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On Kato's local ϵ-isomorphism conjecture for rank-one Iwasawa modules

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This paper contains a complete proof of Fukaya and Kato's ϵ-isomorphism conjecture for invertible Λ-modules (the case of $V = V_0(r)$, where V_0 is unramified of dimension 1). Our results rely heavily on Kato's proof, in an unpublished set of lecture notes, of (commutative) ϵ-isomorphisms for one-dimensional representations of $G_{\mathbb{Q}_p}$, but apart from fixing some sign ambiguities in Kato's notes, we use the theory of (ϕ, Γ)-modules instead of syntomic cohomology. Also, for the convenience of the reader we give a slight modification or rather reformulation of it in the language of Fukuya and Kato and extend it to the (slightly noncommutative) semiglobal setting. Finally we discuss some direct applications concerning the Iwasawa theory of CM elliptic curves, in particular the local Iwasawa Main Conjecture for CM elliptic curves E over the extension of $\mathbb{Q}_p$ which trivialises the p-power division points $E(p)$ of E. In this sense the paper is complimentary to our work with Bouganis (*Asian J. Math.* **14**:3 (2010), 385–416) on noncommutative Main Conjectures for CM elliptic curves.

1. Introduction

The significance of local ϵ-factors à la Deligne and Tate, or more generally that of the (conjectural) ϵ-isomorphism suggested in [Fukaya and Kato 2006, §3] is at least twofold. First, they are important ingredients to obtain a precise functional equation for L-functions or more generally for (conjectural) ζ-isomorphisms [loc. cit., §2] of motives in the context of equivariant or noncommutative Tamagawa number conjectures (see, e.g., Theorem 4.1). Secondly, they are essential in interpolation formulae of (actual) p-adic L-functions and for the relation between ζ-isomorphisms and (conjectural, not necessarily commutative) p-adic L-functions as discussed in [loc. cit., §4]. Of course the two occurrences are closely related; for a survey on these ideas see also [Venjakob 2007].

Our motivation for writing this article stems from Theorem 8.4 of [Burns and Venjakob 2011] (see Theorem 4.2), which describes under what conditions the

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validity of a (noncommutative) Iwasawa Main Conjecture for a critical (ordinary at p) motive M over some p-adic Lie extension F_∞ of $\mathbb{Q}$ implies parts of the equivariant Tamagawa number conjecture (ETNC) by Burns and Flach for M with respect to a finite Galois extension $F \subseteq F_\infty$ of $\mathbb{Q}$. Due to the second above mentioned meaning it requires among others the existence of an ϵ-isomorphism

$$\epsilon_{p,\mathbb{Z}_p[G(F/\mathbb{Q})]}(\hat{\mathbb{T}}_F) : \mathbf{1}_{\mathbb{Z}_p[G_{F/\mathbb{Q}}]} \to \mathbf{d}_{\mathbb{Z}_p[G_{F/\mathbb{Q}}]}(R\Gamma(\mathbb{Q}_p, \hat{\mathbb{T}}_F))\mathbf{d}_{\mathbb{Z}_p[G_{F/\mathbb{Q}}]}(\hat{\mathbb{T}}_F) \qquad (1)$$

in the sense of [Fukaya and Kato 2006, Conjecture 3.4.3], where the Iwasawa module $\hat{\mathbb{T}}_F$ is related to the ordinary condition of M; e.g., for an (ordinary) elliptic curve E it arises from the formal group part of the usual Tate module of E. Unfortunately, very little is known about the existence of such ϵ-isomorphisms in general. To the knowledge of the author it is not even contained in the literature for $\hat{\mathbb{T}}_F$ attached to a CM-elliptic curve E and the trivialising extension $F_\infty := F(E(p))$, where $E(p)$ denotes the group of p-power division points of E. In principle a rough sketch of a proof is contained in [Kato 1993b], which unfortunately has never been published. Moreover there were still some sign ambiguities which we fix in this paper; in particular, it turns out that one has to take $-\mathscr{L}_{K,\epsilon^{-1}}$, that is, -1 times the classical Coleman map (6), in the construction of the epsilon isomorphism (17).

Benois and Berger [2008] have proved the conjecture $C_{EP}(L/K, V)$ for arbitrary crystalline representations V of G_K, where K is an unramified extension of $\mathbb{Q}_p$ and L a finite subextension of $K_\infty = K(\mu(p))$ over K. Although they mention in their introduction that “Les mêmes arguments, avec un peu plus de calculs, permettent de démontrer la conjecture $C_{EP}(L/K, V)$ pour toute extension L/K contenue dans $\mathbb{Q}_p^{ab}$. Cette petite généralisation est importante pour la version équivariante des conjectures de Bloch et Kato”, they leave it as an “exercise” to the reader. In the special case $V = \mathbb{Q}_p(r)$, $r \in \mathbb{Z}$, Burns and Flach [2006] proved a local ETNC using global ingredients in a semilocal setting, while in the above example we need it for $V = \mathbb{Q}_p(\eta)(r)$, where η denotes an unramified character. Also we would like to stress that the existence of the ϵ-isomorphisms à la Fukaya and Kato is a slightly finer statement than the $C_{EP}(L/K, V)$-conjecture or the result of Burns and Flach, because the former one states that a certain family of certain precisely defined units of integral group algebras of finite groups in a certain tower can be interpolated by a unit in the corresponding Iwasawa algebra while in the latter ones “only” a family of lattices is “interpolated” by one over the Iwasawa algebra.

The aim of this article, which also might hopefully serve as a survey into the subject, is to provide detailed and complete arguments for the existence of the ϵ-isomorphism

$$\epsilon_\Lambda(\mathbb{T}(T)) : \mathbf{1}_{\widetilde{\Lambda}} \to \mathbf{d}_\Lambda(R\Gamma(\mathbb{Q}_p, \mathbb{T}(T)))_{\widetilde{\Lambda}}\mathbf{d}_\Lambda(\mathbb{T}(T))_{\widetilde{\Lambda}},$$

where $\Lambda = \Lambda(G)$ is the Iwasawa algebra of $G = G(K_\infty/\mathbb{Q}_p)$ for any (possibly infinite) unramified extension K of $\mathbb{Q}_p$, $T = \mathbb{Z}_p(\eta)(r)$ and $R\Gamma(\mathbb{Q}_p, \mathbb{T}(T))$ denotes the complex calculating local Galois cohomology of $\mathbb{T}(T)$, the usual Iwasawa theoretic deformation of T (see (28)). Furthermore, for an associative ring R with one, $\mathbf{d}_R$ denotes the determinant functor with $\mathbf{1}_R = \mathbf{d}_R(0)$ (see Appendix B) while $\widetilde{\Lambda}$ is defined in (2). We are mainly interested in the case where $G \cong \mathbb{Z}_p^2 \times \Delta$ for a finite group Δ — such extensions arise for example by adjoining the p-power division points of a CM elliptic curve to the base field as above. This corresponds to a (generalised) conjecture $C_{IW}(K_\infty/\mathbb{Q}_p)$ (in the notation of Benois and Berger) originally due to Perrin-Riou. It is the first example of an ϵ-isomorphism associated with a two-dimensional p-adic Lie group extension. Following Kato's approach we construct a *universal* ϵ-isomorphism $\epsilon_\Lambda(\mathbb{T}(\mathbb{Z}_p(1)))$, from which all the others arise by suitable twists and descent. But while Kato constructs it first over cyclotomic $\mathbb{Z}_p$-extensions and then takes limits, here we construct it directly over $(\mathbb{Z}_p^2 \times \Delta)$-extensions (and then take limits). To show that they satisfy the right interpolation property with respect to Artin (Dirichlet) characters of G, we use the theory of (ϕ, Γ)-modules and the explicit formulae in [Berger 2003], instead of the much more involved syntomic cohomology and Kato's reciprocity laws for formal groups. In contrast to Kato's unpublished preprint, in which he uses the language of étale sheaves and cohomology, we prefer Galois cohomology as used also in [Fukaya and Kato 2006]. In order to work out in detail Kato's reduction argument [1993b] to the case of trivial η we have to show a certain twist compatibility of Perrin-Riou's exponential map/Coleman map for T versus $\mathbb{Z}_p(r)$ over a trivialising extension K_∞ for η, see Lemma A.4. Going over to semilocal settings we obtain the first ϵ-isomorphism over a (slightly) noncommutative ring. In a forthcoming paper [Loeffler et al. 2013], using the techniques of [Benois and Berger 2008] and [Loeffler and Zerbes 2011], we are going to extend these results to the case of arbitrary crystalline representations for the same tower of local fields as above. Of course it would be most desirable to extend the existence of ϵ-isomorphism also to nonabelian local extensions, but this seems to require completely new ideas and to be out of reach at present (see [Izychev 2012] for some examples). Some evidence in that direction has been provided by Fukaya (unpublished).

Combined with Yasuda's work [2009] concerning ϵ-isomorphisms for $l \neq p$, we also obtain in principle a purely local proof of the Burns–Flach result for $V = \mathbb{Q}_p(r)$.

2. Kato's proof for one-dimensional representations

Let p be a prime and let K be any unramified (possibly infinite) Galois extension of $\mathbb{Q}_p$. We set $K_n := K(\mu_{p^n})$ for $0 \leq n \leq \infty$ and

$$\Gamma = G(\mathbb{Q}_{p,\infty}/\mathbb{Q}_p) \cong \mathbb{Z}_p^\times.$$

Recall that the maximal unramified extension $\mathbb{Q}_p^{ur}$ and the maximal abelian extension $\mathbb{Q}_p^{ab}$ of $\mathbb{Q}_p$ are given as $\mathbb{Q}_p(\mu(p'))$ and $\mathbb{Q}_p(\mu) = \mathbb{Q}_p^{ur}(\mu(p))$, where $\mu(p)$ and $\mu(p')$ denote the p-primary and prime-to-p part of μ, the group of all roots of unity, respectively. In particular, we have the canonical decomposition

$$G(\mathbb{Q}_p^{ab}/\mathbb{Q}_p) = G(\mathbb{Q}_p^{ur}/\mathbb{Q}_p) \times G(\mathbb{Q}_{p,\infty}/\mathbb{Q}_p) = \hat{\mathbb{Z}} \times \mathbb{Z}_p^{\times},$$

under which by definition τ_p corresponds to $(\phi, 1)$ (and by abuse of notation also to its image in G below), where $\phi := \mathrm{Frob}_p$ denotes the arithmetic Frobenius $x \mapsto x^p$. We put

$$H := H_K := G(K/\mathbb{Q}_p) = \overline{\langle\phi\rangle}$$

and

$$G := G(K_\infty/\mathbb{Q}_p) \cong H \times \Gamma.$$

Assume that G is a p-adic Lie group, that is, H is the product of a finite abelian group of order prime to p with a (not necessarily strict) quotient of $\mathbb{Z}_p$. By

$$\Lambda := \Lambda(G) := \mathbb{Z}_p[\![G]\!]$$

we denote as usual the Iwasawa algebra of G. Also we write $\widehat{\mathbb{Z}_p^{ur}}$ for the ring of Witt vectors $W(\overline{\mathbb{F}_p})$ with its natural action by ϕ and we set

$$\widetilde{\Lambda} = \Lambda \widehat{\otimes}_{\mathbb{Z}_p} \widehat{\mathbb{Z}_p^{ur}} = \widehat{\mathbb{Z}_p^{ur}}[\![G]\!]. \tag{2}$$

By

$$\mathbb{T}_{un} := \Lambda^{\sharp}(1)$$

we denote the free Λ-module of rank one with the Galois action

$$\chi_{un} : G_{\mathbb{Q}_p} \to \Lambda^{\times}, \quad \sigma \mapsto [\mathbb{T}_{un}, \sigma] := \bar{\sigma}^{-1}\kappa(\sigma),$$

where $\bar{\ } : G_{\mathbb{Q}_p} \twoheadrightarrow G$ is the natural projection map and $\kappa : G_{\mathbb{Q}_p} \twoheadrightarrow \mathbb{Z}_p^{\times}$ is the p-cyclotomic character. Furthermore, we write

$$\mathbb{U}(K_\infty) := \varprojlim_{L,i} \mathcal{O}_L^{\times}/p^i$$

for the Λ-module of local units, where L and i run through the finite subextensions of $K_\infty/\mathbb{Q}_p$ and the natural numbers, respectively, and the transition maps are induced by the norm. Finally we fix once and for all a $\mathbb{Z}_p$-basis $\epsilon = (\epsilon_n)_n$ of $\mathbb{Z}_p(1) = \varprojlim_n \mu_{p^n}$.

We set

$$\Lambda_a = \{x \in \widetilde{\Lambda} \mid (1 \otimes \phi)(x) = (a \otimes 1) \cdot x\} \quad \text{for } a \in \Lambda^{\times} = K_1(\Lambda).$$

Proposition 2.1. *For $a = [\mathbb{T}_{un}, \tau_p]^{-1} = \tau_p$ there is a canonical isomorphism*

$$\Lambda_a \cong \begin{cases} \mathcal{O}_K[\![\Gamma]\!] & \text{if } H \text{ is finite,} \\ \varprojlim_{\mathbb{Q}_p \subseteq K' \subseteq K \textit{finite}, \mathrm{Tr}} \mathcal{O}_{K'}[\![\Gamma]\!] & \text{if } H \text{ is infinite,} \end{cases}$$

as Λ-modules. All modules are free of rank one.

Proof. We first assume $H = \langle \tau_p \rangle$ to be finite of order d and replace Γ by a finite quotient without changing the notation. Then any element $x \in \widetilde{\Lambda} = \widehat{\mathbb{Z}_p^{ur}}[\Gamma][H]$ can be uniquely written as $\sum_{i=0}^{d-1} a_i \tau_p^i$ with $a_i \in \widehat{\mathbb{Z}_p^{ur}}[\Gamma]$ and ϕ acts coefficientwise on the latter elements. The calculation

$$\begin{aligned}(1 \otimes \phi)(x) - (\tau_p \otimes 1)x &= \sum_{i=0}^{d-1} \phi(a_i)\tau_p^i - \sum_{i=0}^{d-1} a_i \tau_p^{i+1} \\ &= \sum_{i=0}^{d-1} \big(\phi(a_i) - a_{i-1}\big)\tau_p^i\end{aligned}$$

with $a_{-1} := a_{d-1}$ shows that x belongs to Λ_a if and only if $\phi^d(a_i) = a_i$ and $\phi^{-i}(a_0) = a_i$ for all i. As $\widehat{\mathbb{Z}_p^{ur}}^{\phi^d=1} = \mathcal{O}_K$, the canonical map

$$\Lambda_a \cong \mathcal{O}_K[\Gamma], \quad \sum a_i \tau_p^i \mapsto a_0,$$

is an isomorphism of Λ-modules, the inverse of which is

$$x \mapsto \sum_{h \in H} h \otimes h^{-1}(x)$$

and which is obviously functorial in Γ, whence the same result follows for the original (infinite) Γ.

Now, for a surjection $\pi : H'' \twoheadrightarrow H'$ it is easy to check that the trace $\mathrm{Tr}_{K''/K'} : \mathcal{O}_{K''} \to \mathcal{O}_{K'}$ induces a commutative diagram

$$\begin{tikzcd} \Lambda''_{a''} \arrow[r, "\cong"] \arrow[d, "\pi"'] & \mathcal{O}_{K''}[\![\Gamma]\!] \arrow[d, "\mathrm{Tr}_{K''/K'}"] \\ \Lambda'_{a'} \arrow[r, "\cong"] & \mathcal{O}_{K'}[\![\Gamma]\!], \end{tikzcd} \tag{3}$$

whence the first claim follows. From the normal basis theorem for finite fields we obtain (noncanonical) isomorphisms

$$\mathcal{O}_{K'} \cong \mathbb{Z}_p[H_{K'}],$$

which are compatible with trace and natural projection maps. Indeed, the sets $S_{K'} := \{a \in \mathcal{O}_{K'} \mid \mathbb{Z}_p[H_{K'}]a = \mathcal{O}_{K'}\} \cong \mathbb{Z}_p[H_{K'}]^\times$ are compact, since $1+\mathrm{Jac}(\mathbb{Z}_p[H_{K'}])$

for the Jacobson radical $\mathrm{Jac}(\mathbb{Z}_p[H_{K'}])$ is open in $\mathbb{Z}_p[H_{K'}]^\times$, and thus $\varprojlim_{K'} S_{K'}$ is nonempty. Hence the trace maps induce (noncanonical) isomorphisms

$$\varprojlim_{K'} \mathbb{O}_{K'} \cong \mathbb{Z}_p[\![H]\!] \quad \text{and} \quad \varprojlim_{K'} \mathbb{O}_{K'}[\![\Gamma]\!] \cong \mathbb{Z}_p[\![G]\!]. \qquad \square$$

We now review Coleman's exact sequence [1979; 1983], which is one crucial ingredient in the construction of the ϵ-isomorphism.

Assume first that $K/\mathbb{Q}_p$ *is finite*. Then $\mathbb{U}(K_\infty) := \varprojlim_{n,i} \mathbb{O}_{K_n}^\times/p^i$ with $K_n := K(\mu_{p^n})$, and the sequence

$$0 \longrightarrow \mathbb{Z}_p(1) \xrightarrow{\iota} \mathbb{U}(K_\infty) \xrightarrow{\mathrm{Col}} \mathbb{O}_K[\![\Gamma]\!] \xrightarrow{\pi} \mathbb{Z}_p(1) \longrightarrow 0 \tag{4}$$

of Λ-modules is exact, where the maps are defined as follows:

- $\iota(\epsilon) = \epsilon$.
- $\mathrm{Col}(u) := \mathrm{Col}_\epsilon(u)$ is defined by the rule

$$\mathcal{L}(g_u) := \Big(1 - \frac{\varphi}{p}\Big)\log(g_u) = \frac{1}{p}\log\frac{g_u^p}{\varphi(g_u)} = \mathrm{Col}(u)\cdot(X+1) \tag{5}$$

  in $\mathbb{O}_K[\![X]\!]$, with $g_u := g_{u,\epsilon} \in \mathbb{O}_K[\![X]\!]$ the Coleman power series satisfying $g^{\phi^{-n}}(\epsilon_n - 1) = u_n$ for all n. Here ϕ is acting coefficientwise on $g_u = g_u(X)$, while $\varphi : \mathbb{O}_K[\![X]\!] \to \mathbb{O}_K[\![X]\!]$ is induced by $X \mapsto (X+1)^p - 1$ and the action of ϕ on the coefficients. Furthermore, the $\mathbb{O}_K$-linear action of $\mathbb{O}_K[\![\Gamma]\!]$ on $\mathbb{O}_K[\![X]\!]$ is induced by $\gamma \cdot X = (1+X)^{\kappa(\gamma)} - 1$.
- π is the composite of $\mathbb{O}_K[\![\Gamma]\!] \to \mathbb{O}_K$, $\gamma \mapsto \kappa(\gamma)$, followed by the trace $\mathrm{Tr}_{K/\mathbb{Q}_p} : \mathbb{O}_K \to \mathbb{Z}_p$ (and strictly speaking followed by $\mathbb{Z}_p \to \mathbb{Z}_p(1)$, $c \mapsto c\epsilon$).

Using Proposition 2.1 and the isomorphism

$$\Lambda_{[\mathbb{T}_{un},\tau_p]^{-1}} \cong \mathbb{T}_{un} \otimes_\Lambda \Lambda_{[\mathbb{T}_{un},\tau_p]^{-1}}, \quad a \mapsto (1 \otimes \epsilon) \otimes a,$$

we thus obtain an exact sequence of Λ-modules

$$0 \longrightarrow \mathbb{Z}_p(1) \longrightarrow \mathbb{U}(K_\infty) \xrightarrow{\mathcal{L}_{K,\epsilon}} \mathbb{T}_{un}(K_\infty) \otimes_\Lambda \Lambda_{[\mathbb{T}_{un},\tau_p]^{-1}} \longrightarrow \mathbb{Z}_p(1) \longrightarrow 0. \tag{6}$$

In the end we actually shall need the analogous exact sequence

$$0 \to \mathbb{Z}_p(1) \to \mathbb{U}(K_\infty) \xrightarrow{-\mathcal{L}_{K,-\epsilon}} \mathbb{T}_{un}(K_\infty) \otimes_\Lambda \Lambda_{[\mathbb{T}_{un},\tau_p]^{-1}} \to \mathbb{Z}_p(1) \to 0, \tag{7}$$

where we replace ϵ by $-\epsilon$ everywhere in the construction and where we multiply (only) the middle map by -1. Note that the maps involving $\mathbb{Z}_p(1)$ do not change compared with (6).

To deal with the case where $K/\mathbb{Q}_p$ *is infinite*, that is, $p^\infty \mid [K:\mathbb{Q}_p]$, consider finite intermediate extensions $\mathbb{Q}_p \subseteq L \subseteq L' \subseteq K$. We claim that the diagram

$$\begin{array}{ccccccccc} 0 \longrightarrow & \mathbb{Z}_p(1) & \longrightarrow & \mathbb{U}(L_\infty) & \xrightarrow{\mathscr{L}_{L',\epsilon}} & \mathbb{T}_{un}(L'_\infty)\otimes_\Lambda \Lambda_{[\mathbb{T}_{un},\tau_p]^{-1}} & \longrightarrow & \mathbb{Z}_p(1) & \longrightarrow 0 \\ & {\scriptstyle N_{L'_\infty/L_\infty}=[L':L]\cdot}\downarrow & & {\scriptstyle N_{L'_\infty/L_\infty}}\downarrow & & {\scriptstyle \mathrm{pr}_{L'/L}}\downarrow & & {\scriptstyle =}\downarrow & \\ 0 \longrightarrow & \mathbb{Z}_p(1) & \longrightarrow & \mathbb{U}(L_\infty) & \xrightarrow{\mathscr{L}_{L,\epsilon}} & \mathbb{T}_{un}(L_\infty)\otimes_\Lambda \Lambda_{[\mathbb{T}_{un},\tau_p]^{-1}} & \longrightarrow & \mathbb{Z}_p(1) & \longrightarrow 0 \end{array} \tag{8}$$

commutes, where the norm maps $N_{L'_\infty/L_\infty} = N_{L'/L}$ are induced by $N_{L'_n/L_n}$ for all n, which on $\mathbb{Z}_p(1)$ amounts to multiplication by $[L':L]$ while $N_{L'_\infty/L_\infty}:\mathbb{U}(L'_\infty)\to\mathbb{U}(L_\infty)$ is nothing else than the projection on the corresponding inverse (sub)system. Recalling (3) this is equivalent to the commutativity of

$$\begin{array}{ccccccccc} 0 \longrightarrow & \mathbb{Z}_p(1) & \longrightarrow & \mathbb{U}(L'_\infty) & \xrightarrow{\mathrm{Col}_{L',\epsilon}} & \mathbb{O}_{L'}[[\Gamma]] & \longrightarrow & \mathbb{Z}_p(1) & \longrightarrow 0 \\ & {\scriptstyle N_{L'_\infty/L_\infty}=[L':L]\cdot}\downarrow & & {\scriptstyle N_{L'_\infty/L_\infty}}\downarrow & & {\scriptstyle \mathrm{Tr}_{L'/L}}\downarrow & & {\scriptstyle =}\downarrow & \\ 0 \longrightarrow & \mathbb{Z}_p(1) & \longrightarrow & \mathbb{U}(L_\infty) & \xrightarrow{\mathrm{Col}_{L,\epsilon}} & \mathbb{O}_{L}[[\Gamma]] & \longrightarrow & \mathbb{Z}_p(1) & \longrightarrow 0 \end{array} \tag{9}$$

where $\mathrm{Tr}_{L'/L}:\mathbb{O}_{L'}[[\Gamma]]\to\mathbb{O}_L[[\Gamma]]$ is induced by the trace on the coefficients. While the left and right square obviously commute, we sketch how to check this for the middle one.

By the uniqueness of the Coleman power series we have

$$N_{L'/L}(g_{u'}) = g_{N_{L'/L}(u')} \quad \text{for } u'\in\mathbb{U}(L'_\infty),$$

where $N_{L'/L}:\mathbb{O}_{L'}[[X]]\to\mathbb{O}_L[[X]]$ is defined by $f(X)\mapsto\prod_{\sigma\in G(L'/L)} f^\sigma(X)$, where σ acts coefficientwise on f (see the proof of Lemma 2 in [Yager 1982] for a similar argument). Next, one has

$$\mathscr{L}(N_{L'/L}(g)) = \mathrm{Tr}_{L'/L}\mathscr{L}(g)$$

for $g\in\mathbb{O}_{L'}[[X]]^\times$, since $N_{L'/L}$ and ϕ commute. So far we have seen that

$$\mathrm{Tr}_{L'/L}\mathscr{L}(g_{u'}) = \mathscr{L}(g_{N_{L'/L}(u')}),$$

which implies the claim

$$\mathrm{Tr}_{L'/L}(\mathrm{Col}(u')) = \mathrm{Col}(g_{N_{L'/L}(u')})$$

using the defining equation (5) and the compatibility of $\mathrm{Tr}_{L'/L}$ with the Mahler transform $\mathfrak{M}:\mathbb{O}_K[[\Gamma]]\to\mathbb{O}_K[[X]]$, $\lambda\mapsto\lambda\cdot(1+X)$.

Taking inverse limits of (8) we obtain the exact sequence

$$0\to\mathbb{U}(K_\infty)\xrightarrow{\mathscr{L}_{K,\epsilon}}\mathbb{T}_{un}(K_\infty)\otimes_\Lambda\Lambda_{[\mathbb{T}_{un},\tau_p]^{-1}}\longrightarrow\mathbb{Z}_p(1)\to 0. \tag{10}$$

Similarly, starting with (7) we obtain the exact sequence

$$0 \to \mathbb{U}(K_\infty) \xrightarrow{-\mathcal{L}_{K,-\epsilon}} \mathbb{T}_{un}(K_\infty) \otimes_\Lambda \Lambda_{[\mathbb{T}_{un},\tau_p]^{-1}} \longrightarrow \mathbb{Z}_p(1) \to 0. \tag{11}$$

Galois cohomology. The complex $R\Gamma(\mathbb{Q}_p, \mathbb{T}_{un}(K_\infty))$ of continuous cochains has only nontrivial cohomology groups for $i = 1, 2$:

$$\mathrm{H}^1(\mathbb{Q}_p, \mathbb{T}_{un}(K_\infty)) = \varprojlim_{\mathbb{Q}_p \subseteq L \subseteq K_\infty \text{finite}} \mathrm{H}^1(L, \mathbb{Z}_p(1)) = \varprojlim_L (L^\times)^{\wedge p} \tag{12}$$

by Kummer theory and

$$\mathrm{H}^2(\mathbb{Q}_p, \mathbb{T}_{un}(K_\infty)) = \varprojlim_{\mathbb{Q}_p \subseteq L \subseteq K_\infty \text{finite}} \mathrm{H}^2(L, \mathbb{Z}_p(1)) = \mathbb{Z}_p \tag{13}$$

by local Tate duality; here the sign of the trace map $\mathrm{tr} : \mathrm{H}^2(\mathbb{Q}_p, \mathbb{T}_{un}(K_\infty)) \cong \mathbb{Z}_p$ is normalised according to [Kato 1993a, Chapter II, §1.4] as follows: If $\theta \in \mathrm{H}^1(\mathbb{Q}_p, \Lambda)$ denotes the character $G_{\mathbb{Q}_p} \xrightarrow{w} \hat{\mathbb{Z}} \xrightarrow{\text{canon}} \Lambda$, where w is the map which sends Frob_p to 1 and the inertia subgroup to 0, then we have a commutative diagram

$$\begin{tikzcd} \mathbb{Q}_p^\times \arrow[r, "v"] \arrow[d, "\delta"'] & \mathbb{Z} \arrow[r, "\text{canon}"] & \mathbb{Z}_p \\ \mathrm{H}^1(\mathbb{Q}_p, \mathbb{Z}_p(1)) \arrow[rr, "-\cup\theta"] & & \mathrm{H}^2(\mathbb{Q}_p, \mathbb{Z}_p(1)), \arrow[u, "\cong"', "\mathrm{tr}"] \end{tikzcd} \tag{14}$$

where v denotes the normalised valuation map and δ is the Kummer map. The first isomorphism (12) induces

- a canonical exact sequence

$$0 \to \mathbb{U}(K_\infty) \longrightarrow \mathrm{H}^1(\mathbb{Q}_p, \mathbb{T}_{un}(K_\infty)) \xrightarrow{-\hat{v}} \mathbb{Z}_p \to 0, \tag{15}$$

  if $K/\mathbb{Q}_p$ is finite, $\hat{v}$ being induced from the valuation maps $v_L : L^\times \to \mathbb{Z}$ (the sign before $\hat{v}$ will become evident by the descent calculation (54));
- an isomorphism

$$\mathbb{U}(K_\infty) \cong \mathrm{H}^1(\mathbb{Q}_p, \mathbb{T}_{un}(K_\infty)), \tag{16}$$

  if $p^\infty \mid [K : \mathbb{Q}_p]$.

Determinants. Now we assume that $K/\mathbb{Q}_p$ is *infinite*. Then

$$G \cong G' \times \Delta,$$

where Δ is a finite abelian group of order d prime to p and $G' \cong \mathbb{Z}_p^2$. Thus

$$\Lambda(G) = \mathbb{Z}_p[\Delta][[\mathbb{Z}_p^2]]$$

is a product of regular, hence Cohen–Macaulay, rings. Set

$$\mathcal{O} := \mathbb{Z}_p[\mu_d].$$

Then

$$\Lambda(G) \subseteq \Lambda_{\mathbb{O}}(G) = \prod_{\chi \in \mathrm{Irr}_{\overline{\mathbb{Q}}_p}(\Delta)} \Lambda_{\mathbb{O}}(G')e_\chi,$$

where e_χ denotes the idempotent corresponding to χ, while $\mathrm{Irr}_{\overline{\mathbb{Q}}_p}(\Delta)$ denotes the set of $\overline{\mathbb{Q}}_p$-rational characters of Δ. Since regular rings are normal (or by Wedderburn theory) it follows that there is a product decomposition into local regular integral domains

$$\Lambda(G) = \prod_{\chi \in \mathrm{Irr}_{\mathbb{Q}_p}(\Delta)} \Lambda_{\mathbb{O}_\chi}(G')e_\chi,$$

where now $\mathrm{Irr}_{\mathbb{Q}_p}(\Delta)$ denotes the set of $\mathbb{Q}_p$-rational characters and $\mathbb{O}_\chi$ is the ring of integers of $K_\chi := \mathrm{End}_{\mathbb{Z}_p[\Delta]}(\chi)$.

For the various rings R showing up like $\Lambda(G)$ for different G, we fix compatible determinant functors $\mathbf{d}_R : \mathrm{D}^{\mathrm{p}}(R) \to \mathcal{P}_R$ from the category of perfect complexes of R-modules (consisting of (bounded) complexes of finitely generated R-modules quasi-isomorphic to strictly perfect complexes, that is, bounded complexes of finitely generated projective R-modules) into the Picard category $\mathcal{P}_R$ with unit object $\mathbf{1}_R = \mathbf{d}_R(0)$, see Appendix B) for the yoga of determinants used in this article.

Lemma 2.2. *For all $r \in \mathbb{Z}$ there exists a canonical isomorphism*

$$\mathbf{1}_\Lambda \xrightarrow{\mathrm{can}_{\mathbb{Z}_p(r)}} \mathbf{d}_\Lambda(\mathbb{Z}_p(r)).$$

Remark 2.3. The proof will show that the same result holds for $G \cong \mathbb{Z}_p^k \times \Delta$, $k \geq 2$ and any $\Lambda(G)$-module M of Krull codimension at least 2.

Proof. Since

$$\mathrm{Ext}^i_{\Lambda(G)}(\mathbb{Z}_p(r), \Lambda(G)) \cong \mathrm{Ext}^i_{\Lambda(G')}(\mathbb{Z}_p(r), \Lambda(G')) = 0$$

for $i \neq k$ (≥ 2) we see that the codimension of $\mathbb{Z}_p(r)$ equals $k+1-1 = k \geq 2$. Setting $M = \mathbb{Z}_p(r)$ we first show that the class $[M]$ in $G_0(\Lambda) = K_0(\Lambda)$ vanishes; i.e., there exists an isomorphism $c_0 : \mathbf{1} \cong \mathbf{d}(M)$ by the definition of $\mathcal{P}_R$ in [Fukaya and Kato 2006]. Since

$$K_0(\Lambda) = \bigoplus_\chi K_0(\Lambda_{\mathbb{O}_\chi}(G')) \cong \bigoplus_\chi \mathbb{Z},$$

where the last map is given by the rank, the claim follows because the $e_\chi M$ are torsion $\Lambda_{\mathbb{O}_\chi}(G')$-modules. By the knowledge of the codimension we have $M_{\mathfrak{p}} = 0$ for all prime ideals $\mathfrak{p} \subset \Lambda$ of height at most 1. In particular, we obtain canonical isomorphisms

$$c_{\mathfrak{p}} : \mathbf{1}_{\Lambda_{\mathfrak{p}}} \cong \mathbf{d}_{\Lambda_{\mathfrak{p}}}(M_{\mathfrak{p}}).$$

Since $\mathrm{Mor}(\mathbf{1}_{\Lambda_{\mathfrak{p}}}, \mathbf{d}_{\Lambda_{\mathfrak{p}}}(M_{\mathfrak{p}}))$ is a (nonempty) $K_1(\Lambda_{\mathfrak{p}})$-torsor, there exists for each $\mathfrak{p}$ a unique $\lambda_{\mathfrak{p}} \in \Lambda_{\mathfrak{p}}^{\times} = K_1(\Lambda_{\mathfrak{p}})$ such that

$$c_{\mathfrak{p}} = (c_0)_{\mathfrak{p}} \cdot \lambda_{\mathfrak{p}},$$

where $(c_0)_{\mathfrak{p}} = \Lambda_{\mathfrak{p}} \otimes_{\Lambda} c_0$. Now let $\mathfrak{q} = \mathfrak{q}_{\chi}$ be a prime of height zero corresponding to $\chi \in \mathrm{Irr}_{\mathbb{Q}_p}(\Delta)$. Then

$$\begin{aligned} c_{\mathfrak{q}} &= \Lambda_{\mathfrak{q}} \otimes_{\Lambda_{\mathfrak{p}}} c_{\mathfrak{p}} \\ &= \Lambda_{\mathfrak{q}} \otimes_{\Lambda} c_0 \cdot \lambda_{\mathfrak{p}} = (c_0)_{\mathfrak{q}} \lambda_{\mathfrak{p}} \end{aligned}$$

for all prime ideals $\mathfrak{p} \supset \mathfrak{q}$ of height one, whence

$$\lambda_{\mathfrak{p}} = \lambda_{\mathfrak{q}}.$$

Thus

$$\lambda_{\mathfrak{q}} \in \bigcap_{\mathfrak{p} \supset \mathfrak{q}, \mathrm{ht}(\mathfrak{p})=1} \Lambda_{\mathfrak{p}}^{\times} = \Lambda_{\mathbb{O}_{\chi}}(G')^{\times}$$

($\Lambda_{\mathbb{O}_{\chi}}(G')$ being regular, that is, $\bigcap_{\mathfrak{p} \supset \mathfrak{q}, \mathrm{ht}(\mathfrak{p})=1} \Lambda_{\mathfrak{p}} = \Lambda_{\mathbb{O}_{\chi}}(G')$) and

$$\mathrm{can}_M := (c_0 \cdot \lambda_{\mathfrak{q}_{\chi}})_{\chi} : \mathbf{1}_{\Lambda} \to \mathbf{d}_{\Lambda}(M)$$

is unique and independent of the choice of c_0. Here we used the canonical decomposition $K_1(\Lambda(G)) \cong \bigoplus_{\chi} K_1(\Lambda_{\mathbb{O}_{\chi}}(G'))$. $\square$

Now we can finally define the ϵ-isomorphism for the pair $(\Lambda(G), \mathbb{T}_{un})$:

$$\epsilon_{\Lambda}(\mathbb{T}_{un}) := \epsilon_{\Lambda,\epsilon}(\mathbb{T}_{un}) : \mathbf{1}_{\Lambda} \to \mathbf{d}_{\Lambda}(R\Gamma(\mathbb{Q}_p, \mathbb{T}_{un}))\mathbf{d}_{\Lambda}(\mathbb{T}_{un} \otimes_{\Lambda} \Lambda_{\tau_p}). \tag{17}$$

Since Λ is regular we obtain, by property (B.h) in the Appendix,

$$\begin{aligned} \mathbf{d}_{\Lambda}(R\Gamma(\mathbb{Q}_p, \mathbb{T}_{un}))^{-1} &\cong \mathbf{d}_{\Lambda}(\mathrm{H}^1(\mathbb{Q}_p, \mathbb{T}_{un}))\mathbf{d}_{\Lambda}(\mathrm{H}^2(\mathbb{Q}_p, \mathbb{T}_{un}))^{-1} \\ &\cong \mathbf{d}_{\Lambda}(\mathbb{U}(K_{\infty}))\mathbf{d}_{\Lambda}(\mathbb{Z}_p)^{-1} \\ &\cong \mathbf{d}_{\Lambda}(\mathbb{T}_{un} \otimes_{\Lambda} \Lambda_{\tau_p})\mathbf{d}_{\Lambda}(\mathbb{Z}_p(1))^{-1}\mathbf{d}_{\Lambda}(\mathbb{Z}_p)^{-1} \\ &\cong \mathbf{d}_{\Lambda}(\mathbb{T}_{un} \otimes_{\Lambda} \Lambda_{\tau_p}); \end{aligned}$$

here we have used (13) and (16) for the second isomorphism, regularity and the sequence (11) with its map $-\mathscr{L}_{K,\epsilon^{-1}}$ (sic!) for the third, and the identifications $\mathrm{can}_{\mathbb{Z}_p(1)}$ and $\mathrm{can}_{\mathbb{Z}_p}$ in the last step. This induces (17).

