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We give a geometric interpretation of the inner product on the modified quantum group of $\widehat{\mathfrak{sl}}_n$. We also give some applications of this interpretation, including a positivity result for the inner product, and a new geometric construction of the canonical basis.

1. Introduction

Let \mathbf{U} be a quantized universal enveloping algebra. The positive part \mathbf{U}^+ of \mathbf{U} is well known to possess a canonical basis [Lusztig 1990; Kashiwara 1991; Lusztig 1991]. In contrast, there is no natural basis for the algebra \mathbf{U} itself. However Lusztig [1992] has defined a variant of the quantized enveloping algebra known as the modified quantum group. This algebra has essentially the same representation theory, and can be given a canonical basis \mathbf{B} which packages together natural bases of the tensor product of a highest and lowest weight \mathbf{U} -module (when such modules exist), in the same way that the canonical basis \mathbf{B} of \mathbf{U}^+ packages together natural bases of highest weight representations. Just as for \mathbf{B} (see [Grojnowski and Lusztig 1993; Kashiwara 1991]) it is possible to characterise this basis, up to sign, in terms of an involution and an inner product.

In [Beilinson et al. 1990] the quantized enveloping algebra of \mathfrak{gl}_n was constructed geometrically as a limit of certain convolution algebras. Subsequently Lusztig [1993, Part 4, Notes 1; 1999], and independently Ginzburg and Vasserot [1993], observed that this construction could be extended to the case of quantum affine \mathfrak{sl}_n . More precisely, one can define a sequence of algebra \mathcal{A}_D , and maps $\psi_D : \mathcal{A}_D \rightarrow \mathcal{A}_{D-n}$, along with compatible maps ϕ_D from the modified quantum group $\dot{\mathbf{U}}(\widehat{\mathfrak{sl}}_n)$ such that the resulting map into the inverse limit of the system $(\mathcal{A}_D, \psi_D)_{D \in \mathbb{N}}$ is injective (in fact we will give a new proof of this injectivity statement in Section 7). One of the main results of the present paper is a construction of the inner product on

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the modified quantum group $\hat{\mathbf{U}}(\widehat{\mathfrak{sl}}_n)$ as a kind of limit of a natural family of inner products on the algebras $(\mathfrak{A}_D)_{D \in \mathbb{N}}$. We do this both in the context of function on \mathbb{F}_q -rational points and perverse sheaves, giving two proofs of the fact that our construction yields the inner product on the modified quantum group — the first is elementary, using explicit formulae for multiplication in $\mathfrak{A}_{D,n,n}$, while the second, although requiring more machinery, gives a more conceptual explanation in terms of equivariant cohomology. As applications of these results we give a new construction of the canonical basis of $\hat{\mathbf{U}}$ in this context (which was already shown in [Schiffmann and Vasserot 2000] using crystal basis techniques), and a prove a positivity property for the inner product of elements of $\hat{\mathbf{B}}$ which is conjectured to hold for arbitrary types.

2. Background

We begin by recalling the setup of [Lusztig 1999]. Fix a positive integer n . Let D be a positive integer, ϵ an indeterminate, \mathbb{k} a finite field with q elements and v a square root of q . Given a free $\mathbb{k}[\epsilon, \epsilon^{-1}]$ -module V of rank D , a *lattice* in V is a free $\mathbb{k}[\epsilon]$ -submodule of V , of rank D . Let \mathcal{F}^n denote the set of n -step periodic lattices in V , that is, \mathcal{F}^n consists of sequences of lattices $\mathbf{L} = (L_i)_{i \in \mathbb{Z}}$ where $L_{i-1} \subset L_i$, and $L_{i-n} = \epsilon L_i$ for all $i \in \mathbb{Z}$. We will also write \mathcal{F}_D^n if we wish to emphasise the rank of V . Throughout this paper, if X is a finite set, we will write $|X|$ for the cardinality of X .

The group $G = \text{Aut}(V)$ of automorphisms of V acts on \mathcal{F}^n in the natural way. We shall be interested in functions supported on \mathcal{F}^n and its square which are invariant with respect to the action of G (where G acts diagonally on $\mathcal{F}^n \times \mathcal{F}^n$). Thus we first describe the orbits of G on these spaces. Let $\mathfrak{S}_{D,n}$ be the finite set of all $\mathbf{a} = (a_i)_{i \in \mathbb{Z}}$ such that

- $a_i \in \mathbb{N}$;
- $a_i = a_{i+n}$ for all $i \in \mathbb{Z}$;
- $a_i + a_{i+1} + \dots + a_{i+n-1} = D$ for all $i \in \mathbb{Z}$.

For $\mathbf{L} \in \mathcal{F}^n$, let $|\mathbf{L}| \in \mathfrak{S}_{D,n}$ be given by $|\mathbf{L}|_i = \dim(L_i/L_{i-1})$. The G -orbits on \mathcal{F}^n are indexed by this graded dimension: for $\mathbf{a} \in \mathfrak{S}_{D,n}$ set $\mathcal{F}_{\mathbf{a}} = \{\mathbf{L} \in \mathcal{F}^n : |\mathbf{L}| = \mathbf{a}\}$; then the $\mathcal{F}_{\mathbf{a}}$ are precisely the G -orbits on \mathcal{F}^n . The G -orbits on $\mathcal{F}^n \times \mathcal{F}^n$ are indexed, slightly more elaborately, by the set $\mathfrak{S}_{D,n,n}$ of matrices $A = (a_{i,j})_{i,j \in \mathbb{Z}}$ such that

- $a_{i,j} \in \mathbb{N}$;
- $a_{i,j} = a_{i+n,j+n}$ for all $i, j \in \mathbb{Z}$;
- $a_{i,*} + a_{i+1,*} + \dots + a_{i+n-1,*} = D$ for any $i \in \mathbb{Z}$;
- $a_{*,j} + a_{*,j+1} + \dots + a_{*,j+n-1} = D$ for any $j \in \mathbb{Z}$.

Here

$$a_{i,*} = \sum_{j \in \mathbb{Z}} a_{i,j} \quad \text{and} \quad a_{*,j} = \sum_{i \in \mathbb{Z}} a_{i,j}.$$

For $A \in \mathfrak{S}_{D,n,n}$ set

$$r(A) = (a_{i,*})_{i \in \mathbb{Z}} \in \mathfrak{S}_{D,n} \quad \text{and} \quad c(A) = (a_{*,j})_{j \in \mathbb{Z}} \in \mathfrak{S}_{D,n}.$$

For $A \in \mathfrak{S}_{D,n,n}$ the corresponding G -orbit \mathbb{O}_A consists of pairs (L, L') such that

$$a_{i,j} = \dim \left(\frac{L_i \cap L'_j}{(L_{i-1} \cap L'_j) + (L_i \cap L'_{j-1})} \right),$$

thus $L \in \mathfrak{F}_{r(A)}$ and $L' \in \mathfrak{F}_{c(A)}$.

Let $\mathfrak{A}_{D;q}$ be the space of integer-valued G -invariant functions on $\mathfrak{F}^n \times \mathfrak{F}^n$ supported on a finite number of orbits. If e_A denotes the characteristic function of an orbit \mathbb{O}_A , the set $\{e_A : A \in \mathfrak{S}_{D,n,n}\}$ is a basis of $\mathfrak{A}_{D;q}$. The space $\mathfrak{A}_{D;q}$ has a natural convolution product which gives it the structure of an associative algebra. With respect to the basis of characteristic functions the structure constants are given as follows. For $A, B, C \in \mathfrak{S}_{D,n,n}$, let $\eta_{A,B;q}^C$ be the coefficient of e_C in the product $e_A e_B$. Then $\eta_{A,B;q}^C$ is zero unless $c(A) = r(B)$, $r(A) = r(C)$ and $c(B) = c(C)$. Now suppose these conditions are satisfied and fix $(L, L'') \in \mathbb{O}_C$. Then $\eta_{A,B;q}^C$ is the number of points in the set

$$\{L' \in \mathfrak{F}_{c(A)} : (L, L') \in \mathbb{O}_A, (L', L'') \in \mathbb{O}_B\}.$$

Clearly this is independent of the choice of (L, L'') , and moreover it can be shown that these structure constants are polynomial in q , allowing us to construct an algebra $\mathfrak{A}_{D,\mathbb{Z}[t]}$ over $\mathbb{Z}[t]$, which is a free $\mathbb{Z}[t]$ -module on a basis $\{e_A : A \in \mathfrak{S}_{D,n,n}\}$ such that $\mathfrak{A}_{D,\mathbb{Z}[t]|_{t=q}} = \mathfrak{A}_{D;q}$. In fact we will use the algebra $\mathfrak{A}_{D,\mathcal{A}}$ which is obtained from $\mathfrak{A}_{D,\mathbb{Z}[t]}$ by extending scalars to $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$ where $v^2 = t$ and the $\mathbb{Q}(v)$ -algebra \mathfrak{A}_D obtained by extending scalars to $\mathbb{Q}(v)$ (we will, by deliberate misuse, treat v as both an indeterminate and a square root of q , depending on the context). The algebra \mathfrak{A}_D is sometimes known as the affine q -Schur algebra. In what follows it will be more convenient to use a rescaled version of the basis $\{e_A\}_{A \in \mathfrak{S}_{D,n,n}}$ of $\mathfrak{A}_{D,\mathcal{A}}$, with elements $[A] = v^{-d_A} e_A$, where

$$d_A = \sum_{\substack{i \geq k \\ j < l \\ 1 \leq i \leq n}} a_{ij} a_{kl}.$$

Note that if we define $\Psi([A]) = [A^t]$, where $(A^t)_{ij} = a_{ji}$, then it is straightforward to check that Ψ is an algebra antiautomorphism (see [Lusztig 1999, Lemma 1.11] for details), which we will sometimes call the transpose antiautomorphism.

Next we introduce quantum groups. In order to do this we recall the notion of a root datum from [Lusztig 1993].

Definition 2.1. A *Cartan datum* is a pair (I, \cdot) consisting of a finite set I and a \mathbb{Z} -valued symmetric bilinear pairing on the free abelian group $\mathbb{Z}[I]$, such that

- $i \cdot i \in \{2, 4, 6, \dots\}$ and
- $2 \frac{i \cdot j}{i \cdot i} \in \{0, -1, -2, \dots\}$, for $i \neq j$.

A *root datum* of type (I, \cdot) is a pair Y, X of finitely generated free abelian groups and a perfect pairing $\langle \cdot, \cdot \rangle : Y \times X \rightarrow \mathbb{Z}$, together with embeddings $I \subset X$ ($i \mapsto i$) and $I \subset Y$ ($i \mapsto i'$) such that $\langle i', j \rangle = 2(i \cdot j)/(i \cdot i)$.

Given a root datum, we may define an associated quantum group \mathbf{U} . Since it is the only case we need, we will assume that our datum is symmetric and simply laced so that $i \cdot i = 2$ for each $i \in I$, and $i \cdot j \in \{0, -1\}$ if $i \neq j$. In this case, \mathbf{U} is generated as an algebra over $\mathbb{Q}(v)$ by symbols $E_i, F_i, K_\mu, i \in I, \mu \in Y$, subject to the following relations.

- (1) $K_0 = 1, K_{\mu_1} K_{\mu_2} = K_{\mu_1 + \mu_2}$ for $\mu_1, \mu_2 \in Y$.
- (2) $K_\mu E_i K_\mu^{-1} = v^{\langle \mu, i' \rangle} E_i, \quad K_\mu F_i K_\mu^{-1} = v^{-\langle \mu, i' \rangle} F_i$ for all $i \in I, \mu \in Y$.
- (3) $E_i F_j - F_j E_i = \delta_{i,j} (K_i - K_i^{-1}) / (v - v^{-1})$.
- (4) $E_i E_j = E_j E_i, \quad F_i F_j = F_j F_i$, for $i, j \in I$ with $i \cdot j = 0$.
- (5) $E_i^2 E_j + (v + v^{-1}) E_i E_j E_i + E_j E_i^2 = 0$ for $i, j \in I$ with $i \cdot j = -1$.
- (6) $F_i^2 F_j + (v + v^{-1}) F_i F_j F_i + F_j F_i^2 = 0$ for $i, j \in I$ with $i \cdot j = -1$.

Thus \mathbf{U} is naturally X -graded, $\mathbf{U} = \bigoplus_{\nu \in X} \mathbf{U}_\nu$. The subalgebras \mathbf{U}^+ and \mathbf{U}^- generated by the E_i s and F_i s respectively are isomorphic to each other, and indeed are isomorphic to the $\mathbb{Q}(v)$ -algebra \mathfrak{f} generated by symbols $\{\theta_i : i \in I\}$ subject only to the relation

$$\begin{aligned} \theta_i \theta_j - \theta_j \theta_i &= 0 \text{ for } i, j \in I \text{ with } i \cdot j = 0, \\ \theta_i^2 \theta_j + (v + v^{-1}) \theta_i \theta_j \theta_i + \theta_j \theta_i^2 &= 0 \text{ for } i, j \in I \text{ with } i \cdot j = -1. \end{aligned}$$

Note that the algebra \mathfrak{f} depends only on the Cartan datum.

We also need to consider the modified quantum group $\dot{\mathbf{U}}$. This is defined by

$$\dot{\mathbf{U}} = \bigoplus_{\lambda \in X} \mathbf{U} 1_\lambda, \quad \mathbf{U} 1_\lambda = \mathbf{U} / \sum_{\mu \in Y} \mathbf{U} (K_\mu - v^{\langle \mu, \lambda \rangle}),$$

where the multiplicative structure is given in the natural way by the formulae

$$\begin{aligned} 1_\lambda x &= x 1_{\lambda - \nu} \quad \text{for } x \in \mathbf{U}_\nu, \\ 1_\lambda 1_{\lambda'} &= \delta_{\lambda, \lambda'} 1_\lambda. \end{aligned}$$

Let $\dot{\mathbf{U}}_{\mathcal{A}}$ be the \mathcal{A} -subalgebra of $\dot{\mathbf{U}}$ generated by $\{E_i^{(n)} 1_\lambda, F_i^{(n)} 1_\lambda : n \in \mathbb{Z}_{\geq 0}, \lambda \in X\}$. It is known [Lusztig 1993] that $\dot{\mathbf{U}}_{\mathcal{A}}$ is an \mathcal{A} -form of $\dot{\mathbf{U}}$, and moreover it is a free \mathcal{A} -module.

To describe the connection between our convolution algebra and quantum groups, we will need the following notation. For $\mathbf{a} \in \mathfrak{S}_{D,n}$ let $\mathbf{i}_a \in \mathfrak{S}_{D,n,n}$ be the diagonal matrix with $(\mathbf{i}_a)_{i,j} = \delta_{i,j} a_i$. Let $E^{i,j} \in \mathfrak{S}_{1,n,n}$ be the matrix with $(E^{i,j})_{k,l}$ equal to 1 if $k = i + sn$ and $l = j + sn$ for some $s \in \mathbb{Z}$, and to 0 otherwise. Let \mathfrak{S}^n be the set of all $\mathbf{b} = (b_i)_{i \in \mathbb{Z}}$ such that $b_i = b_{i+n}$ for all $i \in \mathbb{Z}$. Let $\mathfrak{S}^{n,n}$ denote the set of all matrices $A = (a_{i,j})$, $i, j \in \mathbb{Z}$, with entries in \mathbb{Z} such that

- $a_{i,j} \geq 0$ for all $i \neq j$;
- $a_{i,j} = a_{i+n,j+n}$ for all $i, j \in \mathbb{Z}$;
- for any $i \in \mathbb{Z}$ the set $\{j \in \mathbb{Z} : a_{i,j} \neq 0\}$ is finite;
- for any $j \in \mathbb{Z}$ the set $\{i \in \mathbb{Z} : a_{i,j} \neq 0\}$ is finite.

