Algebra & Number Theory

Volume 6

2012

No. 6

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mathematical sciences publishers

ALGEBRA AND NUMBER THEORY 6:6 (2012)
dx.doi.org/10.2140/ant.2012.6.1061

The smallest prime that does not split completely in a number field

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We study the problem of bounding the least prime that does not split completely in a number field. This is a generalization of the classic problem of bounding the least quadratic nonresidue. Here, we present two distinct approaches to this problem. The first is by studying the behavior of the Dedekind zeta function of the number field near 1, and the second by relating the problem to questions involving multiplicative functions. We derive the best known bounds for this problem for all number fields with degree greater than 2. We also derive the best known upper bound for the residue of the Dedekind zeta function in the case where the degree is small compared to the discriminant.

1. Introduction

1.1. ***Historical background.*** Let $\mathcal{N}$ denote the least quadratic nonresidue modulo a prime p. An old and difficult problem in number theory is to find good upper bounds for $\mathcal{N}$. Much work has been done on this problem, and we will only mention a small selection of that here.

The best result known arises from considerations of cancellation in character sums. To be more specific, let χ be the quadratic character with modulus p. Then we say that χ exhibits cancellation at $x = x(p)$ if $\sum_{n \le x} \chi(n) = o(x)$. Thus, the well known bound of Pólya and Vinogradov for character sums implies that cancellation occurs for $x = p^{1/2+o(1)}$; see [Davenport 2000]. Vinogradov [1927] proved that such cancellation implies that the least quadratic nonresidue is $\mathcal{N} \ll p^{1/(2\sqrt{e})+o(1)}$. Burgess [1957] showed that cancellation occurs at $x = p^{1/4+o(1)}$, and this implied that

$$\mathcal{N} \ll p^{1/(4\sqrt{e})+o(1)}, \tag{1}$$

which apart from different quantifications of $o(1)$ is the best result known.

The author is partially supported by a NSERC PGS-D award.
MSC2000: primary 11N60; secondary 11R42.
Keywords: primes, split, number fields, Dedekind zeta function.

Vinogradov conjectured that $\mathcal{N} \ll_\epsilon p^\epsilon$ for any $\epsilon > 0$. This is very reasonable since the Riemann hypothesis for $L(s, \chi)$ implies the stronger bound of

$$\mathcal{N} \ll \log^2 p. \tag{2}$$

The true bound is suspected to be $\mathcal{N} \ll \log p \log\log p$, arising from probabilistic considerations.

1.2. ***Generalization.*** This problem is the same as finding the least prime which does not split completely in a quadratic field. A generalization is to find upper bounds for the least prime which does not split completely in an arbitrary number field. Let K be a number field of degree l with discriminant d_K, $\mathcal{N}$ the least prime which does not split, and let $\zeta_K(s)$ denote its Dedekind zeta function. Then $\zeta_K(s)$ is analytic on the complex plane except for a simple pole at $s = 1$. Moreover, the Euler product

$$\zeta_K(s) = \prod_{\mathfrak{p}} \left(1 - \frac{1}{N(\mathfrak{p})^s}\right)^{-1}$$

holds for $\Re s > 1$, where the product is over prime ideals $\mathfrak{p}$ and $N(\mathfrak{p})$ denotes the norm of $\mathfrak{p}$. We note that, if all integer primes split over K, the Euler product for $\zeta_K(s)$ would be the same as that for $\zeta(s)^l$, where as usual $\zeta(s)$ denotes the Riemann zeta function. Since $\zeta(s)^l$ has a pole of order l at $s = 1$ and $\zeta_K(s)$ has only a simple pole at $s = 1$, we see that not all primes split. This also leads to quantifications of the statement that the least prime which does not split cannot be too large and even suggests that stronger results should be available as l grows. Using this approach, K. Murty [1994] showed, assuming GRH for $\zeta_K(s)$, that $\mathcal{N} \ll ((\log d_K)/(l-1))^2$, which is analogous to (2). Unconditionally, Murty notes in a remark in the same paper that his method would give a bound with a main term that is of the form

$$\mathcal{N} \ll d_K^{\frac{1}{2(l-1)}}. \tag{3}$$

This type of result was explicitly proved later using essentially elementary methods by Vaaler and Voloch [2000]. Their result is that

$$\mathcal{N} \le 26 l^2 d_K^{\frac{1}{2(l-1)}}, \tag{4}$$

provided that

$$d_K \ge \tfrac{1}{8} e^{2(l-1)\max(105,\, 25\log^2 l)}.$$

Vaaler and Voloch note that this result is an improvement on the more general result of Lagarias, Montgomery, and Odlyzko [Lagarias et al. 1979]. The latter condition on the size of d_K is artificial, and there is reason to expect even better results when d_K is small compared to l.

Can this result be improved by some generalization of Vinogradov's method? Interestingly enough, we will show in Theorem 3 that this is not the case. In fact, the best result from Vinogradov's approach is also a bound of the same form. Our Lemma 1 and the discussion immediately preceding it give an alternate fourth proof of the $d_K^{1/(2(l-1))}$ bound.

It thus appears that $d_K^{1/(2(l-1))}$ is a natural barrier. However, using some ideas involving basic information on the zeros of $\zeta_K(s)$, we prove in Theorem 1 a result of the form

$$\mathcal{N} \ll d_K^{\frac{1}{4(l-1)}(1+o(1))}.$$

We also show that approaching the problem with multiplicative functions does pay dividends in some cases, which appear in Theorems 4 and 5, where we derive good bounds for $\mathcal{N}$ in the cases where K is cubic or biquadratic. The idea here is to study how certain multiplicative functions interact with one another and take advantage of the behavior of extremal quadratic characters. The behaviour of extremal quadratic characters has appeared previously in [Diamond et al. 2006], which reproduces unpublished work of Heath-Brown. It is also contained in [Granville and Soundararajan $\geq$ 2012]. In Lemma 12, we quantify what it means for a quadratic character to be almost extremal, which may be of independent interest.

In the cubic case, a consideration of the multiplicative functions involved will immediately generate a bound of $\mathcal{N} \ll d_K^{1/(4\sqrt{e})+\epsilon}$, where $4\sqrt{e} = 6.59\ldots$. By studying almost extremal quadratic characters, we will show a modest improvement, to $\mathcal{N} \ll d_K^{1/6.64}$. We also give the following simple example in the biquadratic case here. Given moduli q_1 and q_2, where for simplicity we assume that $q_1 \asymp q_2 \asymp q$ for some q, the least quadratic nonresidue for either q_1 or q_2 is $\ll q^{(1-\delta)/(4\sqrt{e})}$ for some $\delta > \frac{7}{100}$.

1.3. ***On residues.*** This discussion is related to another interesting problem — that of finding upper bounds on the residue κ of $\zeta_K(s)$ at $s=1$. We remind the reader that the class number formula relates κ to various algebraic invariants of K. Specifically, let r_1 and $2r_2$ denote the number of real and complex embeddings of K, h the class number, R the regulator, and ω the number of roots of unity. Then

$$\kappa = \frac{2^{r_1}(2\pi)^{r_2} h R}{\omega\sqrt{d_K}}.$$

The best known explicit upper bound is due to Louboutin [2000], who showed that

$$\kappa \leq \left(\frac{e \log d_K}{2(l-1)}\right)^{l-1}. \tag{5}$$

We also refer to [Louboutin 2000; 2001] for applications and connections of this type of result to other questions as well as references to previous works from Siegel as well as Lavrik and Egorov. We will show a result of the form

$$\kappa \le \Big(\frac{(1+o(1))e^{\gamma}\log d_K}{4l}\Big)^{l-1},$$

when $l/\log d_K = o(1)$ is small, and where $\gamma = 0.577\ldots$ is Euler's constant. See Theorem 2 for the exact result.

1.4. ***Statement of results.*** We consider these problems from two different vantage points. The first is via analysis of L-functions attached to the number field K, and the other stems from Vinogradov's work and work on multiplicative functions as in [Granville and Soundararajan 2001]. It is interesting that the latter method, which gives us the best known bounds in the quadratic case, is not optimal for number fields of large degree. Indeed, the first method will give us the best known upper bounds on the least prime that does not split for number fields of large degree and will also lead to such a result on the residue of the Dedekind zeta function. Specifically, we will show in Section 2:

Theorem 1. *Let K be a number field of degree l and discriminant d_K. Let $\mathcal{N}$ be the least prime that does not split completely in K. Then*

$$\mathcal{N} \ll_\epsilon d_K^{\frac{(1+\epsilon)}{4A(l-1)}}.$$

Here $A = \sup_{\lambda\ge 0} \dfrac{1-\frac{l}{l-1}e^{-\lambda}}{\lambda}$ *satisfies* $A \ge 1-\sqrt{\dfrac{2}{l-1}} = 1+O\Big(\dfrac{1}{\sqrt{l}}\Big) \to 1$ *as* $l\to\infty$.

The dependence on ϵ may be quantified explicitly by

$$\mathcal{N} \ll \Big(\frac{\log d_K}{l}\Big)^2 d_K^{\frac{1+o(1)}{4A(l-1)}}.$$

Here $o(1)$ denotes a quantity that tends to 0 as either l or d_K grows. It is illustrative here to consider two examples. First, if we consider some sequence of number fields such that $d_K \le C^l$ for some constant C, we see that the least prime that does not split must be bounded by a constant. This case does not appear in [Vaaler and Voloch 2000]. Secondly, in the opposite case where $(\log d_K)/l \to \infty$, we obtain $\mathcal{N} \ll d_K^{(1+o(1))/(4A)}$.

Remark 1. The value of A may be calculated for small l. The result above beats the bound $d_K^{1/(2(l-1))}$ when $l \ge 4$. We comment that the best result in the case $l = 2$ is still of the form (1). The best result available in the case $l = 3$ is also not of the form $d_K^{1/(2(l-1))}$ but is the one described below in Theorem 4.

Moreover, we have the following upper bound for the residue of the Dedekind zeta function.

Theorem 2. *Let κ be the residue at $s = 1$ of the Dedekind zeta function of K, and let $d = \log d_K^{1/l}$. Then*

$$\kappa \ll \left(\frac{\left(\frac{1}{4} + B\right) e^{\gamma + \sqrt{2/l}} \log d_K}{l} \right)^{l-1},$$

where $B = (2 \log\log d)/(\log d) + O(1/\log d)$.

In the case where d_K grows faster than an exponential[1] in l, we have $B = o(1)$. Note also that since d_K grows at least exponentially in l, B is usually small. However, the result above is not optimal for d_K very small. Rather, results like those of Hoffstein [1979] and Bessassi [2003] optimize that particular case.

Remark 2. The above results can be made explicit if desired, but we choose not to do so for ease of exposition. Improvements are possible in the coefficient in B above as well as quantifications of the ϵ appearing in Theorem 1.

Also, by applying a result of Stechkin [1970], it is possible to prove the above results more explicitly, but replacing $\frac{1}{4}$ with $(1 - 1/\sqrt{5})/2 = 0.276\ldots > \frac{1}{4}$. See Lemma 1 and environs for details.

The utility of Vinogradov's method in the context of number fields has not been well understood. We show in Section 3:

Theorem 3. *Let K be a number field of degree l and discriminant d_K. Let $f(n)$ be a real multiplicative function satisfying $0 \le f(p) \le l$ on the primes and such that*

$$\sum_n \frac{f(n)}{n^s} = \zeta(s) \sum_n \frac{g(n)}{n^s},$$

valid for $\Re(s) > 1$, for some multiplicative function $g(n)$ such that

$$\sum_{n \le x} g(n) = o(x)$$

for all $x > d_K^{1/2+o(1)}$. Then there exists some $p < d_K^{\frac{1+O(l^{-1/2+\epsilon})}{2(l-1)}}$ such that $f(p) \ne l$.

Moreover, this is essentially the best possible result for large l. To be specific, there exists a real multiplicative function satisfying all the properties above such that $f(p) = l$ for all

$$p < d_K^{\frac{1+O(l^{1/2+\epsilon})}{2(l-1)}}.$$

Thus, the technique behind Theorem 1 is aware of information that cannot be matched solely through the multiplicative functions approach, despite the fact that this approach gives the best known result for the quadratic case $l = 2$.

[1] By this, we mean that the statement $d_K \ll C^l$ is not true for any $C > 0$. An example would be the condition of Vaaler and Voloch immediately following (4).