In the spirit of Fukaya and Kato, this can be reformulated in a way that also covers noncommutative rings Λ later. For any $a \in K_1(\widetilde{\Lambda})$ define

$$K_1(\Lambda)_a := \{x \in K_1(\widetilde{\Lambda}) \mid (1 \otimes \phi)_*(x) = a \cdot x\},$$

which is nonempty by [Fukaya and Kato 2006, Proposition 3.4.5]. If Λ is the Iwasawa algebra of an abelian p-adic Lie group, that is, $K_1(\widetilde{\Lambda}) = \widetilde{\Lambda}^{\times}$, this implies in particular that $\Lambda_a \cap \widetilde{\Lambda}^\times = K_1(\Lambda)_a \neq \varnothing$, whence we obtain an isomorphism of $\widetilde{\Lambda}$-modules

$$\Lambda_a \otimes_\Lambda \widetilde{\Lambda} \cong \widetilde{\Lambda}, \quad x \otimes y \mapsto x \cdot y. \tag{18}$$

Thus, one immediately sees that the map

$$\mathbb{U}(K_\infty) \to \mathbb{T}_{un} \otimes_\Lambda \Lambda_{\tau_p} \subseteq \mathbb{T}_{un} \otimes_\Lambda \widetilde{\Lambda}$$

extends to an exact sequence of $\widetilde{\Lambda}$-modules

$$0 \to \mathbb{U}(K_\infty) \otimes_\Lambda \widetilde{\Lambda} \longrightarrow \mathbb{T}_{un} \otimes_\Lambda \widetilde{\Lambda} \longrightarrow \widehat{\mathbb{Z}_p^{ur}}(1) \to 0, \tag{19}$$

which in fact is canonically isomorphic to the base change of (10) from Λ- to $\widetilde{\Lambda}$-modules. Therefore base changing (17) by $\widetilde{\Lambda} \otimes_\Lambda -$ and using (18) (tensored with $\mathbb{T}_{un}(K_\infty)$) we obtain

$$\epsilon'_\Lambda(\mathbb{T}_{un}) := \epsilon'_{\Lambda,\epsilon}(\mathbb{T}_{un}) : \mathbf{1}_{\widetilde{\Lambda}} \to \mathbf{d}_\Lambda(R\Gamma(\mathbb{Q}_p, \mathbb{T}_{un}))_{\widetilde{\Lambda}} \mathbf{d}_\Lambda(\mathbb{T}_{un})_{\widetilde{\Lambda}}, \tag{20}$$

which actually arises as base change from some

$$\epsilon_0 : \mathbf{1}_\Lambda \to \mathbf{d}_\Lambda(R\Gamma(\mathbb{Q}_p, \mathbb{T}_{un}(K_\infty))\mathbf{d}_\Lambda(\mathbb{T}_{un}(K_\infty))$$

plus a twisting by an element $\delta \in K_1(\Lambda)_{\tau_p}$, that is,

$$\epsilon'_\Lambda(\mathbb{T}_{un}) \in \mathrm{Mor}(\mathbf{1}_\Lambda, \mathbf{d}_\Lambda(R\Gamma(\mathbb{Q}_p, \mathbb{T}_{un}(K_\infty))\mathbf{d}_\Lambda(\mathbb{T}_{un}(K_\infty))) \times^{K_1(\Lambda)} K_1(\Lambda)_{\tau_p}.$$

Indeed, fixing an isomorphism $\psi : \Lambda \cong \Lambda_{\tau_p}$ (see Proposition 2.1) sending 1 to δ, (18) implies that $\delta \in K_1(\Lambda)_{\tau_p}$ and the claim follows from the commutative diagram

$$\begin{tikzcd}
\mathbb{T}_{un} \otimes_\Lambda \widetilde{\Lambda} \arrow[r, "\mathbb{T}_{un}\otimes\delta^{-1}"] & \mathbb{T}_{un} \otimes_\Lambda \widetilde{\Lambda} \\
\mathbb{T}_{un} \otimes_\Lambda \Lambda_{\tau_p} \arrow[u, hook] \arrow[r, "\mathbb{T}_{un}\otimes\psi^{-1}"] & \mathbb{T}_{un} \otimes_\Lambda \Lambda \arrow[u, hook]
\end{tikzcd}$$

($\epsilon'_\Lambda(\mathbb{T}_{un})$ equals δ times the base change of $\epsilon_0 := (\mathbb{T}_{un} \otimes \psi^{-1}) \circ \epsilon_\Lambda(\mathbb{T}_{un})$).

Twisting. We recall the following definition from [Fukaya and Kato 2006, §1.4]:

Definition 2.4. A ring R is *of type* 1 if there exists a two-sided ideal I of R such that R/I^n is finite of order a power of p for any $n \geq 1$ and such that $R \cong \varprojlim_n R/I^n$.

A ring R is *of type* 2 if it is the matrix algebra $M_n(L)$ of some finite extension L over $\mathbb{Q}_p$, for some $n \geq 1$.

By Lemma 1.4.4 in the same work, R is of type 1 if and only if the defining condition above holds for the Jacobson ideal $J = J(R)$. Such rings are always semilocal and R/J is a finite product of matrix algebras over finite fields.

Now let R be a commutative ring of type 1 and let $\mathbb{T} = \mathbb{T}_\chi$ be a free R-module of rank one with Galois action given by

$$\chi = \chi_{\mathbb{T}} : G_{\mathbb{Q}_p} \to R^\times$$

which factors through G. By $\tilde{\chi}_{\mathbb{T}}$ we denote the induced ring homomorphism $\Lambda(G) \to R$. Furthermore let $Y = Y_\chi$ be the $(R, \Lambda(G))$-bimodule which is R as R-module and where $\Lambda(G)$ is acting via

$$\chi_Y := \tilde{\chi}_{\mathbb{T}}^{-1} \chi_{cyc} : \Lambda(G) \to R$$

(from the right), where

$$\chi_{cyc} : \Lambda(G) \to \mathbb{Z}_p \to R$$

is induced by the cyclotomic character and the unique ring homomorphism $\mathbb{Z}_p \to R$. Then the map

$$Y \otimes_{\Lambda(G)} \mathbb{T}_{un} \xrightarrow{\cong} \mathbb{T}, \qquad y \otimes t \mapsto y \cdot \chi_Y(t),$$

is an isomorphism of R-modules which is Galois equivariant, where the Galois action on the tensor product is given by $\sigma(y \otimes t) = y \otimes \sigma(t)$ for $\sigma \in G_{\mathbb{Q}_p}$.

Let $\widetilde{R}$ and R_a be defined in the same way as for Λ. Then, using the isomorphisms

$$Y \otimes_\Lambda \mathbf{d}_\Lambda(R\Gamma(\mathbb{Q}_p, \mathbb{T}_{un})) \cong \mathbf{d}_R(R\Gamma(\mathbb{Q}_p, Y \otimes_\Lambda \mathbb{T}_{un})) \cong \mathbf{d}_R(R\Gamma(\mathbb{Q}_p, \mathbb{T}))$$

by [Fukaya and Kato 2006, 1.6.5] and

$$R \otimes_\Lambda \Lambda_a \cong R_{\chi(a)},$$

where $\chi : \Lambda \to R$ denotes a continuous ring homomorphism, we may define the following ϵ-isomorphisms:

Definition 2.5. In the above situation we set

$$\epsilon_R(\mathbb{T}) := \epsilon_{R,\epsilon}(\mathbb{T}) := Y \otimes_\Lambda \epsilon_{\Lambda,\epsilon}(\mathbb{T}_{un}) : \mathbf{1}_R \to \mathbf{d}_R(R\Gamma(\mathbb{Q}_p, \mathbb{T}))\mathbf{d}_R(\mathbb{T} \otimes_R R_{\chi(\tau_p)})$$

and

$$\epsilon'_R(\mathbb{T}) := \epsilon'_{R,\epsilon}(\mathbb{T}) := Y \otimes_\Lambda \epsilon'_{\Lambda,\epsilon}(\mathbb{T}_{un}) : \mathbf{1}_{\widetilde{R}} \to \mathbf{d}_R(R\Gamma(\mathbb{Q}_p, \mathbb{T}))_{\widetilde{R}} \mathbf{d}_R(\mathbb{T})_{\widetilde{R}}.$$

By definition we have an important twist invariance property: if R and R' are commutative rings of type 1 or 2 and Y' is any (R', R)-bimodule that is projective as an R'-module and satisfies $Y' \otimes_R \mathbb{T} \cong \mathbb{T}'$, we have

$$Y' \otimes_R \epsilon_R(\mathbb{T}) = \epsilon_{R'}(\mathbb{T}') \quad \text{and} \quad Y' \otimes_R \epsilon'_R(\mathbb{T}) = \epsilon'_{R'}(\mathbb{T}'). \tag{21}$$

Indeed, to this end the definition extends to all pairs $(R, \mathbb{T})$, where R is a (not necessarily commutative) ring of type 1 or 2 and $\mathbb{T}$ stands for a projective R-module such that there exists a (R, Λ)-bimodule Y which is projective as R-module and such that $\mathbb{T} \cong Y \otimes_\Lambda \mathbb{T}_{un}$. In this context we denote by $[\mathbb{T}, \sigma]$, $\sigma \in G_{\mathbb{Q}_p}$, the element in $K_1(R)$ induced by the action of $G_{\mathbb{Q}_p}$ on $\mathbb{T}$; note that this induces a homomorphism $[\mathbb{T}, -] : G(\mathbb{Q}_p^{ab}) \to K_1(R)$.

Example 2.6. Let $\psi : G_F \to \mathbb{Z}_p^\times$ be a Grössencharacter of an imaginary quadratic field F such that p is split in F and assume that its restriction to G_{F_ν}, ν a place above p, factors through G. We write $\mathbb{T}_\psi$ for the free rank-one $\Lambda(G)$-module with Galois action given by $\sigma(\lambda) = \lambda\bar{\sigma}^{-1}\psi(\sigma)$. Then we also write $\epsilon_\Lambda(\psi)$ for $\epsilon_\Lambda(\mathbb{T}_\psi)$.

The ϵ-conjecture. We fix $K/\mathbb{Q}_p$ infinite and recall that $G = G(K_\infty/\mathbb{Q}_p)$ as well as $\Lambda = \Lambda(G)$ and $\Lambda_\mathbb{O} = \Lambda_\mathbb{O}(G)$ for $\mathbb{O} = \mathbb{O}_L$ the ring of integers of some finite extension L of $\mathbb{Q}_p$. If $\chi : G \to \mathbb{O}_L^\times$ denotes any continuous character such that the representation

$$V_\chi := L(\chi),$$

whose underlying vector space is just L and whose $G_{\mathbb{Q}_p}$-action is given by χ, is de Rham, hence potentially semistable by [Serre 1968] (in this classical case) or by [Berger 2002] (in general) then we have

$$L \otimes_{\mathbb{O}_L} \epsilon'_{\mathbb{O}_L}(\mathbb{T}_\chi) = \epsilon'_L(V_\chi)$$

by definition. The ϵ-isomorphism conjecture (Conjecture 3.4.3 of [Fukaya and Kato 2006]) states that

$$\epsilon'_L(V_\chi) = \Gamma_L(V_\chi) \cdot \epsilon_{L,\epsilon,dR}(V_\chi) \cdot \theta_L(V_\chi), \tag{22}$$

where, for any de Rham p-adic representation V of G, the notation used is as follows:

(a) $\Gamma_L(V) := \prod_\mathbb{Z} \Gamma^*(j)^{-h(-j)}$ with $h(j) = \dim_L gr^j D_{dR}(V)$ and

$$\Gamma^*(j) = \begin{cases} (-1)^j(-j)!^{-1} & \text{for } j \le 0, \\ \Gamma(j) & \text{for } j > 0, \end{cases}$$

denotes the leading coefficient of the Γ-function.

(b) The map

$$\epsilon_{dR}(V) := \epsilon_{L,\epsilon,dR}(V) : \mathbf{1}_{\widetilde{L}} \to \mathbf{d}_{\widetilde{L}}(V)\mathbf{d}_{\widetilde{L}}(D_{dR}(V))^{-1},$$

with $\widetilde{L} := \widehat{\mathbb{Q}_p^{ur}} \otimes_{\mathbb{Q}_p} L$, is defined in [Fukaya and Kato 2006, Proposition 3.3.5]. We shall recall its definition after the proof of Lemma A.5.

(c) $\theta_L(V)$ is defined as follows: Firstly, $R\Gamma_f(\mathbb{Q}_p, V)$ is defined as a certain subcomplex of the local cohomology complex $R\Gamma(\mathbb{Q}_p, V)$, concentrated in degrees 0 and 1, whose image in the derived category is isomorphic to

$$R\Gamma_f(\mathbb{Q}_p, V) \cong \big[D_{\mathrm{cris}}(V) \xrightarrow{(1-\varphi_p,1)} D_{\mathrm{cris}}(V) \oplus D_{dR}(V)/D^0_{dR}(V)\big]. \tag{23}$$

Here φ_p denotes the usual Frobenius homomorphism and the induced map $t(V) := D_{dR}(V)/D^0_{dR}(V) \to H^1_f(\mathbb{Q}_p, V)$ is the exponential map $\exp_{BK}(V)$ of Bloch–Kato, where we write $\mathrm{H}^n_f(\mathbb{Q}_p, V)$ for the cohomology of $R\Gamma_f(\mathbb{Q}_p, V)$. Now

$$\theta_L(V) : \mathbf{1}_L \to \mathbf{d}_L(R\Gamma(\mathbb{Q}_p, V)) \cdot \mathbf{d}_L(D_{dR}(V)) \tag{24}$$

is by definition induced from $\eta_p(V) \cdot \overline{(\eta_p(V^*(1))^*)}$ (see Remark B.1 for the notation) — with

$$\eta_p(V) : \mathbf{1}_L \to \mathbf{d}_L(R\Gamma_f(\mathbb{Q}_p, V))\mathbf{d}_L(t(V)) \tag{25}$$

arising by trivialising $D_{\mathrm{cris}}(V)$ in (23) by the identity — followed by an isomorphism induced by local Tate duality

$$R\Gamma_f(\mathbb{Q}_l, V) \cong \big(R\Gamma(\mathbb{Q}_l, V^*(1))/R\Gamma_f(\mathbb{Q}_l, V^*(1))\big)^*[-2] \tag{26}$$

and using $D^0_{dR}(V) = t(V^*(1))^*$.

More explicitly, $\theta_L(V)$ is obtained from applying the determinant functor to the following exact sequence:

$$0 \to \mathrm{H}^0(\mathbb{Q}_p, V) \to D_{\mathrm{cris}}(V) \to D_{\mathrm{cris}}(V) \oplus t(V) \xrightarrow{\exp_{BK}(V)} \mathrm{H}^1(\mathbb{Q}_p, V)$$
$$\xrightarrow{\exp_{BK}(V^*(1))^*} D_{\mathrm{cris}}(V^*(1))^* \oplus t(V^*(1))^* \to D_{\mathrm{cris}}(V^*(1))^* \to \mathrm{H}^2(\mathbb{Q}_p, V) \to 0,$$

which arises from joining the defining sequences of $\exp_{BK}(V)$ with the dual sequence for $\exp_{BK}(V^*(1))$ by local duality (26).

Remark 2.7. (a) The ϵ-conjecture may analogously be formulated using $\epsilon_R(\mathbb{T})$ instead of $\epsilon'_R(\mathbb{T})$. In the following we will amply switch between the two versions.
(b) Since by definition of $\epsilon_{\mathcal{O}_L}(\mathbb{T}_\chi)$ we have

$$L \otimes_{\mathcal{O}_L} \epsilon'_{\mathcal{O}_L}(\mathbb{T}_\chi) = L \otimes_{\mathcal{O}_L} \big(Y_\chi \otimes_\Lambda \epsilon'_\Lambda(\mathbb{T}_{un})\big) = (L \otimes_{\mathcal{O}_L} Y_\chi) \otimes_\Lambda \epsilon'_\Lambda(\mathbb{T}_{un}),$$

proving (22) amounts to showing that

$$L \otimes_\Lambda \epsilon_\Lambda(\mathbb{T}_{un}) = \epsilon_L(V_\chi), \tag{27}$$

where Λ acts on L via $\chi^{-1}\chi_{cyc} : \Lambda(G) \to \mathcal{O}_L \subseteq L$. Once we have shown (27) for all possible χ as above, it follows immediately by twisting that for example $\epsilon_\Lambda(\mathbb{T}_{K_\infty}(T))$ for $T = \mathbb{Z}_p(\eta)(r)$ as below satisfies the descent property

$$V_\rho \otimes_\Lambda \epsilon_\Lambda(\mathbb{T}_{K_\infty}(T)) = \epsilon_L(V(\rho^*))$$

with $V(\rho^*) := V \otimes_{\mathbb{Q}_p} V_{\rho^*}$ for all one-dimensional representations V_ρ arising from some continuous $\rho : G \to \mathcal{O}_L^\times$ and its contragredient representation V_{ρ^*}.

Note that by [Serre 1968] any V_χ as above is of the form

$$W = L(\eta\rho)(r) = Lt_{\rho\eta,r},$$

where r is some integer, $\eta : G \to \mathbb{O}_L^\times$ is an unramified character and $\rho : G \twoheadrightarrow G(K'_m/\mathbb{Q}_p) \to \mathbb{O}_L^\times$ denotes an Artin character for some finite subextension K' of $K/\mathbb{Q}_p$ and with $m = a(\rho)$ chosen minimal, that is, $p^{a(\rho)}$ is the p-part of the conductor of ρ.

In the following we fix η and r and we set $T := \mathbb{Z}_p(\eta)(r)$, $V := T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ and

$$\mathbb{T}_{K_\infty} = \mathbb{T}_{K_\infty}(T) = \Lambda^\sharp \otimes_{\mathbb{Z}_p} T, \tag{28}$$

the free Λ-module on which $\sigma \in G_{\mathbb{Q}_p}$ acts as $\bar{\sigma}^{-1}\eta\kappa^r(\sigma)$.

Now we are going to make the map (24) explicit. First we describe the local cohomology groups:

$$\mathrm{H}^0(\mathbb{Q}_p, W) = \begin{cases} L & \text{if } r = 0 \text{ and } \rho\eta = \mathbb{1}, \\ 0 & \text{otherwise.} \end{cases} \tag{29}$$

By local Tate duality we have

$$\mathrm{H}^2(\mathbb{Q}_p, W) \cong \mathrm{H}^0(\mathbb{Q}_p, W^*(1))^* = \begin{cases} L & \text{if } r = 1 \text{ and } \rho\eta = \mathbb{1}, \\ 0 & \text{otherwise.} \end{cases} \tag{30}$$

From the local Euler–Poincaré characteristic formula one immediately obtains

$$\dim_L \mathrm{H}^1(\mathbb{Q}_p, W) = \dim_L \mathrm{H}^1(\mathbb{Q}_p, W^*(1)) = \begin{cases} 2 & \text{if } r = 0 \text{ or } 1 \text{ and } \rho\eta = \mathbb{1}, \\ 1 & \text{otherwise.} \end{cases} \tag{31}$$

Following the same reasoning used for Lemma 1.3.1 of [Benois and Nguyen Quang Do 2002], one sees that

$$\begin{aligned} \mathrm{H}^1_f(\mathbb{Q}_p, W) &\cong \big(\mathrm{H}^1(\mathbb{Q}_p, W^*(1))/\mathrm{H}^1_f(\mathbb{Q}_p, W^*(1))\big)^* \\ &= \begin{cases} \mathrm{H}^1(\mathbb{Q}_p, W) & \text{if } r \geq 2, \text{ or } r = 1 \text{ and } \rho\eta \neq \mathbb{1}, \\ \mathrm{im}\big(\mathbb{U}(\mathbb{Q}_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to \mathrm{H}^1(\mathbb{Q}_p, \mathbb{Q}_p(1))\big) & \text{if } r = 1 \text{ and } \rho\eta = \mathbb{1}, \\ \mathrm{H}^1(\mathbb{F}_p, \mathbb{Q}_p) & \text{if } r = 0 \text{ and } \rho\eta = \mathbb{1}, \\ 0, & r \leq -1, \text{ or } r = 0 \text{ and } \rho\eta \neq \mathbb{1}, \end{cases} \end{aligned}$$

where the map in the second line is the Kummer map. Hence we call the cases where $r = 0$ or 1 and $\rho\eta = \mathbb{1}$ *exceptional* and all the others *generic*.

For the tangent space we have by (61)

$$t(W) = \begin{cases} D_{dR}(W) = L & \text{if } r > 0, \\ 0 & \text{if } r \leq 0, \end{cases} \tag{32}$$

$$t(W^*(1)) = \begin{cases} 0 & \text{if } r > 0, \\ D_{dR}(W^*(1)) = L & \text{if } r \leq 0, \end{cases} \tag{33}$$

and

$$D_{\text{cris}}(W) = \begin{cases} 0 & \text{if } a(\rho) \neq 0, \\ Le_{\rho\eta,r} & \text{otherwise,} \end{cases} \tag{34}$$

with Frobenius action given as $\phi(e_{\rho\eta,r}) = p^{-r}\rho\eta(\tau_p^{-1})e_{\rho\eta,r}$.

The case $r \geq 1$. In this case we have $\Gamma_L(W) = \Gamma(r)^{-1} = \dfrac{1}{(r-1)!}$ and $\mathrm{H}^0(\mathbb{Q}_p, W) = 0$, whence

$$1 - \phi : D_{\text{cris}}(W) \to D_{\text{cris}}(W) \tag{35}$$

and

$$\exp(W) : D_{dR}(W) \cong H^1_f(\mathbb{Q}_p, W) \tag{36}$$

are bijections. Combined with the exact sequences

$$\begin{aligned} 0 \longrightarrow \mathrm{H}^1_f(\mathbb{Q}_p, W^*(1))^* &\xrightarrow{\exp(W^*(1))^*} D_{\text{cris}}(W^*(1))^* \\ &\xrightarrow{1-\phi^*} D_{\text{cris}}(W^*(1))^* \longrightarrow \mathrm{H}^2(\mathbb{Q}_p, W) \longrightarrow 0 \end{aligned} \tag{37}$$

and

$$0 \to \mathrm{H}^1_f(\mathbb{Q}_p, W) \longrightarrow \mathrm{H}^1(\mathbb{Q}_p, W) \longrightarrow \mathrm{H}^1_f(\mathbb{Q}_p, W^*(1))^* \to 0,$$

they induce the following isomorphism corresponding to $\theta_L(W)^{-1}$:

$$\mathbf{d}_L(D_{dR}(W)) \to \mathbf{d}_L(R\Gamma(\mathbb{Q}_p, W))^{-1}.$$

In the *generic* case this decomposes as

$$\mathbf{d}_L(\exp(W)) : \mathbf{d}_L(D_{dR}(W)) \to \mathbf{d}_L(\mathrm{H}^1(\mathbb{Q}_p, W)) = \mathbf{d}_L(R\Gamma(\mathbb{Q}_p, W))^{-1}$$

times

$$\frac{\det(1-\phi^* \mid D_{\text{cris}}(W^*(1))^*)}{\det(1-\phi \mid D_{\text{cris}}(W))} : \mathbf{1}_L \to \mathbf{1}_L,$$

which equals

$$\frac{\det(1-\phi \mid D_{\text{cris}}(W^*(1)))}{\det(1-\phi \mid D_{\text{cris}}(W))} = \begin{cases} \dfrac{1-p^{r-1}\rho\eta(\tau_p)}{1-p^{-r}\rho\eta(\tau_p^{-1})} & \text{if } a(\rho) = 0, \\ 1 & \text{otherwise.} \end{cases} \tag{38}$$

Now let $r = 1$ and $\rho\eta = \mathbb{1}$, that is, we consider the *exceptional* case $W = \mathbb{Q}_p(1)$. As now $\det(1-\phi \mid D_{\text{cris}}(W^*(1))) = 0$ and the two occurrences of $D_{\text{cris}}(W^*(1))^*$ in (37) are identified via the identity, the map $\theta_L(W)^{-1}$ is also induced by (35), (36) together with the (second) exact sequence in the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}_p^\times \otimes \mathbb{Q}_p \arrow[r] \arrow[d, "\cong"', "\delta"] & \widehat{\mathbb{Q}_p^\times} \otimes \mathbb{Q}_p \arrow[r, "-\hat{v}\otimes\mathbb{Q}_p"] \arrow[d, "\cong"', "\delta"] & \mathbb{Q}_p \arrow[r] & 0 \\
0 \arrow[r] & \mathrm{H}^1_f(\mathbb{Q}_p, \mathbb{Q}_p(1)) \arrow[r] & \mathrm{H}^1(\mathbb{Q}_p, \mathbb{Q}_p(1)) \arrow[r, dotted] & \mathrm{H}^2(\mathbb{Q}_p, \mathbb{Q}_p(1)) \arrow[r] \arrow[u, "\cong", "\mathrm{Tr}"'] & 0
\end{tikzcd} \tag{39}$$

where the first two vertical maps δ are induced by Kummer theory, v denotes the normalised valuation map and the dotted arrow is defined by commutativity; that is, $\theta_L(W)^{-1}$ arises from

$$\mathbf{d}_{\mathbb{Q}_p}(D_{dR}(\mathbb{Q}_p(1))) \xrightarrow{\exp_{\mathbb{Q}_p(1)}} \mathbf{d}_{\mathbb{Q}_p}(H^1_f(\mathbb{Q}_p, \mathbb{Q}_p(1))) \cong \mathbf{d}_{\mathbb{Q}_p}(R\Gamma(\mathbb{Q}_p, W))^{-1} \tag{40}$$

times

$$\det(1-\phi \mid D_{\mathrm{cris}}(\mathbb{Q}_p(1))) = (1-p^{-1}). \tag{41}$$

Combining (39), (40) and (41) this can rephrased as follows:

Proposition 2.8. *The map $\theta(\mathbb{Q}_p(1))$ is just induced by the single exact sequence*

$$0 \to t(\mathbb{Q}_p(1)) \cong \mathbb{Q}_p \xrightarrow{(1-p^{-1})^{-1}\exp_{\mathbb{Q}_p(1)}} \mathrm{H}^1(\mathbb{Q}_p, \mathbb{Q}_p(1)) \xrightarrow{-\hat{v}\otimes\mathbb{Q}_p} \mathrm{H}^2(\mathbb{Q}_p, \mathbb{Q}_p(1)) \to 0.$$

Proof. Since $t(\mathbb{Q}_p) = 0$, it follows directly from its definition as a connecting homomorphism that

$$\exp_{\mathbb{Q}_p} : \mathbb{Q}_p = D_{\mathrm{cris}}(\mathbb{Q}_p) \to \mathrm{H}^1_f(\mathbb{Q}_p, \mathbb{Q}_p) \subseteq \mathrm{H}^1(\mathbb{Q}_p, \mathbb{Q}_p) \tag{42}$$

sends $\alpha \in \mathbb{Q}_p$ to the character $\chi_\alpha : G_{\mathbb{Q}_p} \to \mathbb{Q}_p$, $g \mapsto (g-1)c$, where $c \in \widehat{\mathbb{Q}_p^{nr}}$ satisfies $(1-\varphi)c = \alpha$, that is, $\chi_\alpha(\phi) = -\alpha$. As noted in [Benois and Nguyen Quang Do 2002, Lemma 1.3.1], we thus may identify $\mathrm{H}^1_f(\mathbb{Q}_p, \mathbb{Q}_p) = \mathrm{H}^1(\mathbb{F}_p, \mathbb{Q}_p)$. Identifying the copies of $D_{\mathrm{cris}}(\mathbb{Q}_p)$ (in the dual of (37)) gives rise to a map

$$\psi : \mathbb{Q}_p = \mathrm{H}^0(\mathbb{Q}_p, \mathbb{Q}_p) \to \mathrm{H}^1_f(\mathbb{Q}_p, \mathbb{Q}_p), \quad \alpha \mapsto \chi_\alpha.$$

By local Tate duality

$$\begin{tikzcd}
\mathrm{H}^1(\mathbb{Q}_p, \mathbb{Q}_p(1))/\mathrm{H}^1_f(\mathbb{Q}_p, \mathbb{Q}_p(1)) \arrow[d, "\psi^*"'] & \times\ \mathrm{H}^1_f(\mathbb{Q}_p, \mathbb{Q}_p) \arrow[r] & \mathrm{H}^2(\mathbb{Q}_p, \mathbb{Q}_p(1)) \cong \mathbb{Q}_p \arrow[d, equal] \\
\mathrm{H}^2(\mathbb{Q}_p, \mathbb{Q}_p(1)) & \times\ \mathrm{H}^0(\mathbb{Q}_p, \mathbb{Q}_p) \arrow[r] \arrow[u, "\psi"'] & \mathrm{H}^2(\mathbb{Q}_p, \mathbb{Q}_p(1)) \cong \mathbb{Q}_p
\end{tikzcd}$$

we obtain for the dual map ψ^* using the normalisation (14)

$$\mathrm{tr}(\psi^*(\delta(p)) = \mathrm{tr}(\delta(p) \cup \chi_1) = \chi_1(\phi) = -1.$$

The dotted arrow in (39) being ψ^*, this diagram commutes as claimed. □

The case $r \leq 0$. This case is dual to the previous one, replacing W by $W^*(1)$.

The descent. Let K be infinite. In order to describe the descent of $\mathscr{L}_{K,\epsilon^{-1}}$ in (10) we set

$$\mathscr{L}_{\mathbb{T}} := \mathscr{L}_{\mathbb{T},\epsilon^{-1}} := Y \otimes_{\Lambda} \mathscr{L}_{K,\epsilon^{-1}} \tag{43}$$

if the projective left Λ'-module Y (with commuting right Λ-module structure) satisfies $Y \otimes_{\Lambda} \mathbb{T}_{un} \cong \mathbb{T}$ as Λ'-modules. Since $\mathscr{L}_{K,\epsilon^{-1}}$ is the crucial ingredient in the definition of $\epsilon'_{\Lambda}(\mathbb{T})$, the following descent diagram will be important:

For fixed ρ as before we choose $K' \subseteq K$ and $n \geq \max\{1, a(\rho)\}$ such that ρ factorises over $G_n := G(K'_n/\mathbb{Q}_p)$. Setting $\Lambda' := \mathbb{Q}_p[G_n]$ and $V' := \mathbb{Q}_p[G_n]^{\sharp} \otimes \mathbb{Q}_p(\eta)(r)$ we first note that

$$\mathrm{H}^i(\mathbb{Q}_p, V') \cong \mathrm{H}^i(K'_n, \mathbb{Q}_p(\eta)(r))$$

by Shapiro's lemma. Also, let Y' be the (Λ', Λ)-bimodule such that $Y' \otimes_{\Lambda} \mathbb{T}_{un} \cong V'$. We write $e_{\chi} := (1/\#G_n) \sum_{g \in G_n} \chi(g^{-1})g$ for the usual idempotent, which induces a canonical decomposition $\Lambda' \cong \prod L_{\chi}$ into a product of finite extensions L_{χ} of $\mathbb{Q}_p$. In particular, for $L = L_{\rho}$ we have $W \cong e_{\rho^{-1}} V' = L_{\rho}(\rho\eta)(r)$.

Then, for $r \geq 1$ and with $\Gamma(V') := \bigoplus_{\chi} \Gamma(e_{\chi} V')$, we have a commutative diagram

$$\begin{tikzcd}
Y' \otimes_{\Lambda} \mathrm{H}^1(\mathbb{Q}_p, \mathbb{T}_{un}) \arrow[rr, "-\mathscr{L}_{V'} = -Y' \otimes_{\Lambda} \mathscr{L}_{K,\epsilon^{-1}}"] \arrow[d, "\mathrm{pr}_n"'] & & Y' \otimes_{\Lambda} \mathbb{T}_{un} \otimes_{\Lambda} \Lambda_{[\mathbb{T}_{un}, \tau_p^{-1}]} \arrow[d, "\cong"', "\mathrm{pr}_n"] \\
\mathrm{H}^1(K'_n, \mathbb{Q}_p(\eta)(r)) & D_{dR}(V') \arrow[l, "\Gamma(V')^{-1} \exp_{V'}"'] & V' \otimes_{\Lambda'} (\Lambda')_{[V', \tau_p^{-1}]} \arrow[l]
\end{tikzcd}$$

of Λ'-modules as will be explained in the Appendix, Proposition A.6.