Thus we have $\mathfrak{S}_{D,n,n} \subset \mathfrak{S}^{n,n}$ for all D . For $i \in \mathbb{Z}/n\mathbb{Z}$, let $\mathbf{i} \in \mathfrak{S}^n$ be given by $\mathbf{i}_k = 1$ if $k = i \pmod n$, $\mathbf{i}_k = -1$ if $k = i + 1 \pmod n$, and $\mathbf{i}_k = 0$ otherwise. We write $\mathbf{a} \cup_i \mathbf{a}'$ if $\mathbf{a} = \mathbf{a}' + \mathbf{i}$. For such \mathbf{a}, \mathbf{a}' set

$$\begin{aligned} {}_a \mathbf{e}_{a'} &= \mathbf{i}_a - E^{i,i} + E^{i,i+1} \in \mathfrak{S}^{n,n}, \\ {}_{a'} \mathbf{f}_a &= \mathbf{i}_{a'} - E^{i+1,i+1} + E^{i+1,i} \in \mathfrak{S}^{n,n}. \end{aligned}$$

Note that if $\mathbf{a}, \mathbf{a}' \in \mathfrak{S}_{D,n}$ then ${}_a \mathbf{e}_{a'}, {}_{a'} \mathbf{f}_a \in \mathfrak{S}_{D,n,n}$. For $i \in \mathbb{Z}/n\mathbb{Z}$ set

$$E_i(D) = \sum [{}_a \mathbf{e}_{a'}], \quad F_i(D) = \sum [{}_{a'} \mathbf{f}_a],$$

where the sum is taken over all \mathbf{a}, \mathbf{a}' in $\mathfrak{S}_{D,n}$ such that $\mathbf{a} \cup_i \mathbf{a}'$. For $\mathbf{a} \in \mathfrak{S}^n$ set

$$K_a(D) = \sum_{\mathbf{b} \in \mathfrak{S}_{D,n}} v^{a \cdot \mathbf{b}} [i_{\mathbf{b}}]$$

where $\mathbf{a} \cdot \mathbf{b} = \sum_{i=1}^n a_i b_i \in \mathbb{Z}$ for any $\mathbf{a}, \mathbf{b} \in \mathfrak{S}^n$. If we let $X = Y = \mathfrak{S}^n$, and $I = \mathbb{Z}/n\mathbb{Z}$, with the embedding of $I \subset X = Y$ given by $i \mapsto \mathbf{i}$ and pairing as given above, we obtain a symmetric simply laced root datum. We call the quantum group associated to it $\mathbf{U}(\widehat{\mathfrak{gl}}_n)$ (in fact this is the degenerate, or level zero, form of the affine quantum group associated to $\widehat{\mathfrak{gl}}_n$). It can be shown [Lusztig 1999] that the elements $E_i(D), F_i(D), K_a(D)$ generate a subalgebra \mathbf{U}_D which is a quotient of the quantum group $\mathbf{U}(\widehat{\mathfrak{gl}}_n)$, via the map that the notation suggests. In particular this gives the algebra \mathfrak{A}_D the structure of a $\mathbf{U}(\widehat{\mathfrak{gl}}_n)$ -module. Since $E_i(D), F_i(D), K_a(D)$ all lie in $\mathfrak{A}_{D,\mathcal{A}}$ we similarly have an \mathcal{A} -subalgebra $\mathbf{U}_{D,\mathcal{A}}$. Note that the idea that one may extend the construction of [Beilinson et al. 1990] to the affine context is mentioned already in [Lusztig 1993, Notes on Part IV] and [Ginzburg and Vasserot 1993], but

note that the surjectivity claimed in Theorem 9.2 of [Ginzburg and Vasserot 1993] is false, as shown in [Lusztig 1999].

Similarly, letting $\mathbf{b}_0 = (\dots, 1, 1, \dots) \in \mathfrak{S}^n$, we have a root datum given by $X' = \mathfrak{S}^n / \mathbb{Z}\mathbf{b}_0$, $Y' = \{\mathbf{a} \in \mathfrak{S}^n : \mathbf{a} \cdot \mathbf{b}_0 = 0\}$ with embeddings $I \subset X', Y'$ induced by the above embeddings into X and Y . This new root datum is associated to (again the degenerate form of) the quantum group $U(\widehat{\mathfrak{sl}}_n)$. We have an algebra map $\phi_D : \dot{U}(\widehat{\mathfrak{sl}}_n) \rightarrow \mathfrak{A}_D$ given by $E_i^{(n)} 1_\lambda \mapsto E_i(D)^{(n)}[\mathbf{i}_a]$ where $\mathbf{a} \in \mathfrak{S}_{D,n}$ satisfies $\mathbf{a} \equiv \lambda \pmod{\mathbb{Z}\mathbf{b}_0}$ if such an \mathbf{a} exists, and $E_i^{(n)} 1_\lambda \mapsto 0$ otherwise, and similarly for the $F_i^{(n)} 1_\lambda$. Clearly ϕ_D restricts to a map between the integral forms $\dot{U}(\widehat{\mathfrak{sl}}_n)_{\mathcal{A}}$ and $\mathfrak{A}_{D,\mathcal{A}}$. It can be readily checked, using a Vandermonde determinant argument, that the image of ϕ_D is exactly U_D (see [Lusztig 1999, Lemma 2.8]).

3. Inner product on U_D

Definition 3.1. We define a bilinear form

$$(\cdot, \cdot)_D : \mathfrak{A}_{D;q} \times \mathfrak{A}_{D;q} \rightarrow \overline{\mathbb{Q}}_l$$

by

$$(f, \tilde{f})_D = \sum_{L, L'} v^{\sum |L|_i^2 - \sum |L'|_i^2} f(L, L') \tilde{f}(L, L'), \tag{3-1}$$

for f and \tilde{f} in $\mathfrak{A}_{D,q}$, where L runs over \mathcal{F}^n and L' runs over a set of representatives for the G -orbits on \mathcal{F}^n .

Let \mathcal{O}_A be a G -orbit on $\mathcal{F}^n \times \mathcal{F}^n$, and let

$$X_A^L = \{L' \in \mathcal{F}^n : (L, L') \in \mathcal{O}_A\}.$$

It is easy to check that

$$2d_A - 2d_{A'} = \sum_{i=1}^n a_{i,*}^2 - \sum_{j=1}^n a_{*,j}^2. \tag{3-2}$$

Thus if A, A' are in $\mathfrak{S}_{D,n,n}$ we find that

$$(e_A, e_{A'})_D = \delta_{A,A'} q^{d_A - d_{A'}} |X_{A'}^{L'}|,$$

where L' is any lattice in $\mathcal{F}_{c(A)}$. Thus the basis $\{e_A : A \in \mathfrak{S}_{D,n,n}\}$ is orthogonal for our inner product, and hence (\cdot, \cdot) is nondegenerate. If $\{\eta_{A,B;q}^C\}$ are the structure constants of $\mathfrak{A}_{D;q}$ with respect to the basis $\{[A] : A \in \mathfrak{S}_{D,n,n}\}$, then we have

$$([A], [A'])_D = \delta_{A,A'} v^{-d_{A'}} \eta_{A',A;q}^{i_{c(A)}} \tag{3-3}$$

We therefore obtain an inner product on $\mathfrak{A}_{D,\mathcal{A}}$ taking values in \mathcal{A} by defining

$$([A], [A'])_D = \delta_{A,A'} v^{-d_{A'}} \eta_{A',A}^{i_{c(A)}} \in \mathbb{Z}[v, v^{-1}]. \tag{3-4}$$

By extending scalars, we obtain a $\mathbb{Q}(v)$ -valued symmetric bilinear form on \mathfrak{A}_D . We now give some basic properties of this inner product:

Proposition 3.2. *Let $A \in \mathfrak{S}_{D,n}$, and let $f, \tilde{f} \in \mathfrak{A}_D$. Then*

$$([A]f, \tilde{f})_D = v^{d_A - d_{A'}} (f, [A^t] \tilde{f}).$$

Proof. Clearly it suffices to establish this equation in the algebra $\mathfrak{A}_{D;q}$. Since the characteristic functions of G-orbits form a basis of $\mathfrak{A}_{D;q}$, we may assume that $f = e_B$ and $\tilde{f} = e_C$, moreover we may assume that

$$r(A) = r(C), \quad c(A) = r(B), \quad c(B) = c(C). \tag{3-5}$$

as both sides are zero otherwise. It follows immediately that

$$[A] \cdot e_B = v^{-d_A} e_A \cdot e_B, \quad v^{d_A - d_{A'}} [A^t] \cdot e_C = v^{d_A - 2d_{A'}} e_{A'} \cdot e_C.$$

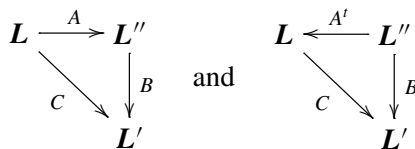
Hence, if $(\tilde{L}, L') \in \mathbb{O}_C$ is fixed,

$$\begin{aligned} ([A] \cdot e_B, e_C)_D &= q^{d_C - d_{C'}} |X_{C'}^{L'}| \cdot v^{-d_A} |\{L'' : (\tilde{L}, L'') \in \mathbb{O}_A, (L'', L') \in \mathbb{O}_B\}| \\ &= v^\alpha |\{L, L'' : (L, L'') \in \mathbb{O}_A, (L'', L') \in \mathbb{O}_B, (L, L') \in \mathbb{O}_C\}|, \end{aligned} \tag{3-6}$$

where $\alpha = 2d_C - 2d_{C'} - d_A$. Similarly, if $(\tilde{L}'', L') \in \mathbb{O}_B$ is fixed,

$$\begin{aligned} v^{d_A - d_{A'}} (e_B, [A^t] \cdot e_C)_D &= q^{d_B - d_{B'}} |X_{B'}^{L'}| \cdot v^{d_A - 2d_{A'}} |\{L : (\tilde{L}'', L) \in \mathbb{O}_{A'}, (L, L') \in \mathbb{O}_C\}| \\ &= v^\beta |\{L, L'' : (L'', L) \in \mathbb{O}_{A'}, (L'', L') \in \mathbb{O}_B, (L, L') \in \mathbb{O}_C\}|, \end{aligned} \tag{3-7}$$

where $\beta = 2d_B - 2d_{B'} + d_A - 2d_{A'}$. But now considering the diagrams



it is clear that the last line of (3-6) is the same as the last line of (3-7) if $\alpha = \beta$, that is, if

$$2d_C - 2d_{C'} - d_A = 2d_B - 2d_{B'} + d_A - 2d_{A'} \tag{3-8}$$

But this follows directly from (3-2) and (3-5). □

We have the following easy consequence:

Corollary 3.3. *Let $i \in \mathbb{Z}$, and let $f, \tilde{f} \in \mathfrak{A}_D$ and $\mathbf{c} \in \mathfrak{S}^n$. Then*

- (1) $(E_i(f), \tilde{f})_D = (f, vK_i F_i(\tilde{f}))_D,$
- (2) $(F_i(f), \tilde{f})_D = (f, vK_{-i} E_i(\tilde{f}))_D,$
- (3) $(K_{\mathbf{c}}(f), \tilde{f})_D = (f, K_{\mathbf{c}}(\tilde{f}))_D.$

Proof. We may assume that $f = e_A$ and $\tilde{f} = e_B$. The third equation can then be checked immediately from the formulas above. The second equation follows from the symmetry of the inner product and the other two, so it only remains to prove the first. We may assume that $r(A) = r(B) - \mathbf{i}$ and $c(A) = c(B)$, as both sides are zero otherwise. Set $\mathbf{a} = r(A)$, $\mathbf{b} = r(B)$. Now from the definitions we have

$$E_i(e_A) = [{}_b \mathbf{e}_a] \cdot e_A, \quad vK_i F_i(e_B) = v^{1+i \cdot \mathbf{a}} [{}_a \mathbf{f}_b] \cdot e_B.$$

Since ${}_b \mathbf{e}_a = {}_a \mathbf{f}_b^t$ and $d_{{}_b \mathbf{e}_a} - d_{{}_a \mathbf{f}_b} = 1 + \mathbf{i} \cdot \mathbf{a}$, the result now follows immediately from the previous proposition. □

Remark 3.4. There is a unique algebra antiautomorphism $\rho : \mathbf{U}(\widehat{\mathfrak{gl}}_n) \rightarrow \mathbf{U}(\widehat{\mathfrak{gl}}_n)$ such that

$$\rho(E_i) = vK_i F_i, \quad \rho(F_i) = vK_{-i} E_i \quad \rho(K_i) = K_i$$

With this we may state the result of the previous corollary in the form

$$(u(f), \tilde{f})_D = (f, \rho(u)\tilde{f})_D \quad \text{for } u \in \mathbf{U}(\widehat{\mathfrak{gl}}_n) \text{ and } f, \tilde{f} \in \mathfrak{A}_D. \tag{3-9}$$

Note also that another natural choice¹ of definition for an inner product on $\mathfrak{A}_{D,q}$ would be given by taking the sum in (3-1) over a set of representative of the $\text{Aut}(V)$ -orbits on \mathcal{F} in the first factor, and all lattices in the second (the opposite of our choice). This inner product, which we denote by $(\cdot, \cdot)_D^t$ is obtained from the one we use via the transpose antiautomorphism Ψ , that is,

$$(f, \tilde{f})_D^t = (\Psi(f), \Psi(\tilde{f}))_D,$$

and thus obeys “transposed” versions of the properties established in this section so that

$$(fu, \tilde{f})_D^t = (f, \tilde{f}\bar{\rho}(u))_D^t, \quad u \in \mathbf{U}(\widehat{\mathfrak{gl}}_n), \quad f, \tilde{f} \in \mathfrak{A}_D, \tag{3-10}$$

where $\bar{\rho}$ is given by $\bar{\rho}(x) = \overline{\rho(\bar{x})}$. The precise relation between $(\cdot, \cdot)_D$ and $(\cdot, \cdot)_D^t$ can be given as follows: if $f, \tilde{f} \in [{}_i \mathbf{a}]\mathfrak{A}_D[{}_i \mathbf{b}]$ then

$$v^{\sum_{i=1}^n b_i^2} ([\mathbf{b}]\!)^{-1} (f, \tilde{f})_D = v^{\sum_{i=1}^n a_i^2} ([\mathbf{a}]\!)^{-1} (f, \tilde{f})_D^t. \tag{3-11}$$

where we define $[\mathbf{a}]\! = \prod_{i=1}^n \prod_{j=1}^{a_i} (1 - v^{-2j})$. In the finite type case of [Beilinson et al. 1990] this follows easily from considering the orbits on the product of the

¹It is clear that one needs to restrict one factor to run over representatives of the G -orbits to avoid infinity sums — in the finite type case considered in [Beilinson et al. 1990] this issue doesn’t arise.

space of n -step flags with itself, while in the affine case it requires some more care. Since we will not use this fact we do not include the details.

Lemma 3.5. (1) For $A \in \mathfrak{S}_{D,n,n}$, we have $([A], [A])_D \in 1 + v^{-1}\mathbb{Z}[v^{-1}]$.

(2) For $A, A' \in \mathfrak{S}_{D,n,n}$ and $A \neq A'$, we have $([A], [A'])_D = 0$.

Proof. The second part of the statement is obvious. For the first, note that by [Lusztig 1999, 4.3] the set $X_{A'}^{L'}$ is an irreducible variety of dimension $d_{A'}$. Since

$$([A], [A'])_D = \delta_{A,A'} q^{-d_{A'}} |X_{A'}^{L'}|,$$

the Lang–Weil estimates [1954] then show that $([A], [A])_D$ lies in $1 + v^{-1}\mathbb{Z}[v^{-1}]$, as required. \square

Remark 3.6. The results of this section are analogues of the results of [Lusztig 1999, §7]; however our inner product is *not* the same as that of [Lusztig 1999, 7.1], and this makes the proofs somewhat simpler. Lusztig’s choice of inner product is compatible with the left and right \mathbf{U} -module structure, in the sense that Equations (3-9) and (3-10) both hold. Although this is not quite proved in [Lusztig 1999] it follows from our above discussion using (3-11). However, it does not produce the inner product on $\dot{\mathbf{U}}$ in the way we need.

