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Squareful numbers in hyperplanes

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Let $n \geqslant 4$. In this article, we will determine the asymptotic behavior of the size of the set of integral points $(a_0 : \cdots : a_n)$ on the hyperplane $\sum_{i=0}^{n} X_i = 0$ in $\mathbb{P}^n$ such that a_i is squareful (an integer a is called squareful if the exponent of each prime divisor of a is at least two) and $|a_i| \leqslant B$ for each $i \in \{0, \ldots, n\}$, when B goes to infinity. For this, we will use the classical Hardy–Littlewood method. The result obtained supports a possible generalization of the Batyrev–Manin program to Fano orbifolds.

1. Introduction

The problem we consider can be related to a question Campana posed concerning rational points on orbifolds. A good overview is given for example in [Abramovich 2009; Poonen 2006; Campana 2005]. Examining the orbifold $(\mathbb{P}^1, \Delta)$ with $\mathbb{Q}$-divisor $\Delta = 1/2 \cdot [0] + 1/2 \cdot [1] + 1/2 \cdot [\infty]$, it is explained for example in [Poonen 2006] why it is reasonable to expect that the set

$$\{(a_1, a_2, a_3) \in \mathbb{Z}^3 : a_1 + a_2 = a_3,\ a_1, a_2, a_3 \text{ are squareful}, \\ \max\{|a_1|, |a_2|, |a_3|\} \leqslant B, \gcd(a_1, a_2, a_3) = 1\}$$

will asymptotically behave as $C \cdot B^{1/2}$ as B tends to infinity.

Since this question turns out to be too difficult at the moment, we generalize to a higher-dimensional analogue $(\mathbb{P}^{n-1}, \Delta)$, where now Δ is the $\mathbb{Q}$-divisor $\Delta = 1/2 \cdot [H_0] + \cdots + 1/2 \cdot [H_n]$ with H_i the hyperplane defined by $X_i = 0$ for $i \in \{0, \ldots, n-1\}$ and H_n defined by $X_0 + \cdots + X_{n-1} = 0$. In analogy with the one-dimensional case, a point $P = (a_0 : \cdots : a_{n-1}) \in \mathbb{P}^{n-1}(\mathbb{Q})$ (we assume $a_0, \ldots, a_{n-1} \in \mathbb{Z}$ and $\gcd(a_0, \ldots, a_{n-1}) = 1$) will be called a rational point in Campana's sense on $(\mathbb{P}^{n-1}, \Delta)$ if for every $i \in \{0, \ldots, n\}$ and every prime p for which the reduction of P is contained in the reduction of H_i modulo p, we have $i_p(P, H_i) \geqslant 2$, where $i_p(P, H_i)$ denotes the intersection number of P and H_i above the prime p. These conditions will be satisfied if a_i is squareful for every

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$i \in \{0, \dots, n-1\}$ and if $\sum_{i=0}^{n-1} a_i$ is also squareful. We denote the set of all such rational points by $(\mathbb{P}^{n-1}, \Delta)(\mathbb{Q})$. Using the height function

$$H(x_0 : \cdots : x_{n-1}) = \max\left\{|x_0|, \dots, |x_{n-1}|, \left|\sum_{i=0}^{n-1} x_i\right|\right\},$$

the set of points $P \in (\mathbb{P}^{n-1}, \Delta)(\mathbb{Q})$ of bounded height is denoted $(\mathbb{P}^{n-1}, \Delta)(\mathbb{Q})_{\leqslant B}$.

Defining the canonical divisor of the orbifold $(\mathbb{P}^{n-1}, \Delta)$ as

$$K_{(\mathbb{P}^{n-1}, \Delta)} = K_{\mathbb{P}^{n-1}} + \Delta,$$

we have $K_{(\mathbb{P}^{n-1}, \Delta)} \sim (-(n-1)/2) \cdot H$ in $\mathrm{Pic}(\mathbb{P}^{n-1})_{\mathbb{Q}}$, where H is the hyperplane class of $\mathbb{P}^{n-1}$. Since the height function we use is associated to H, a very naïve generalization of Manin's conjecture would predict that $\#(\mathbb{P}^{n-1}, \Delta)(\mathbb{Q})_{\leqslant B} \sim C \cdot B^{(n-1)/2}$ for some constant $C > 0$, as B tends to infinity. Our main goal is to prove the following theorem.

Theorem 1.1. *For $n \geqslant 4$, there exists a $\delta > 0$ so that*

$$\#(\mathbb{P}^{n-1}, \Delta)(\mathbb{Q})_{\leqslant B} = C \cdot B^{(n-1)/2} + O(B^{(n-1)/2-\delta})$$

for some constant $C > 0$.

In Section 5 we will give an explicit description of the constant C and examine the distribution of rational points on the orbifold $(\mathbb{P}^{n-1}, \Delta)$.

2. Description of the proof

Throughout the article, we will use the following notation.

We will denote the $(n+1)$-tuple $(x_0, \dots, x_n) \in A^{n+1}$ for any ring A by $\underline{x}$. For the nonzero integers we use the notation $\mathbb{Z}_0$, that is $\mathbb{Z}_0 = \mathbb{Z} \setminus \{0\}$. If there exists a constant $C > 0$ such that $|f(x)| \leqslant Cg(x)$ for real-valued functions f and g with g only taking positive values, we write $f(x) \ll g(x)$ or $f(x) = O(g(x))$. If C depends on other parameters, this will be denoted explicitly when this dependence is important for the computations. We will write $f(x) \sim g(x)$ if $f(x)/g(x)$ tends to one if x goes to infinity. Also, we allow the small positive constant ε to take different values at different points of the arguments. Finally, for any $\alpha \in \mathbb{R}$ we will write $e(\alpha) = \exp(2\pi i \alpha)$.

To prove Theorem 1.1, we first restrict ourselves to the set of points

$$(a_0 : \cdots : a_{n-1}) \in (\mathbb{P}^{n-1}, \Delta)(\mathbb{Q})$$

for which $a_i \neq 0$ for each $i \in \{0, \dots, n-1\}$ and $\sum_{i=0}^{n-1} a_i \neq 0$. We denote this subset by $(\mathbb{P}^{n-1}, \Delta)(\mathbb{Q})^+$. Also, $(\mathbb{P}^{n-1}, \Delta)(\mathbb{Q})^+_{\leqslant B}$ indicates the intersection of $(\mathbb{P}^{n-1}, \Delta)(\mathbb{Q})^+$ with $(\mathbb{P}^{n-1}, \Delta)(\mathbb{Q})_{\leqslant B}$.

By the definition of $(\mathbb{P}^{n-1}, \Delta)(\mathbb{Q})$, we can identify $(\mathbb{P}^{n-1}, \Delta)(\mathbb{Q})^+_{\leqslant B}$ with the set

$$\left\{(a_0:\ldots:a_n)\in H(\mathbb{Q}) : a_i\in\mathbb{Z}_0,\, a_i \text{ is squareful},\, \gcd(a_0,\ldots,a_n)=1,\, \max_{0\leqslant i\leqslant n}|a_i|\leqslant B\right\},$$

where $H\subset\mathbb{P}^n$ is the hyperplane defined by $X_0+\cdots+X_n=0$.

Since a squareful integer can be written uniquely (up to the sign of x) as x^2y^3, where y is squarefree, the latter set in turn corresponds to

$$\Big\{(x_0^2y_0^3:\cdots:x_n^2y_n^3)\in H(\mathbb{Q}) : x_i,\, y_i\in\mathbb{Z}_0 \text{ and } y_i \text{ is squarefree}, \\ \gcd(x_0y_0,\ldots,x_ny_n)=1,\, \max_{0\leqslant i\leqslant n}|x_i^2y_i^3|\leqslant B\Big\}. \quad (1)$$

Definition. We define $M(B)$ as the set

$$\Big\{(\underline{x},\underline{y})\in\mathbb{Z}_0^{2n+2} : \sum_{i=0}^n x_i^2y_i^3=0,\ \gcd(x_0y_0,\ldots,x_ny_n)=1, \\ \max_{0\leqslant i\leqslant n}|x_i^2y_i^3|\leqslant B,\ \prod_{i=0}^n\mu^2(|y_i|)=1\Big\}.$$

(Note that for any integer $y\in\mathbb{Z}$, the condition $\mu^2(|y_i|)=1$ means that y_i is squarefree.) Also, we denote by $M_{\underline{a},t}(B)$ the set

$$\left\{(\underline{x},\underline{y})\in\mathbb{Z}_0^{2n+2} : \sum_{i=0}^n a_ix_i^2y_i^3=t,\ \max_{0\leqslant i\leqslant n}|a_ix_i^2y_i^3|\leqslant B,\ \prod_{i=0}^n\mu_i'(y_i)=1\right\},$$

where $a_0,\ldots,a_n,t\in\mathbb{Z}$ are fixed, $\gcd(a_0,\ldots,a_n)=1$ and $\prod_{i=0}^n a_i\neq 0$. Here, μ_i' denotes an arbitrary function $\mathbb{Z}_0\to\{0,1\}$, for each $i\in\{0,\ldots,n\}$.

As a first step in the proof, we will use the classical Hardy–Littlewood circle method to determine an expression for the cardinality of the set $M_{\underline{a},t}(B)$. Notice that in the definition of $M_{\underline{a},t}(B)$, we replaced the function $\mu^2(\cdot)$ in the definition of $M(B)$ with the more general function $\mu_i'(\cdot)$. We shall see that applying the circle method is independent of this condition, but nevertheless necessary to derive an asymptotic formula for $\#M(B)$ since squarefree conditions on multiples of the y_i will appear as we will explain below. We see that $M(B)$ is a subset of $M_{(1,\ldots,1),0}(B)$ (if we take $\mu_i'(\cdot)$ to be $\mu^2(\cdot)$ for each i), with the additional gcd condition $\gcd(x_0y_0,\ldots,x_ny_n)=1$ on the solutions. We will take this gcd condition into account using an adapted version of the Möbius inversion.

Identifying $(\mathbb{P}^{n-1},\Delta)(\mathbb{Q})^+_{\leqslant B}$ with (1), it readily follows that

$$\#(\mathbb{P}^{n-1},\Delta)(\mathbb{Q})^+_{\leqslant B}=\frac{1}{2^{n+2}}\#M(B),$$

which implies that an asymptotic formula for $\#M(B)$ induces an asymptotic formula for $\#(\mathbb{P}^{n-1},\Delta)(\mathbb{Q})^+_{\leqslant B}$.