Applying the exact functor $V_{\rho^*} \otimes_{\Lambda'} -$ leads to the final commutative descent diagram — at least for $W \neq \mathbb{Q}_p(1)$ —

$$\begin{tikzcd}
Y'' \otimes_{\Lambda} \mathrm{H}^1(\mathbb{Q}_p, \mathbb{T}_{un}) \arrow[rr, "-\mathscr{L}_{W} = -Y'' \otimes_{\Lambda} \mathscr{L}_{K,\epsilon^{-1}}"] \arrow[d, "\mathrm{pr}_n"'] & & Y'' \otimes_{\Lambda} \mathbb{T}_{un} \otimes_{\Lambda} \Lambda_{[\mathbb{T}_{un}, \tau_p^{-1}]} \arrow[d, "\cong"', "\mathrm{pr}_n"] \\
\mathrm{H}^1(\mathbb{Q}_p, W) & D_{dR}(W) \arrow[l, "\Gamma(W)^{-1} \exp_W"'] & W \otimes_L L_{[W, \tau_p^{-1}]}, \arrow[l, "\frac{\det(1-\varphi|D_{\mathrm{cris}}(W^*(1)))}{\det(1-\varphi|D_{\mathrm{cris}}(W))} \cdot \epsilon_{L,\epsilon,dR}(W)^{-1}"']
\end{tikzcd} \tag{44}$$

where $Y'' := V_{\rho^*} \otimes_{\Lambda'} Y' = V_{\rho^*} \otimes_{\Lambda} Y$ is a (L, Λ)-bimodule. For $W = \mathbb{Q}_p(1)$ the Euler factor in the denominator and the map pr_0 become zero, so we shall instead apply a direct descent calculation in Lemma 2.9 using semisimplicity and a Bockstein homomorphism.

For the descent we need

- the long Tor-exact sequence by applying $Y'' \otimes_{\Lambda(G)} -$ to the defining sequence (10) for $-\mathscr{L}_{K,\epsilon^{-1}}$;

- the convergent cohomological spectral sequence

$$E_2^{i,j} := \mathrm{Tor}_{-i}^{\Lambda}(Y'', H^j(\mathbb{Q}_p, \mathbb{T}_{un})) \Rightarrow \mathrm{H}^{i+j}(\mathbb{Q}_p, W), \tag{45}$$

  which is induced from the isomorphism

$$Y'' \otimes^{\mathbb{L}} R\Gamma(\mathbb{Q}_p, \mathbb{T}_{un}) \cong R\Gamma(\mathbb{Q}_p, Y'' \otimes_{\Lambda} \mathbb{T}_{un}),$$

  proved in [Fukaya and Kato 2006] and using $W \cong Y'' \otimes \mathbb{T}_{un}$;
- and the fact that the determinant functor is compatible with both these ingredients [Venjakob 2012].

For $\mathbb{T} = \mathbb{T}(T) := \Lambda^{\sharp} \otimes_{\mathbb{Z}_p} T \cong Y \otimes_{\Lambda} \mathbb{T}_{un}$, we have

$$\mathrm{H}^i(\mathbb{Q}_p, \mathbb{T}) \cong \begin{cases} T & \text{if } i = 0 \text{ and } r = 0, \eta = \mathbb{1}, \\ \mathrm{H}^1(\mathbb{Q}_p, \mathbb{T}) \neq 0 & \text{if } i = 1, \\ T(-1) & \text{if } i = 2 \text{ and } r = 1, \eta = \mathbb{1}, \\ 0 & \text{otherwise.} \end{cases} \tag{46}$$

Hence we obtain for $r \geq 1$ the following exact sequence of terms in lower degree:

$$\begin{aligned} 0 \to \mathrm{Tor}_1^{\Lambda}(Y'', \mathrm{H}^1(\mathbb{Q}_p, \mathbb{T}_{un})) \to \mathrm{H}^0(\mathbb{Q}_p, W) \to \mathrm{Tor}_2^{\Lambda}(Y'', \mathrm{H}^2(\mathbb{Q}_p, \mathbb{T}_{un})) \\ \to Y'' \otimes_{\Lambda} \mathrm{H}^1(\mathbb{Q}_p, \mathbb{T}_{un}) \to \mathrm{H}^1(\mathbb{Q}_p, W) \to \mathrm{Tor}_1^{\Lambda}(Y'', \mathrm{H}^2(\mathbb{Q}_p, \mathbb{T}_{un})) \to 0, \end{aligned} \tag{47}$$

and we also obtain

$$\mathrm{Tor}_2^{\Lambda}(Y'', \mathrm{H}^1(\mathbb{Q}_p, \mathbb{T}_{un})) = 0 \quad \text{and} \quad Y'' \otimes_{\Lambda} \mathrm{H}^2(\mathbb{Q}_p, \mathbb{T}_{un}) \cong \mathrm{H}^2(\mathbb{Q}_p, W).$$

Since $Y'' \otimes_{\Lambda}^{\mathbb{L}} R\Gamma(\mathbb{Q}_p, \mathbb{T}_{un}) \cong V_{\rho^*} \otimes_{\Lambda}^{\mathbb{L}} \big(Y \otimes_{\Lambda} R\Gamma(\mathbb{Q}_p, \mathbb{T}_{un})\big) \cong V_{\rho^*} \otimes_{\Lambda}^{\mathbb{L}} R\Gamma(\mathbb{Q}_p, \mathbb{T})$, the preceding sequence is canonically isomorphic to

$$\begin{aligned} 0 \to \mathrm{Tor}_1^{\Lambda}(V_{\rho^*}, \mathrm{H}^1(\mathbb{Q}_p, \mathbb{T})) \to \mathrm{H}^0(\mathbb{Q}_p, W) \to \mathrm{Tor}_2^{\Lambda}(V_{\rho^*}, \mathrm{H}^2(\mathbb{Q}_p, \mathbb{T})) \\ \to V_{\rho^*} \otimes_{\Lambda} \mathrm{H}^1(\mathbb{Q}_p, \mathbb{T}) \to \mathrm{H}^1(\mathbb{Q}_p, W) \to \mathrm{Tor}_1^{\Lambda}(V_{\rho^*}, \mathrm{H}^2(\mathbb{Q}_p, \mathbb{T})) \to 0, \end{aligned} \tag{48}$$

and we get

$$\mathrm{Tor}_2^{\Lambda}(V_{\rho^*}, \mathrm{H}^1(\mathbb{Q}_p, \mathbb{T})) = 0 \quad \text{and} \quad V_{\rho^*} \otimes_{\Lambda} \mathrm{H}^2(\mathbb{Q}_p, \mathbb{T}) \cong \mathrm{H}^2(\mathbb{Q}_p, W).$$

In the *generic case* the spectral sequence boils down to the isomorphism

$$Y'' \otimes_{\Lambda} \mathrm{H}^1(\mathbb{Q}_p, \mathbb{T}_{un}) \cong \mathrm{H}^1(\mathbb{Q}_p, W). \tag{49}$$

Considering the support of $\mathbb{Z}_p(1)$, one easily sees that $\mathrm{Tor}_i^{\Lambda}(Y'', \mathbb{Z}_p(1)) = 0$ for all $i \geq 0$. Hence the long exact Tor-sequence associated with (10) combined with

(16) degenerates to

$$Y'' \otimes_\Lambda \mathrm{H}^1(\mathbb{Q}_p, \mathbb{T}_{un}) \xrightarrow[\cong]{-\mathscr{L}_W} W \otimes_L L_{[W,\tau_p]^{-1}}, \tag{50}$$

while for all $i \geq 0$

$$\mathrm{Tor}_i^\Lambda(Y'', \mathrm{H}^2(\mathbb{Q}_p, \mathbb{T}_{un})) = \mathrm{Tor}_i^\Lambda(Y'', \mathbb{Z}_p) = 0. \tag{51}$$

Thus the conjectured equation (22) holds by (44), (49), (50) and the definition (17) with $-\mathscr{L}_{K,\epsilon^{-1}}$.

For the *exceptional case* $W = \mathbb{Z}_p(1)$ we set $R = \Lambda(\Gamma)_\mathfrak{p}$, where $\mathfrak{p}$ denotes the augmentation ideal of $\Lambda(\Gamma)$ and recall that R is a discrete valuation ring with uniformising element $\pi := 1 - \gamma_0$, where γ_0 is a fixed element in Γ sent to 1 under

$$\Gamma \xrightarrow{\kappa} \mathbb{Z}_p^\times \xrightarrow{\log_p} \mathbb{Z}_p,$$

and residue field $R/\pi = \mathbb{Q}_p$. The commutative diagram of homomorphisms of rings

$$\begin{tikzcd} \Lambda = \Lambda(G) \arrow[r] \arrow[d] & \Lambda(\Gamma) \arrow[r] \arrow[d] & R \arrow[d] \\ \mathbb{Z}_p \arrow[r] & \mathbb{Z}_p \arrow[r] & \mathbb{Q}_p \end{tikzcd}$$

induces with $Y'' := R/\pi$ the isomorphism

$$\begin{aligned} R\Gamma(\mathbb{Q}_p, \mathbb{Q}_p(1)) &\cong Y'' \otimes^{\mathbb{L}}_\Lambda R\Gamma(\mathbb{Q}_p, \mathbb{T}_{un}(K_\infty)) \\ &\cong \mathbb{Q}_p \otimes^{\mathbb{L}}_{\Lambda(\Gamma)} \left(\Lambda(\Gamma) \otimes^{\mathbb{L}}_\Lambda R\Gamma(\mathbb{Q}_p, \mathbb{T}_{un}(K_\infty))\right) \\ &\cong \mathbb{Q}_p \otimes^{\mathbb{L}}_R R \otimes_{\Lambda(\Gamma)} \otimes^{\mathbb{L}}_\Lambda R\Gamma(\mathbb{Q}_p, \mathbb{T}_{un}(\mathbb{Q}_{p,\infty})) \\ &\cong \mathbb{Q}_p \otimes^{\mathbb{L}}_R R\Gamma(\mathbb{Q}_p, \mathbb{T}_{un}(\mathbb{Q}_{p,\infty}))_\mathfrak{p}. \end{aligned} \tag{52}$$

In particular, the descent calculation factorises over the cyclotomic level; that is,

$$\epsilon'_{\mathbb{Q}_p}(\mathbb{Q}_p(1)) = R/\pi \otimes_R \epsilon'_R(R \otimes_{\Lambda(\Gamma)} \mathbb{T}_{un}(\mathbb{Q}_{p,\infty}))$$

is induced by $\epsilon'_R(R \otimes_{\Lambda(\Gamma)} \mathbb{T}_{un}(\mathbb{Q}_{p,\infty}))$, which in turn is induced by the localisation at $\mathfrak{p}$ of the exact sequences (6) and (15) for $K = \mathbb{Q}_p$, which are respectively

$$\mathbb{U}(\mathbb{Q}_{p,\infty})_\mathfrak{p} \xrightarrow[\cong]{-\mathscr{L}_{\mathbb{T}_{un}(\mathbb{Q}_{p,\infty})_\mathfrak{p}}} \mathbb{T}_{un}(\mathbb{Q}_{p,\infty})_\mathfrak{p} \otimes_R R_{[\mathbb{T}_{un},\tau_p]^{-1}} \tag{53}$$

(this arises as the long exact Tor-sequence from (10)) and

$$0 \to \mathbb{U}(\mathbb{Q}_{p,\infty})_\mathfrak{p} \to \mathrm{H}^1(\mathbb{Q}_p, \mathbb{T}_{un}(\mathbb{Q}_{p,\infty}))_\mathfrak{p} \xrightarrow{-\hat{v}} \mathbb{Q}_p \cong \mathrm{H}^2(\mathbb{Q}_p, \mathbb{T}_{un}(\mathbb{Q}_{p,\infty}))_\mathfrak{p} \to 0. \tag{54}$$

This last sequence arises from an analogue of the spectral sequence (45) above — which gives with $\mathcal{H} = G(K_\infty/\mathbb{Q}_{p,\infty})$ an exact sequence

$$0 \to \mathrm{H}^1(\mathbb{Q}_p, \mathbb{T}_{un}(K_\infty))_{\mathcal{H}} \to \mathrm{H}^1(\mathbb{Q}_p, \mathbb{T}_{un}(\mathbb{Q}_{p,\infty})) \to \mathrm{H}^2(\mathbb{Q}_p, \mathbb{T}_{un}(K_\infty))^{\mathcal{H}} \to 0,$$

and

$$\mathrm{H}^2(\mathbb{Q}_p, \mathbb{T}_{un}(K_\infty))_{\mathcal{H}} = \mathrm{H}^2(\mathbb{Q}_p, \mathbb{T}_{un}(\mathbb{Q}_{p,\infty}))$$

— combined with (13) and an identification of $\mathrm{H}^2(\mathbb{Q}_p, \mathbb{T}_{un}(K_\infty))^{\mathcal{H}} = \mathbb{Z}_p$ with $\mathrm{H}^2(\mathbb{Q}_p, \mathbb{T}_{un}(K_\infty))_{\mathcal{H}} = \mathbb{Z}_p$ induced by the base change of $\mathrm{can}_{\mathbb{Z}_p}$. Indeed, it is easy to check that the long exact $\mathcal{H}$-homology ($= \mathrm{Tor}_i^{\Lambda}(\Lambda(\Gamma), -)$) sequence associated with (10) recovers (6), in particular $\mathrm{H}^1(\mathbb{Q}_p, \mathbb{T}_{un}(K_\infty))_{\mathcal{H}} \cong \mathbb{U}(K_\infty)_H \cong \mathbb{U}(\mathbb{Q}_{p,\infty})$. Moreover, the composite

$$\begin{aligned}\tilde{\beta} : \mathrm{H}^1(\mathbb{Q}_p, \mathbb{T}_{un}(\mathbb{Q}_{p,\infty})) \to \mathrm{H}^2(\mathbb{Q}_p, \mathbb{T}_{un}(K_\infty))^{\mathcal{H}} &= \mathrm{H}^2(\mathbb{Q}_p, \mathbb{T}_{un}(K_\infty))_{\mathcal{H}} \\ &= \mathrm{H}^2(\mathbb{Q}_p, \mathbb{T}_{un}(\mathbb{Q}_{p,\infty}))\end{aligned}$$

is via restriction and taking $G(K_\infty^{\mathcal{H}'}/\mathbb{Q}_{p,\infty})$-invariants by construction induced by the Bockstein homomorphism β associated to the exact triangle in the derived category

$$R\Gamma(\mathbb{Q}_p, \mathbb{T}_{un}(K_\infty)) \xrightarrow{1-h_0} R\Gamma(\mathbb{Q}_p, \mathbb{T}_{un}(K_\infty)) \longrightarrow R\Gamma(\mathbb{Q}_p, \mathbb{T}_{un}(K_\infty^{\mathcal{H}'})),$$

where $\mathcal{H}'$ is the maximal pro-p quotient of $\mathcal{H}$ and h_0 is the image of ϕ. By [Flach 2004, Lemma 5.9] (and the argument following directly afterwards using the projection formula for the cup product) it follows that $\tilde{\beta}$ is given by the cup product $\theta \cup -$, where

$$\theta : G_{\mathbb{Q}_{p,\infty}} \twoheadrightarrow \mathcal{H}' \cong \mathbb{Z}_p$$

is the unique character such that h_0 is sent to 1 under the second isomorphism. Using our above convention of the trace map (14) one finds according to [Kato 1993a, Chapter II, §1.4.2] that the above composite equals $-\hat{v}$. Indeed

$$\mathrm{tr}(\tilde{\beta}(\delta(p))) = \mathrm{tr}(\theta \cup \delta(p)) = -\theta(\phi) = -\theta(h_0) = -1.$$

Now consider the element

$$u := (1 - \epsilon_n^{-1})_n \in \varprojlim_n (\mathbb{Q}_p(\mu_{p^n})^\times)^\wedge \cong \mathrm{H}^1(\mathbb{Q}_p, \mathbb{T}_{un}(\mathbb{Q}_{p,\infty}))$$

and its image $u_\mathfrak{p}$ in $\mathrm{H}^1(\mathbb{Q}_p, \mathbb{T}_{un}(K_\infty))_\mathfrak{p}$.

Lemma 2.9. $\mathrm{H}^1(\mathbb{Q}_p, \mathbb{T}_{un}(K_\infty))_\mathfrak{p} \cong Ru_\mathfrak{p}$ *is a free R-module of rank one and* $\mathcal{L}_{\mathbb{T}_{un}(\mathbb{Q}_{p,\infty})_\mathfrak{p}}$ *induces modulo* π *a canonical isomorphism*

$$t(\mathbb{Q}_p(1)) \xleftarrow{-\mathcal{L}_{\mathbb{Q}_p(1)}} \mathbb{U}(\mathbb{Q}_{p,\infty})_\mathfrak{p}/\pi \longrightarrow \mathbb{Q}_p \longrightarrow \mathrm{H}^1(\mathbb{Q}_p, \mathbb{T}_{un}(K_\infty))_\mathfrak{p}/\pi \qquad (55)$$

which sends $(1-p^{-1})e \in \mathbb{Q}_p e = t(\mathbb{Q}_p(1))$ *to* $\bar{u}$, *the image of* $u_{\mathfrak{p}}$ (*but which is of course not induced by the map* $\mathbb{U}(\mathbb{Q}_{p,\infty})_{\mathfrak{p}} \to \mathrm{H}^1(\mathbb{Q}_p, \mathbb{T}_{un}(K_\infty))_{\mathfrak{p}}$ *as the latter map becomes trivial modulo* π!).

Proof. The natural inclusion $\mathrm{H}^1(\mathbb{Q}_p, \mathbb{T}_{un}(\mathbb{Q}_{p,\infty}))_{\mathfrak{p}}/\pi \subseteq \mathrm{H}^1(\mathbb{Q}_p, \mathbb{Q}_p(1))$ maps $\bar{u}$ to the image of p under $(\mathbb{Q}_p^\times)^\wedge \otimes \mathbb{Q}_p \cong \mathrm{H}^1(\mathbb{Q}_p, \mathbb{Q}_p(1))$, the isomorphism of Kummer theory, because p is the image of the elements $1-\epsilon_n^{-1}$ under the norm maps. In particular, $\bar{u}$ is nonzero. By (15) the element u^{γ_0-1} belongs to $\mathbb{U}(\mathbb{Q}_{p,\infty})$. In order to calculate the image of the class $\overline{u^{\gamma_0-1}}$ of $u_{\mathfrak{p}}^{\gamma_0-1}$ modulo π under $-\mathscr{L}_{\mathbb{Q}_p(1)}$ we note that

$$g(X) = g_{u^{\gamma_0-1},-\epsilon}(X) = \frac{(1+X)^{\kappa(\gamma_0)}-1}{X} \equiv \kappa(\gamma_0) \bmod (X),$$

whence we obtain from setting $X=0$ in $-(5)$ (i.e., Equation (5) multiplied by -1) that

$$-(1-p^{-1}) = -(1-p^{-1})\log(\kappa(\gamma_0)) = -\mathrm{Col}_{-\epsilon}(u^{\gamma_0-1})\cdot 1$$

equals the image of $\overline{u^{\gamma_0-1}}$ in $\mathbb{Q}_p = R/\pi \otimes_{\Lambda(\Gamma)} \mathbb{T}_{un}(\mathbb{Q}_{p,\infty}) \otimes_{\Lambda(\Gamma)} \Lambda(\Gamma)_{[\mathbb{T}_{un},\tau_p]^{-1}}$ under $-\mathscr{L}_{\mathbb{Q}_p(1)}$. In particular, $\overline{u^{\gamma_0-1}}$ is a basis of $\mathbb{U}(\mathbb{Q}_{p,\infty})_{\mathfrak{p}}/\pi$, which is mapped to zero in $\mathrm{H}^1(\mathbb{Q}_p, \mathbb{T}_{un}(\mathbb{Q}_{p,\infty}))_{\mathfrak{p}}/\pi$, whence the long exact Tor-sequence associated with (54) induces the isomorphisms

$$\mathrm{H}^1(\mathbb{Q}_p, \mathbb{T}_{un}(\mathbb{Q}_{p,\infty}))_{\mathfrak{p}}/\pi \xrightarrow{-v} \mathbb{Q}_p, \quad \bar{u} \mapsto -1$$

(since $v(p)=1$) and

$$\mathbb{Q}_p \longrightarrow \mathbb{U}(\mathbb{Q}_{p,\infty})_{\mathfrak{p}}/\pi, \quad 1 \mapsto \overline{u^{\gamma_0-1}},$$

where the latter formula follows from the snake lemma. By the first isomorphism and Nakayama's lemma the first statement is proven and therefore

$$\mathrm{H}^1(\mathbb{Q}_p, \mathbb{T}_{un}(\mathbb{Q}_{p,\infty}))_{\mathfrak{p}}[\pi] = \mathbb{U}(\mathbb{Q}_{p,\infty})_{\mathfrak{p}}[\pi] = 0.$$

The second claim follows now from the composition of these isomorphisms. $\square$

Finally, the exact triangle in the derived category of R-modules

$$R\Gamma(\mathbb{Q}_p, \mathbb{T}_{un}(\mathbb{Q}_{p,\infty}))_{\mathfrak{p}} \xrightarrow{1-\gamma_0} R\Gamma(\mathbb{Q}_p, \mathbb{T}_{un}(\mathbb{Q}_{p,\infty}))_{\mathfrak{p}} \longrightarrow \mathbb{Z}_p \otimes^{\mathbb{L}}_{\Lambda(\Gamma)} R\Gamma(\mathbb{Q}_p, \mathbb{T}_{un}(\mathbb{Q}_{p,\infty}))_{\mathfrak{p}} \longrightarrow$$

combined with (52) induces the Bockstein map $\beta = \theta\cup$ sitting in the canonical exact sequence (depending on γ_0)

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{H}^1(\mathbb{Q}_p, \mathbb{T}_{un}(K_\infty))_{\mathfrak{p}}/\pi \arrow[r] & \mathrm{H}^1(\mathbb{Q}_p, \mathbb{Q}_p(1)) \arrow[r, "\beta"] \arrow[d, "\cong"'] & \mathrm{H}^2(\mathbb{Q}_p, \mathbb{Q}_p(1)) \arrow[r] \arrow[d, "\cong"] & 0, \\ & & (\mathbb{Q}_p^\times)^\wedge \otimes \mathbb{Q}_p \arrow[r, "\log_p"] & \mathbb{Q}_p & \end{tikzcd} \tag{56}$$

where θ denotes the composite $G_{\mathbb{Q}_p} \xrightarrow{\kappa} \mathbb{Z}_p^\times \xrightarrow{\log_p} \mathbb{Z}_p$ considered as an element of $\mathrm{H}^1(\mathbb{Q}_p, \mathbb{Z}_p)$ (see [Flach 2004, Lemma 5.7-9; Burns and Venjakob 2006, §3.1; Burns and Flach 2006, §5.3] and, for the commutativity of the square, [Kato 1993a, Ch. II, 1.4.5]). The last zero on the upper line comes from $\mathrm{H}^3(\mathbb{Q}_p, \mathbb{T}_{un}(K_\infty))_{\mathfrak{p}}[\pi] = 0$. Combining with (55) it follows that $\epsilon'_{\mathbb{Q}_p}(\mathbb{Q}_p(1))$ is induced from the exact sequence

$$0 \to t(\mathbb{Q}_p(1)) \cong \mathbb{Q}_p \xrightarrow{(1-p^{-1})^{-1}\mathrm{im}(p)} \mathrm{H}^1(\mathbb{Q}_p, \mathbb{Q}_p(1)) \xrightarrow{\beta=\log_p} \mathrm{H}^2(\mathbb{Q}_p, \mathbb{Q}_p(1)) \to 0,$$

which does *not* coincide at all with the sequence of Proposition 2.8 (not even up to sign). Nevertheless they induce the same map on determinants: both induce a map

$$\begin{aligned} \mathbf{d}_{\mathbb{Q}_p}(\mathrm{H}^1(\mathbb{Q}_p, \mathbb{Q}_p(1))) \to \mathbf{d}_{\mathbb{Q}_p}(\mathrm{H}^2(\mathbb{Q}_p, \mathbb{Q}_p(1))) \otimes \mathbf{d}_{\mathbb{Q}_p}(t(\mathbb{Q}_p(1)) \\ \cong \mathbf{d}_{\mathbb{Q}_p}(\mathbb{Q}_p) \otimes \mathbf{d}_{\mathbb{Q}_p}(\mathbb{Q}_p) \end{aligned} \tag{57}$$

sending $(1-p^{-1})^{-1}\exp(1) \wedge -\mathrm{im}(p) = (1-p^{-1})^{-1}\mathrm{im}(p) \wedge \exp(1)$ to $1 \wedge 1$. This completes the proof in the exceptional case.

For $r \leq 0$ one has symmetric calculations — at least in the generic case — using a descent diagram analogous to (44), except that the left map on the bottom is now induced by the dual Bloch–Kato exponential map $\Gamma(V)\exp^*_{V^*(1)}$ as indicated in (67) (left to the reader). The exceptional case can be dealt with by using the duality principle (generalised reciprocity law) as follows:

Let $\mathbb{T}$ be a free R-module of rank one with compatible $G_{\mathbb{Q}_p}$-action as above. Then

$$\mathbb{T}^* := \mathrm{Hom}_R(\mathbb{T}, R)$$

is a free R°-module of rank one — for the action $h \mapsto h(-)r$, r in the opposite ring R° of R — with compatible $G_{\mathbb{Q}_p}$-action given by $h \mapsto h \circ \sigma^{-1}$. Recall that in Iwasawa theory we have the canonical involution $\iota : \Lambda^\circ \to \Lambda$, induced by $g \mapsto g^{-1}$, which allows us to consider (left) Λ°-modules again as (left) Λ-modules; for example, one has $\mathbb{T}^*(T)^\iota \cong \mathbb{T}(T^*)$ as $(\Lambda, G_{\mathbb{Q}_p})$-modules, where $M^\iota := \Lambda \otimes_{\iota,\Lambda^\circ} M$ denotes the Λ-module with underlying abelian group M, but on which $g \in G$ acts as g^{-1} for any Λ°-module M.

Given $\epsilon'_{R^\circ,-\epsilon}(\mathbb{T}^*(1))$ we may apply the dualising functor $-^*$ (compare (B.j) in Appendix B) to obtain an isomorphism

$$\epsilon'_{R^\circ,-\epsilon}(\mathbb{T}^*(1))^* : (\mathbf{d}_{R^\circ}(R\Gamma(\mathbb{Q}_p, \mathbb{T}^*(1)))_{\widetilde{R^\circ}})^*(\mathbf{d}_{R^\circ}(\mathbb{T}^*(1))_{\widetilde{R^\circ}})^* \to \mathbf{1}_{\widetilde{R^\circ}},$$

while the local Tate-duality isomorphism [Fukaya and Kato 2006, §1.6.12]

$$\psi(\mathbb{T}) : R\Gamma(\mathbb{Q}_p, \mathbb{T}) \cong R\mathrm{Hom}_{R^\circ}(R\Gamma(\mathbb{Q}_p, \mathbb{T}^*(1)), R^\circ)[-2]$$

induces an isomorphism

$$\overline{\mathbf{d}_R(\psi(\mathbb{T}))_{\widetilde{R}}}^{-1} : \left((\mathbf{d}_{R^\circ}(R\Gamma(\mathbb{Q}_p, \mathbb{T}^*(1)))_{\widetilde{R^\circ}})^*\right)^{-1}$$
$$\cong \mathbf{d}_R(R\mathrm{Hom}_{R^\circ}(R\Gamma(\mathbb{Q}_p, \mathbb{T}^*(1)), R^\circ))_{\widetilde{R}}^{-1} \to \mathbf{d}_R(R\Gamma(\mathbb{Q}_p, \mathbb{T}))_{\widetilde{R}}^{-1}, \quad (58)$$

in the notation of Remark B.1. Consider the product

$$\epsilon'_{R,\epsilon}(\mathbb{T}) \cdot \epsilon'_{R^\circ,-\epsilon}(\mathbb{T}^*(1))^* \cdot \overline{\mathbf{d}_R(\psi(\mathbb{T}))_{\widetilde{R}}}^{-1} : \mathbf{d}_R(\mathbb{T}(-1))_{\widetilde{R}} \cong \mathbf{d}_R(\mathbb{T}^*(1)^*)_{\widetilde{R}} \to \mathbf{d}_R(\mathbb{T})_{\widetilde{R}}$$

and the isomorphism $\mathbb{T}(-1) \xrightarrow{\cdot\epsilon} \mathbb{T}$ that sends $t \otimes \epsilon^{\otimes -1}$ to t.

Proposition 2.10 (duality). *Let $\mathbb{T}$ be as above and such that $\mathbb{T} \cong Y \otimes_\Lambda \mathbb{T}_{un}$ for some (R, Λ)-bimodule Y that is projective as R-module. Then*

$$\epsilon'_{R,\epsilon}(\mathbb{T}) \cdot \epsilon'_{R^\circ,-\epsilon}(\mathbb{T}^*(1))^* \cdot \overline{\mathbf{d}_R(\psi(\mathbb{T}))_{\widetilde{R}}}^{-1} = \mathbf{d}_R\big(\,\mathbb{T}(-1) \xrightarrow{\cdot\epsilon} \mathbb{T}\,\big)_{\widetilde{R}}.$$

Proof. The statement is stable under applying $Y' \otimes_R -$ for some (R', R)-bimodule Y', which is projective as an R'-module by the functoriality of local Tate duality and the lemma below. This reduces the proof to the case $(R, \mathbb{T}) = (\Lambda, \mathbb{T}(T))$, where $T = \mathbb{Z}_p(r)(\eta)$ is generic. Since the morphisms between $\mathbf{d}_R(\mathbb{T}(-1))_{\widetilde{R}}$ and $\mathbf{d}_R(\mathbb{T})_{\widetilde{R}}$ form a $K_1(\widetilde{\Lambda})$-torsor and the kernel

$$SK_1(\widetilde{\Lambda}) := \ker\left(K_1(\widetilde{\Lambda}) \to \prod_{\rho\in\mathrm{Irr}\,G} K_1(\widetilde{L}_\rho)\right) = 1$$

is trivial (because G is abelian), it suffices to check the statement for all $(L, V(\rho))$, which is nothing else than the content of [Fukaya and Kato 2006, Proposition 3.3.8]. Here $\mathrm{Irr}\,G$ denotes the set of $\overline{\mathbb{Q}}_p$-valued irreducible representations of G with finite image. □

Lemma 2.11. *Let Y be a (R', R)-bimodule such that $Y \otimes_R \mathbb{T} \cong \mathbb{T}'$ as $(R', G_{\mathbb{Q}_p})$-module and let $Y^* = \mathrm{Hom}_{R'}(Y, R')$ the induced (R'°, R°)-bimodule. Then there is a natural equivalence of functors*

$$Y \otimes_R \mathrm{Hom}_{R^\circ}(-, R^\circ) \cong \mathrm{Hom}_{R'^\circ}(Y^* \otimes_{R^\circ} -, R'^\circ)$$

on $P(R^\circ)$, and a natural isomorphism $Y^* \otimes_{R^\circ} \mathbb{T}^* \cong (\mathbb{T}')^*$ of $(R'^\circ, G_{\mathbb{Q}_p})$-modules.

Proof. This is easily checked using the adjointness of Hom and $\otimes$. □

Proposition 2.12 (Change of ϵ). *Let $c \in \mathbb{Z}_p^\times$ and let σ_c be the unique element of the inertia subgroup of $G(\mathbb{Q}_p^{ab}/\mathbb{Q}_p)$ such that $\sigma_c(\epsilon) = c\epsilon$ (in the $\mathbb{Z}_p$-module $\mathbb{Z}_p(1)$, whence written additively). Then*

$$\epsilon'_{R,c\epsilon}(\mathbb{T}) = [\mathbb{T}, \sigma_c]\epsilon'_{R,\epsilon}(\mathbb{T}).$$

Proof. As in the proof of Proposition 2.10 this is easily reduced to the pairs $(L, V(\rho))$, for which the statement follows from the functorial properties of ϵ-constants [Fukaya and Kato 2006, §3.2.2(2)]. □

Altogether we have proved this:

Theorem 2.13 (Kato, ϵ-isomorphisms). *Let $\mathbb{T}$ be such that $\mathbb{T} \cong Y \otimes_\Lambda \mathbb{T}_{un}$ as $(R, G_{\mathbb{Q}_p})$-modules for some (R, Λ)-bimodule Y which is projective as R-module, where $\Lambda = \Lambda(G)$ with $G = G(L/\mathbb{Q}_p)$ for any $L \subseteq \mathbb{Q}_p^{ab}$. Then a unique epsilon isomorphism $\epsilon'_R(\mathbb{T})$ exists satisfying the twist invariance property* (21), *the descent property* (22), *the "change of ϵ" relation* (*Proposition 2.12*) *and the duality relation* (*Proposition 2.10*). *In particular $\epsilon'_\Lambda(\mathbb{T})$ exists for all pairs $(\Lambda, \mathbb{T})$ with $\mathbb{T} \cong \Lambda$ one-dimensional* (*free*) *as a Λ-module.*

Proof. For G a two-dimensional p-adic Lie group this has been shown explicitly above. The general case follows by taking limits. □

We will indicate shortly how this result implies the validity of a local Main Conjecture in this context. Here again we restrict to the universal case $\mathbb{T}_{un}$, but we point out that similar statements hold for general $\mathbb{T}$ as in the above theorem by the twisting principle; in particular it applies to $\mathbb{T}_E$ for the local representation given by a CM elliptic curve as in Example 3.1 below.