4. Inner product on $\dot{\mathbf{U}}$

In this section we will write $\dot{\mathbf{U}}$ to denote the modified quantum group $\dot{\mathbf{U}}(\widehat{\mathfrak{sl}}_n)$ associated to the root datum (X', Y') defined in Section 2. We wish to obtain an inner product on $\dot{\mathbf{U}}$ using those on the family of algebras $\{\mathbf{U}_D\}_{D \in \mathbb{N}}$.

We begin with some technical lemmas. Given $A \in \mathfrak{S}^{n,n}$ let $a_{i, \geq s} = \sum_{j \geq s} a_{i,j}$, and $a_{i, > s}$, $a_{i, \leq s}$, etc. similarly.

Lemma 4.1. (a) Let $A \in \mathfrak{S}_{D,n,n}$ and $\mathbf{a}' = r(A)$. If there is an $\mathbf{a} \in \mathfrak{S}_{D,n}$ such that $\mathbf{a} \cup_i \mathbf{a}'$ (i.e., if $a'_{i+1} > 0$), then

$$[\mathbf{a} \mathbf{e} \mathbf{a}'] [A] = \sum_{\substack{s \in \mathbb{Z} \\ a_{i+1,s} \geq 1}} v^{a_{i, \geq s} - a_{i+1, > s}} \left(\frac{1 - v^{-2(a_{i,s} + 1)}}{1 - v^{-2}} \right) [A + E^{i,s} - E^{i+1,s}], \quad (4-1)$$

where $A = (a_{i,j})$.

(b) Let $A' \in \mathfrak{S}_{D,n,n}$ and $\mathbf{a} = r(A')$. If there is an $\mathbf{a}' \in \mathfrak{S}_{D,n}$ such that $\mathbf{a} \cup_i \mathbf{a}'$ (i.e., if $a_i > 0$), then

$$[\mathbf{a}' \mathbf{f} \mathbf{a}] [A'] = \sum_{\substack{s \in \mathbb{Z} \\ a'_{i,s} \geq 1}} v^{a'_{i+1, \leq s} - a'_{i, < s}} \left(\frac{1 - v^{-2(a'_{i+1,s} + 1)}}{1 - v^{-2}} \right) [A' - E^{i,s} + E^{i+1,s}], \quad (4-2)$$

where $A' = (a'_{i,j})$.

Proof. This follows by rescaling the statement of [Lusztig 1999, Proposition 3.5]. \square

Let \mathcal{R} be the subring of $\mathbb{Q}(v)[u]$ generated by $\{v^j : j \in \mathbb{Z}\}$ and by the elements

$$\prod_{i=1}^t \frac{v^{-2(a-i)}u^2 - 1}{v^{-2i} - 1} \quad \text{for } a \in \mathbb{Z}, t \geq 1.$$

For $A \in \mathfrak{S}^{n,n}$ let ${}_pA$ be the matrix with $({}_pA)_{i,j} = a_{i,j} + p\delta_{i,j}$. We have the following partial analogue of [Beilinson et al. 1990, 4.2].

Lemma 4.2. *Let A_1, A_2, \dots, A_k be matrices of the form $\mathbf{a}e_{\mathbf{a}'}$ or $\mathbf{a}\mathbf{f}_{\mathbf{a}'}$, for $\mathbf{a}, \mathbf{a}' \in \mathfrak{S}^n$, and A any element of $\mathfrak{S}^{n,n}$. Then there exist matrices $Z_1, Z_2, \dots, Z_m \in \mathfrak{S}^{n,n}$, and $G_1, G_2, \dots, G_m \in \mathcal{R}$ and an integer $p_0 \in \mathbb{Z}$ such that*

$$[{}_pA_1][{}_pA_2] \dots [{}_pA_k][{}_pA] = \sum_{i=1}^m G_i(v, v^{-p})[{}_pZ_i], \tag{4-3}$$

for all $p \geq p_0$.

Proof. This follows using the same argument as in the proof of Proposition 4.2 in [Beilinson et al. 1990] (where a similar but stronger result is proved for the finite-type case). One uses induction on k . When $k = 1$ the result follows from the previous lemma, once we note that both $a_{i,\geq s} - a_{i+1,>s}$ and $a_{i+1,\leq s} - a_{i,<s}$ are unchanged when A is replaced with $A + pI$. \square

Recall from Section 2 that there is a surjective homomorphism $\phi_D : \dot{\mathbf{U}} \rightarrow \mathbf{U}_D$ which, for $\lambda \in X$, sends $E_i 1_\lambda \mapsto E_i(D)[i_\lambda]$ and $F_i 1_\lambda \mapsto F_i(D)[i_\lambda]$ if there is an \mathbf{a} in $\mathfrak{S}_{D,n}$ such that $\mathbf{a} = \lambda \bmod \mathbb{Z}\mathbf{b}_0$, otherwise both $E_i 1_\lambda, F_i 1_\lambda$ are sent to zero. Let \mathbf{f} be the algebra defined in Section 2. Pick a monomial basis of \mathbf{f} , $\{\zeta_i : i \in J\}$ say. The triangular decomposition for $\dot{\mathbf{U}}$ [Lusztig 1992, 23.2.1] shows that $\mathfrak{B} = \{\zeta_i^+ \zeta_j^- 1_\lambda : i, j \in J, \lambda \in X\}$ is a basis of $\dot{\mathbf{U}}$, where $+ : \mathbf{f} \rightarrow \mathbf{U}^+$, and $- : \mathbf{f} \rightarrow \mathbf{U}^-$ are the standard isomorphisms defined by $\theta_i \mapsto E_i$ and $\theta_i \mapsto F_i$ respectively. Define a bilinear pairing $\langle \cdot, \cdot \rangle_D$ on $\dot{\mathbf{U}}$ via ϕ_D as follows:

$$\langle x, y \rangle_D = (\phi_D(x), \phi_D(y))_D.$$

Proposition 4.3. *Let $k \in \{0, 1, \dots, n - 1\}$, then if $x, y \in \dot{\mathbf{U}}$*

$$\langle x, y \rangle_{k+pn}$$

converges in $\mathbb{Q}((v^{-1}))$, as $p \rightarrow \infty$, to an element of $\mathbb{Q}(v)$.

Proof. We may assume that x, y are elements of \mathfrak{B} . Then we need to show that

$$\langle \zeta_{i_1}^+ \zeta_{j_1}^- 1_\lambda, \zeta_{i_2}^+ \zeta_{j_2}^- 1_\mu \rangle_{k+pn}, \quad \text{for } i_1, i_2, j_1, j_2 \in J; \lambda, \mu \in X,$$

converges as $p \rightarrow \infty$. Let $\iota : \mathfrak{f} \rightarrow \mathfrak{f}$ be the $\mathbb{Q}(v)$ -algebra antiautomorphism fixing the generators θ_i , $1 \leq i \leq n$. Using [Corollary 3.3](#), it is easy to see that this inner product differs from

$$\langle 1_\lambda, \iota(\zeta_{j_1})^+ \iota(\zeta_{i_1})^- \zeta_{i_2}^+ \zeta_{j_2}^- 1_\mu \rangle_{k+pn} \tag{4-4}$$

by a power of v which is independent of p . But then the definition of the inner product and [Lemma 4.2](#) show that (4-4) may be written as $G(v, v^{-p})$ for some $G \in \mathcal{R}$. The result then follows immediately from the definition of \mathcal{R} . \square

Remark 4.4. The proof of the last proposition actually allows us to conclude that

$$(\phi_D(\zeta_i^+ \zeta_j^- 1_\lambda), [{}_p A])_{k+pn}$$

converges to an element of $\mathbb{Q}(v)$, as $p \rightarrow \infty$, for any $A \in \mathfrak{S}^{n,n}$. We will need this in the next section.

Definition 4.5. We define

$$\langle \cdot, \cdot \rangle : \dot{\mathbf{U}} \times \dot{\mathbf{U}} \rightarrow \mathbb{Q}(v),$$

a symmetric bilinear form on $\dot{\mathbf{U}}$ given by

$$\langle x, y \rangle = \sum_{k=0}^{n-1} \lim_{p \rightarrow \infty} \langle x, y \rangle_{k+pn}.$$

Remark 4.6. Although the inner products $(\cdot, \cdot)_D$ satisfy only (3-9) and not (3-10), the formula (3-11) which relates $(\cdot, \cdot)_D$ and $(\cdot, \cdot)_D^t$ can be used to show that our limiting inner product $\langle \cdot, \cdot \rangle$ satisfies the analogue of both equations, as indeed Lusztig shows for his inner product on $\dot{\mathbf{U}}$ in [[Lusztig 1993](#), Proposition 26.1.3].

5. Comparison of inner products

Lusztig has shown that the algebra $\dot{\mathbf{U}}$ has a natural inner product which characterised by the following result. (Again, in this section $\dot{\mathbf{U}}$ denotes the modified quantum group attached to the root datum (X', Y') .)

Theorem 5.1 (Lusztig). *There exists a unique $\mathbb{Q}(v)$ bilinear pairing*

$$(\cdot, \cdot) : \dot{\mathbf{U}} \times \dot{\mathbf{U}} \rightarrow \mathbb{Q}(v)$$

such that

- (1) $(1_{\lambda_1} x 1_{\lambda_2}, 1_{\mu_1} y 1_{\mu_2}) = 0$ for all $x, y \in \dot{\mathbf{U}}$ unless $\lambda_1 = \mu_1, \lambda_2 = \mu_2$;
- (2) $(ux, y) = (x, \rho(u)y)$ for all $x, y \in \dot{\mathbf{U}}$ and $u \in \mathbf{U}$;
- (3) $(x^- 1_\lambda, y^- 1_\lambda) = (x, y)$ for all $x, y \in \mathfrak{f}$ and $\lambda \in X$.

Here (x, y) is the standard inner product on \mathbf{f} (see [Lusztig 1993, 1.2.5]). The resulting inner product is automatically symmetric.

Proof. See [Lusztig 1993, 26.1.2]. \square

Theorem 5.2. *The inner products $\langle \cdot, \cdot \rangle$ of Section 4 and (\cdot, \cdot) of Theorem 5.1 coincide.*

The remainder of this section is devoted to a proof of this theorem. Property (1) in Theorem 5.1 clearly holds for $\langle \cdot, \cdot \rangle$, as the representatives for elements of X in $\mathfrak{S}_{D,n}$ are distinct when they exist. Property (2) follows from Corollary 3.3; thus it only remains to verify (3).

Fix $\lambda \in X$. The algebra \mathbf{f} is naturally graded: $\mathbf{f} = \bigoplus_{\nu \in \mathbb{N}I} \mathbf{f}_\nu$. For $\nu \in \mathbb{Z}[I]$ with $\nu = \sum_{i \in I} \nu_i i$, let $\text{tr}(\nu) = \sum_{i \in I} \nu_i$. If z is homogeneous we set $|z| = \nu$, where $z \in \mathbf{f}_\nu$. Thus for the third property we may assume that $x, y \in \mathbf{f}$ are homogeneous, i.e., $x, y \in \mathbf{f}_\nu$ for some ν , and proceed by induction on $N = \text{tr}(\nu)$. If $N = 0$ then we are reduced to the equation

$$\langle 1_\lambda, 1_\lambda \rangle = 1,$$

which holds trivially. Now suppose that $N > 0$ and the result is known for $x, y \in \mathbf{f}_\nu$ when $\text{tr}(\nu) < N$. If x, y are in \mathbf{f}_ν and $\text{tr}(\nu) = N$, then we may assume that they are monomials, and $y = \theta_i z$ for some $z \in \mathbf{f}_{\nu-i}$. Thus we have

$$\begin{aligned} \langle x^{-1} 1_\lambda, y^{-1} 1_\lambda \rangle &= \langle x^{-1} 1_\lambda, F_i z^{-1} 1_\lambda \rangle \\ &= \langle \nu K_{-i} E_i x^{-1} 1_\lambda, z^{-1} 1_\lambda \rangle, \end{aligned}$$

using property (2) of the inner product (which we have already seen holds for both (\cdot, \cdot) and $\langle \cdot, \cdot \rangle$). Now using standard commutation formulas (see [Lusztig 1993, 3.1.6]) this becomes

$$\langle \nu K_{-i} x^{-1} E_i 1_\lambda, z^{-1} 1_\lambda \rangle + \frac{1}{1 - \nu^{-2}} \langle (i r(x)^{-} - \nu K_{-i} r_i(x)^{-} K_{-i}) 1_\lambda, z^{-1} 1_\lambda \rangle$$

where $i r$ and r_i are the twisted derivations defined in [Lusztig 1993, 1.2.13]. Tidying this up we get

$$\frac{1}{1 - \nu^{-2}} \langle i r(x)^{-} 1_\lambda, z^{-1} 1_\lambda \rangle + \left\langle \nu^{i \cdot |x| - i \cdot \lambda - 1} \left(x^{-1} E_i - \frac{\nu^{-i \cdot \lambda}}{\nu - \nu^{-1}} r_i(x)^{-} \right) 1_\lambda, z^{-1} 1_\lambda \right\rangle$$

But $i r(x), z \in \mathbf{f}_{\nu-i}$, hence by induction we have $\langle i r(x) 1_\lambda, z 1_\lambda \rangle = (i r(x)^{-}, z)$, and by standard properties of the inner product (\cdot, \cdot) on \mathbf{f} we know that

$$\frac{1}{1 - \nu^{-2}} (i r(x), z) = (x, \theta_i z);$$

thus we are done by induction if we can show that for any $x \in \mathfrak{f}_v$ the element

$$u(x) = \left(x^- E_i - \frac{v^{-i \cdot \lambda}}{v - v^{-1}} r_i(x)^- \right) 1_\lambda \tag{5-1}$$

annihilates $U^{-1}\lambda$. To see this we need some (rather technical) lemmas about multiplication in \mathfrak{A}_D .

Lemma 5.3. *Let $A \in \mathfrak{S}^{n,n}$ be such that $a_{r,s} = 0$ for $r < s$ unless $r = s - 1$ and $r = i \pmod n$, when $a_{r,r+1} \in \{0, 1\}$; then the following hold for p sufficiently large.*

(1) *For $j \neq i$ we have*

$$F_j[pA] = \sum_{k=1}^m g_k(v)[pZ_k]$$

where $g_k(v) \in \mathfrak{A}$ and $Z_k \in \mathfrak{S}^{n,n}$ ($1 \leq k \leq m$) and moreover $g_k(v)$ is independent both of p and $\{a_{r,s} : r \leq s\}$, and we have $(Z_k)_{r,s} = a_{r,s}$ for $r < s$.

(2)
$$F_i[pA] = \sum_{k=1}^m g_k(v)[pZ_k] + v^{1-i \cdot r(A)} \left(\frac{1 - v^{-2(a_{i+1,i+1}+1+p)}}{1 - v^{-2}} \right) [p(A + E^{i+1,i+1} - E^{i,i+1})],$$

where $g_k(v) \in \mathfrak{A}$ is independent of p and $\{a_{r,s} : r \leq s\}$, and we have $(Z_k)_{r,s} = a_{r,s}$ for $r < s$, and the final term occurs only if $a_{i,i+1} = 1$.

