}\mathfrak{S}_{\underline{y},\underline{a},t}\mathfrak{I}_{\underline{\varepsilon}}}{\prod_{i=0}^n |a_i y_i^3|^{1/2}} \cdot B^{(n-1)/2} \\ &\quad + O\Bigg(\frac{\prod_{i=0}^n |a_i y_i^3|^{\varepsilon}}{\operatorname{lcm}(a_0 y_0^3, \ldots, a_n y_n^3)^{1/2}} \cdot B^{(n-1)/2} P^{\Delta(-n+3)/2} \\ &\qquad + \frac{B^{(n-2)/2} + P^{(\Delta-2)(1-n)/2}}{\prod_{i=0}^n |a_i y_i^3|^{1/2}} + \frac{\sum_{i=0}^n |a_i y_i^3|^{1/2}}{\prod_{i=0}^n |a_i y_i^3|^{1/2}} \cdot B^{(n+2)/2} P^{5\Delta-4}\Bigg).\end{aligned} \tag{19}$$

For this expression to be nontrivial, we have to determine $P = P(B)$ and Δ properly (under the condition $BP^{\Delta-2} \geqslant 1$) so that the error term is $O_{\underline{y},\underline{a}}(B^{(n-1)/2-\delta})$ for some $\delta > 0$. Taking $P = B^{1/2}$ and $0 < \Delta < 1/5$ is satisfactory. □

We can now prove our estimate for the major arcs.

Theorem 3.8. *For $n \geqslant 4$ and for any Δ with $0 < \Delta < 1/15$, there exists a $\delta > 0$ so that*

$$\int_{\mathfrak{M}(\Delta)} E(\alpha)\, d\alpha = C_{\underline{a},t} \cdot B^{(n-1)/2} + O\big(B^{(n-1)/2-\delta}\big),$$

where

$$C_{\underline{a},t} = 2^{n+1} \sum_{\underline{y} \in \mathbb{Z}_0^{n+1}} \left(\prod_{i=0}^n \mu_i'(y_i)\right) \frac{\mathfrak{S}_{\underline{y},\underline{a},t}\mathfrak{I}_{\underline{\varepsilon}}}{\prod_{i=0}^n |a_i y_i^3|^{1/2}},$$

with $\mathfrak{S}_{\underline{y},\underline{a},t}$ and $\mathfrak{I}_{\underline{\varepsilon}}$ as defined above.

Proof. We sum (19) over all admitted y_i such that $1 \leqslant |y_i^3| \leqslant B$, $i \in \{0, \dots, n\}$, and denote the sum of the coefficients of the main term by $C_{\underline{a},t}(B)$.

We obtain, using (11),

$$\frac{\mathfrak{S}_{\underline{y},\underline{a},t}}{\prod_{i=0}^n |a_i y_i^3|^{1/2}} \ll \frac{\prod_{i=0}^n |a_i y_i^3|^{\varepsilon}}{\operatorname{lcm}(a_0 y_0^3, \dots, a_n y_n^3)^{1/2}} \tag{20}$$

for any $\varepsilon > 0$. We have

$$\begin{aligned}
\sum_{\substack{\max\limits_{0\leqslant i\leqslant n} |y_i^3| \geqslant B}} \frac{\prod_{i=0}^n |a_i y_i^3|^{\varepsilon}}{\operatorname{lcm}(a_0 y_0^3, \dots, a_n y_n^3)^{1/2}} &\ll \sum_{\substack{\max\limits_{0\leqslant i\leqslant n} |y_i^3| \geqslant B}} \frac{1}{\operatorname{lcm}(a_0 y_0^3, \dots, a_n y_n^3)^{1/2-(n+1)\varepsilon}} \\
&\ll \sum_{\substack{\max\limits_{0\leqslant i\leqslant n} |y_i^3| \geqslant B}} \frac{1}{\operatorname{lcm}(y_0, \dots, y_n)^{3/2-3(n+1)\varepsilon}} \\
&\ll \sum_{N^3 \geqslant B} \frac{\#\{(y_0, \dots, y_n) : \operatorname{lcm}(y_0, \dots, y_n) = N\}}{N^{3/2-3(n+1)\varepsilon}} \\
&\ll B^{-1/6+(n+1)\varepsilon}
\end{aligned} \tag{21}$$

for any $\varepsilon > 0$. This allows us to replace $C_{\underline{a},t}(B)$ by $C_{\underline{a},t}$.

We now turn to the error term in (19), summing over all admitted values of $\underline{y}$ and putting $P = B^{1/2}$ as before.

The first error term can be treated as the main term. The coefficients of the third and fourth error terms will also converge without any extra conditions. Moreover, the upper bound can be made independent of the a_i. For the last error term however, the coefficient will asymptotically contribute $O(B^{1/3})$.

This means the extra condition

$$\frac{1}{3} + \frac{n+2}{2} + \frac{5\Delta - 4}{2} < \frac{n-1}{2} \Leftrightarrow \Delta < \frac{1}{15}$$

has to be satisfied for the error term to behave properly. □

Note that (20) and (21) also provides a uniform upper bound of $C_{\underline{a},t}$, that is, $C_{\underline{a},t} \leqslant C$, independently of $\underline{a}$ and t.

Minor arcs. The goal of this section is to prove the following theorem.

Theorem 3.9. *For $n \geqslant 4$, there exists a $\delta > 0$ so that*

$$\int_{\mathfrak{m}(\Delta)} E(\alpha)\, d\alpha = O\big(B^{(n-1)/2-\delta}\big).$$

To treat the integral over the minor arcs, we will not fix $\underline{y}$ but examine the whole equation at once. Recall that

$$E(\alpha) = e(-\alpha t)\prod_{i=0}^{n} S_i(\alpha) = e(-\alpha t)\prod_{i=0}^{n} \sum_{1\leqslant|a_i x^2 y^3|\leqslant B} \mu_i'(y)e(\alpha a_i x^2 y^3).$$

Using Hölder's inequality repeatedly, we get for $n \geqslant 4$,

$$\left|\int_{\mathfrak{m}(\Delta)} E(\alpha)\,d\alpha\right| \leqslant \sup_{\alpha\in\mathfrak{m}(\Delta)} (|S_0(\alpha)|\cdots|S_{n-4}(\alpha)|) \max_{j=n-3,\dots,n} \int_0^1 |S_j(\alpha)|^4 d\alpha. \quad (22)$$

To obtain a good upper bound of this expression, we first examine $\int_0^1 |S_j(\alpha)|^4 d\alpha$.

Lemma 3.10. *For any* $\varepsilon > 0$, *we have*

$$\int_0^1 |S_j(\alpha)|^4 d\alpha \ll_\varepsilon B^{1+\varepsilon}.$$

Proof. From now on, we will concentrate on the part of the sum where the variables are positive. This will suffice to prove the theorem because of the symmetry.

Let

$$S_Y(\alpha) = \sum_{Y<y\leqslant 2Y} \mu_j'(y) \sum_{1\leqslant x\leqslant B_{a_j,y}} e(\alpha a_j x^2 y^3)$$

be the contribution to $S_j(\alpha)$ for $Y < y \leqslant 2Y$. Using Cauchy's inequality, it follows that

$$\begin{aligned}\int_0^1 |S_Y(\alpha)|^4 d\alpha &\ll Y \int_0^1 |S_Y(\alpha)|^2 \sum_{Y<y\leqslant 2Y} \mu_j'(y)\left|\sum_{1\leqslant x\leqslant B_{a_j,y}} e(\alpha a_j x^2 y^3)\right|^2 d\alpha \\ &\ll Y \sum_{Y<y_1,y_2,y_3\leqslant 2Y} \sum_{\substack{1\leqslant x_1\leqslant B_{a_j,y_1}\\ 1\leqslant x_2\leqslant B_{a_j,y_2}\\ 1\leqslant x_3,x_4\leqslant B_{a_j,y_3}}} \int_0^1 e(\alpha a_j G(\underline{x},\underline{y}))\,d\alpha \\ &\leqslant Y\cdot \#Z(Y,B),\end{aligned}$$

with $G(\underline{x},\underline{y}) = y_3^3(x_4^2-x_3^2)+x_1^2y_1^3-x_2^2y_2^3$ and $Z(Y,B)=\{(\underline{x},\underline{y})\in\mathbb{Z}_0^7 : y_3^3(x_3^2-x_4^2) = x_1^2y_1^3-x_2^2y_2^3,\ 1\leqslant x_i < B_Y,\ Y<y_j\leqslant 2Y\}$, where $B_Y = (B/Y^3)^{1/2}$.

If we make a distinction between solutions $(\underline{x},\underline{y})\in\mathbb{Z}_0^7$ of $G(\underline{x},\underline{y})=0$ for which $x_1^2y_1^3 - x_2^2y_2^3 = 0$ or not, it follows that both sets contain $O(Y^{-1}\cdot B^{1+\varepsilon})$ solutions. Hence, we conclude that $\#Z(Y,B) \ll_\varepsilon Y^{-1}\cdot B^{1+\varepsilon}$, and thus

$$\int_0^1 |S_Y(\alpha)|^4 d\alpha \ll_\varepsilon B^{1+\varepsilon}.$$

Summing over all intervals $(Y, 2Y]$ with $Y = 2^k \ll B^{1/3}$ and applying Cauchy's inequality twice on $|S_j(\alpha)|^4 = |\sum_{Y=2^k \ll B^{1/3}} S_Y(\alpha)|^4$, we get

$$\int_0^1 |S_j(\alpha)|^4 d\alpha \ll B^{3\varepsilon'} \sum_{Y=2^k \ll B^{1/3}} \int_0^1 |S_Y(\alpha)|^4 d\alpha \ll B^{3\varepsilon'} \sum_{Y=2^k \ll B^{1/3}} B^{1+\varepsilon} = B^{1+\varepsilon''},$$

which completes the proof. □

Remark 3.11. Recalling the expression for $\#M_{\underline{a},t}(B)$ in (2) and putting $n = 3$, $\underline{a} = (1, 1, 1, 1)$, $t = 0$ and $\mu'_i(\cdot) = \mu^2(\cdot)$ for each i, this lemma implies that the equation $n_1 + n_2 = n_3 + n_4$, where n_i is squareful and $1 \leqslant |n_i| \leqslant B$ for each $i \in \{1, 2, 3, 4\}$, has $O(B^{1+\varepsilon})$ solutions.

In order to handle the first part of (22), namely $\sup_{\alpha \in \mathfrak{m}(\Delta)}(|S_0(\alpha)| \cdots |S_{n-4}(\alpha)|)$, we will prove the following proposition.

Proposition 3.12. *Let $\alpha \in \mathfrak{m}(\Delta)$. Then there exists a $\delta > 0$ such that*

$$|S_i(\alpha)| \ll B^{1/2-\delta}.$$

Proof. Let $\psi > 0$. We may henceforth assume that $|a_i| \leqslant B^{\psi}$, since otherwise the trivial upper bound yields

$$|S_i(\alpha)| \leqslant \sum_{y=1}^{\infty} \sqrt{\frac{B}{a_i y^3}} \ll B^{(1-\psi)/2},$$

which is satisfactory. Similarly, we may assume that $y \leqslant B^{\psi}$ in $S_i(\alpha)$. Thus, we have

$$|S_i(\alpha)| \ll B^{(1-\psi)/2} + \sum_{y \leqslant B^{\psi}} \mu'_i(y) \big|T_{\underline{y}}(\alpha)\big|,$$

with, if we set $X = \sqrt{B/(|a_i|y^3)}$,

$$T_{\underline{y}}(\alpha) = \sum_{x \leqslant X} e(\alpha a_i y^3 x^2).$$

Since $|a_i|y^3x^2 \leqslant B$, in particular $X \geqslant B^{1/2-2\psi}$. Using the usual squaring and differencing approach (see for example [Davenport 2005, Chapter 3]), we obtain

$$\begin{aligned} \big|T_{\underline{y}}(\alpha)\big|^2 &\leqslant \sum_{|h| \leqslant X} \Bigg| \sum_{\substack{x \\ x, x+h \leqslant X}} e(2\alpha a_i y^3 h x) \Bigg| \\ &\ll \sum_{|h| \leqslant X} \min\{X, \|2\alpha a_i y^3 h\|^{-1}\} \ll X + B^{\varepsilon} \cdot \sum_{y \leqslant Y} \min\{X, \|\alpha y\|^{-1}\}, \end{aligned}$$

where $Y = 2|a_i|y^3 X$ and $\|a\| = \min\{|\beta| \in \mathbb{R} : \beta \equiv a \bmod 1\}$ for any real number a.

In order to estimate the sum over y, we will use the following lemma.

Lemma 3.13 (Separation lemma). *Let P, $Q \geqslant 1$ be reals, $\alpha \in T$ and $a, q \in \mathbb{Z}$ with* $\gcd(a, q) = 1$ *and* $|\alpha - a/q| < q^{-2}$. *Then*

$$\sum_{x \leqslant P} \min\left\{\frac{PQ}{x}, \|\alpha x\|^{-1}\right\} \ll PQ\big(q^{-1} + Q^{-1} + q(PQ)^{-1}\big)\log(2qP).$$

Proof. A full proof is given in [Vaughan 1997, Lemma 2.2]. □

Choosing $P = Y$ and $Q = X$, Lemma 3.13 implies

$$\begin{aligned}\big|T_{\underline{y}}(\alpha)\big|^2 &\ll X + XYB^{\varepsilon}\Big(\frac{1}{q} + \frac{1}{X} + \frac{q}{XY}\Big)\\ &\ll XYB^{2\varepsilon}\Big(\frac{1}{q} + \frac{1}{X} + \frac{q}{XY}\Big)\\ &\ll B^{1+2\varepsilon}\Big(\frac{1}{q} + B^{2\psi-1/2}\Big) + qB^{2\varepsilon},\end{aligned}$$

since $X \leqslant Y$ and $XY = 2|a_i|y^3X^2 = 2B$. Hence,

$$|S_i(\alpha)| \ll B^{1/2-2\psi} + B^{1/2+\varepsilon+\psi}\Big(\frac{1}{\sqrt{q}} + B^{\psi-1/4}\Big) + \sqrt{q}B^{\varepsilon+\psi}. \tag{23}$$

According to Dirichlet, we can find $a, q \in \mathbb{Z}$ with $\gcd(a, q) = 1$ and $q \leqslant B^{(2-\Delta)/4}$ such that $|\alpha q - a| < 1/B^{(2-\Delta)/4} = B^{(\Delta-2)/4}$. (Note we also have $|\alpha - a/q| < 1/q^2$.) Furthermore, it is necessary that $q > B^{\Delta/2}$: otherwise, we would have $\alpha \in \mathfrak{M}(\Delta)$. With these boundaries for q in (23), a suitable small choice for ψ in terms of Δ leads to the statement. □

We are now able to prove Theorem 3.9.