We place ourselves in the situation described at the bottom of page 2376; in particular, G is a two-dimensional p-adic Lie group. Denote by

$$S := \{\lambda \in \Lambda \mid \Lambda/\Lambda\lambda \text{ is finitely generated over } \Lambda(G(K_\infty/\mathbb{Q}_{p,\infty}))\}$$

the canonical Ore set of Λ (see [Coates et al. 2005]) and by $\widetilde{S}$ the canonical Ore set of $\widetilde{\Lambda}$. Fix an element u of $\mathbb{U}(K_\infty) = \mathrm{H}^1(\mathbb{Q}_p, \mathbb{T}_{un}(K_\infty))$ such that the map $\Lambda \to \mathrm{H}^1(\mathbb{Q}_p, \mathbb{T}_{un}(K_\infty))$ taking 1 to u becomes an isomorphism after base change to $\widetilde{\Lambda}_{\widetilde{S}}$ (such "generators" exist according to (19) and Proposition 2.1). Then, with $\mathbb{L} := -\mathscr{L}_{K,\epsilon^{-1}}$, the map

$$\epsilon'_\Lambda(\mathbb{T}_{un}) : \mathbf{1}_{\widetilde{\Lambda}} \to \mathbf{d}_\Lambda(R\Gamma(\mathbb{Q}_p, \mathbb{T}_{un}))_{\widetilde{\Lambda}} \mathbf{d}_\Lambda(\mathbb{T}_{un})_{\widetilde{\Lambda}}$$

induces a map

$$\mathbf{1}_{\widetilde{\Lambda}} \to \mathbf{d}_\Lambda(\mathrm{H}^1(\mathbb{Q}_p, \mathbb{T}_{un})/\Lambda u)^{-1}_{\widetilde{\Lambda}} \mathbf{d}_\Lambda(\mathrm{H}^2(\mathbb{Q}_p, \mathbb{T}_{un}))_{\widetilde{\Lambda}} \mathbf{d}_\Lambda(\mathbb{T}_{un}/\mathbb{L}(u))_{\widetilde{\Lambda}} \tag{59}$$

whose base change followed by the canonical trivialisations

$$\begin{aligned}\mathbf{1}_{\widetilde{\Lambda}_{\widetilde{S}}} \to \mathbf{d}_\Lambda(\mathrm{H}^1(\mathbb{Q}_p, \mathbb{T}_{un})/\Lambda u)^{-1}_{\widetilde{\Lambda}_{\widetilde{S}}} \mathbf{d}_\Lambda(\mathrm{H}^2(\mathbb{Q}_p, \mathbb{T}_{un}))_{\widetilde{\Lambda}_{\widetilde{S}}} \mathbf{d}_\Lambda(\mathbb{T}_{un}/\Lambda\mathbb{L}(u))_{\widetilde{\Lambda}_{\widetilde{S}}} \\ \cong \mathbf{d}_{\widetilde{\Lambda}_{\widetilde{S}}}(\widehat{\mathbb{Z}_p^{ur}}) \mathbf{d}_{\widetilde{\Lambda}_{\widetilde{S}}}(\widehat{\mathbb{Z}_p^{ur}}(1)) \to \mathbf{1}_{\widetilde{\Lambda}_{\widetilde{S}}}\end{aligned}$$

(here all arguments on the right are $\tilde{S}$-torsion modules!) equals the identity in

$\mathrm{Aut}(\mathbf{1}_{\widetilde{\Lambda}_{\widetilde{S}}}) = K_1(\widetilde{\Lambda}_{\widetilde{S}})$ by Lemma 2.2. Let $\mathscr{E}_u$ be the element in $K_1(\widetilde{\Lambda}_{\widetilde{S}})$ such that

$$\mathbb{L}(u) = \mathscr{E}_u^{-1} \cdot (1 \otimes \epsilon).$$

Consider the connecting homomorphism ∂ in the exact localisation sequence

$$K_1(\widetilde{\Lambda}) \longrightarrow K_1(\widetilde{\Lambda}_{\widetilde{S}}) \xrightarrow{\partial} K_0(\widetilde{S}\text{-tor}) \longrightarrow 0,$$

where $\widetilde{S}$-tor denotes the category of finitely generated $\widetilde{\Lambda}$-modules which are $\widetilde{S}$-torsion. Then we obviously have

$$\partial(\mathscr{E}_u) = -[\mathbb{T}_{un}/\Lambda\mathbb{L}(u)] = [\mathrm{H}^2(\mathbb{Q}_p, \mathbb{T}_{un})] - [\mathrm{H}^1(\mathbb{Q}_p, \mathbb{T}_{un})/\Lambda u]$$

in $K_0(\widetilde{S}\text{-tor})$. Moreover one can evaluate $\mathscr{E}_u$ at Artin characters ρ of G as in [Coates et al. 2005] and derive an interpolation property for $\mathscr{E}(\rho)$ from Theorem 2.13 by the techniques of [Fukaya and Kato 2006, Lemma 4.3.10]; this is carried out in [Schmitt $\geq$ 2013]. These two properties build the local Main Conjecture as suggested by Fukaya and Kato in a much more general, not necessarily commutative setting. Kato (unpublished) has shown that $\widetilde{\Lambda}_{\widetilde{S}} \otimes_\Lambda \mathbb{U}(K_\infty) \cong \widetilde{\Lambda}_{\widetilde{S}}$ does hold in vast generality for p-adic Lie extensions.

3. The semilocal case

Let $F_\infty/\mathbb{Q}$ be a p-adic Lie extension with Galois group G and ν be any place of F_∞ above p such that $G_\nu = G(F_{\infty,\nu}/\mathbb{Q}_p)$ is the decomposition group at ν. For any free $\mathbb{Z}_p$-module T of finite rank with continuous Galois action by $G_\mathbb{Q}$ we define the free $\Lambda(G)$-module

$$\mathbb{T} := \mathbb{T}(T)_{F_\infty} := \Lambda(G)^\natural \otimes_{\mathbb{Z}_p} T$$

with the usual diagonal $G_\mathbb{Q}$-action. Similarly, we define the free $\Lambda(G_\nu)$-module

$$\mathbb{T}^{\mathrm{loc}} := \Lambda(G_\nu)^\natural \otimes_{\mathbb{Z}_p} T$$

with the usual diagonal $G_{\mathbb{Q}_p}$-action. Then we have the canonical isomorphism of $(\Lambda(G), G_{\mathbb{Q}_p})$-bimodules

$$\mathbb{T} \cong \Lambda(G) \otimes_{\Lambda(G_\nu)} \mathbb{T}^{\mathrm{loc}}.$$

Thus we might define

$$\begin{aligned}&\epsilon_{\Lambda(G)}(\mathbb{Q}_p, \mathbb{T})\\ &\quad := \Lambda(G) \otimes_{\Lambda(G_\nu)} \epsilon'_{\Lambda(G_\nu)}(\mathbb{T}^{\mathrm{loc}}) : \mathbf{1}_{\widetilde{\Lambda(G)}} \to \mathbf{d}_{\Lambda(G)}(R\Gamma(\mathbb{Q}_p, \mathbb{T}))_{\widetilde{\Lambda(G)}} \mathbf{d}_{\Lambda(G)}(\mathbb{T})_{\widetilde{\Lambda(G)}}.\end{aligned}$$

Now let $\rho : G \to GL_n(\mathbb{O}_L)$ be a continuous map and ρ_ν its restriction to G_ν, where L is a finite extension of $\mathbb{Q}_p$. By abuse of notation we shall denote the induced ring homomorphisms $\Lambda(G) \to M_n(\mathcal{O}_L)$ and $\Lambda(G_\nu) \to M_n(\mathcal{O}_L)$ by the same letters. Since we have a canonical isomorphism

$$L^n \otimes_{\rho,\Lambda(G)} \mathbb{T} \cong L^n \otimes_{\rho_\nu,\Lambda(G_\nu)} \mathbb{T}^{\mathrm{loc}}$$

of $(L, G_{\mathbb{Q}_p})$-bimodules, we obtain

$$\begin{aligned} &L^n \otimes_{\rho,\Lambda(G)} \epsilon_{\Lambda(G)}(\mathbb{Q}_p, \mathbb{T}) \\ &\qquad = L^n \otimes_{\rho_\nu,\Lambda(G_\nu)} \epsilon'_{\Lambda(G_\nu)}(\mathbb{T}^{\mathrm{loc}}) : \mathbf{1}_{\widetilde{L}} \to \mathbf{d}_L(R\Gamma(\mathbb{Q}_p, V(\rho^*)))_{\widetilde{L}} \mathbf{d}_L(V(\rho^*))_{\widetilde{L}}, \end{aligned}$$

where $V = T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.

Example 3.1. Let E be a elliptic curve defined over $\mathbb{Q}$ with CM by the ring of integers of an imaginary quadratic extension $K \subseteq F_\infty$ of $\mathbb{Q}$ and let ψ denote the Grössencharacter associated to E. Then $T_E \cong \mathrm{Ind}_{\mathbb{Q}}^K T_\psi$, which is isomorphic to $T_\psi \oplus T_{\psi^c}$ as representation of G_K. Here T_ψ equals $\mathbb{Z}_p$ on which $G_{\mathbb{Q}}$ acts via ψ, while ψ^c is the conjugate of ψ by complex multiplication $c \in G(K/\mathbb{Q})$.

Assuming $K_\nu = \mathbb{Q}_p$ and setting $\mathbb{T}_E := \mathbb{T}$, $\mathbb{T}_E^{\mathrm{loc}} := \mathbb{T}^{\mathrm{loc}}$ for $T = T_E$ as well as $\mathbb{T}_\psi := \Lambda(G)^\natural \otimes_{\mathbb{Z}_p} T_\psi$, $\mathbb{T}_\psi^{\mathrm{loc}} := \Lambda(G_\nu)^\natural \otimes_{\mathbb{Z}_p} T_\psi$ we obtain

$$\mathbb{T}_E \cong \mathbb{T}_\psi \oplus \mathbb{T}_{\psi^c}$$

as $(\Lambda(G), G_K)$-modules and hence

$$\begin{aligned} \epsilon_{\Lambda(G)}(\mathbb{Q}_p, \mathbb{T}_E) &= \epsilon_{\Lambda(G)}(\mathbb{Q}_p, \mathbb{T}_\psi)\epsilon_{\Lambda(G)}(\mathbb{Q}_p, \mathbb{T}_{\psi^c}) \\ &= \Lambda(G) \otimes_{\Lambda(G_\nu)} \left(\epsilon_{\Lambda(G_\nu)}(\mathbb{Q}_p, \mathbb{T}_\psi^{\mathrm{loc}})\epsilon_{\Lambda(G_\nu)}(\mathbb{Q}_p, \mathbb{T}_{\psi^c}^{\mathrm{loc}})\right). \end{aligned}$$

If F is a number field and F_∞ a p-adic Lie extension of F again with Galois group G, then, for a place $\mathfrak{p}$ above p and a projective $\Lambda(G)$-module $\mathbb{T}$ with continuous $G_{F_\mathfrak{p}}$-action, we define a corresponding ϵ-isomorphism

$$\epsilon_{\Lambda(G)}(F_\mathfrak{p}, \mathbb{T}) : \mathbf{1}_{\widetilde{\Lambda(G)}} \to \mathbf{d}_{\Lambda(G)}(R\Gamma(F_\mathfrak{p}, \mathbb{T}))_{\widetilde{\Lambda(G)}} \mathbf{d}_{\Lambda(G)}(\mathbb{T})_{\widetilde{\Lambda(G)}}^{[F_\mathfrak{p}:\mathbb{Q}_p]}$$

to be induced from

$$\begin{aligned} &\epsilon_{\Lambda(G)}(\mathbb{Q}_p, \mathbb{Z}[G_{\mathbb{Q}_p}] \otimes_{\mathbb{Z}[G_{F_\mathfrak{p}}]} \mathbb{T}) \\ &: \mathbf{1}_{\widetilde{\Lambda(G)}} \to \mathbf{d}_{\Lambda(G)}(R\Gamma(\mathbb{Q}_p, \mathbb{Z}[G_{\mathbb{Q}_p}] \otimes_{\mathbb{Z}[G_{F_\mathfrak{p}}]} \mathbb{T}))_{\widetilde{\Lambda(G)}} \mathbf{d}_{\Lambda(G)}(\mathbb{Z}[G_{\mathbb{Q}_p}] \otimes_{\mathbb{Z}[G_{F_\mathfrak{p}}]} \mathbb{T})_{\widetilde{\Lambda(G)}}. \end{aligned}$$

Finally we put

$$\epsilon_\Lambda(F \otimes_{\mathbb{Q}} \mathbb{Q}_p, \mathbb{T}) = \epsilon_\Lambda\left(\mathbb{Q}_p, \bigoplus_{\mathfrak{p}|p} \mathbb{Z}[G_{\mathbb{Q}_p}] \otimes_{\mathbb{Z}[G_{F_\mathfrak{p}}]} \mathbb{T}\right) = \prod_{\mathfrak{p}|p} \epsilon_\Lambda(F_\mathfrak{p}, \mathbb{T}),$$

where $\mathfrak{p}$ runs through the places of F above p.

4. Global functional equation

In this section we would like to explain the applications addressed in the introduction. In the same setting as in Example 3.1 we assume that p is a prime of good ordinary reduction for the CM elliptic curve E and we set $F_\infty = \mathbb{Q}(E(p))$, as well as $G = G(F_\infty/\mathbb{Q})$ and $\Lambda := \Lambda(G)$. We write $M = h^1(E)(1)$ for the motive attached to E and set $\epsilon_{p,\Lambda}(M) = \epsilon_\Lambda(\mathbb{Q}_p, \mathbb{T}_E)$. Using [Yasuda 2009] one obtains similarly ϵ-isomorphisms over $\mathbb{Q}_l$, $l \neq p$, which we call analogously $\epsilon_{l,\Lambda}(M)$. Finally, one can define $\epsilon_{\infty,\Lambda}(M)$ also at the place at infinity; this is done in [Fukaya and Kato 2006, §3.5, Conjecture] and, with a slightly different normalisation, at the end of [Venjakob 2007, §5]. We choose the latter normalisation. Let S be the finite set of places of $\mathbb{Q}$ consisting of p, ∞ as well as the places of bad reduction of M.

Now, according to the conjectures of [Fukaya and Kato 2006] there exists a ζ-isomorphism

$$\zeta_\Lambda(M) := \zeta_\Lambda(\mathbb{T}_E) : \mathbf{1}_\Lambda \to \mathbf{d}_\Lambda(R\Gamma_c(U, \mathbb{T}_E))^{-1}$$

which is the global analogue of the ϵ-isomorphism concerning special L-values (at motivic points in the sense of [Flach 2009]) instead of ϵ- and Γ-factors; here $R\Gamma_c(U, \mathbb{T}_E)$ denotes the perfect complex calculating étale cohomology with compact support of $\mathbb{T}_E$ with respect to $U = \mathrm{Spec}(\mathbb{Z}) \setminus S$. Good evidence for the existence of $\zeta_\Lambda(M)$ is given in (loc. cit.) although Flach concentrates on the commutative case, that is, he considers $\Lambda(G(F_\infty/K))$ instead of $\Lambda(G)$; from this the noncommutative version probably follows by similar techniques as in [Bouganis and Venjakob 2010], but as a detailed discussion would lead us too far away from the topic of this article, we just assume the existence here for simplicity. Then we obtain the following:

Theorem 4.1. *There is the functional equation*

$$\zeta_\Lambda(M) = (\overline{\zeta_\Lambda(M)^*})^{-1} \cdot \prod_{v \in S} \epsilon_{v,\Lambda}(M).$$

This result is motivated by [Fukaya and Kato 2006, Conjecture 3.5.5]; for more details see [Venjakob 2007, Theorem 5.11], and compare with [Burns and Flach 2001, §5]. Observe that we used the self-duality $M = M^*(1)$ of M here.

Finally we want to address the application towards the descent result with Burns mentioned in the introduction. If ω denotes the Neron differential of E, we obtain the usual real and complex periods $\Omega_\pm = \int_{\gamma^\pm} \omega$ by integrating along paths $\gamma^\pm$ which generate $H_1(E(\mathbb{C}), \mathbb{Z})^\pm$. We set $R = \{q \text{ prime} : |j(E)|_q > 1\} \cup \{p\}$ and let u, w be the roots of the characteristic polynomial of the action of Frobenius on the Tate module T_E of E, which is

$$1 - a_p T + pT^2 = (1 - uT)(1 - wT), \quad u \in \mathbb{Z}_p^\times.$$

Further let $p^{\mathfrak{f}_p(\rho)}$ be the p-part of the conductor of an Artin representation ρ, while $P_p(\rho, T) = \det(1 - \mathrm{Frob}_p^{-1} T | V_\rho^{I_p})$ describes the Euler factor of ρ at p. We also set $d^{\pm}(\rho) = \dim_{\mathbb{C}} V_\rho^{\pm}$ and denote by ρ^* the contragredient representation of ρ. By $e_p(\rho)$ we denote the local ϵ-factor of ρ at p. In the notation of [Tate 1979] this is $e_p(\rho, \psi(-x), dx_1)$, where ψ is the additive character of $\mathbb{Q}_p$ defined by $x \to \exp(2\pi i x)$ and dx_1 is the Haar measure that gives volume 1 to $\mathbb{Z}_p$. Moreover, we write $R_\infty(\rho^*)$ and $R_p(\rho^*)$ for the complex and p-adic regulators of E twisted by ρ^*. Finally, in order to express special values of complex L-functions in the p-adic world, we fix embeddings of $\bar{\mathbb{Q}}$ into $\mathbb{C}$ and $\mathbb{C}_p$, the completion of an algebraic closure of $\mathbb{Q}_p$.

In [Bouganis and Venjakob 2010, Theorem 2.14] we have shown that as a consequence of the work of Rubin and Yager there exists $\mathscr{L}_E \in K_1(\Lambda_{\mathbb{Z}_p}(G)_S)$ satisfying the interpolation property

$$\mathscr{L}_E(\rho) = \frac{L_R(E, \rho^*, 1)}{\Omega_+^{d^+(\rho)} \Omega_-^{d^-(\rho)}} e_p(\rho) \frac{P_p(\rho, u^{-1})}{P_p(\rho^*, w^{-1})} u^{-\mathfrak{f}_p(\rho)}$$

for all Artin representations ρ of G. Moreover the (slightly noncommutative) Iwasawa Main Conjecture (see [Coates et al. 2005] or Conjecture 1.4 in [loc. cit.]) is true provided that the $\mathfrak{M}_H(G)$ conjecture (see [Coates et al. 2005] or Conjecture 1.2 in [loc. cit.]) holds; for CM elliptic curves this conjecture is equivalent to the vanishing of the cyclotomic μ-invariant of E. In [Burns and Venjakob 2011, Conjecture 7.4/9 and Proposition 7.8] a refined Main Conjecture was formulated requiring the following p-adic BSD-type formula:

At each Artin representation ρ of G (with coefficients in L) the leading term $\mathscr{L}_E^*(\rho)$ of $\mathscr{L}_E$ (as defined in [Burns and Venjakob 2006]) equals

$$(-1)^{r(E)(\rho^*)} \frac{L_R^*(E, \rho^*) R_p(\rho^*)}{\Omega_+^{d_+(\rho)} \Omega_-^{d_-(\rho)} R_\infty(\rho^*)} e_p(\rho) \frac{P_p(\rho, u^{-1})}{P_p(\rho^*, w^{-1})} u^{-\mathfrak{f}_p(\rho)}, \tag{60}$$

where $L_R^*(E, \rho^*)$ is the leading coefficient at $s = 1$ of the L-function $L_R(E, \rho^*, s)$ obtained from the Hasse–Weil L-function of E twisted by ρ^* by removing the Euler factors at R. Here the number $r(E)(\rho^*)$ is defined in [Burns and Venjakob 2011, (51)] (with $M = h^1(E)(1)$) and equals $\dim_{\mathbb{C}_p}(e_{\rho^*}(\mathbb{C}_p \otimes_{\mathbb{Z}} E(K^{\ker(\rho)})))$ if the Tate–Shafarevich group $Ш(E/F_\infty^{\ker(\rho)})$ is finite.

We write $X(E/F_\infty)$ for the Pontryagin dual of the (p-primary) Selmer group of E over F_∞.

Theorem 4.2. *Let F be a number field contained in F_∞ and assume that*

(i) *the $\mathfrak{M}_H(G)$ conjecture holds*,

(ii) *$\mathscr{L}_E$ satisfies the refined interpolation property* (60), *and*

(iii) $X(E/F_\infty)$ *is semisimple at all* ρ *in* $\operatorname{Irr} G_{F/\mathbb{Q}}$ (in the sense of [Burns and Venjakob 2006, Definition 3.11]).

The p-part of the equivariant Tamagawa number conjecture for $(E, \mathbb{Z}[G(F/\mathbb{Q})])$ *is true in this situation. If, moreover, the Tate–Shafarevich group* $\text{Ш}(E/F)$ *of* E *over* F *is finite, this implies the p-part of a Birch–Swinnerton-Dyer type formula* (see, for example, [Venjakob 2007, §3.1]).

For more details on the "p-part" of the ETNC and the proof of this result, which uses the existence of (1) as shown in this paper, see [Burns and Venjakob 2011, Theorem 8.4]. Note that due to our semisimplicity assumption combined with Remark 7.6 and Proposition 7.8 of [loc. cit.], formula (60) coincides with that of [loc. cit., Conjecture 7.4]. Also Assumption (W) of Theorem 8.4 is valid for weight reasons. Finally we note that by [Burns and Venjakob 2006, Lemma 3.13, 6.7] $X(E/F_\infty)$ is semisimple at ρ if and only if the p-adic height pairing

$$h_p(V_p(E)\otimes\rho^*) : H^1_f(\mathbb{Q}, V_p(E)\otimes\rho^*)\times H^1_f(\mathbb{Q}, V_p(E)\otimes\rho)\to L$$

from [Nekovář 2006, §11] (see also [Schneider 1982] or [Perrin-Riou 1992]) is nondegenerate, where $V_p(E)=\mathbb{Q}_p\otimes T_E$ is the usual p-adic representation attached to E. As far as we are aware, the only theoretical evidence for nondegeneracy is a result in [Bertrand 1982] that for an elliptic curve with complex multiplication, the height of a point of infinite order is nonzero. Computationally, however, a lot of work has been done recently by Stein and Wuthrich [Wuthrich 2004].

Appendix A: p-adic Hodge theory and (φ, Γ)-modules

As before in the local situation K denotes a (finite) unramified extension of $\mathbb{Q}_p$. Let $\eta : G_{\mathbb{Q}_p}\to\mathbb{Z}_p^\times$ (here $\mathbb{Z}_p^\times$ can also be replaced by $\mathcal{O}_L^\times$, but for simplicity of notation we won't do that in this exposition) be an unramified character and let T_0 be the free $\mathbb{Z}_p$-module with basis $t_{\eta,0}$ such that $\sigma\in G_{\mathbb{Q}_p}$ acts via $\sigma t_{\eta,0}=\eta(\sigma)t_{\eta,0}$. More generally, for $r\in\mathbb{Z}$, we consider the $G_{\mathbb{Q}_p}$-module

$$T := T_0(r),$$

which is free as a $\mathbb{Z}_p$-module with basis $t_{\eta,r} := t_{\eta,0}\otimes\epsilon^{\otimes r}$, where $\epsilon=(\epsilon_n)_n$ denotes a fixed generator of $\mathbb{Z}_p(1)$, that is, $\epsilon_n^p=\epsilon_{n-1}$ for all $n\geq 1$, $\epsilon_0=1$ and $\epsilon_1\neq 1$. Thus we have $\sigma(t_{\eta,r})=\eta(\sigma)\kappa^r(\sigma)t_{\eta,r}$, where $\kappa : G_{\mathbb{Q}_p}\to\mathbb{Z}_p^\times$ denotes the p-cyclotomic character. Setting $V := \mathbb{Q}_p\otimes T = V_0(r)$ we obtain for its de Rham filtration

$$D^i_{dR}(V)=\begin{cases} D_{dR}(V)\cong Ke_{\eta,r} & \text{if } i\leq -r,\\ 0 & \text{otherwise,}\end{cases} \tag{61}$$

where $e_{\eta,r} := at^{-r}\otimes t_{\eta,r}$ with a unique $a=a_\eta\in\widehat{\mathbb{Z}_p^{ur}}^\times$, such that $\tau_p(a)=\eta^{-1}(\tau_p)a$, see [Serre 1968, Theorem 1, p. III-31]. Here as usual $t=\log[\epsilon]\in B_{\text{cris}}\subseteq B_{dR}$

denotes the p-adic period analogous to $2\pi i$. Furthermore we have

$$D_{\mathrm{cris}}(V) = K e_{\eta,r}$$

with

$$\varphi(e_{\eta,r}) = p^{-r}\eta^{-1}(\tau_p)e_{\eta,r}.$$

If η is trivial, we also write t_r and e_r for $t_{\eta,r}$ and $e_{\eta,r}$, respectively.

Now consider the $\mathbb{O}_K$-lattices

$$M_0 := \mathbb{O}_K e_{\eta,0} = (\widehat{\mathbb{Z}_p^{ur}} \otimes_{\mathbb{Z}_p} T_0)^{G_K} \subseteq D_{\mathrm{cris}}(V_0)$$

and

$$M := (t^{-r} \otimes \epsilon^{\otimes r})M_0 = \mathbb{O}_K e_{\eta,r} \subseteq D_{\mathrm{cris}}(V).$$

Using the variable $X = [\epsilon] - 1$ we have $t = \log(1 + X)$ and on the rings

$$\mathbb{O}_K[\![X]\!] \subseteq B^+_{\mathrm{rig},K} := \left\{ f(X) = \sum_{k\geq 0} a_k X^k \;\middle|\; a_k \in K,\ f(X) \text{ converges on } \{x \in \mathbb{C}_p \mid |x|_p < 1\} \right\}$$

we have the following operations: φ is induced by the usual action of ϕ on the coefficients and by $\varphi(X) := (1+X)^p - 1$, while $\gamma \in \Gamma$ acts trivially on coefficients and by $\gamma(X) = (1+X)^{\kappa(\gamma)} - 1$; letting $H_K = G(K/\mathbb{Q}_p)$ act just on the coefficients we obtain a $\Lambda(G)$-module structure on $\mathbb{O}_K[\![X]\!]$. Moreover, φ has a left inverse operator ψ uniquely determined via $\varphi \circ \psi(f) = (1/p)\sum_{\zeta^p=1} f(\zeta(1+X)-1)$. The differential operator $D := (1+X)\,d/dX$ satisfies

$$D\varphi f = p\varphi Df \quad \text{and} \quad D\gamma f = \kappa(\gamma)\gamma Df. \tag{62}$$

It is well-known [Perrin-Riou 1994, Lemma 1.1.6] that D induces an isomorphism of $\mathbb{O}_K[\![X]\!]^{\psi=0}$. Furthermore, setting $\Delta_i f := D^i f(0)$ for $f \in \mathbb{O}_K[\![X]\!]^{\psi=0}$, we have an exact sequence [loc. cit., §2.2.7, (2.1)]

$$\begin{aligned} 0 \longrightarrow t^r \otimes D_{\mathrm{cris}}(V)^{\varphi=p^{-r}} \longrightarrow (B^+_{\mathrm{rig},K} \otimes_K D_{\mathrm{cris}}(V))^{\psi=1} \xrightarrow{1-\varphi} \\ (B^+_{\mathrm{rig},K})^{\psi=0} \otimes_K D_{\mathrm{cris}}(V) \xrightarrow{\Delta_r} (D_{\mathrm{cris}}(V)/(1-p^r\varphi))(r) \longrightarrow 0, \end{aligned} \tag{63}$$

where φ (and ψ) acts diagonally on $B^+_{\mathrm{rig},K} \otimes_K D_{\mathrm{cris}}(V)$, while D operates just on the first tensor factor. We set

$$\mathfrak{D}_M := \mathbb{O}_K[\![X]\!]^{\psi=0} \otimes_{\mathbb{O}_K} M,$$

and denote by

$$D(T) = (\mathbb{A} \otimes_{\mathbb{Z}_p} T)^{G_{K_\infty}}$$

the (φ, Γ)-module attached to T, where the definition of the ring $\mathbb{A}$ together with its φ- and Γ-action can be found for example in [Berger 2003]. Here we only recall that $\mathbb{A}_K^+ \cong \mathbb{O}_K[\![X]\!]$ and $\mathbb{A}_K \cong (\mathbb{O}_K[\![X]\!][1/X])^{\wedge p\text{-adic}}$ is the p-adic completion of the Laurent series ring.

Remark A.1. (i) Let η be nontrivial. From [Berger 2003, Theorem A.3] and its proof one sees immediately that for the Wach module $N(T_0)$, which according to Proposition A.1 of [loc. cit.] equals $\mathbb{O}_K[\![X]\!] \otimes_{\mathbb{O}_K} M_0$, the natural inclusion $N(T_0) \hookrightarrow \mathbb{A}_K \otimes_{\mathbb{A}_K^+} N(T_0)$ induces an isomorphism

$$(\mathbb{O}_K[\![X]\!] \otimes_{\mathbb{O}_K} M_0)^{\psi=1} \xrightarrow{\cong} N(T_0)^{\psi=1} \xrightarrow{\cong} \left(\mathbb{A}_K \otimes_{\mathbb{A}_K^+} N(T_0)\right)^{\psi=1} \cong D(T_0)^{\psi=1}.$$

(ii) If η is trivial, one has similarly $N(\mathbb{Z}_p) = \mathbb{A}_K^+ = \mathbb{O}_K[\![X]\!]$ by the same Proposition A.1, whence $N(\mathbb{Z}_p(1)) = X^{-1}\mathbb{A}_K^+ \otimes t_1 = X^{-1}\mathbb{O}_K[\![X]\!] \otimes t_1$ by the usual twist behaviour of Wach modules. We obtain

$$\begin{aligned} D(\mathbb{Z}_p(1))^{\psi=1} \cong N(\mathbb{Z}_p(1))^{\psi=1} &= (X^{-1}\mathbb{O}_K[\![X]\!] \otimes t_1)^{\psi=1} \\ &= \mathbb{Z}_p X^{-1} \otimes t_1 \oplus (\mathbb{O}_K[\![X]\!] \otimes t_1)^{\psi=1}, \end{aligned}$$

but $N(\mathbb{Z}_p)^{\psi=1} \ncong D(\mathbb{Z}_p)^{\psi=1}$ according to Proposition A.3 of [loc. cit.].

We define $\tilde{D}(\mathbb{Z}_p(r))^{\psi=1} = (\mathbb{O}_K[\![X]\!] \otimes t_r)^{\psi=1}$ and $\tilde{D}(T)^{\psi=1} = D(T)^{\psi=1}$ for nontrivial η and obtain a canonical isomorphism

$$(\mathbb{O}_K[\![X]\!] \otimes_{\mathbb{O}_K} M)^{\psi=p^r} \cong \tilde{D}(T)^{\psi=1} \tag{64}$$

induced by multiplication with t^r:

$$f(X) \otimes (oat^{-r} \otimes t_{\eta,r}) \mapsto f(X)oa \otimes t_{\eta,r},$$

where $o \in \mathbb{O}_K$ and a is as before.

Setting $\mathbb{T}_{K_\infty} := \mathbb{T}_{K_\infty}(T) := \Lambda(G(K_\infty/\mathbb{Q}_p))^\sharp \otimes_{\mathbb{Z}_p} T$ we recall that there is a canonical isomorphism due to Fontaine

$$D(T)^{\psi=1} \cong \mathrm{H}^1(\mathbb{Q}_p, \mathbb{T}_{K_\infty}), \tag{65}$$

which for example is called $\{h^1_{K_n,V}\}_n$ in [Berger 2003] and its inverse $\mathrm{Log}^*_{T^*(1)}$ in [Cherbonnier and Colmez 1999, Remark II.1.4].

I am very grateful to Denis Benois for parts of the proof of the following proposition, which has been stated in [Perrin-Riou 1994, Proposition 4.1.3] in a slightly different form, but without proof.[1]

Proposition A.2. (i) *There is a canonical exact sequence of $\mathbb{O}_K$-modules*

$$0 \to 1 \otimes M^{\varphi=p^{-r}} \to (\mathbb{O}_K[\![X]\!] \otimes_{\mathbb{O}_K} M)^{\psi=p^r} \to \mathfrak{D}_M \xrightarrow{\Delta_{M,r}} M/(1-p^r\varphi)M \to 0,$$

where the map in the middle is induced by $1-\varphi$ *up to twisting* (see the first diagram

[1] As twisting with the cyclotomic character starting from $\mathbb{Q}_p(1)$ only recovers the representations $V = \mathbb{Q}_p(r)$, the general case where V_0 is nontrivial is not covered in that reference.

in the proof below).

(ii) *Assume that η is nontrivial. Then, using the isomorphisms* (64) *and* (65) *we obtain the following commutative diagram of $\Lambda(G)$-modules, in which the maps $\mathscr{C}(\mathbb{T}_{K_\infty})$ $(= (D^{-r} \otimes t^{-r})(1-\varphi))$ and $\mathscr{L}_0(\mathbb{T}_{K_\infty})$ are defined by the property that the rows become isomorphic to the exact sequence in* (i):

$$\begin{tikzcd}
0 \arrow[r] & D(T)^{\varphi=1} \arrow[r] \arrow[d, "\cong"] & D(T)^{\psi=1} \arrow[r, "\mathscr{C}(\mathbb{T}_{K_\infty})"] \arrow[d, "\cong", "\mathrm{Log}^*_{T^*(1)}"'] & \mathfrak{D}_M \arrow[r, "\Delta_{M,r}"] \arrow[d, "="] & M/(1-p^r\varphi)M \arrow[r] \arrow[d, "="] & 0 \\
0 \arrow[r] & \mathrm{H}^1(\mathbb{Q}_p, \mathbb{T}_{K_\infty})_{tors} \arrow[r] & \mathrm{H}^1(\mathbb{Q}_p, \mathbb{T}_{K_\infty}) \arrow[r, "\mathscr{L}_0(\mathbb{T}_{K_\infty})"] & \mathfrak{D}_M \arrow[r, "\Delta_{M,r}"] & M/(1-p^r\varphi)M \arrow[r] & 0.
\end{tikzcd}$$

(iii) *The sequence* (15) *can be interpreted in terms of (φ, Γ)-modules by the commutative diagram*

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{U}(K_\infty) \arrow[r, "\delta"] \arrow[d, "\mathrm{Dlog}\, g_-", "\cong"'] & \mathrm{H}^1(\mathbb{Q}_p, \mathbb{T}_{un}) \cong \varprojlim_n \widehat{K_n^\times} \arrow[r, "-\hat{v}"] \arrow[d, "\cong"'] & \mathbb{Z}_p \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \tilde{D}(\mathbb{Z}_p(1))^{\psi=1} \arrow[r] & D(\mathbb{Z}_p(1))^{\psi=1} \arrow[r] & \mathbb{Z}_p \arrow[r] & 0,
\end{tikzcd}$$

where δ denotes the Kummer map and $\hat{v}$ is induced from the normalised valuation map. Furthermore, we obtain again a commutative diagram of $\Lambda(G)$-modules, in which the maps $\mathscr{C}(\mathbb{T}_{K_\infty})$ $(= (D^{-r} \otimes t^{-r})(1-\varphi))$ and $\mathscr{L}_0(\mathbb{T}_{K_\infty})$ are defined by the property that the rows become isomorphic to the exact sequence in (i):

$$\begin{tikzcd}
0 \arrow[r] & \tilde{D}(\mathbb{Z}_p(r))^{\varphi=1} \arrow[r] & \tilde{D}(\mathbb{Z}_p(r))^{\psi=1} \arrow[r, "\mathscr{C}(\mathbb{T}_{K_\infty})"] & \mathfrak{D}_M \arrow[r, "\Delta_{M,r}"] & M/(1-p^r\varphi)M \arrow[r] & 0 \\
0 \arrow[r] & \mathbb{Z}_p(r) \arrow[r] \arrow[u, "\cong"] & \mathbb{U}(K_\infty)(r-1) \arrow[r, "\mathscr{L}_0(\mathbb{T}_{K_\infty})"] \arrow[u, "\mathrm{Dlog}\, g_-", "\cong"'] & \mathfrak{D}_M \arrow[r, "\Delta_{M,r}"] \arrow[u, equal] & \mathbb{Z}_p(r) \arrow[r] \arrow[u, "\cong"'] & 0.
\end{tikzcd}$$

Hence, using the map $\mathfrak{M} \otimes e_1 : \mathbb{O}_K[\![\Gamma]\!] \xrightarrow{\cong} \mathfrak{D}_M$, $\lambda \mapsto \lambda \cdot (1+X) \otimes e_1$, *where $\mathfrak{M}$ denotes the Mahler* (*or p-adic Mellin*) *transform* (see [Coates and Sujatha 2006, Theorem 3.3.3]), *the lower sequence can be canonically identified with Coleman's exact sequence* (4): $\mathscr{L}_0(\mathbb{T}_{K_\infty}) = (\mathfrak{M} \otimes e_1) \circ \mathrm{Col}_\epsilon$.