Finally, we will explain why this result suffices to prove Theorem 1.1.

3. Calculating $\#M_{\underline{a},t}(B)$

Let us first fix the framework of the circle method.

Let T be $\mathbb{R}/\mathbb{Z}$. For $0 < \Delta \leqslant 1$ and $P \geqslant 1$ (we always suppose $B \geqslant 1$), we define $\mathfrak{M}(\Delta, q, a)$ as the image in T of $\{\alpha \in \mathbb{R} : |\alpha - a/q| < P^{\Delta-2}\}$ with $a, q \in \mathbb{Z}$ and

$$\mathfrak{M}(\Delta) = \bigcup_{\substack{1 \leqslant a \leqslant q \leqslant P^{\Delta} \\ \gcd(a,q)=1}} \mathfrak{M}(\Delta, q, a).$$

We call $\mathfrak{M}(\Delta)$ the union of the *major arcs* and $T \setminus \mathfrak{M}(\Delta) = \mathfrak{m}(\Delta)$ the union of the *minor arcs*. We shall clarify the constraint on the constant Δ and the dependence of P on B in Proposition 3.7 and Theorem 3.8.

The circle method calculates $\#M_{\underline{a},t}(B)$ by integrating an exponential sum over T, namely

$$\#M_{\underline{a},t}(B) = \int_T \sum_{\substack{1 \leqslant |a_i x_i^2 y_i^3| \leqslant B \\ i=0,\dots,n}} \biggl(\prod_{i=0}^{n} \mu_i'(y_i)\biggr) e(\alpha f(\underline{x}, \underline{y}))\, d\alpha, \tag{2}$$

where $f(\underline{x}, \underline{y}) = \sum_{i=0}^{n} a_i x_i^2 y_i^3 - t$. We will denote the integrand of (2) by $E(\alpha)$ and will set

$$S_i(\alpha) = \sum_{1 \leqslant |a_i x^2 y^3| \leqslant B} \mu_i'(y) e(\alpha a_i x^2 y^3).$$

Therefore,

$$E(\alpha) = e(-\alpha t) \prod_{i=0}^{n} S_i(\alpha).$$

As usual, the integral over $\mathfrak{M}(\Delta)$ will provide the main term while the integral over $\mathfrak{m}(\Delta)$ will only contribute to the error term.

Major arcs. We refer to [Schmidt 1984, Section 5; Davenport 2005, Chapter 4] for avoid conflict with theorems. (Many authors improperly cite a detailed description of the circle method over the major arcs for the classical case of diagonal equations. In order to apply this to $\int_{\mathfrak{M}(\Delta)} E(\alpha)\, d\alpha$, we will first fix $\underline{y}$ and thus consider the diagonal equation $f(\underline{x}, \underline{y}) = f_{\underline{y}}(\underline{x}) = 0$; afterwards we will take the sum of the obtained expression over all admitted $\underline{y}$.

Since we fix $\underline{y}$, we only look at x_i satisfying $1/|a_i y_i^3|^{1/2} \leqslant |x_i| \leqslant (B/|a_i y_i^3|)^{1/2}$. Most of the time, it suffices to consider only positive x_i; we will denote the corresponding interval for positive x_i with D_i, that is,

$$D_i = \bigl[1/|a_i y_i^3|^{1/2}, B^{1/2}/|a_i y_i^3|^{1/2}\bigr]. \tag{3}$$

We will also use the notation

$$B_{a_i,y_i} = B^{1/2}/|a_i y_i^3|^{1/2}. \tag{4}$$

Note that since we consider only $\underline{y}$ with $1 \leqslant |y_i^3| \leqslant B$, we have $1 \leqslant B_{a_i,y_i} \leqslant B^{1/2}$ for each $i \in \{0, \dots, n\}$.

Because we first wish to examine the exponential sum $E(\alpha)$ (for $\alpha \in \mathfrak{M}(\Delta)$) for some $\underline{y}$ fixed, we denote this part of $E(\alpha)$ by

$$E_{\underline{y}}(\alpha) = \sum_{\substack{1/|a_i y_i^3|^{1/2} \leqslant |x_i| \leqslant B_{a_i,y_i} \\ i=0,\dots,n}} e(\alpha f_{\underline{y}}(\underline{x})).$$

Furthermore, for every positive integer q and every integer a relatively prime to q, we define

$$\sigma_{\underline{y}}\Big(\frac{a}{q}\Big) = q^{-(n+1)} \sum_{\underline{z} \in (\mathbb{Z}/q\mathbb{Z})^{n+1}} e\Big(\frac{a f_{\underline{y}}(\underline{z})}{q}\Big), \tag{5}$$

and for every $\beta \in \mathbb{R}$,

$$\tau_{\underline{y},B}(\beta) = \int_{D_0} \cdots \int_{D_n} e(\beta f_{\underline{y}}(\underline{x}))\, d\underline{x}. \tag{6}$$

Proposition 3.1. *For $\alpha = a/q + \beta \in \mathfrak{M}(\Delta; q, a)$, we have*

$$E_{\underline{y}}(\alpha) = 2^{n+1} \sigma_{\underline{y}}\Big(\frac{a}{q}\Big) \tau_{\underline{y},B}(\beta) + O\bigg(q \frac{\sum_{i=0}^n |a_i y_i^3|^{1/2}}{\prod_{i=0}^n |a_i y_i^3|^{1/2}} B^{(n+2)/2} P^{\Delta-2}\bigg)$$

under the condition $B P^{\Delta-2} \geqslant 1$ on P and Δ.

Proof. Combining positive and negative signs of x_i, we have

$$E_{\underline{y}}(\alpha) = 2^{n+1} e(-\alpha t) \prod_{i=0}^n \sum_{x_i \in D_i} e(\alpha a_i x_i^2 y_i^3). \tag{7}$$

For $\alpha = a/q + \beta$, the inner sum over x_i equals

$$\sum_{1 \leqslant z_i \leqslant q} e\bigg(\frac{a a_i z_i^2 y_i^3}{q}\bigg) \sum_{\substack{v_i \in \mathbb{Z} \\ q v_i + z_i \in D_i}} e(\beta a_i (q v_i + z_i)^2 y_i^3). \tag{8}$$

Euler's summation formula (in its simplest version) implies

$$\sum_{X \leqslant qv+z \leqslant Y} e(\zeta (qv+z)^2) = \frac{1}{q} \int_X^Y e(\zeta \eta^2)\, d\eta + O\Big(1 + \frac{Y}{q} |\zeta| q Y\Big)$$

for any real numbers $0 \leqslant X < Y$, $\zeta \in \mathbb{R}$, $q, z \in \mathbb{N}$. Taking $Y = B_{a_i,y_i}$, $\zeta = \beta a_i y_i^3$

and recalling the definition of D_i in (3), we can rewrite (8) as

$$\sum_{1\leqslant z_i\leqslant q} e\Big(\frac{aa_iz_i^2y_i^3}{q}\Big)\Big(\frac{1}{q}\int_{D_i} e(\beta a_ix_i^2y_i^3)\,dx_i + O(1+|\beta|B)\Big).$$

We substitute these expressions successively back into (7) and obtain the desired main term. Using the trivial upper bounds

$$\Big|\sum_{x_i\in D_i} e(\alpha a_ix_i^2y_i^3)\Big| + \Big|\frac{1}{q}\sum_{1\leqslant z_i\leqslant q} e\Big(\frac{aa_iz_i^2y_i^3}{q}\Big)\int_{D_i} e(\beta a_ix_i^2y_i^3)\,dx_i\Big| \ll B_{a_i,y_i},$$

we get the total error term $O\big(q(1+|\beta|B)\max_{0\leqslant i\leqslant n}\prod_{j\neq i} B_{a_j,y_j}\big)$. Using (4) and $1+|\beta|B \ll P^{\Delta-2}B$, we complete the proof. □

From this result, we can now derive an expression for the integral of $E_{\underline{y}}(\alpha)$ over $\mathfrak{M}(\Delta)$ by first integrating the expression for $E_{\underline{y}}(\alpha)$ obtained in Proposition 3.1 over $\mathfrak{M}(\Delta; q, a)$ and then summing over all admitted a and q.

We first define

$$\mathfrak{I}_{\underline{\varepsilon},t,B}(L) = \int_{|\gamma|<L} e(-\gamma t/B)\,d\gamma \int_{[B^{-1/2},1]^{n+1}} e\Big(\gamma\sum_{i=0}^{n}\varepsilon_i x_i'^2\Big)\,d\underline{x}',$$

(where $\varepsilon_i = \operatorname{sgn}(a_iy_i)$) and

$$\mathfrak{S}_{\underline{y},\underline{a},t}(L) = \sum_{q\leqslant L}\sum_{\substack{0<\frac{a}{q}\leqslant 1\\ \gcd(a,q)=1}} \sigma_{\underline{y}}\Big(\frac{a}{q}\Big).$$

We have

$$\int_{|\beta|<P^{\Delta-2}} \tau_{\underline{y},B}(\beta)\,d\beta = \frac{B^{(n-1)/2}}{\prod_{i=0}^{n}|a_iy_i^3|^{1/2}}\,\mathfrak{I}_{\underline{\varepsilon},t,B}(BP^{\Delta-2}),$$

and therefore

$$\int_{\mathfrak{M}(\Delta)} E_{\underline{y}}(\alpha)\,d\alpha = \frac{2^{n+1}\mathfrak{S}_{\underline{y},\underline{a},t}(P^{\Delta})\mathfrak{I}_{\underline{\varepsilon},t,B}(BP^{\Delta-2})}{\prod_{i=0}^{n}|a_iy_i^3|^{1/2}}\cdot B^{(n-1)/2} + O\bigg(\frac{\sum_{i=0}^{n}|a_iy_i^3|^{1/2}}{\prod_{i=0}^{n}|a_iy_i^3|^{1/2}}\,B^{(n+2)/2}P^{5\Delta-4}\bigg). \tag{9}$$

Note that the integral $\mathfrak{I}_{\underline{\varepsilon},t,B}(L)$ only depends on the signs of $\underline{y}$ and $\underline{a}$ and no longer on their actual values.