However, the natural extension of Vinogradov's method and in particular, the structure in [Granville and Soundararajan 2001] has the advantage that it can utilize more information about the interaction between different multiplicative functions. This allows us to improve bounds on $\mathcal{N}$ in the case of cubic and biquadratic fields. Specifically, we will show in Section 4:

Theorem 4. *Let notation be as in Theorem 1. If K is a cubic field, then*

$$\mathcal{N} \ll d_K^{1/6.64}.$$

A similar idea will enable us to show in Section 4 that:

Theorem 5. *Let K be biquadratic with moduli q_1 and q_2. Then*

$$\mathcal{N} \ll (q_1 q_2)^{0.146/2}.$$

Furthermore, if $q_1 \asymp q_2$,

$$\mathcal{N} \ll (q_1 q_2)^{0.141/2}.$$

As we explain in Section 4, these results should be compared to the trivial bounds of $d_K^{1/(4\sqrt{e})+\epsilon}$ in the cubic case and $(q_1q_2)^{1/(8\sqrt{e})}$ in the biquadratic case. Numerically, the results above are respectable, but have not been completely optimized. We would like to exhibit that an interesting interaction between multiplicative functions leads to better bounds, rather than to push for the best possible numerical result.

1.5. ***Notation.*** In the following, when we write $f = O(g)$, or equivalently $f \ll g$, for functions f and g, we shall mean that there exists a constant C such that $|f| \leq C|g|$. In the case where g is a function of ϵ where as usual, ϵ denotes an arbitrary positive number, C is allowed to depend on ϵ. Unless otherwise stated, C is absolute, and in particular, C never depends on the number field K. We will also use $o(1)$ to denote a quantity which tends to 0 as either $d_k \to \infty$ or $l \to \infty$ except in §3, where we are not concerned with uniformity in l and $o(1)$ shall denote a quantity which tends to 0 as $d_K \to \infty$ and $l/\log d_K \to 0$.

2. Working with the Dedekind zeta function

As usual, write $s = \sigma + it$. In this section, we will usually denote by $\rho = \beta + i\gamma$ a zero of the Dedekind zeta function. Let

$$F(s) = \Re \sum_\rho \frac{1}{s-\rho} = \sum_\rho \frac{\sigma - \beta}{(\sigma-\beta)^2 + (t-\gamma)^2},$$

defined for all $s \neq \rho$. As before, let $l = r_1 + 2r_2$ denote the degree of K over $\mathbb{Q}$ and r_1 and $2r_2$ be the number of real and complex embeddings of K respectively. Let

$$\xi_K(s) = s(s-1)\Big(\frac{d_K}{4^{r_2}\pi^l}\Big)^{s/2}\Gamma(s/2)^{r_1}\Gamma(s)^{r_2}\zeta_K(s).$$

Then $\xi_K(s)$ is entire of order 1 and has a Hadamard product of the form

$$\xi_K(s) = e^{A+Bs} \prod_{\rho} \Big(1 - \frac{s}{\rho}\Big) e^{s/\rho}.$$

Logarithmically differentiating $\xi(s)$ gives that

$$F(s) = \Re\Big(\tfrac{1}{2} \log \frac{d_K}{2^{2r_2}\pi^l} + \frac{\zeta_K'}{\zeta_K}(s) + G(s) + \frac{1}{s} + \frac{1}{s-1}\Big), \tag{6}$$

where

$$G(s) = \Re\Big(\frac{r_1}{2}\frac{\Gamma'}{\Gamma}\Big(\frac{s}{2}\Big) + r_2\frac{\Gamma'}{\Gamma}(s)\Big).$$

Here we have used that $\Re B = -\Re \sum_\rho \frac{1}{\rho}$. (See (11) in [Davenport 2000, p. 82] in the case of $\zeta(s)$. The proof for the general case is the same.) We have

$$-\frac{\zeta_K'}{\zeta_K}(s) = \sum_{n\ge 1} \frac{\Lambda_K(n)}{n^s},$$

where $\Lambda_K(n) = 0$ if n is not a power of a prime, and $0 \le \Lambda_K(p^r) \le l \log p$. Rewriting (6) for $s = \sigma > 1$ gives

$$\sum_{n\ge 1} \frac{\Lambda_K(n)}{n^\sigma} = \frac{1}{2}\log\frac{d_K}{2^{2r_2}\pi^l} + \frac{1}{\sigma-1} - F(\sigma) + G(\sigma) + \frac{1}{\sigma}. \tag{7}$$

Then $F(\sigma) > 0$ and $\zeta_K'/\zeta_K(\sigma) < 0$ for $\sigma > 1$. This observation led Stark [1975] to his lower bounds on discriminants, and this will be our starting point. Indeed, if we use that $F(\sigma) > 0$ and that $G(\sigma) < 0$ for σ close to 1, we have that, for $1 < \sigma < \frac{5}{4}$,

$$\sum_{n\ge 1} \frac{\Lambda_K(n)}{n^\sigma} \le \frac{1}{2}\log\frac{d_K}{2^{2r_2}\pi^n} + \frac{1}{\sigma-1} + 1. \tag{8}$$

Note that $\Lambda_K(n)$ is maximized when n is a prime that splits completely in K, and the inequality above is a statement of the form that $\Lambda_K(n)$ cannot be too large for many n. With some work, this already leads to a bound of the form

$$\mathcal{N} \ll d_K^{\frac{1+O(1/\sqrt{l})}{2(l-1)}},$$

which is similar to the results of [Murty 1994] and [Vaaler and Voloch 2000]. Specifically:

Lemma 1. *Suppose that for some quantity $c > 0$, the bound*

$$\sum_{n\ge 1} \frac{\Lambda_K(n)}{n^\sigma} \le c \log d_K + \frac{1}{\sigma-1} \tag{9}$$

holds for all σ *in the range* $1+\frac{1}{\log d_K} \le \sigma \le 1+\frac{10\sqrt{l}}{\log d_K}$. *Also let*

$$a(\lambda) = \frac{1-\frac{l}{l-1}e^{-\lambda}}{\lambda},$$

and set $A = \sup_{\lambda \ge 0} a(\lambda)$. *Then*

$$\mathcal{N} \ll d_K^{\frac{c}{A(l-1)}(1+o(1))}.$$

Proof. If all primes split completely up to $x > 2$, then $\Lambda_K(n) = l\Lambda(n)$ for all $n \le x$, where $\Lambda(n)$ is the usual von Mangoldt function. Then, by the prime number theorem for $\mathbb{Q}$,

$$\sum_{n\le x} \frac{\Lambda_K(n)}{n^\sigma} = \sum_{n\le x} \frac{l\Lambda(n)}{n^\sigma} = l\left(\int_1^x \frac{1}{t^\sigma} dt + O(1)\right) = \frac{l}{\sigma-1} - \frac{lx^{1-\sigma}}{\sigma-1} + O(l).$$

Thus we have from (9) that

$$\frac{l-1}{\sigma-1} - \frac{lx^{1-\sigma}}{\sigma-1} \le c\log d_K + O(l)$$

Set $\sigma = 1+\frac{\lambda}{\log x}$. Then the expression above is the same as

$$\frac{(l-1)(\log x + O(1))}{\lambda}\left(1-\frac{l}{l-1}e^{-\lambda}\right) \le c\log d_K + O(l).$$

We may assume that $O(1) = o(\log x)$, since otherwise the result is obvious. Rearranging we have

$$\log x \le \frac{c+o(1)}{a(\lambda)(l-1)} \log d_K + O(1).$$

We note that $a(\lambda)$ has a global maximum for $\lambda > 0$. If we let A be that maximum, the result follows immediately. □

A corresponding statement on upper bounds for κ also results from considerations of this type. This conforms to the intuition that in order to maximize κ, we should put as much weight as possible on the small primes in the sum in (8). In other words, the worst-case scenario is when all the small primes split. To this end, we prove the following lemma.

Lemma 2. *Assume that* (9) *holds as in Lemma 1 for some* $\sigma = 1+\frac{\alpha}{\log d_K}$, *and that there exists some* T *such that*

$$l\sum_{n\le T} \frac{\Lambda(n)}{n^\sigma} \ge c\log d_K + \frac{1}{\sigma-1}.$$

Then

$$\log\kappa \le c\alpha + l\sum_{n\le T}\frac{\Lambda(n)}{n^\sigma\log n} + \log(\sigma-1).$$

Proof. We first show that

$$\log \zeta_K(\sigma) \le l \sum_{n \le T} \frac{\Lambda(n)}{n^\sigma \log n}. \tag{10}$$

To this end, let

$$S(t) = \sum_{n \le t} \Lambda_K(n)/n^\sigma \quad \text{and} \quad \tilde{S}(t) = \min\left\{ \sum_{n \le t} \frac{l\Lambda(n)}{n^\sigma},\ c \log d_K + \frac{1}{\sigma - 1} \right\}.$$

Essentially, $\tilde{S}(t)$ is the version of $S(t)$ that grows at the fastest rate possible, and visibly $S(t) \le \tilde{S}(t)$. Note also that $\tilde{S}(t) = c \log d_K + 1/(\sigma - 1)$ is constant for $t \ge T$. Since

$$\log \zeta_K(\sigma) = \sum_{n \ge 1} \frac{\Lambda_K(n)}{n^\sigma \log n},$$

by partial summation,

$$\begin{aligned} \log \zeta_K(\sigma) &= \int_1^\infty \frac{S(t)}{t \log^2 t}\, dt \\ &\le \int_1^\infty \frac{\tilde{S}(t)}{t \log^2 t}\, dt = \int_1^T \frac{\tilde{S}(t)}{t \log^2 t}\, dt + \frac{\tilde{S}(T)}{\log T} = l \sum_{n \le T} \frac{\Lambda(n)}{n^\sigma \log n}, \end{aligned}$$

and this proves (10). By (9), we have by integration that

$$\log \kappa - \log(\sigma - 1)\zeta_K(\sigma) \le c(\sigma - 1) \log d_K = c\alpha,$$

as desired. □

Again, since (9) follows from (8) with $c = \frac{1}{2}$, with some work the lemma above gives us a bound roughly of the form

$$\kappa \ll \left(\frac{(1 + o(1)) e^\gamma \log d_K}{2(l - 1)} \right)^{l-1},$$

at least when d_K is large when compared to l. This is already an improvement over Louboutin's result when $l/\log d_K$ is small.

It is clear from both Lemma 1 and 2 that we gain information on both $\mathcal{N}$ and κ if we were able to extract nontrivial contribution from $F(\sigma)$ in (7). However, the discussion immediately following (7) neglected the contribution of the zeros entirely. We now proceed to rectify that situation. There are a number of possible approaches to this, and the best seems to be due to Heath-Brown [1992] in the case of the Dirichlet L-functions. There are some minor technicalities in our case, which we resolve with the help of the following lemma.

Lemma 3. *Let* $\sigma_0 > 1 + 1/\log d_K$ *and* $\frac{1}{4} < R < \frac{1}{2}$. *Let* C_1 *be the half-circle of radius* R *centered at* σ_0 *with real part to the right of* σ_0. *Let*

$$\mathfrak{D} = \log \frac{\log d_K}{l}.$$

Then

$$\frac{1}{\pi R} \int_{C_1} |\log(s-1)\zeta_K(s)| ds \leq l\mathfrak{D} + O(l).$$

Proof. Let $s = \sigma + it$, where $\sigma > 1 + \dfrac{1}{\log d_K}$. Then

$$|\log \zeta_K(s)| \leq |\log \zeta_K(\sigma)| \leq \log \zeta_K \Big(1 + \frac{1}{\log d_K}\Big).$$

These inequalities follow upon comparing Dirichlet series and since the coefficients of $\log \zeta_K(\sigma)$ are positive. We now claim that

$$\log \zeta_K \Big(1 + \frac{1}{\log d_K}\Big) \leq l\mathfrak{D} + O(l).$$

Our calculations in Lemma 2 gives us this bound almost immediately. Specifically, we have from (8) and (10) in the proof of Lemma 2 that

$$\log \zeta_K(\sigma) \leq l \sum_{n \leq T} \frac{\Lambda(n)}{n^\sigma \log n},$$

provided that

$$l \sum_{n \leq T} \frac{\Lambda(n)}{n^\sigma} \geq \frac{\log d_K}{2} + \frac{1}{\sigma - 1} + 1.$$

Say that $\sigma = 1 + \dfrac{1}{\log d_K}$. Then

$$l \sum_{n \leq T} \frac{\Lambda(n)}{n^\sigma} \geq l e^{-\frac{\log T}{\log d_K}} \sum_{n \leq T} \frac{\Lambda(n)}{n} \geq l e^{-\frac{\log T}{\log d_K}} \log T + O(l).$$

Thus there is some constant[2] C such that

$$l \sum_{n \leq T} \frac{\Lambda(n)}{n^\sigma} \geq \frac{\log d_K}{2} + \frac{1}{\sigma - 1}$$

for $T = d_K^{C/l}$. Hence

$$\log \zeta_K(\sigma) \leq l \log\log T + O(l) = l \log \frac{C \log d_K}{l} + O(l) = l \log \frac{\log d_K}{l} + O(l).$$

Note that our bounds here hold uniformly in d_K and l. □

[2]Later on, we will have a specific value of C when we prove the Theorem 2, but for our present purposes, it suffices to note that this is possible for some absolute constant C.

