Algebra & Number Theory

Volume 7

2013

No. 1

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ALGEBRA AND NUMBER THEORY 7:1 (2013)
dx.doi.org/10.2140/ant.2013.7.1

Powers of ideals and the cohomology of stalks and fibers of morphisms

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À Jean-Pierre Jouanolou, avec admiration et amitié

We first provide here a very short proof of a refinement of a theorem of Kodiyalam and Cutkosky, Herzog and Trung on the regularity of powers of ideals. This result implies a conjecture of Hà and generalizes a result of Eisenbud and Harris concerning the case of ideals primary for the graded maximal ideal in a standard graded algebra over a field. It also implies a new result on the regularities of powers of ideal sheaves. We then compare the cohomology of the stalks and the cohomology of the fibers of a projective morphism to the effect of comparing the maximums over fibers and over stalks of the Castelnuovo–Mumford regularities of a family of projective schemes.

1. Introduction

An important result of Kodiyalam and Cutkosky, Herzog and Trung states that the Castelnuovo–Mumford regularity of the power I^t of an ideal over a standard graded algebra is eventually a linear function in t. The leading term of this function has been determined by Kodiyalam in his proof.

This result was first obtained for standard graded algebras over a field, and later extended by Trung and Wang to standard graded algebras over a Noetherian ring.

We first provide here a very short proof of a refinement of this result.

Theorem 1.1. *Let A be a positively graded Noetherian algebra, $M \neq 0$ be a finitely generated graded A-module, I be a graded A-ideal, and set*

$$d := \min\{\mu \mid \textit{there exists } p,\ (I_{\leq\mu})I^pM = I^{p+1}M\}.$$

Then

$$\lim_{t\to\infty}(\operatorname{end}(H^i_{A_+}(I^tM)) + i - td) \in \mathbb{Z} \cup \{-\infty\}$$

exists for any i, and is at least equal to the initial degree of M for some i.

MSC2000: primary 13D02; secondary 13A30, 13D45, 14A15.
Keywords: cohomology, stalks, Rees algebras, fibers of morphisms, powers of ideals, Castelnuovo–Mumford regularity.

The end of a graded module H is $\operatorname{end}(H) := \sup\{\mu \mid H_\mu \neq 0\}$ if $H \neq 0$ and $-\infty$ otherwise. Recall that for a graded A-module N, $\operatorname{reg}(N) = \max_i\{\operatorname{end}(H^i_{A_+}(N)) + i\}$.

Very interesting examples showing hectic behavior of the value of

$$a^i(t) := \operatorname{end}(H^i_{A_+}(I^t))$$

as t varies were given in [Cutkosky 2000]. These examples point out that the existence of the limit quoted above does not imply that all of the functions $a^i(t)$ are eventually linear functions of t. It only implies that at least one of them is eventually linear in t. For instance, in the examples given by Cutkosky, the limit in the theorem is $-\infty$ for all $i \neq 0$.

More recently, Eisenbud and Harris proved that in the case of a standard graded algebra A over a field, for a graded ideal that is A_+-primary and generated in a single degree, the constant term in the linear function is the maximum of the regularity of the fibers of the morphism defined by a set of minimal generators. In a recent preprint, Huy Tài Hà [2011, 1.3] generalized this result by proving that if an ideal is generated in a single degree d, a variant of the regularity (the a^*-invariant) satisfies $a^*(I^t) = dt + a$ for $t \gg 0$, where a can be expressed in terms of the maximum of the values of a^* on the stalks of the projection π from the closure of the graph of the map defined by the generators to its image. He conjectures that a similar result holds for the regularity.

In Theorem 5.3 we prove this conjecture of Hà. More precisely, we show that the limit in the theorem above is the maximum of the end degree of the i-th local cohomology of the stalks of π, for ideals generated in a single degree. This holds for graded ideals in a Noetherian positively graded algebra.

An interesting, and perhaps surprising, consequence of this result is the following result on the limit of the regularity of saturation of powers, or equivalently of powers of ideal sheaves, in a positively graded Noetherian algebra:

Corollary 1.2. *Let I be a graded ideal generated in a single degree d. Then,*

$$\lim_{t\to\infty} (\operatorname{reg}((I^t)^{\text{sat}}) - dt)$$

exists and the following are equivalent:

(i) *the limit is nonnegative*,

(ii) *the limit is not $-\infty$*,

(iii) *the projection π from the closure of the graph of the function defined by minimal generators of I to its image admits a fiber of positive dimension.*

This can be applied to ideals generated in degree at most d, replacing I by $I_{\geq d}$.

It gives a simple geometric criterion for an ideal I generated in degree (at most) d to satisfy $\operatorname{reg}((I^t)^{\text{sat}}) = dt + b$ for $t \gg 0$: This holds if and only if there exists a subvariety V of the closure of the graph that is contracted in its projection to the closure of the image (that is, $\dim(\pi(V)) < \dim V$). A very simple example is the following. In a polynomial ring in $n+1$ variables, any graded ideal generated by n forms of the same degree d satisfies $\operatorname{reg}((I^t)^{\mathrm{sat}}) = dt + b$ for $t \gg 0$, with $b \geq 0$. The same result holds if a reduction of the ideal is generated by at most n elements (in other words, if the analytic spread of I is at most n).

The result of Eisenbud and Harris is stated in terms of regularity of fibers. For a finite morphism, there is no difference between the regularity of stalks and the regularity of fibers. This follows from the following result that is likely part of folklore, but that we didn't find in several of the classical references in the field:

Lemma 1.3. *Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring, $S := R[X_1, \ldots, X_n]$ be a polynomial ring over R with $\deg X_i > 0$ and $\mathbb{M}$ be a finitely generated graded S-module. Set $d := \dim(\mathbb{M} \otimes_R k)$. Then $H^i_{S_+}(\mathbb{M}) = 0$ for $i > d$ and the natural graded map $H^d_{S_+}(\mathbb{M}) \otimes_R k \to H^d_{S_+}(\mathbb{M} \otimes_R k)$ is an isomorphism.*

For morphisms that are not finite or flat, the situation is more subtle — see Proposition 6.3. We show that for families of projective schemes that are close to being flat (if the Hilbert polynomial of any two fibers differ at most by a constant, in the standard graded situation), the maximum of the regularities of stalks and the maximum of the regularities of fibers agree. Also the maximum regularity of stalks bounds above the one for fibers under a weaker hypothesis. Putting this together provides a collection of results that covers the results obtained in [Eisenbud and Harris 2010; Hà 2011]. See Theorem 6.11.

To simplify the statements, we introduce the notion of regularity over a scheme, generalizing the usual notion of regularity with reference to a polynomial extension of a ring. This is natural in our situation: The family of schemes given by the closure of the graph over the parameter space given by the closure of the image of our map, considered as a projective scheme, is a key ingredient of this study.

2. Notation and general setup

Let R be a commutative ring and S a polynomial ring over R in finitely many variables.

If S is $\mathbb{Z}$-graded, $R \subset S_0$, and $X_1, \ldots, X_n$ are the variables with positive degrees, the Čech complex $\mathscr{C}^\bullet_{(S_+)}(M)$ with

$$\mathscr{C}^0_{(S_+)}(M) = M \quad \text{and} \quad \mathscr{C}^i_{(S_+)}(M) = \bigoplus_{j_1 < \cdots < j_i} M_{X_{j_1} \cdots X_{j_i}} \quad \text{for } i > 0$$

is graded whenever M is a graded S-module.