Proof. The exactness in (i) for M_0 can be checked as follows. Let $f(X) \otimes e_{\eta,0}$ be in $\mathfrak{D}_{M_0}{}^{\Delta_{M_0,0}=0}$, that is, $f(0)e_{\eta,0} = (1-\varphi)b$ for some $b \in M_0$. Hence

$$(f(X) - f(0)) \otimes e_{\eta,0} = Xg(X) \otimes e_{\eta,0}$$

for some $g \in \mathbb{O}_K[\![X]\!]$ and

$$F' := (1-\varphi)^{-1}(Xg(X) \otimes e_{\eta,0}) := \sum_{i \geq 0} \varphi^i (Xg(X) \otimes e_{\eta,0}) \in \mathbb{O}_K[\![X]\!] \otimes M_0$$

is a well-defined element. Setting $F := F' + b$ we have $(1-\varphi)F = f(X) \otimes e_{\eta,0}$ as desired. Now exactness follows from (63). The general case follows from the following commutative "twist diagram" of $\mathbb{O}_K$-modules:

$$\begin{tikzcd} 0 \arrow[r] & 1 \otimes M^{\varphi=p^{-r}} \arrow[r] \arrow[d, "1\otimes(t^r\otimes\epsilon^{\otimes -r})"', "\cong"] & (\mathbb{O}_K[[X]] \otimes_{\mathbb{O}_K} M)^{\psi=p^r} \arrow[r] \arrow[d, "1\otimes(t^r\otimes\epsilon^{\otimes -r})"', "\cong"] & \mathfrak{D}_M \arrow[r, "\Delta_{M,r}"] \arrow[d, "D^r\otimes(t^r\otimes\epsilon^{\otimes -r})"', "\cong"] & M/(1-p^r\varphi)M \arrow[r] \arrow[d, "t^r\otimes\epsilon^{\otimes -r}"', "\cong"] & 0 \\ 0 \arrow[r] & 1 \otimes M_0^{\varphi=1} \arrow[r] & (\mathbb{O}_K[[X]] \otimes_{\mathbb{O}_K} M_0)^{\psi=1} \arrow[r, "1-\varphi"] & \mathfrak{D}_{M_0} \arrow[r, "\Delta_{M_0,0}"] & M_0/(1-p^r\varphi)M_0 \arrow[r] & 0. \end{tikzcd}$$

Item (ii) is clear from the fact that $D(T) = \mathbb{A}_K \cdot a \otimes t_{\eta,r}$, which can either be calculated directly or deduced from the above remark. The statement about the torsion (first vertical isomorphism) follows from [Colmez 2004, Theorem 5.3.15].

For (iii) first note that by [Cherbonnier and Colmez 1999, Proposition V.3.2(iii)] we have a commutative diagram

$$\begin{tikzcd} \mathbb{U}(K_\infty) \arrow[r, "\Upsilon"] \arrow[dr, "\delta"'] & D(\mathbb{Z}_p(1))^{\psi=1} \arrow[r, equal] \arrow[d, "\mathrm{Log}^*_{\mathbb{Q}_p}"] & D(\mathbb{Z}_p)^{\psi=1}(1) \\ & \mathrm{H}^1(\mathbb{Q}_p, \mathbb{T}_{K_\infty}(\mathbb{Z}_p(1))), & \end{tikzcd}$$

where Υ maps u to $(Dg_u/g_u) \otimes t_1 = \mathrm{Dlog}\, g_u \otimes t_1$. The statements concerning the first diagram follow easily, see also [Colmez 2004, §7.2]. The second diagram follows as above. By construction the composite

$$\mathbb{U}(K_\infty) \to D(\mathbb{Z}_p(1))^{\psi=1} \to \mathfrak{D}_M$$

maps $u = (u_n)_n$ to

$$\begin{aligned} \big(D^{-1}(1-\varphi)\,\mathrm{Dlog}\, g_u\big) \otimes e_1 &= \big((1-p^{-1}\varphi)\log g_u\big) \otimes e_1 \\ &= (1-\varphi)\,(\log g_u \otimes e_1) \\ &= \mathcal{L}(g_u) \otimes e_1 \\ &= \mathrm{Col}(u) \cdot (1+X) \otimes e_1, \end{aligned}$$

where $\mathcal{L}$ was defined in (5). This implies the last statement. $\square$

Now let K be again a finite extension of degree d_K over $\mathbb{Q}_p$. For a uniform treatment we define

$$\tilde{\mathrm{H}}^1(\mathbb{Q}_p, \mathbb{T}_{K_\infty}(T)) := \begin{cases} \mathrm{H}^1(\mathbb{Q}_p, \mathbb{T}_{K_\infty}(T)) & \text{if } \eta \neq \mathbb{1}, \\ \mathbb{U}(K_\infty)(r-1) & \text{if } T = \mathbb{Z}_p(r). \end{cases}$$

Now set

$$\mathcal{H}_M := \{F \in B^+_{\mathrm{rig},K} \otimes_{\mathbb{O}_K} M \mid (1-\varphi)f \in \mathfrak{D}_M\}.$$

Using [Berger 2003, Theorem II.11] and the commutativity of the diagram

$$\begin{tikzcd}
\mathcal{H}_M \arrow[r, "1-\varphi"] \arrow[d, "D^r\otimes(t^r\otimes\epsilon^{\otimes -r})"'] & \mathcal{D}_M^{\Delta_{M,r}=0} \arrow[d, "D^r\otimes(t^r\otimes\epsilon^{\otimes -r})"] \\
(\mathcal{O}_K[\![X]\!]\otimes_{\mathcal{O}_K} M_0)^{\psi=1} \arrow[r, "1-\varphi"] & \mathcal{D}_{M_0}^{\Delta_{M_0,0}=0}
\end{tikzcd}$$

we see that the map $\mathcal{L}_0(\mathbb{T}_{K_\infty})$ coincides with the "inverse" of Perrin-Riou's [1999] large exponential map

$$\Omega_{T,r}:\mathcal{D}_M^{\Delta_{M,r}=0}\to D(T)^{\psi=1}/T^{H_K}\quad(\cong \mathrm{H}^1(\mathbb{Q}_p,\mathbb{T}_{K_\infty})/T^{H_K}),$$

(which is $(-1)^{r-1}$ times the one in [Perrin-Riou 1994]). This map sends f to $(D^r\otimes t^r)F$, where $F\in\mathcal{H}_M$ satisfies $(1-\varphi)F=f$. Here "$D^r\otimes t^r$" denotes the composite

$$\mathcal{H}_M\xrightarrow{D^r\otimes(t^r\otimes\epsilon^{\otimes -r})}(\mathcal{O}_K[\![X]\!]\otimes_{\mathcal{O}_K}M_0)^{\psi=1}\xrightarrow{1\otimes(t^{-r}\otimes\epsilon^{\otimes r})}(\mathcal{O}_K[\![X]\!]\otimes_{\mathcal{O}_K}M)^{\psi=p^r}\xrightarrow{t^r}D(T)^{\psi=1}$$

and corresponds to the operator $\nabla_{r-1}\circ\cdots\circ\nabla_0$ in [Berger 2003] for $r\geq 1$. In particular, by Theorem II.10/13 of the same reference we obtain the following descent diagram for $r,n\geq 1$, where the maps $\Xi_{M,n}=\Xi^{\epsilon}_{M,n}$ are recalled in (71):

$$\begin{tikzcd}
\tilde{\mathrm{H}}^1(\mathbb{Q}_p,\mathbb{T}_{K_\infty}(T)) \arrow[r, "\mathcal{L}_0(\mathbb{T}_{K_\infty}(T))"] \arrow[d, "\mathrm{pr}_n"'] & \mathcal{D}_M \arrow[d, "\Xi_{M,n}"] \\
\mathrm{H}^1(K_n,V) & K_n\cong D_{dR,K_n}(V), \arrow[l, "(-1)^{r-1}(r-1)!\exp_{K_n,V}"']
\end{tikzcd}\tag{66}$$

while for $r\leq 0$

$$\begin{tikzcd}
\tilde{\mathrm{H}}^1(\mathbb{Q}_p,\mathbb{T}_{K_\infty}(T)) \arrow[r, "\mathcal{L}_0(\mathbb{T}_{K_\infty}(T))"] \arrow[d, "\mathrm{pr}_n"'] & \mathcal{D}_M \arrow[d, "\Xi_{M,n}"] \\
\mathrm{H}^1(K_n,V) \arrow[r, "(-r)!\exp^*_{K_n,V^*(1)}"] & K_n\cong D_{dR,K_n}(V).
\end{tikzcd}\tag{67}$$

Remark A.3. In particular, for $T=\mathbb{Z}_p(1)$ we have the following commutative descent diagram for $n\geq 1$:

$$\begin{tikzcd}
\mathbb{U}(K_\infty) \arrow[rr, "\mathcal{L}_0(\mathbb{T}_{K_\infty}(\mathbb{Z}_p(1)))"] \arrow[d, "\mathrm{pr}_n"'] & & \mathfrak{D}_M \arrow[d, "\Xi_n"] \\
\mathbb{Q}_p \otimes_{\mathbb{Z}_p} U_n \arrow[dr, "\delta"] & & K_n \cong D_{dR,K_n}(\mathbb{Q}_p(1)), \arrow[ll, "\exp"'] \arrow[dl, "\exp_{K_n,\mathbb{Q}_p(1)}"'] \\
& \mathrm{H}^1(K_n, \mathbb{Q}_p(1)) &
\end{tikzcd}$$

where exp denotes the usual p-adic exponential (series), while Ξ_n maps the element $((1-p^{-1}\varphi)\log g_u)\otimes e_1$ to $\log g_u^{\phi^{-n}}(\epsilon_n - 1) = \log u_n$.

In order to arrive at a morphism

$$\mathcal{L}(\mathbb{T}_{K_\infty}(T)) : \tilde{\mathrm{H}}^1(\mathbb{Q}_p, \mathbb{T}_{K_\infty}(T)) \to \mathbb{T}_{K_\infty}(T) \otimes_\Lambda \Lambda_{[\mathbb{T}(T),\tau_p]^{-1}},$$

where $[\mathbb{T}, \tau_p]^{-1} = \tau_p \eta^{-1}(\tau_p)$, generalising $\mathcal{L}_{K,\epsilon}$ in (6), we compose $\mathcal{L}_0(\mathbb{T}_{K_\infty}(T))$ with the following canonical isomorphisms:

$$\mathfrak{D}_M = \mathcal{O}_K[\![X]\!]^{\psi=0} \otimes_{\mathcal{O}_K} M \xleftarrow[\cong]{\Psi_M} \mathcal{O}_K[\![\Gamma]\!] \otimes_{\mathcal{O}_K} M \xrightarrow[\cong]{\Theta_M} \mathbb{T}_{K_\infty} \otimes_\Lambda \Lambda_{[\mathbb{T},\tau_p]^{-1}}, \quad (68)$$

where the left one, $\Psi_M(\lambda \otimes m) = \lambda \cdot (1+X) \otimes m$, is induced by $\mathfrak{M}$, while the right one is given by

$$\begin{aligned}
\Theta_M(\lambda \otimes (at^{-r} \otimes t_{\eta,r})) &= (1 \otimes t_{\eta,r}) \otimes \left(\sum_{i=0}^{d_K-1} \tau_p^i \otimes \eta^{-i}(\tau_p)\phi^{-i}(\lambda a) \right) \\
&= (1 \otimes t_{\eta,r}) \otimes \left(\sum_i \tau_p^i \otimes \phi^{-i}(\lambda) a \right).
\end{aligned}$$

Similarly to the original Coleman map Col in (4), the homomorphisms $\mathcal{C}(\mathbb{T}_{K_\infty})$, $\mathcal{L}_0(\mathbb{T}_{K_\infty})$ and $\mathcal{L}(\mathbb{T}_{K_\infty})$ are norm compatible when enlarging K within $\mathbb{Q}_p^{ur}$. Thus, by taking inverse limits we may and do define them also for infinite unramified extensions K of $\mathbb{Q}_p$. Then we have the following twist and descent properties:

Lemma A.4. *Let $K' \subseteq K$ be (possibly infinite) unramified extensions of $\mathbb{Q}_p$ and Y a $(\Lambda(G(K'_\infty/\mathbb{Q}_p)), \Lambda(G(K_\infty/\mathbb{Q}_p))$-module such that $Y \otimes_{\Lambda(G(K_\infty/\mathbb{Q}_p))} \mathbb{T}_{K_\infty}(T) \cong \mathbb{T}_{K'_\infty}(T')$ as $\Lambda(G(K'_\infty/\mathbb{Q}_p))$-modules with compatible $G_{\mathbb{Q}_p}$-action. Then*

$$Y \otimes_{\Lambda(G(K_\infty/\mathbb{Q}_p))} \mathcal{L}_0(\mathbb{T}_{K_\infty}(T)) = \mathcal{L}_0(\mathbb{T}_{K'_\infty}(T'))$$

and

$$Y \otimes_{\Lambda(G(K_\infty/\mathbb{Q}_p))} \mathcal{L}(\mathbb{T}_{K_\infty}(T)) = \mathcal{L}(\mathbb{T}_{K'_\infty}(T')).$$

In particular, $\mathcal{L}(\mathbb{T}_{K'_\infty}(T)) = \mathcal{L}_{\mathbb{T}_{K'_\infty}(T),\epsilon}$ in (43).

Proof. The proof can be divided into a twist statement, where $K' = K$ and $T' \cong T \otimes_{\mathbb{Z}_p} T''$, such that $G_{\mathbb{Q}_p}$ acts diagonally on the tensor product and T' is a rank-one $\mathbb{Z}_p$-representation of G, and a descent statement. One first proves the twist statement for T''/p^n, n fix, and all finite subextensions K' of K, such that $G(K/K')$ acts trivially on T''/p^n. Afterwards one takes limits over K' obtaining the twist statement for T''/p^n. Then, taking the projective limit with respect to n (see [Berger 2004] for the correct behaviour of (φ, Γ)-modules under such limits) one shows the full twist statement (compare with the well-known twisting for H^i_{IW}). The descent statement then follows easily from the norm compatibility and the fact that the twisted analogue of the exact sequence (10)

$$0 \to \tilde{\mathrm{H}}^1(\mathbb{Q}_p, \mathbb{T}_{K_\infty}(T)) \xrightarrow{\mathcal{L}(\mathbb{T}_{K_\infty})} \mathbb{T}_{K_\infty}(T) \otimes_\Lambda \Lambda_{[\mathbb{T}(T),\tau_p]^{-1}} \longrightarrow T \to 0$$

recovers (for finite extension K' of $\mathbb{Q}_p$) the exact sequence

$$0 \to T^{G(K/K')} \to \tilde{\mathrm{H}}^1(\mathbb{Q}_p, \mathbb{T}_{K'_\infty}(T)) \xrightarrow{\mathcal{L}(\mathbb{T}_{K'_\infty})} \mathbb{T}_{K'_\infty}(T) \otimes_\Lambda \Lambda_{[\mathbb{T}(T),\tau_p]^{-1}} \to T \to 0$$

by taking $G(K/K')$-coinvariants. We explain the unramified twist in more detail (the cyclotomic twist being well known): Assume that η factorises over $G(K/\mathbb{Q}_p)$, that is, $a = a_\eta \in \mathcal{O}_K^\times$, and let $N := \mathcal{O}_K e_r \subseteq D_{\mathrm{cris}}(\mathbb{Q}_p(r))$ be the lattice associated to $\mathbb{Q}_p(r)$. Then we have the following commutative diagram of Λ-modules:

$$\begin{tikzcd}
\big(\mathcal{O}_K[\![X]\!]^{\psi=0} \otimes_{\mathcal{O}_K} N\big) \otimes_{\mathbb{Z}_p} T_0 \arrow[d, "a^{-1}\otimes a \otimes 1"'] & (\mathcal{O}_K[\![\Gamma]\!] \otimes N) \otimes_{\mathbb{Z}_p} T_0 \arrow[l, "\Psi_N \otimes T_0"'] \arrow[r, "\Theta_N \otimes T_0"] \arrow[d, "a^{-1}\otimes a \otimes 1"'] & \Lambda \otimes_{\Lambda, f} \mathbb{T}(\mathbb{Z}_p(r)) \otimes \Lambda_{\tau_p} \arrow[d, "\vartheta \otimes \tilde{f}"] \\
\mathcal{O}_K[\![X]\!]^{\psi=0} \otimes_{\mathcal{O}_K} M & \mathcal{O}_K[\![\Gamma]\!] \otimes M \arrow[l, "\Psi_M"'] \arrow[r, "\Theta_M"] & \mathbb{T}(T) \otimes_\Lambda \Lambda_{\tau_p \eta(\tau_p)^{-1}},
\end{tikzcd}$$

where in the top line the Λ-action is induced by the diagonal G-action and via left multiplication on Λ, respectively,

$$\Theta_N \otimes T_0(\lambda \otimes (t^{-r} \otimes t_r) \otimes t_{\eta,0}) = 1 \otimes 1 \otimes t_r \otimes \sum_i \tau_p^i \otimes \phi^{-i}(\lambda)$$

and $\tilde{f} := f \otimes 1$ on $\Lambda \widehat{\otimes} \widehat{\mathbb{Z}_p^{ur}}$ is induced by $f : \Lambda \to \Lambda$, $g \mapsto \eta(g)^{-1} g$, while

$$\vartheta : \Lambda \otimes_{\Lambda, f} \mathbb{T}(\mathbb{Z}_p(r)) \to \mathbb{T}(T), \quad a \otimes (b \otimes t_r) \mapsto a f(b) \otimes t_{\eta,r}.$$

Here $\Lambda \otimes_{\Lambda, f} -$ indicates that the tensor product is formed with respect to f. Also we have the commutative diagram

$$\begin{tikzcd}
D(\mathbb{Z}_p(r))^{\psi=1} \otimes T_0 \arrow[r, "\mathcal{C}(\mathbb{T}_{K_\infty}(\mathbb{Z}_p(r)))"] \arrow[d, "\cong"'] & \big(\mathcal{O}_K[\![X]\!]^{\psi=0} \otimes_{\mathcal{O}_K} N\big) \otimes_{\mathbb{Z}_p} T_0 \arrow[d, "a^{-1}"] \\
D(T)^{\psi=1} \arrow[r, "\mathcal{C}(\mathbb{T}_{K_\infty}(T))"] & \mathcal{O}_K[\![X]\!]^{\psi=0} \otimes_{\mathcal{O}_K} M.
\end{tikzcd}$$

□

As on page 2386 we set $\Lambda' = \mathbb{Q}_p[G_n]$.

Lemma A.5. *There are natural isomorphisms*

(i) $\Sigma_{M,n} : K'_n \otimes M = K'_n(at^{-r} \otimes t_{r,\eta}) \cong D_{dR}(V')$ *of* Λ'*-modules*;

(ii) $1 \otimes \Sigma_{M,n} : V_{\rho^*} \otimes_{\Lambda'} K'_n \otimes M \cong V_{\rho^*} \otimes_{\Lambda'} D_{dR}(V') \cong D_{dR}(W)$ *of* L*-vector spaces.*

Proof. The canonical isomorphism (which makes explicit the general formula $(\mathrm{Ind}_G^H(B \otimes V)) \cong (B \otimes \mathrm{Ind} V))$

$$\mathbb{Q}_p[G_{\mathbb{Q}_p}] \otimes_{\mathbb{Q}_p[G_{K'_n}]} \big(B_{dR} \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(\eta)(r)\big) \cong B_{dR} \otimes_{\mathbb{Q}_p} \mathbb{Q}_p[G_n]^\sharp \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(\eta)(r),$$

which maps $g \otimes a \otimes b$ to $ga \otimes \bar{g}^{-1} \otimes gb$ with $g \in G_{\mathbb{Q}_p}$ induces the isomorphism (via the general isomorphism $N^H \cong (\mathrm{Ind}_G^H N)^G$, $n \mapsto \sum_{\bar{g} \in G/H} g \otimes n$)

$$K'_n \cdot (at^{-r} \otimes t_{r,\eta}) = \big(B_{dR} \otimes \mathbb{Q}_p(\eta)(r)\big)^{G_{K'_n}} \cong D_{dR}(V'),$$

which maps $x \cdot at^{-r} \otimes t_{r,\eta}$ to

$$\sum_{g \in G_n} g(xat^{-r}) \otimes g^{-1} \otimes gt_{r,\eta} = \sum_{g \in G_n} g(x)at^{-r} \otimes g^{-1} \otimes t_{r,\eta}. \tag{69}$$

Putting $e_{\eta,r} := at^{-r} \otimes t_{r,\eta}$ we similarly obtain the isomorphism in (ii) sending $l \otimes x \otimes e_{\eta,r}$ to

$$\sum_{g \in G_n} g(x)at^{-r} \otimes \rho(g)l \otimes t_{r,\eta},$$

where this element is regarded in $B_{dR} \otimes_{\mathbb{Q}_p} W = B_{dR} \otimes_{\mathbb{Q}_p} L \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(\eta)(r)$. Alternatively we can read it in $(B_{dR} \otimes_{\mathbb{Q}_p} L) \otimes_L W$ as

$$\# G_n at^{-r} e_{\rho^*}(x) l \otimes t_{\rho\eta,r}. \qquad \square$$

Any embedding $\sigma : L_\rho \to \overline{\mathbb{Q}}_p$ induces a map $A_\rho := \mathbb{Q}_p^{nr} \otimes_{\mathbb{Q}_p} L_\rho \to \overline{\mathbb{Q}}_p$ taking $x \otimes y$ to $x\sigma(y)$; we still call this map σ.

Consider the Weil group $W(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$, which fits into a short exact sequence

$$1 \to I \longrightarrow W(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \xrightarrow{\ v\ } \mathbb{Z} \to 0,$$

and let D be the linearised $W(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$-module associated to $D_{pst}(W) = A_\rho e_{\eta,r}(\rho)$, that is, $g \in W(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ acts as $g_{\mathrm{old}}\varphi^{-v(g)}$ or explicitly via the character

$$\chi_D(g) := \rho(g)\eta(\tau_p)^{v(g)} p^{rv(g)}.$$

For an embedding σ we write $\bar{D}_\sigma := \overline{\mathbb{Q}}_p \otimes_{A_\rho,\sigma} D \cong \overline{\mathbb{Q}}_p e_{\eta,r}(\rho^\sigma)$, where σ acts coefficientwise on ρ. If $n \geq 0$ is minimal with the property that $G(\overline{\mathbb{Q}}_p/\mathbb{Q}_p^{nr}(\mu(p^n)))$

acts trivially on $\bar{D}_\sigma$, then by properties (3) and (7) in [Fukaya and Kato 2006, §3.2.2][2] we obtain for the epsilon constant attached to $\bar{D}_\sigma$ (see loc. cit.)

$$\epsilon(\bar{D}_\sigma, -\psi) = 1$$

if $n = 0$, while for $n \geq 1$

$$\begin{aligned}\epsilon(\bar{D}_\sigma, -\psi) &= \epsilon(\bar{D}^*_\sigma(1), \psi)^{-1} \\ &= \Big((\rho^\sigma \eta(\tau_p) p^{r-1})^n \sum_{\gamma\in\Gamma_n} \rho^\sigma(\gamma)\gamma \cdot \epsilon_n\Big)^{-1} \\ &= \big((\rho^\sigma \eta(\tau_p) p^{r-1})^n \tau(\rho^\sigma, \epsilon_n)\big)^{-1}.\end{aligned}$$

Here $\Gamma_n := G(K_n/K)$, $\psi : \mathbb{Q}_p \to \bar{\mathbb{Q}}_p^{\times}$ corresponds to the compatible system $(\epsilon_n)_n$, that is $\psi(1/p^n) = \epsilon_n$, and $\bar{D}^*_\sigma(1)$ denotes the linearised Kummer dual of $\bar{D}_\sigma$, that is,

$$\chi_{\bar{D}^*_\sigma(1)}(g) = \rho^\sigma(g)^{-1} \eta(\tau_p)^{-v(g)} p^{-(r-1)v(g)},$$

while

$$\tau(\rho^\sigma, \epsilon_n) := \sum_{\gamma\in\Gamma_n} \rho^\sigma(\gamma)\gamma \cdot \epsilon_n = \#\Gamma_n e^{\Gamma_n}_{\rho^*} \epsilon_n$$

denotes the usual Gauss sum. Furthermore

$$\epsilon_L(D, -\psi) = \big(\epsilon(\bar{D}_\sigma, -\psi)\big)_\sigma \in \prod_\sigma \mathbb{Q}_p^\times \cong (\overline{\mathbb{Q}}_p \otimes_{\mathbb{Q}_p} L)^\times \subseteq (B_{dR} \otimes_{\mathbb{Q}_p} L)^\times$$

is the ϵ-element as defined in [Fukaya and Kato 2006, §3.3.4]. We may assume that L contains $\mathbb{Q}_p(\mu_{p^n})$; then $\epsilon_L(D, -\psi)$ can be identified with

$$1 \otimes (\rho\eta(\tau_p) p^{r-1})^{-n} \tau(\rho, \epsilon_n)^{-1}.$$

Hence the comparison isomorphism renormalised by $\epsilon_L(D, -\psi)$

$$\epsilon_{L,-\epsilon,dR}(W)^{-1} : W \otimes L_{[W,\tau_p^{-1}]} \to D_{dR}(W) \subseteq B_{dR} \otimes_{\mathbb{Q}_p} L \otimes_L W$$

is explicitly given as

$$x \otimes l \mapsto \epsilon_L(D, -\psi)^{-1} (-t)^r l \otimes x = (-1)^r (\rho\eta(\tau_p) p^{r-1})^n \tau(\rho, \epsilon_n) t^r l \otimes x, \quad (70)$$

[2]Apparently, the formula in §3.2.2 (7) of [Fukaya and Kato 2006] is not compatible with Deligne as claimed: Deligne identifies $W(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ via class field theory with $\mathbb{Q}_p^\times$ by sending the *geometric* Frobenius automorphism to p, which induces, by a standard calculation applied to Definition (3.4.3.2) for epsilon constants of quasicharacters of $\mathbb{Q}_p^\times$ in [Deligne 1973] (see for example [Hida 1993, §8.5 between (4a) and (4b)]), the formula $\epsilon(V_\chi, \psi) = \chi(\tau_p)^{-n} \sum_{\sigma\in\Gamma_n} \chi(\sigma)^{-1}\sigma\epsilon_n$, while in [Fukaya and Kato 2006] the factor is just $\chi(\tau_p)^n$. Here $\chi : W(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to E^\times$ is a character which gives the action on the E-vector space V_χ.

where $\epsilon_L(D,-\psi)^{-1}(-t)^r l$ is considered as an element of $B_{dR}\otimes_{\mathbb{Q}_p} L$.

In order to deduce the descent diagram (44) from (66), for $n\geq 1$, we have to add a commutative diagram of the form

$$\begin{tikzcd} \mathfrak{D}_M \arrow[rr] \arrow[d, "\Xi_{M,n}"'] & & \mathbb{T}_{K_\infty}\otimes_\Lambda \Lambda_{[\mathbb{T},\tau_p]^{-1}} \arrow[d, "Y\otimes_\Lambda -"] \\ K_n\otimes M\cong D_{dR,K_n}(\mathbb{Q}_p(\eta)(r))/D_{\mathrm{cris}}(\mathbb{Q}_p(\eta)(r))^{\varphi=1} & & V'\otimes_{\Lambda'}(\Lambda')_{[V',\tau_p^{-1}]}, \arrow[ll] \end{tikzcd}$$

where

$$\Xi_{M,n}(f)=\Xi^{\epsilon}_{M,n}(f)=p^{-n}(\phi\otimes\varphi)^{-n}\,(F)\,(\epsilon_n-1)=p^{-n}(\varphi\otimes\varphi)^{-n}(F)(0),\quad (71)$$

with $F\in\mathcal{H}_M$ such that $(1-\varphi)F=f=\tilde f\otimes e_{\eta,r}$ (recall that φ acts as $\varphi\otimes\varphi$ here) on ${\mathfrak{D}_M}^{\Delta=0}$; and more generally we have, mod $D_{\mathrm{cris}}(\mathbb{Q}_p(\eta)(r))^{\varphi=1}$ (recalling that $D_{\mathrm{cris}}(\mathbb{Q}_p(\eta)(r))^{\varphi=1}=0$ in the generic case),

$$\begin{aligned}\Xi_{M,n}(f)&=p^{-n}\bigg(\sum_{k=1}^{n}(\phi\otimes\varphi)^{-k}\,(f(\epsilon_k-1))+(1-\phi\otimes\varphi)^{-1}(f(0))\bigg)\\ &=p^{-n}\bigg(\sum_{k=1}^{n}p^{kr}\eta(\tau_p)^k\tilde f^{\phi^{-k}}(\epsilon_k-1)+(1-p^{-r}\eta(\tau_p)^{-1}\phi)^{-1}\tilde f(0)\bigg)\otimes e_{\eta,r}\end{aligned}$$

(see [Benois and Berger 2008, Lemma 4.9], where $f(0)$ is considered in $D_{\mathrm{cris}}(V)$ and hence the last summand above equals $(1-\varphi)^{-1}f(0)$ there by the ϕ-linearity of φ). Here, for any $H(X)=\tilde H(X)\otimes e\in B^+_{\mathrm{rig},K}\otimes_{\mathbb{O}_K} M$ we consider $H(\epsilon_k-1)=\tilde H(\epsilon_k-1)\otimes e$, $k\leq n$, as an element in $K_n\otimes_{\mathbb{O}_K}M$, on which $\phi\otimes\varphi$ acts naturally.

First we note that for $n\geq 1$ we have a commutative diagram

$$\begin{tikzcd} \mathfrak{D}_M \arrow[d, "\Xi_{M,n}"] & \mathbb{O}_K[\![\Gamma]\!]\otimes M \arrow[l, "\Psi_M"'] \arrow[r, "\Theta_M"] \arrow[d, "\mathrm{pr}_n\otimes id"'] & \mathbb{T}_{K_\infty}\otimes_\Lambda \Lambda_{[\mathbb{T},\tau_p]^{-1}} \arrow[d, "Y\otimes_\Lambda -"] \\ K_n\otimes_{\mathbb{O}_K} M/D_{\mathrm{cris}}(\mathbb{Q}_p(\eta)(r))^{\varphi=1} & K[\Gamma_n]\otimes M \arrow[l, "\Psi_{M,n}"'] \arrow[r] & V'\otimes_{\Lambda'}(\Lambda')_{[V',\tau_p^{-1}]}, \end{tikzcd}\quad (72)$$

where

$$\begin{aligned}&\Psi_{M,n}(\mu\otimes e_{\eta,r})\\ &\quad=\Psi^{\epsilon}_{M,n}(\mu\otimes e_{\eta,r})\\ &\quad=p^{-n}\bigg(\sum_{k=1}^{n}\epsilon_k^{\phi^{-k}(\mu)}\otimes\varphi^{-k}(e_{\eta,r})+(1-\phi\otimes\varphi)^{-1}(1^{\mu}\otimes e_{\eta,r})\bigg)\\ &\quad=\bigg(\sum_{k=1}^{n}p^{kr-n}\eta(\tau_p)^k\epsilon_k^{\phi^{-k}(\mu)}+p^{-n}(1-p^{-r}\eta(\tau_p)^{-1}\phi)^{-1}(1^{\mu})\bigg)\otimes e_{\eta,r}\qquad (73)\end{aligned}$$

modulo $D_{\mathrm{cris}}(\mathbb{Q}_p(\eta)(r))^{\varphi=1}$. Here ϕ acts coefficientwise on $K[\Gamma_n]$ and 1^μ is the same as the image of μ under the augmentation map $\mathbb{O}_K[\Gamma_n] \to \mathbb{O}_K$.

Proposition A.6. (i) *For $n \geq \max\{1, a(\rho)\}$ and $W \neq \mathbb{Q}_p(1)$, the following diagram is commutative*:

$$\begin{tikzcd}
& V_{\rho^*} \otimes_{\mathbb{Q}_p[G_n]} K[\Gamma_n] \otimes M \arrow[dl, "1\otimes\Psi_{M,n}"'] \arrow[dr, "1\otimes\Theta_{M,n}"] & \\
V_{\rho^*} \otimes_{\mathbb{Q}_p[G_n]} K_n \otimes M \arrow[d, "1\otimes\Sigma_{M,n}"'] & & V_{\rho^*} \otimes_{\mathbb{Q}_p[G_n]} V' \otimes_{\mathbb{Q}_p[G_n]} \Lambda'_{[V',\tau_p^{-1}]} \arrow[d, "\cong"] \\
V_{\rho^*} \otimes_{\mathbb{Q}_p[G_n]} D_{dR}(V') \cong D_{dR}(W) & D_{dR}(W) \arrow[l, "\Phi_W"'] & W \otimes_L L_{[W,\tau_p^{-1}]}, \arrow[l, "(-1)^r \epsilon_{L,-\epsilon,dR}(W)^{-1}"']
\end{tikzcd}$$

where

$$\Phi_W := \begin{cases} \mathrm{id}_{D_{dR}(W)} & \text{if } a(\rho) \neq 0, \\ \dfrac{\det(1-\varphi \mid D_{\mathrm{cris}}(W^*(1)))}{\det(1-\varphi \mid D_{\mathrm{cris}}(W))} & \text{otherwise.} \end{cases}$$

(ii) *For $W \neq \mathbb{Q}_p(1)$ the diagram* (44) *commutes.*

Proof. Let b denote a normal basis of $\mathbb{O}_K$, that is, $\mathbb{O}_K = \mathbb{Z}_p[\overline{H}]b$ with $\overline{H} = G(K/\mathbb{Q}_p)$, which can be lifted from the residue field, K being unramified, and $e := e_{\eta,r}$. Then $1 \otimes b \otimes e = 1 \otimes e_{\rho^*} b \otimes e$ is a basis of $V_{\rho^*} \otimes_{\mathbb{Q}_p[G_n]} K[\Gamma_n] \otimes M$ as L-vector space (in general $\rho(g)$ does not lie in K, but using $V_{\rho^*} \otimes_{\mathbb{Q}_p[G_n]} K[\Gamma_n] \cong V_{\rho^*} \otimes_{L[G_n]} L[G_n] \otimes_{\mathbb{Q}_p[G_n]} K[\Gamma_n] \cong V_{\rho^*} \otimes_{L[G_n]} (L \otimes_{\mathbb{Q}_p} K[\Gamma_n])$ one can make sense of it). We calculate (going clockwise in the above diagram)

$$\begin{aligned}
&1 \otimes \Theta_{M,n}(1 \otimes b \otimes e) \\
&\quad = 1 \otimes (1 \otimes t_{\eta,r}) \otimes \sum_{i=0}^{d_K-1} \tau_p^i \otimes \phi^{-i}(b)a \quad (\subseteq V_{\rho^*} \otimes_{\mathbb{Q}_p[G_n]} V' \otimes_{\mathbb{Q}_p[G_n]} \Lambda'_{[V',\tau_p^{-1}]}) \\
&\quad = t_{\rho\eta,r} \otimes \sum_{i=0}^{d_K-1} \rho(\tau_p)^{-i} \rho^*(\phi^{-i}(b))a \qquad (\subseteq W \otimes_L L_{[W,\tau_p^{-1}]}) \\
&\quad = t_{\rho\eta,r} \otimes \sum_{i=0}^{d_K-1} \rho(\tau_p)^{-i} \phi^{-i}(b)a \\
&\quad = t_{\rho\eta,r} \otimes \varsigma(\rho, b)a,
\end{aligned}$$

with

$$\varsigma(\rho, b) := \sum_{i=0}^{d_K-1} \rho(\tau_p)^{-i} \phi^{-i}(b) = d_K e_{\rho^*}^{\overline{H}} b$$

a Gauss-like sum, where $e_{\rho^*}^{\overline{H}}=\dfrac{1}{\#\overline{H}}\sum_{h\in H}\rho(h)h$. The image of this element under $(-1)^r\epsilon_{L,-\epsilon,dR}(W)$ is

$$(-1)^r\epsilon_L(D,-\psi)^{-1}(-t)^{-r}\varsigma(\rho,b)a\otimes t_{\rho\eta,r}\\ =p^{mr-m}(\rho\eta)(\tau_p^m)\tau(\rho,\epsilon_m)\varsigma(\rho,b)at^{-r}\otimes t_{\rho\eta,r}\quad (74)$$

in $D_{dR}(W)$, where we used (70) with $m=a(\rho)$.