Lemma 4. *Assume that*

$$1+\frac{1}{\log d_K} < \sigma_0 \le 1+\frac{10\sqrt{l}}{\log d_K} \quad \textit{and} \quad \mathfrak{D}=\log\frac{\log d_K}{l}.$$

Then

$$-\frac{\zeta_K'}{\zeta_K}(\sigma_0) \le \left(\tfrac{1}{4}+o(1)\right)\log d_K + \frac{1}{\sigma_0-1} + 2l\mathfrak{D} + O(l)$$

uniformly in σ_0.

Proof. Let $f(s)=(s-1)\zeta_K(s)$. Let C_R denote the circle of radius R with center σ_0 with no zeros of $f(s)$ on C_R. Then $f(s)$ is analytic and we apply Lemma 3.2 in [Heath-Brown 1992] to get that

$$-\Re\frac{f'}{f}(\sigma_0) = \sum_{\rho}{}'\left(\frac{1}{\sigma_0-\rho}-\frac{\sigma_0-\rho}{R^2}\right) - \frac{1}{\pi R}\int_0^{2\pi}\cos\theta\log\left|f(\sigma_0+Re^{i\theta})\right|d\theta$$

where $\sum'$ denotes a sum over all zeros of f within C_R. This is related to Jensen's formula and we refer the reader to [Heath-Brown 1992] for a proof.

We now need to bound the integral above, which we split into two ranges. The first is when $0\le\theta\le\pi/2$ and $3\pi/2\le\theta\le 2\pi$. The second is when $\pi/2\le\theta\le 3\pi/2$. In the first range Lemma 3 tells us that

$$\frac{1}{\pi R}\int_{C_1}\log\left|f(\sigma_0+Re^{i\theta})\right| \le l\mathfrak{D}+O(l).$$

In the second range, we use the convexity bound $\zeta_K(\sigma+it) \ll d_K^{(1-\sigma)/2}e^{l\mathfrak{D}+Cl}$ for some $C>0$. Since $\cos\theta\le 0$, we have

$$\begin{aligned}\cos\theta\log\left|f(\sigma_0+Re^{i\theta})\right| &\ge \cos\theta\frac{1-\sigma_0-R\cos\theta}{2}\log d_K(1+o(1))+\cos\theta(l\mathfrak{D}+Cl)\\ &\ge -\cos\theta\frac{R\cos\theta}{2}\log d_K(1+o(1))+\cos\theta(l\mathfrak{D}+Cl).\end{aligned}$$

Here we have used that $1-\sigma_0=o(1)$. Now, we may assume that $2/\pi < R < 1$ so the contribution of the second term to the integral is at most $l\mathfrak{D}+Cl$.

The contribution of the first term to the integral is at most

$$\frac{\log d_K+o(1)}{2\pi R}\left(\int_{\pi/2}^{3\pi/2} R\cos^2\theta d\theta + o(1)\right) = \left(\tfrac{1}{4}+o(1)\right)\log d_K.$$

Hence

$$\begin{aligned}-\frac{\zeta_K'}{\zeta_K}(\sigma_0) &\le \frac{1}{\sigma_0-1} - \Re\sum_{\rho}{}'\left(\frac{1}{\sigma_0-\rho}-\frac{\sigma_0-\rho}{R^2}\right) + \frac{1+o(1)}{4}\log d_K + 2l\mathfrak{D}+O(l)\\ &\le \frac{1}{\sigma_0-1} + \frac{1+o(1)}{4}\log d_K + 2l\mathfrak{D}+O(l),\end{aligned}$$

where we have used that

$$\Re\Big(\frac{1}{\sigma_0-\rho}-\frac{\sigma_0-\rho}{R^2}\Big)=(\sigma_0-\beta)\Big(\frac{1}{|\sigma_0-\rho|^2}-\frac{1}{R^2}\Big)\geq 0. \qquad \square$$

2.1. ***Proof of Theorem 1.*** Theorem 1 now follows immediately from Lemma 1 and Lemma 4 with $c=\frac{1}{4}+o(1)+(2l\mathfrak{D}+O(l))/\log d_K$, where $\mathfrak{D}=\log(\log d_K/l)$ as before. For $d=d_K^{1/l}$ we have

$$\frac{2l\mathfrak{D}+O(l)}{\log d_K}=2\frac{\log\log d}{\log d}+O\Big(\frac{1}{\log d}\Big).$$

Also

$$d_K^{\frac{2\log\log d+O(1)}{(l-1)\log d}}\ll(\log d)^2.$$

We further need to verify that

$$A=\sup_{\lambda\geq 0}\frac{1-\frac{l}{l-1}e^{-\lambda}}{\lambda}\geq 1-\sqrt{\frac{2}{l-1}}.$$

We have

$$\frac{1-\frac{l}{l-1}e^{-\lambda}}{\lambda}=\frac{1-e^{-\lambda}}{\lambda}-\frac{e^{-\lambda}}{(l-1)\lambda}\geq 1-\frac{\lambda}{2}-\frac{1}{(l-1)\lambda}=1-\sqrt{\frac{2}{l-1}},$$

upon setting $\lambda=\sqrt{2/(l-1)}$.

2.2. ***Proof of Theorem 2.*** It remains to prove the upper bound on the residue κ in Theorem 2. As before, set $d=\log d_K^{1/l}$. We already have from Lemma 2 that with $\sigma=1+\alpha/\log d_K$ and for any T such that $l\sum_{n\leq T}\Lambda(n)/n^\sigma\geq c\log d_K+1/(\sigma-1)$ with $c=\frac{1}{4}+2\log\log d/\log d+O(1/\log d)+o(1)$, then

$$\begin{aligned}\log\kappa&\leq c\alpha+l\sum_{n\leq T}\frac{\Lambda(n)}{n^\sigma\log n}+\log(\sigma-1)\\&\leq c\alpha+\log(\sigma-1)+l\Big(\log\log T+\gamma+\frac{2}{\log^2 T}\Big),\end{aligned}$$

where the latter line follows from taking logarithms in [Rosser and Schoenfeld 1962, (3.27)]. Set $\alpha=4\sqrt{l}$ and recall that $\sigma=1+\alpha/\log d_K$. We need to find the smallest admissible value of T. Let $S(x)=\sum_{n\leq x}\Lambda(n)/n=\log x-C+E(x)$ for some constant C. From [Rosser and Schoenfeld 1962], we know that $-1/\log x<E(x)<1/\log x$. We have

$$\begin{aligned}\sum_{n\leq T}\frac{\Lambda(n)}{n^\sigma}&=\int_{2^-}^{T}\frac{1}{x^{\sigma-1}}d(S(x))=\int_{2^-}^{T}\frac{1}{x^\sigma}dx+\frac{E(T)}{T^{\sigma-1}}\\&=\frac{1}{\sigma-1}(2^{\sigma-1}-T^{\sigma-1})+\frac{E(T)}{T^{\sigma-1}}=\log T+\frac{E(T)}{T^{\sigma-1}}+O((\sigma-1)T^{\sigma-1})\end{aligned}$$

We see easily that $T \ll d_K^{1/l}$, so $(\sigma-1)T^{\sigma-1} = o(1)$. Thus

$$\log T = \frac{\log d_K}{l}\Big(c + \frac{1}{\alpha}\Big) + R(T),$$

where $|R(T)| < 1/\log T$. If $\log T \ge (\log d_K)/(4l)$, we may absorb $R(T)$ into the $O(l/\log d_K)$ term inside c and write

$$\log T = \frac{\log d_K}{l}\Big(c + \frac{1}{\alpha}\Big).$$

Otherwise, $\log T \le \frac{\log d_K}{4l} \le \frac{\log d_K}{l}\Big(c + \frac{1}{\alpha}\Big)$. Either way, we have

$$\kappa \le \exp(\sqrt{l})\frac{4\sqrt{l}}{\log d_K}(e^{\gamma}\log T)^l \le \frac{4e^{\gamma}c}{\sqrt{l}}\Big(ce^{\gamma+2/\sqrt{l}}\,\frac{\log d_K}{l}\Big)^{l-1},$$

where we have written $c + 1/\alpha \le ce^{1/c\alpha}$. Let $B = \dfrac{2\log\log d}{\log d} + O\Big(\dfrac{l}{\log d}\Big)$. Then we have also

$$\kappa \ll \Big(\big(\tfrac{1}{4} + B\big)e^{\gamma+2/\sqrt{l}}\,\frac{\log d_K}{l}\Big)^{l-1}$$

Since d_K grows at least as fast as an exponential in l, B is always bounded. As mentioned before, we are most interested here in the case when d grows, so that $B = o(1)$.

3. On multiplicative functions

3.1. ***Preliminaries.*** Let $\zeta_K(s) = \sum_{n\ge 1} a(n)/n^s$ be the Dirichlet series for $\zeta_K(s)$. For this section, let $f(n)$ be the multiplicative function such that

$$\frac{\zeta_K(s)}{\zeta(s)} = \zeta_K(s)\prod_p\Big(1 - \frac{1}{p^s}\Big) = \sum_n \frac{f(n)}{n^s},$$

for $\Re s > 1$. At primes, $f(p) = a(p) - 1$. We first note that $f(n)$ exhibits cancellation at $d_K^{1/2+o(1)}$. This argument is a standard one wherein we examine the Dirichlet series

$$D(s) := \frac{\zeta_K(s)}{\zeta(s)} = \sum_{n\ge 1}\frac{f(n)}{n^s}.$$

Then the standard zero-free region for $\zeta(s)$ is sufficient to find cancellation using Perron's formula.

The question of bounding the least nonsplit prime can be converted to a more general question involving $f(n)$. To be precise, knowing that $f(n)$ exhibits cancellation at $d_K^{1/2+o(1)}$, what is the maximum y such that $f(p) = l-1$ for all $p \le y$?

We now collect some facts about multiplicative functions that will be useful for the remainder of this section. Since the applications will be towards proving Theorems 3, 4 and 5, we will not take the same care to prove uniformity in l as in the previous results. The following material is essentially culled from [Granville and Soundararajan 2001]; the results there are proved for the case where $|f(n)| \le 1$, but the proofs extend to our case with very minor modifications. We summarize below the results and the required modifications to the proofs.

Let $f(n)$ be the multiplicative function defined above, with $-1 \le f(p) \le k := l-1$, where we recall that l is the degree of our number field K. Fix some $y \ge 2$ such that $f(p) = k$ for all $p \le y$. This implies that all y smooth numbers n satisfy $f(n) = d_k(n)$, where the latter is the number of ways of writing n as a product of k numbers. Then define

$$\sigma(u) = \frac{1}{y^u \log^{k-1} y} \sum_{n \le y^u} f(n)$$

and

$$P(u) = \frac{1}{y^u} \sum_{p \le y^u} f(p) \log p.$$

There are two related ways to express the relationship between $\sigma(u)$ and $P(u)$.

First say that $\tilde{\sigma}$ satisfies the convolution equation

$$u\tilde{\sigma}(u) = \tilde{\sigma} * P(u) = \int_0^u \tilde{\sigma}(u-t) P(t)\, dt, \tag{11}$$

for $u > 1$ subject to $\tilde{\sigma}(u) = u^{k-1}$ for $u \le 1$. Then for our case, we will have $\tilde{\sigma}(u) = \sigma(u) + o(1)$. The proof of this when $|f(n)| \le 1$ is contained in [Granville and Soundararajan 2001, Section 4], and the proof for our case is almost the same. There (proof of Proposition 4.1), one defines the multiplicative function $g(n)$ by $g(p^k) = f(p^k) - f(p^{k-1})$ for all prime powers. The nonnegative function $|g(n)|$ still satisfies the hypothesis of Theorem 2 in [Halberstam and Richert 1979], which gives

$$\sum_{n \le x} |g(n)| \le k \frac{x}{\log x} \sum_{n \le x} \frac{|g(n)|}{n} \left(1 + O\left(\frac{1}{\log x}\right)\right).$$

The only modification in the proofs thereafter is to replace error terms of the form $O(A)$ by $O(kA)$.