Proof of Theorem 3.9. Combining Proposition 3.12 and Lemma 3.10 in (22), we obtain

$$\left|\int_{\mathfrak{m}(\Delta)} E(\alpha)\, d\alpha\right| \ll B^{(1/2-\delta)(n-3)} \cdot B^{1+\varepsilon} \leqslant B^{(n-1)/2-\delta+\varepsilon} < B^{(n-1)/2}$$

for any $0 < \varepsilon < \delta$. □

4. Towards the main problem

Combining the previous results, we are able to prove the following theorem.

Theorem 4.1. *For $n \geqslant 4$, there exists a $\delta > 0$ so that*

$$\#M_{\underline{a},t}(B) = C_{\underline{a},t} \cdot B^{(n-1)/2} + O\big(B^{(n-1)/2-\delta}\big),$$

with the constant $C_{\underline{a},t}$ described in Theorem 3.8.

Proof. This follows directly from Theorem 3.8, Theorem 3.9 and (2). □

Remark 4.2. Note that the error term is independent of $\underline{a}$ and t and recall we also proved $C_{\underline{a},t}$ can be bounded uniformly independent of $\underline{a}$ and t. This implies that $\#M_{\underline{a},t}(B) \leqslant C \cdot B^{(n-1)/2}$ for some constant $C > 0$. Indeed, when $B < 1$, $M_{\underline{a},t}(B) = \varnothing$, and for $B \geqslant 1$, it follows from Theorem 4.1 that

$$\#M_{\underline{a},t}(B) \leqslant C' \cdot B^{(n-1)/2} + C'' \cdot B^{(n-1)/2-\delta} \leqslant C \cdot B^{(n-1)/2},$$

where $C = 2\max\{C', C''\}$.

Going back to $M(B)$ (see definition on page 1021), we will now prove the following theorem.

Theorem 4.3. *For $n \geqslant 4$, there exists an explicit constant D and a $\delta > 0$ such that*

$$\#M(B) = D \cdot B^{(n-1)/2} + O\big(B^{(n-1)/2-\delta}\big)$$

as B goes to infinity.

(The definition of the constant D is given in Lemma 4.5; in the next section, we will give some indications about the interpretation of D.)

The only problem still left in proving Theorem 4.3 is to understand how we can tackle the additional gcd condition $\gcd(x_0y_0, \dots, x_ny_n) = 1$ on the solutions. Note that the Möbius inversion at hand leads to divisibility conditions on both x_i and y_i which have to be handled with care.

Let $\underline{e} = (e_0, \dots, e_n) \in \mathbb{N}_0^{n+1}$ and $\underline{f} = (f_0, \dots, f_n) \in \mathbb{N}_0^{n+1}$, where f_i is squarefree for each $i \in \{0, \dots, n\}$.

Definition. We denote the set

$$\Big\{(\underline{x}, \underline{y}) \in \mathbb{Z}_0^{2n+2} \sum_{i=0}^{n} x_i^2 y_i^3 = 0,\ \max_{0\leqslant i\leqslant n} |x_i^2 y_i^3| \leqslant B,\ e_i|x_i,\ f_i|y_i \text{ and } \prod_{i=0}^{n} \mu^2(|y_i|) = 1\Big\}$$

by $N_{(\underline{e},\underline{f})}(B)$.

Demanding that solutions in $N_{(\underline{1},\underline{1})}(B)$ satisfy $\gcd(x_0y_0, \dots, x_ny_n) = 1$ means we wish to leave out those solutions of $N_{(\underline{1},\underline{1})}(B)$ for which there exists a prime p and a subset $I \subset \{0, \dots, n\}$ such that $p|x_i$ if $i \in I$ and $p|y_i$ if $i \notin I$ (or $i \in I^c$, where I^c denotes the complement of I in $\{0, \dots, n\}$) in order to get to $M(B)$. Defining for a prime p and subsets $I, J \subset \{0, \dots, n\}$ the couple $(\underline{e}^{p,I}, \underline{f}^{p,J})$ by $e_i^{p,I} = p$ for $i \in I$ and $e_i^{p,I} = 1$ otherwise and analogously for $\underline{f}^{p,J}$, it hence follows that

$$M(B) = N_{(\underline{1},\underline{1})}(B) \setminus \bigcup_{(p,I)} N_{(\underline{e}^{p,I},\underline{f}^{p,I^c})}(B). \tag{24}$$

Notice that in this last union only a finite number of sets are nonempty since for a prime $p \geqslant \sqrt{B}$, we get $N_{(\underline{e}^{p,I},\underline{f}^{p,I^c})}(B) = \varnothing$.

Definition. Let S be a finite set of couples (p, I). We can associate to S a couple $(\underline{e}, \underline{f})$ as follows: defining for each prime p the index sets $I_p = \cup_{(p,I)\in S} I$ and $J_p = \cup_{(p,I)\in S} I^c$, the associated couple is given by $e_i = \prod_{\{p|i\in I_p\}} p$ and $f_i = \prod_{\{p|i\in J_p\}} p$.

We then define

$$\mu(\underline{e}, \underline{f}) = \sum_{n\geqslant 0}(-1)^n \,\#\big\{\text{sets } S \text{ of cardinality } n \text{ with associated couple } (\underline{e}, \underline{f})\big\}.$$

Observing (24) together with this definition, we have

$$\#M(B) = \sum_{e=1}^{\infty} \sum_{\substack{(\underline{e},\underline{f})\in\mathbb{N}^{2n+2}\\ e=\gcd(e_i f_i,\ i=0,\dots,n)}} \mu(\underline{e}, \underline{f}) \cdot \#N_{(\underline{e},\underline{f})}(B). \tag{25}$$

The following lemma collects some properties of μ.

Lemma 4.4. *There exists a function* $\widetilde{\mu} : \mathbb{Z}^{2n+2} \to \mathbb{Z}$ *such that*

(i) $\mu(\underline{e}, \underline{f}) = \prod_p \widetilde{\mu}(v_p(\underline{e}), v_p(\underline{f}))$, *where* $v_p(\underline{e}) = (v_p(e_0), \dots, v_p(e_n))$ *(and analogously for* $v_p(\underline{f})$*),*

(ii) $\widetilde{\mu}(\underline{m}, \underline{n}) = 0$ *if* $m_i = n_i = 0$ *and* $(\underline{m}, \underline{n}) \neq (\underline{0}, \underline{0})$ *or if* $m_i > 1$ *for some* i,

(iii) $\sum_{I\cup J=\{0,\dots,n\}} |\widetilde{\mu}(I, J)| \leqslant 2^{2^{n+1}}$, *where, for subsets* $I, J \subset \{0, \dots, n\}$, $\widetilde{\mu}(I, J)$ *denotes* $\widetilde{\mu}(m_0^I, \dots, m_n^I, m_0'^J, \dots, m_n'^J)$ *with* $m_i^I = 1$ *if* $i \in I$ *and* $m_i^I = 0$ *otherwise and* $m_i'^J = 1$ *if* $i \in J$ *and* $m_i'^J = 0$ *otherwise.*

Proof. (i) and (ii) follow directly from the definition of μ immediately above. From the same definition, it follows, if $I \cup J = \{0, \dots, n\}$, and denoting by T a finite set of subsets $I \subset \{0, \dots, n\}$, that

$$\widetilde{\mu}(I, J) = \sum_m (-1)^m \,\#\big\{\text{sets } T \text{ of cardinality } m \text{ such that } I = \cup_{K\in T} K \text{ and } J = \cup_{K\in T} K^c\big\}.$$

If we sum over all possible I and J such that $I \cup J = \{0, \dots, n\}$, we get (iii). □

Consider now $N_{(\underline{e},\underline{f})}(B)$ for a couple $(\underline{e}, \underline{f})$ for which $\mu(\underline{e}, \underline{f}) \neq 0$ and

$$\gcd(e_i f_i,\ i = 0, \dots, n) = e,$$

i.e., a subset with nontrivial contribution to $\#M(B)$ (recall (25)). Since $\#N_{(\underline{e},\underline{f})}(B) = \#M_{\underline{e^2 f^3},0}(B)$, choosing $\mu_i'(y_i) = \mu^2(f_i|y_i|)$ (where $\underline{e^2 f^3} = (e_0^2 f_0^3, \dots, e_n^2 f_n^3)$), we know by Theorem 4.1 that

$$\#N_{(\underline{e},\underline{f})}(B) = C_{\underline{e^2 f^3},0} \cdot B^{(n-1)/2} + O(B^{(n-1)/2-\delta}).$$

Since e divides $e_i f_i$, we can write $e_i^2 f_i^3 = v_i e^2$ for some $v_i \in \mathbb{N}$ for each i in $\{0, \dots, n\}$. Making the substitutions $x_i' = x_i/e_i$ and $y_i' = y_i/f_i$, we see that $N_{(\underline{e},\underline{f})}(B)$ corresponds to the set

$$\Big\{(\underline{x}', \underline{y}') \in \mathbb{Z}_0^{2n+2} : \sum_{i=0}^n v_i x_i'^2 y_i'^3 = 0,\ \max_{0\leqslant i\leqslant n} |v_i x_i'^2 y_i'^3| \leqslant \frac{B}{e^2} \text{ and } \prod_{i=0}^n \mu^2(f_i|y_i'|) = 1\Big\},$$

where we eliminated e^2 in the equation, and hence $\#N_{(\underline{e},\underline{f})}(B) = \#M_{\underline{v},0}(B/e^2)$. Letting B go to infinity, this implies that the main terms in the asymptotic formulas of $\#N_{\underline{e},\underline{f}}(B)$ and $\#M_{\underline{v},0}(B/e^2)$ are equal, and in particular that

$$\#N_{(\underline{e},\underline{f})}(B) - C_{\underline{e^2 f^3},0} \cdot B^{(n-1)/2} = O\Big(\frac{B^{(n-1)/2-\delta}}{e^{n-1-2\delta}}\Big). \tag{26}$$

Notice we also obtain (recall Remark 4.2) that

$$\#N_{(\underline{e},\underline{f})}(B) \leqslant C \cdot \frac{B^{(n-1)/2}}{e^{n-1}} \quad \text{and} \quad C_{\underline{e^2 f^3},0} \leqslant \frac{C}{e^{n-1}}. \tag{27}$$

From these results, we can now prove:

Lemma 4.5. *The series* $D = \sum_{e=1}^{\infty} \sum_{\substack{(\underline{e},\underline{f})\in\mathbb{N}^{2n+2} \\ \gcd(e_i f_i,\ i=0,\dots,n)=e}} \mu(\underline{e}, \underline{f}) \cdot C_{\underline{e^2 f^3},0}$ *converges.*

Proof. Substituting (27) into the definition of D and using the properties of μ in Lemma 4.4, we get

$$\begin{aligned} |D| &\ll \sum_{e=1}^{\infty} \sum_{\substack{(\underline{e},\underline{f})\in\mathbb{N}^{2n+2} \\ \gcd(e_i f_i,\ i=0,\dots,n)=e}} \frac{|\mu(\underline{e}, \underline{f})|}{e^{n-1}} \\ &\leqslant \prod_p \sum_{k=0}^{2} \sum_{\substack{(v_p(\underline{e}),v_p(\underline{f}))\in\mathbb{N}^{2n+2} \\ \min_i\{v_p(e_i)+v_p(f_i)\}=k}} \frac{|\mu_p(v_p(\underline{e}), v_p(\underline{f}))|}{p^{k(n-1)}} \leqslant \prod_p \Big(1 + 2\frac{2^{2^{n+1}}}{p^{n-1}}\Big), \end{aligned}$$

which converges since $n \geqslant 4$. □

Proof of Theorem 4.3. From the definition of D and (26), it follows that

$$\big|\#M(B) - D \cdot B^{(n-1)/2}\big| \ll \sum_{e=1}^{\infty} \sum_{\substack{(\underline{e},\underline{f})\in\mathbb{N}^{2n+2} \\ \gcd(e_i f_i,\ i=0,\dots,n)=e}} \big|\mu(\underline{e}, \underline{f})\big| \cdot \frac{B^{(n-1)/2-\delta}}{e^{(n-1)-2\delta}}.$$

Following the same reasoning as in Lemma 4.5, we then get

$$\big|\#M(B) - D \cdot B^{(n-1)/2}\big| \ll B^{(n-1)/2-\delta} \cdot \prod_p \Big(1 + 2\frac{2^{2^{n+1}}}{p^{n-1-2\delta}}\Big),$$

where the product converges for $\delta > 0$ small enough since $n \geqslant 4$. This proves the theorem. □

5. Rational points on the orbifold $(\mathbb{P}^{n-1}, \Delta)$

We can now prove our main theorem.

Theorem 5.1. *For $n \geqslant 4$, there exists a $\delta > 0$ such that*

$$\#(\mathbb{P}^{n-1}, \Delta)(\mathbb{Q})_{\leqslant B} = C \cdot B^{(n-1)/2} + O\big(B^{(n-1)/2-\delta}\big).$$

Here,

$$C = \frac{1}{2^{n+1}} \sum_{e=1}^{\infty} \sum_{\substack{(\underline{e}, \underline{f}) \in \mathbb{N}^{2n+2} \\ \gcd(e_i f_i,\ i=0,\ldots,n)=e}} \mu(\underline{e}, \underline{f}) \sum_{\substack{\underline{y} \in \mathbb{Z}_0^{n+1}/\{\pm 1\} \\ f_i y_i \text{ squarefree}}} \frac{2^{n+1} \mathfrak{S}_{\underline{y}, \underline{e^2 f^3}, 0} \mathfrak{I}_{\underline{\varepsilon}}}{\prod_{i=0}^{n} (e_i^2 f_i^3 |y_i^3|)^{1/2}},$$

with $\mathfrak{S}_{\underline{y},\underline{a},t}$, $\mathfrak{I}_{\underline{\varepsilon}}$ and the function μ as defined before. (By $\underline{y} \in \mathbb{Z}_0^{n+1}/\{\pm 1\}$, we denote the $(n+1)$-tuples $(y_0, \ldots, y_n) \in \mathbb{Z}_0^{n+1}$, defined up to sign as an $(n+1)$-tuple.)

Proof. The connection between $(\mathbb{P}^{n-1}, \Delta)(\mathbb{Q})^{+}_{\leqslant B}$ and the set $M(B)$ given by (1), together with Theorem 4.3, implies that the theorem holds for $\#(\mathbb{P}^{n-1}, \Delta)(\mathbb{Q})^{+}_{\leqslant B}$. It remains to prove that, for $n \geqslant 4$, the set of points $(a_0 : \cdots : a_n) \in (\mathbb{P}^{n-1}, \Delta)(\mathbb{Q})_{\leqslant B}$ with at least one zero coordinate (whose cardinality is $\ll \#(\mathbb{P}^{n-2}, \Delta)(\mathbb{Q})_{\leqslant B}$), is asymptotically negligible compared to $(\mathbb{P}^{n-1}, \Delta)(\mathbb{Q})^{+}_{\leqslant B}$.