Proof. By Lemma 4.1, for any $A \in \mathfrak{S}^{n,n}$ and p large enough we have

$$F_j[pA] = \sum_{k:(pA)_{j,k} \geq 1} v^{a_{j+1,\leq k} - a_{j,<k}} \left(\frac{1 - v^{-2(a_{j+1,k} + p\delta_{j+1,k+1})}}{1 - v^{-2}} \right) [pA + E^{j+1,k} - E^{j,k}].$$

We claim that in our case the coefficients are independent of p and of $\{a_{r,s} : r \leq s\}$, unless $j \equiv i \pmod n$. Indeed then $(pA)_{j,k} \geq 1$ implies that $j \geq k$, and hence the coefficient of $[pA + E^{j+1,k} - E^{j,k}]$ in the sum above is

$$v^{a_{j+1,\leq k} - a_{j,<k}} \left(\frac{1 - v^{-2(a_{j+1,k+1})}}{1 - v^{-2}} \right),$$

which evidently involves only entries $a_{r,s}$ of A with $r > s$, thus establishing the first part of the lemma.

For the second part, if $j \equiv i \pmod n$ then we get the same conclusion except when $k = j + 1$, if $a_{j,j+1} = 1$ in which case $a_{j+1,\leq j+1} = a_{j+1,*}$ and $a_{j,<j+1} = a_{j,*} - 1$ by our assumptions, so that the term $a_{j+1,\leq k} - a_{j,<k} = 1 - i \cdot r(A)$, and this yields the final term in second part, as required. \square

Fix $\lambda \in X'$. Then $\sum_{i=1}^n \lambda_i = k \pmod n$ for a well-defined $k \in \{0, 1, \dots, n-1\}$. If $D = k + pn$, then there is a unique $\mathbf{a} \in X$ which is a representative of $\lambda \in X'$ satisfying $\sum_{i=1}^n a_i = D$ (that is, $\mathbf{a} \in \mathfrak{S}_{D,n}$).

Lemma 5.4. *Let $\sum_{j=1}^n \lambda_j = k \pmod n$, where $k \in \{0, 1, \dots, n-1\}$, and suppose that $D = k + pn$ for some p . Then if x is a monomial in the generators $\{\theta_i : i \in I\}$ we have, for sufficiently large p ,*

$$\begin{aligned} \phi_D(x^{-1} E_i 1_\lambda) &= \sum_{k=1}^{m_1} a_k(v) [{}_p B_k] + \sum_{k=1}^{m_2} (b_k(v) + v^{-2p} c_k(v)) [{}_p H_k], \\ \phi_D(x^{-1} 1_\lambda) &= \sum_{k=1}^{m_1} a_k(v) [{}_p B_k + E^{i+1, i+1} - E^{i, i+1}] \end{aligned} \tag{5-2}$$

for some $B_k, H_k \in \mathfrak{S}^{n,n}$ independent of p , where $(B_k)_{r,s} = (H_k)_{r,s} = 0$ for $r < s$ unless $r = i \pmod n$ when $(B_k)_{i, i+1} = 1$, and the coefficients a_k, b_k, c_k are independent of p , with $a_k \in \mathcal{A}$ and $b_k, c_k \in (v - v^{-1})^{-1} \mathcal{A}$. Moreover, we have

$$\frac{v^{-i \cdot \lambda}}{v - v^{-1}} \phi_D(r_i(x)^{-1} 1_\lambda) = \sum_{k=1}^{m_2} b_k(v) [{}_p H_k]$$

Proof. We use induction on $N = \text{tr}(|x|)$. If $N = 0$ then the result is clear, since we have $\phi_D(E_i 1_\lambda) = [i_{\mathbf{a}} + E^{i, i+1} - E^{i+1, i+1}]$ and $\phi_E(1_\lambda) = [i_{\mathbf{a}}]$ (thus in this case we have $m_1 = 1$ and $m_2 = 0$). Now suppose the result is known for all y with $\text{tr}(|y|) < N$. We may write $x = \theta_j z$ where $\text{tr}(|z|) = N - 1$.

By induction, we have

$$\begin{aligned} \phi_D(x E_i 1_\lambda) &= F_j \phi_D(z E_i 1_\lambda) \\ &= F_j \left(\sum_{k=1}^{m'_1} a'_k(v) [{}_p B'_k] + \sum_{k=1}^{m'_2} (b'_k(v) + v^{-2p} c'_k(v)) [{}_p H'_k] \right), \end{aligned}$$

with $a'_k, b'_k, c'_k, B'_k, H'_k$ as in the statement of the lemma (for $x = z$). Now applying [Lemma 5.3](#) to each of these terms, we find that

$$\begin{aligned} \phi_D(x E_i 1_\lambda) &= \\ \sum_{k=1}^{m_1} a_k(v) [{}_p B_k] &+ \delta_{i,j} v^{1-i \cdot r(B'_k)} \left(\frac{1 - v^{-2((B'_k)_{i+1, i+1} + 1 + p)}}{1 - v^{-2}} \right) a'_k(v) [{}_p B'_k + E^{i+1, i+1} - E^{i, i+1}] \\ &+ \sum_{k=1}^{m''_2} (b_k(v) + v^{-2p} c_k(v)) [{}_p H_k], \end{aligned} \tag{5-3}$$

where $(B_k)_{r,s} = 0$ if $r < s$ unless $r = i \pmod n$ and $s = r + 1$, and similarly $(H_k)_{r,s} = 0$ if $r < s$. Thus the first formula of the lemma is established by induction once we

note that when $i \equiv j \pmod n$ we may write

$$\begin{aligned} v^{1-i \cdot r(B'_k)} & \left(\frac{1 - v^{-2((B'_k)_{i+1,i+1+1+p})}}{1 - v^{-2}} \right) a'_k(v) [{}_p B'_k + E^{i+1,i+1} - E^{i,i+1}] \\ & = \frac{v^{-i \cdot r(B'_k)}}{v - v^{-1}} (1 - v^{-2p} v^{-2((B'_k)_{i+1,i+1+1})}) a'_k(v) [{}_p B'_k + E^{i+1,i+1} - E^{i,i+1}] \\ & = (b_{m''_2+k} + v^{-2p} c_{m''_2+k}) [H_{m''_2+k}], \end{aligned}$$

where the last line defines $b_{m''_2+k}$, $c_{m''_2+k}$ and $H_{m''_2+k}$, and by induction $b_{m''_2+k}$, $c_{m''_2+k}$ lie in $(v - v^{-1})^{-1} \mathcal{A}$, since $a'_k(v) \in \mathcal{A}$. Setting $m_2 = m''_2 + \delta_{i,j} m_1$ the first formula is therefore established.

To show the second formula, we again use induction so that we have

$$\phi_D(x^{-1} 1_\lambda) = F_j \cdot \sum_{k=1}^{m'_1} a'_k(v) [{}_p B'_k + E^{i+1,i+1} - E^{i,i+1}]$$

Now by [Lemma 5.3](#) (in particular the independence of the coefficients from the values of $\{a_{r,s} : r \leq s\}$ in all but the final term of the second formula) this is equal to

$$\sum_{k=1}^{m_1} a_k(v) [{}_p B_k + E^{i+1,i+1} - E^{i,i+1}],$$

as required. Finally, to see the “moreover” part of the lemma, note that by definition we have

$$\begin{aligned} \left(\frac{v^{-i \cdot \lambda}}{v - v^{-1}} \right) \phi_D(r_i(x)^{-1} 1_\lambda) & = \left(\frac{v^{-i \cdot \lambda}}{v - v^{-1}} \right) \phi_D(r_i(\theta_j z)^{-1} 1_\lambda) \\ & = \left(\frac{v^{-i \cdot \lambda}}{v - v^{-1}} \right) (\delta_{i,j} v^{i \cdot |z|} \phi_D(z^{-1} 1_\lambda) + F_j (\phi_D(r_i(z)^{-1} 1_\lambda))). \end{aligned}$$

Comparing this with [\(5-3\)](#), we note that $r(B'_k) = \mathbf{a} + \mathbf{i} - |z|$, hence $1 - \mathbf{i} \cdot r(B'_k) = \mathbf{i} \cdot (|z| - \lambda) - 1$, so that

$$v^{1-i \cdot r(B'_k)} \left(\frac{1 - v^{-2((B'_k)_{i+1,i+1+1+p+1})}}{1 - v^{-2}} \right) = \left(\frac{v^{i \cdot (|z| - \lambda)}}{v - v^{-1}} \right) (1 - v^{-2p} v^{-2((B'_k)_{i+1,i+1+1})}).$$

Hence the result follow once again by induction. □

Having established these technical lemmas, it is now straightforward to complete the proof of [Theorem 5.2](#).

Definition 5.5. Let

$$\mathfrak{A}_D^- = \text{span}\{[A] : a_{r,s} = 0 \text{ for all } r < s\}$$

and let $\pi_D : \mathfrak{A}_D \rightarrow \mathfrak{A}_D^-$ be the orthogonal projection; thus its kernel is spanned by the elements $[A]$ such that $a_{r,s} > 0$ for some $r, s \in \mathbb{Z}$ with $r < s$. Note that [Lemma 5.3](#) shows that $\phi_D(x^{-1}\lambda) \in \mathfrak{A}_D^-$ for any $x \in \mathfrak{f}$. Let $s_D : \mathfrak{f} \rightarrow \mathfrak{A}_D^-$ be given by

$$x \mapsto \pi_D(\phi_D(x^{-1}E_i 1_\lambda))$$

and define $r_D : \mathfrak{f} \rightarrow \mathfrak{A}_D^-$ by setting

$$x \mapsto \frac{v^{-i\lambda}}{v - v^{-1}} \phi_D(r_i(x)^{-1} 1_\lambda).$$

Corollary 5.6. *Let $x \in \mathfrak{f}$.*

$$s_D(x) - r_D(x) = \frac{v^{-2p}}{v - v^{-1}} \left(\sum_{k=1}^m c_k(v) [{}_p Z_k] \right)$$

for some $Z_k \in \mathfrak{S}^{n,n}$ and $c_k \in \mathcal{A}$, independent of p . Hence the element $u(x)$ of (5-1) is orthogonal to $\mathbf{U}^{-1}\lambda$.

Proof. Let $y \in \mathfrak{f}$ be a monomial. Then we have

$$\langle u, y^{-1}\lambda \rangle = \lim_{p \rightarrow \infty} \langle u, y^{-1}\lambda \rangle_{k+pn},$$

and by definition

$$\langle u, y^{-1}\lambda \rangle_{k+pn} = (s_{k+pn}(x) - r_{k+pn}(x), \phi_{k+pn}(y^{-1}\lambda))_{k+pn}. \tag{5-4}$$

By [Lemma 5.4](#),

$$s_{k+pn}(x) - r_{k+pn}(x) = v^{-2p} \left(\sum_{j=1}^m c_j(v) [{}_p Z_j] \right), \quad Z_j \in \mathfrak{S}^{n,n},$$

and by [Remark 4.4](#), we know that $([{}_p Z_j], \phi_{k+pn}(y^{-1}\lambda))_{k+pn}$ converges in $\mathbb{Q}((v^{-1}))$ as $p \rightarrow \infty$. Thus the right-hand side of (5-4) tends to zero as required. \square

6. Geometric interpretation

Recall from [\[Lusztig 1999, §4\]](#) that \mathfrak{A}_D possesses a canonical basis \mathfrak{B}_D consisting of elements $\{A\}$, $A \in \mathfrak{S}_{D,n,n}$. To define these elements we must assume \mathbb{k} is algebraically closed (either the algebraic closure of \mathbb{F}_q , in which case we must use sheaves in the étale topology, or \mathbb{C} in which case we use the analytic topology). Fix $A \in \mathfrak{S}_{D,n}$, and $\mathbf{L} \in \mathfrak{F}_{r(A)}$.

The space \mathfrak{F}^n can be given the structure of an ind-scheme such that each set $X_A^{\mathbf{L}}$ (see [Section 3](#)) lies naturally in a projective algebraic variety. This follows from the fact that if we fix $i_0, j_0 \in \mathbb{Z}$, then the subsets

$$\mathfrak{F}_{\mathbf{b},\mathbf{L}}^p = \{\mathbf{L}' \in \mathfrak{F}_{\mathbf{b}} : \varepsilon^p L_{i_0} \subset L'_{j_0} \subset \varepsilon^{-p} L_{i_0}\}$$

(for $p = 1, 2, \dots$) are naturally projective algebraic varieties each embedded in the next, and for any fixed $A \in \mathfrak{S}_{D,n,n}$ there is a $p_0 \in \mathbb{Z}$ such that X_A^L is a locally closed subset of $\mathcal{F}_{b,L}^p$ for all $p \geq p_0$. Thus its closure \bar{X}_A^L is naturally a projective algebraic variety. Let \mathcal{I}_A^L (or sometimes for convenience just \mathcal{I}_A) denote the simple perverse sheaf on \bar{X}_A^L whose restriction to X_A^L is $\mathbb{C}[d_A]$. Let $\mathcal{H}^s(\mathcal{I}_A^L)$ be the s -th cohomology sheaf of \mathcal{I}_A^L . For $A_1 \in \mathfrak{S}_{D,n,n}$ such that $X_{A_1}^L \subset \bar{X}_A^L$ we write $A_1 \leq A$, and set

$$\Pi_{A_1,A} = \sum_{s \in \mathbb{Z}} \dim(\mathcal{H}_y^{s-d_{A_1}}(\mathcal{I}_A^L))v^s \in \mathbb{Z}[v^{-1}],$$

where $\mathcal{H}_y^{s-d_{A_1}}(\mathcal{I}_A^L)$ is the stalk of $\mathcal{H}^{s-d_{A_1}}(\mathcal{I}_A^L)$ at a point $y \in \bar{X}_{A_1}^L$ (since \mathcal{I}_A^L is constructible with respect to the stratification of \bar{X}_A^L given by $\{X_{A_1}^L : A_1 < A\}$, this is independent of the choice of y). We let

$$\{A\} = \sum_{A_1; A_1 \leq A} \Pi_{A_1,A}[A_1].$$

The next result is an immediate consequence of the definitions and [Lemma 3.5](#). (It is the analogue for our inner product of [\[Lusztig 1999, Lemma 7.5\]](#)).

Lemma 6.1. *Let $A, A' \in \mathfrak{S}_{D,n,n}$. Then*

$$(\{A\}, \{A'\})_D \in \delta_{A,A'} + v^{-1}\mathbb{Z}[v^{-1}]. \quad \square$$

The algebra \mathcal{A}_D may be viewed as a convolution algebra of (equivariant) complexes on \mathcal{F}^n . One must be slightly careful here since one cannot (at least straightforwardly) consider convolution on $\mathcal{F}^n \times \mathcal{F}^n$ as the ‘‘complexes’’ one would then have to consider would have infinite-dimensional support. However, [\[Lusztig 1999, 4.2\]](#) gives one way in which this difficulty can be avoided: given $A, B \in \mathfrak{S}_{D,n,n}$ we may consider the set

$$Z = \{(L', L'') \in \mathcal{F}_b \times \mathcal{F}_c : L' \in \bar{X}_A^L, L'' \in \bar{X}_B^L\}.$$

As with \bar{X}_A^L we see that Z is naturally a projective variety and the projection π to the second factor gives a proper map $Z \rightarrow \mathcal{F}_c$. The group $G_L \subset \text{Aut}(V)$ of automorphisms stabilising L acts on \mathcal{F}_b through a quotient which is naturally an algebraic group, and thus it makes sense to speak of G_L -equivariant perverse sheaves on \mathcal{F}_b and Z . If \mathcal{I} denotes the middle extension of the constant sheaf on the smooth locus of Z , then \mathcal{I} has a canonical G_L -equivariant structure, and so by the decomposition theorem its push-forward along π is a direct sum of (shifted) perverse sheaves of the form \mathcal{I}_C , ($C \in \mathfrak{S}_{D,n,n}$). We denote this push-forward by $\mathcal{I}_A * \mathcal{I}_B$. If $\mathcal{H}_{D,n,n}$ denotes the free \mathcal{A} -module on the set $[\mathcal{I}_A^L]$ of isomorphism classes of the sheaves \mathcal{I}_A^L (as L runs over a set of $\text{Aut}(V)$ -orbit representatives on \mathcal{F}) then the convolution $*$ gives $\mathcal{H}_{D,n,n}$ an associative \mathcal{A} -algebra structure, which

is shown in [Lusztig 1999, 4.4] to be isomorphic to $\mathfrak{A}_{D,\mathcal{A}}$ via the map Θ given by $\Theta([\mathcal{F}_A]) = \{A\}$. Moreover, Lusztig has shown that the submodule $K_{D,n,n}$ of $\mathcal{H}_{D,n,n}$ spanned by the elements $\{[\mathcal{F}_A] : A \in \mathfrak{S}_{D,n,n}^{\text{ap}}\}$ is precisely the preimage under Θ of the subalgebra $\mathbf{U}_{D,\mathcal{A}}$, where $\mathfrak{S}_{D,n,n}^{\text{ap}}$ is the subset of $\mathfrak{S}_{D,n,n}$ consisting of those matrices $A \in \mathfrak{S}_{D,n,n}$ for which, given any $p \in \mathbb{Z} \setminus \{0\}$, there is a $k \in \mathbb{Z}$ with $a_{k,k+p} = 0$.