Now we determine the image of $1\otimes b\otimes e=1\otimes e_{\rho^*}b\otimes e$ anticlockwise. First note that the idempotent e_{ρ^*} decomposes as $e_{\rho^*}^{\Gamma_n}\cdot e_{\rho^*}^{\overline{H}}$.

Hence, for $n\geq a(\rho)\geq 1$, where $p^{a(\rho)}$ denotes the conductor of ρ restricted to Γ_n, we have

$$\begin{aligned}(1\otimes\Psi_{M,n})(1\otimes b\otimes e)&=1\otimes e_{\rho^*}\Psi_{M,n}(b\otimes e)\\ &=1\otimes p^{nr-n}\eta(\tau_p)^n\phi^{-n}(e_{\rho^*}^{\overline{H}}b)e_{\rho^*}^{\Gamma_n}\cdot\epsilon_n\otimes e\\ &=1\otimes p^{nr-n}\eta(\tau_p)^n\rho^*(\tau_p^{-n})e_{\rho^*}^{\overline{H}}be_{\rho^*}^{\Gamma_n}\cdot\epsilon_n\otimes e\\ &=1\otimes\frac{p^{nr-n}}{\#G_n}(\rho\eta)(\tau_p^n)\varsigma(\rho,b)\tau(\rho,\epsilon_n)\otimes e,\end{aligned}$$

where we have used the explicit formula (73) and the following fact about Gauss sums, valid for $k\leq n$ (see for example [Burns and Flach 2006, Lemma 5.2]):

$$e_\rho^{\Gamma_n}(\epsilon_k)=\begin{cases}e_\rho^{\Gamma_n}(\epsilon_k) & \text{if } a(\rho)=k,\\ (1-p)^{-1} & \text{if } a(\rho)=0 \text{ and } k=1,\\ 0 & \text{otherwise.}\end{cases}$$

Now from the end of the proof of Lemma A.5 we see that $\Sigma_{M,n}$ sends this element, which already "lies in the right eigenspace" to

$$at^{-r}p^{nr-n}(\rho\eta)(\tau_p^n)\tau(\rho,\epsilon_n)\varsigma(\rho,b)\otimes t_{\rho\eta,r}\\ =p^{nr-n}(\rho\eta)(\tau_p^n)\tau(\rho,\epsilon_n)\varsigma(\rho,b)at^{-r}\otimes t_{\rho\eta,r},$$

that is, to the same element as in (74), whence the result follows if $a(\rho)\neq 0$.

Now assume that $a(\rho)=0$, that is, $\rho\mid\Gamma_n$, the restriction to Γ_n, is trivial. Setting $n=1$ we then have

$$\begin{aligned}&(1\otimes\Psi_{M,1})(1\otimes b\otimes e)\\ &\quad=1\otimes\Psi_{M,1}(e_{\rho^*}b\otimes e)\\ &\quad=1\otimes\big(p^{r-1}\eta(\tau_p)\epsilon_1^{\phi^{-1}(e_{\rho^*}b)}+p^{-1}(1-p^{-r}\eta(\tau_p)^{-1}\phi)^{-1}(e_{\rho^*}^{\overline{H}}b)\big)\otimes e\\ &\quad=1\otimes\big(p^{r-1}\eta(\tau_p)\phi^{-1}(e_{\rho^*}^{\overline{H}}b)e_{\rho^*}^{\Gamma_1}\cdot\epsilon_1+p^{-1}(1-p^{-r}\rho\eta(\tau_p)^{-1})^{-1}(e_{\rho^*}^{\overline{H}}b)\big)\otimes e\\ &\quad=1\otimes\big(p^{r-1}\rho\eta(\tau_p)(1-p)^{-1}+p^{-1}(1-p^{-r}\rho\eta(\tau_p)^{-1})^{-1}\big)\,\frac{\varsigma(\rho,b)}{d_K}\otimes e\end{aligned}$$

$$= 1 \otimes \left(\frac{1 - p^{r-1} \rho\eta(\tau_p)}{1 - p^{-r} \rho\eta(\tau_p^{-1})} \right) \frac{\varsigma(\rho, b)}{d_K(p-1)} \otimes e,$$

which is sent under $\Sigma_{M,1}$ to

$$\left(\frac{\det(1-\varphi \mid D_{\mathrm{cris}}(W^*(1)))}{\det(1-\varphi \mid D_{\mathrm{cris}}(W))} \right) \varsigma(\rho, b) a t^{-r} \otimes t_{\rho\eta,r},$$

while (74) becomes just

$$\varsigma(\rho, b) a t^{-r} \otimes t_{\rho\eta,r}.$$

Upon replacing ϵ by $-\epsilon = \epsilon^{-1}$ (we have used both the additive and multiplicative notation!) the second statement follows from (66), (72) and the diagram in part (i) of the proposition, combined with the isomorphism (68) and Lemma A.4. □

Appendix B: Determinant functors

In this appendix we recall some details of the formalism of determinant functors introduced in [Fukaya and Kato 2006] (see also [Venjakob 2007]).

We fix an associative unital noetherian ring R. We write $B(R)$ for the category of bounded complexes of (left) R-modules, $C(R)$ for the category of bounded complexes of finitely generated (left) R-modules, $P(R)$ for the category of finitely generated projective (left) R-modules and $C^{\mathrm{p}}(R)$ for the category of bounded (cohomological) complexes of finitely generated projective (left) R-modules. By $D^{\mathrm{p}}(R)$ we denote the category of perfect complexes as a full triangulated subcategory of the derived category $D^{\mathrm{b}}(R)$ of $B(R)$. We write $(C^{\mathrm{p}}(R), \mathrm{quasi})$ for the subcategory of quasi-isomorphisms of $C^{\mathrm{p}}(R)$ and $(D^{\mathrm{p}}(R), \mathrm{isom})$ for the subcategory of isomorphisms of $D^{\mathrm{p}}(R)$.

For each complex $C = (C^\bullet, d_C^\bullet)$ and each integer r we define the r-fold shift $C[r]$ of C by setting $C[r]^i = C^{i+r}$ and $d^i_{C[r]} = (-1)^r d_C^{i+r}$ for each integer i.

We first recall that there exists a Picard category $\mathscr{C}_R$ and a determinant functor $\mathbf{d}_R : (C^{\mathrm{p}}(R), \mathrm{quasi}) \to \mathscr{C}_R$ with the following properties (for objects C, C' and C'' of $\mathrm{C}^{\mathrm{p}}(R)$):

(B.a) $\mathscr{C}_R$ has an associative and commutative product structure $(M, N) \mapsto M \cdot N$ (which we often write more simply as MN) with canonical unit object $\mathbf{1}_R = \mathbf{d}_R(0)$. If P is any object of $P(R)$, then in $\mathscr{C}_R$ the object $\mathbf{d}_R(P)$ has a canonical inverse $\mathbf{d}_R(P)^{-1}$. Every object of $\mathscr{C}_R$ is of the form $\mathbf{d}_R(P) \cdot \mathbf{d}_R(Q)^{-1}$ for suitable objects P and Q of $P(R)$.

(B.b) All morphisms in $\mathscr{C}_R$ are isomorphisms and elements of the form $\mathbf{d}_R(P)$ and $\mathbf{d}_R(Q)$ are isomorphic in $\mathscr{C}_R$ if and only if P and Q correspond to the same element of the Grothendieck group $K_0(R)$. There is a natural identification $\mathrm{Aut}_{\mathscr{C}_R}(\mathbf{1}_R) \cong K_1(R)$ and if $\mathrm{Mor}_{\mathscr{C}_R}(M, N)$ is nonempty then it is a $K_1(R)$-torsor, where each element α of $K_1(R) \cong \mathrm{Aut}_{\mathcal{C}_R}(\mathbf{1}_R)$ acts on $\phi \in \mathrm{Mor}_{\mathcal{C}_R}(M, N)$ to give

$$\alpha\phi : M = \mathbf{1}_R \cdot M \xrightarrow{\alpha\cdot\phi} \mathbf{1}_R \cdot N = N.$$

(B.c) $\mathbf{d}_R$ preserves the product structure: specifically, for each P and Q in $P(R)$ one has $\mathbf{d}_R(P \oplus Q) = \mathbf{d}_R(P) \cdot \mathbf{d}_R(Q)$.

(B.d) If $C' \to C \to C''$ is a short exact sequence of complexes, there is a canonical isomorphism $\mathbf{d}_R(C) \cong \mathbf{d}_R(C')\mathbf{d}_R(C'')$ in $\mathcal{C}_R$ (which we usually take to be an identification).

(B.e) If C is acyclic, the quasi-isomorphism $0 \to C$ induces a canonical isomorphism $\mathbf{1}_R \to \mathbf{d}_R(C)$.

(B.f) For any integer r one has $\mathbf{d}_R(C[r]) = \mathbf{d}_R(C)^{(-1)^r}$.

(B.g) The functor $\mathbf{d}_R$ factorises over the image of $C^{\mathrm{p}}(R)$ in $D^{\mathrm{p}}(R)$ and extends (uniquely up to unique isomorphisms) to $(D^{\mathrm{p}}(R), \mathrm{isom})$. Moreover, if R is regular, also property (B.d) extends to all distinguished triangles.

(B.h) For each C in $D^{\mathrm{b}}(R)$ we write $\mathrm{H}(C)$ for the complex which has $\mathrm{H}(C)^i = H^i(C)$ in each degree i and in which all differentials are 0. If $\mathrm{H}(C)$ belongs to $D^{\mathrm{p}}(R)$ (in which case one says that C is *cohomologically perfect*), then C belongs to $D^{\mathrm{p}}(R)$ and there are canonical isomorphisms

$$\mathbf{d}_R(C) \cong \mathbf{d}_R(\mathrm{H}(C)) \cong \prod_{i\in\mathbb{Z}} \mathbf{d}_R(H^i(C))^{(-1)^i}.$$

(For an explicit description of the first isomorphism see [Knudsen and Mumford 1976, §3] or [Breuning and Burns 2005, Remark 3.2].)

(B.i) If R' is another (associative unital noetherian) ring and Y an (R', R)-bimodule that is both finitely generated and projective as an R'-module then the functor $Y \otimes_R - : P(R) \to P(R')$ extends to a commutative diagram

$$\begin{tikzcd}
(\mathrm{D}^p(R), \mathrm{isom}) \arrow[r, "\mathbf{d}_R"] \arrow[d, "Y\otimes^{\mathbb{L}}_R -"'] & \mathcal{C}_R \arrow[d, "Y\otimes_R -"] \\
(\mathrm{D}^p(R'), \mathrm{isom}) \arrow[r, "\mathbf{d}_{R'}"] & \mathcal{C}_{R'}.
\end{tikzcd}$$

In particular, if $R \to R'$ is a ring homomorphism and C is in $\mathrm{D}^{\mathrm{p}}(R)$ then we often simply write $\mathbf{d}_R(C)_{R'}$ in place of $R' \otimes_R \mathbf{d}_R(C)$.

(B.j) Let R° be the opposite ring of R. The functor $\mathrm{Hom}_R(-, R)$ induces an antiequivalence between $\mathcal{C}_R$ and $\mathcal{C}_{R^\circ}$, with quasi-inverse induced by $\mathrm{Hom}_{R^\circ}(-, R^\circ)$;

both functors will be denoted by $-^*$. This extends to give a diagram

$$\begin{tikzcd}
(\mathrm{D}^p(R), \mathrm{isom}) \arrow[r, "\mathbf{d}_R"] \arrow[d, "\mathrm{RHom}_R(-,R)"'] & \mathcal{C}_R \arrow[d, "-^*"] \\
(\mathrm{D}^p(R^\circ), \mathrm{isom}) \arrow[r, "\mathbf{d}_{R^\circ}"] & \mathcal{C}_{R^\circ}
\end{tikzcd}$$

which commutes (up to unique isomorphism); similarly we have such a commutative diagram for $\mathrm{RHom}_{R^\circ}(-, R^\circ)$.

For the handling of the determinant functor the following considerations are important in practice:

Remark B.1. (i) For objects $A, B \in \mathcal{C}_R$ we often identify a morphism $f : A \to B$ with the induced morphism

$$\mathbf{1}_R = A \cdot A^{-1} \xrightarrow{f \cdot \mathrm{id}_{A^{-1}}} B \cdot A^{-1}.$$

Then for morphisms $f : A \to B$ and $g : B \to C$ in $\mathcal{C}_R$, the composition $g \circ f : A \to C$ is identified with the product $g \cdot f : \mathbf{1}_R \to C \cdot A^{-1}$ of $g : \mathbf{1}_R \to C \cdot B^{-1}$ and $f : \mathbf{1}_R \to B \cdot A^{-1}$. Also, by this identification a map $f : A \to A$ corresponds uniquely to an element in $K_1(R) = \mathrm{Aut}_{\mathcal{C}_R}(\mathbf{1}_R)$. Furthermore, for a map $f : A \to B$ in $\mathcal{C}_R$, we write $\bar{f} : B \to A$ for its inverse with respect to composition, while $f^{-1} =: \overline{\mathrm{id}_{B^{-1}} \cdot f \cdot \mathrm{id}_{A^{-1}}} : A^{-1} \to B^{-1}$ for its inverse with respect to the multiplication in $\mathcal{C}_R$, that is $f \cdot f^{-1} = \mathrm{id}_{\mathbf{1}_R}$. Obviously, for a map $f : A \to A$ both inverses $\bar{f}$ and f^{-1} coincide if all maps are considered as elements of $K_1(R)$ as above.

Convention B.2. If $f : \mathbf{1} \to A$ is a morphism and B an object in $\mathcal{C}_R$, we write $\bullet f : B \to B \cdot A$ for the morphism $\mathrm{id}_B \cdot f$. In particular, any morphism $f : B \to A$ can be written as $\bullet(\mathrm{id}_{B^{-1}} \cdot f) : B \to A$.

(ii) The determinant of the complex $C = [P_0 \xrightarrow{\phi} P_1]$ (in degrees 0 and 1) with $P_0 = P_1 = P$ is by definition $\mathbf{d}_R(C) = \mathbf{1}_R$; it is defined even if ϕ is not an isomorphism (in contrast to $\mathbf{d}_R(\phi)$). But if ϕ happens to be an isomorphism, i.e., if C is acyclic, then by (B.e) there is also a canonical map $\mathrm{acyc} : \mathbf{1}_R \to \mathbf{d}_R(C)$, which is none other than

$$\mathbf{1}_R = \mathbf{d}_R(P_1)\mathbf{d}_R(P_1)^{-1} \xrightarrow{\mathbf{d}(\phi)^{-1} \cdot \mathrm{id}_{\mathbf{d}(P_1)^{-1}}} \mathbf{d}_R(P_0)\mathbf{d}_R(P_1)^{-1} = \mathbf{d}_R(C)$$

(and which depends on ϕ, in contrast with the first identification). Hence, the composite

$$\mathbf{1}_R \xrightarrow{\mathrm{acyc}} \mathbf{d}_R(C) \stackrel{\mathrm{def}}{=} \mathbf{1}_R$$

corresponds to $\mathbf{d}_R(\phi)^{-1} \in K_1(R)$ according to the first remark. In order to distinguish the above identifications between $\mathbf{1}_R$ and $\mathbf{d}_R(C)$ we also say that C is *trivialised by the identity* when we refer to $\mathbf{d}_R(C) = \mathbf{1}_R$ (or its inverse with respect to composition). For $\phi = \mathrm{id}_P$ both identifications obviously agree.

We end this section by considering the example where $R = K$ is a field and V a finite-dimensional vector space over K. Then, according to [Fukaya and Kato 2006, 1.2.4], $\mathbf{d}_K(V)$ can be identified with the highest exterior product $\bigwedge^{\mathrm{top}} V$ of V and for an automorphism $\phi : V \to V$ the determinant $\mathbf{d}_K(\phi) \in K^\times = K_1(K)$ can be identified with the usual determinant $\det_K(\phi)$. In particular, we identify $\mathbf{d}_K = K$ with canonical basis 1. Then a map $\psi : \mathbf{1}_K \to \mathbf{1}_K$ corresponds uniquely to the value $\psi(1) \in K^\times$.

Remark B.3. Note that every *finite* $\mathbb{Z}_p$-module A possesses a free resolution C; that is, $\mathbf{d}_{\mathbb{Z}_p}(A) \cong \mathbf{d}_{\mathbb{Z}_p}(C)^{-1} = \mathbf{1}_{\mathbb{Z}_p}$. Then modulo $\mathbb{Z}_p^\times$ the composite

$$\mathbf{1}_{\mathbb{Q}_p} \xrightarrow{\text{acyc}} \mathbf{d}_{\mathbb{Z}_p}(C)_{\mathbb{Q}_p} \overset{\text{def}}{=\!=} \mathbf{1}_{\mathbb{Q}_p}$$

corresponds to the cardinality $|A|^{-1} \in \mathbb{Q}_p^\times$.

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Polyhedral adjunction theory

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In this paper we offer a combinatorial view on the adjunction theory of toric varieties. Inspired by classical adjunction theory of polarized algebraic varieties we explore two convex-geometric notions: the $\mathbb{Q}$-codegree and the nef value of a rational polytope P. We prove a structure theorem for lattice polytopes P with large $\mathbb{Q}$-codegree. For this, we define the adjoint polytope $P^{(s)}$ as the set of those points in P whose lattice distance to every facet of P is at least s. It follows from our main result that if $P^{(s)}$ is empty for some $s < 2/(\dim P + 2)$, then the lattice polytope P has lattice width one. This has consequences in Ehrhart theory and on polarized toric varieties with dual defect. Moreover, we illustrate how classification results in adjunction theory can be translated into new classification results for lattice polytopes.

Introduction

Let $P \subseteq \mathbb{R}^n$ be a rational polytope of dimension n. Any such polytope P can be described in a unique minimal way as

$$P = \{x \in \mathbb{R}^n : \langle a_i, x\rangle \geqslant b_i,\ i = 1, \dots, m\},$$

where the a_i are primitive rows of an $m \times n$ integer matrix A and $b \in \mathbb{Q}^m$.

For any $s \geqslant 0$ we define the *adjoint polytope* $P^{(s)}$ as

$$P^{(s)} := \{x \in \mathbb{R}^n : Ax \geqslant b + s\mathbb{1}\},$$

where $\mathbb{1} = (1, \dots, 1)^{\mathrm{T}}$.

We call the study of such polytopes $P^{(s)}$ *polyhedral adjunction theory*.

Adjunction theory is an area of algebraic geometry which has played a fundamental role in the classification of projective algebraic varieties [Batyrev and Tschinkel 1998; Beltrametti and Di Termini 2003; Beltrametti et al. 1992; Beltrametti and

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Figure 1. Two examples of polyhedral adjunction. See page 2426 for discussion.

Sommese 1994; 1995; Fania and Sommese 1989; Fujita 1987; 1992; 1996; 1997; Nakamura 1997; Sommese 1986]. The correspondence between polarized toric varieties and lattice polytopes provides a natural ground for an adjunction theory of lattice polytopes, as suggested in [Dickenstein et al. 2009].

The main purpose of this article is to convince the reader that polyhedral adjunction theory is an exciting area of research with many open questions connecting toric geometry, polyhedral combinatorics and geometry of numbers.

By the toric dictionary between convex geometry of polytopes and geometry of projective toric varieties, a lattice polytope P defines a toric variety X_P polarized by an ample line bundle L_P. The pair (X_P, L_P) is often referred to as a *polarized toric variety*. Sometimes the pair (X, L) is replaced by the equivariant embedding $X \hookrightarrow \mathbb{P}^N$ defined by a suitable multiple of the line bundle L. Adjunction theory provides tools to characterize and classify the pairs (X, L) by looking at the behavior of the *adjoint systems* $|uK_X + vL|$, for integers u, v, where K_X is the canonical divisor in X. We refer to Section 4 for details. If P is the polytope defined by the line bundle L on X, then $(vP)^{(u)}$ is the polytope defined by the line bundle $uK_X + vL$.

In adjunction theory the *nef value* $\tau(L)$ and the *unnormalized spectral value* $\mu(L)$ (sometimes called the canonical threshold) measure the positivity of the adjoint systems. In Section 4 an account of these notions is given. An "integral" version of the unnormalized spectral value for lattice polytopes has been present in the literature for quite some time (even though it was never defined this way) under the name *codegree*, denoted by $\mathrm{cd}(P)$ — see Definition 1.7. This notion appeared in connection with Ehrhart theory and was studied by Batyrev and Nill [2007].

A "rational" version, again for lattice polytopes, has recently been introduced in [Dickenstein et al. 2009]. Let c be the maximal rational number for which $P^{(c)}$ is nonempty. Its reciprocal $\mu(P) := 1/c$ equals precisely the unnormalized spectral value $\mu(L_P)$. It is called the $\mathbb{Q}$-*codegree* of P (Definition 1.5).

A long-standing conjecture in algebraic geometry states that general polarized varieties should have unnormalized spectral values that are bounded above by approximately half their dimension. In particular, as discussed more fully in Remark 4.10, we have the following conjecture:

Conjecture 1 [Beltrametti and Sommese 1994]. If an n-dimensional polarized variety X is smooth, then $\mu(L) > (n+1)/2$ implies that X is a fibration.

Let us consider lattice polytopes again. A Cayley sum of $t+1$ polytopes is a polytope (denoted by $P_0 * \cdots * P_t$) built by assembling the polytopes P_i along the vertices of a t-dimensional simplex — see Definition 3.1. For $t = 0$, the condition of being a Cayley sum is vacuous. So when we say that P has a Cayley structure we mean a nontrivial one with $t > 0$. For example, for $t = 1$, the condition is known in the literature as P having *lattice width one*. From an (apparently) unrelated perspective Batyrev and Nill conjectured that there is a function $f(n)$ such that, if $\mathrm{cd}(P) \geqslant f(n)$, the polytope has a nontrivial Cayley structure. This can be sharpened:

Conjecture 2 [Dickenstein and Nill 2010]. If an n-dimensional lattice polytope P satisfies $\mathrm{cd}(P) > (n+2)/2$, then P decomposes as a Cayley sum of lattice polytopes of dimension at most $2(n+1-\mathrm{cd}(P))$.

The polarized toric variety associated to a Cayley polytope is birationally fibered in projective spaces, as explained on page 2441. It follows that Conjecture 2 could be considered an "integral-toric" version of Conjecture 1 extended to singular varieties. It also suggests that geometrically it would make sense to replace $\mathrm{cd}(P)$ by $\mu(P)$ and use the bound $(n+1)/2$ from Conjecture 1. This leads to a reformulation (we note that $\mu(P) \leqslant \mathrm{cd}(P)$):

Conjecture 3. If an n-dimensional lattice polytope P satisfies $\mu(P) > (n+1)/2$, then P decomposes as a Cayley sum of lattice polytopes of dimension at most $\lfloor 2(n+1-\mu(P)) \rfloor$.

The main result of this paper is Theorem 3.4. It implies a slightly weaker version of Conjecture 3, with $\mu(P) > (n+1)/2$ replaced by $\mu(P) \geqslant (n+2)/2$ — see Corollary 3.7.

Despite much work both Conjectures 1 and 2 are still open in their original generality. It is known that $f(n)$ can be chosen quadratic in n [Haase et al. 2009] and that Conjecture 2 is true for smooth polytopes [Dickenstein et al. 2009; Dickenstein and Nill 2010]. The results in [Dickenstein et al. 2009; Dickenstein and Nill 2010] also imply that for toric polarized manifolds Conjecture 1 holds for $\mu(L) > (n+2)/2$.

Besides the underlying geometric intuition and motivation, polyhedral adjunction theory and the results of this paper have connections with other areas.

Geometry of numbers. It follows from the definition of the $\mathbb{Q}$-codegree that $\mu(P) > 1$ implies that P is *lattice-free*, that is, it has no interior lattice points. Lattice-free polytopes are of importance in geometry of numbers and integer linear optimization — see [Averkov et al. 2011; Nill and Ziegler 2011] for recent results. Lattice-free simplices turn up naturally in singularity theory [Morrison and Stevens 1984]. Most prominently, the famous flatness theorem states that n-dimensional lattice-free convex bodies have bounded lattice width (we refer to [Barvinok 2002] for details). Cayley polytopes provide the most special class of lattice-free polytopes: they have lattice width one, that is, the vertices of the polytope lie on two parallel affine hyperplanes that do not have any lattice points lying strictly between them. Our main result, Theorem 3.4, shows that lattice polytopes with sufficiently large $\mathbb{Q}$-codegree have to be Cayley polytopes. This hints at a close and not yet completely understood relation between the $\mathbb{Q}$-codegree and the lattice width of a lattice polytope.

Let us remark that for $n \geqslant 3$ Corollary 3.7 only provides a sufficient criterion for P to be a Cayley polytope. For instance, $P = [0, 1]^n$ has lattice width one, but $\mu(P) = 2 < (n+2)/2$. Still, for even n the choice of $(n+2)/2$ is tight. Let $P = 2\Delta_n$, where $\Delta_n := \operatorname{conv}(0, e_1, \ldots, e_n)$ is the unimodular n-simplex. Here, P does not have lattice width one, since every edge contains a lattice point in the middle. On the other hand, we have $\mu(P) = (n+1)/2$. Since for n even we have $\operatorname{cd}(P) = (n+2)/2$, this example also shows that the bound $(n+2)/2$ in Conjecture 2 is sharp.

Projective duality. There is evidence that the unnormalized spectral value is connected to the behavior of the associated projective variety under projective duality. An algebraic variety is said to be *dual defective* if its dual variety has codimension strictly larger than 1. The study of dual defective projective varieties is a classical area of algebraic geometry (starting from Bertini) and a growing subject in combinatorics and elimination theory, as it is related to discriminants [Gelfand et al. 1994]. It is known that nonsingular dual-defective polarized varieties necessarily satisfy $\mu > (n+2)/2$ [Beltrametti et al. 1992]. On the other hand, in [Dickenstein and Nill 2010; Di Rocco 2006] it was shown that a polarized nonsingular toric variety corresponding to a lattice polytope P as above is dual defective if and only if $\mu > (n+2)/2$. It was conjectured in [Dickenstein and Nill 2010] that also in the singular toric case $\mu > (n+2)/2$ would imply (X_P, L_P) to be dual defective. Theorem 3.4 gives significant evidence in favor of this conjecture, as it was shown in [Curran and Cattani 2007; Esterov 2010] that the lattice points in such a dual defective lattice polytope lie on two parallel hyperplanes. Moreover, using our main result we verify a weaker version of this conjecture (Proposition 4.11).

Classification of polytopes and adjunction theory beyond $\mathbb{Q}$-Gorenstein varieties. We believe that polyhedral adjunction theory can help to develop useful intuition for problems in (not necessarily toric) classical adjunction theory, when no algebro-geometric tools or results exist so far. For instance, defining μ makes sense in the polyhedral setting even if the canonical divisor of the toric variety is not $\mathbb{Q}$-Cartier.

How to read this paper. Sections 1–3, as well as the Appendix, are kept purely combinatorial, no prior knowledge of algebraic or toric geometry is assumed. The algebro-geometrically inclined reader may jump directly to Section 4. We refer the reader who is unfamiliar with polytopes to [Ziegler 1995].

In Section 1 we introduce the two main players: the $\mathbb{Q}$-codegree and the nef value of a rational polytope. Section 2 proves useful results about how these invariants behave under (natural) projections. These results should be viewed as a toolbox for future applications. Section 3 contains the main theorem and its proof. The algebro-geometric background and implications are explained in Section 4. In the Appendix we include a combinatorial translation of some well-known algebro-geometric classification results by Fujita which we think may be of interest to combinatorialists.

1. The $\mathbb{Q}$-codegree, the codegree, and the nef value

Throughout let $P \subseteq \mathbb{R}^n$ be an n-dimensional rational polytope.

Preliminaries. Let us recall that P is a *rational polytope* if the vertices of P lie in $\mathbb{Q}^n$. Moreover, P is a *lattice polytope* if its vertices lie in $\mathbb{Z}^n$. We consider lattice polytopes up to lattice-preserving affine transformations. Let us denote by $\langle \cdot , \cdot \rangle$ the pairing between $\mathbb{Z}^n$ and its dual lattice $(\mathbb{Z}^n)^*$.

There exists a natural lattice distance function d_P on $\mathbb{R}^n$ such that for $x \in \mathbb{R}^n$ the following holds: $x \in P$ (respectively, $x \in \operatorname{int}(P)$) if and only if $d_P(x) \geqslant 0$ (respectively, $d_P(x) > 0$).

Definition 1.1. Let P be given by the inequalities

$$\langle a_i , \cdot \rangle \geqslant b_i \quad \text{for } i = 1, \ldots, m, \tag{$*$}$$

where $b_i \in \mathbb{Q}$ and the $a_i \in (\mathbb{Z}^n)^*$ are primitive (i.e., they are not the multiple of another lattice vector). We consider the a_i as the rows of an $m \times n$ integer matrix A. Further, we assume all inequalities to define facets F_i of P. Then for $x \in \mathbb{R}^n$ we define the *lattice distance* from F_i by

$$d_{F_i}(x) := \langle a_i , x \rangle - b_i$$

and the *lattice distance* with respect to ∂P by

$$d_P(x) := \min_{i=1,\ldots,m} d_{F_i}(x).$$

For $s > 0$ we define the adjoint polytope as

$$P^{(s)} := \{x \in \mathbb{R}^n : d_P(x) \geqslant s\}.$$

Remark 1.2. We remark that it is important to assume that all F_i are *facets*, as the following two-dimensional example shows. Let $a_1 := (-1, 1)$, $a_2 := (1, 2)$, $a_3 := (0, -1)$, $a_4 := (0, 1)$. We set $b_1 := 0$, $b_2 := 0$, $b_3 := -1$, $b_4 := 0$. This defines

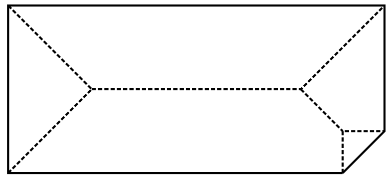


Figure 2. The skeleton of vertices of the adjoint polytopes.

the lattice triangle $P := \operatorname{conv}\big((0,0),(1,1),(-2,1)\big)$ having facets F_1, F_2, F_3, while $F_4 := \{x \in P : \langle a_4, x\rangle = 0\}$ is just the vertex $(0,0)$. Then the point $x := (-1/6, 1/4)$ satisfies $d_P(x) = 1/3$, however $\langle a_4, x\rangle = 1/4$. Note that a_4 is a strict convex combination of $(0,0)$, a_1 and a_2. It can be shown that such a behavior cannot occur for canonical rational polytopes in the sense of Definition 2.4 below.

Remark 1.3. As the parameter s varies, the vertices of the adjoint polytopes trace out a skeleton of straight line segments (compare Figure 2 and Lemma 1.12). In computational geometry there are similar constructions such as the medial axis and the straight skeleton [Aichholzer et al. 1995; Eppstein and Erickson 1999], which are of importance in many applications from geography to computer graphics. "Roof constructions" such as $M(P)$ in Proposition 1.14 are also intensively studied in this context (compare Figure 4). The skeleton proposed here is different, since it uses a distance function which is invariant under lattice-preserving affine transformations and not defined in terms of Euclidean distance or angles.

Let us note some elementary properties of *polyhedral adjunction*:

Proposition 1.4. *Let* $s \geqslant 0$.

(1) *Each facet of* $P^{(s)}$ *is of the form*

$$F^{(s)} := \{x \in P^{(s)} : d_F(x) = s\}$$

for some facet F *of* P.

(2) *Assume* $P^{(s)}$ *has dimension* n, *and let* $x \in P^{(s)}$. *Then* $d_{P^{(s)}}(x) = d_P(x) - s$. *Moreover, if* $x \in \operatorname{int}(P^{(s)})$ *and* $d_P(x) = d_F(x)$ *for a facet* F *of* P, *then* $F^{(s)}$ *is a facet of* $P^{(s)}$, *and* $d_{P^{(s)}}(x) = d_{F^{(s)}}(x)$.

(3) *Assume* $P^{(s)}$ *has dimension* n, *and let* $r \geqslant 0$. *Then*

$$\big(P^{(s)}\big)^{(r)} = P^{(s+r)}.$$

(4) *For* $r > 0$ *we have* $r(P^{(s)}) = (rP)^{(rs)}$.

Proof. Property (1) follows directly from the definition. For (2), we first prove the second statement. Let $x \in \operatorname{int}(P^{(s)})$, and let F be a facet of P with $d_P(x) = d_F(x)$. If we set $\lambda := s/d_F(x)$, we have $\lambda x + (1-\lambda)F \subseteq F^{(s)}$: all elements y of the left-hand side satisfy $d_F(y) = s$ and $d_G(y) \geqslant s$ for facets G of P other than F. This shows that $F^{(s)}$ is indeed $(n-1)$-dimensional.

This also shows that

$$d_P(x) = d_F(x) = d_{F^{(s)}}(x) + s \geqslant d_{P^{(s)}}(x) + s.$$

On the other hand, pick a facet G of P such that $G^{(s)}$ is a facet of $P^{(s)}$ and that $d_{G^{(s)}}(x) = d_{P^{(s)}}(x)$. Then $d_P(x) \leqslant d_G(x) = d_{G^{(s)}}(x) + s = d_{P^{(s)}}(x) + s$.

Finally, if x sits on the boundary of $P^{(s)}$, then the desired equality reads $0 = 0$.

Now (3) follows directly from (2), and (4) is immediate from the definition. □

The $\mathbb{Q}$-codegree. We now define the invariant we are most interested in. The reciprocal is used to keep the notation consistent with already existing algebro-geometric terminology.

Definition 1.5. We define the $\mathbb{Q}$-*codegree* of P as

$$\mu(P) := \bigl(\sup\{s > 0\colon\ P^{(s)} \neq \varnothing\}\bigr)^{-1},$$

and the *core* of P is $\operatorname{core}(P) := P^{(1/\mu(P))}$.

As the following proposition shows, the supremum is actually a maximum. Moreover, since P is a rational polytope, $\mu(P)$ is a positive rational number.

Proposition 1.6. *The following quantities coincide*:

(1) $\mu(P)$,

(2) $\bigl(\max\{s > 0\colon\ P^{(s)} \neq \varnothing\}\bigr)^{-1}$,

(3) $\bigl(\sup\{s > 0\colon\ \dim(P^{(s)}) = n\}\bigr)^{-1}$,

(4) $\min\{p/q > 0\colon\ p, q \in \mathbb{Z}_{>0},\ (pP)^{(q)} \neq \varnothing\}$,

(5) $\inf\{p/q > 0\colon\ p, q \in \mathbb{Z}_{>0},\ \dim((pP)^{(q)}) = n\}$,

(6) $\min\{p/q > 0\colon\ p, q \in \mathbb{Z}_{>0},\ (pP)^{(q)} \cap \mathbb{Z}^n \neq \varnothing\}$.