Next, we make the coefficient of $B^{(n-1)/2}$ in this expression independent of B. We first focus on the factor $\mathfrak{S}_{\underline{y},\underline{a},t}(P^{\Delta})$.

The singular series.

Lemma 3.2. *We have*

$$\left|\sigma_{\underline{y}}\left(\frac{a}{q}\right)\right| \ll q^{-(n+1)/2} \cdot \prod_{i=0}^{n} \gcd(a_i y_i^3, q)^{1/2}.$$

Proof. Using elementary properties of generalized Gauss sums (see for example [Berndt et al. 1998, Chapter 1]), we obtain for positive integers a and c that

$$\left|\sum_{n=0}^{c-1} e\left(\frac{an^2}{c}\right)\right| \ll \gcd(a, c)^{1/2}\sqrt{c}.$$

Applying this to (5) implies the statement. □

Corollary 3.3. *For $n \geqslant 4$, the series*

$$\mathfrak{S}_{\underline{y},\underline{a},t} = \sum_{q=1}^{\infty} \sum_{\substack{0<a/q\leqslant 1 \\ \gcd(a,q)=1}} \sigma_{\underline{y}}\left(\frac{a}{q}\right), \tag{10}$$

called the singular series, converges absolutely. In particular, we have

$$\mathfrak{S}_{\underline{y},\underline{a},t} \ll \frac{\prod_{i=0}^{n} |a_i y_i^3|^{1/2+\varepsilon}}{\operatorname{lcm}(a_0 y_0^3, \dots, a_n y_n^3)^{1/2}} \tag{11}$$

and

$$\mathfrak{S}_{\underline{y},\underline{a},t}(P^{\Delta}) = \mathfrak{S}_{\underline{y},\underline{a},t} + O\left(\frac{\prod_{i=0}^{n} |a_i y_i^3|^{1/2+\varepsilon}}{\operatorname{lcm}(a_0 y_0^3, \dots, a_n y_n^3)^{1/2}} \cdot P^{\Delta(-n+3)/2}\right) \tag{12}$$

for any $\varepsilon > 0$.

Proof. From the previous lemma, we deduce that

$$\begin{aligned}
\mathfrak{S}_{\underline{y},\underline{a},t} &\ll \sum_{q=1}^{\infty} q^{-(n-1)/2} \prod_{i=0}^{n} \gcd(a_i y_i^3, q)^{1/2} \\
&\ll \sum_{\substack{d_i \mid a_i y_i^3 \\ i=0,\dots,n}} (d_0 \cdots d_n)^{1/2} \sum_{\substack{q=1 \\ \operatorname{lcm}(d_0,\dots,d_n)\mid q}}^{\infty} q^{-(n-1)/2} \\
&\ll \sum_{\substack{d_i \mid a_i y_i^3 \\ i=0,\dots,n}} \frac{(d_0 \cdots d_n)^{1/2}}{\operatorname{lcm}(d_0, \dots, d_n)^{(n-1)/2}} \sum_{q=1}^{\infty} q^{-(n-1)/2}.
\end{aligned}$$

Since $n \geqslant 4$, the latter expression converges and we get

$$\mathfrak{S}_{\underline{y},\underline{a},t} \ll \sum_{\substack{d_i \mid a_i y_i^3 \\ i=0,\dots,n}} \frac{(d_0 \cdots d_n)^{1/2}}{\operatorname{lcm}(d_0, \dots, d_n)^{(n-1)/2}}$$
$$\ll \frac{\prod_{i=0}^n |a_i y_i^3|^{1/2+\varepsilon}}{\operatorname{lcm}(a_0 y_0^3, \dots, a_0 y_0^3)^{1/2}}$$

for any $\varepsilon > 0$. Moreover, we obtain in the same way that

$$\left|\mathfrak{S}_{\underline{y},\underline{a},t} - \mathfrak{S}_{\underline{y},\underline{a},t}(P^\Delta)\right| \leqslant \sum_{q > P^\Delta} q^{-(n-1)/2} \prod_{i=0}^n \gcd(a_i y_i^3, q)^{1/2}$$
$$\ll \sum_{\substack{d_i \mid a_i y_i^3 \\ 0 \leqslant i \leqslant n}} \frac{(d_0 \cdots d_n)^{1/2}}{\operatorname{lcm}(d_0, \dots, d_n)^{(n-1)/2}} \sum_{\substack{q > P^\Delta \\ \operatorname{lcm}(d_0,\dots,d_n) \mid q}}^{\infty} q^{-(n-1)/2}$$
$$\ll \frac{\prod_{i=0}^n |a_i y_i^3|^{1/2+\varepsilon}}{\operatorname{lcm}(a_0 y_0^3, \dots, a_n y_n^3)^{1/2}} \cdot P^{\Delta(-n+3)/2}. \qquad \square$$

Remark 3.4. One can prove (see for example [Davenport 2005, Lemmas 5.2-5.3]) for $n \geqslant 4$ that $\mathfrak{S}_{\underline{y},\underline{a},t}$ can be written as an Euler product of p-adic densities

$$\lim_{l \to \infty} \frac{\#\{(x_0, \dots, x_n) \in (\mathbb{Z}/p^l\mathbb{Z})^{n+1} : \sum_{i=0}^n a_i y_i^3 x_i^2 \equiv t \bmod p^l\}}{p^{ln}}.$$

The singular integral. Examining $\mathfrak{I}_{\underline{\varepsilon},t,B}(BP^{\Delta-2})$ in (9), we have the following proposition.

Proposition 3.5. *For $n \geqslant 3$, we have*

$$\mathfrak{I}_{\underline{\varepsilon},t,B}(BP^{\Delta-2}) = \mathfrak{I}_{\underline{\varepsilon},t,B} + O\big(B^{(1-n)/2} P^{(\Delta-2)(1-n)/2}\big) \tag{13}$$

with

$$\mathfrak{I}_{\underline{\varepsilon},t,B} = \int_{-\infty}^{+\infty} e(-\gamma t/B)\, d\gamma \int_{[B^{-1/2},1]^{n+1}} e\Big(\gamma \sum_{i=0}^n \varepsilon_i x_i^2\Big)\, d\underline{x}$$

under the condition $BP^{\Delta-2} \geqslant 1$.

Proof. As proved in [Davenport 2005, Proof of Theorem 4.1], we have

$$\left| \int_{B^{-1/2}}^1 e(\gamma \varepsilon_i x_i^2)\, dx_i \right| \ll \min\{1, |\gamma|^{-1/2}\},$$

and thus

$$\left| \int_{[B^{-1/2},1]^{n+1}} e\Big(\gamma \sum_{i=0}^n \varepsilon_i x_i^2\Big)\, d\underline{x} \right| \ll \min\{1, |\gamma|^{-1/2}\}^{n+1}. \tag{14}$$

This implies that the integral $\mathfrak{I}_{\underline{\varepsilon},t,B}$ converges, since

$$\left|\mathfrak{I}_{\underline{\varepsilon},t,B}\right| \ll \int_{-\infty}^{+\infty} \min\{1, |\gamma|^{-1/2}\}^{n+1} d\gamma < +\infty.$$

Also,

$$\begin{aligned}\left|\mathfrak{I}_{\underline{\varepsilon},t,B}(BP^{\Delta-2}) - \mathfrak{I}_{\underline{\varepsilon},t,B}\right| &\ll \int_{|\gamma|>BP^{\Delta-2}} |\gamma|^{-(n+1)/2} d\gamma \\ &\ll B^{(1-n)/2} P^{(\Delta-2)(1-n)/2}.\end{aligned}$$

□

Defining *the singular integral* as

$$\mathfrak{I}_{\underline{\varepsilon}} = \int_{-\infty}^{+\infty} d\gamma \int_{[0,1]^{n+1}} e\Big(\gamma \sum_{i=0}^{n} \varepsilon_i x_i^2\Big)\, d\underline{x}, \tag{15}$$

it follows from the last proof that this integral is also convergent.

Lemma 3.6. *It holds that $\mathfrak{I}_{\underline{\varepsilon},t,B} \to \mathfrak{I}_{\underline{\varepsilon}}$ as B goes to infinity.*

Proof. We have

$$\begin{aligned}\left|\mathfrak{I}_{\underline{\varepsilon},t,B} - \mathfrak{I}_{\underline{\varepsilon}}\right| &\leqslant \int_{-\infty}^{+\infty} \left|(e(-\gamma t/B) - 1)\right| d\gamma \left|\int_{[B^{-1/2},1]^{n+1}} e\Big(\gamma \sum_{i=0}^{n} \varepsilon_i x_i^2\Big)\, d\underline{x}\right| \\ &\qquad + \int_{-\infty}^{+\infty} d\gamma \left|\int_{([B^{-1/2},1]^{n+1})^c} e\Big(\gamma \sum_{i=0}^{n} \varepsilon_i x_i^2\Big)\, d\underline{x}\right| \\ &= I_1(B,t) + I_2(B),\end{aligned}$$

where $\big([B^{-1/2}, 1]^{n+1}\big)^c$ denotes the complement of $[B^{-1/2}, 1]^{n+1}$ in the hypercube $[0, 1]^{n+1}$.

Since $|(e(-\gamma t/B) - 1)| = 2|\sin(\pi\gamma t B^{-1})| \leqslant \min\{2, 2\pi|\gamma||t|B^{-1}\}$, we obtain the following for $I_1(B, t)$, recalling (14):

$$I_1(B,t) \ll \int_{-\infty}^{+\infty} \min\{1, \pi|\gamma||t|B^{-1}\} \cdot \min\{1, |\gamma|^{-1/2}\}^{n+1} d\gamma.$$

Splitting up the latter integral into three parts according to the appropriate range of γ, we get $I_1(B, t) \ll |t|B^{-1}$ for B big enough.