There is an isomorphism $H^i_{(S_+)}(M) \simeq H^i(\mathscr{C}^\bullet_{(S_+)}(M))$ for all i, which is graded if M is. One then defines two invariants attached to such a graded S-module M:

$$a^i(M) := \sup\{\mu \mid H^i_{(S_+)}(M)_\mu \neq 0\}$$

if $H^i_{(S_+)}(M) \neq 0$ and $a^i(M) := -\infty$ otherwise, and

$$b_j(M) := \sup\{\mu \mid \mathrm{Tor}^S_j(M, S/(S_+))_\mu \neq 0\}$$

if $\mathrm{Tor}^S_j(M, S/(S_+)) \neq 0$ and $b_j(M) := -\infty$ otherwise. Notice that $a^i(M) = -\infty$ for $i > n$ and $b_j(M) = -\infty$ for $j > n$. The Castelnuovo–Mumford regularity of a graded S-module M is then defined as

$$\mathrm{reg}(M) := \max_i\{a^i(M) + i\} = \max_j\{b_j(M) - j\} + n - \sigma$$

where σ is the sum of the degrees of the variables with positive degrees. Other options are possible, in particular when S is not standard graded (when $\sigma \neq n$). Another related invariant is

$$a^*(M) := \max_i\{a^i(M)\} = \max_j\{b_j(M)\} - \sigma.$$

The following classical result is usually stated for positive grading.

Theorem 2.1. *Let S be a finitely generated $\mathbb{Z}$-graded algebra over a Noetherian ring $R \subseteq S_0$ and M be a finitely generated graded S-module. Assume S is generated over R by elements of nonzero degree. Then, for any i,*

(i) $a^i(M) \in \{-\infty\} \cup \mathbb{Z}$,

(ii) *the R-module $H^i_{(S_+)}(M)_\mu$ is finitely generated for any $\mu \in \mathbb{Z}$.*

Proof. S is an epimorphic image of a polynomial ring S' over R by a graded morphism. Considering M as an S'-module, one has $H^i_{(S_+)}(M) \simeq H^i_{(S'_+)}(M)$ via the natural induced map, so that we may replace S by S' and assume that

$$S = R[Y_1, \ldots, Y_m, X_1, \ldots, X_n]$$

with $\deg Y_i \leq -1$ and $\deg X_j \geq 1$ for all i and j. We recall that $H^i_{(S_+)}(S) = 0$ for $i < n$ and $H^n_{(S_+)}(S) = (X_1 \cdots X_n)^{-1} R[Y_1, \ldots, Y_m, X_1^{-1}, \ldots, X_n^{-1}]$, and notice that $H^n_{(S_+)}(S)_\mu$ is a finitely generated free R-module for any μ.

Let $F_\bullet$ be a graded free S-resolution of M with F_i finitely generated. Both spectral sequences associated to the double complex $\mathscr{C}^\bullet_{(S_+)} F_\bullet$ degenerate at step 2 and provide graded isomorphisms

$$H^i_{(S_+)}(M) \simeq H_{n-i}(H^n_{(S_+)}(F_\bullet)),$$

which shows that $H^i_{(S_+)}(M)_\mu$ is a subquotient of $H^n_{(S_+)}(F_{n-i})_\mu$ and hence a finitely generated R-module that is zero in degrees greater than $-n+b_{n-i}$, where b_j is the highest degree of a basis element of F_j over S. □

3. Regularity over a scheme

Local cohomology and the torsion functor commute with localization on the base R, providing natural graded isomorphisms for a graded S-module M:

$$H^i_{(S\otimes_R R_\mathfrak{p})_+}(M\otimes_R R_\mathfrak{p}) \simeq H^i_{S_+}(M)\otimes_R R_\mathfrak{p}$$

and

$$\mathrm{Tor}_i^{S\otimes_R R_\mathfrak{p}}(M\otimes_R R_\mathfrak{p}, R_\mathfrak{p}) \simeq \mathrm{Tor}_i^S(M, R)\otimes_R R_\mathfrak{p}.$$

Hence $a^i(M) = \sup_{\mathfrak{p}\in\mathrm{Spec}(R)} a^i(M\otimes_R R_\mathfrak{p})$ and $b_j(M) = \sup_{\mathfrak{p}\in\mathrm{Spec}(R)} b_j(M\otimes_R R_\mathfrak{p})$. It follows that the regularity is a local notion on R:

$$\mathrm{reg}(M) = \sup_{\mathfrak{p}\in\mathrm{Spec}(R)} \mathrm{reg}(M\otimes_R R_\mathfrak{p}).$$

These supremums are maximums whenever $\mathrm{reg}(M) < +\infty$, for instance if R is Noetherian and M is finitely generated. The same holds for $a^*(M)$.

In the following, this definition is extended in a natural way to the case where the base is a scheme.

Definition 3.1. Let Y be a scheme, $\mathscr{E}$ be a locally free $\mathscr{O}_Y$-module of finite rank, and $\mathscr{F}$ be a graded sheaf of $\mathrm{Sym}_Y(\mathscr{E})$-modules. Then

$$a^i(\mathscr{F}) := \sup_{y\in Y} a^i(\mathscr{F}\otimes_{\mathscr{O}_Y}\mathscr{O}_{Y,y}) \quad \text{and} \quad \mathrm{reg}(\mathscr{F}) := \max_i\{a^i(\mathscr{F})+i\}.$$

If $\mathscr{E}$ is free, $\mathrm{Sym}_Y(\mathscr{E}) = \mathscr{O}_Y[X_1, \ldots, X_n]$, and the definition of regularity above makes sense for nonstandard grading.

A closed subscheme Z of $\mathrm{Proj}(\mathrm{Sym}_Y(\mathscr{E}))$ corresponds to $\mathscr{I}_Z$, a unique graded $\mathrm{Sym}_Y(\mathscr{E})$-ideal sheaf saturated with respect to $\mathrm{Sym}_Y(\mathscr{E})_+$. We set

$$a^i(Z) := \sup_{y\in Y} a^i(\mathscr{O}_{Y,y}[X_0, \ldots, X_n]/(\mathscr{I}_Z\otimes_{\mathscr{O}_Y}\mathscr{O}_{Y,y}))$$

(notice that $a^0(Z) = -\infty$) and $\mathrm{reg}(Z) := \max_i\{a^i(Z)+i\}$.

The following proposition is immediate from the definition and the corresponding results over an affine scheme.

Proposition 3.2. *Assume Y is Noetherian, $\mathscr{E}$ is a locally free coherent sheaf on Y and $\mathscr{F}\neq 0$ is a coherent graded sheaf of* $\mathrm{Sym}_Y(\mathscr{E})$*-modules. Then* $\mathrm{reg}(\mathscr{F})\in\mathbb{Z}$*. If $Z\neq\varnothing$ is a closed subscheme of $\mathbb{P}^{n-1}_Y$, then* $\mathrm{reg}(Z)\geq 0$.

4. First result on cohomology of powers

We now prove the first statement of our text on cohomology of powers of ideals. It refines earlier results on the regularity of powers [Kodiyalam 2000; Cutkosky et al. 1999; Trung and Wang 2005]. The argument is based on Theorem 2.1 applied to a Rees algebra and a lemma due to Kodiyalam.