Moreover, $\operatorname{core}(P)$ *is a rational polytope of dimension* $< n$.

Proof. (1), (2), (4) and (6) coincide by Proposition 1.4(4). For the remaining statements, note that for $s > 0$, the adjoint polytope $P^{(s)}$ contains a full-dimensional ball if and only if there exists some small $\varepsilon > 0$ such that $P^{(s+\varepsilon)} \neq \varnothing$. □

The codegree. The $\mathbb{Q}$-codegree is a rational variant of the *codegree*, which came up in Ehrhart theory of lattice polytopes [Batyrev and Nill 2007]. However, the definition also makes sense for rational polytopes.

Definition 1.7. Let P be a rational polytope. We define the *codegree* as

$$\operatorname{cd}(P) := \min\{k \in \mathbb{N}_{\geqslant 1}\colon\ \operatorname{int}(kP) \cap \mathbb{Z}^n \neq \varnothing\}.$$

Lemma 1.8. *Let l be the common denominator of all right-hand sides b_i given in the inequality description of P as in $(*)$ of Definition 1.1. Then*

$$\mathrm{int}(lP) \cap \mathbb{Z}^n = (lP)^{(1)} \cap \mathbb{Z}^n.$$

In particular, $\mu(P) \leqslant l\ \mathrm{cd}(P)$.

Proof. Let $x \in \mathrm{int}(lP) \cap \mathbb{Z}^n$. Then $\mathbb{Z} \ni \langle a_i, x\rangle > lb_i \in \mathbb{Z}$ for all $i = 1, \dots, m$. Hence, $\langle a_i, x\rangle \geqslant lb_i + 1$, as desired. The other inclusion is clear. The last statement follows from Proposition 1.6(6). □

Note that for a *lattice* polytope P, we automatically have $l = 1$, so

$$\mu(P) \leqslant \mathrm{cd}(P) \leqslant n + 1,$$

where the last inequality is well-known (take the sum of $n+1$ affinely independent vertices of P).

The nef value. The third invariant we are going to define is a finite number only if the polytope is not too singular. Let us make this precise.

Definition 1.9. A rational cone $\sigma \subset (\mathbb{R}^n)^*$ with primitive generators $v_1, \dots, v_m$ in $(\mathbb{Z}^n)^*$ is $\mathbb{Q}$*-Gorenstein of index* r_σ if there is a primitive point $u_\sigma \in \mathbb{Z}^n$ with $\langle v_i, u_\sigma\rangle = r_\sigma$ for all i.

The normal fan $\mathcal{N}(P)$ of P is $\mathbb{Q}$*-Gorenstein of index* r if the maximal cones are $\mathbb{Q}$-Gorenstein and $r = \mathrm{lcm}(r_\sigma : \sigma \in \mathcal{N}(P))$.

Such a cone/fan is called *Gorenstein* if the index is 1. Moreover, we say that P is *smooth* if for any maximal cone of $\mathcal{N}(P)$ the primitive ray generators form a lattice basis. Clearly, P smooth implies $\mathcal{N}(P)$ Gorenstein.

In other words, $\mathcal{N}(P)$ is $\mathbb{Q}$-Gorenstein if the primitive ray generators of any maximal cone lie in an affine hyperplane and the index equals the least common multiple of the lattice distance of these hyperplanes from the origin. For instance, any *simple* polytope is $\mathbb{Q}$-Gorenstein because every cone in the normal fan is simplicial.

Definition 1.10. The *nef value* of P is given as

$$\tau(P) := \big(\sup\{s > 0 \colon \mathcal{N}(P^{(s)}) = \mathcal{N}(P)\}\big)^{-1} \in \mathbb{R}_{>0} \cup \{\infty\}.$$

Note that in contrast to the definition of the $\mathbb{Q}$-codegree, here the supremum is never a maximum.

Definition 1.11. Assume $\mathcal{N}(P)$ is $\mathbb{Q}$-Gorenstein, and v is a vertex of P. Assume that in the inequality description of P as in $(*)$ of Definition 1.1, the vertex v satisfies equality precisely for $i \in I$. That is, the normal cone of v is $\sigma = \mathrm{pos}(a_i : i \in I)$. For $s \geqslant 0$, define the point $v(s)$ by $v(s) = v + (s/r_\sigma)u_\sigma$, where u_σ and r_σ are defined in Definition 1.9. Note that $\langle a_i, v(s)\rangle = b_i + s$ for $i \in I$.

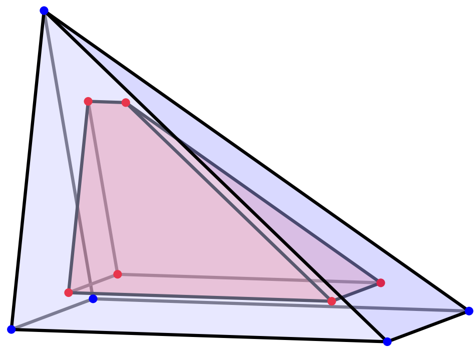


Figure 3. $P^{(1/5)} \subseteq P$ for a 3-dimensional lattice polytope P.

The following lemma collects various ways to compute the nef value τ of a polytope, if the normal fan is $\mathbb{Q}$-Gorenstein.

Lemma 1.12. *$\mathcal{N}(P)$ is $\mathbb{Q}$-Gorenstein if and only if $\tau(P) < \infty$. Assume this condition holds. Then for $s \in [0, \tau(P)^{-1}]$ we have $P^{(s)} = \operatorname{conv}(v(s) \colon v$ vertex of $P)$. Consequently, the following quantities coincide*:

1. $\tau(P)^{-1}$,
2. $\max\{s \in \mathbb{Q}_{>0} \colon v(s) \in P^{(s)}$ *for all vertices v of P*$\}$,
3. $\min\{s \in \mathbb{Q}_{>0} \colon v(s) = v'(s)$ *for two different vertices v, v' of P*$\}$,
4. $\min\{s \in \mathbb{Q}_{>0} \colon P^{(s)}$ *is combinatorially different from P*$\}$,
5. $\max\{s \in \mathbb{Q}_{>0} \colon \mathcal{N}(P)$ *refines* $\mathcal{N}(P^{(s)})\}$.

Proof. The first assertion follows by Definition 1.11. Notice that $\mathcal{N}(P) = \mathcal{N}(P^{(s)})$ if and only if $v(s) \neq v'(s)$ for any two different vertices v, v' of P. This implies the assertions (1) $\iff$ (3) $\iff$ (4). Let now $\xi = \max\{s \in \mathbb{Q}_{>0} \colon v(s) \in P^{(s)}\}$. As remarked in Definition 1.11 it is $\tau(P)^{-1} \leqslant \xi$. On the other hand the existence of an $s \in \mathbb{Q}$ such that $\xi < s < \tau(P)^{-1}$ would lead to a contradiction. In fact it would imply that $\mathcal{N}(P) = \mathcal{N}(P^{(s)})$ and the existence of a vertex $v \in P$ for which $v(s) \notin P^{(s)}$. This proves (1) $\iff$ (2) $\iff$ (5). □

Figure 3 shows a three-dimensional lattice polytope P whose normal fan is not $\mathbb{Q}$-Gorenstein ($\tau(P) = \infty$). Note that P has 5 vertices, while the adjoint polytope $P^{(c)}$ (for $0 < c < 1/\mu(P)$) has 6 vertices.

By definition, we have $\mu(P) \leqslant \tau(P)$. We also want to compare the codegree and the nef value.

Proposition 1.13. *Let P be a lattice polytope with $\mathbb{Q}$-Gorenstein normal fan of index r. If $s \geqslant r\tau(P)$ is an integer, then $(sP)^{(r)}$ is a lattice polytope. In particular,*

$$\operatorname{cd}(P) - 1 < r\tau(P).$$

Proof. By Lemma 1.12 every vertex of $P^{(r/s)}$ is of the form $v\big(\frac{r}{s}\big) = v + \frac{r}{r_\sigma s} u_\sigma$ for some vertex v of P. Hence, every vertex of $(sP)^{(r)}$ is given as $sv\big(\frac{r}{s}\big) = sv + \frac{r}{r_\sigma} u_\sigma$, a lattice point. For the last statement it suffices to observe that $(\mathrm{cd}(P)-1)P$ does not have interior lattice points. $\square$

The "mountain" and $\mathbb{Q}$-normality. We now give a graphical description of the nef value and the $\mathbb{Q}$-codegree, one that provides an efficient way to compute these invariants. Let the *mountain* $M(P) \subseteq \mathbb{R}^{n+1}$ be defined as

$$M(P) := \{(x,s) \colon x \in P,\ 0 \leqslant s \leqslant d_P(x)\}.$$

Proposition 1.14. *Assume that P has an inequality description as in formula* $(*)$ *of Definition 1.1. Then*

$$M(P) = \{(x,s) \in \mathbb{R}^{n+1} \colon (A \mid -\mathbb{1})(x,s)^{\mathrm{T}} \geqslant b,\ s \geqslant 0\}.$$

Therefore, $M(P)$ is a rational polytope with $M(P) \cap \mathbb{R}^n \times \{s_0\} = P^{(s_0)} \times \{s_0\}$. Moreover,

(1) $\qquad \mu(P)^{-1} = \max(s\colon$ *there is a vertex of $M(P)$ with last coordinate s*$)$.

If $\mathcal{N}(P)$ is $\mathbb{Q}$-Gorenstein, then

(2) $\qquad \tau(P)^{-1} = \min(s > 0\colon$ *there is a vertex of $M(P)$ with last coordinate s*$)$.

Proof. Set $q := \mu(P)^{-1}$. By Proposition 1.6(2), we have $q = \max\{s > 0 \colon P^{(s)} \neq \varnothing\}$. By the definition of $P^{(s)}$, this is the maximal positive s such that there is an $x \in P$ which satisfies $d_F(x) \geqslant s$ for all facets F of P. This shows (1).

Let us prove (2). Suppose $\mathcal{N}(P)$ is $\mathbb{Q}$-Gorenstein, and abbreviate $t := \tau(P)^{-1}$. For every vertex v of P and $s > 0$ define $v(s)$ as in Definition 1.11. At every vertex $(v, 0)$ of the bottom facet $P \times \{0\}$ of $M(P)$ there is a unique upwards edge towards $(v(s), s)$ for small s. By Lemma 1.12(3) there are two vertices v, v' of P so that $v(t) = v'(t)$. The corresponding point $(v(t), t) = (v'(t), t)$ in $M(P)$ is a vertex, as it is incident to at least two edges. $\square$

Let us consider the example given on the right-hand side of Figure 1, and take a look at its mountain — see Figure 4. The height of the mountain equals the reciprocal of the $\mathbb{Q}$-codegree, while the height of the first nontrivial vertex is the reciprocal of the nef value.

This motivates the following definition (see [Dickenstein et al. 2009]).

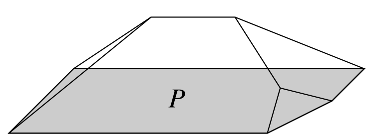


Figure 4. The lattice distance mountain $M(P)$.

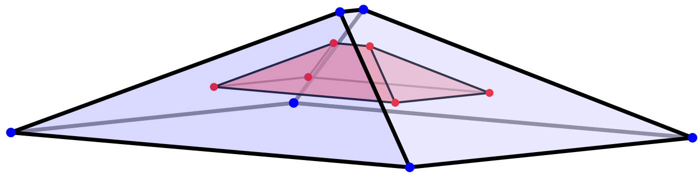


Figure 5. $P^{(4)} \subseteq P$ for a 3-dimensional lattice polytope P.

Definition 1.15. We say that P is $\mathbb{Q}$-*normal* if $\mu(P) = \tau(P)$.

To get the correct intuition for this notion, let us note that P is $\mathbb{Q}$-normal if and only if all *vertices* of P survive under polyhedral adjunction (as long as the adjoint polytope is full-dimensional). For $n \geqslant 3$ it is not enough that all *facets* of P survive, as Figure 5 illustrates (where $\tau(P)^{-1} = 2$, $\mu(P)^{-1} = 6$ and $\mathrm{core}(P)$ is an interval).

2. Natural projections

Throughout let $P \subseteq \mathbb{R}^n$ be an n-dimensional rational polytope.

The core and the natural projection. Recall that $\mathrm{core}(P) := P^{(1/\mu(P))}$ is a rational polytope of dimension $< n$.

Definition 2.1. Let $K(P)$ be the linear space parallel to $\mathrm{aff}(\mathrm{core}(P))$. We call $\pi_P : \mathbb{R}^n \to \mathbb{R}^n / K(P)$ the *natural projection* associated with P.

Lemma 2.2. *Let $x \in \mathrm{relint}(\mathrm{core}(P))$. Let us denote by $F_1, \dots, F_t$ the facets of P with $d_{F_i}(x) = \mu(P)^{-1}$. Then their primitive inner normals $a_1, \dots, a_t$ positively span the linear subspace $K(P)^{\perp}$.*

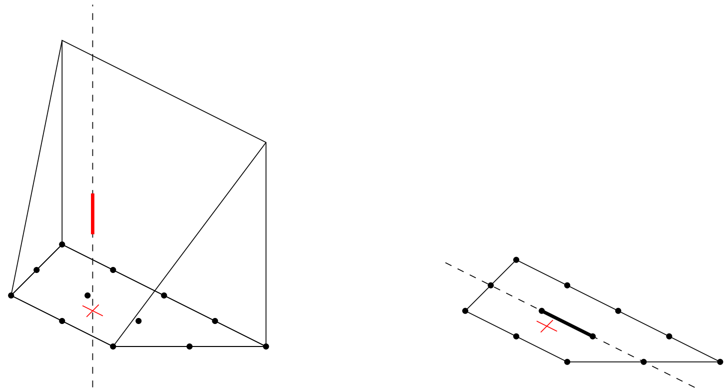
Moreover, if $\mathrm{core}(P) = \{x\}$, *then*

$$\{y \in \mathbb{R}^n : d_{F_i}(y) \geqslant 0 \textit{ for all } i = 1, \dots, t\}$$

is a rational polytope containing P.

Proof. We set $s := \mu(P)^{-1}$. Let $i \in \{1, \dots, t\}$. Since $d_{F_i}(x) = s$ and $x \in \mathrm{relint}(P^{(s)})$, we have $d_{F_i}(y) = s$ for all $y \in P^{(s)}$. This shows $C := \mathrm{pos}(a_1, \dots, a_t) \subseteq K(P)^{\perp}$. Assume that this inclusion were strict. Then there exists some $v \in \mathbb{R}^n$ such that $\langle v, C \rangle \geqslant 0$ and v does not vanish on the linear subspace $K(P)^{\perp}$. In particular, for any $i \in \{1, \dots, t\}$ one gets $\langle v, a_i \rangle \geqslant 0$, so $d_{F_i}(x + \varepsilon v) \geqslant d_{F_i}(x) = s$ for any $\varepsilon > 0$. Moreover, if we choose ε small enough, then $d_G(x + \varepsilon v) \approx d_G(x) > s$ for any other facet G of P. Hence, $x + \varepsilon v \in P^{(s)}$. But this means $v \in K(P)$, and v must vanish on $K(P)^{\perp}$, a contradiction.

Finally, notice that if $P^{(s)} = \{x\}$, then $a_1, \dots, a_t$ positively span $(\mathbb{R}^n)^*$. In particular, $\mathrm{conv}(a_1, \dots, a_t)$ contains a small full-dimensional ball around the origin. Dually, $\{y \in \mathbb{R}^n : \langle a_i, y \rangle \geqslant b_i,\ i = 1, \dots, t\}$ is contained in a large ball. Hence, it is a bounded rational polyhedron, thus a rational polytope. $\square$



$$P = \mathrm{conv}\begin{bmatrix} 0 & 2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 4 & 2 & 0 & 4 \\ 0 & 0 & 0 & 0 & h & h \end{bmatrix} \qquad Q = \mathrm{conv}\begin{bmatrix} 0 & 2 & 0 & 2 \\ 0 & 0 & 4 & 2 \end{bmatrix}$$

Figure 6. The $\mathbb{Q}$-codegree projection $\pi_P : P \to Q$.

The $\mathbb{Q}$-codegree under natural projections. We begin with a key observation.

Proposition 2.3. *The image $Q := \pi_P(P)$ of the natural projection of P is a rational polytope satisfying $\mu(Q) \geqslant \mu(P)$. Moreover, if $\mu(Q) = \mu(P)$, then* $\mathrm{core}(Q)$ *is the point* $\pi_P(\mathrm{core}(P))$.

Proof. Let t, x, F_i, a_i as in Lemma 2.2 and $s := \mu(P)^{-1}$. Q is a rational polytope with respect to the lattice $L := \mathbb{Z}^n/(K(P) \cap \mathbb{Z}^n)$. The dual lattice of L is $(\mathbb{Z}^n)^* \cap K(P)^\perp$. In particular, any a_i for $i \in \{1, \dots, t\}$ is still a primitive normal vector of a facet of Q. In particular, $Q^{(s)} \subseteq \pi_P(P^{(s)}) = \{\pi_P(x)\}$. Therefore, $\mu(Q)^{-1} \leqslant s$. $\square$

The example in Figure 6 shows that this projection can be quite peculiar. The dashed lines are the affine hulls along which we are projecting, while the fat line segments are the cores of P and Q. On the left side we only drew the lattice points on the bottom face for clarity. Here, π_P projects onto the bottom face Q. If we assume that the height h of P is large enough, then the adjoint polytope $\mathrm{core}(P)$ is a line segment projecting onto the point $x = (\frac{4}{3}, \frac{4}{3}, 0)$ marked on the bottom. Note that this point doesn't even lie in the line segment $\mathrm{core}(Q)$. Essentially, the reason for this behavior is that the preimage of one of the two facets of Q defining the affine hull of $\mathrm{core}(Q)$ is not a facet of P. Moreover, $\mu(Q) = 1 > \frac{3}{4} = \mu(P)$.

Projections of α-canonical polytopes.

Definition 2.4. Let σ be a rational cone with primitive generators $v_1, \dots, v_m$. Then the height function associated with σ is the piecewise linear function

$$\mathrm{ht}_\sigma(x) := \max\left\{ \sum_{i=1}^{m} \lambda_i \ : \ \lambda_i \geqslant 0 \text{ for } i = 1, \dots, m \ , \ \sum_{i=1}^{m} \lambda_i v_i = x \right\}$$

on σ. For $\alpha > 0$, we say that σ is *α-canonical* if $\mathrm{ht}_\sigma(x) \geqslant \alpha$ for every nonzero $x \in \sigma \cap \mathbb{Z}^n$. A 1-canonical cone is said to be *canonical*.

A rational polytope is (α-)*canonical* if all cones of its normal fan are.

This is a generalization to the non-$\mathbb{Q}$-Gorenstein case of canonical singularities in algebraic geometry. Note that a $\mathbb{Q}$-Gorenstein cone of index r is $(1/r)$-canonical. In particular, rational polytopes with Gorenstein normal fan are canonical.

Lemma 2.5. *Let $\pi : P \to Q$ be a polytope projection, and assume P is α-canonical. Then $\alpha d_P(x) \leqslant d_Q(\pi(x))$ for all $x \in P$.*

Proof. Let $\langle a, \cdot \rangle \geqslant b$ be a facet of Q realizing $d_Q(\pi(x))$. That is, $\langle a, \pi(x)\rangle = b + d_Q(\pi(x))$. Then the integral linear functional $\pi^* a$ belongs to some cone $\sigma \in \mathcal{N}(P)$ with primitive generators $a_1, \ldots, a_m$. Write $\pi^* a = \sum_{i=1}^m \lambda_i a_i$ with $\lambda_i \geqslant 0$ for $i = 1, \ldots, m$ and $\sum_{i=1}^m \lambda_i = \mathrm{ht}_\sigma(\pi^* a)$. Then $b = \sum_{i=1}^m \lambda_i b_i$, and $\sum_{i=1}^m \lambda_i \geqslant \alpha$. Thus

$$\begin{aligned} d_Q(\pi(x)) &= \langle a, \pi(x)\rangle - b = \langle \pi^* a, x\rangle - b \\ &= \sum_{i=1}^m \lambda_i(\langle a_i, x\rangle - b_i) \geqslant \sum_{i=1}^m \lambda_i d_P(x) \geqslant \alpha d_P(x). \end{aligned}$$ □

Corollary 2.6. *Let $\pi : P \to Q$ be a polytope projection, and assume P is α-canonical. Then $\mu(P) \geqslant \alpha \mu(Q)$.*

In particular, if P is canonical, then $\mu(P) \geqslant \mu(Q)$.

This shows that for *canonical* polytopes the natural projection in Proposition 2.3 is $\mathbb{Q}$-codegree preserving! In particular, the polytope Q has the nice property that $\mathrm{core}(Q)$ is a point.

Example 2.7. Unfortunately, it is in general not true that being α-canonical is preserved under the natural projection, as can be seen from the following example. Consider the polytope

$$P = \mathrm{conv}\begin{bmatrix} 14 & 8 & 0 & -8 & 14 & 0 & 0 & -14 & -14 \\ 7 & 1 & 0 & 1 & 7 & 21 & 21 & 7 & 7 \\ -21 & -3 & 0 & 3 & 21 & 21 & -21 & 21 & -21 \end{bmatrix}$$

This is a three-dimensional lattice polytope. Its core has the vertices $(0, 7, 7)$ and $(0, 7, -7)$, so the natural projection π maps onto a two-dimensional lattice polytope by projecting onto the first two coordinates. The projection is

$$\pi(P) = \mathrm{conv}\begin{bmatrix} 14 & 8 & 0 & -8 & 0 & -14 \\ 7 & 1 & 0 & 1 & 21 & 7 \end{bmatrix}$$

All but one normal cone of P is canonical. The exception is the normal cone at the origin. Its primitive rays are $(-1, -5, -1)$, $(1, -5, 1)$, $(0, -3, -1)$ and $(0, -3, 1)$. The ray $(0, -1, 0)$ is in the cone, and its height is $\frac{1}{3}$. So P is $\frac{1}{3}$-canonical. The normal cones of the natural projection Q are again canonical with one exception.

The normal cone at the origin is generated by the rays $(1, -8)$ and $(-1, -8)$. It contains the ray $(0, -1)$, so Q is only $\frac{1}{8}$-canonical. The computations were done with `polymake` [Joswig et al. 2009].

$\mathbb{Q}$-normality under natural projections.

Proposition 2.8. *Let P be $\mathbb{Q}$-normal. Then its image Q under the natural projection is $\mathbb{Q}$-normal, its core is the point* $\operatorname{core}(Q) = \pi_P(\operatorname{core}(P))$, *and* $\mu(Q) = \mu(P)$. *Moreover, if P is α-canonical, then Q is α-canonical.*

Proof. If P is $\mathbb{Q}$-normal, then the normal fan of P refines the normal fan of $\operatorname{core}(P) = P^{(1/\tau(P))}$. In particular, the face $K(P)^\perp$ of $\mathcal{N}(\operatorname{core}(P))$ is a union of faces of $\mathcal{N}(P)$. Therefore, being α-canonical is preserved. On the other hand, $\mathcal{N}(Q) = \mathcal{N}(P) \cap K(P)^\perp$ for any polytope projection $P \to Q$. That means that every facet F of Q lifts to a facet $\pi_P^* F$ of P. Together with $d_F(\pi_P(x)) = d_{\pi_P^* F}(x)$ (for $x \in P$) this implies $Q^{(s)} = \pi(P^{(s)})$ for any $s \geqslant 0$. This yields the statements. $\square$

If a rational polytope is $\mathbb{Q}$-normal and its core is a point, then the generators of its normal fan form the vertex set of a lattice polytope. Such a fan corresponds to a *toric Fano variety*; see, e.g., [Debarre 2003; Nill 2005].

3. Cayley decompositions

Throughout let $P \subseteq \mathbb{R}^n$ be an n-dimensional lattice polytope.

Lattice width, Cayley polytopes and codegree. We recall that the *lattice width* of a polytope P is defined as the minimum of $\max_{x \in P} \langle u, x \rangle - \min_{x \in P} \langle u, x \rangle$ over all nonzero integer linear forms u. We are interested in lattice polytopes of lattice width one, which we also call (nontrivial) *Cayley polytopes* or Cayley polytopes of length $\geqslant 2$.

Definition 3.1. Given lattice polytopes $P_0, \dots, P_t$ in $\mathbb{R}^k$, the *Cayley sum* $P_0 * \cdots * P_t$ is defined to be the convex hull of $(P_0 \times 0) \cup (P_1 \times e_1) \cup \cdots \cup (P_t \times e_t)$ in $\mathbb{R}^k \times \mathbb{R}^t$ for the standard basis $e_1, \dots, e_t$ of $\mathbb{R}^t$.

We say that $P \subseteq \mathbb{R}^n$ is a *Cayley polytope of length $t + 1$* if there exists an affine lattice basis of $\mathbb{Z}^n \cong \mathbb{Z}^k \times \mathbb{Z}^t$ identifying P with the Cayley sum $P_0 * \cdots * P_t$ for some lattice polytopes $P_0, \dots, P_t$ in $\mathbb{R}^k$.

This definition can be reformulated [Batyrev and Nill 2008, Proposition 2.3].

Lemma 3.2. *Let $\sigma \subseteq \mathbb{R}^{n+1}$ be the cone spanned by $P \times 1$. Then the following statements are equivalent*:

(1) *P is a Cayley polytope $P_0 * \cdots * P_t$ of length $t + 1$.*

(2) *There is a lattice projection P onto a unimodular t-simplex.*

(3) *There are nonzero* $x_1, \dots, x_{t+1} \in \sigma^\vee \cap (\mathbb{Z}^{n+1})^*$ *such that*

$$x_1 + \cdots + x_{t+1} = e_{n+1}.$$

Since the t-th multiple of a unimodular t-simplex contains no interior lattice points, we conclude from Lemma 3.2(2) that

$$\operatorname{cd}(P_0 * \cdots * P_t) \geqslant t + 1.$$

Conversely, Conjecture 2 states that having large codegree implies being a Cayley polytope. To get the reader acquainted with Conjecture 2, we include a simple observation.

Lemma 3.3. *If* $\operatorname{cd}(P) > \lceil (n+1)/2 \rceil$, *then through every vertex there is an edge whose only lattice points are its two vertices.*

Proof. Assume otherwise. Then there exists an injective lattice homomorphism f mapping $2\Delta_n \to P$. Therefore, Stanley's monotonicity theorem [Stanley 1993; Batyrev and Nill 2007] yields $n + 1 - \operatorname{cd}(f(2\Delta_n)) \leqslant n + 1 - \operatorname{cd}(P)$, hence $\operatorname{cd}(P) \leqslant \operatorname{cd}\big(f(2\Delta_n)\big) \leqslant \operatorname{cd}(2\Delta_n) = \lceil (n+1)/2 \rceil$. This yields a contradiction to our assumption. □

The decomposition theorem. Let P, P' be n-dimensional lattice polytopes. We will say that P and P' are *unimodularly equivalent* ($P \cong P'$) if there exists an affine lattice automorphism of $\mathbb{Z}^n$ mapping the vertices of P onto the vertices of P'. It is a well-known result (see, for example, [Batyrev and Nill 2007]) that $P \cong \Delta_n$ if and only if $\operatorname{cd}(P) = n + 1$. Since $\mu(P) \leqslant \operatorname{cd}(P) \leqslant n + 1$ and $\mu(\Delta_n) = n + 1$, we deduce that $P \cong \Delta_n$ if and only if $\mu(P) = n + 1$.

We next prove a general structure result on lattice polytopes of high $\mathbb{Q}$-codegree. We set

$$d(P) := \begin{cases} 2(n - \lfloor \mu(P) \rfloor) & \text{if } \mu(P) \notin \mathbb{N}, \\ 2(n - \mu(P)) + 1 & \text{if } \mu(P) \in \mathbb{N}. \end{cases}$$

If we exclude the special situation $P \cong \Delta_n$, we have $1 \leqslant d(P) < 2(n + 1 - \mu(P))$.

Theorem 3.4. *Let P be an n-dimensional lattice polytope with $P \ncong \Delta_n$. If $n > d(P)$, then P is a Cayley sum of lattice polytopes in $\mathbb{R}^m$ with $m \leqslant d(P)$.*

For the proof we recall the following folklore result.

Lemma 3.5. *Let $P \subseteq \mathbb{R}^n$ be an n-dimensional lattice polytope. Let*

$$z \in \operatorname{pos}(P \times \{1\}) \cap \mathbb{Z}^{n+1}.$$

Then there exist (not necessarily different) vertices $v_1, \dots, v_g$ of P and a lattice point $p \in (jP) \cap \mathbb{Z}^n$ with

$$z = (v_1, 1) + \cdots + (v_g, 1) + (p, j)$$

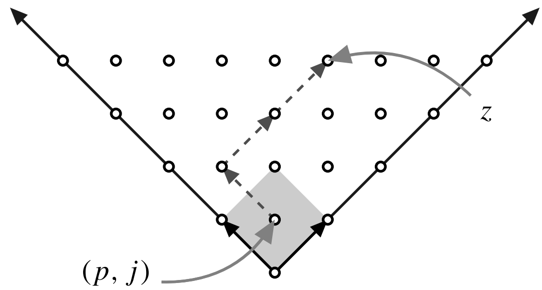


Figure 7. Decomposing z in the proof of Lemma 3.5.

such that $(p, j) = (0, 0)$ *or* $1 \leqslant j \leqslant n + 1 - \operatorname{cd}(P)$.

Proof. There exists an m-dimensional simplex S in P with vertices $v_1, \ldots, v_{m+1}$ in the vertex set of P such that $z \in \operatorname{pos}\big((v_1, 1), \ldots, (v_{m+1}, 1)\big)$. We can write

$$z = \sum_{i=1}^{m+1} k_i(v_i, 1) \; + \; \sum_{i=1}^{m+1} \lambda_i(v_i, 1) \quad \text{for } k_i \in \mathbb{N} \text{ and } \lambda_i \in [0, 1).$$

See also Figure 7. The lattice point $\sum_{i=1}^{m+1} \lambda_i(v_i, 1)$ is an element of the fundamental parallelepiped of the simplex S. By [Beck and Robins 2007, Corollary 3.11], its height j equals at most the degree of the so-called Ehrhart h^*-polynomial. Ehrhart–Macdonald reciprocity implies that this degree is given by $m + 1 - \operatorname{cd}(S)$. We refer to [Batyrev and Nill 2007] for more details. Now, the result follows from $j \leqslant m + 1 - \operatorname{cd}(S) \leqslant n + 1 - \operatorname{cd}(P)$ by Stanley's monotonicity theorem [1993]. □

Proof of Theorem 3.4. By successive application of Proposition 2.3, we can find a lattice projection $P \to Q$ with $\dim(Q) = n' \leqslant n$ such that $\mu(P) \leqslant \mu(Q)$ and $Q^{(s)} = \{x\}$ for $s := \mu(Q)^{-1}$. By observing that $d(Q) + (n - n') \leqslant d(P)$, we see that $d(P) < n$ implies $d(Q) < n'$ and, moreover, if the desired statement holds for Q, then it also holds for P. Hence, we may assume that $s = \mu(P)^{-1}$ and $P^{(s)} = \{x\}$.

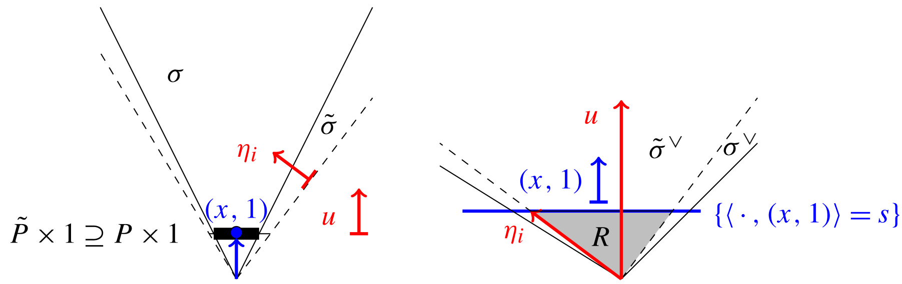
By Lemma 2.2, P is contained in a rational polytope $\tilde{P}$ with $s = \mu(\tilde{P})^{-1}$ and $\tilde{P}^{(s)} = \{x\}$ so that all facets of $\tilde{P}$ have distance s from x. Let $\sigma \subseteq \tilde{\sigma} \subseteq \mathbb{R}^{n+1}$ be the (full-dimensional, pointed) cones over $P \times \{1\} \subseteq \tilde{P} \times \{1\}$, and let $u \in (\mathbb{R}^{n+1})^*$ be the last coordinate functional. As u evaluates positively on all vertices of $\tilde{P} \times \{1\}$, we have $u \in \operatorname{int} \tilde{\sigma}^\vee \subseteq \operatorname{int} \sigma^\vee$. Let us define the lattice polytope

$$R := \operatorname{conv}\big(\{0\} \cup \{\eta \colon \eta \text{ primitive facet normal of } \tilde{\sigma}\}\big) \subseteq (\mathbb{R}^{n+1})^*.$$

In order to invoke Lemma 3.2(3), we will show that R has high codegree so that u can be decomposed into a sum of many lattice points in $\tilde{\sigma}^\vee \subseteq \sigma^\vee$ by Lemma 3.5.

To this end, observe that $\langle \eta, (x, 1)\rangle = s$ for every primitive facet normal η of $\tilde{\sigma}$, so that R is an $(n+1)$-dimensional pyramid with apex 0:

$$R = \tilde{\sigma}^\vee \cap \{y \in (\mathbb{R}^{n+1})^* : \langle y, (x, 1)\rangle \leqslant s\}.$$



Let us bound the height of an interior lattice point of $\tilde{\sigma}^\vee$. Assume there is some $y \in \operatorname{int}\tilde{\sigma}^\vee \cap (\mathbb{Z}^{n+1})^*$ such that $\langle y, (x, 1)\rangle < 1$. Because $x \in P$ is a convex combination of vertices there is some vertex $w \in P \times \{1\}$ such that $\langle y, w\rangle < 1$. However, $y \in \operatorname{int}\tilde{\sigma}^\vee \subseteq \operatorname{int}\sigma^\vee$ implies $0 < \langle y, w\rangle$. This contradicts $\langle y, w\rangle \in \mathbb{Z}$. Now, $\langle \cdot, (x, 1)\rangle \leqslant s$ is a valid inequality for R, and by the above, $\operatorname{int}(kR) \cap (\mathbb{Z}^{n+1})^* = \varnothing$ for $k \leqslant s^{-1} = \mu(P)$.

On the other hand, u is a lattice point in $\operatorname{int}\tilde{\sigma}^\vee$ with $\langle u, (x, 1)\rangle = 1$. So u is in $\operatorname{int}(kR) \cap (\mathbb{Z}^{n+1})^*$ for $k > \mu(P)$. Hence, $r := \operatorname{cd}(R) = \lfloor \mu(P)\rfloor + 1$.

From Lemma 3.5 applied to R and $(u, r) \in \operatorname{pos}(R \times \{1\}) \cap (\mathbb{Z}^{n+2})^*$ we conclude that

$$(u, r) = k(0, 1) + (\eta_1, 1) + \cdots + (\eta_g, 1) + (p, j)$$

for a natural number k, for (not necessarily different) nonzero vertices $\eta_1, \ldots, \eta_g$ of R and for a lattice point $p \in (jR) \cap (\mathbb{Z}^{n+1})^*$ with the property that $(p, j) = (0, 0)$ or $1 \leqslant j \leqslant n + 2 - r$.