We will verify this for $n = 4$; by induction, the statement follows for $n > 4$.

As mentioned in Remark 3.11, it follows from Lemma 3.10 that

$$\#(\mathbb{P}^{2}, \Delta)(\mathbb{Q})^{+}_{\leqslant B} \ll B^{1+\varepsilon}.$$

Combining this with the trivial upper bound $\#(\mathbb{P}^{1}, \Delta)(\mathbb{Q})_{\leqslant B} \ll B$, we obtain

$$\#(\mathbb{P}^{2}, \Delta)(\mathbb{Q})_{\leqslant B} \ll B^{1+\varepsilon} < B^{3/2}$$

for $\varepsilon > 0$ sufficiently small. □

Description of the constant. An alternative description of $(\mathbb{P}^{n-1}, \Delta)(\mathbb{Q})^{+}_{\leqslant B}$ can be obtained as follows. Consider $\underline{y} \in \mathbb{Z}_0^{n+1}/\{\pm 1\}$ with each y_i squarefree. For such $\underline{y}$, let $Q_{\underline{y}}$ denote the smooth quadric defined by the homogeneous polynomial $F_{\underline{y}}(\underline{x}) = y_0^3 X_0^2 + \ldots + y_n^3 X_n^2 \in \mathbb{Z}[X_0, \ldots, X_n]$. Furthermore, define the morphism

$$\begin{array}{rccc} \pi_{\underline{y}} : & Q_{\underline{y}} & \to & H \\ & (x_0 : \cdots : x_n) & \mapsto & (y_0^3 x_0^2 : \cdots : y_n^3 x_n^2). \end{array} \tag{28}$$

We will consider points $(x_0 : \cdots : x_n) \in Q_{\underline{y}}(\mathbb{Q})$ with $x_i \in \mathbb{Z}$ such that $\prod_{i=0}^{n} x_i \neq 0$ and $\gcd(x_0 y_0, \ldots, x_n y_n) = 1$. We denote this subset of $Q_{\underline{y}}(\mathbb{Q})$ by $Q_{\underline{y}}(\mathbb{Q})^{+}$. This set is mapped into $(\mathbb{P}^{n-1}, \Delta)(\mathbb{Q})^+$ by $\pi_{\underline{y}}$ and, keeping in mind (1), we have

$$(\mathbb{P}^{n-1}, \Delta)(\mathbb{Q})^+ = \coprod_{\substack{\underline{y}\in\mathbb{Z}_0^{n+1}/\{\pm1\} \\ y_i \text{ squarefree}}} \pi_{\underline{y}}(Q_{\underline{y}}(\mathbb{Q})^+). \tag{29}$$

This implies

$$\#(\mathbb{P}^{n-1}, \Delta)(\mathbb{Q})^+_{\leqslant B} = \frac{1}{2^{n+1}} \sum_{\substack{\underline{y}\in\mathbb{Z}_0^{n+1}/\{\pm1\} \\ y_i \text{ squarefree}}} \#\big\{(x_0:\dots:x_n)\in Q_{\underline{y}}(\mathbb{Q})^+ : \max_{0\leqslant i\leqslant n} |x_i^2 y_i^3| \leqslant B\big\}.$$

For a fixed $\underline{y}$, an asymptotic expression for each of the latter sets using the classical circle method is known (see [Davenport 2005, Chapter 8]) and a Möbius inversion for the gcd condition $\gcd(x_0y_0, \dots, x_ny_n) = 1$.

Moreover, from Lemma 4.5, it follows that we can change the order of summation for e and $\underline{y}$ in the constant C from Theorem 5.1 and thus, defining

$$C_{Q_{\underline{y}}} = \sum_{e=1}^{\infty} \sum_{\substack{(\underline{e},\underline{f})\in\mathbb{N}^{2n+2} \\ \gcd(e_i f_i,\ i=0,\dots,n)=e \\ f_i\mid y_i}} \mu(\underline{e},\underline{f}) \frac{2^{n+1}\mathfrak{S}_{\underline{y},\underline{e}^2,0}\mathfrak{J}_{\underline{\varepsilon}}}{\prod_{i=0}^{n}(e_i|y_i|^{3/2})}, \tag{30}$$

we have, for $n \geqslant 4$,

$$\#(\mathbb{P}^{n-1}, \Delta)(\mathbb{Q})_{\leqslant B} \sim \Bigg(\frac{1}{2^{n+1}} \sum_{\substack{\underline{y}\in\mathbb{Z}_0^{n+1}/\{\pm1\} \\ y_i \text{ squarefree}}} C_{Q_{\underline{y}}}\Bigg) \cdot B^{(n-1)/2}$$

as B goes to infinity.

This constant $C_{Q_{\underline{y}}}$ can be given a more geometrical interpretation using the adelic space $Q_{\underline{y}}(\mathbb{A}_{\mathbb{Q}})$ of the quadric $Q_{\underline{y}}$, as explained in [Peyre 1995, §5]. Here, it has been shown that the refined version of the Manin conjecture is compatible with the circle method for smooth quadrics in $\mathbb{P}^n_{\mathbb{Q}}$ and moreover, that rational points on smooth quadrics are equidistributed. Considering the Tamagawa measure $\boldsymbol{\omega}_{H_{\underline{y}}}$ (corresponding to the height function $H_{\underline{y}}$ defined as $H_{\underline{y}}(P) = \max_{0\leqslant i\leqslant n} |x_i^2 y_i^3|$ where $P = (x_0 : \dots : x_n) \in Q_{\underline{y}}(\mathbb{Q})$) on $\mathbb{Q}_{\underline{y}}(\mathbb{A}_{\mathbb{Q}})$, the equidistribution of the rational points on $Q_{\underline{y}}$ implies that for every good open subset W (that is, an open subset W for which $\boldsymbol{\omega}_{H_{\underline{y}}}(\partial W) = 0$, where $\partial W = \overline{W} \setminus W$) of $Q_{\underline{y}}(\mathbb{A}_{\mathbb{Q}})$, we have

$$\frac{\#\big\{P \in Q_{\underline{y}}(\mathbb{Q})^+ \cap W \mid H_{\underline{y}}(P) \leqslant B\big\}}{\#\big\{P \in Q_{\underline{y}}(\mathbb{Q})^+ \mid H_{\underline{y}}(P) \leqslant B\big\}} \to \frac{\boldsymbol{\omega}_{H_{\underline{y}}}(W)}{\boldsymbol{\omega}_{H_{\underline{y}}}(Q_{\underline{y}}(\mathbb{A}_{\mathbb{Q}}))}$$

as B goes to infinity. We refer to [Peyre 1995] for more details on this matter. This implies we can obtain a description of the constant $C_{Q_{\underline{y}}}$ in terms of the measure $\boldsymbol{\omega}_{H_{\underline{y}}}$

of a certain subset of the adelic space $Q_{\underline{y}}(\mathbb{A}_{\mathbb{Q}})$ of the quadric $Q_{\underline{y}}$. More precisely, it follows that

$$C_{Q_{\underline{y}}} = \boldsymbol{\omega}_{H_{\underline{y}}}(Q_{\underline{y}}(\mathbb{A}_{\mathbb{Q}})^{\dagger})/(n-1),$$

where $Q_{\underline{y}}(\mathbb{A}_{\mathbb{Q}})^{\dagger}$ denotes the good open subset of $Q_{\underline{y}}(\mathbb{A}_{\mathbb{Q}})$ defined by the gcd condition $\gcd(x_0 y_0, \dots, x_n y_n) = 1$ we imposed on $Q_{\underline{y}}(\mathbb{Q})$. (Note that imposing the open condition $\prod_{i=0}^{n} x_i \neq 0$ does not change the measure.) We obtain the following corollary.

Corollary 5.2. *For $n \geqslant 4$, we have*

$$\#(\mathbb{P}^{n-1}, \Delta)(\mathbb{Q})_{\leqslant B} \sim \Bigg(\frac{1}{2^{n+1}} \sum_{\substack{\underline{y} \in \mathbb{Z}_0^{n+1}/\{\pm 1\} \\ y_i \text{ squarefree}}} C_{Q_{\underline{y}}}\Bigg) \cdot B^{(n-1)/2} \tag{31}$$

as B goes to infinity, where $C_{Q_{\underline{y}}} = \boldsymbol{\omega}_{H_{\underline{y}}}(Q_{\underline{y}}(\mathbb{A}_{\mathbb{Q}})^{\dagger})/(n-1)$.

***The adelic space of the orbifold* $(\mathbb{P}^{n-1}, \Delta)$.** In order to define the adelic space of the orbifold properly, we first have to explain how we can translate the definition of "squarefulness" to the different completions of $\mathbb{Q}$.

At each finite place $v = p$, a p-adic integer $a \in \mathbb{Z}_p$ is squareful if $v_p(a) \neq 1$. Due to the structure of $\mathbb{Q}_p^{\times}$, this means that we can write a squareful p-adic integer a uniquely as $x^2 y^3$ with $x \in \mathbb{Z}_p^{\times}$ and $y \in \mathbb{Z}$ squarefree.

On the other hand, any real number $a \in \mathbb{R}$ can be written as $(\pm 1)^3 x^2$ and ought to be considered as squareful.

Since we identified $(\mathbb{P}^{n-1}, \Delta)(\mathbb{Q})$ with $\{(u_0 : \cdots : u_n) \in H(\mathbb{Q}) : u_i \text{ squareful}\}$ (recall $H \subset \mathbb{P}^n$ is the hyperplane defined by $X_0 + \cdots + X_n = 0$), we have, for each $v \in \mathrm{Val}(\mathbb{Q})$, that

$$\begin{aligned}(\mathbb{P}^{n-1}, \Delta)(\mathbb{Q}_v) &= \{(u_0 : \cdots : u_n) \in H(\mathbb{Q}_v) : u_i \text{ squareful}\} \\ &= \{(x_{0,v}^2 y_0^3 : \cdots : x_{n,v}^2 y_n^3) \in H(\mathbb{Q}_v) : \underline{y} \in \mathbb{Z}_0^{n+1}/\{\pm 1\},\ y_i \text{ squarefree}\}.\end{aligned}$$

This implies, recalling the definition of $\pi_{\underline{y}}$ in (28),

$$(\mathbb{P}^{n-1}, \Delta)(\mathbb{Q}_v) = \bigcup_{\substack{\underline{y} \in \mathbb{Z}_0^{n+1}/\{\pm 1\} \\ y_i \text{ squarefree}}} \pi_{\underline{y}}(Q_{\underline{y}}(\mathbb{Q}_v)^{\dagger}), \tag{32}$$

where for a finite place $v = p$, $Q_{\underline{y}}(\mathbb{Q}_p)^{\dagger}$ is the open subset of $Q_{\underline{y}}(\mathbb{Q}_p)$ defined by the condition $\min_{0 \leqslant i \leqslant n}(v_p(x_{i,p} y_i)) = 0$, and where $Q_{\underline{y}}(\mathbb{R})^{\dagger} = Q_{\underline{y}}(\mathbb{R})$.

Note that the union considered is not disjoint, but that the image for different $\underline{y}$ and $\underline{y}'$ either coincides or is disjoint. Hence, it follows that, at each place $v \in \mathrm{Val}(\mathbb{Q})$, $(\mathbb{P}^{n-1}, \Delta)(\mathbb{Q}_v)$ can be described as a finite disjoint union of sets $\pi_{\underline{y}}(Q_{\underline{y}}(\mathbb{Q}_v)^{\dagger})$ for specified $\underline{y} \in \mathbb{Z}_0^{n+1}/\{\pm 1\}$.

Definition. We define the adelic space $(\mathbb{P}^{n-1}, \Delta)(\mathbb{A}_{\mathbb{Q}})$ as

$$(\mathbb{P}^{n-1}, \Delta)(\mathbb{A}_{\mathbb{Q}}) = \prod_{v \in \mathrm{Val}(\mathbb{Q})} (\mathbb{P}^{n-1}, \Delta)(\mathbb{Q}_v).$$

Remark 5.3. One may prove that $(\mathbb{P}^{n-1}, \Delta)(\mathbb{Q})$ is dense in $(\mathbb{P}^{n-1}, \Delta)(\mathbb{A}_{\mathbb{Q}})$. This follows from the fact that weak approximation holds for smooth quadrics.

***Distribution of rational points on* $(\mathbb{P}^{n-1}, \Delta)$.** We can now consider the probability measure

$$\mu_{H \leqslant B}^{(\mathbb{P}^{n-1},\Delta)} = \frac{1}{\#(\mathbb{P}^{n-1}, \Delta)(\mathbb{Q})_{\leqslant B}} \sum_{\substack{P \in (\mathbb{P}^{n-1},\Delta)(\mathbb{Q}) \\ H(P) \leqslant B}} \delta_P \tag{33}$$

on $(\mathbb{P}^{n-1}, \Delta)(\mathbb{A}_{\mathbb{Q}})$. Here, we will investigate the convergence of $\mu_{H \leqslant B}^{(\mathbb{P}^{n-1},\Delta)}$ to a specific measure on the adelic space of the orbifold, which we have yet to define, when B goes to infinity. Keeping in mind the description of $(\mathbb{P}^{n-1}, \Delta)(\mathbb{A}_{\mathbb{Q}})$ we gave above, we can define this measure in the following natural way.

Definition. We define the measure $\boldsymbol{\omega}_{(\mathbb{P}^{n-1},\Delta)}$ on $(\mathbb{P}^{n-1}, \Delta)(\mathbb{A}_{\mathbb{Q}})$ as

$$\boldsymbol{\omega}_{(\mathbb{P}^{n-1},\Delta)}(U) = \sum_{\substack{\underline{y} \in \mathbb{Z}_0^{n+1}/\{\pm 1\} \\ y_i \text{ squarefree}}} \boldsymbol{\omega}_{H_{\underline{y}}}(\pi_{\underline{y}}^{-1}(U)), \tag{34}$$

where U is an open subset of $(\mathbb{P}^{n-1}, \Delta)(\mathbb{A}_{\mathbb{Q}})$ (which is equipped with the subspace topology coming from $H(\mathbb{A}_{\mathbb{Q}})$) and $\pi_{\underline{y}} : Q_{\underline{y}}(\mathbb{A}_{\mathbb{Q}})^{\dagger} \to (\mathbb{P}^{n-1}, \Delta)(\mathbb{A}_{\mathbb{Q}})$. (Note that the morphisms $\pi_{\underline{y}}$ introduced in (28) define continuous maps $\pi_{\underline{y}} : Q_{\underline{y}}(\mathbb{A}_{\mathbb{Q}}) \to H(\mathbb{A}_{\mathbb{Q}})$ which map $Q_{\underline{y}}(\mathbb{A}_{\mathbb{Q}})^{\dagger}$ into $(\mathbb{P}^{n-1}, \Delta)(\mathbb{A}_{\mathbb{Q}})$.)