For $I_2(B)$, one has

$$\left|\int_0^1 e(\gamma\varepsilon_i x_i^2)\, dx_i\right| \ll \min\{1, |\gamma|^{-1/2}\}$$

and

$$\left|\int_0^{B^{-1/2}} e(\gamma\varepsilon_i x_i^2)\, dx_i\right| \ll \min\{B^{-1/2}, |\gamma|^{-1/2}\}.$$

Applying the exclusion-inclusion principle to $I_2(B)$ and observing the symmetric form of the integrand, we get

$$I_2(B) \ll \sum_{i=1}^{n+1} \int_{-\infty}^{+\infty} \min\{B^{-1/2}, |\gamma|^{-1/2}\}^i \cdot \min\{1, |\gamma|^{-1/2}\}^{n+1-i} d\gamma.$$

It follows that $I_2(B) \ll B^{-1/2}$. Hence,

$$\left|\mathfrak{I}_{\underline{\varepsilon},t,B} - \mathfrak{I}_{\underline{\varepsilon}}\right| \ll_t B^{-1/2} \tag{16}$$

for B big enough, completing the proof. □

Note that from Proposition 3.5 and (16), one has

$$\mathfrak{I}_{\underline{\varepsilon},t,B}(BP^{\Delta-2}) = \mathfrak{I}_{\underline{\varepsilon}} + O\big(B^{-1/2} + B^{(1-n)/2} P^{(\Delta-2)(1-n)/2}\big). \tag{17}$$

We now return to the integral of $E_{\underline{y}}(\alpha)$ over the major arcs.

Proposition 3.7. *For $n \geqslant 4$ and for any Δ with $0 < \Delta < 1/5$, there exists a $\delta > 0$ so that*

$$\int_{\mathfrak{M}(\Delta)} E_{\underline{y}}(\alpha)\, d\alpha = \frac{2^{n+1} \mathfrak{S}_{\underline{y},\underline{a},t} \mathfrak{I}_{\underline{\varepsilon}}}{\prod_{i=0}^n |a_i y_i^3|^{1/2}} \cdot B^{(n-1)/2} + O_{\underline{y},\underline{a}}\big(B^{(n-1)/2-\delta}\big). \tag{18}$$

Proof. Substituting (12) and (17) into formula (9), we get

$$\begin{aligned}\int_{\mathfrak{M}(\Delta)} E_{\underline{y}}(\alpha)\, d\alpha &= \frac{2^{n+1} \mathfrak{S}_{\underline{y},\underline{a},t} \mathfrak{I}_{\underline{\varepsilon}}}{\prod_{i=0}^n |a_i y_i^3|^{1/2}} \cdot B^{(n-1)/2} \\ &\quad + O\bigg(\frac{\prod_{i=0}^n |a_i y_i^3|^{\varepsilon}}{\operatorname{lcm}(a_0 y_0^3, \ldots, a_n y_n^3)^{1/2}} \cdot B^{(n-1)/2} P^{\Delta(-n+3)/2} \\ &\qquad + \frac{B^{(n-2)/2} + P^{(\Delta-2)(1-n)/2}}{\prod_{i=0}^n |a_i y_i^3|^{1/2}} + \frac{\sum_{i=0}^n |a_i y_i^3|^{1/2}}{\prod_{i=0}^n |a_i y_i^3|^{1/2}} \cdot B^{(n+2)/2} P^{5\Delta-4}\bigg). \end{aligned} \tag{19}$$

For this expression to be nontrivial, we have to determine $P = P(B)$ and Δ properly (under the condition $BP^{\Delta-2} \geqslant 1$) so that the error term is $O_{\underline{y},\underline{a}}(B^{(n-1)/2-\delta})$ for some $\delta > 0$. Taking $P = B^{1/2}$ and $0 < \Delta < 1/5$ is satisfactory. □

We can now prove our estimate for the major arcs.

Theorem 3.8. *For $n \geqslant 4$ and for any Δ with $0 < \Delta < 1/15$, there exists a $\delta > 0$ so that*

$$\int_{\mathfrak{M}(\Delta)} E(\alpha)\, d\alpha = C_{\underline{a},t} \cdot B^{(n-1)/2} + O\big(B^{(n-1)/2-\delta}\big),$$

where

$$C_{\underline{a},t} = 2^{n+1} \sum_{\underline{y} \in \mathbb{Z}_0^{n+1}} \bigg(\prod_{i=0}^n \mu_i'(y_i)\bigg) \frac{\mathfrak{S}_{\underline{y},\underline{a},t} \mathfrak{I}_{\underline{\varepsilon}}}{\prod_{i=0}^n |a_i y_i^3|^{1/2}},$$

with $\mathfrak{S}_{\underline{y},\underline{a},t}$ and $\mathfrak{I}_{\underline{\varepsilon}}$ as defined above.

Proof. We sum (19) over all admitted y_i such that $1 \leqslant |y_i^3| \leqslant B$, $i \in \{0, \dots, n\}$, and denote the sum of the coefficients of the main term by $C_{\underline{a},t}(B)$.

We obtain, using (11),

$$\frac{\mathfrak{S}_{\underline{y},\underline{a},t}}{\prod_{i=0}^n |a_i y_i^3|^{1/2}} \ll \frac{\prod_{i=0}^n |a_i y_i^3|^{\varepsilon}}{\operatorname{lcm}(a_0 y_0^3, \dots, a_n y_n^3)^{1/2}} \tag{20}$$

for any $\varepsilon > 0$. We have

$$\begin{aligned}
\sum_{\substack{\max\limits_{0\leqslant i\leqslant n} |y_i^3|\geqslant B}} \frac{\prod_{i=0}^n |a_i y_i^3|^{\varepsilon}}{\operatorname{lcm}(a_0 y_0^3, \dots, a_n y_n^3)^{1/2}} &\ll \sum_{\substack{\max\limits_{0\leqslant i\leqslant n} |y_i^3|\geqslant B}} \frac{1}{\operatorname{lcm}(a_0 y_0^3, \dots, a_n y_n^3)^{1/2-(n+1)\varepsilon}} \\
&\ll \sum_{\substack{\max\limits_{0\leqslant i\leqslant n} |y_i^3|\geqslant B}} \frac{1}{\operatorname{lcm}(y_0, \dots, y_n)^{3/2-3(n+1)\varepsilon}} \\
&\ll \sum_{N^3 \geqslant B} \frac{\#\{(y_0, \dots, y_n) : \operatorname{lcm}(y_0, \dots, y_n) = N\}}{N^{3/2-3(n+1)\varepsilon}} \\
&\ll B^{-1/6+(n+1)\varepsilon}
\end{aligned} \tag{21}$$

for any $\varepsilon > 0$. This allows us to replace $C_{\underline{a},t}(B)$ by $C_{\underline{a},t}$.

We now turn to the error term in (19), summing over all admitted values of $\underline{y}$ and putting $P = B^{1/2}$ as before.

The first error term can be treated as the main term. The coefficients of the third and fourth error terms will also converge without any extra conditions. Moreover, the upper bound can be made independent of the a_i. For the last error term however, the coefficient will asymptotically contribute $O(B^{1/3})$.

This means the extra condition

$$\frac{1}{3} + \frac{n+2}{2} + \frac{5\Delta - 4}{2} < \frac{n-1}{2} \Leftrightarrow \Delta < \frac{1}{15}$$

has to be satisfied for the error term to behave properly. □

Note that (20) and (21) also provides a uniform upper bound of $C_{\underline{a},t}$, that is, $C_{\underline{a},t} \leqslant C$, independently of $\underline{a}$ and t.

Minor arcs. The goal of this section is to prove the following theorem.

Theorem 3.9. *For $n \geqslant 4$, there exists a $\delta > 0$ so that*

$$\int_{\mathfrak{m}(\Delta)} E(\alpha)\, d\alpha = O\big(B^{(n-1)/2-\delta}\big).$$

To treat the integral over the minor arcs, we will not fix $\underline{y}$ but examine the whole equation at once. Recall that

$$E(\alpha) = e(-\alpha t) \prod_{i=0}^{n} S_i(\alpha) = e(-\alpha t) \prod_{i=0}^{n} \sum_{1 \leqslant |a_i x^2 y^3| \leqslant B} \mu_i'(y) e(\alpha a_i x^2 y^3).$$

Using Hölder's inequality repeatedly, we get for $n \geqslant 4$,

$$\left| \int_{\mathfrak{m}(\Delta)} E(\alpha)\, d\alpha \right| \leqslant \sup_{\alpha \in \mathfrak{m}(\Delta)} (|S_0(\alpha)| \cdots |S_{n-4}(\alpha)|) \max_{j=n-3,\dots,n} \int_0^1 |S_j(\alpha)|^4 d\alpha. \quad (22)$$

To obtain a good upper bound of this expression, we first examine $\int_0^1 |S_j(\alpha)|^4 d\alpha$.

Lemma 3.10. *For any* $\varepsilon > 0$, *we have*

$$\int_0^1 |S_j(\alpha)|^4 d\alpha \ll_\varepsilon B^{1+\varepsilon}.$$

Proof. From now on, we will concentrate on the part of the sum where the variables are positive. This will suffice to prove the theorem because of the symmetry.

Let

$$S_Y(\alpha) = \sum_{Y < y \leqslant 2Y} \mu_j'(y) \sum_{1 \leqslant x \leqslant B_{a_j,y}} e(\alpha a_j x^2 y^3)$$

be the contribution to $S_j(\alpha)$ for $Y < y \leqslant 2Y$. Using Cauchy's inequality, it follows that

$$\begin{aligned} \int_0^1 |S_Y(\alpha)|^4 d\alpha &\ll Y \int_0^1 |S_Y(\alpha)|^2 \sum_{Y < y \leqslant 2Y} \mu_j'(y) \left| \sum_{1 \leqslant x \leqslant B_{a_j,y}} e(\alpha a_j x^2 y^3) \right|^2 d\alpha \\ &\ll Y \sum_{Y < y_1, y_2, y_3 \leqslant 2Y} \sum_{\substack{1 \leqslant x_1 \leqslant B_{a_j,y_1} \\ 1 \leqslant x_2 \leqslant B_{a_j,y_2} \\ 1 \leqslant x_3, x_4 \leqslant B_{a_j,y_3}}} \int_0^1 e(\alpha a_j G(\underline{x}, \underline{y}))\, d\alpha \\ &\leqslant Y \cdot \#Z(Y, B), \end{aligned}$$

with $G(\underline{x}, \underline{y}) = y_3^3(x_4^2 - x_3^2) + x_1^2 y_1^3 - x_2^2 y_2^3$ and $Z(Y, B) = \{(\underline{x}, \underline{y}) \in \mathbb{Z}_0^7 : y_3^3(x_3^2 - x_4^2) = x_1^2 y_1^3 - x_2^2 y_2^3,\ 1 \leqslant x_i < B_Y,\ Y < y_j \leqslant 2Y\}$, where $B_Y = (B/Y^3)^{1/2}$.