Remark 6.2. Note also that this isomorphism yields the existence of an \mathcal{A} -antilinear involution on $\mathfrak{A}_{D,\mathcal{A}}$ which fixes the basis elements $\{A\}$, ($A \in \mathfrak{S}_{D,n,n}$), by transporting via Θ the action of the Verdier duality functor. We will write this involution as $x \mapsto \bar{x}$. Since it fixes the generators $E_i(D)$, $F_i(D)$, K_a it preserves the subalgebra $\mathbf{U}_{D,\mathcal{A}}$ and is compatible with the bar involution on $\dot{\mathbf{U}}$ (see [Lusztig 1999, 4.13] for more details). Moreover, Lusztig [Lusztig 1999, Proposition 4.12] shows that the antiautomorphism Ψ corresponds to the map on $\mathcal{H}_{D,n,n}$ which sends $[\mathcal{F}_A]$ to $[\mathcal{F}_{A'}]$.

We wish to give an interpretation of the inner product of Section 3 in the context of the algebra $\mathcal{H}_{D,n,n}$. Suppose that $A, B \in \mathfrak{S}_{D,n,n}$. We want to describe $(\{A\}, \{B\})$. We may assume that $r(A) = r(B) = \mathbf{a}$ and $c(A) = c(B) = \mathbf{b}$. Let $L' \in \mathcal{F}_{\mathbf{b}}$. Let $\mathcal{F}_{A'}^{L'}$ and $\mathcal{F}_{B'}^{L'}$ denote the simple perverse sheaves on $\bar{X}_{A'}^{L'}$ and $\bar{X}_{B'}^{L'}$ respectively. Then define

$$\langle \mathcal{F}_A, \mathcal{F}_B \rangle^D = \sum_{i \in \mathbb{Z}} \dim(H_c^i(\mathcal{F}_{\mathbf{a}}, \mathcal{F}_{A'}^{L'} \otimes \mathcal{F}_{B'}^{L'}))v^i. \tag{6-1}$$

(Here as usual \otimes denotes the derived tensor product.) Clearly $\langle \cdot, \cdot \rangle^D$ extends to an inner product on the whole of \mathfrak{A}_D (viewed as an algebra of equivariant complexes on \mathcal{F}^n). We want to show that it is the same as the inner product $(\cdot, \cdot)_D$ of Section 3, at least on the subalgebra \mathbf{U}_D . We start by showing that $\langle \cdot, \cdot \rangle^D$ satisfies the properties of Corollary 3.3.

Lemma 6.3. *Let $A, B, C \in \mathfrak{S}_{D,n,n}$, and suppose that \mathbb{O}_A is a closed orbit. Then*

$$\langle \mathcal{F}_A * \mathcal{F}_B, \mathcal{F}_C \rangle^D = v^{d_A - d_{A'}} \langle \mathcal{F}_B, \mathcal{F}_{A'} * \mathcal{F}_C \rangle^D.$$

Proof. Both sides are obviously zero unless $r(A) = r(C) = \mathbf{a}$, $c(A) = r(B) = \mathbf{b}$ and $c(B) = c(C) = \mathbf{c}$; thus we assume these equalities from now on. Pick $L^0 \in \mathcal{F}_{\mathbf{c}}$, and pick a subset $Y = \mathcal{F}_{\mathbf{b},L^0}^{L^0}$ of $\mathcal{F}_{\mathbf{b}}$ large enough that Y is a smooth projective variety containing $X_{B'}^{L^0}$. Let

$$Z_A = \{(L, L') \in \mathbb{O}_A : L' \in Y\},$$

We have maps $p_1 : Z_A \rightarrow \mathcal{F}_{\mathbf{a}}$ and $p_2 : Z \rightarrow Y$, the first and second projections respectively. The map p_1 is clearly proper (as the fibre is $X_A^L \cap Y$) and the map p_2 is smooth with fibre dimension $d_{A'}$. It follows that the complex \mathcal{F} used in the definition of $\mathcal{F}_A * \mathcal{F}_B$ is the pull-back $p_2^*(\mathcal{F}_{B'})[d_{A'}]$, and hence we have

$$(\mathcal{F}_A * \mathcal{F}_B)^t = (p_1)_! p_2^*(\mathcal{F}_{B'})[d_{A'}],$$

and thus in particular

$$\begin{aligned} (\mathcal{F}_A * \mathcal{F}_B)^t \otimes \mathcal{F}_{C^t} &= (p_1)_! p_2^*(\mathcal{F}_{B^t})[d_{A^t}] \otimes \mathcal{F}_{C^t} \\ &= (p_1)_!(p_2^*(\mathcal{F}_{B^t}) \otimes p_1^*(\mathcal{F}_{C^t})[d_{A^t}]), \end{aligned} \tag{6-2}$$

where we use the projection formula in the second equality.

On the other hand, to compute the product $\mathcal{F}_{A^t} * \mathcal{F}_C$ we may similarly pick a smooth projective variety $W \subset \mathcal{F}_a$ which contains $X_{C^t}^{L^0}$, and consider the variety

$$Z_{A^t} = \{(L, L') \in \mathbb{O}_A : L \in W\}.$$

As above there are projection maps p_1, p_2 , and the product is given by

$$(\mathcal{F}_{A^t} * \mathcal{F}_C)^t = (p_2)_! p_1^*(\mathcal{F}_{C^t})[d_A].$$

so that

$$\begin{aligned} \mathcal{F}_{B^t} \otimes (\mathcal{F}_{A^t} * \mathcal{F}_C)^t &= \mathcal{F}_{B^t} \otimes (p_2)_! p_1^*(\mathcal{F}_{C^t})[d_{A^t}] \\ &= (p_2)_!(p_2^*(\mathcal{F}_{B^t}) \otimes p_1^*(\mathcal{F}_{C^t})) [d_A] \end{aligned}$$

where we again use the projection formula. Now since tensor product is local, we may restrict to $Z_A \cap Z_{A^t}$, and then it is clear that both inner products are given by the compactly supported cohomologies of $p_2^*(\mathcal{F}_{B^t}) \otimes p_1^*(\mathcal{F}_{C^t})$ up to shift, with the difference in shifts being $d_A - d_{A^t}$ as required. \square

Lemma 6.4. *Let $A, B \in \mathfrak{S}_{D,n,n}$, and $c \in \mathfrak{S}^n$. Then*

- (1) $\langle E_i\{A\}, \{B\} \rangle^D = \langle \{A\}, vK_i F_i\{B\} \rangle^D,$
- (2) $\langle F_i\{A\}, \{B\} \rangle^D = \langle \{A\}, vK_{-i} E_i\{B\} \rangle^D,$
- (3) $\langle K_c\{A\}, \{B\} \rangle^D = \langle \{A\}, K_c\{B\} \rangle^D.$

Proof. This follows from the previous lemma exactly as in the proof of [Corollary 3.3](#), since the varieties $X_{a+i e_a}^L$ are closed. \square

The algebra \mathbf{U}_D is spanned by elements of the form $T_1 T_2 \dots T_N [\mathbf{i}_a]$ where T_s is either E_i or F_i for some i . Thus, by [Corollary 3.3](#), in order to show that the inner products $(\)_D$ and $\langle \cdot, \cdot \rangle^D$ coincide via the isomorphism the previous lemma shows we need only check that

$$\langle T_1 T_2 \dots T_N [\mathbf{i}_a], [\mathbf{i}_a] \rangle^D = (T_1 T_2 \dots T_N [\mathbf{i}_a], [\mathbf{i}_a])_D$$

But this will follow if we can show that

$$\langle \{A\}, [\mathbf{i}_a] \rangle^D = (\{A\}, [\mathbf{i}_a])_D$$

for all $A \in \mathfrak{S}_{D,n,n}$, as $\{\{A\} : A \in \mathfrak{S}_{D,n,n}\}$ is a basis of \mathfrak{A}_D . The simple perverse sheaf corresponding to $\mathbf{i}_a = [\mathbf{i}_a]$ is just the skyscraper sheaf at the point L' ; hence this last equality follows from directly from the definitions. We have therefore shown the following result.

Proposition 6.5. *On the algebra U_D the inner products $\langle \cdot, \cdot \rangle^D$ and $(\cdot, \cdot)_D$ coincide. \square*

Remark 6.6. It can be shown that the algebra \mathfrak{A}_D is generated by the elements $\{A\}$ for which X_A^L is closed, and so the above argument adapts to show that the inner products in fact agree on the whole of \mathfrak{A}_D , but we will not need this. Henceforth we will use the notation $(\cdot, \cdot)_D$ when referring to the inner product on \mathfrak{A}_D in either of its incarnations.

We now give a second proof of the agreement of the limit of the inner products on U_D with Lusztig’s inner product on \dot{U} . Recall that [Theorem 5.1](#) characterises the inner product by three properties, the first two of which are evident for our limiting inner product. The difficulty then is establishing the third property, which relates the inner product on \dot{U} to that on \mathfrak{f} . We give a proof of this property, which while less elementary than the proof in [Section 5](#) is more conceptual. In the remainder of this section we will assume that our base field \mathbb{k} is the field of complex numbers \mathbb{C} , and work with sheaves in the analytic topology. We thus briefly review some basics of the equivariant derived category, following the approach of [\[Bernstein and Lunts 1994\]](#).

We first need to recall the geometric construction of the algebra \mathfrak{f} and its inner product (at least in the case of the cyclic quiver). Let Q be the cyclic quiver $1 \rightarrow 2 \rightarrow \dots \rightarrow n \rightarrow 1$. A representation of Q is a $\mathbb{Z}/n\mathbb{Z}$ -graded vector space $W = \bigoplus_{i \in \mathbb{Z}/n\mathbb{Z}} W_i$ equipped with linear maps $y_i : W_i \rightarrow W_{i+1}$ (where $i \in \mathbb{Z}/n\mathbb{Z}$). The space of such representations is denoted E_W . Such a representation is *nilpotent* if there is an $N > 0$ such that all compositions of y_i s of length greater than N are equal to zero, and we write E_W^{nil} for the subvariety of nilpotent representations. The group $G_W = \prod_{i \in \mathbb{Z}/n\mathbb{Z}} \text{GL}(V_i)$ acts on E_W^{nil} with finitely many orbits. The algebra \mathfrak{f} associated to the Cartan datum of affine type $\widehat{\mathfrak{sl}}_n$ is then given as a convolution algebra of semisimple perverse sheaves which are G_W -equivariant and are supported on E_W^{nil} ; thus, since the stabiliser of a nilpotent representation is connected, the simple objects are labelled by the G_W -orbits on E_W^{nil} , that is, by the isomorphism classes of nilpotent representations.

Given a pair $(t, m) \in \mathbb{Z} \times \mathbb{Z}_{\geq 0}$ we have a representation $V_{t,m}$ of the cyclic quiver with basis $\{e_j : t \leq j \leq t + p - 1\}$ where e_j has degree $j \bmod n$, and $e_t \rightarrow e_{t+1} \rightarrow \dots \rightarrow e_{t+p-1} \rightarrow 0$. The representations $V_{t,p}$ are a complete set of representatives for the isomorphism classes of indecomposable nilpotent representations of the cyclic quiver; thus, since any nilpotent representation is a direct sum of indecomposables, we can record the isomorphism class of any such representation by a tableau $(\mu_{t,p})_{t,p \in \mathbb{Z}}$ where the entry $\mu_{t,p}$ records the multiplicity of $V_{t,m}$ in the representation. We therefore have a natural parametrisation of the canonical basis \mathbf{B} by tableaux $(\mu_{t,p})_{t \in \mathbb{Z}, p \in \mathbb{N}}$, where $\mu_{t,p} \in \mathbb{N}$ and $\mu_{t,p} = \mu_{t-n,p}$ for all $t \in \mathbb{Z}$ and for fixed t only finitely many of the $\mu_{t,p}$ are nonzero.

If $\dim(W_i) = v_i$, ($i \in \mathbb{Z}/n\mathbb{Z}$), the orbits of G_W on E_W^{nil} correspond to the isomorphism classes of nilpotent representations of dimension v hence they are labelled by the set Σ_v consisting of those tableaux $(\mu_{t,p})$ for which

$$\sum_{\substack{t,p \\ t \leq k < t+p}} \mu_{t,p} = v_k,$$

and so this same set indexes the v -homogeneous part of \mathbf{f} .

The inner product on \mathbf{f} is defined in [Lusztig 1993, §12.2]. Let A_1, A_2 be G_W -equivariant simple perverse sheaves on E_W^{nil} . Note that the definition there is simply an explicit calculation of

$$(A_1, A_2) = \sum_{j \in \mathbb{Z}} \dim(H_{G,c}^j(A_1 \otimes A_2))v^{-j}. \tag{6-3}$$

where $H_{G,c}^*$ denotes equivariant cohomology with compact supports (see below for more details).

Definition 6.7. Let X be a variety with a G -action (or more compactly, a G -variety). A *resolution* of X is a map $p : P \rightarrow X$ where P is smooth G -variety on which G acts freely (so that $\bar{P} = P/G$ is a smooth variety also). Let $\pi : P \rightarrow \bar{P}$ denote the quotient map. The category $D_G^b(X, P)$ consists of triples $(\mathcal{F}, \mathcal{G}, \phi)$ where \mathcal{F} is an object in $D^b(\bar{P})$ and \mathcal{G} is an object in $D^b(X)$ and $\phi : \pi^*(\mathcal{F}) \rightarrow p^*(\mathcal{G})$ is an isomorphism.

We will also need to recall the notion of an n -acyclic map.

Definition 6.8. A map $f : Y \rightarrow X$ is said to be n -acyclic if it has the following properties:

- (i) For any sheaf F on Y the adjunction morphism $B \rightarrow R^0 f_* f^*(B)$ is an isomorphism, and $R^i f_* f^*(F) = 0$ for $0 < i \leq n$.
- (ii) For any base change $\tilde{X} \rightarrow X$ the induced map $\tilde{f} : \tilde{Y} = Y \times_X \tilde{X} \rightarrow \tilde{X}$ has property (i).

If we write $\tau_{\leq n}$ for the truncation functor on the derived category $D^b(Y)$, the first condition may be rewritten as saying that the adjunction map $F \rightarrow \tau_{\leq n} Rf_* f^*(F)$ in $D^b(Y)$ is an isomorphism for any sheaf F (thought of as an complex in $D^b(Y)$ concentrated in degree 0).