Theorem 4.1. *Let A be a positively graded Noetherian algebra, $M \neq 0$ be a finitely generated graded A-module, I be a graded A-ideal, and set*

$$d := \min\{\mu \mid \textit{there exists } p,\ (I_{\leq\mu})I^pM = I^{p+1}M\}.$$

Then

$$\lim_{t\to\infty}(a^i(I^tM)+i-td) \in \mathbb{Z}\cup\{-\infty\}$$

exists for any i, and is at least equal to $\operatorname{indeg}(M)$ *for some i.*

Proof. Set $J := I_{\leq d}$ and write $J = (g_1, \dots, g_s)$ with $\deg g_i = d$ for $1 \leq i \leq m$ and $\deg g_i < d$ otherwise. Let

$$\mathcal{R}_J := \bigoplus_{t\geq 0} J(d)^t = \bigoplus_{t\geq 0} J^t(td) \quad \text{and} \quad \mathcal{R}_I := \bigoplus_{t\geq 0} I(d)^t = \bigoplus_{t\geq 0} I^t(td),$$

and $S_0 := A_0[T_1, \dots, T_m]$, $S := S_0[T_{m+1}, \dots, T_s, X_1, \dots, X_n]$, with $\deg(T_i) := \deg(g_i) - d$. Setting $\operatorname{bideg}(T_i) := (\deg(T_i), 1)$ and $\operatorname{bideg}(X_j) := (\deg(X_j), 0)$, one has $J_{\deg(g_i)} = (\mathcal{R}_J)_{\deg g_i - d, 1}$ and hence a bigraded onto map

$$S \to \mathcal{R}_J, \quad T_i \mapsto g_i.$$

As $M\mathcal{R}_I$ is finite over $\mathcal{R}_J$ according to the definition of d, the bigraded embedding $\mathcal{R}_J \to \mathcal{R}_I$ makes $M\mathcal{R}_I$ a finitely generated bigraded S-module.

The equality of graded A-modules $H^i_{(S_+)}(M\mathcal{R}_I)_{(*,t)} = H^i_{A_+}(M\mathcal{R}_I)_{(*,t)}$ shows that

$$H^i_{(S_+)}(M\mathcal{R}_I)_{(\mu,t)} = H^i_{A_+}((M\mathcal{R}_I)_{(*,t)})_\mu = H^i_{A_+}(MI^t)_{\mu+td}.$$

By Theorem 2.1(i), $a^i(M\mathcal{R}_I) < +\infty$ and the equalities above show

$$a^i(MI^t) \leq td + a^i(M\mathcal{R}_I),$$

and that equality holds for some t.

Furthermore, Theorem 2.1(ii) shows that $K_{i,\mu} := H^i_{(S_+)}(M\mathcal{R}_I)_{(\mu,*)}$ is a finitely generated graded S_0-module (for the standard grading $\deg(T_i) = 1$). It follows that $H^i_{(S_+)}(M\mathcal{R}_I)_{(\mu,t)} = 0$ for $t \gg 0$ if and only if $K_{i,\mu}$ is annihilated by a power of $\mathfrak{n} := (T_1, \dots, T_m)$. Hence

$$\lim_{t\to+\infty}(a^i(MI^t) - td) = -\infty$$

if $K_{i,\mu}$ is annihilated by a power of $\mathfrak{n}$ for every $\mu \leq a^i(M\mathcal{R}_I)$, and otherwise

$$\lim_{t\to+\infty}(a^i(MI^t)-td)=\max\{\mu \mid K_{i,\mu}\neq H^0_{\mathfrak{n}}(K_{i,\mu})\}.$$

As $\operatorname{reg}(MI^t)\geq \operatorname{end}(MI^t/A_+MI^t)$, the last claim follows from the next lemma, due to Kodiyalam. □

Lemma 4.2. *With the hypotheses of Theorem 4.1,*

$$\operatorname{end}(MI^t/A_+MI^t)\geq \operatorname{indeg}(M)+td \quad \textit{for all } t.$$

Proof. The proof goes along the same lines as in the proof of [Kodiyalam 2000, Proposition 4]. The needed graded version of Nakayama's lemma does apply. □

5. Cohomology of powers and cohomology of stalks

The following result is a more elaborated, and more technical, version of Theorem 4.1 that essentially follows from its proof. It implies a conjecture of Hà on the regularity of powers of ideals, and refines the main result in [Hà 2011]. We will see later that, combined with a result on the regularity of stalks and fibers of a morphism, it also implies the result in [Eisenbud and Harris 2010].

Proposition 5.1. *Let A be a positively graded Noetherian algebra, M be a finitely generated graded A-module, I be a graded A-ideal and $J\subseteq I$ be a graded ideal such that $JI^pM=I^{p+1}M$ for some p.*

Assume that the ideal J is generated by r forms $f_1,\dots,f_r$ of respective degrees $d_1=\cdots=d_m>d_{m+1}\geq\cdots\geq d_r$. Set $d:=d_1$, $\deg(T_i):=\deg(f_i)-d$, $\operatorname{bideg}(T_i):=(\deg(T_i),1)$ and $\operatorname{bideg}(a):=(\deg(a),0)$ for $a\in A$. Consider the natural bigraded morphism of bigraded A_0-algebras

$$S:=A[T_1,\dots,T_r]\xrightarrow{\psi}\mathcal{R}_I:=\bigoplus_{t\geq0}I(d)^t=\bigoplus_{t\geq0}I^t(dt),$$

sending T_i to f_i, and the bigraded map of S-modules

$$M[T_1,\dots,T_r]\xrightarrow{1_M\otimes_A\psi}M\mathcal{R}_I:=\bigoplus_{t\geq0}MI^t(dt).$$

Let $B:=A_0[T_1,\dots,T_m]$ and $B':=B/\operatorname{ann}_B(\ker(1_M\otimes_A\psi))$.

Then,

$$\lim_{t\to+\infty}(a^i(MI^t)-td)=\max_{\mathfrak{q}\in\operatorname{Proj}(B')}\{a^i(M\mathcal{R}_I\otimes_{B'}B'_{\mathfrak{q}})\}.$$

Proof. First remark that in the proof of Theorem 4.1 we only need the equality $JI^pM=I^{p+1}M$ for some p (as a consequence, for all p big enough). We have shown there that

$$\lim_{t\to+\infty}(a^i(MI^t)-td)=-\infty, \tag{$*$}$$

if and only if the finitely generated B-module $H^i_{(S_+)}(M\mathcal{R}_I)_{(\mu,*)}$ is supported in $V(T_1,\dots,T_m)$ for any μ. As local cohomology commutes with flat base change and elements in B have degree 0,

$$H^i_{(S_+)}(M\mathcal{R}_I)_{(\mu,*)}\otimes_{B'}B'_{\mathfrak{q}} = H^i_{(S_+)}(M\mathcal{R}_I\otimes_{B'}B'_{\mathfrak{q}})_{(\mu,*)};$$

hence $(*)$ holds if and only if $H^i_{(S_+)}(M\mathcal{R}_I\otimes_{B'}B'_{\mathfrak{q}})=0$ for any $\mathfrak{q}\in\operatorname{Proj}(B')$. On the other hand, if this does not hold, there exists μ_0 the maximum value such that $H^i_{(S_+)}(M\mathcal{R}_I)_{(\mu_0,*)}$ is not supported in $V(T_1,\dots,T_m)$, and choosing $\mathfrak{q}\in\operatorname{Proj}(B')\cap\operatorname{Supp}(H^i_{(S_+)}(M\mathcal{R}_I)_{(\mu_0,*)})$ shows that both members in the asserted equality are equal to μ_0. $\square$