If we make a distinction between solutions $(\underline{x}, \underline{y}) \in \mathbb{Z}_0^7$ of $G(\underline{x}, \underline{y}) = 0$ for which $x_1^2 y_1^3 - x_2^2 y_2^3 = 0$ or not, it follows that both sets contain $O(Y^{-1} \cdot B^{1+\varepsilon})$ solutions. Hence, we conclude that $\#Z(Y, B) \ll_\varepsilon Y^{-1} \cdot B^{1+\varepsilon}$, and thus

$$\int_0^1 |S_Y(\alpha)|^4 d\alpha \ll_\varepsilon B^{1+\varepsilon}.$$

Summing over all intervals $(Y, 2Y]$ with $Y = 2^k \ll B^{1/3}$ and applying Cauchy's inequality twice on $|S_j(\alpha)|^4 = |\sum_{Y=2^k \ll B^{1/3}} S_Y(\alpha)|^4$, we get

$$\int_0^1 |S_j(\alpha)|^4 d\alpha \ll B^{3\varepsilon'} \sum_{Y=2^k \ll B^{1/3}} \int_0^1 |S_Y(\alpha)|^4 d\alpha \ll B^{3\varepsilon'} \sum_{Y=2^k \ll B^{1/3}} B^{1+\varepsilon} = B^{1+\varepsilon''},$$

which completes the proof. □

Remark 3.11. Recalling the expression for $\#M_{\underline{a},t}(B)$ in (2) and putting $n = 3$, $\underline{a} = (1, 1, 1, 1)$, $t = 0$ and $\mu_i'(\cdot) = \mu^2(\cdot)$ for each i, this lemma implies that the equation $n_1 + n_2 = n_3 + n_4$, where n_i is squareful and $1 \leqslant |n_i| \leqslant B$ for each $i \in \{1, 2, 3, 4\}$, has $O(B^{1+\varepsilon})$ solutions.

In order to handle the first part of (22), namely $\sup_{\alpha \in \mathfrak{m}(\Delta)}(|S_0(\alpha)| \cdots |S_{n-4}(\alpha)|)$, we will prove the following proposition.

Proposition 3.12. *Let $\alpha \in \mathfrak{m}(\Delta)$. Then there exists a $\delta > 0$ such that*

$$|S_i(\alpha)| \ll B^{1/2-\delta}.$$

Proof. Let $\psi > 0$. We may henceforth assume that $|a_i| \leqslant B^{\psi}$, since otherwise the trivial upper bound yields

$$|S_i(\alpha)| \leqslant \sum_{y=1}^{\infty} \sqrt{\frac{B}{a_i y^3}} \ll B^{(1-\psi)/2},$$

which is satisfactory. Similarly, we may assume that $y \leqslant B^{\psi}$ in $S_i(\alpha)$. Thus, we have

$$|S_i(\alpha)| \ll B^{(1-\psi)/2} + \sum_{y \leqslant B^{\psi}} \mu_i'(y) \big|T_{\underline{y}}(\alpha)\big|,$$

with, if we set $X = \sqrt{B/(|a_i| y^3)}$,

$$T_{\underline{y}}(\alpha) = \sum_{x \leqslant X} e(\alpha a_i y^3 x^2).$$

Since $|a_i| y^3 x^2 \leqslant B$, in particular $X \geqslant B^{1/2-2\psi}$. Using the usual squaring and differencing approach (see for example [Davenport 2005, Chapter 3]), we obtain

$$\begin{aligned}\big|T_{\underline{y}}(\alpha)\big|^2 &\leqslant \sum_{|h| \leqslant X} \Bigg| \sum_{\substack{x \\ x, x+h \leqslant X}} e(2\alpha a_i y^3 h x) \Bigg| \\ &\ll \sum_{|h| \leqslant X} \min\{X, \|2\alpha a_i y^3 h\|^{-1}\} \ll X + B^{\varepsilon} \cdot \sum_{y \leqslant Y} \min\{X, \|\alpha y\|^{-1}\},\end{aligned}$$

where $Y = 2|a_i| y^3 X$ and $\|a\| = \min\{|\beta| \in \mathbb{R} : \beta \equiv a \mod 1\}$ for any real number a.

In order to estimate the sum over y, we will use the following lemma.

Lemma 3.13 (Separation lemma). *Let $P, Q \geqslant 1$ be reals, $\alpha \in T$ and $a, q \in \mathbb{Z}$ with $\gcd(a, q) = 1$ and $|\alpha - a/q| < q^{-2}$. Then*

$$\sum_{x \leqslant P} \min\left\{\frac{PQ}{x}, \|\alpha x\|^{-1}\right\} \ll PQ\left(q^{-1} + Q^{-1} + q(PQ)^{-1}\right) \log(2qP).$$

Proof. A full proof is given in [Vaughan 1997, Lemma 2.2]. □

Choosing $P = Y$ and $Q = X$, Lemma 3.13 implies

$$\begin{aligned} \left|T_{\underline{y}}(\alpha)\right|^2 &\ll X + XYB^{\varepsilon}\left(\frac{1}{q} + \frac{1}{X} + \frac{q}{XY}\right) \\ &\ll XYB^{2\varepsilon}\left(\frac{1}{q} + \frac{1}{X} + \frac{q}{XY}\right) \\ &\ll B^{1+2\varepsilon}\left(\frac{1}{q} + B^{2\psi-1/2}\right) + qB^{2\varepsilon}, \end{aligned}$$

since $X \leqslant Y$ and $XY = 2|a_i|y^3X^2 = 2B$. Hence,

$$|S_i(\alpha)| \ll B^{1/2-2\psi} + B^{1/2+\varepsilon+\psi}\left(\frac{1}{\sqrt{q}} + B^{\psi-1/4}\right) + \sqrt{q}B^{\varepsilon+\psi}. \tag{23}$$

According to Dirichlet, we can find $a, q \in \mathbb{Z}$ with $\gcd(a, q) = 1$ and $q \leqslant B^{(2-\Delta)/4}$ such that $|\alpha q - a| < 1/B^{(2-\Delta)/4} = B^{(\Delta-2)/4}$. (Note we also have $|\alpha - a/q| < 1/q^2$.) Furthermore, it is necessary that $q > B^{\Delta/2}$: otherwise, we would have $\alpha \in \mathfrak{M}(\Delta)$. With these boundaries for q in (23), a suitable small choice for ψ in terms of Δ leads to the statement. □

We are now able to prove Theorem 3.9.

Proof of Theorem 3.9. Combining Proposition 3.12 and Lemma 3.10 in (22), we obtain

$$\left|\int_{\mathfrak{m}(\Delta)} E(\alpha)\, d\alpha\right| \ll B^{(1/2-\delta)(n-3)} \cdot B^{1+\varepsilon} \leqslant B^{(n-1)/2-\delta+\varepsilon} < B^{(n-1)/2}$$

for any $0 < \varepsilon < \delta$. □

4. Towards the main problem

Combining the previous results, we are able to prove the following theorem.

Theorem 4.1. *For $n \geqslant 4$, there exists a $\delta > 0$ so that*

$$\#M_{\underline{a},t}(B) = C_{\underline{a},t} \cdot B^{(n-1)/2} + O\left(B^{(n-1)/2-\delta}\right),$$

with the constant $C_{\underline{a},t}$ described in Theorem 3.8.

Proof. This follows directly from Theorem 3.8, Theorem 3.9 and (2). □

Remark 4.2. Note that the error term is independent of $\underline{a}$ and t and recall we also proved $C_{\underline{a},t}$ can be bounded uniformly independent of $\underline{a}$ and t. This implies that $\#M_{\underline{a},t}(B) \leqslant C \cdot B^{(n-1)/2}$ for some constant $C > 0$. Indeed, when $B < 1$, $M_{\underline{a},t}(B) = \varnothing$, and for $B \geqslant 1$, it follows from Theorem 4.1 that

$$\#M_{\underline{a},t}(B) \leqslant C' \cdot B^{(n-1)/2} + C'' \cdot B^{(n-1)/2-\delta} \leqslant C \cdot B^{(n-1)/2},$$

where $C = 2\max\{C', C''\}$.

Going back to $M(B)$ (see definition on page 1021), we will now prove the following theorem.

Theorem 4.3. *For $n \geqslant 4$, there exists an explicit constant D and a $\delta > 0$ such that*

$$\#M(B) = D \cdot B^{(n-1)/2} + O\big(B^{(n-1)/2-\delta}\big)$$

as B goes to infinity.

(The definition of the constant D is given in Lemma 4.5; in the next section, we will give some indications about the interpretation of D.)

The only problem still left in proving Theorem 4.3 is to understand how we can tackle the additional gcd condition $\gcd(x_0 y_0, \dots, x_n y_n) = 1$ on the solutions. Note that the Möbius inversion at hand leads to divisibility conditions on both x_i and y_i which have to be handled with care.

Let $\underline{e} = (e_0, \dots, e_n) \in \mathbb{N}_0^{n+1}$ and $\underline{f} = (f_0, \dots, f_n) \in \mathbb{N}_0^{n+1}$, where f_i is squarefree for each $i \in \{0, \dots, n\}$.

Definition. We denote the set

$$\Big\{(\underline{x}, \underline{y}) \in \mathbb{Z}_0^{2n+2} \sum_{i=0}^{n} x_i^2 y_i^3 = 0,\ \max_{0 \leqslant i \leqslant n} |x_i^2 y_i^3| \leqslant B,\ e_i | x_i,\ \ f_i | y_i \text{ and } \prod_{i=0}^{n} \mu^2(|y_i|) = 1\Big\}$$

by $N_{(\underline{e},\underline{f})}(B)$.