Remark 5.4. From this definition of the measure $\boldsymbol{\omega}_{(\mathbb{P}^{n-1},\Delta)}$, it follows that its support consists of the (disjoint) union of

$$\pi_{\underline{y}}(Q_{\underline{y}}(\mathbb{A}_{\mathbb{Q}})^{\dagger}) \tag{35}$$

for all $\underline{y} \in \mathbb{Z}_0^{n+1}/\{\pm 1\}$ with y_i squarefree for each $i \in \{0, \dots, n\}$. This is a proper subset of $(\mathbb{P}^{n-1}, \Delta)(\mathbb{A}_{\mathbb{Q}})$.

In order to say something about the convergence of $\mu_{H \leqslant B}^{(\mathbb{P}^{n-1},\Delta)}$, we first define elementary open subsets of $(\mathbb{P}^{n-1}, \Delta)(\mathbb{A}_{\mathbb{Q}})$.

An elementary open subset W of $H(\mathbb{A}_{\mathbb{Q}})$ can be defined as

$$W = \prod_{v \in \mathrm{Val}(\mathbb{Q})} W_v,$$

such that $W_v \subset H(\mathbb{Q}_v)$ is defined at finitely many finite places as $W_p = \mathrm{red}_M^{-1}(X_p)$, where $X_p \subset H(\mathbb{Z}/p^M\mathbb{Z})$ and $\mathrm{red}_M : H(\mathbb{Q}_p) \to H(\mathbb{Z}/p^M\mathbb{Z})$; $W_p = H(\mathbb{Q}_p)$ for any other finite place. Furthermore, at the infinite place $v = \infty$, we require $W_\infty = \bigcap_{i,j}(\lambda_{i,j} x_i < x_j) \subset H(\mathbb{R})$ fixing one of the coordinates x_i to one. Here, $\lambda_{i,j} \in \mathbb{R}_{>0}$ depending on i and j.

To construct elementary open subsets on $(\mathbb{P}^{n-1}, \Delta)(\mathbb{A}_\mathbb{Q})$, we can take the intersection with elementary open subsets of $H(\mathbb{A}_\mathbb{Q})$.

We will now prove the following theorem.

Theorem 5.5. *For every elementary open subset U of $(\mathbb{P}^{n-1}, \Delta)(\mathbb{A}_\mathbb{Q})$, we have*

$$\mu_{H \leqslant B}^{(\mathbb{P}^{n-1},\Delta)}(U) \to \frac{\boldsymbol{\omega}_{(\mathbb{P}^{n-1},\Delta)}(U)}{\boldsymbol{\omega}_{(\mathbb{P}^{n-1},\Delta)}((\mathbb{P}^{n-1}, \Delta)(\mathbb{A}_\mathbb{Q}))}$$

as B goes to infinity.

Proof. Straightforward calculations show that for each admitted $\underline{y}$, the inverse image $\pi_{\underline{y}}^{-1}(U)$ of an elementary open subset U of $(\mathbb{P}^{n-1}, \Delta)(\mathbb{A}_\mathbb{Q})$ defines a good open subset of $Q_{\underline{y}}(\mathbb{A}_\mathbb{Q})^\dagger$.

Now let U be an elementary open subset of $(\mathbb{P}^{n-1}, \Delta)(\mathbb{A}_\mathbb{Q})$. Recalling (33), the partition of $(\mathbb{P}^{n-1}, \Delta)(\mathbb{Q})^+$ in (29), and Theorem 5.1, we get

$$\begin{aligned}\mu_{H \leqslant B}^{(\mathbb{P}^{n-1},\Delta)}(U) &= \frac{\#\big\{(u_0 : \cdots : u_n) \in (\mathbb{P}^{n-1}, \Delta)(\mathbb{Q}) \cap U : \max_{0 \leqslant i \leqslant n} |u_i| \leqslant B\big\}}{\#(\mathbb{P}^{n-1}, \Delta)(\mathbb{Q})_{\leqslant B}} \\ &\sim \frac{\sum_{\underline{y}} \#\big\{(x_0 : \cdots : x_n) \in Q_{\underline{y}}(\mathbb{Q})^+ \cap \pi_{\underline{y}}^{-1}(U) : \max_{0 \leqslant i \leqslant n} |y_i^3 x_i^2| \leqslant B\big\}}{\sum_{\underline{y}} \#\big\{(x_0 : \cdots : x_n) \in Q_{\underline{y}}(\mathbb{Q})^+ : \max_{0 \leqslant i \leqslant n} |y_i^3 x_i^2| \leqslant B\big\}}.\end{aligned}$$

(Here, we used the abbreviated notation $\sum_{\underline{y}}$ to sum over all admitted $\underline{y} \in \mathbb{Z}_0^{n+1}$.)

Combining the fact that rational points on smooth quadrics are equidistributed, the definition of the measure in (34), and Theorem 5.1 enables us to complete the proof. □

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A denominator identity for affine Lie superalgebras with zero dual Coxeter number

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We prove a denominator identity for nontwisted affine Lie superalgebras with zero dual Coxeter number.

Introduction

0.1. Let $\mathfrak{g}$ be a complex finite-dimensional contragredient Lie superalgebra. These algebras were classified by V. Kac [1977] and the list (excluding Lie algebras) consists of four series: $A(m|n)$, $B(m|n)$, $C(m)$, $D(m|n)$ and the exceptional algebras $D(2,1,a)$, $F(4)$, $G(3)$. The finite-dimensional contragredient Lie superalgebras with zero Killing form (or, equivalently, with dual Coxeter number equal to zero) are $A(n|n)$, $D(n|n+1)$ and $D(2,1,a)$.

Denote by Δ_{+0} (resp., Δ_{+1}) the set of positive even (resp., odd) roots of $\mathfrak{g}$. The Weyl denominator R and the affine Weyl denominator $\hat{R}$ are given by the formulas

$$R=\frac{R_0}{R_1},\quad \hat{R}=\frac{\hat{R}_0}{\hat{R}_1},$$

where

$$R_0:=\prod_{\alpha\in\Delta_{+0}}(1-e^{-\alpha}),\quad \hat{R}_0:=R_0\cdot\prod_{k=1}^{\infty}(1-q^k)^{\operatorname{rank}\mathfrak{g}}\prod_{\alpha\in\Delta_0}(1-q^ke^{-\alpha}),$$

$$R_1:=\prod_{\alpha\in\Delta_{+1}}(1+e^{-\alpha}),\quad \hat{R}_1:=R_1\cdot\prod_{k=1}^{\infty}\prod_{\alpha\in\Delta_1}(1+q^ke^{-\alpha}).$$

Let $\hat{\mathfrak{g}}$ be the nontwisted affinization of $\mathfrak{g}$, $\hat{\mathfrak{h}}$ be the Cartan subalgebra of $\hat{\mathfrak{g}}$ and $\hat{\Delta}_+$ be the set of positive roots of $\hat{\mathfrak{g}}$. The affine Weyl denominator is the Weyl denominator of $\hat{\mathfrak{g}}$. Let $\hat{\rho}\in\hat{\mathfrak{h}}$ be such that $2(\hat{\rho},\alpha)=(\alpha,\alpha)$ for each simple root $\alpha\in\hat{\Delta}_+$.

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If $\mathfrak{g}$ has a nonzero Killing form, the affine denominator identity, stated in [Kac and Wakimoto 1994] and proven there and in [Gorelik 2011], takes the form

$$\hat{R}e^{\hat{\rho}} = \sum_{w\in T'} w(Re^{\hat{\rho}}), \tag{1}$$

where T' is the affine translation group corresponding to the *"largest"* root subsystem of Δ_0. The affine denominator identity for strange Lie superalgebras $Q(n)$, which are not contragredient, was stated in [Kac and Wakimoto 1994] and proven in [Zagier 2000].

For a parameter q and a formal variable x we introduce, after [De Sole and Kac 2005], the infinite products

$$(1+x)_q^\infty := \prod_{k=0}^{\infty}(1+q^k x) \quad \text{and} \quad (1-x)_q^\infty := \prod_{k=0}^{\infty}(1-q^k x).$$

These infinite products converge for any $x \in \mathbb{C}$ if the parameter q is a real number $0 < q < 1$. In particular, they are well defined for $0 < x = q < 1$ and $(1 \pm q)_q^\infty := \prod_{n=1}^{\infty}(1 \pm q^n)$.

For $A(n-1|n-1) = \mathfrak{gl}(n|n)$ denote by $\mathfrak{str}$ the restriction of the supertrace to the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ (thus $\mathfrak{str} \in \mathfrak{h}^*$).

In this paper we will prove the following theorem.

0.2. Theorem. *Let $\mathfrak{g}$ be a complex finite-dimensional contragredient Lie superalgebra with zero Killing form. One has*

$$\begin{array}{ll} \hat{R}e^{\hat{\rho}} \cdot f(q, e^{\mathfrak{str}}) = \sum_{w\in T'} w(Re^{\hat{\rho}}) & \textit{for } A(n|n), \\ \hat{R}e^{\hat{\rho}} \cdot f(q) = \sum_{w\in T'} w(Re^{\hat{\rho}}) & \textit{for } D(n+1|n),\ D(2,1,a), \end{array} \tag{2}$$

where T' is the affine translation group corresponding to the "smallest" *root subsystem of Δ_0 (see 0.4 below) and $f(q, e^{\mathfrak{str}})$, $f(q)$ are given by the following formulas*

$$\begin{array}{ll} f(q, e^{\mathfrak{str}}) = \frac{(1-q(-1)^n e^{\mathfrak{str}})_q^\infty \cdot (1-q(-1)^n e^{-\mathfrak{str}})_q^\infty}{((1-q)_q^\infty)^2} & \textit{for } \mathfrak{gl}(n|n), \\ f(q) = \left((1-q)_q^\infty\right)^{-1} & \textit{for } D(n+1|n). \end{array} \tag{3}$$

0.3. The affine denominator identity for $\mathfrak{gl}(2|2)$ was stated by V. Kac and M. Wakimoto [1994] and proven in [Gorelik 2010] (with a proof different from the one presented below).

As pointed by P. Etingof, the terms $f(q, e^{\mathfrak{str}})$, $f(q)$ can be interpreted using "degenerate" cases $n = 1$; for example, for $\mathfrak{gl}(1|1)$ we obtain the formula

$$\hat{R}e^{\hat{\rho}} = \frac{((1-q)_q^\infty)^2}{(1+qe^{\mathfrak{str}})_q^\infty \cdot (1+qe^{-\mathfrak{str}})_q^\infty} Re^{\hat{\rho}},$$

which is trivial since $\mathfrak{gl}(1|1)$ has the only positive root $\beta = \mathfrak{str}$, which is odd.

Since $\mathfrak{sl}(n|n) = \{a \in \mathfrak{gl}(n|n) : \mathfrak{str}(a) = 0\}$ and

$$\operatorname{rank} \mathfrak{sl}(n|n) = 2n - 1 = \operatorname{rank} \mathfrak{gl}(n|n) - 1,$$

one has

$$f(q) = \begin{cases} (1-q)_q^\infty & \text{for } \mathfrak{sl}(2n|2n), \\ \dfrac{((1+q)_q^\infty)^2}{(1-q)_q^\infty} & \text{for } \mathfrak{sl}(2n+1|2n+1). \end{cases}$$

The root datum of $D(2, 1, a)$ is the same as the root datum of $D(2|1)$ so the affine denominator identity for $D(2, 1, a)$ is the same as the affine denominator identity for $D(2|1)$.

As it is shown in [Kac and Wakimoto 1994], the evaluation of the affine denominator identity (2) for $A(1|1)$ gives the following Jacobi identity [1829]:

$$\Box(q)^8 = 1 + 16 \sum_{j,k=1}^{\infty} (-1)^{(j+1)k} k^3 q^{jk}, \tag{4}$$

where $\Box(q) = \sum_{j \in \mathbb{Z}} q^{j^2}$ and thus the coefficient of q^m in the power series expansion of $\Box(q)^8$ is the number of representation of a given integer as a sum of 8 squares (taking into the account the order of summands).

0.4. In order to define T' for $A(n|n)$, $D(n+1|n)$ we present the set of even roots in the form $\Delta_0 = \Delta' \amalg \Delta''$, where

$$\begin{array}{ll} \Delta' \cong \Delta'' = A_{n-1} & \text{for } A(n-1|n-1) = \mathfrak{gl}(n|n), \\ \Delta' = C_n, \ \Delta'' = D_{n+1} & \text{for } D(n+1|n). \end{array}$$

Let W' be the Weyl group of Δ' and $\hat{W}'$ be the corresponding affine Weyl group. Then $\hat{W}' = W' \ltimes T'$, where T' is a translation group, see [Kac 1990, Chapter VI]. By contrast to Lie superalgebras with nonzero Killing form, for $D(n+1|n)$ the rank of root system Δ' is smaller than the rank of Δ''. It is not possible to change T' to T'' in (1) and in (2) for $D(n+1|n)$, since the sum $\sum_{w \in T''} w(Re^{\hat{\rho}})$ is not well defined if $\Delta' \ncong \Delta''$ (see Remark 2.1.4).

The key point of our proof of Theorem 0.2 is Proposition 2.3.2, where it is shown that the expansion of $Y := \hat{R}^{-1} e^{-\hat{\rho}} \sum_{w \in T'} w(Re^{\hat{\rho}})$ contains only $\hat{W}$-invariant elements. This implies that $Y = f(q)$ for $\mathfrak{g} = D(n+1|n)$ and $Y = f(q, e^{-\mathfrak{str}})$ for $\mathfrak{gl}(n|n)$. We determine $f(q)$ and $f(q, e^{\mathfrak{str}})$ using suitable evaluations.

1. Preliminaries

One readily sees (for instance, [Gorelik 2011, 1.5]) that $Re^{\hat{\rho}}$ and $\hat{R}e^{\hat{\rho}}$ do not depend on the choice of set of positive roots Δ_+. As a result, in order to prove Theorem 0.2, it is enough to establish the identity (2) for one choice of Δ_+. Similarly, it is enough to establish the identity for one choice of A_{n-1} for $\mathfrak{gl}(n|n)$. In Section 1.1 we describe our choice of the set of positive roots for $\mathfrak{gl}(n|n)$, $D(n+1|n)$. In Section 1.2 we introduce notation for affine Lie superalgebra $\hat{\mathfrak{g}}$. In Section 1.3 we introduce the algebra $\mathcal{R}$ of formal power series in which we expand R and $\hat{R}$.