For sufficiently acyclic resolutions P (i.e., resolutions $p : P \rightarrow X$ with p an n -acyclic map for n large), the cohomologies of objects in the category $D_G^b(X, P)$ can be used to calculate the cohomologies in $D_G^b(X)$ as indeed Bernstein and Lunts take a limit of resolution of X to obtain their definition of the equivariant derived category. The construction in [Lusztig 1993, §12.2] gives an explicit construction

of a collection of G -resolutions of a variety which can be made arbitrarily highly connected (and hence his definition is the same as that of (6-3)) however for our comparison result we need a more flexible context.

To compare the inner products on \mathbf{f} and \mathbf{U}_D we need to relate the geometry of periodic lattices to the cyclic quiver. The description of the relation we need goes back to [Lusztig 1990, §11], and is also used in [Ginzburg and Vasserot 1993]. Here we follow the presentation of [Lusztig 1999]. Suppose that \mathbf{L} is a fixed lattice, and $\mathbf{a} \in \mathfrak{S}^n$ is such that $\dim(V_i) = a_i$ (where on the left-hand side of this equality i is understood to be taken modulo n). Consider the following spaces:

- $\mathcal{X}_{\mathbf{a},v}^{\mathbf{L}} = \{ \mathbf{L}' \in \mathfrak{F}_{\mathbf{a}} : \mathbf{L}'_i \subseteq \mathbf{L}_i \text{ and } \dim(\mathbf{L}_i/\mathbf{L}'_i) = v_i \text{ for all } i \in \mathbb{Z} \}$.
- $\tilde{\mathcal{X}}_{\mathbf{a},v}^{\mathbf{L}} = \{ (\mathbf{L}', (\phi_i)_{i \in \mathbb{Z}} : \mathbf{L}' \in \mathcal{X}_{\mathbf{a},v}^{\mathbf{L}} \text{ and } \phi_i : \mathbf{L}_i/\mathbf{L}'_i \rightarrow V_i \text{ is an isomorphism} \}$.
- $\mathcal{U}_{\mathbf{a}} \subset E_W^{\text{nil}}$ consists of those representations with label $(\mu_{t,p})$ such that

$$\mu_{t,1} + \mu_{t,2} + \dots \leq a_t \quad \text{for all } t \in \mathbb{Z}.$$

Both $\mathcal{X}_{\mathbf{a},v}^{\mathbf{L}}$ and $\tilde{\mathcal{X}}_{\mathbf{a},v}^{\mathbf{L}}$ can be given a natural structure of algebraic variety (with $\mathcal{X}_{\mathbf{a},v}^{\mathbf{L}}$ projective), in the same fashion as for $\bar{X}_{\mathbf{L}}^A$ above, and the variety $\mathcal{U}_{\mathbf{a}}$ is an open subset of E_W^{nil} (see [Lusztig 1999, Lemma 5.8]). We then have the correspondence

$$\mathcal{X}_{\mathbf{a},v}^{\mathbf{L}} \xleftarrow{\alpha} \tilde{\mathcal{X}}_{\mathbf{a},v}^{\mathbf{L}} \xrightarrow{\beta} \mathcal{U}_{\mathbf{a}},$$

where the map α is given by $(\mathbf{L}, \phi) \mapsto \mathbf{L}$, while the map β is given by sending (\mathbf{L}, ϕ) to the element $(y_i) \in E_V^{\text{nil}}$ where y_i given by the composition

$$V_i \xrightarrow{\phi_i^{-1}} \mathbf{L}_i/\mathbf{L}'_i \longrightarrow \mathbf{L}_{i+1}/\mathbf{L}'_{i+1} \xrightarrow{\phi_{i+1}} V_{i+1},$$

with the middle map induced by the inclusion $\mathbf{L}_i \subseteq \mathbf{L}_{i+1}$ (the point $(y_i)_{i \in \mathbb{Z}/n\mathbb{Z}}$ is automatically nilpotent as a representation of Q by the periodicity of the flags \mathbf{L}, \mathbf{L}'). The map α is clearly a principal G_W -bundle, while the map β is smooth with connected fibres of dimension $\sum_{1 \leq i \leq n} a_i v_i$ (see [Lusztig 1999, Lemma 5.11]).

Notice that if \mathbf{a} has a_i large enough for all i , then we have $E_W^{\text{nil}} = \mathcal{U}_{\mathbf{a}}$. In what follows we will always assume that this is the case. Moreover, the groups $G_{\mathbf{L}}$ (that is, the group of automorphisms of V which preserve the lattice \mathbf{L}) and G_W act naturally on $\tilde{\mathcal{X}}_{\mathbf{a},b}^{\mathbf{L}}$, making the maps α and β equivariant (for the actions of $G_{\mathbf{L}}$ on $\mathcal{X}_{\mathbf{a},\mathbf{L}}$ and G_W on E_W^{nil}). Thus since $\tilde{\mathcal{X}}_{\mathbf{a},v}^{\mathbf{L}}$ is free G_W -space (using the map α) it is a resolution of E_W^{nil} .

Now Lusztig has shown in [Lusztig 1999, §5] that if b is an element of the canonical basis which corresponds to the simple perverse sheaf P on E_W^{nil} with associated G_W -orbit corresponding to the tableau $(\mu_{t,p})$ then $\phi_D(b)[\mathbf{i}_{\mathbf{a}}] \in \mathfrak{A}_{D,n,n}$ is the element $\{B\}$ where $b_{i,i+j} = \mu_{i,j}$ and $b_{ii} = a_i - \sum_{p>0} \mu_{i,p}$, and $b_{ij} = 0$ if

$i > j$. Moreover, we have

$$\alpha^*(\mathcal{F}_B) \cong \beta^*(P)$$

Picking an isomorphism θ (which is unique up to a scalar since A is simple) we therefore obtain an element $\tilde{P} = (\alpha^*(\mathcal{F}_B), P, \theta)$ of $D_G^b(E_W^{\text{nil}}, \tilde{\mathcal{X}}_{a,v}^L)$. Thus if b_1, b_2 are elements of \mathbf{B}_v with associated tableau $(\mu_{t,p})$ and $(\rho_{t,p})$, and P_1, P_2 the corresponding perverse sheaves on E_W^{nil} , and B_1 and B_2 are the associated elements of $\mathfrak{S}_{D,n,n}^-$ then we may choose elements $\tilde{P}_k = (\mathcal{F}_{B_k}, P_k, \theta_k)$ (where $k = 1, 2$) in the category $D_G^b(E_W^{\text{nil}}, \tilde{\mathcal{X}}_{a,v}^L)$ such that

$$(\mathcal{F}_A, \mathcal{F}_B)_D = \sum_{j \in \mathbb{Z}} H_c^j(\tilde{P}_1 \otimes \tilde{P}_2)v^{-j},$$

where $H_c^j(\tilde{P}_1 \otimes \tilde{P}_2)$ denotes the cohomology with compact supports of the object $\tilde{P}_1 \otimes \tilde{P}_2$ in the category $D_G^b(E_W^{\text{nil}}, \tilde{\mathcal{X}}_{a,v}^L)$. It follows that if we consider $a' = a + pb_0$ instead of a for larger and larger p , this inner product will converge to (b_1, b_2) provided the resolutions $\tilde{\mathcal{X}}_{a',v}^L$ become more and more highly connected as the $a'_i \rightarrow \infty$. Thus the compatibility of the inner products is reduced to showing that the maps $\beta : \tilde{X}_{a,v}^L \rightarrow E_W^{\text{nil}}$ is k -connected where $k \rightarrow \infty$ as $\min\{a_i\} \rightarrow \infty$. The rest of this section will be devoted to a proof of this result.

We wish to use a general lemma which gives a criterion for a map to be n -acyclic. Since we cannot find a precise reference for what we need, we sketch the result, though it is presumably well-known to the experts. The statement is a version of the Vietoris–Begle theorem proved in [Kashiwara and Schapira 1994, Proposition 2.7.8].

Lemma 6.9. *Suppose we have a map $f : Y \rightarrow X$ which has k -connected fibres, and that we may exhaust $Y = \bigcup_n Y_n$, by closed subsets Y_n such that $Y_n \subset \text{Int}(Y_{n+1})$ and the restriction of f to Y_n is proper with k -connected fibres for all n , then $\tau_{\leq k} Rf_* \circ f^* \cong \text{id}$.*

Proof. In fact the reference [Kashiwara and Schapira 1994, Proposition 2.7.8] more is proved under the assumption that the fibres of f are contractible, but the weaker statement that we need is precisely what follows from the proof given there. The key point is that in the case where f is proper, one may use proper base change to conclude the vanishing of the functors $Rf_*^j f^*$ in the appropriate range from the k -connectedness of the fibres. The extension to the noncompact case then follows via the Mittag-Leffler condition. \square

Since the hypotheses of Lemma 6.9 are preserved by base change, it yields a criterion for a map to be n -acyclic. We now use the above lemma to show that β is a k -acyclic map for $k = \min_{0 \leq i \leq n-1} \{(a_i - v_i)\}$.

Lemma 6.10. *The fibres of $\beta : \tilde{\mathcal{X}}_{a,v}^L \rightarrow \mathcal{U}_a$ are k -connected for $k = 2 \min_{1 \leq i \leq n} \{(a_i - v_i)\}$.*

pt

Proof. First note that we may view $X = \tilde{\mathcal{X}}_{a,v}^L$ as the set

$$\{(\varphi_i)_{i \in \mathbb{Z}} : \varphi_i : L_i / \ker(\varphi_{i-1}) \rightarrow W_i\},$$

where φ_i is surjective, $\ker(\varphi_i)$ is a lattice, and $\varphi_{i-n} = \epsilon \varphi_i \epsilon^{-1}$. The corresponding pair $(L', (\phi_i))$ is given by $L' = (\ker(\varphi_i))_{i \in \mathbb{Z}}$ with the isomorphisms $\phi_i : L_i / L'_i \rightarrow W_i$ induced by the surjections φ_i .

Now suppose that $y = (y_i)_{i \in \mathbb{Z}/n\mathbb{Z}} \in E_W^{\text{nil}}$ is a nilpotent representation of the cyclic quiver, and that $(\varphi_i)_{i \in \mathbb{Z}}$ is in the fibre of y . Considering the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ker(\varphi_{i+1}) & \longrightarrow & L_{i+1} & \xrightarrow{\varphi_{i+1}} & W_{i+1} & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow y_i & & \\ 0 & \longrightarrow & \ker(\varphi_i) & \longrightarrow & L_i & \xrightarrow{\varphi_i} & W_{i+1} & \longrightarrow & 0 \end{array}$$

we see that the restriction of φ_{i+1} is determined on L_i as it is given by $y_i \circ \varphi_i$ there. Thus given φ_i , the collection of φ_{i+1} s which induce y_i is given by choosing a surjection $\pi : L_{i+1}/L_i \rightarrow W_{i+1}/\text{im}(y_i)$, and then picking a lift of the pair of maps $(\pi, y_i \circ \varphi_i)$ to a map $\varphi_{i+1} : L_i/\ker(\varphi_i) \rightarrow W_{i+1}$ (since any such lift will be a surjective map). Thus the space of such choices is homotopy equivalent to the space of surjections from L_{i+1}/L_i to $W_{i+1}/\text{im}(y_i)$. This is a complex Stiefel manifold, and hence $2(a_i - v_i + \text{rank}(y_i))$ connected.

Thus if we set $k = 2 \min_{1 \leq i \leq n} \{(a_i - v_i)\}$ we may view $\beta^{-1}(y)$ as an iterated sequence of fibre bundles over the space B of surjections $\{\varphi_0 : L_0 \rightarrow W_0\}$ where in each case the fibres are at least k -connected. Thus by the standard long exact sequence $\beta^{-1}(y)$ will be k -connected provided we can show that B is. Now B is the space of surjective linear maps from L_0 to W_0 which intertwine the action of ϵ with the composition $\theta = y_n y_{n-1} y_{n-2} \dots y_0 : W_0 \rightarrow W_0$ (note that since the action of ϵ is nilpotent, this shows the representation y must be also) and the following lemma shows that in fact B is at least $2(D - v_0) \geq 2(a_0 - v_0)$ connected, so we are done. □

Lemma 6.11. *Let (U, θ) be a finite-dimensional vector space U equipped with nilpotent endomorphism θ of Jordan type λ , and let L be a free $\mathbb{C}[\epsilon]$ -module of rank D . Viewing U as a $\mathbb{C}[\epsilon]$ -module via $\epsilon \mapsto \theta$, the space of $\mathbb{C}[\epsilon]$ -module surjections $\varphi : L \rightarrow U$ is d -connected, where $d = 2(D - \ell(\lambda))$ and $\ell(\lambda)$ is the length of the partition λ .*

Proof. A $k[\epsilon]$ -module map $\varphi : L \rightarrow U$ is surjective if and only if the induced k -linear map $\bar{\varphi} : L/\epsilon(L) \rightarrow U/\theta(U)$ is surjective. Moreover, since the space of surjections from $L/\epsilon(L)$ to $U/\theta(U)$ is a complex Stiefel manifold of k -frames in a D dimensional space, it is $2(D - \ell(\lambda))$ -connected.

Now suppose that ψ is a \mathbb{C} -linear map from $L/\epsilon(L) \rightarrow U/\theta(U)$. Let K denotes the D -dimensional \mathbb{C} -vector space spanned by a set of $\mathbb{C}[\epsilon]$ -generators of L , say $\{e_1, \dots, e_D\}$. A $k[\epsilon]$ -module map $\varphi : L \rightarrow U$ with $\bar{\varphi} = \psi$ is completely determined by its restriction to K , and the induced map $\varphi' : K \rightarrow U$ is given by a choice of lifts for the vectors $\{\psi(e_i)\}$ in U (where by abuse of notation we denote by ψ the composition of $L \rightarrow L/\epsilon(L) \rightarrow U/\theta(U)$), and hence the space of such surjections is clearly a vector bundle over the Stiefel manifold of surjections from $L/\epsilon L$ to $U/\theta(U)$ which proves the lemma. \square

Proposition 6.12. *The map $\beta : \tilde{\mathcal{X}}_{a,v}^L \rightarrow E_W^{nil}$ is k -connected.*

Proof. To show β is a k -connected map we wish to apply [Lemma 6.9](#). By [Lemma 6.10](#) we know that the fibres of β are k -connected, and hence we must show that we can filter $\tilde{\mathcal{X}}_{a,v}^L$ by subvarieties $\{Y_i\}_{i \in \mathbb{N}}$ such that $\beta|_{Y_i}$ is proper while ensuring that the fibres remain k -connected. To do this we simply note that the topology of the fibres of β all come from Stiefel manifolds, and these deformation retract on to the compact Stiefel manifolds. Moreover the retraction can be done via the Gram–Schmidt process, once we endow our vector spaces with a Hermitian inner product.

More precisely, we may equip V with an Hermitian inner product (here we will assume that $k = \mathbb{C}$, as in the rest of this section) so that ϵ is a unitary map (e.g., take a $\mathbb{C}[\epsilon]$ -basis $\{e_1, e_2, \dots, e_D\}$ of L_0 and define the Hermitian product $\langle \cdot, \cdot \rangle^D$ by setting

$$\langle \epsilon^l e_j, \epsilon^m e_k \rangle = \delta_{j,k} \delta_{l,m}, \quad (l, m \in \mathbb{Z}, 1 \leq j, k \leq D),$$

Similarly we may equip the $\{W_i\}$ with Hermitian inner products. Then we have a norm function N on $\tilde{\mathcal{X}}_{a,v}^L$ given by

$$N(\mathbf{L}, (\phi_i)) = \max_{0 \leq i \leq n-1} \{ \sup \{ \|\phi_i(u)\| : u \in L_i/L'_i, \|u\| = 1 \} \}$$

where the norm on L_i/L'_i is induced from that on V via the canonical isomorphism $(L'_i)^\perp \cong L_i/L'_i$. Now we may set $Y_i = \{(\mathbf{L}, (\phi_i)) : i^{-1} \leq N(\mathbf{L}, (\phi_i)) \leq i\}$. Since the map α from $\tilde{\mathcal{X}}_{a,v}^L$ to $\mathcal{X}_{a,v}^L$ is a principal G_W -bundle over a projective variety, and the norm condition defining Y_i clearly cuts out a compact subset of the fibres of α , it follows that Y_i is compact, and so in particular $\beta|_{Y_i}$ is proper. Thus it remains to check that the fibres of $\beta|_{Y_i}$ are still k -connected.