Demanding that solutions in $N_{(\underline{1},\underline{1})}(B)$ satisfy $\gcd(x_0 y_0, \dots, x_n y_n) = 1$ means we wish to leave out those solutions of $N_{(\underline{1},\underline{1})}(B)$ for which there exists a prime p and a subset $I \subset \{0, \dots, n\}$ such that $p | x_i$ if $i \in I$ and $p | y_i$ if $i \notin I$ (or $i \in I^c$, where I^c denotes the complement of I in $\{0, \dots, n\}$) in order to get to $M(B)$. Defining for a prime p and subsets $I, J \subset \{0, \dots, n\}$ the couple $(\underline{e}^{p,I}, \underline{f}^{p,J})$ by $e_i^{p,I} = p$ for $i \in I$ and $e_i^{p,I} = 1$ otherwise and analogously for $\underline{f}^{p,J}$, it hence follows that

$$M(B) = N_{(\underline{1},\underline{1})}(B) \setminus \bigcup_{(p,I)} N_{(\underline{e}^{p,I}, \underline{f}^{p,I^c})}(B). \tag{24}$$

Notice that in this last union only a finite number of sets are nonempty since for a prime $p \geqslant \sqrt{B}$, we get $N_{(\underline{e}^{p,I}, \underline{f}^{p,I^c})}(B) = \varnothing$.

Definition. Let S be a finite set of couples (p, I). We can associate to S a couple $(\underline{e}, \underline{f})$ as follows: defining for each prime p the index sets $I_p = \cup_{(p,I)\in S} I$ and $J_p = \cup_{(p,I)\in S} I^c$, the associated couple is given by $e_i = \prod_{\{p|i\in I_p\}} p$ and $f_i = \prod_{\{p|i\in J_p\}} p$.
We then define

$$\mu(\underline{e}, \underline{f}) = \sum_{n\geqslant 0} (-1)^n \,\#\big\{\text{sets } S \text{ of cardinality } n \text{ with associated couple } (\underline{e}, \underline{f})\big\}.$$

Observing (24) together with this definition, we have

$$\#M(B) = \sum_{e=1}^{\infty} \sum_{\substack{(\underline{e},\underline{f})\in\mathbb{N}^{2n+2} \\ e=\gcd(e_i f_i,\ i=0,\dots,n)}} \mu(\underline{e}, \underline{f}) \cdot \#N_{(\underline{e},\underline{f})}(B). \tag{25}$$

The following lemma collects some properties of μ.

Lemma 4.4. *There exists a function* $\widetilde{\mu} : \mathbb{Z}^{2n+2} \to \mathbb{Z}$ *such that*

(i) $\mu(\underline{e}, \underline{f}) = \prod_p \widetilde{\mu}(v_p(\underline{e}), v_p(\underline{f}))$, *where* $v_p(\underline{e}) = (v_p(e_0), \dots, v_p(e_n))$ *(and analogously for* $v_p(\underline{f})$*),*

(ii) $\widetilde{\mu}(\underline{m}, \underline{n}) = 0$ *if* $m_i = n_i = 0$ *and* $(\underline{m}, \underline{n}) \neq (\underline{0}, \underline{0})$ *or if* $m_i > 1$ *for some* i,

(iii) $\sum_{I\cup J=\{0,\dots,n\}} |\widetilde{\mu}(I, J)| \leqslant 2^{2^{n+1}}$, *where, for subsets* $I, J \subset \{0, \dots, n\}$, $\widetilde{\mu}(I, J)$ *denotes* $\widetilde{\mu}(m_0^I, \dots, m_n^I, {m'_0}^J, \dots, {m'_n}^J)$ *with* $m_i^I = 1$ *if* $i \in I$ *and* $m_i^I = 0$ *otherwise and* ${m_i'}^J = 1$ *if* $i \in J$ *and* ${m'_i}^J = 0$ *otherwise.*

Proof. (i) and (ii) follow directly from the definition of μ immediately above. From the same definition, it follows, if $I \cup J = \{0, \dots, n\}$, and denoting by T a finite set of subsets $I \subset \{0, \dots, n\}$, that

$$\widetilde{\mu}(I, J) = \sum_m (-1)^m \,\#\big\{\text{sets } T \text{ of cardinality } m \\ \text{such that } I = \cup_{K\in T} K \text{ and } J = \cup_{K\in T} K^c\big\}.$$

If we sum over all possible I and J such that $I \cup J = \{0, \dots, n\}$, we get (iii). □

Consider now $N_{(\underline{e},\underline{f})}(B)$ for a couple $(\underline{e}, \underline{f})$ for which $\mu(\underline{e}, \underline{f}) \neq 0$ and

$$\gcd(e_i f_i,\ i = 0, \dots, n) = e,$$

i.e., a subset with nontrivial contribution to $\#M(B)$ (recall (25)). Since $\#N_{(\underline{e},\underline{f})}(B) = \#M_{\underline{e^2 f^3},0}(B)$, choosing $\mu'_i(y_i) = \mu^2(f_i|y_i|)$ (where $\underline{e^2 f^3} = (e_0^2 f_0^3, \dots, e_n^2 f_n^3)$), we know by Theorem 4.1 that

$$\#N_{(\underline{e},\underline{f})}(B) = C_{\underline{e^2 f^3},0} \cdot B^{(n-1)/2} + O(B^{(n-1)/2-\delta}).$$

Since e divides $e_i f_i$, we can write $e_i^2 f_i^3 = v_i e^2$ for some $v_i \in \mathbb{N}$ for each i in $\{0, \ldots, n\}$. Making the substitutions $x_i' = x_i/e_i$ and $y_i' = y_i/f_i$, we see that $N_{(\underline{e},\underline{f})}(B)$ corresponds to the set

$$\left\{(\underline{x}', \underline{y}') \in \mathbb{Z}_0^{2n+2} : \sum_{i=0}^{n} v_i x_i'^2 y_i'^3 = 0,\ \max_{0 \leqslant i \leqslant n} |v_i x_i'^2 y_i'^3| \leqslant \frac{B}{e^2} \text{ and } \prod_{i=0}^{n} \mu^2(f_i |y_i'|) = 1\right\},$$

where we eliminated e^2 in the equation, and hence $\#N_{(\underline{e},\underline{f})}(B) = \#M_{\underline{v},0}(B/e^2)$. Letting B go to infinity, this implies that the main terms in the asymptotic formulas of $\#N_{\underline{e},\underline{f}}(B)$ and $\#M_{\underline{v},0}(B/e^2)$ are equal, and in particular that

$$\#N_{(\underline{e},\underline{f})}(B) - C_{\underline{e^2 f^3},0} \cdot B^{(n-1)/2} = O\left(\frac{B^{(n-1)/2-\delta}}{e^{n-1-2\delta}}\right). \tag{26}$$

Notice we also obtain (recall Remark 4.2) that

$$\#N_{(\underline{e},\underline{f})}(B) \leqslant C \cdot \frac{B^{(n-1)/2}}{e^{n-1}} \quad \text{and} \quad C_{\underline{e^2 f^3},0} \leqslant \frac{C}{e^{n-1}}. \tag{27}$$

From these results, we can now prove:

Lemma 4.5. *The series* $D = \sum_{e=1}^{\infty} \sum_{\substack{(\underline{e},\underline{f}) \in \mathbb{N}^{2n+2} \\ \gcd(e_i f_i,\ i=0,\ldots,n)=e}} \mu(\underline{e}, \underline{f}) \cdot C_{\underline{e^2 f^3},0}$ *converges.*

Proof. Substituting (27) into the definition of D and using the properties of μ in Lemma 4.4, we get

$$\begin{aligned} |D| &\ll \sum_{e=1}^{\infty} \sum_{\substack{(\underline{e},\underline{f}) \in \mathbb{N}^{2n+2} \\ \gcd(e_i f_i,\ i=0,\ldots,n)=e}} \frac{|\mu(\underline{e}, \underline{f})|}{e^{n-1}} \\ &\leqslant \prod_p \sum_{k=0}^{2} \sum_{\substack{(v_p(\underline{e}), v_p(\underline{f})) \in \mathbb{N}^{2n+2} \\ \min_i \{v_p(e_i) + v_p(f_i)\} = k}} \frac{|\mu_p(v_p(\underline{e}), v_p(\underline{f}))|}{p^{k(n-1)}} \leqslant \prod_p \left(1 + 2\frac{2^{2^{n+1}}}{p^{n-1}}\right), \end{aligned}$$

which converges since $n \geqslant 4$. ☐

Proof of Theorem 4.3. From the definition of D and (26), it follows that

$$\left|\#M(B) - D \cdot B^{(n-1)/2}\right| \ll \sum_{e=1}^{\infty} \sum_{\substack{(\underline{e},\underline{f}) \in \mathbb{N}^{2n+2} \\ \gcd(e_i f_i,\ i=0,\ldots,n)=e}} \left|\mu(\underline{e}, \underline{f})\right| \cdot \frac{B^{(n-1)/2-\delta}}{e^{(n-1)-2\delta}}.$$

Following the same reasoning as in Lemma 4.5, we then get

$$\left|\#M(B) - D \cdot B^{(n-1)/2}\right| \ll B^{(n-1)/2-\delta} \cdot \prod_p \left(1 + 2\frac{2^{2^{n+1}}}{p^{n-1-2\delta}}\right),$$

where the product converges for $\delta > 0$ small enough since $n \geqslant 4$. This proves the theorem. $\square$

5. Rational points on the orbifold $(\mathbb{P}^{n-1}, \Delta)$

We can now prove our main theorem.

Theorem 5.1. *For $n \geqslant 4$, there exists a $\delta > 0$ such that*

$$\#(\mathbb{P}^{n-1}, \Delta)(\mathbb{Q})_{\leqslant B} = C \cdot B^{(n-1)/2} + O\big(B^{(n-1)/2-\delta}\big).$$

Here,

$$C = \frac{1}{2^{n+1}} \sum_{e=1}^{\infty} \sum_{\substack{(\underline{e},\underline{f}) \in \mathbb{N}^{2n+2} \\ \gcd(e_i f_i,\ i=0,\dots,n)=e}} \mu(\underline{e},\underline{f}) \sum_{\substack{\underline{y} \in \mathbb{Z}_0^{n+1}/\{\pm 1\} \\ f_i y_i \text{ squarefree}}} \frac{2^{n+1} \mathfrak{S}_{\underline{y},\underline{e^2 f^3},0} \mathfrak{J}_{\underline{\varepsilon}}}{\prod_{i=0}^{n} (e_i^2 f_i^3 |y_i^3|)^{1/2}},$$

with $\mathfrak{S}_{\underline{y},\underline{a},t}$, $\mathfrak{J}_{\underline{\varepsilon}}$ and the function μ as defined before. (By $\underline{y} \in \mathbb{Z}_0^{n+1}/\{\pm 1\}$, we denote the $(n+1)$-tuples $(y_0, \dots, y_n) \in \mathbb{Z}_0^{n+1}$, defined up to sign as an $(n+1)$-tuple.)