Next, we also have an inclusion-exclusion relationship. To be specific, let

$$I_j(u) = \int_{\substack{t_1 + \cdots + t_j \le u \\ t_i \ge 1}} \left(\frac{u - \sum_{i=1}^{j} t_i}{u}\right)^{k-1} \prod_{i=1}^{j} \frac{k - P(t_i)}{t_i}\, dt_1 \cdots dt_j. \tag{12}$$

Then

$$\tilde{\sigma}(u) = u^{k-1} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} I_j(u), \tag{13}$$

where we set $I_0 = 1$. This sum is finite since $I_j(u) = 0$ for $u \le j$.

(We digress briefly to elucidate this inclusion-exclusion relationship. We have $\tilde{\sigma}(u) \le u^{k-1} + o(1)$, since $f(n) \le d_k(n)$. Now, if $p \ge y$, note that

$$\sum_{\substack{n \le y^u \\ p \mid n \\ p \ge y}} 1 = \Bigg(\sum_{\substack{n \le y^u \\ p \mid n,\ p^2 \nmid n \\ p \ge y}} 1 \Bigg) (1 + O(1/y)),$$

so

$$\begin{aligned}\sum_{n \le y^u} f(n) &\ge y^u \log^{k-1}(y^u)(1+o(1)) - \sum_{y \le p \le y^u} \sum_{\substack{n \le y^u \\ p \mid n}} (d_k(n) - f(n)) \\ &\ge y^u \log^{k-1}(y^u)(1+o(1)) - \sum_{y \le p \le y^u} \sum_{m \le y^u/p} d_k(m)(k - f(p)) \\ &\ge y^u \log^{k-1}(y^u)(1+o(1)) - \sum_{y \le p \le y^u} (k - f(p)) \frac{y^u}{p} \left(\log \frac{y^u}{p} \right)^{k-1}.\end{aligned}$$

An appropriate application of summation by parts brings this to

$$\sigma(u) \ge u^{k-1}(1 - I_1(1) + o(1)),$$

and one can derive (13) in this manner. However, we will relate this independently to the convolution equation (11).)

Now, for fixed $P(t)$, the solution $\tilde{\sigma}(u)$ to (11) is unique by the same proof as Theorem 3.3 in [Granville and Soundararajan 2001]. Thus to prove (13), it suffices to show that

$$u^{k-1} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} I_j(u)$$

satisfies the convolution (11). The calculation here is similar to [ibid., Lemma 3.2] and the main step is checking that

$$k * J_j(u) = u J_j(u) - j((k - P) * J_{j-1})(u), \tag{14}$$

where $J_j(u) = u^{k-1} I_j(u)$. This is because (14) immediately implies that

$$u \sum_{j=1}^{\infty} \frac{(-1)^j}{j!} J_j(u) + u^k = u^k + k * \sum_{j=1}^{\infty} \frac{(-1)^j}{j!} J_j(u) - \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} ((k-P) * J_{j-1})(u)$$

which becomes (13) upon noting that $u^k = k * J_0$.

Some of the details in proving (14) differ slightly from those in [Granville and Soundararajan 2001], so we will provide the proof in the lemma below.

Lemma 5. *For $J_j(u)$ defined as above, $k * J_j(u) = u I_j(u) - j((k-P) * J_{j-1})(u)$.*

Proof. For notational convenience, set $S = \sum_{i=1}^{j} t_i$. Then

$$\begin{aligned}
k * J_j(u) &= \int_0^u k \int_{\substack{S \le t \\ t_i \ge 1}} (t-S)^{k-1} \prod_{i=1}^{j} \frac{k - P(t_i)}{t_i} \, dt_1 \cdots dt_j \, dt \\
&= \int_{\substack{S \le u \\ t_i \ge 1}} \prod_{i=1}^{j} \frac{k - P(t_i)}{t_i} \int_S^u k(t-S)^{k-1} \, dt \, dt_1 \cdots dt_j \\
&= \int_{\substack{S \le u \\ t_i \ge 1}} \prod_{i=1}^{j} \frac{k - P(t_i)}{t_i} (u-S)^{k-1} (u-S) \, dt_1 \cdots dt_j \\
&= u J_j(u) - j \int_{\substack{t_1 + \cdots + t_{j-1} \\ \le u - t_j \le u \\ t_i \ge 1}} t_j \prod_{i=1}^{j} \frac{k - P(t_i)}{t_i} \left(u - t_j - \sum_{i=1}^{j-1} t_i \right)^{k-1} dt_1 \cdots dt_j \\
&= u J_j(u) - j(k - P) * J_{j-1}(u). \qquad \square
\end{aligned}$$

Henceforth, by an abuse of notation, we write $\sigma(u)$ for $\tilde{\sigma}(u)$ as well, and suppress the $o(1)$ error. Frequently, it will be useful to know that the minimal value of $P(t)$ gives the earliest cancellation in $\sigma(t)$. The following proposition tells us this. For an alternate proof, see also [Granville and Soundararajan 2001, Lemma 3.4].

Proposition 1. *Suppose that we have two multiplicative functions f and $f^\sharp$. Let*

$$P(u) = \frac{1}{y^u} \sum_{p \le y^u} f(p) \log p \quad \text{and} \quad P^\sharp(u) = \frac{1}{y^u} \sum_{p \le y^u} f^\sharp(p) \log p.$$

Define $\sigma(u)$ and $\sigma^\sharp(u)$ to be the solutions to (11) for $P(u)$ and $P^\sharp(u)$ respectively. Further suppose that $P(u) = P^\sharp(u)$ for $u \le 1$, and that $P(u) \le P^\sharp(u)$ always. Let u_0 be the first zero of $\sigma(u)$. Then $0 \le \sigma(u) \le \sigma^\sharp(u)$ for $u \le u_0$.

Proof. We use $I_j(u)$ and $I_j^\sharp(u)$ to denote the various integrals defined as in (12). Further let $1_{(a,a+\epsilon)}(t)$ be the indicator function of the small interval $(a, a+\epsilon)$. Without loss of generality, it suffices to prove the result in the case where $P^\sharp(t) = P(t) + \delta 1_{(a,a+\epsilon)}(t)$ for all $\delta > 0$, all $a > 1$ and ϵ arbitrarily small. This is because linear combinations of functions of the form $\delta 1_{(a,a+\epsilon)}(t)$ are L^2 dense. For notational convenience, set

$$S(t,u) = S(t) = \frac{k - P(t)}{t} \quad \text{and} \quad Q(t,u) = Q(t) = \frac{\delta 1_{(a,a+\epsilon)}(t)}{t}.$$

We may also assume that $u > 1 + a$, since otherwise $\sigma(u) = \sigma^\sharp(u)$. Now fix some $1 + a < u < u_0$, and say that $N \ge u$ is the smallest such integer. We have

$$\begin{aligned}
\sigma^\sharp(u)-\sigma(u) & \\
&= u^{k-1}\sum_{j=0}^{N}\frac{(-1)^j}{j!}(I_j^\sharp(u)-I_j(u)) \\
&= \sum_{j=1}^{N}\biggl(\frac{(-1)^j}{j!} \\
&\qquad \times\int_{\substack{t_1+\cdots+t_j\le u\\ t_i\ge 1}}\Bigl(u-\sum_{i=1}^{j}t_i\Bigr)^{k-1}\Bigl(\prod_{i=1}^{j}(S(t_i)-Q(t_i))-\prod_{i=1}^{j}S(t_i)\Bigr)\,dt_1\cdots dt_j\biggr) \\
&= \sum_{j=1}^{N}\frac{(-1)^{j-1}}{(j-1)!}\bigl(\mathcal{T}_j+O(\epsilon^2)\bigr),
\end{aligned}$$

where

$$\mathcal{T}_j=\int_{\substack{t_1+\cdots+t_j\le u\\ t_i\ge 1}}Q(t_1)\Bigl(u-\sum_{i=1}^{j}t_i\Bigr)^{k-1}\prod_{i=2}^{j}S(t_i)\,dt_1\cdots dt_j.$$

Here, we have used that integrals containing two factors of Q like

$$\int_{\substack{t_1+\cdots+t_j\le u\\ t_i\ge 1}}Q(t_1)Q(t_2)\prod_{i=3}^{j}S(t_i)\,dt_1\cdots dt_j$$

are $O(\epsilon^2)$. The terms containing one factor of Q are the same by symmetry. We now note that

$$\begin{aligned}
\mathcal{T}_j &= \int_a^{a+\epsilon}Q(t_1)\Bigl(u-\sum_{i=1}^{j}t_i\Bigr)^{k-1}\int_{\substack{t_1+\cdots+t_j\le u\\ t_i\ge 1}}\prod_{i=2}^{j}S(t_i)\,dt_1\cdots dt_j \\
&= \int_a^{a+\epsilon}Q(t_1)\,dt_1\,\biggl(\int_{\substack{t_2+\cdots+t_j\le u-a\\ t_i\ge 1}}\Bigl(u-a-\sum_{i=2}^{j}t_i\Bigr)^{k-1}\prod_{i=2}^{j}S(t_i)\,dt_2\cdots dt_j+O(\epsilon)\biggr) \\
&= \int_a^{a+\epsilon}Q(t_1)\,dt_1\,\bigl(u^{k-1}I_{j-1}(u-a)+O(\epsilon)\bigr).
\end{aligned}$$

Here the $O(\epsilon)$ arises from replacing instances of t_1 by a and using that $a\le t_1\le a+\epsilon$. Combining the above with the previous equation gives us

$$\sigma^\sharp(u)-\sigma(u)=\int_a^{a+\epsilon}Q(t_1)\,dt_1\Bigl(\frac{u^{k-1}}{(u-a)^{k-1}}\sigma(u-a)+O(\epsilon)\Bigr).$$

If we pick ϵ to be sufficiently small, the latter is positive since $\int_a^{a+\epsilon}Q(t_1)\,dt_1>0$ and $\sigma(u-a)>0$. □

Remark 3. Actually, wherever we use this result, we have $f^\sharp(p) \geq f(p)$. When this is true, there is an alternative argument, which we now sketch. Let $g(n)$ be the multiplicative function defined by $f^\sharp = f * g$, that is, $f^\sharp(n) = \sum_{d|n} f(d)g(n/d)$. Then since $f^\sharp(p) = f(p) + g(p)$, we must have $g(p) \geq 0$. Hence

$$\sum_{n\leq x} f^\sharp(n) = \sum_{n\leq x}\sum_{d|n} f(d)g(n/d) = \sum_{d\leq x} f(d) \sum_{n\leq x/d} g(n).$$

One may then argue that the contribution from values of g on the prime powers is benign and so the latter is an upper bound for $\sum_{n\leq x} f(n)$.

3.2. ***Generalization of Vinogradov's method.*** By Proposition 1, we only need consider the case where $P(u) = k$ for $u \leq 1$, and $P(u) = -1$ otherwise.

By the convolution (11), we get that $\sigma(u)$ satisfies the following differential difference equation:

$$u\sigma'(u) + (1-k)\sigma(u) + (k+1)\sigma(u-1) = 0. \tag{15}$$

Lemma 6. *Let u_0 be a zero of $\sigma(u)$. Then $u_0 \gg k/\log k$.*

Proof. Without loss of generality, we may suppose that u_0 is minimal. By a change of variables $\tau(u) = \sigma(u)u^{1-k}$, we derive from (15) that

$$\tau'(u) = -(k+1)\Big(1-\frac{1}{u}\Big)^k \tau(u-1).$$

We see immediately that τ is positive and decreasing on $[0, u_0)$, so $-\tau'(u) \leq (k+1)(1-1/u)^k \ll (k+1)e^{-k/u}$. The result follows since by mean value theorem, $1 \ll (u_0-1)(k+1)e^{-k/u}$ for some $u \in [1, u_0)$. □

This allows us to say that cancellation occurs later than $k/\log k$ but we require finer analysis in order to obtain that it must occur very near k. For this, we use the saddle-point method.

