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We prove a denominator identity for nontwisted affine Lie superalgebras with zero dual Coxeter number.

Introduction

0.1. Let $\mathfrak{g}$ be a complex finite-dimensional contragredient Lie superalgebra. These algebras were classified by V. Kac [1977] and the list (excluding Lie algebras) consists of four series: $A(m|n)$, $B(m|n)$, $C(m)$, $D(m|n)$ and the exceptional algebras $D(2,1,a)$, $F(4)$, $G(3)$. The finite-dimensional contragredient Lie superalgebras with zero Killing form (or, equivalently, with dual Coxeter number equal to zero) are $A(n|n)$, $D(n|n+1)$ and $D(2,1,a)$.

Denote by Δ_{+0} (resp., Δ_{+1}) the set of positive even (resp., odd) roots of $\mathfrak{g}$. The Weyl denominator R and the affine Weyl denominator $\hat{R}$ are given by the formulas

$$R=\frac{R_0}{R_1},\quad \hat{R}=\frac{\hat{R}_0}{\hat{R}_1},$$

where

$$R_0:=\prod_{\alpha\in\Delta_{+0}}(1-e^{-\alpha}),\quad \hat{R}_0:=R_0\cdot\prod_{k=1}^{\infty}(1-q^k)^{\operatorname{rank}\mathfrak{g}}\prod_{\alpha\in\Delta_0}(1-q^k e^{-\alpha}),$$

$$R_1:=\prod_{\alpha\in\Delta_{+1}}(1+e^{-\alpha}),\quad \hat{R}_1:=R_1\cdot\prod_{k=1}^{\infty}\prod_{\alpha\in\Delta_1}(1+q^k e^{-\alpha}).$$

Let $\hat{\mathfrak{g}}$ be the nontwisted affinization of $\mathfrak{g}$, $\hat{\mathfrak{h}}$ be the Cartan subalgebra of $\hat{\mathfrak{g}}$ and $\hat{\Delta}_+$ be the set of positive roots of $\hat{\mathfrak{g}}$. The affine Weyl denominator is the Weyl denominator of $\hat{\mathfrak{g}}$. Let $\hat{\rho}\in\hat{\mathfrak{h}}$ be such that $2(\hat{\rho},\alpha)=(\alpha,\alpha)$ for each simple root $\alpha\in\hat{\Delta}_+$.

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If $\mathfrak{g}$ has a nonzero Killing form, the affine denominator identity, stated in [Kac and Wakimoto 1994] and proven there and in [Gorelik 2011], takes the form

$$\hat{R}e^{\hat{\rho}}=\sum_{w\in T'}w(Re^{\hat{\rho}}), \tag{1}$$

where T' is the affine translation group corresponding to the *"largest"* root subsystem of Δ_0. The affine denominator identity for strange Lie superalgebras $Q(n)$, which are not contragredient, was stated in [Kac and Wakimoto 1994] and proven in [Zagier 2000].

For a parameter q and a formal variable x we introduce, after [De Sole and Kac 2005], the infinite products

$$(1+x)_q^\infty:=\prod_{k=0}^\infty(1+q^kx)\quad\text{and}\quad(1-x)_q^\infty:=\prod_{k=0}^\infty(1-q^kx).$$

These infinite products converge for any $x\in\mathbb{C}$ if the parameter q is a real number $0<q<1$. In particular, they are well defined for $0<x=q<1$ and $(1\pm q)_q^\infty:=\prod_{n=1}^\infty(1\pm q^n)$.

For $A(n-1|n-1)=\mathfrak{gl}(n|n)$ denote by $\mathfrak{str}$ the restriction of the supertrace to the Cartan subalgebra $\mathfrak{h}\subset\mathfrak{g}$ (thus $\mathfrak{str}\in\mathfrak{h}^*$).

In this paper we will prove the following theorem.

0.2. Theorem. *Let $\mathfrak{g}$ be a complex finite-dimensional contragredient Lie superalgebra with zero Killing form. One has*

$$\begin{aligned}&\hat{R}e^{\hat{\rho}}\cdot f(q,e^{\mathfrak{str}})=\textstyle\sum_{w\in T'}w(Re^{\hat{\rho}}) && \text{for } A(n|n),\\ &\hat{R}e^{\hat{\rho}}\cdot f(q)=\textstyle\sum_{w\in T'}w(Re^{\hat{\rho}}) && \text{for } D(n{+}1|n),\ D(2,1,a),\end{aligned} \tag{2}$$

where T' is the affine translation group corresponding to the "smallest" *root subsystem of Δ_0 (see 0.4 below) and $f(q,e^{\mathfrak{str}})$, $f(q)$ are given by the following formulas*

$$\begin{aligned}&f(q,e^{\mathfrak{str}})=\tfrac{(1-q(-1)^ne^{\mathfrak{str}})_q^\infty\cdot(1-q(-1)^ne^{-\mathfrak{str}})_q^\infty}{((1-q)_q^\infty)^2} && \text{for } \mathfrak{gl}(n|n),\\ &f(q)=\big((1-q)_q^\infty\big)^{-1} && \text{for } D(n{+}1|n).\end{aligned} \tag{3}$$

0.3. The affine denominator identity for $\mathfrak{gl}(2|2)$ was stated by V. Kac and M. Wakimoto [1994] and proven in [Gorelik 2010] (with a proof different from the one presented below).

As pointed by P. Etingof, the terms $f(q,e^{\mathfrak{str}})$, $f(q)$ can be interpreted using "degenerate" cases $n=1$; for example, for $\mathfrak{gl}(1|1)$ we obtain the formula

$$\hat{R}e^{\hat{\rho}}=\frac{((1-q)_q^\infty)^2}{(1+qe^{\mathfrak{str}})_q^\infty\cdot(1+qe^{-\mathfrak{str}})_q^\infty}Re^{\hat{\rho}},$$

which is trivial since $\mathfrak{gl}(1|1)$ has the only positive root $\beta = \mathfrak{str}$, which is odd.

Since $\mathfrak{sl}(n|n) = \{a \in \mathfrak{gl}(n|n) : \mathfrak{str}(a) = 0\}$ and

$$\operatorname{rank} \mathfrak{sl}(n|n) = 2n - 1 = \operatorname{rank} \mathfrak{gl}(n|n) - 1,$$

one has

$$f(q) = \begin{cases} (1-q)_q^\infty & \text{for } \mathfrak{sl}(2n|2n), \\ \dfrac{((1+q)_q^\infty)^2}{(1-q)_q^\infty} & \text{for } \mathfrak{sl}(2n+1|2n+1). \end{cases}$$

The root datum of $D(2, 1, a)$ is the same as the root datum of $D(2|1)$ so the affine denominator identity for $D(2, 1, a)$ is the same as the affine denominator identity for $D(2|1)$.

As it is shown in [Kac and Wakimoto 1994], the evaluation of the affine denominator identity (2) for $A(1|1)$ gives the following Jacobi identity [1829]:

$$\Box(q)^8 = 1 + 16 \sum_{j,k=1}^{\infty} (-1)^{(j+1)k} k^3 q^{jk}, \tag{4}$$

where $\Box(q) = \sum_{j\in\mathbb{Z}} q^{j^2}$ and thus the coefficient of q^m in the power series expansion of $\Box(q)^8$ is the number of representation of a given integer as a sum of 8 squares (taking into the account the order of summands).

0.4. In order to define T' for $A(n|n)$, $D(n+1|n)$ we present the set of even roots in the form $\Delta_0 = \Delta' \amalg \Delta''$, where

$$\begin{array}{ll} \Delta' \cong \Delta'' = A_{n-1} & \text{for } A(n-1|n-1) = \mathfrak{gl}(n|n), \\ \Delta' = C_n, \ \ \Delta'' = D_{n+1} & \text{for } D(n+1|n). \end{array}$$

Let W' be the Weyl group of Δ' and $\hat{W}'$ be the corresponding affine Weyl group. Then $\hat{W}' = W' \ltimes T'$, where T' is a translation group, see [Kac 1990, Chapter VI]. By contrast to Lie superalgebras with nonzero Killing form, for $D(n+1|n)$ the rank of root system Δ' is smaller than the rank of Δ''. It is not possible to change T' to T'' in (1) and in (2) for $D(n+1|n)$, since the sum $\sum_{w\in T''} w(Re^{\hat{\rho}})$ is not well defined if $\Delta' \not\cong \Delta''$ (see Remark 2.1.4).