To do this we may use the Gram–Schmidt process to iteratively deform the linear maps in the fibres in the same manner as we checked k -connectedness via a sequence of fibre bundles. In the case of the choice of the surjection $L_0 \rightarrow W_0$ which defines L'_0 and ϕ_0 , note that we need only apply Gram–Schmidt to the frame defining the map from K to $W_0/\theta(W_0)$, where $K = \text{span}_{\mathbb{C}}\{e_1, \dots, e_D\}$. \square

We can now complete the geometric proof of the equality of our inner product with that in [\[Lusztig 1993\]](#).

Theorem 6.13. *The inner products $\langle \cdot, \cdot \rangle$ and (\cdot, \cdot) on $\hat{\mathbf{U}}$ coincide.*

Proof. As discussed above, it is enough to know that the varieties $\tilde{\mathcal{X}}_{\mathbf{a}+p\mathbf{b}_0, v}^L$ become more and more connected as p tends to infinity. Since the value of $k = \min_{0 \leq i \leq n-1} \{a_i + p - v_i\}$ clearly tends to infinity as p does (where \mathbf{a} is the representative of $\lambda \in X'$ lying in $\mathfrak{S}_{D,n}$) the equality of the inner products follows. □

7. A construction of the canonical basis of $\hat{\mathbf{U}}(\widehat{\mathfrak{sl}}_n)$

Lusztig [2000] defined homomorphisms

$$\psi_D : \mathfrak{A}_D \rightarrow \mathfrak{A}_{D-n},$$

which are characterised, at least on \mathbf{U}_D , by the conditions

- $\psi_D(E_i(D)) = E_i(D - n),$
- $\psi_D(F_i(D)) = F_i(D - n),$
- $\psi_D(K_{\mathbf{a}}(D)) = v^{\mathbf{a} \cdot \mathbf{b}_0} K_{\mathbf{a}}(D - n),$

where \mathbf{b}_0 has all entries equal to 1. It follows that if we work with the root datum (X', Y') , i.e., with $\widehat{\mathfrak{sl}}_n$, then the maps ψ_D and $\phi_D : \hat{\mathbf{U}} \rightarrow \mathfrak{A}_{D,n,n}$ are compatible, that is $\psi_{D+n} \circ \phi_{D+n} = \phi_D$.

Let $\hat{\mathbf{U}} = \varprojlim_D \mathfrak{A}_D$, where the limit is taken over the projective system given by the maps $(\psi_D)_{D \in \mathbb{N}}$ described above. Since the maps ϕ_D are compatible with this system, there is a unique map $\phi : \hat{\mathbf{U}} \rightarrow \hat{\mathbf{U}}$, which factors each of the maps ϕ_D through the canonical map $\hat{\mathbf{U}} \rightarrow \mathfrak{A}_D$. Theorem 5.2 allows us to give an alternative proof of the following injectivity result which is due to Lusztig [2000].

Proposition 7.1. *The homomorphism ϕ is injective.*

Proof. We first note that the inner product on $\hat{\mathbf{U}}$ is nondegenerate. While this is not explicitly stated in the book [Lusztig 1993], it follows easily from the results there. For example, one may use the results of §26.2 and the nondegeneracy of the inner product defined in §19.1 which is established in Lemma 19.1.4 (all references in this sentence are to sections of [Lusztig 1993]). Now suppose that u is in the kernel of ϕ . Then for every D we have $\phi_D(u) = 0$, and hence by Theorem 5.2 we see that u is in the radical of the inner product on $\hat{\mathbf{U}}$, and hence it follows that $u = 0$. □

The modified quantum group $\hat{\mathbf{U}}$ is equipped with a canonical basis $\hat{\mathbf{B}}$ which generalises the canonical basis of \mathbf{U}^- . We now show that the compatibility of the inner products can be used to give an essentially self-contained construction of this basis. Let $\mathbf{A} = \mathbb{Q}(v) \cap \mathbb{Q}[[v^{-1}]]$ and let \mathbb{B}^{\pm} be defined by

$$\mathbb{B}^{\pm} = \{b \in \hat{\mathbf{U}}_{\mathcal{A}} : (b, b) \in 1 + v^{-1}\mathbf{A}, \bar{b} = b\}.$$

We show that this set is a signed basis of $\dot{\mathbf{U}}$. (We will also be able to choose a basis within this set.)

Remark 7.2. Lusztig [1993] showed that $\dot{\mathbf{B}}$ is “almost orthonormal” for the inner product, from which one can also deduce the (weaker) nondegeneracy statement used in the proof of Proposition 7.1. We prefer the argument given above since we wish to give a geometric construction of the canonical basis which does not presuppose its existence.

We begin by showing that \mathbb{B}^\pm is closely related to the bases \mathfrak{B}_D of the algebras \mathfrak{A}_D .

Proposition 7.3. *Let $b \in \mathbb{B}^\pm$. Then there exists λ such that $b \in \dot{\mathbf{U}}1_\lambda$. If k is the residue of $\sum_{i=1}^n \lambda_i \pmod n$ then there is a $p_0 > 0$ such that for all $p > p_0$ we have $\phi_{k+pn}(b) \in \pm\mathfrak{B}_D$. Conversely, if $b \in \dot{\mathbf{U}}1_\lambda$ has $\phi_D(b) \in \pm\mathfrak{B}_{k+pn}$ for all $p > p_1$ (some $p_1 \in \mathbb{N}$) then $b \in \mathbb{B}^\pm$.*

Proof. Suppose that $b \in \mathbb{B}^\pm$. Then since the inner product $\dot{\mathbf{U}}$ is obtained as a limit from the inner products on \mathfrak{A}_D , we see that for large p we have (in the notation of Section 4)

$$\sum_{l=0}^{n-1} \langle b, b \rangle_{l+pn} = 1 \pmod{v^{-1}\mathbb{Z}[v^{-1}]}.$$

Now for each l , ($0 \leq l \leq n - 1$) set $x_{l+pn} = \phi_{l+pn}(b)$. It is clear that x_{l+pn} is bar-invariant (for the bar involution on \mathfrak{A}_{l+pn} , see Remark 6.2), and lies in $\mathfrak{A}_{l+pn, \mathcal{A}}$. Thus we may write $x_{l+pn} = \sum_{i \in I} a_i \{A_i\}$ for some $a_i \in \mathcal{A}$ and $A_i \in \mathfrak{S}_{l+pn, n, n}^{\text{ap}}$, where $\bar{a}_i = a_i$. Now suppose that $a_i \in v^m \mathbb{Z}[v^{-1}]$ for all $i \in I$, and m is minimal with this property. Then, using the “almost orthonormality” property that $(\{A_i\}, \{A_j\}) \in \delta_{i,j} + v^{-1}\mathbb{Z}[v^{-1}]$ (see Lemma 6.1), we see that if $J \subset I$ denotes the subset consisting of those i with $a_i = c_i v^m + \dots$, where $c_i \neq 0$, then

$$(x_{l+pn}, x_{l+pn})_{l+pn} = \left(\sum_{i \in J} c_i^2 \right) v^{2m} + \text{lower order terms}.$$

In particular, since $(x_{l+pn}, x_{l+pn}) \in \mathbb{Z}[v^{-1}]$ we must have $m = 0$. But then since x_{l+pn} is bar-invariant, we must have $a_i \in \mathbb{Z}$ for each $i \in I$. Now since

$$\sum_{k=1}^{n-1} (x_{k+pn}, x_{k+pn}) \in 1 + v^{-1}\mathbb{Z}[v^{-1}],$$

it follows that in fact there is a $k \in \{0, 1, \dots, n - 1\}$ such that $x_{l+pn} = 0$ for $l \neq k$ and $x_{k+pn} = \pm\{A\}$ for some $A \in \mathfrak{S}_{k+pn, n, n}$. Indeed the same argument shows that the signed basis $\pm\mathfrak{B}_D$ is characterised by the properties that its elements are

bar-invariant, integral, and almost orthonormal. Note also that if $\lambda = c(A) \bmod \mathbb{Z}b_0$ it is then easy to see that $b = b1_\lambda$ as claimed in the statement of the lemma.

The converse is easier, since we know that \dot{U} injects into the inverse limit of the \mathfrak{A}_D , so that if $\phi_D(b) \in \mathfrak{A}_{D,\mathcal{A}}$ for all $D \equiv k \pmod n$, then $b \in \dot{U}_{\mathcal{A}}$, and bar invariance and the condition on (b, b) is also evident. \square

To extract a basis from \mathbb{B}^\pm we need to recall some results of Lusztig. For this we need some definitions. Let $\mathfrak{S}_{D,n,n}^-$ be the set of all $B \in \mathfrak{S}_{D,n,n}$ such that $b_{ij} = 0$ for $i > j$. Let $\mathfrak{S}_{D,n,n}^+$ be the set of all $B \in \mathfrak{S}_{D,n,n}$ such that $b_{ij} = 0$ for all $i < j$. Given $A \in \mathfrak{S}_{D,n,n}$ we may define A^+ and A^- in $\mathfrak{S}_{D,n,n}^+$ and $\mathfrak{S}_{D,n,n}^-$ respectively by

$$a_{ij}^- = a_{ij} \text{ if } i < j, \quad a_{ij}^- = 0 \text{ if } i > j, \quad a_{ii}^- = \sum_{j \in \mathbb{Z}, i \geq j} a_{ij},$$

$$a_{ij}^+ = a_{ij} \text{ if } i > j, \quad a_{ij}^+ = 0 \text{ if } i < j, \quad a_{ii}^+ = \sum_{k \in \mathbb{Z}, k \leq i} a_{ki}.$$

Lemma 7.4. *Let $A \in \mathfrak{S}_{D,n,n}$.*

- (1) *If $A \in \mathfrak{S}_{D,n,n}^\pm$ then $\psi_D(\{A\}) = \{A - I\}$.*
- (2) *For any $A \in \mathfrak{S}_{D,n,n}$ we have*

$$\{A^-\}\{A^+\} = \{A\} + \sum_{A' < A} c_{A,A'}\{A'\},$$

where $c_{A,A'} \in \mathbb{Z}[v, v^{-1}]$.

- (3) *For any $A \in \mathfrak{S}_{D,n,n}$ we have*

$$\psi_D(\{A\}) = \{A - I\} + \sum_{A' < A} e_{A,A'}\{A' - I\},$$

where $\{A - I\}$ is interpreted as 0 if $A - I$ does not lie in $\mathfrak{S}_{D,n,n}$, and likewise for $\{A' - I\}$.

Proof. In [Lusztig 2000, §3.7] the elements of $A \in \mathfrak{S}_{D,n,n}^\pm$ are related to perverse sheaves on quiver varieties attached to the cyclic quiver, giving a geometric interpretation of part of the map from \dot{U} to $\mathfrak{A}_{D,n,n}$ (see also Section 6 and [Lusztig 1999, §5] for more details). From this and the compatibility of the maps ϕ_D and ψ_D , part (1) readily follows. Part (2) is [Lusztig 1999, Proposition 4.11]. The last part follows by induction on the partial order $<$ using parts (1) and (2) together with the fact that $(A - I)^\pm = A^\pm - I$. \square

Definition 7.5. Next we note that the partial order \leq has a combinatorial cousin \preceq which we can make more explicit: Given $A, B \in \mathfrak{S}^{n,n}$ say $A \preceq B$ if for all $i < j \in \mathbb{Z}$ we have

$$\sum_{r \leq i; s \geq j} a_{r,s} \leq \sum_{r \leq i; s \geq j} b_{r,s},$$

and for any $i > j$ we have

$$\sum_{r \geq i; s \leq j} a_{r,s} \leq \sum_{r \geq i; s \leq j} b_{r,s}.$$

It is easy to check that if $A, B \in \mathfrak{S}_{D,n,n}$ and $A \leq B$ then $A \preceq B$ (see for example [Beilinson et al. 1990, Lemma 3.6] and [Lusztig 1999, §1.6]). Also, if we write ${}_p A = A + pI$, then it is clear that $A \preceq B$ if and only if ${}_p A \preceq_p B$. Moreover, crucially in what follows, given $A \in \mathfrak{S}^{n,n}$ the set

$$\{B \in \mathfrak{S}^{n,n} : B \preceq A, r(B) = r(A), c(B) = c(A)\}$$

is finite.

We now resolve the ambiguity of signs in the definition of \mathbb{B}^\pm and extract a basis from the signed basis \mathbb{B}^\pm .

Corollary 7.6. *Let*

$$\mathbb{B} = \{b \in \mathbb{B}^\pm : \phi_D(b) \in \mathfrak{B}_D \cup \{0\} \text{ for all } D \gg 0\}.$$

Then $\mathbb{B}^\pm = \mathbb{B} \sqcup (-\mathbb{B})$. Moreover, if $\phi_{k+pn}(b) = \{A_{k+pn}\}$, where $A_{k+pn} \in \mathfrak{S}_{D,n,n}$ for all $p > p_0$ say, then $A_{k+(p+1)n} = A_{k+pn} + I$.

Proof. It is only necessary to show that \mathbb{B} is well-defined. Suppose $b \in \mathbb{B}^\pm$. The previous proposition shows that if $b = b1_\lambda$, and $k = \sum_{i=1}^n \lambda_i$, then for large enough p , say $p \geq p_0$, we have $\phi_{k+pn}(b) \in \pm \mathfrak{B}_{k+pn}$, and moreover $\phi_D(b) = 0$ if D is not congruent to k modulo n . Thus we have $\phi_{k+pn}(b) = \epsilon_{k+pn} \{A_{k+pn}\}$ where $\epsilon_{k+pn} \in \{\pm 1\}$ and $A_{k+pn} \in \mathfrak{S}_{k+pn,n,n}$ for all $p \geq p_0$. But now by part (3) of Lemma 7.4 we have

$$\psi_{k+(p+1)n}(\{A_{k+(p+1)n}\}) = \{A_{k+(p+1)n} - I\} + \sum_{B \preceq A_{k+pn}} e_B \{B\}, \quad (e_B \in \mathcal{A}),$$

whereas $\psi_{k+(p+1)n}(\phi_{k+(p+1)n}(b)) = \phi_{k+pn}(b) = \epsilon_{k+pn} \{A_{k+pn}\}$. Comparing these two expressions we conclude that $\epsilon_{k+(p+1)n} = \epsilon_{k+pn}$ and $\{A_{k+(p+1)n}\} = \{A_{k+pn}\} + I$ as claimed. □

Corollary 7.7. *The set \mathbb{B} is almost orthonormal, that is*

$$(b_1, b_2) \in \delta_{b_1, b_2} + v^{-1} \mathbb{Z} \llbracket v^{-1} \rrbracket.$$

Thus the set \mathbb{B} is linearly independent.