3.3. ***The saddle-point method.*** Let $\hat{\sigma}(s) = \int_0^\infty \sigma(t)e^{-st}dt$ denote the Laplace transform of $\sigma(t)$. In Lemma 7, we will show that $\hat{\sigma}(s)$ can be analytically continued to all of $\mathbb{C}$. Thus, by Laplace inversion,

$$\sigma(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{\sigma}(s)e^{us}\,dt \tag{16}$$

where $s = x + it$ for fixed x. The idea of the saddle-point method is that the integral for $\sigma(u)$ above is dominated by a small interval where the argument of the integrand changes slowly. First, we need to obtain a workable form for $\hat{\sigma}(s)$. Our approach will be similar to the analysis of the classic Dickman's function in [Tenenbaum 1995, Section 5.4].

Lemma 7. *We have*

$$\hat{\sigma}(s) = (k-1)!\, s e^{(k+1)(I(-s)+\gamma)} = \frac{(k-1)!}{s^k} e^{-(k+1)J(s)},$$

where γ is Euler's constant and

$$I(s) = \int_0^s \frac{e^t - 1}{t}\, dt, \qquad J(s) = \int_0^\infty \frac{e^{-(s+t)}}{s+t}\, dt.$$

Note that $J(s)$ only has holomorphic extension to $\mathbb{C} \setminus (-\infty, 0]$; the purpose of writing $\hat{\sigma}(s)$ in terms of $I(-s)$ is to analytically continue the transform to all of $\mathbb{C}$.

Proof. A change of variables $t = v/s$ in the definition of the Laplace transform gives us that

$$s\hat{\sigma}(s) = \int_0^\infty e^{-v} \sigma(v/s)\, dv.$$

Differentiating both sides with respect to s gives

$$\begin{aligned} \frac{d}{ds} s\hat{\sigma}(s) &= \frac{1}{s} \int_0^\infty e^{-v}(-(v/s)\sigma'(v/s))\, dv \\ &= \frac{1}{s} \int_0^\infty e^{-v}\Big((k+1)\sigma\Big(\frac{v}{s} - 1\Big) - (k-1)\sigma(v/s)\Big)\, dv \\ &= (k+1)e^{-s}\hat{\sigma}(s) - (k-1)\hat{\sigma}(s), \end{aligned}$$

upon changing variables again. Solving this differential equation for $s\hat{\sigma}(s)$ gives

$$s\hat{\sigma}(s) = C \frac{e^{-(k+1)J(s)}}{s^{k-1}},$$

for some constant C. We have $\lim_{s\to\infty} J(s) = 0$, so

$$\lim_{s\to\infty} s^k \hat{\sigma}(s) = C.$$

On the other hand,

$$\lim_{s\to\infty} s^k \hat{\sigma}(s) = \lim_{s\to\infty} s^{k-1} \int_0^\infty e^{-v}\Big(\frac{v}{s}\Big)^{k-1} dv = \int_0^\infty e^{-v} v^{k-1} dv = (k-1)!\,,$$

from which it follows that $C = (k-1)!$. Note that the first line follows from the fact that $\sigma(t) = t^{k-1}$ for $t \le 1$, and that e^{-v} decreases rapidly.

By [Tenenbaum 1995, Lemma 7.1 of Section 5.4], we have

$$-J(s) = I(-s) + \gamma + \log s$$

for $s \in \mathbb{C} \setminus (-\infty, 0]$, and this concludes the proof. □

In order to apply the saddle-point method, we first collect some information on the extrema of the integrand in (16). In the sequel, we let $W(x)$ denote the Lambert W function, defined by $x = W(x)e^{W(x)}$. We remind the reader that there exist two real branches of $W(x)$ when $x \geq -1/e$, which we denote by W_0 and W_{-1}, where they are distinguished by $W_0(0) = 0$ and $W_{-1}(0) = -\infty$.

Lemma 8. *Let $\Phi(s) = \hat{\sigma}(s)e^{us}$ and let $\xi(u) = -W((-(k+1)e^{-k/u})/u) - k/u$, where W is a branch of the Lambert W function. Then $\Phi'(\xi) = 0$. If $|u-k| \geq 2\sqrt{k}$, then we may pick $\xi(u)$ to be real. In particular, we pick*

$$\xi(u) = \begin{cases} -W_0\Big(\frac{-(k+1)e^{-k/u}}{u}\Big) - k/u & \text{for } u \leq k - 2\sqrt{k}, \\ -W_{-1}\Big(\frac{-(k+1)e^{-k/u}}{u}\Big) - k/u & \text{for } u > k + 2\sqrt{k}. \end{cases} \tag{17}$$

For this choice of $\xi(u)$, if $|u-k| \gg k^{1/2+\epsilon}$, then $\xi(u) \gg k^{-1/2+\epsilon}$. Moreover, $\xi(u) < 0$ for $u < k - 2\sqrt{k}$ and $\xi(u) > 0$ for $u > k - 2\sqrt{k}$.

Proof. We have that

$$\frac{d}{ds}(se^{(k+1)I(-s)}e^{us}) = e^{(k+1)I(-s)}e^{us}(1 + s(u-(k+1)I'(-s))),$$

and this is 0 when $s = -\xi(u)$ where $\xi(u)$ satisfies

$$(k+1)e^{\xi(u)} = k + u\xi(u).$$

In other words,

$$\xi(u) = -W\Big(\frac{-(k+1)e^{-k/u}}{u}\Big) - k/u, \tag{18}$$

Note that

$$\frac{-(k+1)e^{-k/u}}{u} \geq -1/e \iff (k+1) \leq ue^{(k-u)/u}.$$

We first verify that the latter holds for all $|u-k| \geq 2\sqrt{k}$. Indeed, a little calculus tells us that the function $ue^{(k-u)/u}$ has a global minimum on $[0, \infty)$ at $u = k$. Since it is decreasing on $[0, k)$ and increasing on $[k, \infty)$, it suffices to check the assertion for $|u-k| = 2\sqrt{k}$. But for $|u-k| = 2\sqrt{k}$, we have

$$\begin{aligned} ue^{(k-u)/u} &= k + \frac{(k-u)^2}{2u} + \frac{(k-u)^3}{3!\,u^2} + \cdots \\ &\geq k + \Big(\frac{1}{2} - \frac{1}{3!}\Big)\frac{(k-u)^2}{u} \geq k + \frac{4}{3} > k+1. \end{aligned}$$

Now let $u = k+E$, where $|E| > 2\sqrt{k}$. We examine two cases. First, when $E < 0$, we know that $-W_0(x) \leq 1$ for all $x \leq 0$ so

$$\xi(u) \leq 1 - \frac{k}{k+E} = \frac{E}{k+E} < 0.$$

Next, when $E > 0$, we know that $-W_{-1}(x) \geq 1$ for all $x \leq 0$ so

$$\xi(u) \geq 1 - \frac{k}{k+E} = \frac{E}{k+E} > 0.$$

Note that $|E/(k+E)| \gg 1/k^{1/2-\epsilon}$ if $|E| \gg k^{1/2+\epsilon}$, and that $\xi(u)$ shares the same sign with E. □

Remark 4. To motivate the definition of $\xi(u)$ in this lemma, note that k/u is close to satisfying the equation defining $W(-(k+1)e^{-k/u}/u)$, so k/u must sometimes be close to one of the branches. The idea here is to take the other branch. The sign change for $\xi(u)$ occurs near $u = k$, and this is also when the branches converge to the same value at $-1/e$.

We now need to estimate $\sigma(u)$ by Laplace inversion of $\hat{\sigma}(s)$ on the $\Re s = \Re \xi$ line. For this purpose, we collect the following estimates.

Lemma 9. *Let ξ be as in Lemma 8. Write $s = -\xi + i\tau$, with τ real, and assume $1 < u \leq 10k$ with $|k-u| \gg k^{1/2+\epsilon}$. Then for $|\tau| \geq k + u|\xi|$,*

$$\hat{\sigma}(s) = \frac{(k-1)!}{s^{k-1}}\Big(1 + O\Big(\frac{u\xi+k}{|s|}\Big)\Big). \tag{19}$$

Moreover, there exists $c > 0$ such that for $|\tau| \leq \pi$,

$$\hat{\sigma}(s) \ll (k-1)!\, s e^{(k+1)(\gamma+I(\xi))} e^{-c\frac{k+1}{|\xi|+1}\tau^2}, \tag{20}$$

and for $|\tau| > \pi$,

$$\hat{\sigma}(s) \ll (k-1)!\, s e^{(k+1)(\gamma+I(\xi))} e^{-c\frac{k+1}{|\xi|+1}}. \tag{21}$$

Proof. The first bound follows from $\hat{\sigma}(s) = ((k-1)!/s^{k-1})e^{-(k+1)J(s)}$, and the bound $J(s) \ll |e^{\xi}/s| = |(u\xi+k)/((k+1)s)|$. For the other two cases, set $H(\tau) = I(\xi) - I(-s) = \int_0^1 (e^{h\xi}/h)(1 - e^{-i\tau h})\, dh$. We extract the real part to get that

$$\Re H(\tau) = \int_0^1 \frac{e^{h\xi}}{h}(1 - \cos\tau h)\, dh$$

For (20), note that $1 - \cos h\tau \geq 2\tau^2 h^2/\pi^2$ for $|\tau| \leq \pi$. By the calculation in [Tenenbaum 1995, Lemma 8.2],

$$\Re H(\tau) \geq \frac{\tau^2}{2\pi^2}\left|\int_0^1 e^{h\xi}\, dh\right| \gg \frac{\tau^2}{|\xi|+1}.$$

From this and Lemma 7, we have (20).

To prove the third bound, (21), observe that

$$\Re H(\tau) = \int_0^1 \frac{e^{h\xi}}{h}(1 - \cos\tau h)\, dh \gg \frac{1}{|\xi|+1}.$$

The last inequality follows from considering an open set $E \subset [0, 1]$ of small measure outside of which $(1 - \cos \tau h) \gg 1$. One may make E small enough so that $\int_E e^{h\xi}\, dh$ is bounded by $\int_{[0,1]\setminus E} e^{h\xi}\, dh$. This is possible since $\xi \le C$ for some absolute constant C for u in the specified range. This is true in the case $u < k$ because $-W_0(x) \le 1$ for $x \le 0$ and it is true for $u > k$ since the argument inside W_{-1} is bounded away from 0 when $u \le 10k$. □

We now apply the bounds above to obtain an estimate for $\sigma(t)$. Set

$$\delta = \sqrt{\frac{\log^3(k+1)}{c(k+1)}}$$

where c is the constant appearing in Lemma 9. Let $K(u) = 1/(2\pi) \int_{-\delta}^{\delta} \hat{\sigma}(s) e^{us}\, d\tau$, and $H(u) = 1/(2\pi) \int_{\mathbb{R}\setminus[-\delta,\delta]} \hat{\sigma}(s) e^{us}\, d\tau$. As above, we have written $s = -\xi + i\tau$. We know that $\sigma(u) = K(u) + H(u)$, and we first find an upper bound for $H(u)$.

Lemma 10. *Assume $k \ge 3$ and $u \gg k/\log k$ with $|k-u| \gg k^{1/2+\epsilon}$. Then*

$$H(u) \ll (k-1)!\, e^{(k-1)(\gamma + I(\xi))} \frac{1}{(k+1)^{\log^2 k}}.$$

Proof. First note by (18) that $\xi \ll \log k$ when $u \gg k/\log k$. Now, we split the integral in the definition of $H(u)$ into 3 ranges. First, when $\delta < |\tau| \le \pi$, we have by (20) that the integral is

$$\begin{aligned} &\ll (k-1)!\, e^{(k-1)(\gamma+I(\xi))} \int_{\delta}^{\infty} e^{-c(k+1)\tau^2/\log k} d\tau \\ &\ll (k-1)!\, e^{(k-1)(\gamma+I(\xi))} \frac{1}{(\sqrt{k+1})^{1-\epsilon}} \int_{\log^{3/2}(k+1)}^{\infty} e^{-\tau^2} d\tau \\ &\ll (k-1)!\, e^{(k-1)(\gamma+I(\xi))} \frac{1}{(k+1)^{\log^2 k}}. \end{aligned}$$

Next, when $\pi < |\tau| \le k + u|\xi|$, we get by (21) that the integral is

$$\ll (k-1)!\, e^{(k-1)(\gamma+I(\xi))} e^{-k^{1-\epsilon}},$$

where we have used that $u \ll k$. Lastly, for $|\tau| \ge k + u|\xi|$, we get by (19) that the integral is

$$\ll (k-1)!\, \frac{1}{k^{k-1}},$$

which is tiny. □

Now we are ready to evaluate $K(u)$.