The key point of our proof of Theorem 0.2 is Proposition 2.3.2, where it is shown that the expansion of $Y := \hat{R}^{-1}e^{-\hat{\rho}} \sum_{w\in T'} w(Re^{\hat{\rho}})$ contains only $\hat{W}$-invariant elements. This implies that $Y = f(q)$ for $\mathfrak{g} = D(n+1|n)$ and $Y = f(q, e^{-\mathfrak{str}})$ for $\mathfrak{gl}(n|n)$. We determine $f(q)$ and $f(q, e^{\mathfrak{str}})$ using suitable evaluations.

1. Preliminaries

One readily sees (for instance, [Gorelik 2011, 1.5]) that $Re^{\hat{\rho}}$ and $\hat{R}e^{\hat{\rho}}$ do not depend on the choice of set of positive roots Δ_+. As a result, in order to prove Theorem 0.2, it is enough to establish the identity (2) for one choice of Δ_+. Similarly, it is enough to establish the identity for one choice of A_{n-1} for $\mathfrak{gl}(n|n)$. In Section 1.1 we describe our choice of the set of positive roots for $\mathfrak{gl}(n|n)$, $D(n+1|n)$. In Section 1.2 we introduce notation for affine Lie superalgebra $\hat{\mathfrak{g}}$. In Section 1.3 we introduce the algebra $\mathcal{R}$ of formal power series in which we expand R and $\hat{R}$.

Note that if the dual Coxeter number of $\mathfrak{g}$ is zero, then

$$\hat{\rho}=\rho=\frac{1}{2}\Big(\sum_{\alpha\in\Delta_{+0}}\alpha-\sum_{\alpha\in\Delta_{+1}}\alpha\Big).$$

1.1. ***Root systems.*** Let $\mathfrak{g}$ be $\mathfrak{gl}(n|n)$ or $D(n|n+1)$ and let $\mathfrak{h}$ be its Cartan subalgebra. We fix the following sets of simple roots:

$$\Pi=\begin{cases}\{\varepsilon_1-\delta_1,\,\delta_1-\varepsilon_2,\,\varepsilon_2-\delta_2,\,\dots,\,\varepsilon_n-\delta_n\} & \text{for } \mathfrak{gl}(n|n),\\ \{\varepsilon_1-\delta_1,\,\delta_1-\varepsilon_2,\,\varepsilon_2-\delta_2,\,\dots,\,\varepsilon_n-\delta_n,\,\delta_n\pm\varepsilon_{n+1}\} & \text{for } D(n+1|n).\end{cases}$$

We fix a nondegenerate symmetric invariant bilinear form on $\mathfrak{g}$ and denote by $(-,-)$ the induced nondegenerate symmetric bilinear form on $\mathfrak{h}^*$; we normalize the form in such a way that $-(\varepsilon_i,\varepsilon_j)=(\delta_i,\delta_j)=\delta_{ij}$; notice that $\{\varepsilon_i,\delta_i : 1\le i\le n\}$ (resp., $\{\varepsilon_j,\delta_i : 1\le i\le n,\, 1\le j\le n+1\}$) is an orthogonal basis of $\mathfrak{h}^*$ for $\mathfrak{gl}(n|n)$ (resp., for $D(n+1|n)$).

For this choice one has

$$\Delta_{0+}=\begin{cases}\{\varepsilon_i-\varepsilon_j\}_{1\le i<j\le n}\amalg\{\delta_i-\delta_j\}_{1\le i<j\le n} & \text{for } \mathfrak{gl}(n|n),\\ \Delta_{0+}=\{\varepsilon_i\pm\varepsilon_j\}_{1\le i<j\le n+1}\amalg\{\delta_s\pm\delta_t\}_{1\le s<t\le n}\cup\{2\delta_s\}_{1\le s\le n} & \text{for } D(n+1|n),\end{cases}$$

$$\Delta_{1+}=\begin{cases}\{\varepsilon_i-\delta_j\}_{1\le i\le j\le n}\cup\{\delta_i-\varepsilon_j\}_{1\le i<j\le n} & \text{for } \mathfrak{gl}(n|n),\\ \Delta_{1+}=\{\varepsilon_i-\delta_s\}_{1\le i\le s\le n}\cup\{\delta_s-\varepsilon_j\}_{1\le s<j\le n+1}\cup\{\delta_i+\varepsilon_j\}_{1\le i\le n;1\le j\le n+1} & \\ & \text{for } D(n+1|n).\end{cases}$$

For $D(n+1|n)$ one has $\rho=0$. For $\mathfrak{gl}(n|n)$ one has $\mathfrak{str}=\sum_{i=1}^{n}(\varepsilon_i-\delta_i)$ and $\rho=-\frac{1}{2}\mathfrak{str}$.

Recall that $\mathfrak{sl}(n|n)=\{a\in\mathfrak{gl}(n|n) : \mathfrak{str}(a)=0\}$ and so $\mathfrak{h}^*$ for $\mathfrak{sl}(n|n)$ is the quotient of $\mathfrak{h}^*$ for $\mathfrak{gl}(n|n)$ by $\mathbb{C}\mathfrak{str}$.

By the above, Δ_0 is the union of two irreducible root systems, and we write $\Delta_0=\Delta''\amalg\Delta'$, where Δ'' lies in the span of the ε_i and Δ' lies in the span of the δ_i (this notation is compatible with the notation in Section 0.4).

1.2. ***Nontwisted affinization.*** Let $\mathfrak{g}=\mathfrak{n}_-\oplus\mathfrak{h}\oplus\mathfrak{n}_+$ be any complex finite-dimensional contragredient Lie superalgebra with a fixed triangular decomposition, and let Δ_+ be its set of positive roots. Let $\hat{\mathfrak{g}}$ be the affinization of $\mathfrak{g}$ and let $\hat{\mathfrak{h}}$ be its Cartan subalgebra, see [Kac 1990, Chapter VI]. Let $\hat{\Delta}=\hat{\Delta}_0\amalg\hat{\Delta}_1$ be the set of roots of $\hat{\mathfrak{g}}$. We set

$$\hat{\Delta}^+=\Delta_+\cup\Big(\bigcup_{k=1}^{\infty}\{\alpha+k\delta \mid \alpha\in\Delta\}\Big)\cup\Big(\bigcup_{k=1}^{\infty}\{k\delta\}\Big),$$

where δ is the minimal imaginary root. Let W and $\hat{W}$ be the Weyl groups of Δ_0 and $\hat{\Delta}_0$. One has $(\hat{\mathfrak{h}}^*)^{\hat{W}} = \mathbb{C}\delta$ for $\mathfrak{g} \neq \mathfrak{gl}(n|n)$ and $(\hat{\mathfrak{h}}^*)^{\hat{W}} = \mathbb{C}\delta \oplus \mathbb{C}\mathfrak{str}$ for $\mathfrak{g} = \mathfrak{gl}(n|n)$.

We extend the nondegenerate symmetric invariant bilinear form from $\mathfrak{g}$ to $\hat{\mathfrak{g}}$ and denote by $(-,-)$ the induced nondegenerate symmetric bilinear form on $\hat{\mathfrak{h}}^*$ (the above-mentioned form on $\mathfrak{h}^*$ is induced by this form on $\hat{\mathfrak{h}}^*$). For $A \subset \hat{\mathfrak{h}}^*$ we set $A^{\perp} = \{\mu \in \hat{\mathfrak{h}}^* : \forall \nu \in A,\ (\mu, \nu) = 0\}$.

1.2.1. In Section 1.1 we introduced the root systems Δ', Δ'' for $\mathfrak{g} = \mathfrak{gl}(n|n)$ and $\mathfrak{g} = D(n+1|n)$. Let W' and W'' be the Weyl groups of Δ' and Δ'', respectively. One has $W = W' \times W''$. We denote by $\hat{W}'$ the Weyl group of the affine root system $\hat{\Delta}'$. Recall that $\hat{W}' = W' \ltimes T'$, where T' is a translation group; see [Kac 1990, Chapter VI].

1.2.2. For $N \subset \hat{\mathfrak{h}}^*$ we use the notation $\mathbb{Z}N$ for the set $\sum_{\mu \in N} \mathbb{Z}\mu$. Set

$$Q^+ := \sum_{\mu \in \Delta_+} \mathbb{Z}_{\geq 0}\mu, \quad Q := \mathbb{Z}\Delta_+, \quad \hat{Q}^{\pm} := \pm \sum_{\mu \in \hat{\Delta}_+} \mathbb{Z}_{\geq 0}\mu, \quad \hat{Q} := \mathbb{Z}\hat{\Delta}_+.$$

We introduce the standard partial order on $\hat{\mathfrak{h}}^*$: $\mu \leq \nu$ if $(\nu - \mu) \in \hat{Q}^+$.

1.3. ***The algebra*** $\mathcal{R}$. We are going to use the notation of [Gorelik 2011, 1.4], which we recall below. We retain the notation of Section 1.2.