Proof. The connection between $(\mathbb{P}^{n-1}, \Delta)(\mathbb{Q})^{+}_{\leqslant B}$ and the set $M(B)$ given by (1), together with Theorem 4.3, implies that the theorem holds for $\#(\mathbb{P}^{n-1}, \Delta)(\mathbb{Q})^{+}_{\leqslant B}$. It remains to prove that, for $n \geqslant 4$, the set of points $(a_0 : \dots : a_n) \in (\mathbb{P}^{n-1}, \Delta)(\mathbb{Q})_{\leqslant B}$ with at least one zero coordinate (whose cardinality is $\ll \#(\mathbb{P}^{n-2}, \Delta)(\mathbb{Q})_{\leqslant B}$), is asymptotically negligible compared to $(\mathbb{P}^{n-1}, \Delta)(\mathbb{Q})^{+}_{\leqslant B}$.

We will verify this for $n = 4$; by induction, the statement follows for $n > 4$.

As mentioned in Remark 3.11, it follows from Lemma 3.10 that

$$\#(\mathbb{P}^{2}, \Delta)(\mathbb{Q})^{+}_{\leqslant B} \ll B^{1+\varepsilon}.$$

Combining this with the trivial upper bound $\#(\mathbb{P}^{1}, \Delta)(\mathbb{Q})_{\leqslant B} \ll B$, we obtain

$$\#(\mathbb{P}^{2}, \Delta)(\mathbb{Q})_{\leqslant B} \ll B^{1+\varepsilon} < B^{3/2}$$

for $\varepsilon > 0$ sufficiently small. $\square$

Description of the constant. An alternative description of $(\mathbb{P}^{n-1}, \Delta)(\mathbb{Q})^{+}_{\leqslant B}$ can be obtained as follows. Consider $\underline{y} \in \mathbb{Z}_0^{n+1}/\{\pm 1\}$ with each y_i squarefree. For such $\underline{y}$, let $Q_{\underline{y}}$ denote the smooth quadric defined by the homogeneous polynomial $F_{\underline{y}}(\underline{x}) = y_0^3 X_0^2 + \dots + y_n^3 X_n^2 \in \mathbb{Z}[X_0, \dots, X_n]$. Furthermore, define the morphism

$$\begin{array}{rccc} \pi_{\underline{y}} : & Q_{\underline{y}} & \to & H \\ & (x_0 : \dots : x_n) & \mapsto & (y_0^3 x_0^2 : \dots : y_n^3 x_n^2). \end{array} \tag{28}$$

We will consider points $(x_0 : \dots : x_n) \in Q_{\underline{y}}(\mathbb{Q})$ with $x_i \in \mathbb{Z}$ such that $\prod_{i=0}^{n} x_i \neq 0$ and $\gcd(x_0 y_0, \dots, x_n y_n) = 1$. We denote this subset of $Q_{\underline{y}}(\mathbb{Q})$ by $Q_{\underline{y}}(\mathbb{Q})^{+}$. This set is mapped into $(\mathbb{P}^{n-1},\Delta)(\mathbb{Q})^+$ by $\pi_{\underline{y}}$ and, keeping in mind (1), we have

$$(\mathbb{P}^{n-1},\Delta)(\mathbb{Q})^+ = \coprod_{\substack{\underline{y}\in\mathbb{Z}_0^{n+1}/\{\pm1\}\\ y_i \text{ squarefree}}} \pi_{\underline{y}}(Q_{\underline{y}}(\mathbb{Q})^+). \tag{29}$$

This implies

$$\#(\mathbb{P}^{n-1},\Delta)(\mathbb{Q})^+_{\leqslant B} = \frac{1}{2^{n+1}} \sum_{\substack{\underline{y}\in\mathbb{Z}_0^{n+1}/\{\pm1\}\\ y_i \text{ squarefree}}} \#\Big\{(x_0:\cdots:x_n)\in Q_{\underline{y}}(\mathbb{Q})^+ : \max_{0\leqslant i\leqslant n} |x_i^2 y_i^3| \leqslant B\Big\}.$$

For a fixed $\underline{y}$, an asymptotic expression for each of the latter sets using the classical circle method is known (see [Davenport 2005, Chapter 8]) and a Möbius inversion for the gcd condition $\gcd(x_0y_0,\ldots,x_ny_n)=1$.

Moreover, from Lemma 4.5, it follows that we can change the order of summation for e and $\underline{y}$ in the constant C from Theorem 5.1 and thus, defining

$$C_{Q_{\underline{y}}} = \sum_{e=1}^{\infty} \sum_{\substack{(\underline{e},\underline{f})\in\mathbb{N}^{2n+2}\\ \gcd(e_i f_i,\ i=0,\ldots,n)=e\\ f_i\mid y_i}} \mu(\underline{e},\underline{f}) \frac{2^{n+1}\mathfrak{S}_{\underline{y},\underline{e}^2,0}\mathfrak{J}_{\underline{\varepsilon}}}{\prod_{i=0}^n (e_i|y_i|^{3/2})}, \tag{30}$$

we have, for $n\geqslant 4$,

$$\#(\mathbb{P}^{n-1},\Delta)(\mathbb{Q})_{\leqslant B} \sim \Bigg(\frac{1}{2^{n+1}} \sum_{\substack{\underline{y}\in\mathbb{Z}_0^{n+1}/\{\pm1\}\\ y_i \text{ squarefree}}} C_{Q_{\underline{y}}}\Bigg)\cdot B^{(n-1)/2}$$

as B goes to infinity.

This constant $C_{Q_{\underline{y}}}$ can be given a more geometrical interpretation using the adelic space $Q_{\underline{y}}(\mathbb{A}_{\mathbb{Q}})$ of the quadric $Q_{\underline{y}}$, as explained in [Peyre 1995, §5]. Here, it has been shown that the refined version of the Manin conjecture is compatible with the circle method for smooth quadrics in $\mathbb{P}^n_{\mathbb{Q}}$ and moreover, that rational points on smooth quadrics are equidistributed. Considering the Tamagawa measure $\boldsymbol{\omega}_{H_{\underline{y}}}$ (corresponding to the height function $H_{\underline{y}}$ defined as $H_{\underline{y}}(P)=\max_{0\leqslant i\leqslant n}|x_i^2y_i^3|$ where $P=(x_0:\cdots:x_n)\in Q_{\underline{y}}(\mathbb{Q})$) on $\mathbb{Q}_{\underline{y}}(\mathbb{A}_{\mathbb{Q}})$, the equidistribution of the rational points on $Q_{\underline{y}}$ implies that for every good open subset W (that is, an open subset W for which $\boldsymbol{\omega}_{H_{\underline{y}}}(\partial W)=0$, where $\partial W=\overline{W}\setminus W$) of $Q_{\underline{y}}(\mathbb{A}_{\mathbb{Q}})$, we have

$$\frac{\#\big\{P\in Q_{\underline{y}}(\mathbb{Q})^+\cap W \mid H_{\underline{y}}(P)\leqslant B\big\}}{\#\big\{P\in Q_{\underline{y}}(\mathbb{Q})^+ \mid H_{\underline{y}}(P)\leqslant B\big\}} \to \frac{\boldsymbol{\omega}_{H_{\underline{y}}}(W)}{\boldsymbol{\omega}_{H_{\underline{y}}}(Q_{\underline{y}}(\mathbb{A}_{\mathbb{Q}}))}$$

as B goes to infinity. We refer to [Peyre 1995] for more details on this matter. This implies we can obtain a description of the constant $C_{Q_{\underline{y}}}$ in terms of the measure $\boldsymbol{\omega}_{H_{\underline{y}}}$

of a certain subset of the adelic space $Q_{\underline{y}}(\mathbb{A}_{\mathbb{Q}})$ of the quadric $Q_{\underline{y}}$. More precisely, it follows that

$$C_{Q_{\underline{y}}} = \boldsymbol{\omega}_{H_{\underline{y}}}(Q_{\underline{y}}(\mathbb{A}_{\mathbb{Q}})^{\dagger})/(n-1),$$

where $Q_{\underline{y}}(\mathbb{A}_{\mathbb{Q}})^{\dagger}$ denotes the good open subset of $Q_{\underline{y}}(\mathbb{A}_{\mathbb{Q}})$ defined by the gcd condition $\gcd(x_0y_0, \dots, x_ny_n) = 1$ we imposed on $Q_{\underline{y}}(\mathbb{Q})$. (Note that imposing the open condition $\prod_{i=0}^{n} x_i \neq 0$ does not change the measure.) We obtain the following corollary.

Corollary 5.2. *For $n \geqslant 4$, we have*

$$\#(\mathbb{P}^{n-1}, \Delta)(\mathbb{Q})_{\leqslant B} \sim \Bigg(\frac{1}{2^{n+1}} \sum_{\substack{\underline{y} \in \mathbb{Z}_0^{n+1}/\{\pm 1\} \\ y_i \text{ squarefree}}} C_{Q_{\underline{y}}}\Bigg) \cdot B^{(n-1)/2} \tag{31}$$

as B goes to infinity, where $C_{Q_{\underline{y}}} = \boldsymbol{\omega}_{H_{\underline{y}}}(Q_{\underline{y}}(\mathbb{A}_{\mathbb{Q}})^{\dagger})/(n-1)$.

***The adelic space of the orbifold* $(\mathbb{P}^{n-1}, \Delta)$.** In order to define the adelic space of the orbifold properly, we first have to explain how we can translate the definition of "squarefulness" to the different completions of $\mathbb{Q}$.

At each finite place $v = p$, a p-adic integer $a \in \mathbb{Z}_p$ is squareful if $v_p(a) \neq 1$. Due to the structure of $\mathbb{Q}_p^{\times}$, this means that we can write a squareful p-adic integer a uniquely as x^2y^3 with $x \in \mathbb{Z}_p^{\times}$ and $y \in \mathbb{Z}$ squarefree.

On the other hand, any real number $a \in \mathbb{R}$ can be written as $(\pm 1)^3x^2$ and ought to be considered as squareful.