Lemma 11. *Suppose that $k \ge 3$ and $k/\log k \ll u \le 10k$ with $|k-u| \gg k^{1/2+\epsilon}$. Then*

$$K(u) = \frac{-(k-1)!\, \xi e^{(k+1)(\gamma+I(\xi)) - u\xi}}{\sqrt{2\pi(k+1)I''(\xi)}} \Big(1 + O\Big(\frac{1}{(k+1)^{\epsilon}}\Big)\Big).$$

Proof. We first examine the Taylor expansion of $I(-s)$ about ξ. First note that

$$I'(\xi)=\frac{e^{\xi}-1}{\xi}=\frac{u}{k+1}-\frac{1}{(k+1)\xi},$$

as before. Thus

$$I(-s)=I(\xi)-\frac{i\tau u}{k+1}+\frac{i\tau}{(k+1)\xi}-\frac{\tau^2 I''(\xi)}{2}+O(\tau^3).$$

Since $1/(k^{1/2-\epsilon})\ll\xi\ll\log k$ for $k/\log k\ll u\le 10k$, we have for $|\tau|\le\delta$ that

$$e^{(k+1)I(-s)+us}=e^{(k+1)I(\xi)-u\xi-(k+1)\frac{\tau^2 I''(\xi)}{2}}\Big(1+O\Big(\frac{1}{k^{\epsilon}}\Big)\Big),$$

and so

$$\begin{aligned}K(u)&=(k-1)!\,e^{(k+1)(\gamma+I(\xi))-u\xi}\int_{-\delta}^{\delta}e^{-(k+1)(\tau^2 I''(\xi))/2}(-\xi+i\tau)\,d\tau\,\Big(1+O\Big(\frac{1}{k^{\epsilon}}\Big)\Big)\\&=-(k-1)!\,\xi e^{(k+1)(\gamma+I(\xi))-u\xi}\int_{-\delta}^{\delta}e^{-(k+1)(\tau^2 I''(\xi))/2}\,d\tau\,\Big(1+O\Big(\frac{1}{k^{\epsilon}}\Big)\Big),\end{aligned}$$

by symmetry. Note that

$$I''(\xi)=\frac{\xi e^{\xi}-e^{\xi}+1}{\xi^2}.$$

Then for u in the range specified, $1/\log^2 k\ll I''(\xi)\ll 1$. We also have

$$\begin{aligned}\int_{-\delta}^{\delta}e^{-(k+1)\tau^2 I''(\xi)/2}d\tau&=\int_{-\infty}^{\infty}e^{-(k+1)\tau^2 I''(\xi)/2}d\tau+O\Big(\frac{1}{\sqrt{I''(\xi)}(k+1)^{3/2}}\Big)\\&=\sqrt{\frac{2\pi}{(k+1)I''(\xi)}}\Big(1+O\Big(\frac{1}{(k+1)^{1/2}}\Big)\Big),\end{aligned}$$

as desired. □

Proposition 2. *Say that $k\ge 3$ and $k/\log k\ll u\le 10k$ with $|u-k|\gg k^{1/2+\epsilon}$. Then*

$$\sigma(u)=\frac{-(k-1)!\,\xi e^{(k+1)(\gamma+I(\xi))-u\xi}}{\sqrt{2\pi(k+1)I''(\xi)}}\Big(1+O\Big(\frac{1}{(k+1)^{\epsilon}}\Big)\Big)$$

Moreover, by Lemma 8, the first zero of $\sigma(u)$ must be $k+O(k^{1/2+\epsilon})$.

Proof. The expression for $\sigma(u)=K(u)+H(u)$ follows directly from Lemmas 10 and 11. Note that $I''(\xi)\gg 1/\log^2 k$ for u in the range specified. The last assertion follows from noting that $\sigma(u)$ changes sign when ξ changes sign, and the fact that by Lemma 6, the first zero of $\sigma(u)$ must be $\gg k/\log k$. □

Finally, we note that Theorem 3 follows immediately from the Proposition 2.

4. Cubic and biquadratic fields

We now investigate the question of bounding the least nonsplit prime when K is either cubic or biquadratic. The general philosophy is the same for the two cases, although the technical details are different. There is always a "trivial" bound which arises from considering cancellation in a quadratic character, and our purpose is to show that this bound can be improved. In both cases, we benefit from interaction between a primary multiplicative function of interest and quadratic characters. Simply put, if all the primes split up to the trivial bound, then the quadratic character is extremal and we may predict its behavior far beyond the cancellation point. In this case, the interaction with the primary multiplicative function produces a contradiction. In order to obtain an actual bound, we need to understand what it means for a quadratic character to be close to extremal.

4.1. ***Extremal behavior.*** Let χ denote a quadratic character with modulus q such that $\chi(p) = 1$ for all $p \le y$ whenever $p \nmid q$. We set

$$P(u) = \frac{1}{\nu(y^u)} \sum_{p \le y^u} \chi(p) \log p,$$

where $\nu(x) = \sum_{p \le x} \log p$. Also, let $\sigma(u) = \frac{1}{y^u} \sum_{n \le y^u} \chi(n)$. We further define

$$I_j(u) = \int_{\substack{t_1+\cdots+t_j \le u \\ t_i \ge 1 \,\forall 1 \le i \le j}} \prod_{i=1}^{j} \frac{1 - P(t_i)}{t_i} \, dt_1 \cdots dt_j.$$

We remind the reader that

$$\sigma(u) = \sum_{j \ge 0} \frac{(-1)^j I_j(u)}{j!},$$

where $I_0 \equiv 1$. Note that the sum on the right is finite. Moreover, we have

$$\sum_{j=0}^{2m-1} \frac{(-1)^j I_j(u)}{j!} \le \sigma(u) \le \sum_{j=0}^{2m} \frac{(-1)^j I_j(u)}{j!}$$

for any $m \ge 0$. Once again, we refer the reader to [Granville and Soundararajan 2001] for more details.

Let $A > 0$ be such that $y^A = q^{1/4}$, so that $\sigma(u) = o(1)$ for $u > A$. The reader should think of A as being somewhat larger than $\sqrt{e}$. The simple case when $A = \sqrt{e}$ is the extremal case appearing in the bound (1) and the behavior of $P(t)$ here has been studied by other authors. In their study of Beurling primes, Diamond, Montgomery, and Vorhauer reproduce the unpublished analysis of Heath-Brown on this subject in the appendix of [Diamond et al. 2006]. This is also examined in

[Granville and Soundararajan $\geq$ 2012]. The lemma below quantifies the behavior of $P(t)$ by comparing χ to an extremal character.

Lemma 12. *Suppose that* $\sqrt{e} \leq A \leq 2$, *and set*[3] $E = 2\log A - 1$.

1. *Say that we have some interval* $(a, b) \subset (1, A)$. *Then*

$$\int_a^b \frac{1-P(t)}{t}\,dt \geq 2\log\frac{b}{a} - E + o(1).$$

2. *For all* $t \in [2, 3]$ *but for a set of measure* 0, *we have*

$$\frac{1-P(t)}{t} = \frac{1}{2}\int_1^{t-1} \frac{1-P(u)}{u}\,\frac{1-P(t-u)}{t-u}\,du.$$

Moreover, for all $t \in [2, 4]$ *but for a set of measure* 0, *we have*

$$\frac{1-P(t)}{t} \leq \frac{1}{2}\int_1^{t-1} \frac{1-P(u)}{u}\,\frac{1-P(t-u)}{t-u}\,du.$$

3. *For all* $t \in [2, 1+A]$ *but for a set of measure* 0, *we have*

$$\frac{4}{t}\log(t-1) - 2E \leq \frac{1-P(t)}{t} \leq \frac{4}{t}\log(t-1)$$

4. *For all* $t \in [1+A, 3]$ *but for a set of measure* 0, *we have*

$$\frac{1-P(t)}{t} \geq \frac{4}{t}\log\frac{A}{t-A} - 2E + o(1),$$

   and for $t \in [3, 4]$, *we have*

$$\frac{1-P(t)}{t} \geq \frac{4}{t}\log\frac{A}{t-A} - 2E - \frac{2}{3}(t-3)^2 + o(1).$$

Moreover, for all $t \in [1+A, 2A]$ *but for a set of measure* 0, *we have*

$$\frac{1-P(t)}{t} \leq \frac{4}{t}\log\frac{A}{t-A} + o(1).$$

Proof. Note that $\sigma(u) = o(1)$ for $u > A$. Thus

$$\int_1^A \frac{1-P(t)}{t}\,dt = 1 + o(1)$$

and the first assertion follows since $1 - P(t) \leq 2$.

The second assertion follows from the fact that $P(t)$ is continuous almost everywhere, and when $P(t)$ is continuous,

$$\frac{1-P(t)}{t} = \lim_{\epsilon\to 0}\frac{1}{\epsilon}(I_1(t+\epsilon) - I_1(t)).$$

[3] E measures the deviation of A from $\sqrt{e}$. In particular, $E = 0$ when $A = \sqrt{e}$.

For $t \in [2, 3]$,

$$I_1(t+\epsilon) - I_1(t) = \tfrac{1}{2}(I_2(t+\epsilon) - I_2(t)),$$

and for $t \in [2, 4]$,

$$\begin{aligned} I_1(t+\epsilon) - I_1(t) &= \tfrac{1}{2}(I_2(t+\epsilon) - I_2(t)) - \tfrac{1}{6}(I_3(t+\epsilon) - I_3(t)) \\ &\le \tfrac{1}{2}(I_2(t+\epsilon) - I_2(t)). \end{aligned}$$

Thus, it remains to evaluate

$$\begin{aligned} \lim_{\epsilon\to 0} \frac{1}{2\epsilon}(I_2(t+\epsilon) - I_2(t)) &= \lim_{\epsilon\to 0} \frac{1}{2\epsilon} \int_{\substack{t\le t_1+t_2\le t+\epsilon \\ t_1,t_2\ge 1}} \frac{1-P(t_1)}{t_1}\frac{1-P(t_2)}{t_2}\,dt_1\,dt_2 \\ &= \frac{1}{2}\int_1^{t-1} \frac{1-P(t_1)}{t_1}\frac{1-P(t-t_1)}{t-t_1}\,dt_1, \end{aligned}$$

almost everywhere, as desired.

To prove the upper bound in the third assertion, note that

$$\frac{1}{2}\int_1^{t-1} \frac{1-P(u)}{u}\frac{1-P(t-u)}{t-u}du \le 2\int_1^{t-1}\frac{1}{u}\frac{1}{t-u}du = \frac{4}{t}\log(t-1).$$

To prove the lower bound in the third assertion, we let $f(t) = (1-P(t))/t$ and $m(t) = 2/t \ge f(t)$ for all t. Then for $t \in [2, 1+A]$ we have

$$\begin{aligned} &\int_1^{t-1} f(u)f(t-u)\,du \\ &\quad= \int_1^{t-1}(f(u)-m(u))f(t-u)\,du + \int_1^{t-1} m(u)(f(t-u)-m(t-u))\,du \\ &\qquad + \int_1^{t-1} m(u)m(t-u)\,du \\ &\quad\ge \frac{8}{t}\log(t-1) - 4E. \end{aligned}$$

Here we have bounded both the first two terms from below by $-2E$ using the first assertion and that $f(u) \le m(u) \le 2$ for all $u \in [1, A]$.