1.3.1. Call a $\hat{Q}^+$-*cone* a set of the form $(\lambda - \hat{Q}^+)$, where $\lambda \in \hat{\mathfrak{h}}^*$.

For a formal sum of the form $Y := \sum_{\nu \in \hat{\mathfrak{h}}^*} b_\nu e^\nu$, $b_\nu \in \mathbb{Q}$ define the *support* of Y by $\operatorname{supp}(Y) := \{\nu \in \hat{\mathfrak{h}}^* : b_\nu \neq 0\}$. Let $\mathcal{R}$ be a vector space over $\mathbb{Q}$, spanned by the sums of the form $\sum_{\nu \in \hat{Q}^+} b_\nu e^{\lambda - \nu}$, where $\lambda \in \hat{\mathfrak{h}}^*$, $b_\nu \in \mathbb{Q}$. In other words, $\mathcal{R}$ consists of the formal sums $Y = \sum_{\nu \in \hat{\mathfrak{h}}^*} b_\nu e^\nu$ with the support lying in a finite union of $\hat{Q}^+$-cones.

Clearly, $\mathcal{R}$ has a structure of commutative algebra over $\mathbb{Q}$. If $Y \in \mathcal{R}$ is such that $YY' = 1$ for some $Y' \in \mathcal{R}$, we write $Y^{-1} := Y'$.

1.3.2. *Action of the Weyl group.* For $w \in \hat{W}$ set $w\left(\sum_{\nu \in \hat{\mathfrak{h}}^*} b_\nu e^\nu\right) := \sum_{\nu \in \hat{\mathfrak{h}}^*} b_\nu e^{w\nu}$. By the above, $wY \in \mathcal{R}$ if and only if $w(\operatorname{supp} Y)$ is a subset of a finite union of $\hat{Q}^+$-cones. For each subgroup $\tilde{W}$ of $\hat{W}$ we set $\mathcal{R}_{\tilde{W}} := \{Y \in \mathcal{R} : wY \in \mathcal{R} \text{ for each } w \in \tilde{W}\}$; notice that $\mathcal{R}_{\tilde{W}}$ is a subalgebra of $\mathcal{R}$.

1.3.3. *Infinite products.* An infinite product of the form $Y = \prod_{\nu \in X} (1 + a_\nu e^{-\nu})^{r(\nu)}$, where $a_\nu \in \mathbb{Q}$, $r(\nu) \in \mathbb{Z}_{\geq 0}$ and $X \subset \hat{\Delta}$ is such that the set $X \setminus \hat{\Delta}_+$ is finite, can be naturally viewed as an element of $\mathcal{R}$; clearly, this element does not depend on the order of factors. Let $\mathcal{Y}$ be the set of such infinite products. For any $w \in \hat{W}$ the infinite product

$$wY := \prod_{\nu \in X} (1 + a_\nu e^{-w\nu})^{r(\nu)},$$

is again an infinite product of the above form, since the set $w\hat{\Delta}_+ \setminus \hat{\Delta}_+$ is finite (see for example [Gorelik 2011, Lemma 1.2.8]). Hence $\mathcal{Y}$ is a $\hat{W}$-invariant multiplicative subset of $\mathcal{R}_{\hat{W}}$.

The elements of $\mathcal{Y}$ are invertible in $\mathcal{R}$: using the geometric series we can expand Y^{-1}. For example, $(1-e^{\alpha})^{-1} = -e^{-\alpha}(1-e^{-\alpha})^{-1} = -\sum_{i=1}^{\infty} e^{-i\alpha}$.

1.3.4. *The subalgebra* $\mathcal{R}'$. Denote by $\mathcal{R}'$ the localization of $\mathcal{R}_{\hat{W}}$ by $\mathcal{Y}$. By the above, $\mathcal{R}'$ is a subalgebra of $\mathcal{R}$. Observe that $\mathcal{R}' \not\subset \mathcal{R}_{\hat{W}}$: for example, $(1-e^{-\alpha})^{-1} \in \mathcal{R}'$, but $(1-e^{-\alpha})^{-1} = \sum_{j=0}^{\infty} e^{-j\alpha} \notin \mathcal{R}_{\hat{W}}$. We extend the action of $\hat{W}$ from $\mathcal{R}_{\hat{W}}$ to $\mathcal{R}'$ by setting $w(Y^{-1}Y') := (wY)^{-1}(wY')$ for $Y \in \mathcal{Y}$, $Y' \in \mathcal{R}_{\hat{W}}$.

Notice that an infinite product of the form $Y = \prod_{\nu\in X}(1+a_\nu e^{-\nu})^{r(\nu)}$, where a_ν, X are as above and $r(\nu) \in \mathbb{Z}$, lies in $\mathcal{R}'$ and $wY = \prod_{\nu\in X}(1+a_\nu e^{-w\nu})^{r(\nu)}$. The support $\operatorname{supp}(Y)$ has a unique maximal element (with respect to the standard partial order) and this element is given by the formula

$$\max \operatorname{supp}(Y) = -\sum_{\nu\in X\setminus\hat{\Delta}_+ : a_\nu \neq 0} r_\nu \nu.$$

1.3.5. Let $\tilde{W}$ be a subgroup of $\hat{W}$. For $Y \in \mathcal{R}'$ we say that Y *is $\tilde{W}$-invariant (resp., $\tilde{W}$-anti-invariant)* if $wY = Y$ (resp., $wY = \operatorname{sgn}(w)Y$) for each $w \in \tilde{W}$.

Let $Y = \sum a_\mu e^\mu \in \mathcal{R}_{\tilde{W}}$ be $\tilde{W}$-anti-invariant. Then $a_{w\mu} = (-1)^{\operatorname{sgn}(w)} a_\mu$ for each μ and $w \in \tilde{W}$. In particular, $\tilde{W}\operatorname{supp}(Y) = \operatorname{supp}(Y)$, and, moreover, for each $\mu \in \operatorname{supp}(Y)$ one has $\operatorname{Stab}_{\tilde{W}} \mu \subset \{w \in \tilde{W} : \operatorname{sgn}(w) = 1\}$. The condition $Y \in \mathcal{R}_{\tilde{W}}$ is essential: for example, for $\tilde{W} = \{\mathrm{id}, s_\alpha\}$, the expressions $Y := e^\alpha - e^{-\alpha}$, $Y^{-1} = e^{-\alpha}(1-e^{-2\alpha})^{-1}$ are $\tilde{W}$-anti-invariant, $\operatorname{supp}(Y) = \{\pm\alpha\}$ is s_α-invariant, but $\operatorname{supp}(Y^{-1}) = \{-\alpha, -3\alpha, \ldots\}$ is not s_α-invariant.

For $Y \in \mathcal{R}_{\tilde{W}}$ such that each $\tilde{W}$-orbit in $\hat{\mathfrak{h}}^*$ has a finite intersection with $\operatorname{supp}(Y)$, introduce the sum

$$\mathcal{F}_{\tilde{W}}(Y) := \sum_{w\in\tilde{W}} \operatorname{sgn}(w) wY.$$

This sum is well defined, but does not always belong to $\mathcal{R}$. For $Y = \sum a_\mu e^\mu$ one has $\mathcal{F}_{\tilde{W}}(Y) = \sum b_\mu e^\mu$, where $b_\mu = \sum_{w\in\tilde{W}} \operatorname{sgn}(w) a_{w\mu}$; in particular, $b_\mu = \operatorname{sgn}(w) b_{w\mu}$ for each $w \in \tilde{W}$. One has

$$Y \in \mathcal{R}_{\tilde{W}} \text{ and } \mathcal{F}_{\tilde{W}}(Y) \in \mathcal{R} \implies \begin{cases} \mathcal{F}_{\tilde{W}}(Y) \in \mathcal{R}_{\tilde{W}}, \\ \operatorname{supp}(\mathcal{F}_{\tilde{W}}(Y)) \text{ is } \tilde{W}\text{-stable}, \\ \mathcal{F}_{\tilde{W}}(Y) \text{ is } \tilde{W}\text{-anti-invariant}. \end{cases}$$

We call a vector $\lambda \in \hat{\mathfrak{h}}^*$ *$\tilde{W}$-regular* if $\operatorname{Stab}_{\tilde{W}} \lambda = \{\mathrm{id}\}$, and we say that the orbit $\tilde{W}\lambda$ is $\tilde{W}$-regular if λ is $\tilde{W}$-regular (so the orbit consists of $\tilde{W}$-regular points). If $\tilde{W}$ is an affine Weyl group, then for any $\lambda \in \hat{\mathfrak{h}}^*$ the stabilizer $\operatorname{Stab}_{\tilde{W}} \lambda$ is either trivial or contains a reflection. Thus for $\tilde{W}=\hat{W}',\ \hat{W}''$ one has

$$Y\in\mathcal{R}_{\tilde{W}} \text{ and } \mathcal{F}_{\tilde{W}}(Y)\in\mathcal{R} \implies \operatorname{supp}(\mathcal{F}_{\tilde{W}}(Y)) \text{ is a union of } \tilde{W}\text{-regular orbits.}$$

2. Proof

Unless stated otherwise, $\mathfrak{g}$ is assumed to be one of the algebras $\mathfrak{gl}(n|n)$, $D(n+1|n)$.