Note that if the dual Coxeter number of $\mathfrak{g}$ is zero, then

$$\hat{\rho}=\rho=\frac{1}{2}\Big(\sum_{\alpha\in\Delta_{+0}}\alpha-\sum_{\alpha\in\Delta_{+1}}\alpha\Big).$$

1.1. ***Root systems.*** Let $\mathfrak{g}$ be $\mathfrak{gl}(n|n)$ or $D(n|n+1)$ and let $\mathfrak{h}$ be its Cartan subalgebra. We fix the following sets of simple roots:

$$\Pi=\begin{cases}\{\varepsilon_1-\delta_1,\delta_1-\varepsilon_2,\varepsilon_2-\delta_2,\dots,\varepsilon_n-\delta_n\} & \text{for } \mathfrak{gl}(n|n),\\ \{\varepsilon_1-\delta_1,\delta_1-\varepsilon_2,\varepsilon_2-\delta_2,\dots,\varepsilon_n-\delta_n,\delta_n\pm\varepsilon_{n+1}\} & \text{for } D(n+1|n).\end{cases}$$

We fix a nondegenerate symmetric invariant bilinear form on $\mathfrak{g}$ and denote by $(-,-)$ the induced nondegenerate symmetric bilinear form on $\mathfrak{h}^*$; we normalize the form in such a way that $-(\varepsilon_i,\varepsilon_j)=(\delta_i,\delta_j)=\delta_{ij}$; notice that $\{\varepsilon_i,\delta_i:1\le i\le n\}$ (resp., $\{\varepsilon_j,\delta_i:1\le i\le n,\ 1\le j\le n+1\}$) is an orthogonal basis of $\mathfrak{h}^*$ for $\mathfrak{gl}(n|n)$ (resp., for $D(n+1|n)$).

For this choice one has

$$\Delta_{0+}=\begin{cases}\{\varepsilon_i-\varepsilon_j\}_{1\le i<j\le n}\amalg\{\delta_i-\delta_j\}_{1\le i<j\le n} & \text{for } \mathfrak{gl}(n|n),\\ \Delta_{0+}=\{\varepsilon_i\pm\varepsilon_j\}_{1\le i<j\le n+1}\amalg\{\delta_s\pm\delta_t\}_{1\le s<t\le n}\cup\{2\delta_s\}_{1\le s\le n} & \text{for } D(n+1|n),\end{cases}$$

$$\Delta_{1+}=\begin{cases}\{\varepsilon_i-\delta_j\}_{1\le i\le j\le n}\cup\{\delta_i-\varepsilon_j\}_{1\le i<j\le n} & \text{for } \mathfrak{gl}(n|n),\\ \Delta_{1+}=\{\varepsilon_i-\delta_s\}_{1\le i\le s\le n}\cup\{\delta_s-\varepsilon_j\}_{1\le s<j\le n+1}\cup\{\delta_i+\varepsilon_j\}_{1\le i\le n;1\le j\le n+1} & \\ & \text{for } D(n+1|n).\end{cases}$$

For $D(n+1|n)$ one has $\rho=0$. For $\mathfrak{gl}(n|n)$ one has $\mathfrak{str}=\sum_{i=1}^n(\varepsilon_i-\delta_i)$ and $\rho=-\frac{1}{2}\mathfrak{str}$.

Recall that $\mathfrak{sl}(n|n)=\{a\in\mathfrak{gl}(n|n):\mathfrak{str}(a)=0\}$ and so $\mathfrak{h}^*$ for $\mathfrak{sl}(n|n)$ is the quotient of $\mathfrak{h}^*$ for $\mathfrak{gl}(n|n)$ by $\mathbb{C}\mathfrak{str}$.

By the above, Δ_0 is the union of two irreducible root systems, and we write $\Delta_0=\Delta''\amalg\Delta'$, where Δ'' lies in the span of the ε_i and Δ' lies in the span of the δ_i (this notation is compatible with the notation in Section 0.4).

1.2. ***Nontwisted affinization.*** Let $\mathfrak{g}=\mathfrak{n}_-\oplus\mathfrak{h}\oplus\mathfrak{n}_+$ be any complex finite-dimensional contragredient Lie superalgebra with a fixed triangular decomposition, and let Δ_+ be its set of positive roots. Let $\hat{\mathfrak{g}}$ be the affinization of $\mathfrak{g}$ and let $\hat{\mathfrak{h}}$ be its Cartan subalgebra, see [Kac 1990, Chapter VI]. Let $\hat{\Delta}=\hat{\Delta}_0\amalg\hat{\Delta}_1$ be the set of roots of $\hat{\mathfrak{g}}$. We set

$$\hat{\Delta}^+=\Delta_+\cup\Big(\bigcup_{k=1}^{\infty}\{\alpha+k\delta\mid\alpha\in\Delta\}\Big)\cup\Big(\bigcup_{k=1}^{\infty}\{k\delta\}\Big),$$

where δ is the minimal imaginary root. Let W and $\hat{W}$ be the Weyl groups of Δ_0 and $\hat{\Delta}_0$. One has $(\hat{\mathfrak{h}}^*)^{\hat{W}} = \mathbb{C}\delta$ for $\mathfrak{g} \neq \mathfrak{gl}(n|n)$ and $(\hat{\mathfrak{h}}^*)^{\hat{W}} = \mathbb{C}\delta \oplus \mathbb{C}\mathfrak{str}$ for $\mathfrak{g} = \mathfrak{gl}(n|n)$.

We extend the nondegenerate symmetric invariant bilinear form from $\mathfrak{g}$ to $\hat{\mathfrak{g}}$ and denote by $(-,-)$ the induced nondegenerate symmetric bilinear form on $\hat{\mathfrak{h}}^*$ (the above-mentioned form on $\mathfrak{h}^*$ is induced by this form on $\hat{\mathfrak{h}}^*$). For $A \subset \hat{\mathfrak{h}}^*$ we set $A^{\perp} = \{\mu \in \hat{\mathfrak{h}}^* : \forall \nu \in A,\ (\mu, \nu) = 0\}$.

<u>1.2.1.</u> In Section 1.1 we introduced the root systems Δ', Δ'' for $\mathfrak{g} = \mathfrak{gl}(n|n)$ and $\mathfrak{g} = D(n+1|n)$. Let W' and W'' be the Weyl groups of Δ' and Δ'', respectively. One has $W = W' \times W''$. We denote by $\hat{W}'$ the Weyl group of the affine root system $\hat{\Delta}'$. Recall that $\hat{W}' = W' \ltimes T'$, where T' is a translation group; see [Kac 1990, Chapter VI].

<u>1.2.2.</u> For $N \subset \hat{\mathfrak{h}}^*$ we use the notation $\mathbb{Z}N$ for the set $\sum_{\mu \in N} \mathbb{Z}\mu$. Set

$$Q^+ := \sum_{\mu \in \Delta_+} \mathbb{Z}_{\geq 0}\mu, \quad Q := \mathbb{Z}\Delta_+, \quad \hat{Q}^{\pm} := \pm \sum_{\mu \in \hat{\Delta}_+} \mathbb{Z}_{\geq 0}\mu, \quad \hat{Q} := \mathbb{Z}\hat{\Delta}_+.$$

We introduce the standard partial order on $\hat{\mathfrak{h}}^*$: $\mu \leq \nu$ if $(\nu - \mu) \in \hat{Q}^+$.

1.3. ***The algebra*** $\mathcal{R}$. We are going to use the notation of [Gorelik 2011, 1.4], which we recall below. We retain the notation of Section 1.2.

<u>1.3.1.</u> Call a $\hat{Q}^+$*-cone* a set of the form $(\lambda - \hat{Q}^+)$, where $\lambda \in \hat{\mathfrak{h}}^*$.

For a formal sum of the form $Y := \sum_{\nu \in \hat{\mathfrak{h}}^*} b_\nu e^\nu,\ b_\nu \in \mathbb{Q}$ define the *support* of Y by $\operatorname{supp}(Y) := \{\nu \in \hat{\mathfrak{h}}^* : b_\nu \neq 0\}$. Let $\mathcal{R}$ be a vector space over $\mathbb{Q}$, spanned by the sums of the form $\sum_{\nu \in \hat{Q}^+} b_\nu e^{\lambda - \nu}$, where $\lambda \in \hat{\mathfrak{h}}^*,\ b_\nu \in \mathbb{Q}$. In other words, $\mathcal{R}$ consists of the formal sums $Y = \sum_{\nu \in \hat{\mathfrak{h}}^*} b_\nu e^\nu$ with the support lying in a finite union of $\hat{Q}^+$-cones.

Clearly, $\mathcal{R}$ has a structure of commutative algebra over $\mathbb{Q}$. If $Y \in \mathcal{R}$ is such that $YY' = 1$ for some $Y' \in \mathcal{R}$, we write $Y^{-1} := Y'$.

<u>1.3.2.</u> *Action of the Weyl group.* For $w \in \hat{W}$ set $w\big(\sum_{\nu \in \hat{\mathfrak{h}}^*} b_\nu e^\nu\big) := \sum_{\nu \in \hat{\mathfrak{h}}^*} b_\nu e^{w\nu}$. By the above, $wY \in \mathcal{R}$ if and only if $w(\operatorname{supp} Y)$ is a subset of a finite union of $\hat{Q}^+$-cones. For each subgroup $\tilde{W}$ of $\hat{W}$ we set $\mathcal{R}_{\tilde{W}} := \{Y \in \mathcal{R} : wY \in \mathcal{R} \text{ for each } w \in \tilde{W}\}$; notice that $\mathcal{R}_{\tilde{W}}$ is a subalgebra of $\mathcal{R}$.

<u>1.3.3.</u> *Infinite products.* An infinite product of the form $Y = \prod_{\nu \in X} (1 + a_\nu e^{-\nu})^{r(\nu)}$, where $a_\nu \in \mathbb{Q},\ \ r(\nu) \in \mathbb{Z}_{\geq 0}$ and $X \subset \hat{\Delta}$ is such that the set $X \setminus \hat{\Delta}_+$ is finite, can be naturally viewed as an element of $\mathcal{R}$; clearly, this element does not depend on the order of factors. Let $\mathcal{Y}$ be the set of such infinite products. For any $w \in \hat{W}$ the infinite product

$$wY := \prod_{\nu \in X} (1 + a_\nu e^{-w\nu})^{r(\nu)},$$

is again an infinite product of the above form, since the set $w\hat{\Delta}_+ \setminus \hat{\Delta}_+$ is finite (see for example [Gorelik 2011, Lemma 1.2.8]). Hence $\mathcal{Y}$ is a $\hat{W}$-invariant multiplicative subset of $\mathcal{R}_{\hat{W}}$.

The elements of $\mathcal{Y}$ are invertible in $\mathcal{R}$: using the geometric series we can expand Y^{-1}. For example, $(1-e^{\alpha})^{-1} = -e^{-\alpha}(1-e^{-\alpha})^{-1} = -\sum_{i=1}^{\infty} e^{-i\alpha}$.

1.3.4. *The subalgebra $\mathcal{R}'$.* Denote by $\mathcal{R}'$ the localization of $\mathcal{R}_{\hat{W}}$ by $\mathcal{Y}$. By the above, $\mathcal{R}'$ is a subalgebra of $\mathcal{R}$. Observe that $\mathcal{R}' \not\subset \mathcal{R}_{\hat{W}}$: for example, $(1-e^{-\alpha})^{-1} \in \mathcal{R}'$, but $(1-e^{-\alpha})^{-1} = \sum_{j=0}^{\infty} e^{-j\alpha} \notin \mathcal{R}_{\hat{W}}$. We extend the action of $\hat{W}$ from $\mathcal{R}_{\hat{W}}$ to $\mathcal{R}'$ by setting $w(Y^{-1}Y') := (wY)^{-1}(wY')$ for $Y \in \mathcal{Y},\ Y' \in \mathcal{R}_{\hat{W}}$.

Notice that an infinite product of the form $Y = \prod_{\nu\in X}(1+a_\nu e^{-\nu})^{r(\nu)}$, where a_ν, X are as above and $r(\nu) \in \mathbb{Z}$, lies in $\mathcal{R}'$ and $wY = \prod_{\nu\in X}(1+a_\nu e^{-w\nu})^{r(\nu)}$. The support $\mathrm{supp}(Y)$ has a unique maximal element (with respect to the standard partial order) and this element is given by the formula

$$\max \mathrm{supp}(Y) = -\sum_{\nu\in X\setminus\hat{\Delta}_+ : a_\nu\neq 0} r_\nu \nu.$$

1.3.5. Let $\tilde{W}$ be a subgroup of $\hat{W}$. For $Y \in \mathcal{R}'$ we say that Y *is $\tilde{W}$-invariant (resp., $\tilde{W}$-anti-invariant)* if $wY = Y$ (resp., $wY = \mathrm{sgn}(w)Y$) for each $w \in \tilde{W}$.

Let $Y = \sum a_\mu e^\mu \in \mathcal{R}_{\tilde{W}}$ be $\tilde{W}$-anti-invariant. Then $a_{w\mu} = (-1)^{\mathrm{sgn}(w)} a_\mu$ for each μ and $w \in \tilde{W}$. In particular, $\tilde{W}\,\mathrm{supp}(Y) = \mathrm{supp}(Y)$, and, moreover, for each $\mu \in \mathrm{supp}(Y)$ one has $\mathrm{Stab}_{\tilde{W}}\,\mu \subset \{w \in \tilde{W} : \mathrm{sgn}(w) = 1\}$. The condition $Y \in \mathcal{R}_{\tilde{W}}$ is essential: for example, for $\tilde{W} = \{\mathrm{id}, s_\alpha\}$, the expressions $Y := e^\alpha - e^{-\alpha}$, $Y^{-1} = e^{-\alpha}(1-e^{-2\alpha})^{-1}$ are $\tilde{W}$-anti-invariant, $\mathrm{supp}(Y) = \{\pm\alpha\}$ is s_α-invariant, but $\mathrm{supp}(Y^{-1}) = \{-\alpha, -3\alpha, \ldots\}$ is not s_α-invariant.