Remark 5.2. In the proposition above, as well as in other places in this text, we localize at homogeneous primes $\mathfrak{q}\in\operatorname{Proj}(C)$ for some standard graded algebra C, in other words, at graded prime ideals that do not contain C_+. We may as well replace these localizations by the degree zero part of the localization at such a prime ideal, usually denoted by $C_{(\mathfrak{q})}$: The multiplication by an element $\ell\in C_1\setminus\mathfrak{q}$ induces an isomorphism $(C_{\mathfrak{q}})_\mu\simeq(C_{\mathfrak{q}})_{\mu+1}$ for any μ. Hence, for any C-module M, $M\otimes_C C_{\mathfrak{q}}=0$ if and only if $M\otimes_C C_{(\mathfrak{q})}=0$.

In the equal degree case, the following corollary, which we state in a more geometric fashion, implies the conjecture of Hà [2011].

Theorem 5.3. *Let $A:=A_0[x_0,\dots,x_n]$ be a positively graded Noetherian algebra and I be a graded A-ideal generated by $m+1$ forms of degree d. Set $Y:=\operatorname{Spec}(A_0)$ and $X:=\operatorname{Proj}(A/I)\subset\operatorname{Proj}(A)\subseteq\tilde{\mathbb{P}}^n_Y$. Let $\phi:\tilde{\mathbb{P}}^n_Y\setminus X\to\mathbb{P}^m_Y$ be the corresponding rational map, W be the closure of the image of ϕ, and*

$$\Gamma\subset\tilde{\mathbb{P}}^n_W\subseteq\tilde{\mathbb{P}}^n_{\mathbb{P}^m_Y}=\tilde{\mathbb{P}}^n_Y\times_Y\mathbb{P}^m_Y$$

be the closure of the graph of ϕ. Let $\pi:\Gamma\to W$ be the projection induced by the natural map $\tilde{\mathbb{P}}^n_{\mathbb{P}^m_Y}\to\mathbb{P}^m_Y$. Then

$$\lim_{t\to+\infty}(a^i(I^t)-dt)=a^i(\Gamma).$$

Proof. Choose $J:=I$ and $M:=A$ in Proposition 5.1. The equality

$$\lim_{t\to+\infty}(a^i(I^t)-dt)=a^i(\Gamma)$$

directly follows from the conclusion of Proposition 5.1 according the definition of $a^i(\Gamma)$ for $\Gamma\subset\tilde{\mathbb{P}}^n_W$ given in Definition 3.1. $\square$

6. Cohomology of stalks and cohomology of fibers

We will now compare the cohomology of stalks and of fibers of a projective morphism, in order to compare their Castelnuovo–Mumford regularities. It will need results on the support of Tor modules. These are likely part of folklore. However, we include a proof as we did not find a reference that properly fits our exact need.

Lemma 6.1. *Let $R \to S$ be a homomorphism of Noetherian rings, $\mathbb{M}$ be a finitely generated S-module and N be a finitely generated R-module.*

Then the S-modules $\mathrm{Tor}_q^R(\mathbb{M}, N)$ are finitely generated over S and

(i) $\mathrm{Supp}_S(\mathrm{Tor}_q^R(\mathbb{M}, N)) \subseteq \mathrm{Supp}_S(\mathbb{M} \otimes_R N)$ *for any q,*

(ii) *if further $(R, \mathfrak{m})$ is local, $S = R[X_1, \dots, X_n]$, with $\deg X_i > 0$ and $\mathbb{M}$ is a graded S-module, then $\mathrm{Supp}_S(\mathrm{Tor}_q^R(\mathbb{M}, R/\mathfrak{m})) \subseteq \mathrm{Supp}_S(\mathrm{Tor}_1^R(\mathbb{M}, R/\mathfrak{m}))$ for any $q \geq 1$.*

Proof. First the modules $\mathrm{Tor}_q^R(\mathbb{M}, N)$ are finitely generated over S by [Bourbaki 1980, X §6 N°4 Corollaire]. Second,

$$\mathrm{Supp}_S(\mathbb{M} \otimes_R N) = \mathrm{Supp}_S(\mathbb{M}) \cap \varphi^{-1}(\mathrm{Supp}_R(N)),$$

where $\varphi : \mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is the natural map induced by $R \to S$, by [Bourbaki 1985, II §4 N°4, Propositions 18 and 19], since $\mathbb{M} \otimes_R N = \mathbb{M} \otimes_S (N \otimes_R S)$. For $\mathfrak{P} \in \mathrm{Spec}(S)$, set $\mathfrak{p} := \varphi(\mathfrak{P})$. Then $\mathrm{Tor}_q^R(\mathbb{M}, N)_{\mathfrak{P}} = \mathrm{Tor}_q^{R_{\mathfrak{p}}}(\mathbb{M}_{\mathfrak{P}}, N_{\mathfrak{p}})$ vanishes if either $\mathbb{M}_{\mathfrak{P}} = 0$ or $N_{\mathfrak{p}} = 0$.

For (ii), we can reduce to the case of a local morphism by localizing S and $\mathbb{M}$ at $\mathfrak{m} + S_+$. In this local situation, $\mathrm{Tor}_1^R(\mathbb{M}, R/\mathfrak{m}) = 0$ if and only if $\mathbb{M}$ is A-flat by [André 1974, Lemme 58], which proves our claim by localization at primes $\mathfrak{P}$ such that $\varphi(\mathfrak{P}) = \mathfrak{m}$. $\square$

Let R be a commutative ring, N be a R-module, $S := R[X_1, \dots, X_n]$ be a positively graded polynomial ring over R and $\mathbb{M}$ be a graded S-module. For a S-module $\mathbb{M}$, we will denote by $\mathrm{cd}_{S_+}(\mathbb{M})$ the cohomological dimension of $\mathbb{M}$ with respect to S_+, which is the maximal index i such that $H^i_{S_+}(\mathbb{M}) \neq 0$ (and $-\infty$ if all these local cohomology groups are 0). The following lemma is a natural way for comparing cohomology of stalks to cohomology of fibers.

Lemma 6.2. *There are two converging spectral sequences of graded S-modules with the same abutment $H^\bullet$ and with respective second terms*

$${}'_2E^p_q = H^p_{S_+}(\mathrm{Tor}_q^R(\mathbb{M}, N)) \Rightarrow H^{p-q} \quad \textit{and} \quad {}''_2E^p_q = \mathrm{Tor}_q^R(H^p_{S_+}(\mathbb{M}), N) \Rightarrow H^{p-q}.$$

Let $d := \max\{i \mid H^i_{S_+}(\mathbb{M} \otimes_R N) \neq 0\}$. *If* R *is Noetherian*, N *is finitely generated over* R *and* $\mathbb{M}$ *is finitely generated over* S, *then*

$$H^d_{S_+}(\mathbb{M} \otimes_R N) \simeq H^d_{S_+}(\mathbb{M}) \otimes_R N$$

and $\mathrm{Tor}^R_q(H^i_{S_+}(\mathbb{M}), N) = H^i_{S_+}(\mathrm{Tor}^R_q(\mathbb{M}, N)) = 0$ *for any* q *if* $i > d$.