As it is pointed out in Section 1, it is enough to establish the denominator identity for a particular choice of Δ_+ and we do this for the choice described in Section 1.1. Recall that the group T' was introduced in Section 1.2.1. The steps of the proof are the following.

- In Section 2.1 we check that the sum $\mathcal{F}_{T'}(Re^{\hat{\rho}})$ is well-defined and belongs to $\mathcal{R}$.
- In Section 2.2 we prove the inclusions

$$\operatorname{supp}(\mathcal{F}_{T'}(Re^{\hat{\rho}})),\ \operatorname{supp}(\hat{R}e^{\hat{\rho}})\subset U, \tag{5}$$

where

$$U:=\{\mu\in\hat{\rho}-\hat{Q}^{+}:(\mu,\mu)=(\hat{\rho},\hat{\rho})\}. \tag{6}$$

We remark that (5) holds for simple contragredient Lie superalgebras with nonzero Killing form; see [Gorelik 2011, 2.4].

- In Section 2.3 we show that if the dual Coxeter number of $\mathfrak{g}$ is zero, then the inclusions (5) imply that $\operatorname{supp}\big(\hat{R}^{-1}e^{-\hat{\rho}}\mathcal{F}_{T'}(Re^{\hat{\rho}})\big)\subset\hat{Q}^{\hat{W}}$. As a result, $\hat{R}^{-1}e^{-\hat{\rho}}\mathcal{F}_{T'}(Re^{\hat{\rho}})$ takes the form $f(q)$ for $\mathfrak{g}\neq\mathfrak{gl}(n|n)$ and $f(q,e^{\mathfrak{str}})$ for $\mathfrak{gl}(n|n)$.
- In Section 2.4 we compute $f(q)$ for $D(n+1|n)$ and $f(q,e^{\mathfrak{str}})$ for $\mathfrak{gl}(n|n)$. This completes the proof of the identities (2).

2.1. In this subsection we show that for $\mathfrak{g}=\mathfrak{gl}(n|n)$, $D(n+1|n)$, the sum $\mathcal{F}_{T'}(Re^{\hat{\rho}})$ is a well-defined element of $\mathcal{R}$. Since $\hat{\rho}=\rho$ is $\hat{W}$-invariant, it is enough to verify that $\mathcal{F}_{T'}(R)$ is a well-defined element of $\mathcal{R}$.

Recall that $T'=\mathbb{Z}\{t_{\delta_i-\delta_{i+1}}\}_{i=1}^{n-1}$ for $\mathfrak{gl}(n|n)$ and $T'=\mathbb{Z}\{t_{\delta_i}\}_{i=1}^{n}$ for $D(n+1|n)$, where

$$t_\mu(\alpha)=\alpha-(\alpha,\mu)\delta\ \text{ for any } \alpha\in\hat{Q}. \tag{7}$$

<u>2.1.1.</u> By Section 1.3.4 one has

$$\max\operatorname{supp}(w(R))=-\sum_{\substack{\alpha\in\Delta_{0+}:\\ w\alpha<0}}w\alpha+\sum_{\substack{\beta\in\Delta_{1+}:\\ w\beta<0}}w\beta.$$

For $w\in T'$ write $w=t_\mu$, where $\mu\in\mathbb{Z}\{\delta_i-\delta_{i+1}\}_{1\le i<n}$ for $\mathfrak{gl}(n|n)$ and $\mu\in\mathbb{Z}\{\delta_i\}_{i=1}^{n}$ for $D(n+1|n)$. From (7) we get

$$\{\beta\in\Delta_{i+}|w\beta<0\}=\{\beta\in\Delta_{i+}|(\beta,\mu)>0\}\ \text{ for } i=0,1.$$

We obtain $\max \operatorname{supp}(t_\mu(R)) = -v(\mu) + (v(\mu), \mu)\delta$, where

$$v(\mu) := \sum_{\substack{\beta\in\Delta_{0+}:\\ (\beta,\mu)>0}} \beta - \sum_{\substack{\beta\in\Delta_{1+}:\\ (\beta,\mu)>0}} \beta.$$

In order to prove that $\mathcal{F}_{T'}(R)$ is a well-defined element of $\mathcal{R}$ we verify that

$$\begin{aligned} &\text{(i)} \quad (v(\mu), \mu) \leq 0 && \text{for all } \mu; \\ &\text{(ii)} \quad \{\mu : (v(\mu), \mu) \geq -N\} \text{ is finite} && \text{for all } N > 0. \end{aligned} \tag{8}$$

Condition (ii) ensures that the sum $\mathcal{F}_{T'}(R) = \sum_\mu t_\mu(R)$ is well-defined and condition (i) means that for each μ one has

$$\max \operatorname{supp}(t_\mu(R)) = -v(\mu) \leq \sum_{\beta\in\Delta_{1+}} \beta$$

so $\operatorname{supp}\bigl(\mathcal{F}_{T'}(R)\bigr) \subset \sum_{\beta\in\Delta_{1+}} \beta - \hat{Q}^+$ and thus $\mathcal{F}_{T'}(R) \in \mathcal{R}$.

2.1.2. *Case* $\mathfrak{gl}(n|n)$. Recall that $w \in T'$ has the form $w = t_\mu$, $\mu = \sum_{i=1}^n k_i\delta_i$, where the k_is are integers and $\sum_{i=1}^n k_i = 0$. One has

$$\{\alpha \in \Delta_{+0} : (\alpha, \mu) > 0\} = \{\delta_i - \delta_j : i < j,\ k_i > k_j\},$$
$$\{\alpha \in \Delta_{+1} : (\alpha, \mu) > 0\} = \{\varepsilon_i - \delta_j : k_j < 0,\ i \leq j\} \cup \{\delta_i - \varepsilon_j : k_i > 0,\ i < j\},$$

where $1 \leq i, j \leq n$.

Write $v(\mu) = v' + v''$, where $v' = \sum_{i=1}^n a_i\delta_i$ and v'' lies in the span of the ε_i. By the above, for $k_i > 0$ one has $a_i \leq (n-i) - (n-i) = 0$ and for $k_j < 0$ one has $a_j \geq -(j-1) + j = 1$. Therefore

$$(v(\mu), \mu) = \sum_{i=1}^n a_ik_i \leq \sum_{k_i<0} k_i \leq 0$$

and the set $\{\mu : (v(\mu), \mu) \geq -N\}$ is a subset of the set $\{\mu : \sum_{k_i<0} k_i \geq -N\}$, which is finite for any N, because the k_i are integers and $\sum_{i=1}^n k_i = 0$. This establishes conditions (8).

2.1.3. *Case* $D(n+1|n)$. Recall that $w \in T'$ has the form $w = t_\mu$, $\mu = \sum k_i\delta_i$, where the k_is are integers. One has

$$\{\alpha \in \Delta_{+0} : (\alpha, \mu) > 0\} = \{\delta_i - \delta_j : i < j, k_i > k_j\} \cup \{\delta_i + \delta_j : i \neq j,\ k_i + k_j > 0\} \cup \{2\delta_i : k_i > 0\},$$
$$\{\alpha \in \Delta_{+1} : (\alpha, \mu) > 0\} = \{\varepsilon_s - \delta_j : k_j < 0, s \leq j\} \cup \{\delta_i - \varepsilon_s : k_i > 0, i < s\} \cup \{\delta_i + \varepsilon_s : k_i > 0\},$$

where $1 \leq i, j \leq n$ and $1 \leq s \leq n+1$.

Write $v(\mu) = v' + v''$, where $v' = \sum_{i=1}^{n} a_i \delta_i$ and v'' lies in the span of the ε_i. By the above, for $k_i > 0$ one has $a_i \le (2n+1-i) - (2n+2-i) = -1$ and for $k_j < 0$ one has $a_j \ge -(j-1) + j = 1$. Therefore

$$(v(\mu), \mu) = \sum_{i=1}^{n} a_i k_i \le -\sum_{k_i>0} k_i + \sum_{k_j<0} k_j = -\sum_{1=1}^{n} |k_i| \le 0,$$

so the set $\{\mu : (v(\mu), \mu) \ge -N\}$ is a subset of $\{\mu : \sum_{i=1}^{n} |k_i| \le N\}$, which is finite for any N. This establishes the conditions (8).