For $Y \in \mathcal{R}_{\tilde{W}}$ such that each $\tilde{W}$-orbit in $\hat{\mathfrak{h}}^*$ has a finite intersection with $\mathrm{supp}(Y)$, introduce the sum

$$\mathcal{F}_{\tilde{W}}(Y) := \sum_{w\in\tilde{W}} \mathrm{sgn}(w) wY.$$

This sum is well defined, but does not always belong to $\mathcal{R}$. For $Y = \sum a_\mu e^\mu$ one has $\mathcal{F}_{\tilde{W}}(Y) = \sum b_\mu e^\mu$, where $b_\mu = \sum_{w\in\tilde{W}} \mathrm{sgn}(w) a_{w\mu}$; in particular, $b_\mu = \mathrm{sgn}(w) b_{w\mu}$ for each $w \in \tilde{W}$. One has

$$Y \in \mathcal{R}_{\tilde{W}} \text{ and } \mathcal{F}_{\tilde{W}}(Y) \in \mathcal{R} \implies \begin{cases} \mathcal{F}_{\tilde{W}}(Y) \in \mathcal{R}_{\tilde{W}}, \\ \mathrm{supp}(\mathcal{F}_{\tilde{W}}(Y)) \text{ is } \tilde{W}\text{-stable}, \\ \mathcal{F}_{\tilde{W}}(Y) \text{ is } \tilde{W}\text{-anti-invariant.} \end{cases}$$

We call a vector $\lambda \in \hat{\mathfrak{h}}^*$ *$\tilde{W}$-regular* if $\mathrm{Stab}_{\tilde{W}}\,\lambda = \{\mathrm{id}\}$, and we say that the orbit $\tilde{W}\lambda$ is $\tilde{W}$-regular if λ is $\tilde{W}$-regular (so the orbit consists of $\tilde{W}$-regular points). If $\tilde{W}$ is an affine Weyl group, then for any $\lambda \in \hat{\mathfrak{h}}^*$ the stabilizer $\mathrm{Stab}_{\tilde{W}}\,\lambda$ is either trivial or contains a reflection. Thus for $\tilde{W} = \hat{W}', \ \hat{W}''$ one has

$$Y \in \mathcal{R}_{\tilde{W}} \text{ and } \mathcal{F}_{\tilde{W}}(Y) \in \mathcal{R} \implies \operatorname{supp}(\mathcal{F}_{\tilde{W}}(Y)) \text{ is a union of } \tilde{W}\text{-regular orbits.}$$

2. Proof

Unless stated otherwise, $\mathfrak{g}$ is assumed to be one of the algebras $\mathfrak{gl}(n|n)$, $D(n+1|n)$.

As it is pointed out in Section 1, it is enough to establish the denominator identity for a particular choice of Δ_+ and we do this for the choice described in Section 1.1. Recall that the group T' was introduced in Section 1.2.1. The steps of the proof are the following.

- In Section 2.1 we check that the sum $\mathcal{F}_{T'}(Re^{\hat{\rho}})$ is well-defined and belongs to $\mathcal{R}$.
- In Section 2.2 we prove the inclusions

$$\operatorname{supp}(\mathcal{F}_{T'}(Re^{\hat{\rho}})), \operatorname{supp}(\hat{R}e^{\hat{\rho}}) \subset U, \tag{5}$$

where

$$U := \{\mu \in \hat{\rho} - \hat{Q}^+ : (\mu, \mu) = (\hat{\rho}, \hat{\rho})\}. \tag{6}$$

We remark that (5) holds for simple contragredient Lie superalgebras with nonzero Killing form; see [Gorelik 2011, 2.4].

- In Section 2.3 we show that if the dual Coxeter number of $\mathfrak{g}$ is zero, then the inclusions (5) imply that $\operatorname{supp}\big(\hat{R}^{-1}e^{-\hat{\rho}}\mathcal{F}_{T'}(Re^{\hat{\rho}})\big) \subset \hat{Q}^{\hat{W}}$. As a result, $\hat{R}^{-1}e^{-\hat{\rho}}\mathcal{F}_{T'}(Re^{\hat{\rho}})$ takes the form $f(q)$ for $\mathfrak{g} \neq \mathfrak{gl}(n|n)$ and $f(q, e^{\mathfrak{str}})$ for $\mathfrak{gl}(n|n)$.
- In Section 2.4 we compute $f(q)$ for $D(n+1|n)$ and $f(q, e^{\mathfrak{str}})$ for $\mathfrak{gl}(n|n)$. This completes the proof of the identities (2).

2.1. In this subsection we show that for $\mathfrak{g} = \mathfrak{gl}(n|n)$, $D(n+1|n)$, the sum $\mathcal{F}_{T'}(Re^{\hat{\rho}})$ is a well-defined element of $\mathcal{R}$. Since $\hat{\rho} = \rho$ is $\hat{W}$-invariant, it is enough to verify that $\mathcal{F}_{T'}(R)$ is a well-defined element of $\mathcal{R}$.

Recall that $T' = \mathbb{Z}\{t_{\delta_i - \delta_{i+1}}\}_{i=1}^{n-1}$ for $\mathfrak{gl}(n|n)$ and $T' = \mathbb{Z}\{t_{\delta_i}\}_{i=1}^{n}$ for $D(n+1|n)$, where

$$t_\mu(\alpha) = \alpha - (\alpha, \mu)\delta \ \text{ for any } \alpha \in \hat{Q}. \tag{7}$$

<u>2.1.1.</u> By Section 1.3.4 one has

$$\max \operatorname{supp}(w(R)) = -\sum_{\substack{\alpha \in \Delta_{0+}: \\ w\alpha < 0}} w\alpha + \sum_{\substack{\beta \in \Delta_{1+}: \\ w\beta < 0}} w\beta.$$

For $w \in T'$ write $w = t_\mu$, where $\mu \in \mathbb{Z}\{\delta_i - \delta_{i+1}\}_{1 \le i < n}$ for $\mathfrak{gl}(n|n)$ and $\mu \in \mathbb{Z}\{\delta_i\}_{i=1}^n$ for $D(n+1|n)$. From (7) we get

$$\{\beta \in \Delta_{i+} | w\beta < 0\} = \{\beta \in \Delta_{i+} | (\beta, \mu) > 0\} \ \text{ for } i = 0, 1.$$

We obtain $\max\operatorname{supp}(t_\mu(R)) = -v(\mu) + (v(\mu),\mu)\delta$, where

$$v(\mu) := \sum_{\substack{\beta\in\Delta_{0+}:\\ (\beta,\mu)>0}} \beta - \sum_{\substack{\beta\in\Delta_{1+}:\\ (\beta,\mu)>0}} \beta.$$

In order to prove that $\mathcal{F}_{T'}(R)$ is a well-defined element of $\mathcal{R}$ we verify that

$$\begin{array}{lll} \text{(i)} & (v(\mu),\mu)\le 0 & \text{for all } \mu; \\ \text{(ii)} & \{\mu : (v(\mu),\mu)\ge -N\} \text{ is finite} & \text{for all } N>0. \end{array} \tag{8}$$

Condition (ii) ensures that the sum $\mathcal{F}_{T'}(R) = \sum_\mu t_\mu(R)$ is well-defined and condition (i) means that for each μ one has

$$\max\operatorname{supp}(t_\mu(R)) = -v(\mu) \le \sum_{\beta\in\Delta_{1+}} \beta$$

so $\operatorname{supp}\big(\mathcal{F}_{T'}(R)\big) \subset \sum_{\beta\in\Delta_{1+}} \beta - \hat{Q}^+$ and thus $\mathcal{F}_{T'}(R)\in\mathcal{R}$.

2.1.2. *Case* $\mathfrak{gl}(n|n)$. Recall that $w\in T'$ has the form $w=t_\mu$, $\mu=\sum_{i=1}^n k_i\delta_i$, where the k_is are integers and $\sum_{i=1}^n k_i = 0$. One has

$$\{\alpha\in\Delta_{+0} : (\alpha,\mu)>0\} = \{\delta_i-\delta_j : i<j,\ k_i>k_j\},$$
$$\{\alpha\in\Delta_{+1} : (\alpha,\mu)>0\} = \{\varepsilon_i-\delta_j : k_j<0,\ i\le j\}\cup\{\delta_i-\varepsilon_j : k_i>0,\ i<j\},$$

where $1\le i,\ j\le n$.

Write $v(\mu)=v'+v''$, where $v'=\sum_{i=1}^n a_i\delta_i$ and v'' lies in the span of the ε_i. By the above, for $k_i>0$ one has $a_i\le(n-i)-(n-i)=0$ and for $k_j<0$ one has $a_j\ge -(j-1)+j=1$. Therefore

$$(v(\mu),\mu) = \sum_{i=1}^n a_ik_i \le \sum_{k_i<0} k_i \le 0$$

and the set $\{\mu : (v(\mu),\mu)\ge -N\}$ is a subset of the set $\{\mu : \sum_{k_i<0} k_i\ge -N\}$, which is finite for any N, because the k_i are integers and $\sum_{i=1}^n k_i=0$. This establishes conditions (8).

2.1.3. *Case* $D(n{+}1|n)$. Recall that $w\in T'$ has the form $w=t_\mu$, $\mu=\sum k_i\delta_i$, where the k_is are integers. One has

$$\{\alpha\in\Delta_{+0} : (\alpha,\mu)>0\} = \\ \{\delta_i-\delta_j : i<j, k_i>k_j\}\cup\{\delta_i+\delta_j : i\ne j,\ k_i+k_j>0\}\cup\{2\delta_i : k_i>0\},$$
$$\{\alpha\in\Delta_{+1} : (\alpha,\mu)>0\} = \\ \{\varepsilon_s-\delta_j : k_j<0, s\le j\}\cup\{\delta_i-\varepsilon_s : k_i>0, i<s\}\cup\{\delta_i+\varepsilon_s : k_i>0\},$$

where $1\le i,\ j\le n$ and $1\le s\le n+1$.

Write $v(\mu)=v'+v''$, where $v'=\sum_{i=1}^{n}a_i\delta_i$ and v'' lies in the span of the ε_i. By the above, for $k_i>0$ one has $a_i\le(2n+1-i)-(2n+2-i)=-1$ and for $k_j<0$ one has $a_j\ge-(j-1)+j=1$. Therefore

$$(v(\mu),\mu)=\sum_{i=1}^{n}a_ik_i\le-\sum_{k_i>0}k_i+\sum_{k_j<0}k_j=-\sum_{1=1}^{n}|k_i|\le 0,$$

so the set $\{\mu:(v(\mu),\mu)\ge-N\}$ is a subset of $\{\mu:\sum_{i=1}^{n}|k_i|\le N\}$, which is finite for any N. This establishes the conditions (8).

2.1.4. **Remark.** For $\mathfrak{gl}(n|n)$ one can interchange Δ' and Δ'' so the sum $\mathcal{F}_{T''}(R)$ is well-defined. One readily sees that $\mathcal{F}_{T''}(R)$ is not well-defined for $D(n{+}1|n)$. For instance, for $n>1$, for each $k>0$ one has $v(-2k\varepsilon_1)=0$ so $\max\operatorname{supp}\bigl(t_{-2k\varepsilon_1}(R)\bigr)=0$ and the sum $\sum_{k=1}^{\infty}t_{-2k\varepsilon_1}(R)$ is not well-defined; hence $\mathcal{F}_{T''}(R)$ is not well-defined as well.

2.2. By Section 1.3.3, $\hat{R}$ is an invertible element of $\mathcal{R}'$. From representation theory we know that since $\hat{\mathfrak{g}}$ admits a Casimir element [Kac 1990, Chapter II], the character of the trivial $\hat{\mathfrak{g}}$-module is a linear combination of the characters of Verma $\hat{\mathfrak{g}}$-modules $M(\lambda)$, where $\lambda\in-\hat{Q}$ are such that $(\lambda+\hat\rho,\lambda+\hat\rho)=(\hat\rho,\hat\rho)$. Since the character of $M(\lambda)$ is equal to $\hat{R}^{-1}e^{\lambda}$, we obtain

$$1=\sum_{\substack{\lambda\in\hat{Q}^-\\(\lambda+\hat\rho,\lambda+\hat\rho)=(\hat\rho,\hat\rho)}}a_\lambda\hat{R}^{-1}e^{\lambda},$$

where $a_\lambda\in\mathbb{Z}$. This can be rewritten as

$$\hat{R}e^{\hat\rho}=\sum_{\substack{\lambda\in\hat\rho-\hat{Q}^+\\(\lambda,\lambda)=(\hat\rho,\hat\rho)}}a_\lambda e^{\lambda},$$

that is $\operatorname{supp}(\hat{R})\subset U$, see (6) for notation.

It remains to verify the inclusion $\operatorname{supp}\bigl(\mathcal{F}_{T'}(Re^{\hat\rho})\bigr)\subset U$. The denominator identity for $\mathfrak{g}$ (see [Kac and Wakimoto 1994; Gorelik 2012]) takes the form

$$Re^{\rho}=\mathcal{F}_{W''}\Bigl(\frac{e^{\rho}}{\prod_{\beta\in S}(1+e^{-\beta})}\Bigr),$$

where $S:=\{\varepsilon_i-\delta_i\}_{i=1}^{n}$ (the identity for $\mathfrak{gl}(n|n)$ immediately follows from the identity for $\mathfrak{sl}(n|n)$). Since $\rho=\hat\rho$ is $\hat{W}$-invariant, this implies

$$t_\mu(Re^{\hat\rho})=e^{\hat\rho}\sum_{w\in W''}\operatorname{sgn}(w)\prod_{\beta\in S}(1+e^{-t_\mu w\beta})^{-1}.$$

For each $t_\mu \in T'$ and $w \in W''$ one has

$$\operatorname{supp}\Big(\prod_{\beta\in S}(1+e^{-t_\mu w\beta})^{-1}\Big) \subset V, \text{ where } V := \mathbb{Z}\{t_\mu w\beta : \beta \in S\} \cap \hat{Q}^-.$$

Since $(t_\mu w\beta, t_\mu w\beta') = (\beta, \beta') = (t_\mu w\beta, \hat{\rho}) = (\hat{\rho}, \beta) = 0$ for any $\beta, \beta' \in S$, one has $(V, V) = (V, \hat{\rho}) = 0$. Therefore $V + \hat{\rho} \subset U$ so $\operatorname{supp}\big(t_\mu(Re^{\hat{\rho}})\big) \subset U$ for each μ. This establishes the required inclusion $\operatorname{supp}\big(\mathcal{F}_{T'}(Re^{\hat{\rho}})\big) \subset U$ and completes the proof of (5).

2.3. Let us deduce from (5) that the support of $\hat{R}^{-1}e^{\hat{\rho}} \cdot \mathcal{F}_{T'}(Re^{\hat{\rho}})$ consists of $\hat{W}$-invariant elements of $\hat{Q}^-$. We do this in two steps: first, proving Lemma 2.3.1, which is valid for any simple contragredient Lie superalgebra and for $\mathfrak{gl}(n|n)$, and then, proving Proposition 2.3.2, which uses the fact that $\hat{\rho} = \rho$ for $\mathfrak{g}$ (this is equivalent to the fact that the dual Coxeter number is zero).