Proof. Let $b_1, b_2 \in \mathbb{B}$. Take $\lambda \in X'$ so that $b_1 = b_1 1_\lambda$. Then either $b_2 1_\lambda = 0$, in which case the corollary holds trivially, or $b_2 = b_2 1_\lambda$. In that case, we see from

Corollary 7.6 that we may find a $p_0 \in \mathbb{N}$ and $A, B \in \mathfrak{S}^{n,n}$ so that $\phi_{p_0+pn}(b) = \{pA\}$ and $\phi_{p_0+pn}(b_2) = \{pB\}$ for all $p \geq 0$. But then it follows from **Lemma 6.1** that

$$(\{pA\}, \{pB\}) \in \delta_{A,B} + v^{-1}\mathbb{Z}[[v^{-1}]].$$

for all p , and hence taking the limit we obtain the same result for b_1, b_2 . To see that this implies the linear independence of the set \mathbb{B} , consider a dependence involving the minimal number of elements of \mathbb{B} :

$$\sum_{j=1}^k p_j b_j = 0,$$

where by clearing denominators if necessary we may assume that $p_k \in \mathbb{Z}[v, v^{-1}]$ (and by minimality they are all nonzero) and $b_k \in \mathbb{B}$. We may moreover assume, multiplying through by an appropriate power of v , that $p_i = n_i + v^{-1}\mathbb{Z}[v^{-1}]$, where $n_i \in \mathbb{Z}$, and, reordering if necessary, that $n_1 \neq 0$. Pick D large enough so that

$$\sum_{l=0}^{n-1} (\phi_D(b_r), \phi_D(b_s))_{D+l} = (b_r, b_s) \pmod{v^{-1}\mathbb{Z}[[v^{-1}]]} \quad \text{for } 1 \leq r, s \leq k.$$

and moreover that for each j we have $\phi_D(b_j) = \{B_j\}$ for some $B_j \in \mathfrak{S}_{D,n,n}$. Then

$$\begin{aligned} 0 &= \left(\sum_{r=1}^k p_r b_r, \sum_{s=1}^k p_s b_s \right) = \sum_{1 \leq r, s \leq k} p_r p_s (b_r, b_s) \\ &\equiv \sum_{1 \leq r, s \leq k} p_r p_s (\{B_r\}, \{B_s\})_D \pmod{v^{-1}\mathbb{Z}[v^{-1}]} \\ &\equiv \sum_{1 \leq r \leq k} n_r^2 \pmod{v^{-1}\mathbb{Z}[v^{-1}]}, \end{aligned}$$

which is a contradiction, since $n_1 \neq 0$. □

We now show that if $A \in \mathfrak{S}^{n,n}$ then for large enough p there is a unique $b \in \mathbb{B}$ such that $\phi_D(b) = \{pA\}$ (where $D = \sum_{i \in [1,n], j \in \mathbb{Z}} a_{ij} + pn$), and hence by **Corollary 7.6** it will also follow that for large enough p we have $\psi_D(\{pA\}) = \{p_{-1}A\}$. We need to recall the relation between the canonical basis \mathbf{B} of \mathbf{U}^- and \mathfrak{B}_D . Recall from **Section 6** that the representation theory of the cyclic quiver allows us to parametrise \mathbf{B} by tableaux $(\mu_{t,p})_{t \in \mathbb{Z}, p \in \mathbb{N}}$, where $\mu_{t,p} \in \mathbb{N}$ and $\mu_{t,p} = \mu_{t-n,p}$ for all $t \in \mathbb{Z}$ and for fixed t only finitely many of the $\mu_{t,p}$ are nonzero. The v -graded part \mathbf{B}_v is then indexed by Σ_v .

The correspondence described in **Section 6** gives a bijection between those $(\mu_{t,p})$ in Σ_v satisfying

$$\mu_{i,1} + \mu_{i,2} + \dots \leq a_i \quad \text{for all } i,$$

and the orbits in corresponding to matrices $B \in \mathfrak{S}_{D,n,n}^-$ with $r(B) = \mathbf{a}$. (Note that if the integers a_i are sufficiently positive, this gives an injection from \mathbf{B}_v into $\mathfrak{S}_{D,n,n}^-$.) Using the same correspondence, composed with the transpose map Ψ , we obtain a similar correspondence between (appropriate subsets of) \mathbf{B}_v and elements of $\mathfrak{S}_{D,n,n}^+$. These can be combined to give a correspondence between the set of triples $\mathcal{T} = \{(b_1, b_2, \lambda) : b_1, b_2 \in \mathbf{B}, \lambda \in X\}$ and elements of $\mathfrak{S}^{n,n}$ as follows: the elements b_1, b_2 corresponds to tableau $(\mu_{t,p})$ and $(\rho_{t,p})$ say, and we define $A = A(b_1, b_2, \lambda) \in \mathcal{G}^{n,n}$ by setting

$$a_{ij} = \begin{cases} \mu_{i,j-i} & \text{if } i < j, \\ \rho_{j,i-j} & \text{if } i > j, \\ \lambda_i - \sum_{t \geq 1} \mu_{i,t} - \sum_{s \geq 1} \rho_i & \text{if } i = j. \end{cases}$$

We will write b_A for the element $b_1^+ 1_\lambda b_2^- \in \dot{\mathbf{U}}$, and ${}_D b_A$ for its image under ϕ_D . It follows from the [Lusztig 1999, §5] and [Lusztig 1999, Proposition 4.11] that if $\sum_{i=1}^n \lambda_i = D$ and the entries of $A(b_1, b_2, \lambda)$ are all nonnegative, then

$${}_D b_A = \{pA\} + \sum_{B <_p A} e_{B,pA} \{B\} \tag{7-1}$$

Proposition 7.8. *Let $A \in \mathfrak{S}^{n,n}$. For large enough p we have*

$$\psi_D(\{pA\}) = \{p-1A\}$$

Proof. Via the bijection described above between \mathcal{T} and $\mathfrak{S}^{n,n}$, we may find a subset \mathcal{T}_a of \mathcal{T} such that $\{A(b_1, b_2, \mathbf{a})\}$ is a basis of $\mathfrak{A}_D[\mathbf{i}_a]$ as (b_1, b_2, \mathbf{a}) runs over the set \mathcal{T}_a . Then the elements ${}_D b_A$ are clearly also a basis of $\mathfrak{A}_D[\mathbf{i}_a]$ since they are related to the elements $\{A\}$ by an upper triangular matrix, and moreover they satisfy

- (1) $\overline{{}_D b_A} = {}_D b_A$,
- (2) ${}_D b_A \in \mathbf{U}_{D,\mathcal{A}}$.

As in the proof of Proposition 7.3, the basis $\{\{A\} : A \in \mathfrak{S}_{D,n,n}\}$ is characterised up to sign by the properties of being bar-invariant, integral (that is, contained in $\mathbf{U}_{D,\mathcal{A}}$), and being almost orthonormal, so that

$$(\{A\}, \{B\})_D \in \delta_{A,B} + v^{-1}\mathbb{Z}[v^{-1}].$$

(In fact, Proposition 7.3 shows that less than this characterises $\pm\mathfrak{B}_D$).

We now show that one can obtain $\{A\}$ from ${}_D b_A$ by a Gram–Schmidt style process. Indeed if A is minimal for the ordering \leq , then clearly $\{A\} = b_A$. Thus we consider the following claim:

- For each $A \in \mathfrak{S}^{n,n}$, there is a $p_0 \in \mathbb{Z}$ such that for all $p > p_0$ we have

$$\{pA\} = {}_D b_A + \sum_{A' < A} d_{A',AD} b_{A'}$$

where $d_{A',A} \in \mathcal{A}$ do not depend on p , and $D = pn + \sum_{i,j:1 \leq i \leq n} a_{ij}$.

The proof of the proposition now follows immediately since $\psi_D({}_D b_A) = {}_{D-n} b_A$. We show this by induction on \preceq : if A is minimal, then (7-1) implies that $b_A = \{A\}$, and we are done. Thus suppose that the result is known for all $B \prec A$, and let I be the (finite) set

$$\{B \in \mathfrak{S}^{n,n} : B \preceq A, r(A) = r(B), c(A) = c(B)\}.$$

Now for x in the span of $\{\{B\} : B \in I\}$, set $N(x) = \max\{v(x, \{B\})_D : B \in I, B \neq A\}$, where for $f \in \mathcal{A}$ we let $v(f)$ denote the highest power of v occurring in f . Let $N = N(b_A)$, and suppose that $N \geq 0$. Let J denotes the subset of I for which $v(b_A, \{B\}) = N$, so that if $B \in J$ we have

$$(b_A, \{B\})_D = c_B v^N + \text{lower order terms} \quad (c_B \in \mathbb{Z}).$$

Now $(\{B\}, \{B\})_D \in 1 + v^{-1}\mathbb{Z}[v^{-1}]$, so we may recursively solve for

$$a_B \in v^{-N}\mathbb{Z}[v] \cap \mathbb{Z}[v^{-1}]$$

such that $a_B \cdot (\{B\}, \{B\})_D \in 1 + v^{-N-1}\mathbb{Z}[v^{-1}]$. It follows immediately that we may find $e_B \in \mathcal{A}$ such that $\bar{e}_B = e_B$ and $e_B = c_B v^N a_B \bmod v^{-1}\mathbb{Z}[v^{-1}]$. Then we set

$$b'_A = b_A - \sum_{B \in J} e_B \{B\}.$$

It follows from the almost orthonormality of the $\{B\}$ that $(b'_A, \{B\})_D \in v^{N-1}\mathbb{Z}[v^{-1}]$, and b'_A is again bar-invariant, lies in $\mathbf{U}_{D,\mathcal{A}}$, and satisfies $N(b'_A) < N$. We may thus iterate this construction to obtain an element b''_A which has $N(b''_A) \leq -1$, is bar-invariant, and lies in $\mathbf{U}_{D,\mathcal{A}}$. But then we claim that we must have $b''_A = \{A\}$. Indeed we know from (7-1) that we can write

$$b''_A = \{A\} + \sum_{B < A} f_B \{B\}$$

for some $f_B \in \mathcal{A}$ with $\bar{f}_B = f_B$. If it is not the case that $f_B = 0$ for all B , then there is some B with $v(f_B) \geq 0$ maximal, whence we see that $v(b''_A, \{B\})_D \geq 0$ which is a contradiction. Thus $b''_A = \{A\}$ as required.

Now examining the above process, we see that it uses only the values of $(b_A, \{B\})_D$ down to order to $v^{-N(b_A)}\mathbb{Z}[v^{-1}]$, and by induction we see that these, for large enough p are determined by the values of $(b_A, b_B)_D$, down to some possibly lower order (determined by the coefficients $d_{B,C}$). Since the values of $(b_A, b_B)_D$ converge in $\mathbb{Z}((v^{-1}))$ we see that we may find a large enough p_0 so that $\{pA\}$ is

a linear combination of $\{\phi_D(b_{A'}) : A' \leq A\}$ with coefficients independent of p as required. \square

We can now show that the set \mathbb{B} is a basis of $\dot{\mathbf{U}}$.

Theorem 7.9. \mathbb{B} is a basis of $\dot{\mathbf{U}}$.

Proof. Notice first that given any $b \in \mathbb{B}$, Proposition 7.3 implies that $\phi_D(b) \in \mathfrak{B}_{D \cup \{0\}}$ for large enough D , and is nonzero provided D has a fixed residue modulo n . By Corollary 7.7 we know that the elements of \mathbb{B} are linearly independent, so we need only show that they span $\dot{\mathbf{U}}$. To do this it is enough to show that the element $b_1^+ b_2^- 1_\lambda$ for $b_1, b_2 \in \mathbf{B}$ and $\lambda \in X$ lie in the span of \mathbb{B} , since they form a basis for $\dot{\mathbf{U}}$. But the claim in the proof of Proposition 7.8 shows that we may find an element of \mathbb{B} which is a linear combination of such basis elements with leading coefficient 1, so that the matrix relating the two sets is invertible and \mathbb{B} indeed spans $\dot{\mathbf{U}}$. \square

Remark 7.10. The results of [Lusztig 1993, §26.3] then show that $\mathbb{B} = \dot{\mathbf{B}}$, and thus the results of this section give a new proof of the conjecture made in [Lusztig 1999, §9.3], which was originally proved by Schiffmann and Vasserot [2000]. Our goal here was to construct the canonical basis purely within the context of the inverse system \mathbf{U}_D ; thus, unlike Schiffmann and Vasserot, we do not need to assume the existence of $\dot{\mathbf{B}}$, nor use any properties of crystal bases. It should be noted however that by using results of Kashiwara on global crystal bases, those authors have obtained a more precise result (also conjectured in [2000]) saying that the maps ϕ_D are all compatible with the canonical basis; i.e., if $b \in \dot{\mathbf{B}}$ then $\phi_D(b) \in \mathfrak{B}_D \cup \{0\}$, and moreover the kernel of ϕ_D is spanned by a subset of $\dot{\mathbf{B}}$. The results of this section show that this theorem would also follow if we could show that the maps ψ_D are compatible with the bases \mathbf{B}_D and \mathbf{B}_{D-n} , a question which can be phrased purely geometrically (in terms of perverse sheaves). Note that it is *not* true that the maps ψ_D send \mathfrak{B}_D to $\mathfrak{B}_{D-n} \cup \{0\}$, as was pointed out already in [Lusztig 2000, 1.12]. It is possible to give a construction of the maps ψ_D in the context of perverse sheaves on the ind-varieties \mathcal{F}_a , (i.e., to show that there exists a functor on the derived category that preserves perverse sheaves (up to shift) and induces ψ_D on the Grothendieck group which moreover is compatible with the ‘‘convolution’’ on $\mathcal{H}_{D,n,n}$), but it is not immediately clear why this functor preserves simple objects.

8. A positivity result

We may combine Theorem 5.2 and Proposition 6.5 to prove a positivity result for the inner product of two elements of $\dot{\mathbf{B}}$. This has been conjectured by Lusztig for all types.

Theorem 8.1. Let $b_1, b_2 \in \dot{\mathbf{B}}$. Then

$$(b_1, b_2) \in \mathbb{N}[[v^{-1}]] \cap \mathbb{Q}(v).$$

Proof. We may assume that there is a $\lambda \in X$ such that $b_1 1_\lambda = b_1$, and $b_2 1_\lambda = b_2$. Let $k \in \{0, 1, \dots, n-1\}$ be such that $\sum_{j=1}^n \lambda_j = k \bmod n$. Then

$$(b_1, b_2) = \lim_{p \rightarrow \infty} (\phi_{k+pn}(b_1), \phi_{k+pn}(b_2))_{k+pn}$$

By [Proposition 7.3](#) we know that for all large enough D we have $\phi_D(b_1), \phi_D(b_2)$ are in \mathfrak{B}_D , hence it is clear from [\(6-1\)](#) that

$$(\phi_{k+pn}(b_1), \phi_{k+pn}(b_2))_{k+pn} \in \mathbb{N}[v, v^{-1}].$$

However, it follows also from [Lemma 6.1](#) that the left-hand side is in fact in $\mathbb{N}[v^{-1}]$ (this can also be seen directly, using the definition of intersection cohomology sheaves). Hence (b_1, b_2) is the limit of elements of $\mathbb{N}[v^{-1}]$, and the statement follows. \square

Remark 8.2. All the results of this paper have analogues for the nonaffine case, which can be proved in exactly the same way. The module V is replaced by a D -dimensional vector space over \mathbb{k} , and the space \mathcal{F}^n of n -step periodic lattices should be replaced by the space of n -step flags in that vector space. In this case the algebra corresponding to \mathbf{U}_D is actually equal to the algebra analogous to \mathfrak{A}_D , hence the results are sometimes more straightforward.

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The smallest prime that does not split completely in a number field XIANNAN LI	1061
On the geometric realization of the inner product and canonical basis for quantum affine \mathfrak{sl}_n KEVIN MCGERTY	1097
Combinatorics of the tropical Torelli map MELODY CHAN	1133
On fusion categories with few irreducible degrees SONIA NATALE and JULIA YAEL PLAVNIK	1171
Cusp form motives and admissible G -covers DAN PETERSEN	1199
Ideals of degree one contribute most of the height AARON LEVIN and DAVID MCKINNON	1223
Torsion des modules de Drinfeld de rang 2 et formes modulaires de Drinfeld CÉCILE ARMANA	1239