2.1.4. **Remark.** For $\mathfrak{gl}(n|n)$ one can interchange Δ' and Δ'' so the sum $\mathcal{F}_{T''}(R)$ is well-defined. One readily sees that $\mathcal{F}_{T''}(R)$ is not well-defined for $D(n+1|n)$. For instance, for $n > 1$, for each $k > 0$ one has $v(-2k\varepsilon_1) = 0$ so $\max \operatorname{supp}\big(t_{-2k\varepsilon_1}(R)\big) = 0$ and the sum $\sum_{k=1}^{\infty} t_{-2k\varepsilon_1}(R)$ is not well-defined; hence $\mathcal{F}_{T''}(R)$ is not well-defined as well.

2.2. By Section 1.3.3, $\hat{R}$ is an invertible element of $\mathcal{R}'$. From representation theory we know that since $\hat{\mathfrak{g}}$ admits a Casimir element [Kac 1990, Chapter II], the character of the trivial $\hat{\mathfrak{g}}$-module is a linear combination of the characters of Verma $\hat{\mathfrak{g}}$-modules $M(\lambda)$, where $\lambda \in -\hat{Q}$ are such that $(\lambda + \hat{\rho}, \lambda + \hat{\rho}) = (\hat{\rho}, \hat{\rho})$. Since the character of $M(\lambda)$ is equal to $\hat{R}^{-1} e^{\lambda}$, we obtain

$$1 = \sum_{\substack{\lambda \in \hat{Q}^- \\ (\lambda+\hat{\rho}, \lambda+\hat{\rho}) = (\hat{\rho}, \hat{\rho})}} a_\lambda \hat{R}^{-1} e^{\lambda},$$

where $a_\lambda \in \mathbb{Z}$. This can be rewritten as

$$\hat{R} e^{\hat{\rho}} = \sum_{\substack{\lambda \in \hat{\rho} - \hat{Q}^+ \\ (\lambda, \lambda) = (\hat{\rho}, \hat{\rho})}} a_\lambda e^{\lambda},$$

that is $\operatorname{supp}(\hat{R}) \subset U$, see (6) for notation.

It remains to verify the inclusion $\operatorname{supp}\big(\mathcal{F}_{T'}(Re^{\hat{\rho}})\big) \subset U$. The denominator identity for $\mathfrak{g}$ (see [Kac and Wakimoto 1994; Gorelik 2012]) takes the form

$$Re^{\rho} = \mathcal{F}_{W''}\left(\frac{e^{\rho}}{\prod_{\beta \in S}(1 + e^{-\beta})}\right),$$

where $S := \{\varepsilon_i - \delta_i\}_{i=1}^{n}$ (the identity for $\mathfrak{gl}(n|n)$ immediately follows from the identity for $\mathfrak{sl}(n|n)$). Since $\rho = \hat{\rho}$ is $\hat{W}$-invariant, this implies

$$t_\mu(Re^{\hat{\rho}}) = e^{\hat{\rho}} \sum_{w \in W''} \operatorname{sgn}(w) \prod_{\beta \in S} (1 + e^{-t_\mu w \beta})^{-1}.$$

For each $t_\mu \in T'$ and $w \in W''$ one has

$$\operatorname{supp}\Big(\prod_{\beta\in S}(1+e^{-t_\mu w\beta})^{-1}\Big) \subset V, \text{ where } V := \mathbb{Z}\{t_\mu w\beta : \beta \in S\} \cap \hat{Q}^-.$$

Since $(t_\mu w\beta, t_\mu w\beta') = (\beta, \beta') = (t_\mu w\beta, \hat{\rho}) = (\hat{\rho}, \beta) = 0$ for any $\beta, \beta' \in S$, one has $(V, V) = (V, \hat{\rho}) = 0$. Therefore $V + \hat{\rho} \subset U$ so $\operatorname{supp}\big(t_\mu(Re^{\hat{\rho}})\big) \subset U$ for each μ. This establishes the required inclusion $\operatorname{supp}\big(\mathcal{F}_{T'}(Re^{\hat{\rho}})\big) \subset U$ and completes the proof of (5).

2.3. Let us deduce from (5) that the support of $\hat{R}^{-1}e^{\hat{\rho}} \cdot \mathcal{F}_{T'}(Re^{\hat{\rho}})$ consists of $\hat{W}$-invariant elements of $\hat{Q}^-$. We do this in two steps: first, proving Lemma 2.3.1, which is valid for any simple contragredient Lie superalgebra and for $\mathfrak{gl}(n|n)$, and then, proving Proposition 2.3.2, which uses the fact that $\hat{\rho} = \rho$ for $\mathfrak{g}$ (this is equivalent to the fact that the dual Coxeter number is zero).

The affine root system $\hat{\Delta}'$ is a subsystem of $\hat{\Delta}_0$. Set $\hat{\Delta}'_+ = \hat{\Delta}' \cap \hat{\Delta}_+$ and let $\hat{\Pi}'$ be the corresponding set of simple roots. Fix $\hat{\rho}' \in \hat{\mathfrak{h}}^*$ such that $2(\hat{\rho}', \alpha) = (\alpha, \alpha)$ for each $\alpha \in \hat{\Pi}'$.

2.3.1. **Lemma.** *The term $\hat{R}_1 e^{\hat{\rho}'-\hat{\rho}} \cdot \mathcal{F}_{T'}(Re^{\hat{\rho}})$ is a $\hat{W}'$-anti-invariant element of $\mathcal{R}_{\hat{W}'}$.*

Proof. By Section 2.1.1, $\mathcal{F}_{T'}(Re^{\hat{\rho}}) \in \mathcal{R}$ and thus $\hat{R}_1 e^{\hat{\rho}'-\hat{\rho}} \cdot \mathcal{F}_{T'}(Re^{\hat{\rho}}) \in \mathcal{R}$.

Let R'_0, R''_0 be the Weyl denominators for Δ', Δ'' (i.e., $R'_0 = \prod_{\alpha\in\Delta'_+}(1-e^{-\alpha})$). Notice that $R''_0 e^{\hat{\rho}}/R_1 \in \mathcal{R}'$ so $w\big(R''_0 e^{\hat{\rho}}/R_1\big)$ is well-defined. Below we will show that the sum $\mathcal{F}_{\hat{W}'}\big(R''_0 e^{\hat{\rho}}/R_1\big)$ is a well-defined element of $\mathcal{R}$ and will establish the following formula

$$\mathcal{F}_{T'}(Re^{\hat{\rho}}) = \mathcal{F}_{\hat{W}'}\bigg(\frac{R''_0 e^{\hat{\rho}}}{R_1}\bigg). \tag{9}$$

It is easy to see that $\hat{R}_0 e^{\hat{\rho}'}$, $\hat{R}e^{\hat{\rho}}$ are $\hat{W}'$-anti-invariant elements of $\mathcal{R}'$ (see, for instance, [Gorelik 2011, 1.5.1]). Since $\hat{R}_1 e^{\hat{\rho}'-\hat{\rho}} \in \mathcal{R}'$ and $\hat{R}_1 e^{\hat{\rho}'-\hat{\rho}} \cdot \hat{R}e^{\hat{\rho}} = \hat{R}_0 e^{\hat{\rho}'}$, we conclude that $\hat{R}_1 e^{\hat{\rho}'-\hat{\rho}}$ is a $\hat{W}'$-invariant element of $\mathcal{R}'$. However, by Section 1.3.3, $\hat{R}_1 \in \mathcal{R}_{\hat{W}}$, and thus $\hat{R}_1 e^{\hat{\rho}'-\hat{\rho}}$ is a $\hat{W}'$-invariant element of $\mathcal{R}_{\hat{W}}$. Multiplying both sides of formula (9) by $\hat{R}_1 e^{\hat{\rho}'-\hat{\rho}}$ we obtain

$$\hat{R}_1 e^{\hat{\rho}'-\hat{\rho}} \cdot \mathcal{F}_{T'}(Re^{\hat{\rho}}) = \mathcal{F}_{\hat{W}'}\bigg(\frac{\hat{R}_1}{R_1} \cdot R''_0 e^{\hat{\rho}'}\bigg). \tag{10}$$

By Section 1.3.3, $\hat{R}_1/R_1$ and R''_0 lie in $\mathcal{R}_{\hat{W}}$. In the light of Section 1.3.5, the formula (10) implies the assertion of the lemma.

Let us show that the right-hand side of (9) is well-defined. Since R''_0 and $\hat{\rho}$ are $\hat{W}'$-invariant, it is enough to check that $\mathcal{F}_{\hat{W}'}(R_1^{-1})$ is a well-defined element of $\mathcal{R}$.