Since we identified $(\mathbb{P}^{n-1}, \Delta)(\mathbb{Q})$ with $\{(u_0 : \cdots : u_n) \in H(\mathbb{Q}) : u_i \text{ squareful}\}$ (recall $H \subset \mathbb{P}^n$ is the hyperplane defined by $X_0 + \cdots + X_n = 0$), we have, for each $v \in \mathrm{Val}(\mathbb{Q})$, that

$$\begin{aligned}(\mathbb{P}^{n-1}, \Delta)(\mathbb{Q}_v) &= \{(u_0 : \cdots : u_n) \in H(\mathbb{Q}_v) : u_i \text{ squareful}\} \\ &= \{(x_{0,v}^2y_0^3 : \cdots : x_{n,v}^2y_n^3) \in H(\mathbb{Q}_v) : \underline{y} \in \mathbb{Z}_0^{n+1}/\{\pm 1\},\ y_i \text{ squarefree}\}.\end{aligned}$$

This implies, recalling the definition of $\pi_{\underline{y}}$ in (28),

$$(\mathbb{P}^{n-1}, \Delta)(\mathbb{Q}_v) = \bigcup_{\substack{\underline{y} \in \mathbb{Z}_0^{n+1}/\{\pm 1\} \\ y_i \text{ squarefree}}} \pi_{\underline{y}}(Q_{\underline{y}}(\mathbb{Q}_v)^{\dagger}), \tag{32}$$

where for a finite place $v = p$, $Q_{\underline{y}}(\mathbb{Q}_p)^{\dagger}$ is the open subset of $Q_{\underline{y}}(\mathbb{Q}_p)$ defined by the condition $\min_{0 \leqslant i \leqslant n}(v_p(x_{i,p}y_i)) = 0$, and where $Q_{\underline{y}}(\mathbb{R})^{\dagger} = Q_{\underline{y}}(\mathbb{R})$.

Note that the union considered is not disjoint, but that the image for different $\underline{y}$ and $\underline{y}'$ either coincides or is disjoint. Hence, it follows that, at each place $v \in \mathrm{Val}(\mathbb{Q})$, $(\mathbb{P}^{n-1}, \Delta)(\mathbb{Q}_v)$ can be described as a finite disjoint union of sets $\pi_{\underline{y}}(Q_{\underline{y}}(\mathbb{Q}_v)^{\dagger})$ for specified $\underline{y} \in \mathbb{Z}_0^{n+1}/\{\pm 1\}$.

Definition. We define the adelic space $(\mathbb{P}^{n-1}, \Delta)(\mathbb{A}_\mathbb{Q})$ as

$$(\mathbb{P}^{n-1}, \Delta)(\mathbb{A}_\mathbb{Q}) = \prod_{v \in \mathrm{Val}(\mathbb{Q})} (\mathbb{P}^{n-1}, \Delta)(\mathbb{Q}_v).$$

Remark 5.3. One may prove that $(\mathbb{P}^{n-1}, \Delta)(\mathbb{Q})$ is dense in $(\mathbb{P}^{n-1}, \Delta)(\mathbb{A}_\mathbb{Q})$. This follows from the fact that weak approximation holds for smooth quadrics.

***Distribution of rational points on* $(\mathbb{P}^{n-1}, \Delta)$.** We can now consider the probability measure

$$\mu_{H \leqslant B}^{(\mathbb{P}^{n-1},\Delta)} = \frac{1}{\#(\mathbb{P}^{n-1}, \Delta)(\mathbb{Q})_{\leqslant B}} \sum_{\substack{P \in (\mathbb{P}^{n-1},\Delta)(\mathbb{Q}) \\ H(P) \leqslant B}} \delta_P \tag{33}$$

on $(\mathbb{P}^{n-1}, \Delta)(\mathbb{A}_\mathbb{Q})$. Here, we will investigate the convergence of $\mu_{H \leqslant B}^{(\mathbb{P}^{n-1},\Delta)}$ to a specific measure on the adelic space of the orbifold, which we have yet to define, when B goes to infinity. Keeping in mind the description of $(\mathbb{P}^{n-1}, \Delta)(\mathbb{A}_\mathbb{Q})$ we gave above, we can define this measure in the following natural way.

Definition. We define the measure $\boldsymbol{\omega}_{(\mathbb{P}^{n-1},\Delta)}$ on $(\mathbb{P}^{n-1}, \Delta)(\mathbb{A}_\mathbb{Q})$ as

$$\boldsymbol{\omega}_{(\mathbb{P}^{n-1},\Delta)}(U) = \sum_{\substack{\underline{y} \in \mathbb{Z}_0^{n+1}/\{\pm 1\} \\ y_i \text{ squarefree}}} \boldsymbol{\omega}_{H_{\underline{y}}}(\pi_{\underline{y}}^{-1}(U)), \tag{34}$$

where U is an open subset of $(\mathbb{P}^{n-1}, \Delta)(\mathbb{A}_\mathbb{Q})$ (which is equipped with the subspace topology coming from $H(\mathbb{A}_\mathbb{Q})$) and $\pi_{\underline{y}} : Q_{\underline{y}}(\mathbb{A}_\mathbb{Q})^\dagger \to (\mathbb{P}^{n-1}, \Delta)(\mathbb{A}_\mathbb{Q})$. (Note that the morphisms $\pi_{\underline{y}}$ introduced in (28) define continuous maps $\pi_{\underline{y}} : Q_{\underline{y}}(\mathbb{A}_\mathbb{Q}) \to H(\mathbb{A}_\mathbb{Q})$ which map $Q_{\underline{y}}(\mathbb{A}_\mathbb{Q})^\dagger$ into $(\mathbb{P}^{n-1}, \Delta)(\mathbb{A}_\mathbb{Q})$.)

Remark 5.4. From this definition of the measure $\boldsymbol{\omega}_{(\mathbb{P}^{n-1},\Delta)}$, it follows that its support consists of the (disjoint) union of

$$\pi_{\underline{y}}(Q_{\underline{y}}(\mathbb{A}_\mathbb{Q})^\dagger) \tag{35}$$

for all $\underline{y} \in \mathbb{Z}_0^{n+1}/\{\pm 1\}$ with y_i squarefree for each $i \in \{0, \dots, n\}$. This is a proper subset of $(\mathbb{P}^{n-1}, \Delta)(\mathbb{A}_\mathbb{Q})$.

In order to say something about the convergence of $\mu_{H \leqslant B}^{(\mathbb{P}^{n-1},\Delta)}$, we first define elementary open subsets of $(\mathbb{P}^{n-1}, \Delta)(\mathbb{A}_\mathbb{Q})$.

An elementary open subset W of $H(\mathbb{A}_\mathbb{Q})$ can be defined as

$$W = \prod_{v \in \mathrm{Val}(\mathbb{Q})} W_v,$$

such that $W_v \subset H(\mathbb{Q}_v)$ is defined at finitely many finite places as $W_p = \mathrm{red}_M^{-1}(X_p)$, where $X_p \subset H(\mathbb{Z}/p^M\mathbb{Z})$ and $\mathrm{red}_M : H(\mathbb{Q}_p) \to H(\mathbb{Z}/p^M\mathbb{Z})$; $W_p = H(\mathbb{Q}_p)$ for any other finite place. Furthermore, at the infinite place $v=\infty$, we require $W_\infty = \bigcap_{i,j}(\lambda_{i,j}x_i < x_j) \subset H(\mathbb{R})$ fixing one of the coordinates x_i to one. Here, $\lambda_{i,j} \in \mathbb{R}_{>0}$ depending on i and j.

To construct elementary open subsets on $(\mathbb{P}^{n-1},\Delta)(\mathbb{A}_\mathbb{Q})$, we can take the intersection with elementary open subsets of $H(\mathbb{A}_\mathbb{Q})$.

We will now prove the following theorem.

Theorem 5.5. *For every elementary open subset U of $(\mathbb{P}^{n-1},\Delta)(\mathbb{A}_\mathbb{Q})$, we have*

$$\mu_{H\leqslant B}^{(\mathbb{P}^{n-1},\Delta)}(U) \to \frac{\boldsymbol{\omega}_{(\mathbb{P}^{n-1},\Delta)}(U)}{\boldsymbol{\omega}_{(\mathbb{P}^{n-1},\Delta)}((\mathbb{P}^{n-1},\Delta)(\mathbb{A}_\mathbb{Q}))}$$

as B goes to infinity.

Proof. Straightforward calculations show that for each admitted $\underline{y}$, the inverse image $\pi_{\underline{y}}^{-1}(U)$ of an elementary open subset U of $(\mathbb{P}^{n-1},\Delta)(\mathbb{A}_\mathbb{Q})$ defines a good open subset of $Q_{\underline{y}}(\mathbb{A}_\mathbb{Q})^\dagger$.

Now let U be an elementary open subset of $(\mathbb{P}^{n-1},\Delta)(\mathbb{A}_\mathbb{Q})$. Recalling (33), the partition of $(\mathbb{P}^{n-1},\Delta)(\mathbb{Q})^+$ in (29), and Theorem 5.1, we get

$$\begin{aligned}\mu_{H\leqslant B}^{(\mathbb{P}^{n-1},\Delta)}(U) &= \frac{\#\big\{(u_0:\cdots:u_n)\in(\mathbb{P}^{n-1},\Delta)(\mathbb{Q})\cap U : \max_{0\leqslant i\leqslant n}|u_i|\leqslant B\big\}}{\#(\mathbb{P}^{n-1},\Delta)(\mathbb{Q})_{\leqslant B}} \\ &\sim \frac{\sum_{\underline{y}}\#\big\{(x_0:\cdots:x_n)\in Q_{\underline{y}}(\mathbb{Q})^+\cap\pi_{\underline{y}}^{-1}(U) : \max_{0\leqslant i\leqslant n}|y_i^3x_i^2|\leqslant B\big\}}{\sum_{\underline{y}}\#\big\{(x_0:\cdots:x_n)\in Q_{\underline{y}}(\mathbb{Q})^+ : \max_{0\leqslant i\leqslant n}|y_i^3x_i^2|\leqslant B\big\}}.\end{aligned}$$

(Here, we used the abbreviated notation $\sum_{\underline{y}}$ to sum over all admitted $\underline{y}\in\mathbb{Z}_0^{n+1}$.)

Combining the fact that rational points on smooth quadrics are equidistributed, the definition of the measure in (34), and Theorem 5.1 enables us to complete the proof. □

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