Proof. Let $F_\bullet$ be a free R-resolution of N. Consider the double complex

$$\mathscr{C}^\bullet_{S_+}(\mathbb{M} \otimes_R F_\bullet) = \mathscr{C}^\bullet_{S_+}(\mathbb{M}) \otimes_R F_\bullet,$$

totalizing to $T^\bullet$ with $T^i = \bigoplus_{p-q=i} \mathscr{C}^p_{S_+}(\mathbb{M}) \otimes_R F_q$. It gives rise to two spectral sequences abutting to the homology $H^\bullet$ of $T^\bullet$.

One has first terms $\mathscr{C}^p_{S_+}(\mathrm{Tor}^R_q(\mathbb{M}, N))$ and second terms $H^p_{S_+}(\mathrm{Tor}^R_q(\mathbb{M}, N))$.

The other spectral sequence has first terms $H^p_{S_+}(\mathbb{M}) \otimes_R F_q$ and second terms $\mathrm{Tor}^R_q(H^p_{S_+}(\mathbb{M}), N)$. It provides the quoted spectral sequences.

Recall that if P is a finitely presented S-module, one has $\mathrm{cd}_{S_+}(P') \leq \mathrm{cd}_{S_+}(P)$ whenever $\mathrm{Supp}(P') \subseteq \mathrm{Supp}(P)$. This is proved in [Divaani-Aazar et al. 2002, 2.2] under the assumption that S is Noetherian and P' is finitely generated, which is enough for our purpose.

By Lemma 6.1(i), $\mathrm{Supp}(\mathrm{Tor}^R_q(\mathbb{M}, N)) \subseteq \mathrm{Supp}(\mathbb{M} \otimes_R N)$ for any q, which implies that $H^i_{S_+}(\mathrm{Tor}^R_q(\mathbb{M}, N)) = 0$ for any q if $i > d$. It follows that $H^d = H^d_{S_+}(\mathbb{M} \otimes_R N)$ and $H^i = 0$ for $i > d$.

On the other hand, choose i maximal such that $H^i_{S_+}(\mathbb{M}) \otimes_R N \neq 0$. Then $\mathrm{Tor}^R_q(H^p_{S_+}(\mathbb{M}), N) = 0$ for any q if $p > i$, because $H^p_{S_+}(\mathbb{M})_\mu$ is a finitely generated R-module for every μ, and hence $H^i = H^i_{S_+}(\mathbb{M}) \otimes_R N \neq 0$ and $H^j = 0$ for $j > i$. The conclusion follows. □

The following statement extends a classical result on the cohomology of fibers in a flat family; see for instance [Hartshorne 1977, III 9.3]. The hypothesis on the cohomological dimension of Tor modules that appears in (ii) will be connected to the variation of the Hilbert polynomial of fibers in the corresponding family of sheaves in Lemma 6.6; it is a weakening of the flatness condition for this family.

Proposition 6.3. *Let* $(R, \mathfrak{m}, k)$ *be a Noetherian local ring*, $S := R[X_1, \ldots, X_n]$ *be a polynomial ring over* R, *with* $\deg X_i > 0$ *for all* i, *and* $\mathbb{M}$ *be a finitely generated graded* S*-module. Set* $\mathrm{M} := \mathbb{M} \otimes_R k$ *and* $d := \dim \mathrm{M}$. *Then one has the following*:

(i) *The natural graded map* $H^d_{S_+}(\mathbb{M}) \otimes_R k \to H^d_{S_+}(\mathrm{M})$ *is an isomorphism and* $d = \max\{i \mid H^i_{S_+}(\mathbb{M}) \neq 0\}$. *In particular*,

$$a^d(\mathbb{M}) = a^d(\mathrm{M}) \in \mathbb{Z}.$$

(ii) *For any integers* μ *and* ℓ, *if* $\mathrm{cd}_{S_+}(\mathrm{Tor}^R_1(\mathbb{M}, k)) \leq \ell + 1$ *then*

$$\{H^i_{S_+}(\mathbb{M})_\mu = 0 \ \textit{for all } i \geq \ell\} \quad \textit{implies} \quad \{H^i_{S_+}(\mathrm{M})_\mu = 0 \ \textit{for all } i \geq \ell\},$$

and both conditions are equivalent if $\mathrm{cd}_{S_+}(\mathrm{Tor}_1^R(\mathbb{M}, k)) \leq \ell$. *In particular*, $\mathrm{reg}(\mathrm{M}) \leq \mathrm{reg}(\mathbb{M})$ *if* $\mathrm{cd}_{S_+}(\mathrm{Tor}_1^R(\mathbb{M}, k)) \leq 1$ *and equality holds if* $\mathrm{depth}_{S_+}(\mathbb{M}) > 0$.

Proof. We consider the two spectral sequences in Lemma 6.2,

$${}_2'E_q^p = H_{S_+}^p(\mathrm{Tor}_q^R(\mathbb{M}, k)) \Rightarrow H^{p-q} \quad \text{and} \quad {}_2''E_q^p = \mathrm{Tor}_q^R(H_{S_+}^p(\mathbb{M}), k) \Rightarrow H^{p-q}.$$

Let $B := k[X_1, \dots, X_n]$. The module $\mathrm{Tor}_q^R(\mathbb{M}, k)$ is a $R[X_1, \dots, X_n]$-module of finite type, annihilated by $\mathfrak{m}$ and $\mathrm{ann}_S(\mathbb{M})$. Hence M is a graded B-module of finite type and $\mathrm{Tor}_q^R(\mathbb{M}, k)$ is a graded $(B/\,\mathrm{ann}_B(\mathrm{M}))$-module of finite type, for any q.

Notice that $d = \mathrm{cd}_{S_+}(\mathrm{M}) = \mathrm{cd}_{B_+}(\mathrm{M})$. It follows that ${}_2'E_q^p = 0$ if $p > d$, and ${}_2'E_0^d \neq 0$.

By Lemma 6.2, ${}_2''E_q^p = 0$ for all q if $p > d$, in particular $H_{S_+}^p(\mathbb{M})_\mu \otimes_R k = 0$ for any μ if $p > d$. Hence $H_{S_+}^p(\mathbb{M})_\mu = 0$ for any μ if $p > d$. In other words, $H_{S_+}^p(\mathbb{M}) = 0$ for any $p > d$.

The same lemma shows that $H_{S_+}^d(\mathrm{M}) = H_{S_+}^d(\mathbb{M}) \otimes_R k$, and finishes the proof of (i).

For (ii), let μ be an integer. We prove the result by descending induction on ℓ from the case $\ell = d$, which we already proved.

Assume the results hold for $\ell + 1$. Recall that, for any p, the maps

$${}_r'd_{1-r}^{p-r} : {}_r'E_{1-r}^{p-r} \to {}_r'E_0^p \quad \text{and} \quad {}_r''d_0^p : {}_r''E_0^p \to {}_r''E_{-r}^{p+1-r}$$

are the zero map for $r \geq 2$ and $r \geq 1$, respectively.