The affine root system $\hat{\Delta}'$ is a subsystem of $\hat{\Delta}_0$. Set $\hat{\Delta}'_+ = \hat{\Delta}' \cap \hat{\Delta}_+$ and let $\hat{\Pi}'$ be the corresponding set of simple roots. Fix $\hat{\rho}' \in \hat{\mathfrak{h}}^*$ such that $2(\hat{\rho}', \alpha) = (\alpha, \alpha)$ for each $\alpha \in \hat{\Pi}'$.

2.3.1. **Lemma.** *The term $\hat{R}_1 e^{\hat{\rho}'-\hat{\rho}} \cdot \mathcal{F}_{T'}(Re^{\hat{\rho}})$ is a $\hat{W}'$-anti-invariant element of $\mathcal{R}_{\hat{W}'}$.*

Proof. By Section 2.1.1, $\mathcal{F}_{T'}(Re^{\hat{\rho}}) \in \mathcal{R}$ and thus $\hat{R}_1 e^{\hat{\rho}'-\hat{\rho}} \cdot \mathcal{F}_{T'}(Re^{\hat{\rho}}) \in \mathcal{R}$.

Let R'_0, R''_0 be the Weyl denominators for Δ', Δ'' (i.e., $R'_0 = \prod_{\alpha\in\Delta'_+}(1-e^{-\alpha})$). Notice that $R''_0 e^{\hat{\rho}}/R_1 \in \mathcal{R}'$ so $w\big(R''_0 e^{\hat{\rho}}/R_1\big)$ is well-defined. Below we will show that the sum $\mathcal{F}_{\hat{W}'}\big(R''_0 e^{\hat{\rho}}/R_1\big)$ is a well-defined element of $\mathcal{R}$ and will establish the following formula

$$\mathcal{F}_{T'}(Re^{\hat{\rho}}) = \mathcal{F}_{\hat{W}'}\bigg(\frac{R''_0 e^{\hat{\rho}}}{R_1}\bigg). \tag{9}$$

It is easy to see that $\hat{R}_0 e^{\hat{\rho}'}$, $\hat{R}e^{\hat{\rho}}$ are $\hat{W}'$-anti-invariant elements of $\mathcal{R}'$ (see, for instance, [Gorelik 2011, 1.5.1]). Since $\hat{R}_1 e^{\hat{\rho}'-\hat{\rho}} \in \mathcal{R}'$ and $\hat{R}_1 e^{\hat{\rho}'-\hat{\rho}} \cdot \hat{R}e^{\hat{\rho}} = \hat{R}_0 e^{\hat{\rho}'}$, we conclude that $\hat{R}_1 e^{\hat{\rho}'-\hat{\rho}}$ is a $\hat{W}'$-invariant element of $\mathcal{R}'$. However, by Section 1.3.3, $\hat{R}_1 \in \mathcal{R}_{\hat{W}}$, and thus $\hat{R}_1 e^{\hat{\rho}'-\hat{\rho}}$ is a $\hat{W}'$-invariant element of $\mathcal{R}_{\hat{W}}$. Multiplying both sides of formula (9) by $\hat{R}_1 e^{\hat{\rho}'-\hat{\rho}}$ we obtain

$$\hat{R}_1 e^{\hat{\rho}'-\hat{\rho}} \cdot \mathcal{F}_{T'}(Re^{\hat{\rho}}) = \mathcal{F}_{\hat{W}'}\bigg(\frac{\hat{R}_1}{R_1} \cdot R''_0 e^{\hat{\rho}'}\bigg). \tag{10}$$

By Section 1.3.3, $\hat{R}_1/R_1$ and R''_0 lie in $\mathcal{R}_{\hat{W}}$. In the light of Section 1.3.5, the formula (10) implies the assertion of the lemma.

Let us show that the right-hand side of (9) is well-defined. Since R''_0 and $\hat{\rho}$ are $\hat{W}'$-invariant, it is enough to check that $\mathcal{F}_{\hat{W}'}(R_1^{-1})$ is a well-defined element of $\mathcal{R}$.

By Section 1.3.4, for each $w \in \hat{W}'$ one has

$$\max \operatorname{supp}\bigl(w(R_1^{-1})\bigr) = \sum_{\substack{\beta\in\Delta_{1+}:\\ w\beta<0}} w\beta.$$

In particular, $\operatorname{supp}\bigl(w(R_1^{-1})\bigr) \subset \hat{Q}^-$, so, if the term $\mathcal{F}_{\hat{W}'}(R_1^{-1})$ is well-defined, it lies in $\mathcal{R}$. In order to see that $\mathcal{F}_{\hat{W}'}(R_1^{-1})$ is well-defined let us check that for each $\nu \in \hat{Q}^-$ the set

$$X(\nu) := \Bigl\{ w \in \hat{W}' : \sum_{\substack{\beta\in\Delta_{1+}:\\ w\beta<0}} w\beta \geq \nu \Bigr\}$$

is finite. One has

$$X(\nu) \subset \{w \in \hat{W}' : w\beta \geq \nu \text{ for all } \beta \in \Delta_{1+}\}.$$

Write $\nu = -k\delta + \nu'$, where $k \geq 0$, $\nu' \in Q$, and write $w \in X(\nu)$ in the form $w = t_\mu y$, where $t_\mu \in T'$, $y \in W'$. Since $w\beta = y\beta - (y\beta, \mu)\delta$ for $\beta \in \Delta_{1+}$, one has $(y\beta, \mu) \geq -k$ for each $\beta \in \Delta_{1+}$. Since $\{\varepsilon_i - \delta_i, \delta_i - \varepsilon_{i+1}\} \subset \Delta_{1+}$, this gives $|(\mu, y\delta_i)| \leq k$ for $i = 1, \dots, n$. Combining the facts that W' is a subgroup of signed permutation of $\{\delta_j\}_{j=1}^n$ and that (μ, δ_i) is integral for each i, we conclude that $X(\nu)$ is finite. Thus $\mathcal{F}_{\hat{W}'}\bigl(R_0'' e^{\hat{\rho}}/R_1\bigr)$ is a well-defined element of $\mathcal{R}$.

Now let us prove the formula (9). Recall that $\rho = \rho_0' + \rho_0'' - \rho_1$, where

$$\rho_0' := \sum_{\alpha\in\Delta_{0+}'} \alpha/2, \quad \rho_0'' := \sum_{\alpha\in\Delta_{0+}''} \alpha/2, \quad \rho_1 := \sum_{\beta\in\Delta_{1+}} \beta/2.$$

The Weyl denominator identity for Δ_0' takes the form

$$R_0' e^{\rho_0'} = \mathcal{F}_{W'}(e^{\rho_0'}).$$

Since $R_1 e^{\rho_1} = \prod_{\beta\in\Delta_{1+}} (e^{\beta/2} + e^{-\beta/2})$ is W-invariant and $R_0'' e^{\rho_0''}$ is W'-invariant, we get

$$Re^{\rho} = \frac{R_0'' e^{\rho_0''}}{R_1 e^{\rho_1}} \cdot \mathcal{F}_{W'}(e^{\rho_0'}) = \mathcal{F}_{W'}\biggl(\frac{e^{\rho_0'} R_0'' e^{\rho_0''}}{R_1 e^{\rho_1}}\biggr) = \mathcal{F}_{W'}\biggl(\frac{R_0'' e^{\rho}}{R_1}\biggr).$$

Using the W-invariance of $\hat{\rho} - \rho$, we obtain

$$\mathcal{F}_{T'}\bigl(Re^{\hat{\rho}}\bigr) = \mathcal{F}_{T'}\biggl(\mathcal{F}_{W'}\biggl(\frac{R_0'' e^{\hat{\rho}}}{R_1}\biggr)\biggr) = \mathcal{F}_{\hat{W}'}\biggl(\frac{R_0'' e^{\hat{\rho}}}{R_1}\biggr)$$

as required. This completes the proof. □

2.3.2. **Proposition.** *One has*

$$\operatorname{supp}(\hat{R}^{-1}e^{-\hat{\rho}}\cdot\mathcal{F}_{T'}(Re^{\hat{\rho}}))\subset(\hat{Q}^-)^{\hat{W}}=\hat{Q}^-\cap\hat{Q}^{\perp}.$$

Proof. Set

$$Y:=\hat{R}^{-1}e^{-\hat{\rho}}\cdot\mathcal{F}_{T'}(Re^{\hat{\rho}}).$$

By Sections 2.1.1 and 1.3.3, $\mathcal{F}_{T'}(Re^{\hat{\rho}})$, $\hat{R}^{-1}\in\mathcal{R}$. Thus $Y\in\mathcal{R}$. One has

$$\hat{R}_0e^{\hat{\rho}'}Y=\hat{R}_1e^{\hat{\rho}'-\hat{\rho}}\cdot\mathcal{F}_{T'}(Re^{\hat{\rho}}).$$

In the light of Lemma 2.3.1, we obtain

$$\hat{R}_0e^{\hat{\rho}'}Y\ \text{ is a } \hat{W}'\text{-anti-invariant element of } \mathcal{R}_{\hat{W}'}. \tag{11}$$

Write $Y=Y_1+Y_2$, where $\operatorname{supp}(Y_1)=\operatorname{supp}(Y)\cap\hat{Q}^{\perp}$ and $\operatorname{supp}(Y_2)=\operatorname{supp}(Y)\setminus\hat{Q}^{\perp}$. Note that $Y_1, Y_2\in\mathcal{R}$. Assume that $Y_2\neq 0$. Let μ be a maximal element in $\operatorname{supp}(Y_2)$. One has $\operatorname{supp}(\hat{R}^{-1})\subset\hat{Q}^-$ and $\operatorname{supp}\big(\mathcal{F}_{T'}(R)e^{\hat{\rho}}\big)\subset\hat{\rho}-\hat{Q}^+$, by Section 1.3.4 and (5) respectively. Thus $\operatorname{supp}(Y)\subset\hat{Q}^-$ and so $\mu\in\hat{Q}^-$.

Since $\operatorname{supp}(Y_1)\subset\hat{Q}^{\perp}$, Y_1 is a $\hat{W}$-invariant element of $\mathcal{R}_{\hat{W}}$. Recall that $\hat{R}_0e^{\hat{\rho}'}$ is a $\hat{W}'$-anti-invariant element of $\mathcal{R}_{\hat{W}}$. Thus $\hat{R}_0e^{\hat{\rho}'}Y_1$ is a $\hat{W}'$-anti-invariant element of $\mathcal{R}_{\hat{W}'}$. In the light of (11), the product $\hat{R}_0e^{\hat{\rho}'}Y_2$ is also a $\hat{W}'$-anti-invariant element of $\mathcal{R}_{\hat{W}'}$. Clearly, $\hat{\rho}'+\mu$ is a maximal element in the support of $\hat{R}_0e^{\hat{\rho}'}Y_2$. By Section 1.3.5, this support is a union of $\hat{W}'$-regular orbits (recall that regularity means that each element has the trivial stabilizer in $\hat{W}'$), so $\hat{\rho}'+\mu$ is a maximal element in a regular $\hat{W}'$-orbit and thus $2(\hat{\rho}'+\mu,\alpha)/(\alpha,\alpha)\notin\mathbb{Z}_{\leq 0}$ for each $\alpha\in\hat{\Pi}'$. Since $\mu\in\hat{Q}^-$ one has $2(\mu,\alpha)/(\alpha,\alpha)\in\mathbb{Z}$ for each $\alpha\in\hat{\Pi}'$. Taking into account that $2(\hat{\rho}',\alpha)/(\alpha,\alpha)=1$ for each $\alpha\in\hat{\Pi}'$, we obtain

$$\frac{2(\mu,\alpha)}{(\alpha,\alpha)}\in\mathbb{Z}_{\geq 0}\quad\text{for all }\ \alpha\in\hat{\Pi}'. \tag{12}$$

Recall that $\delta=\sum_{\alpha\in\hat{\Pi}'}k_\alpha\alpha$ for some $k_\alpha\in\mathbb{Z}_{>0}$ (see [Kac 1990, Chapter VI]). Since $\mu\in\hat{Q}^-$ one has $(\mu,\delta)=0$. Combining with (12), we get $(\mu,\alpha)=0$ for each $\alpha\in\hat{\Pi}'$ so $\mu\in(\hat{\Delta}')^{\perp}$.

Let us show that $(\mu,\mu)=0$. Since $(\hat{\rho},\hat{Q})=0$, it is equivalent to the equality $(\mu+\hat{\rho},\mu+\hat{\rho})=(\hat{\rho},\hat{\rho})$. Notice that $\mu+\hat{\rho}$ is a maximal element in the support of $\hat{R}e^{\hat{\rho}}Y_2$. Let us check that

$$\operatorname{supp}(\hat{R}e^{\hat{\rho}}Y_2)\subset U=\{\xi\in\hat{\rho}-\hat{Q}^+:(\xi,\xi)=(\hat{\rho},\hat{\rho})\}. \tag{13}$$

Indeed,

$$\hat{R}e^{\hat{\rho}}Y_2=\mathcal{F}_{T'}(Re^{\hat{\rho}})-\hat{R}e^{\hat{\rho}}Y_1$$

and, by (5),

$$\operatorname{supp}\big(\mathcal{F}_{T'}(Re^{\hat{\rho}})\big)\subset U\quad\text{and}\quad\operatorname{supp}\big(\hat{R}e^{\hat{\rho}}\big)\subset U.$$

By construction, $\mathrm{supp}(Y_1) \subset \hat{Q}^{\perp} \cap \hat{Q}^{-}$. Recall that $\hat{\rho} = \rho \in \mathbb{Q}\Delta$, so $U \subset \mathbb{Q} \cdot \hat{Q}$. In particular, we have $(U, \mathrm{supp}(Y_1)) = 0$. Since $(\mathrm{supp}(Y_1), \mathrm{supp}(Y_1)) = 0$, we obtain $(\mathrm{supp}(Y_1) + U) \subset U$ and this establishes the inclusion (13). Hence $(\mu, \mu) = 0$.