By Section 1.3.4, for each $w \in \hat{W}'$ one has

$$\max \operatorname{supp}\big(w(R_1^{-1})\big) = \sum_{\substack{\beta\in\Delta_{1+}:\\ w\beta<0}} w\beta.$$

In particular, $\operatorname{supp}\big(w(R_1^{-1})\big) \subset \hat{Q}^-$, so, if the term $\mathcal{F}_{\hat{W}'}(R_1^{-1})$ is well-defined, it lies in $\mathcal{R}$. In order to see that $\mathcal{F}_{\hat{W}'}(R_1^{-1})$ is well-defined let us check that for each $\nu \in \hat{Q}^-$ the set

$$X(\nu) := \Big\{ w \in \hat{W}' : \sum_{\substack{\beta\in\Delta_{1+}:\\ w\beta<0}} w\beta \geq \nu \Big\}$$

is finite. One has

$$X(\nu) \subset \{w \in \hat{W}' : w\beta \geq \nu \text{ for all } \beta \in \Delta_{1+}\}.$$

Write $\nu = -k\delta + \nu'$, where $k \geq 0$, $\nu' \in Q$, and write $w \in X(\nu)$ in the form $w = t_\mu y$, where $t_\mu \in T'$, $y \in W'$. Since $w\beta = y\beta - (y\beta, \mu)\delta$ for $\beta \in \Delta_{1+}$, one has $(y\beta, \mu) \geq -k$ for each $\beta \in \Delta_{1+}$. Since $\{\varepsilon_i - \delta_i, \delta_i - \varepsilon_{i+1}\} \subset \Delta_{1+}$, this gives $|(\mu, y\delta_i)| \leq k$ for $i = 1, \ldots, n$. Combining the facts that W' is a subgroup of signed permutation of $\{\delta_j\}_{j=1}^n$ and that (μ, δ_i) is integral for each i, we conclude that $X(\nu)$ is finite. Thus $\mathcal{F}_{\hat{W}'}\big(R_0'' e^{\hat{\rho}}/R_1\big)$ is a well-defined element of $\mathcal{R}$.

Now let us prove the formula (9). Recall that $\rho = \rho_0' + \rho_0'' - \rho_1$, where

$$\rho_0' := \sum_{\alpha\in\Delta_{0+}'} \alpha/2, \quad \rho_0'' := \sum_{\alpha\in\Delta_{0+}''} \alpha/2, \quad \rho_1 := \sum_{\beta\in\Delta_{1+}} \beta/2.$$

The Weyl denominator identity for Δ_0' takes the form

$$R_0' e^{\rho_0'} = \mathcal{F}_{W'}(e^{\rho_0'}).$$

Since $R_1 e^{\rho_1} = \prod_{\beta\in\Delta_{1+}} (e^{\beta/2} + e^{-\beta/2})$ is W-invariant and $R_0'' e^{\rho_0''}$ is W'-invariant, we get

$$Re^\rho = \frac{R_0'' e^{\rho_0''}}{R_1 e^{\rho_1}} \cdot \mathcal{F}_{W'}(e^{\rho_0'}) = \mathcal{F}_{W'}\bigg(\frac{e^{\rho_0'} R_0'' e^{\rho_0''}}{R_1 e^{\rho_1}}\bigg) = \mathcal{F}_{W'}\bigg(\frac{R_0'' e^\rho}{R_1}\bigg).$$

Using the W-invariance of $\hat{\rho} - \rho$, we obtain

$$\mathcal{F}_{T'}\big(Re^{\hat{\rho}}\big) = \mathcal{F}_{T'}\bigg(\mathcal{F}_{W'}\bigg(\frac{R_0'' e^{\hat{\rho}}}{R_1}\bigg)\bigg) = \mathcal{F}_{\hat{W}'}\bigg(\frac{R_0'' e^{\hat{\rho}}}{R_1}\bigg)$$

as required. This completes the proof. □

2.3.2. **Proposition.** *One has*

$$\operatorname{supp}(\hat{R}^{-1}e^{-\hat{\rho}}\cdot \mathcal{F}_{T'}(Re^{\hat{\rho}}))\subset (\hat{Q}^-)^{\hat{W}}=\hat{Q}^-\cap\hat{Q}^{\perp}.$$

Proof. Set

$$Y:=\hat{R}^{-1}e^{-\hat{\rho}}\cdot\mathcal{F}_{T'}(Re^{\hat{\rho}}).$$

By Sections 2.1.1 and 1.3.3, $\mathcal{F}_{T'}(Re^{\hat{\rho}})$, $\hat{R}^{-1}\in\mathcal{R}$. Thus $Y\in\mathcal{R}$. One has

$$\hat{R}_0e^{\hat{\rho}'}Y=\hat{R}_1e^{\hat{\rho}'-\hat{\rho}}\cdot\mathcal{F}_{T'}(Re^{\hat{\rho}}).$$

In the light of Lemma 2.3.1, we obtain

$$\hat{R}_0e^{\hat{\rho}'}Y \ \text{ is a } \hat{W}'\text{-anti-invariant element of } \mathcal{R}_{\hat{W}'}. \tag{11}$$

Write $Y=Y_1+Y_2$, where $\operatorname{supp}(Y_1)=\operatorname{supp}(Y)\cap\hat{Q}^{\perp}$ and $\operatorname{supp}(Y_2)=\operatorname{supp}(Y)\setminus\hat{Q}^{\perp}$. Note that $Y_1, Y_2\in\mathcal{R}$. Assume that $Y_2\neq 0$. Let μ be a maximal element in $\operatorname{supp}(Y_2)$. One has $\operatorname{supp}(\hat{R}^{-1})\subset\hat{Q}^-$ and $\operatorname{supp}\big(\mathcal{F}_{T'}(R)e^{\hat{\rho}}\big)\subset\hat{\rho}-\hat{Q}^+$, by Section 1.3.4 and (5) respectively. Thus $\operatorname{supp}(Y)\subset\hat{Q}^-$ and so $\mu\in\hat{Q}^-$.

Since $\operatorname{supp}(Y_1)\subset\hat{Q}^{\perp}$, Y_1 is a $\hat{W}$-invariant element of $\mathcal{R}_{\hat{W}}$. Recall that $\hat{R}_0e^{\hat{\rho}'}$ is a $\hat{W}'$-anti-invariant element of $\mathcal{R}_{\hat{W}}$. Thus $\hat{R}_0e^{\hat{\rho}'}Y_1$ is a $\hat{W}'$-anti-invariant element of $\mathcal{R}_{\hat{W}'}$. In the light of (11), the product $\hat{R}_0e^{\hat{\rho}'}Y_2$ is also a $\hat{W}'$-anti-invariant element of $\mathcal{R}_{\hat{W}'}$. Clearly, $\hat{\rho}'+\mu$ is a maximal element in the support of $\hat{R}_0e^{\hat{\rho}'}Y_2$. By Section 1.3.5, this support is a union of $\hat{W}'$-regular orbits (recall that regularity means that each element has the trivial stabilizer in $\hat{W}'$), so $\hat{\rho}'+\mu$ is a maximal element in a regular $\hat{W}'$-orbit and thus $2(\hat{\rho}'+\mu,\alpha)/(\alpha,\alpha)\notin\mathbb{Z}_{\leq 0}$ for each $\alpha\in\hat{\Pi}'$. Since $\mu\in\hat{Q}^-$ one has $2(\mu,\alpha)/(\alpha,\alpha)\in\mathbb{Z}$ for each $\alpha\in\hat{\Pi}'$. Taking into account that $2(\hat{\rho}',\alpha)/(\alpha,\alpha)=1$ for each $\alpha\in\hat{\Pi}'$, we obtain

$$\frac{2(\mu,\alpha)}{(\alpha,\alpha)}\in\mathbb{Z}_{\geq 0}\quad\text{for all }\ \alpha\in\hat{\Pi}'. \tag{12}$$

Recall that $\delta=\sum_{\alpha\in\hat{\Pi}'}k_\alpha\alpha$ for some $k_\alpha\in\mathbb{Z}_{>0}$ (see [Kac 1990, Chapter VI]). Since $\mu\in\hat{Q}^-$ one has $(\mu,\delta)=0$. Combining with (12), we get $(\mu,\alpha)=0$ for each $\alpha\in\hat{\Pi}'$ so $\mu\in(\hat{\Delta}')^{\perp}$.