If $H_{S_+}^i(\mathbb{M})_\mu = 0$, for all $i \geq \ell$, then $({}_2''E_q^p)_\mu = 0$ for $p \geq \ell$ and all q. As ${}_2''E_q^p = 0$ for $q < 0$, it follows that $({}_2''E_q^p)_\mu = 0$ if $p - q \geq \ell$.

If $\mathrm{cd}_{S_+}(\mathrm{Tor}_1^R(\mathbb{M}, k)) \leq \ell + 1$ then ${}_2'E_q^p = 0$ for $p \geq \ell + 2$ and $q > 0$ by Lemma 6.1(ii), in particular the map

$$({}_r'd_0^\ell)_\mu : ({}_r'E_0^\ell)_\mu \to ({}_r'E_{r-1}^{\ell+r})_\mu$$

is the zero map for any $r \geq 2$, and hence $H_{S_+}^\ell(\mathrm{M})_\mu = ({}_2'E_0^\ell)_\mu = ({}_\infty'E_0^\ell)_\mu = 0$ as claimed.

For the reverse implication, the hypothesis implies that ${}_2'E_q^p = 0$ if $q \geq 1$ and $p \geq \ell + 1$ by Lemma 6.1(ii). Hence $({}_2'E_q^p)_\mu = 0$ for $p - q \geq \ell$ if $H_{S_+}^\ell(\mathrm{M})_\mu = 0$. By induction hypothesis, $H_{S_+}^p(\mathbb{M})_\mu \otimes_R k = 0$ for $p \geq \ell + 1$. Hence

$$({}_2'E_q^p)_\mu = \mathrm{Tor}_q^R(H_{S_+}^p(\mathbb{M})_\mu, k) = 0$$

for $p \geq \ell + 1$ and all q. It implies that $H_{S_+}^\ell(\mathbb{M})_\mu \otimes_R k = ({}_\infty''E_0^\ell)_\mu = 0$, and proves the claimed equivalence.

Finally, recall that $H_{S_+}^i(\mathbb{M}) = 0$ for $i < \mathrm{depth}_{S_+}(\mathbb{M})$. $\square$

Remark 6.4. Notice that $\mathrm{reg}(\mathrm{M}) \leq \mathrm{reg}(\mathbb{M})$ does not hold without the hypothesis $\mathrm{cd}_{S_+}(\mathrm{Tor}_1^R(\mathbb{M}, k)) \leq 1$. To see this, consider generic polynomials of some given degrees $d_1, \dots, d_r$:

$$P_i := \sum_{|\alpha|=d_i} U_{i,\alpha} X^\alpha \in k[U_{i,\alpha}][X_1, \dots, X_n],$$

with $r \leq n$ and a specialization map $\phi : k[U_{i,\alpha}] \to k$ to the field k with kernel $\mathfrak{m}$. Set $R := k[U_{i,\alpha}]_{\mathfrak{m}}$. As the P_i form a regular sequence in $k[U_{i,\alpha}][X_1, \dots, X_n]$, they also form one in $S := R[X_1, \dots, X_n]$ and show that $\mathbb{M} := S/(P_1, \dots, P_r)$ has regularity $d_1 + \cdots + d_r - r$. On the other hand, the regularity of

$$M = k[X_1, \dots, X_n]/(\phi(P_1), \dots, \phi(P_r)),$$

need not be bounded by $d_1 + \cdots + d_r - r$.

For instance, with $n = 4$ and $r = 3$, take

$$\phi(P_1) := X_1^{d-1} X_2 - X_3^{d-1} X_4, \quad \phi(P_2) := X_2^d \quad \text{and} \quad \phi(P_3) := X_4^d$$

(over any field). Then one has $\operatorname{reg}(M) = d^2 - 2$ for $d \geq 3$ (see [Chardin 2007, 1.13.6]), which is bigger than $\operatorname{reg}(\mathbb{M}) = 3d - 3$, and $\operatorname{cd}_{S_+}(\operatorname{Tor}_1^R(\mathbb{M}, k)) = 2$.

Remark 6.5. In the other direction, it may of course be that $\operatorname{reg}(\mathbb{M}) > \operatorname{reg}(M)$. If for instance (R, π, k) is a DVR, one may take $\mathbb{M} := R[X]/(\pi X^d)$, so that $\operatorname{reg}(\mathbb{M}) = d - 1$ and $\operatorname{reg}(M) = 0$, with $\operatorname{cd}_{S_+}(\operatorname{Tor}_1^R(\mathbb{M}, k)) = 1$.

More interesting is the example $R := \mathbb{Q}[a, b]$, $\mathfrak{m} := (a, b)$ and

$$\mathbb{M} := \operatorname{Sym}_R(\mathfrak{m}^3) = R[X_1, \dots, X_4]/(bX_1 - aX_2, bX_2 - aX_3, bX_3 - aX_4).$$

Then for any morphism from R to a field k, $\operatorname{reg}(\mathbb{M} \otimes_R k) = 0$, while $\operatorname{reg}(\mathbb{M}) = 1$.

Similar examples arises from the symmetric algebra of other ideals that are not generated by a proper sequence.

The characterization of flatness in terms of the constancy of the Hilbert polynomial of fibers extends as follows.

Lemma 6.6. *Let p be an integer. In the setting of Proposition 6.3, assume that R is reduced and S is standard graded. Then the following are equivalent*:

(i) $\dim(\operatorname{Tor}_1^R(\mathbb{M}, k)) \leq p$.

(ii) *The Hilbert polynomials of* $\mathbb{M} \otimes_R k$ *and* $\mathbb{M} \otimes_R (R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p})$ *differ at most by a polynomial of degree* $< p$, *for any* $\mathfrak{p} \in \operatorname{Spec}(R)$.

Proof. We induct on p. The result is standard when $p = 0$; see for instance [Hartshorne 1977, III 9.9; Eisenbud 1995, Exercise 20.14].

Assume (i) and (ii) are equivalent for $p - 1 \geq 0$, for any Noetherian local reduced ring, standard graded polynomial ring over it and graded module of finite type.

Set $K := R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$, $M_K := \mathbb{M} \otimes_R K$, $B := k[X_1, \dots, X_n]$ and $C := K[X_1, \dots, X_n]$. Consider variables $U_1, \dots, U_n$ (of degree 0) and let $\ell := U_1 X_1 + \cdots + U_n X_n$. By the Dedekind–Mertens lemma,

(a) $\ker(\mathbb{M}[U] \xrightarrow{\times\ell} \mathbb{M}[U](1)) \subseteq H^0_{S_+}(\mathbb{M})[U]$,

(b) $\ker(M[U] \xrightarrow{\times\ell} M[U](1)) \subseteq H^0_{B_+}(M)[U]$,

(c) $\ker(M_K[U] \xrightarrow{\times\ell} M_K[U](1)) \subseteq H^0_{C_+}(M_K)[U]$, and

(d) $\ker(\mathrm{Tor}_1^R(\mathbb{M}, k)[U] \xrightarrow{\times\ell} \mathrm{Tor}_1^R(\mathbb{M}, k)[U](1)) \subseteq H^0_{B_+}(\mathrm{Tor}_1^R(\mathbb{M}, k))[U]$.