The proof of the fourth assertion is similar. The only difference in the proof of the first and last bounds arises from the fact that $\int_A^2 (1-P(u))/u\,du = o(1)$. Thus for $1+A \le t = 1+A+\delta \le 2A$,

$$\int_1^{t-1}\frac{1-P(u)}{u}\frac{1-P(t-u)}{t-u}du = \int_{1+\delta}^{t-1-\delta}\frac{1-P(u)}{u}\frac{1-P(t-u)}{t-u}du + o(1).$$

For the second bound in the fourth assertion, one also needs to use that

$$\begin{aligned} \frac{1-P(t)}{t} &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\big(\tfrac{1}{2}(I_2(t+\epsilon) - I_2(t)) - \tfrac{1}{6}(I_3(t+\epsilon) - I_3(t))\big) \\ &\ge \tfrac{1}{2}\lim_{\epsilon\to 0}\frac{1}{\epsilon}(I_2(t+\epsilon) - I_2(t)) - \tfrac{2}{3}(t-3)^3. \end{aligned}$$

This follows from the calculation that

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon}(I_3(t+\epsilon)-I_3(t)) = \int_{\substack{t_1+t_2\le t-1\\ t_1,t_2\ge 1}} \frac{1-P(t_1)}{t_1}\frac{1-P(t_2)}{t_2}\frac{1-P(t-t_1-t_2)}{t-t_1-t_2}\,dt_1\,dt_2$$
$$\le 8\int_{\substack{t_1+t_2\le t-1\\ t_1,t_2\ge 1}} \frac{1}{t_1}\frac{1}{t_2}\frac{1}{t-t_1-t_2}\,dt_1\,dt_2 \le 8\frac{(t-3)^2}{2},$$

upon calculating the volume of the region of integration. □

4.2. *Cubic fields.* Let K be a cubic field. In this case, it is easy to see that a much better result than $\mathcal{N} \ll d_K^{1/(2(l-1))}$ is possible. In the case where K is Galois, then K must necessarily be abelian, and so $\zeta_K(s)=\zeta(s)L(s,\chi_1)L(s,\chi_2)$ for some Dirichlet characters χ_1 and χ_2 with conductors q_1 and q_2 respectively. Say that $q_1 \le q_2$. Then since χ_1 has order 3, by [Heath-Brown 1992, Lemma 2.4], $\chi_1(n)$ exhibits cancellation by $q^{1/4+\epsilon}$. Thus $\mathcal{N} \ll_\epsilon q_1^{1/4+\epsilon} \ll d_K^{1/8+\epsilon}$. Clearly, a stronger statement should be possible in the abelian case, but we shall be more interested in the general case here.

For the rest of this section, say that K is not Galois. Then

$$\zeta_K(s)=\zeta(s)L(f,s),$$

where f is a holomorphic modular Hecke eigenform of weight k and level N. Also, the L-function associated to f is of the form

$$L(f,s)=\prod_p\Big(1-\frac{\alpha_p}{p^s}\Big)^{-1}\Big(1-\frac{\beta_p}{p^s}\Big)^{-1}=\prod_p\Big(1-\frac{a(p)}{p^s}+\frac{\chi(p)}{p^{2s}}\Big)^{-1},$$

where χ is a quadratic character with modulus $q \le d_K$. Visibly from the Euler product above, p cannot split in K if $\chi(p)=-1$. Thus,

$$\mathcal{N} \ll d_K^{1/4\sqrt{e}}+o(1). \tag{22}$$

This is the starting point for our investigation.

Let $f(n)$ be the completely multiplicative function with $f(p)=a(p)$ for all primes p. Then $f(n)$ exhibits cancellation by $d_K^{1/2+o(1)}$. We now try to improve the bound of $\mathcal{N} \ll d_K^{1/4\sqrt{e}}$ by leveraging information about the two multiplicative functions $f(n)$ and $\chi(n)$.

Remark 5. Our main focus here is to show that improvements over the bound (22) are possible. For simplicity, we will not attempt to completely optimize our calculations. In particular, we do not use the available subconvexity result for $\zeta_K(s)$ which shows that $f(n)$ exhibits cancellation by $d_K^{1/2-\delta}$ for some $\delta>0$ (see Appendix A of [Einsiedler et al. 2011] for a synopsis of known results).

As in Section 4.1, let $P(t)$ denote the average over primes of $f(p)$ and let $P'(t)$ denote the same average for $\chi(p)$. Let[4] $\sigma(t) = 1/(y^t \log y^t) \sum_{n \le y^t} f(n)$. Also as in Section 4.1, assume that there exists some $y = d_K^A$ such that all primes $p \le y$ split completely, where we may assume that $A > \frac{1}{8}$.

We begin by quantifying the relationship between $f(p)$ and $\chi(p)$.

Lemma 13. *With f and χ as above, we have $f(p) \ge -(\chi(p)+1)/2$ for all unramified primes p. It follows that $P(t) \ge -(P'(t)+1)/2 + o(1)$, where the $o(1)$ is a quantity tending to* 0 *as $d_K \to \infty$ uniformly for $t \ge 1$.*

Proof. This follows from the fact that $f(p) = \alpha_p + \beta_p$ and $\chi(p) = \alpha_p \beta_p$. First assume that p is unramified. There are three possibilities to check, corresponding to the three possibilities for the local factor at p in $\zeta_K(s)$, which is always of the form $\prod_{\mathfrak{p}|p} \bigl(1 - 1/N(\mathfrak{p})^s\bigr)^{-1}$. When p splits completely, the local factor is $\bigl(1 - 1/p^s\bigr)^{-3}$, so $\alpha_p = \beta_p = 1$ whence $f(p) = 2$ and $\chi(p) = 1$. When p is inert, the local factor is of the form $\bigl(1 - 1/p^{3s}\bigr)^{-1}$, so $\alpha_p = 1/\beta_p = e^{\pm 2\pi i/3}$ and $f(p) = -1$ and $\chi(p) = 1$. In the remaining case, p factors as $p = \mathfrak{p}_1 \mathfrak{p}_2$, where the norms of the ideals on the right are p and p^2; thus the local factor is of the form

$$\left(1 - \frac{1}{p^s}\right)\left(1 - \frac{1}{p^{2s}}\right).$$

In this case, then, $\alpha_p = -\beta_p = \pm 1$ and $f(p) = 0$ and $\chi(p) = -1$. In all three cases, we have verified that $f(p) \ge -\frac{1}{2}(\chi(p)+1)$. The statement about the averages $P(t)$ and $P'(t)$ follows by definition, and since the number of ramified primes is bounded by $\log d_K$, and hence contribute at most

$$O\left(\frac{\log^2 d_K}{y}\right) = O\left(\frac{\log^2 d_K}{d_K}\right) = o(1). \qquad \square$$

Outline of proof. Our bound for $\mathcal{N}$ will result from a lower bound for the first zero of $\sigma(t)$, which we know must eventually be identically zero by cancellation. The previous lemma, combined with Proposition 1, tells us that we can instead study the first zero of the solution to (11) with $-\frac{1}{2}(P'(t)+1)$ in place of $P(t)$. We then use our estimates for $P'(t)$ from Lemma 8 to finish the proof.

We let

$$I_j(u) = \int_{\substack{t_1 + \cdots + t_j \le u \\ t_k \ge 1\, \forall 1 \le k \le j}} \left(\frac{u - \sum_{k=1}^{j} t_k}{u}\right) \prod_{k=1}^{j} \frac{(2 - P(t_k))}{t_k} \, dt_1 \cdots dt_j.$$

Then for $u \le 4$,

$$\sigma(u) = 1 - I_1(u) + \frac{I_2(u)}{2} - \frac{I_3(u)}{6}.$$

[4]This definition of $\sigma(t)$ differs from the definition in Section 3 by a factor of t.

Set

$$I_3'(u) = \int_{\substack{t_1+t_2+t_3\le u\\ t_k\ge 1\,\forall 1\le k\le 3}} \frac{u-t_1-t_2-t_3}{ut_1t_2t_3}\,dt_1\,dt_2\,dt_3.$$

Note that

$$\frac{I_3(u)}{6} \le \frac{9}{2} I_3'(u),$$

where we have used the trivial bound $2 - P(t) \le 3$. We thus have

$$\sigma(u) \ge 1 - I_1(u) + \frac{I_2(u)}{2} - \frac{9}{2} I_3'(u). \tag{23}$$

By Proposition 1 and Lemma 13, we know that (23) still holds when $P(t)$ is replaced by $-\frac{1}{2}(P'(t)+1)$. Henceforth, assume that $P(t) = -\frac{1}{2}(P'(t)+1)$ for all $t \ge 1$. Now, we calculate an upper bound for $I_1(u)$.

Lemma 14. *For notational convenience, set*

$$g(t,u) = g(t) = \frac{u-t}{tu}.$$

For all $t \in [A,4]$ but for a set of measure zero, we have $-P(t) \le U(t)$, where

$$U(t) = \begin{cases} 1 & \text{if } A < t \le 2,\\ \min(1, 1 - 2\log(t-1) + Et) & \text{if } 2 < t \le 1+A,\\ \min\bigl(1, 1 - 2\log\frac{A}{t-A} + Et\bigr) & \text{if } 1+A < t \le 3,\\ \min\bigl(1, 1 - 2\log\frac{A}{t-A} + Et + \frac{1}{3}t(t-3)^3\bigr) & \text{if } 3 \le t \le 4.\end{cases}$$

Let $u = 2A \le 4$. Then,

$$\int_1^u (2 - P(t))g(t)\,dt \le 2\int_1^u g(t)\,dt + \int_A^u U(t)g(t)\,dt + \frac{1}{2}\left(\log A - 1 + \frac{1}{u}\Bigl(1 + A - \frac{2A}{\sqrt{e}}\Bigr) + \int_1^A g(t)\,dt\right).$$

Proof. Since we assume that $P(t) = -\frac{1}{2}(P'(t)+1)$ and Lemma 12 applies to $P'(t)$, we see that $-P(t) \le U(t)$ for $A \le t \le u$. Hence,

$$\int_1^u (2-P(t))g(t)\,dt \le 2\int_1^u g(t)\,dt + \int_A^u U(t)g(t)\,dt + \int_1^A \tfrac{1}{2}(1+P'(t))g(t)\,dt,$$

and moreover,

$$\int_1^A \tfrac{1}{2}(1+P'(t))g(t)\,dt = \frac{1}{2}\left(\int_1^A g(t)\,dt + \int_1^A \frac{P'(t)}{t}dt - \int_1^A \frac{P'(t)}{u}dt\right).$$

We know that $\int_1^A \frac{1-P'(t)}{t}\,dt = 1$, so $\int_1^A \frac{P'(t)}{t}\,dt = \log A - 1$. We claim that

$$\int_1^A P'(t)\,dt \geq \int_1^{A/\sqrt{e}} 1\,dt - \int_{A/\sqrt{e}}^A 1\,dt = 2A/\sqrt{e} - 1 - A.$$

To see this, let

$$\gamma(t) = \begin{cases} 1 & \text{if } 1 \leq t \leq A/\sqrt{e}, \\ -1 & \text{if } A/\sqrt{e} < t \leq A. \end{cases}$$

Note that $\int_1^A (\gamma(t)/t)\,dt = \log A - 1$. Let $\lambda(t) : [1, A] \to [-1, 1]$ be any function with $\int_1^A (\lambda(t)/t)\,dt = \log A - 1$ and let $h(t) = \lambda(t) - \gamma(t)$. It suffices to show that $\int_1^A h(t)\,dt \geq 0$. We have

$$A/\sqrt{e} \int_1^{A/\sqrt{e}} \frac{h(t)}{t}\,dt + A/\sqrt{e} \int_{A/\sqrt{e}}^A \frac{h(t)}{t}\,dt = 0.$$

Note that $h(t) \leq 0$ for $1 \leq t \leq A/\sqrt{e}$ and $h(t) \geq 0$ for $A/\sqrt{e} < t \leq A$. Thus we have

$$A/\sqrt{e} \int_1^{A/\sqrt{e}} \frac{h(t)}{t}\,dt \leq \int_1^{A/\sqrt{e}} h(t)\,dt$$

and

$$A/\sqrt{e} \int_{A/\sqrt{e}}^A \frac{h(t)}{t}\,dt \leq \int_{A/\sqrt{e}}^A h(t)\,dt.$$

Adding the two inequalities immediately produces the desired result.

From this, we get that

$$\int_1^A \frac{1+P'(t)}{2} g(t)\,dt \leq \frac{1}{2}\left(\log A - 1 + \frac{1}{u}\left(1 + A - \frac{2A}{\sqrt{e}}\right) + \int_1^A g(t)\,dt\right). \quad \square$$

We now need a lower bound for $I_2(u)$.