Recall that $\mu \in (\hat{\Delta}')^{\perp} \cap \hat{Q}^{-}$. One has

$$(\hat{\Delta}')^{\perp} \cap \hat{Q} = (\hat{Q}^{\perp} \cap \hat{Q}) \oplus \mathbb{Z}\Delta''.$$

For every $\beta \in \hat{Q}^{\perp} \cap \hat{Q}$, $\gamma \in \Delta''$ one has $(\beta, \beta) = (\beta, \gamma) = 0$ and $(\gamma, \gamma) \neq 0$ if $\gamma \neq 0$. Using the equality $(\mu, \mu) = 0$, we get $\mu \in \hat{Q}^{\perp} \cap \hat{Q}$, which contradicts to the construction of Y_2. Hence $Y_2 = 0$ as required. □

2.3.3. **Corollary.** *For $\mathfrak{g} = D(n+1|n)$ one has $f(q) \cdot \hat{R}e^{\hat{\rho}} = \mathcal{F}_{T'}(Re^{\hat{\rho}})$ for some $f(q) = \sum_{k=0}^{\infty} a_k q^k$ $(a_k \in \mathbb{Z})$. For $\mathfrak{g} = \mathfrak{gl}(n|n)$ one has $f(q, e^{\mathfrak{str}}) \cdot \hat{R}e^{\hat{\rho}} = \mathcal{F}_{T'}(Re^{\hat{\rho}})$ for some $f(q, e^{\mathfrak{str}}) = \sum_{k=0}^{\infty} \sum_{m=-\infty}^{\infty} a_{k,m} q^k e^{m \cdot \mathfrak{str}}$ $(a_{k,m} \in \mathbb{Z})$.*

Proof. One has $(\hat{Q})^{\perp} \cap \hat{Q} = \mathbb{Z}\delta + \mathbb{Z}\mathfrak{str}$ for $\mathfrak{gl}(n|n)$ and $(\hat{Q})^{\perp} \cap \hat{Q} = \mathbb{Z}\delta$ for $D(n+1|n)$. □

2.4. In this subsection we complete the proof of the denominator identities (2) by proving the formulas (3). We prove them by taking a suitable evaluation of the term $\hat{R}^{-1} e^{-\hat{\rho}} \mathcal{F}_{T'}(Re^{\hat{\rho}})$. Since $\hat{\rho}$ is $\hat{W}$-invariant, this term is equal to $\hat{R}^{-1}\mathcal{F}_{T'}(R)$, and, by Corollary 2.3.3, it is equal to $f(q)$ for $D(n+1|n)$ and to $f(q, e^{\mathfrak{str}})$ for $\mathfrak{gl}(n|n)$. Now we consider q as a real parameter between 0 and 1. We choose the evaluation in such a way that the evaluation of $\hat{R}^{-1}\mathcal{F}_{T'}(R) = \hat{R}^{-1} \sum_{t \in T'} t(R)$ is equal to the evaluation of $\hat{R}^{-1}R$. As a result, $f(q)$ (resp., $f(q, e^{\mathfrak{str}})$) is equal to the evaluation of $\hat{R}^{-1}R$, which can be easily computed.

2.4.1. *Case $D(n+1|n)$.* Take a complex parameter x and consider the evaluation $e^{-\varepsilon_i} := x^{a_i}$, $e^{-\delta_j} := -x^{b_j}$, where a_i $(i = 1, \ldots, n+1)$ and b_j $(j = 1, \ldots, n)$ are integers such that $a_i \pm b_j \neq 0$, $a_i \pm a_j \neq 0$, $b_i \pm b_j \neq 0$, $b_i \neq 0$ for all indexes i, j. We denote by $\hat{R}$ and $\hat{R}(x)$ the evaluation of R and $R(x)$. The functions $R(x)$ and $\hat{R}(x)$ are meromorphic. One has

$$R(x) = \frac{\prod_{1 \le i < j \le n+1} (1 - x^{a_i \pm a_j}) \cdot \prod_{1 \le i < j \le n} (1 - x^{b_i \pm b_j}) \cdot \prod_{1 \le i \le n} (1 - x^{2b_i})}{\prod_{1 \le i \le j \le n} (1 - x^{a_i \pm b_j}) \prod_{1 \le j < i \le n+1} (1 - x^{b_j \pm a_i})}.$$

One readily sees that $R(x)$ has a pole at $x = 1$ of order $|\Delta_{1+}| - |\Delta_{0+}| = n$.

One has

$$\left.\frac{\hat{R}(x)}{R(x)}\right|_{x=1} = \frac{((1-q)_q^{\infty})^{\dim \mathfrak{g}_0}}{((1-q)_q^{\infty})^{\dim \mathfrak{g}_1}} = ((1-q)_q^{\infty})^{\dim \mathfrak{g}_0 - \dim \mathfrak{g}_1} = (1-q)_q^{\infty}.$$

In particular, $\hat{R}(x)$ also has a pole of order n at $x = 1$.

The evaluation of $(t_{\sum k_i\delta_i}(R))(x)$ is

$$\frac{\prod_{1\le i<j\le n+1}(1-x^{a_i\pm a_j})\cdot\prod_{1\le i\le n}(1-q^{-2k_i}x^{2b_i})\cdot\prod_{1\le i<j\le n}(1-q^{-k_i\mp k_j}x^{b_i\pm b_j})}{\prod_{1\le i\le j\le n}(1-q^{\mp k_j}x^{a_i\pm b_j})\prod_{1\le j<i\le n+1}(1-q^{-k_j}x^{b_j\pm a_i})}$$

which is a meromorphic function. Let s be the number of zeros among $k_1,\dots,k_n$. Then at $x=1$ the order of zero of the numerator is at least is $n(n+1)+s^2$, and the order of zero of the denominator is $2(n+1)s$. Therefore at $x=1$ the function $(t_{\sum k_i\delta_i}(R))(x)$ has the pole of order at most $2(n+1)s-n(n+1)-s^2=n+1-(n+1-s)^2$; in particular, $(t_{\sum k_i\delta_i}(R))(x)$ has the pole of order at most n and it is equal to n if and only if $n=s$ that is $\sum k_i\delta_i=0$ and $(t_{\sum k_i\delta_i}(R))(x)=R(x)$.

We conclude that

$$(\hat{R}(x))^{-1}\cdot\sum_{t\in T':t\neq\mathrm{id}}(t(R))(x)$$

is holomorphic at $x=1$ and its value is zero, and that

$$(\hat{R}(x))^{-1}\cdot\sum_{t\in T'}(t(R))(x)$$

is holomorphic at $x=1$ and its value is $\left.\frac{R(x)}{\hat{R}(x)}\right|_{x=1}$. In the light of Corollary 2.3.3 we obtain

$$f(q)=\left.\frac{R(x)}{\hat{R}(x)}\right|_{x=1}=((1-q)_q^{\infty})^{-1}.$$

2.4.2. *Case* $\mathfrak{gl}(n|n)$. Fix $y>1$. Take a complex parameter x and consider the following evaluation

$$e^{-\varepsilon_1}:=y,\ e^{-\varepsilon_i}:=x^i,\ \text{for } i=2,\dots,n;\ e^{-\delta_i}:=-x^{-i}\ \text{for } i=1,\dots,n.$$

The functions $R(x)$, $\hat{R}(x)$ are meromorphic. One has

$$R(x)=\frac{\prod_{1<i\le n}(1-yx^{-i})\cdot\prod_{1<i<j\le n}(1-x^{i-j})\cdot\prod_{1\le i<j\le n}(1-x^{j-i})}{\prod_{1\le i\le n}(1-yx^{i})\cdot\prod_{1<i\le j\le n}(1-x^{i+j})\cdot\prod_{1\le j<i\le n}(1-x^{-i-j})}.$$

Therefore the function $R(x)$ has a pole of order $n-1$ at $x=1$.

One has

$$\left.\frac{\hat{R}(x)}{R(x)}\right|_{x=1}=\frac{((1-q)_q^{\infty})^{\dim\mathfrak{g}_0-2(n-1)}\cdot((1-qy)_q^{\infty})^{n-1}\cdot((1-qy^{-1})_q^{\infty})^{n-1}}{((1-q)_q^{\infty})^{\dim\mathfrak{g}_1-2n}\cdot((1-qy)_q^{\infty})^{n}\cdot((1-qy^{-1})_q^{\infty})^{n}}.$$

Thus $\hat{R}(x)$ also has a pole of order $n-1$ at $x=1$. Since $\dim\mathfrak{g}_0=\dim\mathfrak{g}_1$ and $e^{\mathfrak{str}}=(-1)^n y^{-1}$ for $x=1$ we obtain

$$\left.\frac{\hat{R}(x)}{R(x)}\right|_{x=1}=\frac{((1-q)_q^{\infty})^2}{(1-q(-1)^n e^{\mathfrak{str}})_q^{\infty}\cdot(1-q(-1)^n e^{-\mathfrak{str}})_q^{\infty}}.$$

One has

$$(t_{\sum k_i\delta_i}(R))(x,y) = \frac{\prod_{1<i\le n}(1-yx^{-i})\cdot\prod_{1<i<j\le n}(1-x^{i-j})\cdot\prod_{1\le i<j\le n}(1-q^{k_j-k_i}x^{j-i})}{\prod_{1\le i\le n}(1-q^{k_i}yx^{i})\cdot\prod_{1<i\le j\le n}(1-q^{k_j}x^{i+j})\cdot\prod_{1\le j<i\le n}(1-q^{-k_j}x^{-i-j})},$$

which is a meromorphic function.

Let s be the number of zeros among $k_1,\dots,k_n$. Then at $x=1$ the order of zero of the numerator is at least

$$\frac{(n-1)(n-2)+s(s-1)}{2},$$

and the order of zero of the denominator is $(n-1)s$. Therefore at $x=1$ the function $(t_{\sum k_i\delta_i}(R))(x,y)$ has a pole of order at most

$$(n-1)s-\frac{(n-1)(n-2)+s(s-1)}{2}=\frac{3n-s-2-(n-s)^2}{2},$$

so the order is at most $n-1$ and it is equal to $n-1$ if and only if $s=n-1,n$. Notice that $s\neq n-1$, since $\sum k_i=0$. Therefore the pole has order $n-1$ if and only if $\sum k_i\delta_i=0$.

We conclude that the function $(\hat{R}(x))^{-1}(\mathcal{F}_{T'}(R))(x)$ is holomorphic at $x=1$ and its value is $(R(x)/\hat{R}(x))\big|_{x=1}$. Using Corollary 2.3.3 we obtain

$$f(q,e^{\mathfrak{str}})=\frac{R(x)}{\hat{R}(x)}\bigg|_{x=1}=\frac{(1-q(-1)^n e^{\mathfrak{str}})_q^\infty\cdot(1-q(-1)^n e^{-\mathfrak{str}})_q^\infty}{((1-q)_q^\infty)^2}.$$

3. Other forms of denominator identity

Recall that the denominator identity for a basic Lie superalgebra can be written in the form

$$Re^{\rho}=\mathcal{F}_{W^\sharp}\bigg(\frac{e^{\rho}}{\prod_{\beta\in S}(1+e^{-\beta})}\bigg), \tag{14}$$

where $S\subset\Pi$ is the maximal isotropic system, and $W^\sharp$ is the Weyl group of the "largest" root subsystem of Δ_0 ($\Delta_0=\Delta'\amalg\Delta''$), see [Kac and Wakimoto 1994; Gorelik 2012]; in particular, $W^\sharp:=W''$ for $\mathfrak{g}=D(n+1|n)$, and $W^\sharp:=W'$ or $W^\sharp:=W''$ for $\mathfrak{g}=\mathfrak{gl}(n|n)$.

If the dual Coxeter number of $\mathfrak{g}$ is nonzero the affine denominator identity for $\mathfrak{g}$ can be written in the form

$$\hat{R}e^{\hat{\rho}}=\mathcal{F}_{\hat{W}^\sharp}\bigg(\frac{e^{\hat{\rho}}}{\prod_{\beta\in S}(1+e^{-\beta})}\bigg),$$

see [Gorelik 2012, 2.1]. In this section we will show that for $\mathfrak{gl}(n|n)$ the denominator identity can be written in a similar form:

$$\hat{R}e^{\rho} = f(q, e^{\mathrm{str}}) \cdot \mathcal{F}_{\hat{W}'}\left(\frac{e^{\rho}}{\prod_{\beta\in S}(1+e^{-\beta})}\right), \tag{15}$$

and that the denominator identities for $D(n+1|n)$ can not be written in a similar form, since the expressions

$$\mathcal{F}_{\hat{W}''}\left(\frac{e^{\rho}}{\prod_{\beta\in S}(1+e^{-\beta})}\right) \quad \text{and} \quad \mathcal{F}_{\hat{W}'}\left(\frac{e^{\rho}}{\prod_{\beta\in S}(1+e^{-\beta})}\right) \tag{16}$$

are not well-defined.

3.1. ***Case $D(n+1|n)$.*** Let us show that the expressions in (16) are not well-defined for $D(n+1|n)$. Fix Π as in Section 1.1 and recall that $\rho = 0$.

We repeat the reasoning of Section 2.1.1. One has

$$\sum_{\beta\in V_S(w)} w\beta \in \operatorname{supp}\left(\frac{1}{\prod_{\beta\in S}(1+e^{-w\beta})}\right) \subset \sum_{\beta\in V_S(w)} w\beta - \hat{Q}^+ \subset \hat{Q}^-,$$

where

$$V_S(w) = \{\beta \in S : w\beta < 0\}.$$

Therefore $1 \in \operatorname{supp}\big(1/\prod_{\beta\in S}(1+e^{-w\beta})\big)$ if and only if $wS \subset \Delta_+$.

Take $S = \{\varepsilon_i - \delta_i\}$; then $t_\mu S \subset \Delta_+$ if $(\varepsilon_i - \delta_i, \mu) < 0$ for all i which holds for all $\mu \in \sum \mathbb{Z}_{<0}\varepsilon_i$ and all $\mu \in \sum \mathbb{Z}_{>0}\delta_i$. Hence the sums in (16) contain infinitely many summands equal to 1 and thus they are not well-defined.

3.2. ***Case $\mathfrak{gl}(n|n)$.*** Fix Π as in Section 1.1; then $S = \{\varepsilon_i - \delta_i\}$.

In order to deduce the formula (15) from (14) and (2) it is enough to verify that the expression

$$\mathcal{F}_{\hat{W}'}\left(\frac{e^{\rho}}{\prod_{\beta\in S}(1+e^{-\beta})}\right) = e^{\rho}\mathcal{F}_{\hat{W}'}\left(\frac{1}{\prod_{\beta\in S}(1+e^{-\beta})}\right)$$

is well-defined (since ρ is $\hat{W}$-invariant). As in Section 2.1.1, this amounts to showing that

$$X_S(\nu) := \left\{ w \in \hat{W}' : \sum_{\beta\in V_S(w)} w\beta \geq -\nu \right\}$$

is finite for any $\nu \in \hat{Q}^+$ (where $V_S(w)$ is defined as in Section 3.1). As in Section 2.1.1, writing $\nu = k\delta + \nu_+$, where $\nu_+ \in \mathbb{Z}\Delta$, we get

$$X_S(\nu) \subset \big\{ t_\mu y : \mu \in T',\, y \in W' \text{ s.t. } (y\beta, \mu) \geq -k \text{ for all } \beta \in S \big\}.$$

Since y permutes δ_is, $t_\mu y \in X_S(\nu)$ forces $(\delta_i, \mu) \geq -k$ for all i. Taking into account that μ lies in the $\mathbb{Z}$-span of δ_i and $(\mu, \sum_{i=1}^n \delta_i) = 0$, we conclude that $X_S(\nu)$ is finite. This establishes (15).

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