Let $R' := R(U)$ be obtained from $R[U]$ by inverting all polynomials whose coefficient ideal is the unit ideal, and denote by N' the extension of scalars from R to R' for the module N. Recall that $R(U)$ is local reduced with maximal ideal $\mathfrak{m}R(U)$, residue field $k' = k(U)$ and that $K' = K(U)$ — see for instance [Nagata 1962, page 17]. As the zero local cohomology modules above vanish in high degrees, (b) and (c) show that $\mathbb{M}'/\ell\mathbb{M}'$ satisfies condition (ii) of the lemma for $p-1$, R' and $R'[X_1, \ldots, X_n]$. Now (a) and (d) provide an exact sequence for $\mu \gg 0$:

$$0 \longrightarrow \mathrm{Tor}_1^R(\mathbb{M}', k')_{\mu-1} \xrightarrow{\times\ell} \mathrm{Tor}_1^{R'}(\mathbb{M}', k')_\mu \longrightarrow \mathrm{Tor}_1^{R'}(\mathbb{M}'/\ell\mathbb{M}', k')_\mu \longrightarrow 0,$$

which shows in particular that

$$\dim \mathrm{Tor}_1^{R'}(\mathbb{M}'/\ell\mathbb{M}', k') = \dim \mathrm{Tor}_1^{R'}(\mathbb{M}', k') - 1 = \dim \mathrm{Tor}_1^R(\mathbb{M}, k) - 1,$$

if $\dim \mathrm{Tor}_1^R(\mathbb{M}, k)$ is positive, and proves our claim by induction. □

Remark 6.7. If the grading is not standard, a quasipolynomial is attached to any finitely generated graded module, and in Lemma 6.6 property (ii) should be replaced by the following:

(ii) The difference between the quasipolynomials of $\mathbb{M} \otimes_R k$ and $\mathbb{M} \otimes_R (R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p})$ is a quasipolynomial of degree $< p$ for any $\mathfrak{p} \in \mathrm{Spec}(R)$.

The degree of a quasipolynomial is the highest degree of the polynomials that define it. The proof of [Hartshorne 1977, III 9.9] extends to this case when $p = 0$, and our proof extends after a slight modification: in the proof that (ii) implies (i), one should take $\ell := U_1 X_1^{w/w_1} + \cdots + U_n X_n^{w/w_n}$, where $w_i := \deg(X_i)$ and $w := \mathrm{lcm}(w_1, \ldots, w_n)$.

The local statement of Lemma 6.6 implies a global statement, by comparing Hilbert functions at generic points of the components and at closed points. We state it below in a ring theoretic form.

Proposition 6.8. *Let p be an integer, R be a reduced commutative ring, S be a Noetherian positively graded polynomial ring over R and $\mathbb{M}$ be a finitely generated graded S-module. Then the following are equivalent*:

(i) $H^i_{S_+}(\mathrm{Tor}_1^R(\mathbb{M}, R/\mathfrak{m})) = 0$ *for all $i > p$ and $\mathfrak{m}$ maximal in* $\mathrm{Spec}(R)$.

(ii) *For any two ideals* $\mathfrak{p} \subset \mathfrak{q}$ *in* $\operatorname{Spec}(R)$, *the quasipolynomials of* $\mathbb{M} \otimes_R R/\mathfrak{p}$ *and* $\mathbb{M} \otimes_R R/\mathfrak{q}$ *differ by a quasipolynomial of degree* $< p$.

(iii) *Over a connected component of* $\operatorname{Spec}(R)$, *the quasipolynomials of two fibers differ by a quasipolynomial of degree* $< p$.

In parallel to the definition of the regularity over a scheme, we define the fiber-regularity $\mathfrak{freg}$ as the maximum over the fibers of their regularity.

Definition 6.9. In the setting of Definition 3.1,

$$\tilde{a}_i(\mathcal{F}) := \sup_{y\in Y} a^i(\mathcal{F} \otimes_{\mathbb{O}_Y} k(y)), \qquad \mathfrak{freg}(\mathcal{F}) := \max_i\{\tilde{a}_i(\mathcal{F}) + i\},$$

and $\mathfrak{freg}(Z) := \max_{i\geq 1}\{\tilde{a}_i(\operatorname{Sym}_Y(\mathscr{E})/\mathscr{I}_Z) + i\}$.

Notice that $\mathfrak{freg}(\mathcal{F})$ is finite if Y is covered by finitely many affine charts and $\mathcal{F}$ is coherent. This holds since the regularity of a graded module over a polynomial ring over a field is bounded in terms of the number of generators and the degrees of generators and relations; see for instance [Chardin et al. 2008, 3.5].

We now return to the problem of studying the ending degree of local cohomologies of powers of a graded ideal I in a positively graded Noetherian algebra A.

From the comparison of cohomology of stalks and cohomology of fibers, we get from Theorem 5.3 the following result. As in Theorem 5.3 we use geometric language and do not introduce a graded module (or a sheaf) to make the exposition more simple. In case a more general statement is needed, it can be easily derived by using Proposition 5.1 in place of Theorem 5.3. The six statements are not independent, but each of them answers a question that is quite natural to ask. Notice that (iv) is essentially equivalent to one of the main results of Eisenbud and Harris [2010, 2.2].

Remark 6.10. It follows from Theorem 5.3 that the dimension of any fiber of the projection π of the graph to its image (see Theorem 5.3 or below for the precise definition of π) is bounded above by the cohomological dimension of A/I with respect to A_+.

Theorem 6.11. *Let* $A := A_0[x_0, \dots, x_n]$ *be a positively graded Noetherian algebra and* I *be a graded* A*-ideal generated by* $m+1$ *forms of degree* d. *Set* $Y := \operatorname{Spec}(A_0)$ *and* $X := \operatorname{Proj}(A/I) \subset \operatorname{Proj}(A) \subseteq \tilde{\mathbb{P}}^n_Y$. *Let* $\phi : \tilde{\mathbb{P}}^n_Y \setminus X \to \mathbb{P}^m_Y$ *be the corresponding rational map*, W *be the closure of the image of* ϕ, *and*

$$\Gamma \subset \tilde{\mathbb{P}}^n_W \subseteq \tilde{\mathbb{P}}^n_{\mathbb{P}^m_Y} = \tilde{\mathbb{P}}^n_Y \times_Y \mathbb{P}^m_Y$$

be the closure of the graph of ϕ. *Let* $\pi : \Gamma \to W$ *be the projection induced by the natural map* $\tilde{\mathbb{P}}^n_{\mathbb{P}^m_Y} \to \mathbb{P}^m_Y$. *Then we have the following*:

(i) $\lim_{t\to+\infty}(\operatorname{reg}((I^t)^{\mathrm{sat}}) - dt) = \max_{i\geq 2}\{a^i(\Gamma) + i\}$.

(ii) *If π admits a fiber $Z \subseteq \tilde{\mathbb{P}}^n_{\operatorname{Spec}\mathfrak{K}}$ of dimension $i-1$, then*

$$\lim_{t\to\infty}(a^i(I^t)+i-td) \geq a^i(Z)+i = \tilde{a}^i(Z)+i \geq 0.$$

(iii) *Let δ be the maximal dimension of a fiber of π. Then,*

$$a^{\delta+1}(I^t)-td = a^{\delta+1}(\Gamma) = \tilde{a}^{\delta+1}(\Gamma) \quad \textit{for all } t \gg 0.$$

(iv) *If π is finite, for instance if $X=\varnothing$, then*

$$\operatorname{reg}(I^t) = a^1(I^t)+1 = \mathfrak{f}\mathrm{reg}(\Gamma)+td \quad \textit{for all } t \gg 0$$

and $\lim_{t\to\infty}(a^i(I^t)-td) = -\infty$ *for* $i \neq 1$.