Lemma 15. *Let*

$$L(t) = \begin{cases} 0 & \text{if } 1 \leq t \leq A, \\ 1 & \text{if } A < t \leq 2, \\ \min(1, 1 - 2\log(t-1)) & \text{if } 2 < t \leq 1 + A, \\ \min\left(1, 1 - 2\log\frac{A}{t-A}\right) & \text{if } 1 + A < t \leq 2A. \end{cases}$$

Then for all $t \in [1, 2A]$ but for a set of measure zero we have $-P(t) \geq L(t)$. Thus, for $u = 2A$,

$$I_2(u) \geq \int_{\substack{t_1+t_2 \leq u \\ t_k \geq 1}} \frac{2 + L(t_1)}{t_1} \, \frac{2 + L(t_2)}{t_2} \, \frac{u - t_1 - t_2}{u}\,dt_1\,dt_2.$$

Proof. The proof is immediate from Lemma 12, and the fact that we have set $P(t) = -\frac{1}{2}(1 + P'(t))$. □

Proof of Theorem 4. Preserve the notation from the lemma above. Since $\sigma(u) = o(1)$ for $u = 2A$, we have

$$o(1) \geq 1 - 2\int_1^u g(t)\,dt + \int_A^u U(t)g(t)\,dt + \int_1^A \frac{1+P'(t)}{2} g(t)\,dt \\ + \frac{1}{2}\int_{\substack{t_1+t_2 \leq u \\ t_k \geq 1}} \frac{(2+L(t_1))}{t_1}\frac{(2+L(t_2))}{t_2}\frac{u-t_1-t_2}{u}\,dt_1\,dt_2 - \frac{9}{2}I_3'(u).$$

Using Maple and the above lemmas, we can check that the right side of the above inequality is positive when $A = 1.6625$. Thus, for the inequality to be true, we must have $A > 1.6625$, so $4A > 6.65$, and since $\mathcal{N} \ll_\epsilon d_K^{1/(4A)+\epsilon}$, we must have

$$\mathcal{N} \ll d_K^{1/6.65}.$$

The number 6.65 should be compared with $4\sqrt{e} = 6.59\ldots$ □

4.3. ***Biquadratic fields.*** We now fix K to be a biquadratic field. Then $\zeta_K(s) = \zeta(s)L(s, \chi_1)L(s, \chi_2)L(s, \chi_1\chi_2)$, where χ_1 and χ_2 are quadratic characters with moduli q_1 and q_2, say. Finding the smallest nonsplit prime is the same as finding the smallest prime which is a quadratic nonresidue for either q_1 or q_2. Clearly, the trivial bound here is of the form $\mathcal{N} \ll_\epsilon \min(q_1, q_2)^{1/(4\sqrt{e})+\epsilon}$ arising immediately from the discussion in the introduction. Our purpose here is to show that more information can be gleaned from considering the behavior of $\chi := \chi_1\chi_2$ in conjunction with that of χ_1 and χ_2. Let $q = \max(q_1, q_2)$; we will only use the fact that both χ_i exhibit cancellation by $q^{1/4} + o(1)$. Note that if q_1 and q_2 are far apart, then we expect to derive little information from the interaction of χ_1 and χ_2. This will be reflected in the discussion at the end of this section.

Assume that all the primes split up to y. (Here the reader may find it helpful to think of y as being a slightly smaller power of q_1q_2 than the trivial bound.) We set

$$P_i(u) = \frac{1}{\nu(y^u)} \sum_{p \leq y^u} \chi_i(p) \log p,$$

for $i = 1, 2$ and where $\nu(x) = \sum_{p \leq x} \log p$. Similarly, we set

$$P(u) = \frac{1}{\nu(y^u)} \sum_{p \leq y^u} \chi(p) \log p.$$

Finally, define $\sigma_i(u)$ for $i \in \{1, 2\}$, and $\sigma(u)$ as in Section 4.1.

We also define

$$I_{i,j}(u) = \int_{\substack{t_1+\cdots+t_j \le u \\ t_k \ge 1\, \forall 1 \le k \le j}} \prod_{k=1}^{j} \frac{1-P_i(t_k)}{t_k}\, dt_1 \cdots dt_j,$$

and similarly

$$I_j(u) = \int_{\substack{t_1+\cdots+t_j \le u \\ t_k \ge 1\, \forall 1 \le k \le j}} \prod_{k=1}^{j} \frac{1-P(t_k)}{t_k}\, dt_1 \cdots dt_j.$$

We begin with the following basic observation.

Lemma 16. *Let*

$$S_1 = \frac{1}{\nu(y^u)} \sum_{\substack{p \le y^u \\ \chi_1(p)=\chi_2(p)=1}} \log p, \quad \textit{and} \quad S_{-1} = \frac{1}{\nu(y^u)} \sum_{\substack{p \le y^u \\ \chi_1(p)=\chi_2(p)=-1}} \log p.$$

Then

$$P(u) = 2S_1 + 2S_{-1} - 1 + o(1).$$

Furthermore, if $P_i(t) \ge \alpha > 0$ for all $i \in \{1, 2\}$, or if $P_i(t) \le -\alpha < 0$ for all $i \in \{1, 2\}$, then

$$P(u) \ge 2\alpha - 1.$$

Proof. Let

$$S_{1,-1}(u) = \frac{1}{\nu(y^u)} \sum_{\substack{p \le y^u \\ \chi_1(p)=-\chi_2(p)=1}} \log p,$$

and similarly define $S_{-1,1}(u)$. Then

$$S_1 + S_{-1} + S_{1,-1} + S_{-1,1} = 1 + o(1),$$

where the $o(1)$ comes from the ramified primes. Since $\chi(p) = \chi_1(p)\chi_2(p)$, we also have

$$P(u) = S_1(u) + S_{-1}(u) - S_{1,-1}(u) - S_{-1,1}(u).$$

Adding the two equations gives the first portion of the lemma. Now say that $P_i(t) \ge \alpha > 0$ for $i \in \{1, 2\}$. Then since $\alpha \le P_1(t) = S_1(t) - S_{-1}(t) + S_{1,-1}(t) - S_{-1,1}(t)$ and $\alpha \le P_2(t) = S_1(t) - S_{-1}(t) - S_{1,-1}(t) + S_{-1,1}(t)$, we have $2\alpha \le 2(S_1(t) - S_{-1}(t)) \le P(t) + 1$, as desired. The remaining assertion is proven in the exact same way. □

Outline of proof. As in Section 4.2, our bound for $\mathcal{N}$ will result from a lower bound for the first zero of $\sigma(t)$, which we know must eventually be identically zero by cancellation. The Lemma above relates the behaviour of $P(t)$ with expressions $P_1(t)$ and $P_2(t)$ which may be estimated by Lemma 8.

Lemma 17. *Let A be such that $y^A = q^{1/4}$, and $B \le 2A$ be such that $y^B = (q_1q_2)^{1/4}$. Then*

$$0 \ge 3 - 4\log A - \int_2^B \frac{1-P(t)}{t}\,dt + o(1).$$

Proof. We have $0 = \sigma_i(u) = 1 - I_{i,1}(u)$ for $A \le u \le 2$. Adding this for $i = 1, 2$, we get

$$\log u - 1 = \int_1^u \frac{P_1(t)+P_2(t)}{2t}\,dt = \int_1^u \frac{S_1(t)-S_{-1}(t)}{t}\,dt.$$

Rearranging, and noting that $S_1(t) \ge 0$, we get that $\int_1^u S_{-1}(t)/t \ge 1 - \log u$. Hence by the previous lemma

$$\int_1^u \frac{P(u)}{u}du \ge \int_1^u \frac{2S_{-1}(t)-1}{t}\,dt \ge 2 - 3\log u.$$

Thus, rearranging again, and setting $u = A$, we get that

$$1 - \int_1^A \frac{1-P(u)}{u}du \ge 3 - 4\log A + o(1).$$

Observe that $\int_A^2 (1-P_i(u))/u\,du = o(1)$ for each i and so $\int_A^2 (1-P(u))/u\,du = o(1)$ also. We thus have

$$o(1) = \sigma(B) \ge 1 - I_1(B) \ge 3 - 4\log A - \int_2^B \frac{1-P(t)}{t}\,dt. \qquad \square$$

Lemma 16 would give us a nontrivial upper bound[5] for $\int_2^B (1-P(t))/t\,dt$ provided that we have sufficient information about χ_1 and χ_2. The latter is furnished by Lemma 12. We collect the calculations and prove the theorem below.

Proof. For $2 \le u \le 1+A$, we have by Lemmas 12 and 16 that $P(t) \ge 1 - 8\log(t-1)$. Hence

$$\int_2^{1+e^{1/4}} \frac{1-P(t)}{t}\,dt \le \int_2^{1+e^{1/4}} \frac{8\log(t-1)}{t}\,dt < 0.13538.$$

In the range $2 \le u \le 1+A$, we have that $P_i(t) \le 1 - 4\log(t-1) + 2Et$ by Lemma 12. By Lemma 16, we have $1 - P(t) \le 4(1 - 2\log(t-1) + Et)$. This

[5] By nontrivial, we mean that it must be smaller than the trivial bound given by $1 - P(t) \le 2$.

bound is only meaningful when the right hand side is ≤ 2. Thus, let $t_0 < 1+A$ be such that $2(1-2\log(t_0-1)+Et_0) = 1$. Then

$$\int_{t_0}^{1+A} \frac{1-P(t)}{t}\,dt \leq 4\int_{t_0}^{1+A}\Big(\frac{1-2\log(t-1)}{t}+E\Big)\,dt.$$

Further, in the range $1+A \leq u \leq 3$, we have by Lemma 12 that $P_i(t) \leq 1 - 4\log(A/(t-A)) + 2Et + o(1)$. By Lemma 16, we have

$$\frac{1-P(t)}{t} \leq 4\frac{1-2\log(A/(t-A))+Et}{t} + o(1).$$

Let $t_1 > 1+A$ be such that $2(1-2\log(A/(t-A))+Et_1) = 1$. Then

$$\int_{1+A}^{t_1} \frac{1-P(t)}{t}\,dt \leq 4\int_{1+A}^{t_1}\left(\frac{1-2\log(A/(t-A))}{t}+E\right)dt + o(1).$$

Let $t_2 = A(1+e^{1/4})/e^{1/4}$. In the range, $t_2 \leq u \leq B \leq 2A$, we have by Lemma 12 that $1-P_i(t) \leq 4\log(A/(t-A)) + o(1)$. Then similarly, we get that

$$\int_{t_2}^{B} \frac{1-P(t)}{t}\,dt \leq 8\int_{t_2}^{B} \frac{\log(A/(t-A))}{t}\,dt.$$

We use the trivial bound of $1-P(t) \leq 2$ for the range not given above. For any given B, the preceding discussion gives us an upper bound for $\int_2^B (1-P(t))/t\,dt$ and we may derive a lower bound for A by Lemma 17 which states that

$$4\log A \geq 3 - \int_2^B \frac{1-P(t)}{t}\,dt + o(1).$$

Without loss of generality, say that for some $\delta \geq 0$ that $q_1 = q^{1-\delta}$ and $q_2 = q$, and note that $B = (2-\delta)A$. If q_1 is much smaller compared to q_2, then we expect to derive little benefit from the above and then our bound will be $\mathcal{N} \ll q^{(1-\delta)/(4\sqrt{e})}$. The rest is a numerical optimization using Maple over values of δ from which we derive that the worst value for δ occurs when $\delta = 0.061\ldots$ and then

$$\mathcal{N} \ll q^{0.142},$$

or equivalently,

$$\mathcal{N} \ll (q_1q_2)^{0.146/2}.$$

When $q_1 \asymp q_2 = q$, $\delta = 0$ and we have

$$\mathcal{N} \ll (q_1q_2)^{0.141/2}.$$

Remark 6. The reader may be curious about whether this result might be improved if we included the $I_2(u)$ and $I_3(u)$ terms, as we did in the cubic case. While we may improve the result with enough care, the possible improvements here are limited.

The reason is because when $1 \le t \le A$, we expect $P_i(t)$ to be close to -1, and when $A < t \le 2$, we have $P_i(t) = 1$. Thus $P(t)$ is close to 1 for $1 \le t \le 2$. Hence for $u \le 4$, it would be reasonable to expect $I_2(u)$ and $I_3(u)$ to be fairly small. □

Acknowledgements

I would like to express my gratitude to Professor Soundararajan for very generously sharing his time and ideas on various topics in this paper, as well as for his constant encouragement throughout. I also wish to thank Vorrapan Chandee for a careful reading of this paper. Finally, I am grateful to the referee for many helpful editorial remarks.

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Communicated by Andrew Granville
Received 2010-06-22 Revised 2011-09-08 Accepted 2011-09-26

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Algebra & Number Theory (ISSN 1937-0652) at Mathematical Sciences Publishers, Department of Mathematics, University of California, Berkeley, CA 94720-3840 is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

ANT peer review and production are managed by EditFLOW® from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

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Typeset in LaTeX

Algebra & Number Theory

Volume 6 No. 6 2012

1937-0652(2012)6:6;1-9