Let us show that $(\mu,\mu)=0$. Since $(\hat{\rho},\hat{Q})=0$, it is equivalent to the equality $(\mu+\hat{\rho},\mu+\hat{\rho})=(\hat{\rho},\hat{\rho})$. Notice that $\mu+\hat{\rho}$ is a maximal element in the support of $\hat{R}e^{\hat{\rho}}Y_2$. Let us check that

$$\operatorname{supp}(\hat{R}e^{\hat{\rho}}Y_2)\subset U=\{\xi\in\hat{\rho}-\hat{Q}^+ : (\xi,\xi)=(\hat{\rho},\hat{\rho})\}. \tag{13}$$

Indeed,

$$\hat{R}e^{\hat{\rho}}Y_2=\mathcal{F}_{T'}(Re^{\hat{\rho}})-\hat{R}e^{\hat{\rho}}Y_1$$

and, by (5),

$$\operatorname{supp}\big(\mathcal{F}_{T'}(Re^{\hat{\rho}})\big)\subset U\quad\text{and}\quad\operatorname{supp}\big(\hat{R}e^{\hat{\rho}}\big)\subset U.$$

By construction, $\operatorname{supp}(Y_1) \subset \hat{Q}^{\perp} \cap \hat{Q}^{-}$. Recall that $\hat{\rho} = \rho \in \mathbb{Q}\Delta$, so $U \subset \mathbb{Q} \cdot \hat{Q}$. In particular, we have $(U, \operatorname{supp}(Y_1)) = 0$. Since $(\operatorname{supp}(Y_1), \operatorname{supp}(Y_1)) = 0$, we obtain $(\operatorname{supp}(Y_1) + U) \subset U$ and this establishes the inclusion (13). Hence $(\mu, \mu) = 0$.

Recall that $\mu \in (\hat{\Delta}')^{\perp} \cap \hat{Q}^{-}$. One has

$$(\hat{\Delta}')^{\perp} \cap \hat{Q} = (\hat{Q}^{\perp} \cap \hat{Q}) \oplus \mathbb{Z}\Delta''.$$

For every $\beta \in \hat{Q}^{\perp} \cap \hat{Q}$, $\gamma \in \Delta''$ one has $(\beta, \beta) = (\beta, \gamma) = 0$ and $(\gamma, \gamma) \neq 0$ if $\gamma \neq 0$. Using the equality $(\mu, \mu) = 0$, we get $\mu \in \hat{Q}^{\perp} \cap \hat{Q}$, which contradicts to the construction of Y_2. Hence $Y_2 = 0$ as required. □

2.3.3. **Corollary.** *For $\mathfrak{g} = D(n+1|n)$ one has $f(q) \cdot \hat{R}e^{\hat{\rho}} = \mathcal{F}_{T'}(Re^{\hat{\rho}})$ for some $f(q) = \sum_{k=0}^{\infty} a_k q^k$ $(a_k \in \mathbb{Z})$. For $\mathfrak{g} = \mathfrak{gl}(n|n)$ one has $f(q, e^{\mathfrak{str}}) \cdot \hat{R}e^{\hat{\rho}} = \mathcal{F}_{T'}(Re^{\hat{\rho}})$ for some $f(q, e^{\mathfrak{str}}) = \sum_{k=0}^{\infty}\sum_{m=-\infty}^{\infty} a_{k,m} q^k e^{m\cdot\mathfrak{str}}$ $(a_{k,m} \in \mathbb{Z})$.*

Proof. One has $(\hat{Q})^{\perp} \cap \hat{Q} = \mathbb{Z}\delta + \mathbb{Z}\mathfrak{str}$ for $\mathfrak{gl}(n|n)$ and $(\hat{Q})^{\perp} \cap \hat{Q} = \mathbb{Z}\delta$ for $D(n+1|n)$. □

2.4. In this subsection we complete the proof of the denominator identities (2) by proving the formulas (3). We prove them by taking a suitable evaluation of the term $\hat{R}^{-1}e^{-\hat{\rho}}\mathcal{F}_{T'}(Re^{\hat{\rho}})$. Since $\hat{\rho}$ is $\hat{W}$-invariant, this term is equal to $\hat{R}^{-1}\mathcal{F}_{T'}(R)$, and, by Corollary 2.3.3, it is equal to $f(q)$ for $D(n+1|n)$ and to $f(q, e^{\mathfrak{str}})$ for $\mathfrak{gl}(n|n)$. Now we consider q as a real parameter between 0 and 1. We choose the evaluation in such a way that the evaluation of $\hat{R}^{-1}\mathcal{F}_{T'}(R) = \hat{R}^{-1}\sum_{t \in T'} t(R)$ is equal to the evaluation of $\hat{R}^{-1}R$. As a result, $f(q)$ (resp., $f(q, e^{\mathfrak{str}})$) is equal to the evaluation of $\hat{R}^{-1}R$, which can be easily computed.

2.4.1. *Case $D(n+1|n)$.* Take a complex parameter x and consider the evaluation $e^{-\varepsilon_i} := x^{a_i}$, $e^{-\delta_j} := -x^{b_j}$, where a_i $(i = 1, \ldots, n+1)$ and b_j $(j = 1, \ldots, n)$ are integers such that $a_i \pm b_j \neq 0$, $a_i \pm a_j \neq 0$, $b_i \pm b_j \neq 0$, $b_i \neq 0$ for all indexes i, j. We denote by $\hat{R}$ and $\hat{R}(x)$ the evaluation of R and $R(x)$. The functions $R(x)$ and $\hat{R}(x)$ are meromorphic. One has

$$R(x) = \frac{\prod_{1 \le i < j \le n+1}(1 - x^{a_i \pm a_j}) \cdot \prod_{1 \le i < j \le n}(1 - x^{b_i \pm b_j}) \cdot \prod_{1 \le i \le n}(1 - x^{2b_i})}{\prod_{1 \le i \le j \le n}(1 - x^{a_i \pm b_j}) \prod_{1 \le j < i \le n+1}(1 - x^{b_j \pm a_i})}.$$

One readily sees that $R(x)$ has a pole at $x = 1$ of order $|\Delta_{1+}| - |\Delta_{0+}| = n$.

One has

$$\left.\frac{\hat{R}(x)}{R(x)}\right|_{x=1} = \frac{((1-q)_q^{\infty})^{\dim \mathfrak{g}_0}}{((1-q)_q^{\infty})^{\dim \mathfrak{g}_1}} = ((1-q)_q^{\infty})^{\dim \mathfrak{g}_0 - \dim \mathfrak{g}_1} = (1-q)_q^{\infty}.$$

In particular, $\hat{R}(x)$ also has a pole of order n at $x = 1$.

The evaluation of $(t_{\sum k_i\delta_i}(R))(x)$ is

$$\frac{\prod_{1\leq i<j\leq n+1}(1-x^{a_i\pm a_j})\cdot\prod_{1\leq i\leq n}(1-q^{-2k_i}x^{2b_i})\cdot\prod_{1\leq i<j\leq n}(1-q^{-k_i\mp k_j}x^{b_i\pm b_j})}{\prod_{1\leq i\leq j\leq n}(1-q^{\mp k_j}x^{a_i\pm b_j})\prod_{1\leq j<i\leq n+1}(1-q^{-k_j}x^{b_j\pm a_i})}$$

which is a meromorphic function. Let s be the number of zeros among $k_1,\dots,k_n$. Then at $x=1$ the order of zero of the numerator is at least is $n(n+1)+s^2$, and the order of zero of the denominator is $2(n+1)s$. Therefore at $x=1$ the function $(t_{\sum k_i\delta_i}(R))(x)$ has the pole of order at most $2(n+1)s-n(n+1)-s^2=n+1-(n+1-s)^2$; in particular, $(t_{\sum k_i\delta_i}(R))(x)$ has the pole of order at most n and it is equal to n if and only if $n=s$ that is $\sum k_i\delta_i=0$ and $(t_{\sum k_i\delta_i}(R))(x)=R(x)$.

We conclude that

$$(\hat{R}(x))^{-1}\cdot\sum_{t\in T':t\neq\mathrm{id}}(t(R))(x)$$

is holomorphic at $x=1$ and its value is zero, and that

$$(\hat{R}(x))^{-1}\cdot\sum_{t\in T'}(t(R))(x)$$

is holomorphic at $x=1$ and its value is $\left.\frac{R(x)}{\hat{R}(x)}\right|_{x=1}$. In the light of Corollary 2.3.3 we obtain

$$f(q)=\left.\frac{R(x)}{\hat{R}(x)}\right|_{x=1}=((1-q)_q^{\infty})^{-1}.$$

2.4.2. *Case* $\mathfrak{gl}(n|n)$. Fix $y>1$. Take a complex parameter x and consider the following evaluation

$$e^{-\varepsilon_1}:=y,\ e^{-\varepsilon_i}:=x^i,\ \text{for } i=2,\dots,n;\ e^{-\delta_i}:=-x^{-i}\ \text{for } i=1,\dots,n.$$

The functions $R(x)$, $\hat{R}(x)$ are meromorphic. One has

$$R(x)=\frac{\prod_{1<i\leq n}(1-yx^{-i})\cdot\prod_{1<i<j\leq n}(1-x^{i-j})\cdot\prod_{1\leq i<j\leq n}(1-x^{j-i})}{\prod_{1\leq i\leq n}(1-yx^{i})\cdot\prod_{1<i\leq j\leq n}(1-x^{i+j})\cdot\prod_{1\leq j<i\leq n}(1-x^{-i-j})}.$$

Therefore the function $R(x)$ has a pole of order $n-1$ at $x=1$.