(v) *If π has fibers of dimension at most one, for instance if the canonical map $X \to Y$ is finite, then*

$$\operatorname{reg}(I^t)-td = \operatorname{reg}(\Gamma) \geq \mathfrak{f}\mathrm{reg}(\Gamma) \quad \textit{for all } t \gg 0,$$

and $\lim_{t\to\infty}(a^i(I^t)-td) = -\infty$ *for* $i \geq 2$.

If furthermore A is standard graded and reduced, π has fibers of dimension one, all of same degree, then $\mathfrak{f}\mathrm{reg}(\Gamma) = \operatorname{reg}(\Gamma)$,

$$\lim_{t\to\infty}(a^1(I^t)-td) \geq \tilde{a}^1(\Gamma)$$

and equality holds if $\operatorname{reg}(I^t) = a^1(I^t)+1$ *for* $t \gg 0$.

(vi) *If A is reduced and, for every connected component T of W, the Hilbert quasipolynomials of fibers of π over any two points in* $\operatorname{Spec}(T)$ *differ by a periodic function, then*

$$\operatorname{reg}(I^t) = \mathfrak{f}\mathrm{reg}(\Gamma)+td \quad \textit{for all } \mu \gg 0.$$

Proof. Part (i) is a direct corollary of Theorem 5.3. Statements (ii), (iii) and (iv) follow from Theorem 5.3 and Proposition 6.3(i).

Statements (v) and (vi) follow from Theorem 5.3, Proposition 6.3(ii) — notice that $\operatorname{depth}_{S_+}(\mathcal{R}_I) \geq 1$ — and the equivalence of (i) and (iii) in Proposition 6.8 applied on the affine charts covering $\pi(\Gamma)$. □

Remark 6.12. Cutkosky, Ein and Lazarsfeld proved in [Cutkosky et al. 2001] that the limit $s(I) := \lim_{t\to\infty}\operatorname{reg}((I^t)^{\mathrm{sat}})/t$ exists and is equal to the inverse of a Seshadri constant, when A_0 is a field and A is standard graded.

Using the existence of c such that $\operatorname{reg}(MI^t) \leq dt+c$ for all t when I is generated in degree at most d and M is finitely generated, one can easily derive the existence of this limit in our more general setting. Indeed, let

$$r_p := \operatorname{reg}((I^p)^{\mathrm{sat}}) \quad \text{and} \quad d_p := \min\{\mu \mid (I^p)^{\mathrm{sat}} = ((I^p)^{\mathrm{sat}}_{\leq\mu})^{\mathrm{sat}}\}.$$

One has $d_{p+q} \leq d_p + d_q$; hence $s := \lim_{p\to\infty}(d_p/p)$ exists. For any p there exists c_p such that

$$\operatorname{reg}(((I^p)^{\mathrm{sat}}_{\leq d_p})^t I^q) \leq td_p + c_p \quad \text{for all } t \geq 1 \text{ and } 0 \leq q < p.$$

The inequalities $d_{pt+q} \leq r_{pt+q} \leq td_p + c_p$ show that $\lim_{p\to\infty}(r_p/p) = s$ and that $d_p \geq ps$ for all p.

The same argument applies to any graded ideal J such that $\operatorname{Proj}(A/J) \to Y$ is finite (that is, $\operatorname{cd}_{A_+}(A/J) \leq 1$). Setting $r_p^J := \operatorname{reg}(I^p :_A J^\infty) \leq \operatorname{reg}(I^p)$ and defining d_p^J similarly to the above,

$$d_p^J := \min\{\mu \mid ((I^p : J^\infty)_{\leq\mu}) : J^\infty = I^p : J^\infty\},$$

the limits of r_p^J/p and d_p^J/p exist and are equal. For example, if X is a scheme with isolated nonlocally complete intersection points, then $\lim_{p\to\infty} \operatorname{reg}(I^{(p)}/p)$ exists, where $I^{(p)}$ denotes the p-th symbolic power of I.

On the other hand, when A/J has cohomological dimension 2 it may be that $\operatorname{reg}(I : J^\infty) > \operatorname{reg}(I)$ for J an embedded prime of I. This shows that the argument above is not directly applicable for symbolic powers in general. It however implies that $s^J := \lim_{p\to\infty}(d_p^J/p)$ exists for any J and is equal to $\lim_{p\to\infty}(\rho_p^J/p)$, where

$$\rho_p^J := \min\{\operatorname{reg}(K) \mid K \subseteq (I^p : J^\infty),\ K : J^\infty = I^p : J^\infty\}.$$

Remark 6.13. If I is generated in degree at most d, Theorem 6.11 implies that $s(I) < d$ if and only if the morphism π corresponding to the ideal (I_d) is finite. More precisely, by Remark 6.12, π is finite if and only if $\operatorname{Proj}(A/I^t)$ is defined by equations of degree $< dt$ for some t, and if not, $\operatorname{reg}((I^t)^{\mathrm{sat}}) - td$ is a nonnegative constant for $t \gg 0$.

This has been remarked in [Niu 2013], using the definition of $s(I)$ as (the inverse of) a Seshadri constant.

Theorem 6.11 also has a consequence on the dimension of the fibers. Assume for simplicity that A_0 is a field. Set $X := \operatorname{Proj}(A/I)$, with I generated in degree at most d and let $0 \leq i \leq \dim X$.

Part (ii) in Theorem 6.11 then shows that the morphism π associated to (I_d) has no fiber of dimension greater than i if there exists $p \geq 1$ and an ideal K, generated in degree less than pd, such that $\operatorname{Proj}(A/I^p)$ and $\operatorname{Proj}(A/K)$ coincide locally at each point $x \in \mathbb{P}^n$ of dimension at least i. Indeed if this happens, then

$$H^j_{A_+}(A/I^{ps}) \simeq H^j_{A_+}(A/K^s) \quad \text{for all } j > i,\ s \geq 1,$$

and therefore there exists c_p such that $a^j(I^{ps}) \leq (pd-1)s + c_p$ for all s and $j \geq i$, showing that $\lim_{t\to\infty}(a^j(I^t) - td) = -\infty$ for $j \geq i$.

Acknowledgments

This work was inspired by results of Huy Tài Hà [2011] and of David Eisenbud and Joe Harris [2010]. Bernd Ulrich made remarks on a very early version of some of these results and motivated my study of the difference between the regularity of stalks and the regularity of fibers, and Joseph Oesterlé provided references concerning Lemma 6.1. It is my pleasure to thank them for their contribution.

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Communicated by Craig Huneke
Received 2010-06-22 Revised 2012-01-10 Accepted 2012-02-07

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Algebra & Number Theory (ISSN 1944-7833 electronic, 1937-0652 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840 is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

ANT peer review and production are managed by EditFLOW® from Mathematical Sciences Publishers.

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Algebra & Number Theory

Volume 7 No. 1 2013

1937-0652(2013)7:1;1-C