One has

$$\left.\frac{\hat{R}(x)}{R(x)}\right|_{x=1}=\frac{((1-q)_q^{\infty})^{\dim\mathfrak{g}_0-2(n-1)}\cdot((1-qy)_q^{\infty})^{n-1}\cdot((1-qy^{-1})_q^{\infty})^{n-1}}{((1-q)_q^{\infty})^{\dim\mathfrak{g}_1-2n}\cdot((1-qy)_q^{\infty})^{n}\cdot((1-qy^{-1})_q^{\infty})^{n}}.$$

Thus $\hat{R}(x)$ also has a pole of order $n-1$ at $x=1$. Since $\dim\mathfrak{g}_0=\dim\mathfrak{g}_1$ and $e^{\mathfrak{str}}=(-1)^n y^{-1}$ for $x=1$ we obtain

$$\left.\frac{\hat{R}(x)}{R(x)}\right|_{x=1}=\frac{((1-q)_q^{\infty})^2}{(1-q(-1)^n e^{\mathfrak{str}})_q^{\infty}\cdot(1-q(-1)^n e^{-\mathfrak{str}})_q^{\infty}}.$$

One has

$$(t_{\sum k_i\delta_i}(R))(x,y) = \frac{\prod_{1<i\le n}(1-yx^{-i})\cdot\prod_{1<i<j\le n}(1-x^{i-j})\cdot\prod_{1\le i<j\le n}(1-q^{k_j-k_i}x^{j-i})}{\prod_{1\le i\le n}(1-q^{k_i}yx^i)\cdot\prod_{1<i\le j\le n}(1-q^{k_j}x^{i+j})\cdot\prod_{1\le j<i\le n}(1-q^{-k_j}x^{-i-j})},$$

which is a meromorphic function.

Let s be the number of zeros among $k_1,\dots,k_n$. Then at $x=1$ the order of zero of the numerator is at least

$$\frac{(n-1)(n-2)+s(s-1)}{2},$$

and the order of zero of the denominator is $(n-1)s$. Therefore at $x=1$ the function $(t_{\sum k_i\delta_i}(R))(x,y)$ has a pole of order at most

$$(n-1)s-\frac{(n-1)(n-2)+s(s-1)}{2}=\frac{3n-s-2-(n-s)^2}{2},$$

so the order is at most $n-1$ and it is equal to $n-1$ if and only if $s=n-1,n$. Notice that $s\neq n-1$, since $\sum k_i=0$. Therefore the pole has order $n-1$ if and only if $\sum k_i\delta_i=0$.

We conclude that the function $(\hat{R}(x))^{-1}(\mathcal{F}_{T'}(R))(x)$ is holomorphic at $x=1$ and its value is $(R(x)/\hat{R}(x))\big|_{x=1}$. Using Corollary 2.3.3 we obtain

$$f(q,e^{\mathfrak{str}})=\frac{R(x)}{\hat{R}(x)}\bigg|_{x=1}=\frac{(1-q(-1)^n e^{\mathfrak{str}})_q^\infty\cdot(1-q(-1)^n e^{-\mathfrak{str}})_q^\infty}{((1-q)_q^\infty)^2}.$$

3. Other forms of denominator identity

Recall that the denominator identity for a basic Lie superalgebra can be written in the form

$$Re^\rho=\mathcal{F}_{W^\sharp}\bigg(\frac{e^\rho}{\prod_{\beta\in S}(1+e^{-\beta})}\bigg), \tag{14}$$

where $S\subset\Pi$ is the maximal isotropic system, and $W^\sharp$ is the Weyl group of the "largest" root subsystem of Δ_0 ($\Delta_0=\Delta'\amalg\Delta''$), see [Kac and Wakimoto 1994; Gorelik 2012]; in particular, $W^\sharp:=W''$ for $\mathfrak{g}=D(n{+}1|n)$, and $W^\sharp:=W'$ or $W^\sharp:=W''$ for $\mathfrak{g}=\mathfrak{gl}(n|n)$.

If the dual Coxeter number of $\mathfrak{g}$ is nonzero the affine denominator identity for $\mathfrak{g}$ can be written in the form

$$\hat{R}e^{\hat{\rho}}=\mathcal{F}_{\hat{W}^\sharp}\bigg(\frac{e^{\hat{\rho}}}{\prod_{\beta\in S}(1+e^{-\beta})}\bigg),$$

see [Gorelik 2012, 2.1]. In this section we will show that for $\mathfrak{gl}(n|n)$ the denominator identity can be written in a similar form:

$$\hat{R}e^{\rho} = f(q, e^{\mathfrak{str}}) \cdot \mathcal{F}_{\hat{W}'}\left(\frac{e^{\rho}}{\prod_{\beta\in S}(1+e^{-\beta})}\right), \tag{15}$$

and that the denominator identities for $D(n+1|n)$ can not be written in a similar form, since the expressions

$$\mathcal{F}_{\hat{W}''}\left(\frac{e^{\rho}}{\prod_{\beta\in S}(1+e^{-\beta})}\right) \quad \text{and} \quad \mathcal{F}_{\hat{W}'}\left(\frac{e^{\rho}}{\prod_{\beta\in S}(1+e^{-\beta})}\right) \tag{16}$$

are not well-defined.

3.1. ***Case $D(n+1|n)$.*** Let us show that the expressions in (16) are not well-defined for $D(n+1|n)$. Fix Π as in Section 1.1 and recall that $\rho = 0$.

We repeat the reasoning of Section 2.1.1. One has

$$\sum_{\beta\in V_S(w)} w\beta \in \operatorname{supp}\left(\frac{1}{\prod_{\beta\in S}(1+e^{-w\beta})}\right) \subset \sum_{\beta\in V_S(w)} w\beta - \hat{Q}^+ \subset \hat{Q}^-,$$

where

$$V_S(w) = \{\beta \in S : w\beta < 0\}.$$

Therefore $1 \in \operatorname{supp}\big(1/\prod_{\beta\in S}(1+e^{-w\beta})\big)$ if and only if $wS \subset \Delta_+$.

Take $S = \{\varepsilon_i - \delta_i\}$; then $t_\mu S \subset \Delta_+$ if $(\varepsilon_i - \delta_i, \mu) < 0$ for all i which holds for all $\mu \in \sum \mathbb{Z}_{<0}\varepsilon_i$ and all $\mu \in \sum \mathbb{Z}_{>0}\delta_i$. Hence the sums in (16) contain infinitely many summands equal to 1 and thus they are not well-defined.

3.2. ***Case $\mathfrak{gl}(n|n)$.*** Fix Π as in Section 1.1; then $S = \{\varepsilon_i - \delta_i\}$.

In order to deduce the formula (15) from (14) and (2) it is enough to verify that the expression

$$\mathcal{F}_{\hat{W}'}\left(\frac{e^{\rho}}{\prod_{\beta\in S}(1+e^{-\beta})}\right) = e^{\rho}\mathcal{F}_{\hat{W}'}\left(\frac{1}{\prod_{\beta\in S}(1+e^{-\beta})}\right)$$

is well-defined (since ρ is $\hat{W}$-invariant). As in Section 2.1.1, this amounts to showing that

$$X_S(\nu) := \left\{ w \in \hat{W}' : \sum_{\beta\in V_S(w)} w\beta \geq -\nu \right\}$$

is finite for any $\nu \in \hat{Q}^+$ (where $V_S(w)$ is defined as in Section 3.1). As in Section 2.1.1, writing $\nu = k\delta + \nu_+$, where $\nu_+ \in \mathbb{Z}\Delta$, we get

$$X_S(\nu) \subset \big\{ t_\mu y : \mu \in T',\, y \in W' \text{ s.t. } (y\beta, \mu) \geq -k \text{ for all } \beta \in S \big\}.$$

Since y permutes δ_is, $t_\mu y \in X_S(\nu)$ forces $(\delta_i, \mu) \geq -k$ for all i. Taking into account that μ lies in the $\mathbb{Z}$-span of δ_i and $(\mu, \sum_{i=1}^n \delta_i) = 0$, we conclude that $X_S(\nu)$ is finite. This establishes (15).

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