From $u \notin (r-2)R$ and $(u, r-2) = (k-2)(0, 1) + (\eta_1, 1) + \cdots + (\eta_g, 1) + (p, j)$, we conclude that $k - 2 < 0$, that is, $k \in \{0, 1\}$. Further, if $k = 1$, then u is in $(r-1)R \setminus \operatorname{int}((r-1)R)$ so that $1 = \langle u, (x, 1)\rangle = (r-1)s$, that is, $\mu(P) \in \mathbb{Z}$.

Let us first consider the case $k = 0$. Since $u \in \operatorname{int}(rR)$, we observe that

$$(u, r) \notin \operatorname{pos}\big((\eta_1, 1), \ldots, (\eta_g, 1)\big),$$

thus $(p, j) \neq (0, 0)$. Therefore, $r = g + j$, and u splits into a sum of at least $g + 1 \geqslant r + 1 - (n + 2 - r) = 2\lfloor \mu(P)\rfloor - n + 1$ nonzero lattice vectors in $\tilde{\sigma}^\vee$. Hence, Lemma 3.2(3) yields that P is a Cayley polytope of lattice polytopes in $\mathbb{R}^m$ with $m \leqslant n + 1 - (g + 1) \leqslant 2(n - \lfloor \mu(P)\rfloor)$.

It remains to deal with the case $k = 1$. Here, we have already observed that $\mu(P) \in \mathbb{Z}$. If $(p, j) = (0, 0)$, then u splits into a sum of at least $g + 1 = r$ nonzero lattice points in $\tilde{\sigma}^\vee$, so Lemma 3.2(3) yields that P is the Cayley polytope of lattice polytopes in $\mathbb{R}^m$ with $m \leqslant n+1-(g+1) \leqslant n+1-\mu(P)$. Finally, if $(p, j) \neq (0, 0)$, then $r = g+1+j$, so we again deduce from Lemma 3.2(3) that P is the Cayley polytope of $g+1 = r-j \geqslant r-(n+2-r) = 2r-n-2$ lattice polytopes in an ambient space of dimension $n+1-(2r-n-2) = 2(n-\mu(P))+1$. □

Remark 3.6. Statement and proof of Theorem 3.4 generalize Theorem 3.1 in [Haase et al. 2009], which proves Conjecture 2 in the case of *Gorenstein polytopes*. A Gorenstein polytope P with codegree c can be characterized by the property that P is a $\mathbb{Q}$-normal lattice polytope with $(cP)^{(1)}$ being a lattice point.

Corollary 3.7. *Let P be an n-dimensional lattice polytope. If n is odd and $\mu(P) > (n+1)/2$, or if n is even and $\mu(P) \geqslant (n+2)/2$, then P is a Cayley polytope.*

There is no obvious analogue for *rational* polytopes. For instance, for $\varepsilon > 0$, the $\mathbb{Q}$-codegree of $(1+\varepsilon)\Delta_n$ equals $(n+1)/(1+\varepsilon)$, so it gets arbitrarily close to $n+1$, however its lattice width is always strictly larger than one.

Theorem 3.4 proves Conjecture 2 if $\lceil \mu(P) \rceil = \operatorname{cd}(P)$. Therefore, we get the following new result using Proposition 1.13.

Corollary 3.8. *Conjecture 2 holds if $\mathcal{N}(P)$ is Gorenstein and P is $\mathbb{Q}$-normal.*

If P is smooth with $\operatorname{cd}(P) > (n+2)/2$, then it was shown in [Dickenstein et al. 2009; Dickenstein and Nill 2010] that $P \cong P_0 * \cdots * P_t$, where $t+1 = \operatorname{cd}(P) = \mu(P)$, and $P_0, \dots, P_t$ have the same normal fan. The proof relies on algebraic geometry; no purely combinatorial proof is known.

A sharper conjecture. We conjecture that in Corollary 3.7 the condition $\mu(P) > (n+1)/2$ should also be sufficient in even dimension. This is motivated by an open question in algebraic geometry — see Remark 4.10. We can prove this conjecture in the case of lattice simplices.

Proposition 3.9. *Let $P \subseteq \mathbb{R}^n$ be an n-dimensional rational simplex. Let a_i be the lattice distance of the i-th vertex of P from the facet of P not containing the vertex. Then*

$$\tau(P) = \mu(P) = \sum_{i=0}^{n} \frac{1}{a_i}.$$

Proof. Let x be the unique point that has the same lattice distance s from each facet. Then $\tau(P)^{-1} = \mu(P)^{-1} = s$. Fix a basis $\{e_0, \dots, e_n\}$ for $\mathbb{R}^{n+1}$ and consider the affine isomorphism

$$P \to \operatorname{conv}(a_0 e_0, \dots, a_n e_n) = \left\{ y \in \mathbb{R}^{n+1}_{\geqslant 0} : \sum_{i=0}^{n} \frac{y_i}{a_i} = 1 \right\} \subset \mathbb{R}^{n+1}$$

given by $y \mapsto \big(d_{F_0}(y), \dots, d_{F_n}(y)\big)$. The point x is mapped to $c := (s, \dots, s)$, so $1/s = \sum_{i=0}^{n} 1/a_i$. □

Corollary 3.10. *Let $P \subseteq \mathbb{R}^n$ be an n-dimensional lattice simplex.*

(1) *If $\mu(P) > (n+1)/2$ (or $\mu(P) = (n+1)/2$ and $a_i \neq 2$ for some i), then P is a lattice pyramid.*

(2) *If $\mu(P) \geqslant (n+1)/2$ and $P \ncong 2\Delta_n$, then P has lattice width one.*

Proof. Assume that P is not a lattice pyramid. Then $a_i \geqslant 2$ for all $i = 0, \dots, n$. Hence,

$$\mu(P) = \sum_{i=0}^{n} \frac{1}{a_i} \leqslant \frac{n+1}{2}.$$

This proves (1). For (2), let us assume that $a_i = 2$ for all $i = 0, \dots, n$. We consider the injective affine map

$$\mathbb{R}^n \to \mathbb{R}^n, \quad y \mapsto (d_{F_1(y)}, \dots, d_{F_n}(y)).$$

Note that the image of P is $2\Delta_n = \operatorname{conv}(0, 2e_1, \dots, 2e_n)$. Let us denote the image of $\mathbb{Z}^n$ by Λ. It satisfies $2\mathbb{Z}^n \subseteq \Lambda \subseteq \mathbb{Z}^n$. If $\Lambda = \mathbb{Z}^n$, then $P \cong 2\Delta_n$. Hence, our assumption yields that the reduction modulo 2 is a proper linear subspace $\Lambda/2\mathbb{Z}^n \subset (\mathbb{Z}/2\mathbb{Z})^n$. Therefore, it must satisfy an equation $\sum_{i\in I} x_i \equiv 0 \bmod 2$ for some subset $\varnothing \neq I \subseteq \{1, \dots, n\}$. The linear functional $\frac{1}{2}\big(\sum_{i\in I} x_i\big)$ defines an element $\lambda \in \Lambda^*$ such that $\lambda(2e_i) = 1$ if $i \in I$ and 0 otherwise. Hence, P has lattice width one in the direction of the pullback of λ. □

Example 3.11. It is tempting to guess that $\mu(P) = (n+1)/2$ and $a_i = 2$ for all i implies that $P \cong 2\Delta_n$. However, here is another example:

$$\operatorname{conv}\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}.$$

A corresponding result for the codegree was proven in [Nill 2008] where it is shown that a lattice n-simplex is a lattice pyramid if $\operatorname{cd}(P) \geqslant \frac{3}{4}(n+1)$. Let us stress that Conjecture 2 is still open for lattice simplices.

4. Adjunction theory of toric varieties

In this section, we explain the connection between the previous combinatorial results and the adjunction theory of toric varieties.

General notation and definitions. Let X be a normal projective algebraic variety of dimension n with canonical class K_X defined over the complex numbers. We assume throughout that X is $\mathbb{Q}$-*Gorenstein* of index r, that is, r is the minimal $r \in \mathbb{N}_{>0}$ such that rK_X is a Cartier divisor. X is called *Gorenstein* if $r = 1$.

Let L be an ample line bundle (we will often use the same symbol for the associated Weil divisor) on X. We use the additive notation to denote the tensor operation in the Picard group $\mathrm{Pic}(X)$. When we consider (associated) $\mathbb{Q}$-divisors, the same additive notation will be used for the operation in the group $\mathrm{Div(X)} \otimes \mathbb{Q}$.

Recall that L is *nef* (respectively, *ample*) if it has nonnegative (respectively, positive) intersection with all irreducible curves in X. Moreover, L is said to be *big* if the global sections of some multiple define a birational map to a projective space. If a line bundle is nef, then being big is equivalent to having positive degree. It follows that every ample line bundle is nef and big. The pair (X, L), where L is an ample line bundle on X, is often called a *polarized algebraic variety*. The linear systems $|K_X + sL|$ are called *adjoint linear systems*. These systems define classical invariants which have been essential tools in the existent classification of projective varieties. In what follows we summarize what is essential to understand the results in this paper. More details can be found in [Beltrametti and Sommese 1995, 1.5.4 and 7.1.1].

Definition 4.1. Let (X, L) be a polarized variety.

(1) The *unnormalized spectral value* of L is defined as

$$\mu(L) := \sup\big\{s \in \mathbb{Q} \colon h^0\big(N(K_X + sL)\big) = 0 \text{ for all positive integers such that } N(K_X + sL) \text{ is an integral Cartier divisor}\big\}.$$

Note that $\mu(L) < \infty$ follows from L being big.

(2) The *nef value* of L is defined as

$$\tau(L) := \min\{s \in \mathbb{R} \colon K_X + sL \text{ is nef}\}.$$

It was proven by Kawamata that $\tau(L) \in \mathbb{Q}$. Moreover if $r\tau = u/v$, where u and v are coprime, then the linear system $|m(vrK_X + uL)|$ is globally generated for a big enough integer m. The corresponding morphism

$$f : X \to \mathbb{P}^M = \mathbb{P}\big(H^0(m(vrK_X + uL))\big)$$

has a Remmert–Stein factorization as $f = p \circ \varphi_\tau$, where $\varphi_\tau : X \to Y$ is a morphism with connected fibers onto a normal variety Y, called the *nef value morphism*. The rationality of $\mu(L)$ was only shown very recently [Birkar et al. 2010, Corollary 1.1.7] as a consequence of the existence of the minimal model program.

Observe that the invariants above can be visualized as follows — see Figure 8. Traveling from L in the direction of the vector K_X in the Neron–Severi space $\mathrm{NS}(X) \otimes \mathbb{R}$ of divisors, $L + (1/\mu(L))K_X$ is the meeting point with the cone of effective divisors $\mathrm{Eff}(X)$ and $L + (1/\tau(L))K_X$ is the meeting point with the cone of nef-divisors $\mathrm{Nef}(X)$. We now summarize some well-known results which will be used in this section.

Proposition 4.2. (1) *$\tau(L)$ is the largest $s \in \mathbb{Q}$ such that $K_X + sL$ is nef but not ample.*

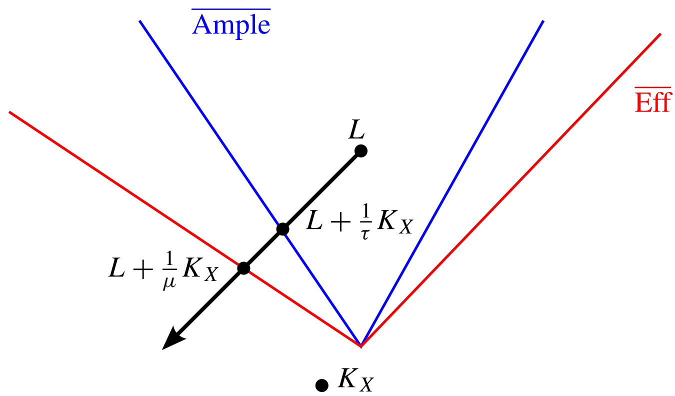


Figure 8. Illustrating $\mu(L)$ and $\tau(L)$.

(2) $\mu(L) \leqslant \tau(L)$, *with equality if and only if* φ_τ *is not birational.*

(3) *Let* $r\tau(L) = u/v$ *with coprime positive integers* u, v. *Then*

$$u \leqslant r(n+1),$$

in particular, $\tau(L) \leqslant r(n+1)$.

(4) $\mu(L) \leqslant n+1$.

Proof. Statement (1) is proven in [Beltrametti and Sommese 1995, 1.5.5]. For (2) observe that the interior of the closure of the effective cone is the big cone, $\overline{\mathrm{Eff}(X)}^{\mathrm{int}} = \mathrm{Big}(X)$. Recall that if a divisor is not big, then the map associated to the global sections has a lower-dimensional image. It follows that the map is birational only when τ and ν do not coincide. A proof can be also found in [Beltrametti and Sommese 1995, 7.1.6]. Statement (3) is part of Kawamata's rationality theorem and (4) is proven in [Beltrametti and Sommese 1995, 7.1.3]. □

Remark 4.3. There are at least three other notions which are related to the unnormalized spectral value. The (nonnegative) *spectral value* $\sigma(L) := n+1-\mu(L)$ was defined by Sommese [1986] (compare this notion with the *degree* of lattice polytopes [Batyrev and Nill 2007]). Fujita [1992] defined the (nonpositive) *Kodaira energy* $\kappa\varepsilon(L)$ as $-\mu(L)$ — see also [Batyrev and Tschinkel 1998]. Furthermore, the reciprocal $\mu(L)^{-1}$ is called the *effective threshold* — see, for example, [Birkar et al. 2010].

There are several classifications of polarized varieties with large nef value. For example:

Theorem 4.4 [Fujita 1987]. *Let* (X, L) *be a polarized normal Gorenstein variety with* $\dim(X) = n$. *Then*:

(1) $\tau(L) \leqslant n$ *unless* $(X, L) = (\mathbb{P}^n, \mathbb{O}_{\mathbb{P}^n}(1))$.

(2) $\tau(L) < n$ *unless we are in one of the following cases*:
   (a) $(X, L) = (\mathbb{P}^n, \mathbb{O}_{\mathbb{P}^n}(1))$.
   (b) X *is a quadric hypersurface and* $L = \mathbb{O}_X(1)$.
   (c) $(X, L) = (\mathbb{P}^2, \mathbb{O}_{\mathbb{P}^n}(2))$.
   (d) $(X, L) = (\mathbb{P}(E), \mathbb{O}(1))$, *where* E *is a vector bundle of rank* n *over a nonsingular curve.*

In the same paper Fujita also classifies the cases $\tau(L) \geqslant n-2$ and $\tau(L) \geqslant n-3$. We will discuss this classification in the toric setting and the induced classification of lattice polytopes with no interior lattice points in the Appendix.

Toric geometry. We refer the reader who is unfamiliar with toric geometry to [Fulton 1993]. In what follows we will assume that X is a $\mathbb{Q}$-*Gorenstein* toric variety of Gorenstein index r and dimension n. Let L be an (equivariant) line bundle on X. Let $N \cong \mathbb{Z}^n$, $\Sigma \subset N \otimes \mathbb{R}$ be the defining fan and denote by $\Sigma(i)$ the set of cones of Σ of dimension i. For $\tau \in \Sigma(i)$, $V(\tau)$ will denote the associated invariant subvariety codimension i.

Recall that L is nef (resp., ample) if and only if $L \cdot V(\rho_j) \geqslant 0$ (resp., > 0) for all $\rho_j \in \Sigma(n-1)$; see [Mustaţă 2002, Theorem 3.1], for example.

There is a one-to-one correspondence between n-dimensional toric varieties polarized by an ample line bundle L and n-dimensional convex lattice polytopes $P_{(X,L)} \subset M \otimes \mathbb{R}$ (up to translations by a lattice vector), where M is the lattice dual to N. Under this correspondence k-dimensional invariant subvarieties of X are associated with k-dimensional faces of $P_{(X,L)}$. More precisely, if

$$P = \{x \in \mathbb{R}^n : Ax \geqslant b\}$$

for an $m \times n$ integer matrix A with primitive rows and $b = (b_1, \dots, b_m) \in \mathbb{Z}^m$, then $L = \sum(-b_i)D_i$, where $D_i = V(\beta_i)$, for $\beta_i \in \Sigma(1)$, are the invariant divisors, generating the Picard group.

More generally, a nef line bundle $\mathcal{L}$ on a toric variety X' defines a polytope $P_{\mathcal{L}} \subset \mathbb{R}^n$, not necessarily of maximal dimension, whose integer points correspond to characters on the torus and form a basis of $H^0(X', \mathcal{L})$. The edges of the polytope $P_{\mathcal{L}}$ correspond to the invariant curves whose intersection with $\mathcal{L}$ is positive. In particular, the normal fan of $P_{\mathcal{L}}$ does not necessarily coincide with the fan of X'. It is the fan of a toric variety X obtained by possibly contracting invariant curves on X'. The contracted curves correspond to the invariant curves having zero intersection with $\mathcal{L}$. Let $\pi : X' \to X$ be the contraction morphism. There is an ample line bundle L on X such that $\pi^*(L) = \mathcal{L}$. Because the dimension of the polytope equals the dimension of the image of the map defined by the global sections one sees immediately that $P_{\mathcal{L}}$ has maximal dimension if and only if $\mathcal{L}$ is big.

Adjoint bundles. (Compare with Section 1.) Let (X, L) be the polarized variety defined by the polytope $P = \{x \in \mathbb{R}^n : Ax \geqslant b\}$. Observe that for any $s \in \mathbb{Q}_{>0}$ the polytope $P^{(s)} := \{x \in \mathbb{R}^n : Ax \geqslant b + s\mathbb{1}\}$, with $\mathbb{1} = (1, \dots, 1)^{\mathrm{T}}$, corresponds to the $\mathbb{Q}$-line bundle $sK_X + L$. With this interpretation it is clear that

$$\mu(P) = \mu(L) \quad \text{and} \quad \tau(P) = \tau(L)$$

Remark 4.5. Proposition 1.14 gives us a geometric interpretation of these invariants. Let $k \in \mathbb{Z}$ such that $kM(P)$ is a lattice polytope and let Y be the associated toric variety. The polytope P is a facet of $M(P)$ and thus the variety X is an invariant divisor of Y. Moreover, the projection $M(P) \twoheadrightarrow P$ induces a rational surjective map $Y \to \mathbb{P}^1$ whose generic fiber (in fact all fibers but the one at ∞) are isomorphic to X.

Remark 4.6. From an inductive viewpoint, it would be desirable to know how "bad" the singularities of $P^{(1)}$ can get if we start out with a "nice" polytope P. However, this seems to be very hard. Traditionally, there is another way, the "onion skinning" of a polytope (see [Haase and Schicho 2009; Ogata 2007]) via the *interior polytope* $P^{[1]} := \operatorname{conv}(\operatorname{int}(P) \cap \mathbb{Z}^n)$. Recall that the lattice points of $P^{(1)}$ correspond to the global sections of $K_X + L_P$. If the line bundle $K_X + L_P$ is globally generated (equivalently nef) then $P^{(1)} = P^{[1]}$, but in general they might be different. Obviously, $P^{[1]} \subseteq P^{(1)}$, with equality if and only if $P^{(1)}$ is a lattice polytope. Ogata [2007] examined the case of smooth polytopes of dimension at most three with interior lattice points. He proves the following:

- *In dimension two*, $P^{(1)}$ equals $P^{[1]}$, and it is even a smooth polytope [Ogata 2007, Lemma 5].
- *In dimension three* [Ogata 2007, Proposition 3], it is claimed that by successively forgetting facet inequalities (corresponding to blow-downs) it is possible to obtain a smooth polytope $P' \supseteq P$ with $P'^{(1)} = P^{(1)} = P^{[1]}$ and $\tau(P') \leqslant 1$. Moreover, while $P^{[1]}$ may not be smooth anymore, Proposition 4 of [Ogata 2007] says that singular points of cones over $(\mathbb{P}^2, O(2))$ and $(\mathbb{P}^1 \times \mathbb{P}^1, O(1, 1))$ are the only possible singularities, occurring at the toric fixpoints of $X_{P^{[1]}}$.

It would be desirable to understand what happens in higher dimensions; for instance, we expect the answer to the following question to be negative:

Let P be a smooth four-dimensional polytope with interior lattice points. Is $P^{(1)}$ still a lattice polytope?

Admissible polarized toric varieties (compare with Section 2). In the language above, Proposition 2.3 states that if (X, L) is a polarized $\mathbb{Q}$-Gorenstein toric variety then there is a finite sequence of maps of toric varieties

$$X_k \to X_{k-1} \to \cdots \to X_2 \to X_1 \to X_0 = X$$

polarized by ample line bundles L_i. In fact, by considering the polytope $P = P_{(X,L)}$, Proposition 2.3 gives a projection $P \twoheadrightarrow Q$ from the linear space $\mathrm{Aff}(P^{(1/\mu(L))})$. The projection defines a map of fans $\Sigma_Q \to \Sigma_P$, and in turn a map of toric varieties $X_1 \to X$. Notice that $\dim(X_1) = \dim(X) - \dim(P^{(1/\mu(L))})$. Let L_1 be the polarization defined by Q on X_1. Starting again with (X_1, L_1) we look at the corresponding projection $Q \twoheadrightarrow Q_1$ and so on. Notice that the sequence will stop when $\mu(X_{k-1}) = \mu(X_k)$ and $\mathrm{core}(Q_k)$ is a single (rational) point. We remark that the $\mathbb{Q}$-codegree has been defined for any polytope while the spectral value is defined only for $\mathbb{Q}$-Gorenstein varieties. In more generality the singularities are quite subtle and it is not at all clear how to proceed within algebraic geometry. For this purpose we will call a polarized $\mathbb{Q}$-Gorenstein toric variety *admissible* if in the sequence above X_i is $\mathbb{Q}$-Gorenstein for every $0 \leqslant i \leqslant k$. Recall that the lattice points of $N\,\mathrm{core}(Q_k)$ correspond to the global sections $H^0\big(N(K_{X_k} + \mu(L_k)L_K)\big)$, for an integer N such that $N(K_{X_k} + \mu(L_k)L_K)$ is an integral line bundle. Then Proposition 2.3 reads as follows:

Proposition 4.7. *Let (X, L) be an admissible polarized $\mathbb{Q}$-Gorenstein toric variety. There is a finite sequence of maps of toric varieties*

$$X_k \to X_{k-1} \to \cdots \to X_2 \to X_1 \to X_0 = X$$

polarized by ample line bundles L_i such that $\mu(L_i) \geqslant \mu(L_{i-1})$ for $1 \leqslant i \leqslant k$ and $H^0\big(N(K_{X_k} + \mu(L_k)L_K)\big)$ consists of a single section for an integer N such that $N(K_{X_k} + \mu(L_k)L_K)$ is an integral line bundle.

Example 4.8. The polytope in Figure 6 defines an admissible polarized $\mathbb{Q}$-Gorenstein toric variety. Let (X, L) be the associated polarized toric variety. The (unnormalized) spectral value satisfies $\mu(L) = \mu(P) = \frac{3}{4}$. The polytope has the following description:

$$P = \left(\begin{pmatrix} x \\ y \\ z \end{pmatrix} \;\middle|\; \begin{array}{rcl} x & \geqslant & 0 \\ y & \geqslant & 0 \\ z & \geqslant & 0 \\ x+y & \leqslant & 4 \\ hx+2z & \leqslant & 2h \end{array} \right) \qquad \text{if } h \text{ odd,}$$

$$P = \left(\begin{pmatrix} x \\ y \\ z \end{pmatrix} \;\middle|\; \begin{array}{rcl} x & \geqslant & 0 \\ y & \geqslant & 0 \\ z & \geqslant & 0 \\ x+y & \leqslant & 4 \\ kx+z & \leqslant & 2k \end{array} \right) \qquad \text{if } h = 2k \text{ for some integer } k.$$

For simplicity let us assume that h is odd. From the polytope one sees that $\mathrm{Pic}(X)$ is generated by $D_1, \ldots, D_5$ with the linear relations

$$D_1 \sim hD_5 + D_4, \quad D_2 \sim D_4, \quad D_3 \sim 2D_5$$

Moreover, $L = 4D_4 + 2hD_5$ and $K_X = -3D_4 - (h+3)D_5$, giving $4K_X + 3L = (2h-12)D_5$, which is effective for $h \geqslant 6$. The first projection onto Q defines in this case an invariant subvariety X_1 which is isomorphic to $\mathbb{P}^2$ blown up at one point. Moreover $L_1 = L|_{X_1} = 4l - 2E$, where l is the pull back of the hyperplane line bundle on $\mathbb{P}^2$ and E is the exceptional divisor. The variety X_1 is smooth and therefore $\mathbb{Q}$-Gorenstein of index 1. Starting again with (X_1, L_1) we have $\nu(L_1) = 1$ and $X_2 \cong \mathbb{P}^1$ with $L_2 = \mathbb{O}_{\mathbb{P}^1}(2)$, which give $\mu(L_2) = 1$ and $H^0(K_{X_2} + L_2) = H^0(\mathbb{O}_{X_2})$.

It would be desirable to have criteria for a toric polarized $\mathbb{Q}$-Gorenstein variety to be admissible.

The main result. (Compare with Section 3.) As explained in [Haase et al. 2009; Dickenstein et al. 2009] the toric variety X, defined by a Cayley polytope

$$P = P_0 * \cdots * P_t,$$

has a prescribed birational morphism to the toric projectivized bundle

$$X = \mathbb{P}(H_0 \oplus H_1 \oplus \cdots \oplus H_t)$$

over a toric variety Y. The variety Y is defined by a common refinement of the inner normal fans of the polytopes P_i. Moreover, the polytopes P_i are associated to the nef line bundles H_i over Y. As a consequence of Theorem 3.4 we get the following result.

Proposition 4.9. *Let (X, L) be a polarized $\mathbb{Q}$-Gorenstein toric variety. Suppose $q \in \mathbb{Q}_{>0}$ such that $2q \leqslant n$ and no multiple of $K_X + (n+1-q)L$ which is Cartier has nonzero global sections. Then there is a proper birational toric morphism $\pi : X' \to X$, where X' is the projectivization of a sum of line bundles on a toric variety of dimension at most $\lfloor 2q \rfloor$ and π^*L is isomorphic to $O(1)$.*

Proof. The assumption $2q \leqslant n$ implies that $\mu(L) \geqslant (n+2)/2$. Theorem 3.4 gives the conclusion. □

Remark 4.10. It is conjectured on page 2434 that $\mu(L) > (n+1)/2$ should suffice in Corollary 3.7. One algebro-geometric statement which hints at this possibility is a conjecture by Beltrametti and Sommese [1995, 7.1.8] stating that $\mu(L) > (n+1)/2$ should imply $\mu(L) = \tau(L)$ when the variety is nonsingular. Moreover, it was also conjectured in [Fania and Sommese 1989] that if $\mu(L) > 1$, then $\mu(L) = p/q$ for integers $0 < q \leqslant p \leqslant n+1$. In particular, $\mu(L) > (n+1)/2$ would again imply $\mu(L) \in \mathbb{Z}$.

Let A be the set of lattice points of a lattice polytope P, and let X_A be the (not necessarily normal) toric variety embedded in $\mathbb{P}^{|A|-1}$. Then there is an irreducible polynomial, called the *A-discriminant*, which is of degree zero if and only if the dual variety X_A^* is not a hypersurface (that is, X_A has *dual defect*) — see [Gelfand et al. 1994].

Proposition 4.11. *Let P be a lattice polytope with $\mu(P) \geqslant (3n+4)/4$ if $\mu(P) \notin \mathbb{N}$, or $\mu(P) \geqslant (3n+3)/4$ if $\mu(P) \in \mathbb{N}$. Then X_A has dual defect.*

Proof. By Theorem 3.4, P is a Cayley polytope of at least $n+1-d$ lattice polytopes in $\mathbb{R}^d$, where the assumptions yield that $n+1-d \geqslant d+2$. Then Proposition 6.1 and Lemma 6.3 in [Dickenstein et al. 2007] imply the desired result. Note that in the notation of [Dickenstein et al. 2007] $m = n+1-d$, $r = d$, and $c = m - r \geqslant 2$. □

For smooth polarized toric varieties it was verified that the assumption $\mu(L) > (n+2)/2$ is equivalent to the variety having dual defect [Dickenstein and Nill 2010]. Moreover, smooth dual-defective varieties are necessarily $\mathbb{Q}$-normal ($\mu(L) = \tau(L)$) by [Beltrametti et al. 1992]. By the results of [Di Rocco 2006; Dickenstein et al. 2009] this implies that the associated lattice polytope is a smooth Cayley polytope with $\mu(L) = \mathrm{cd}(P)$ factors which all share the same normal fan. On the other hand, it has recently been shown [Curran and Cattani 2007; Esterov 2010] that all lattice points in a (possibly singular) dual-defective polytope have to lie on two parallel hyperplanes. However, it is not true that all Cayley polytopes, or polytopes of lattice width 1, are dual defective, even in the nonsingular case. Therefore, the main question is whether the following strengthening of Proposition 4.11 may be true — see [Dickenstein and Nill 2010]:

Question 4.12. Is (X, L) dual defective if $\mu(L) > (n+2)/2$?

Appendix: Fujita's classification results

In this section we provide a translation of the results in [Fujita 1987, Theorem 2 and 3′]. A straightforward corollary gives the classification of smooth polytopes of dimension three with no interior lattice points. One could derive a more extensive classification from the theorems just cited and from later work such as [Beltrametti and Di Termini 2003; Nakamura 1997]. This would require a more elaborate explanation, which goes beyond the scope of this paper.

Theorem A.1 [Fujita 1987]. *Let P be an n-dimensional lattice polytope such that its normal fan is Gorenstein.*

(1) *If $\tau(P) > n$, then $P \cong \Delta_n$.*
(2) *If $n-1 < \tau(P) \leqslant n$, then $P \cong 2\Delta_2$ or $P \cong P_0 * P_1 * \cdots * P_{n-1}$, where the P_i are parallel intervals.*
(3) *If P is smooth and $n-2 < \tau(P) \leqslant n-1$, then P is one of these polytopes*:

(a) *There is a smooth n-dimensional polytope P' and a unimodular simplex $S \not\subseteq P$ such that*

$$P' = P \cup S$$

and $P \cap S$ is a common facet of P and S.

(b) $P^{(1/(n-1))}$ *is a point.*

(c) $P = 2\Delta_3, 3\Delta_3, 2\Delta_4$.

(d) *There is a projection $\pi : P \twoheadrightarrow \Delta_1 \times \Delta_1$*

(e) *There is a projection $\pi : P \twoheadrightarrow 2\Delta_2$ and the polytopes $\pi^{-1}(m_i)$ have the same normal fan, where m_i are the vertices of $2\Delta_2$.*

(f) $P \cong P_0 * P_1 * \cdots * P_{n-2}$, *where the P_i are smooth polygons with the same normal fan.*

Note that in (3)(a) P is given by a vertex truncation of P' (compare with Figure 2), corresponding to a blow-up at a smooth point. The following result is a simple corollary of the previous classification. It was also obtained in a slightly weaker form by Ogata [2007, Proposition 1], using combinatorial methods.

Corollary 4.2. *Let P be a smooth 3-dimensional polytope with no interior lattice points. Then P is of one of the following types.*

(1) $P = \Delta_3, 2\Delta_3, 3\Delta_3$.

(2) *There is a projection $P \twoheadrightarrow \Delta_2$, where any preimage of each vertex is an interval. Equivalently, there are $a, b, c \in \mathbb{Z}$ such that*

$$P = \operatorname{conv}\begin{bmatrix} 0 & 0 & 0 & a & b & c \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

(3) *There is a projection $P \twoheadrightarrow 2\Delta_2$, where any preimage of each vertex is an interval. Equivalently, there are $a, b, c \in \mathbb{Z}$ such that*

$$P = \operatorname{conv}\begin{bmatrix} 0 & 0 & 0 & a & b & c \\ 0 & 2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 2 \end{bmatrix}.$$

(4) *There is a projection $P \twoheadrightarrow \Delta_1 \times \Delta_1$. Equivalently, there are $a, b, c \in \mathbb{Z}$ such that*

$$P = \operatorname{conv}\begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & a & b & c & a+b-c \end{bmatrix}.$$

(5) $P = P_0 * P_1$, *where P_0 and P_1 are smooth polygons with the same normal fan.*

(6) *There is a smooth 3-dimensional polytope P' with no interior lattice points and a unimodular simplex $S \not\subseteq P$ such that*

$$P' = P \cup S$$

and $P \cap S$ is a common facet of P and S.

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Genericity and contragredience in the local Langlands correspondence

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Adams, Vogan, and D. Prasad have given conjectural formulas for the behavior of the local Langlands correspondence with respect to taking the contragredient of a representation. We prove these conjectures for tempered representations of quasisplit real K-groups and quasisplit p-adic classical groups (in the sense of Arthur). We also prove a formula for the behavior of the local Langlands correspondence for these groups with respect to changes of the Whittaker data.

1. Introduction

The local Langlands correspondence is a conjectural relationship between certain representations of the Weil or Weil–Deligne group of a local field F and finite sets, or packets, of representations of a locally compact group arising as the F-points of a connected reductive algebraic group defined over F. In characteristic zero, this correspondence is known for $F=\mathbb{R}$ and $F=\mathbb{C}$ by the work of Langlands [1989] and was later generalized and reinterpreted geometrically by Adams, Barbasch, and Vogan [Adams et al. 1992]. Furthermore, many cases are known when F is a finite extension of the field $\mathbb{Q}_p$ of p-adic numbers. Most notably, the correspondence over p-adic fields is known when the reductive group is GL_n by work of Harris and Taylor [2001] and Henniart [2000], and has very recently been obtained for quasisplit symplectic and orthogonal groups by Arthur [2013]. Other cases include the group U_3 by work of Rogawski, Sp_4 and GSp_4 by work of Gan–Takeda. For general connected reductive groups, there are constructions of the correspondence for specific classes of parameters, including the classical case of unramified representations, the case of representations with Iwahori-fixed vector by work of Kazhdan–Lusztig, unipotent representations by work of Lusztig, and more recently regular depth-zero supercuspidal representations by [DeBacker and Reeder 2009], very cuspidal representations [Reeder 2008], and epipelagic representations [Kaletha 2012].

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The purpose of this paper is to explore how the tempered local Langlands correspondence behaves with respects to two basic operations on the group. The first operation is that of taking the contragredient of a representation. In a recent paper, Adams and Vogan [2012] studied this question for the general (not just tempered) local Langlands correspondence for real groups. They provide a conjecture on the level of L-packets for any connected reductive group over a local field F and prove this conjecture when F is the field of real numbers. One of our main results is the fact that this conjecture holds for the tempered L-packets of symplectic and special orthogonal p-adic groups constructed by Arthur. In fact, inspired by the work of Adams and Vogan, we provide a refinement of their conjecture to the level of representations, rather than packets, for the tempered local Langlands correspondence. We prove this refinement when G is either a quasisplit connected real reductive group (more generally, quasisplit real K-group), a quasisplit symplectic or special orthogonal p-adic group, and in the context of the constructions of [DeBacker and Reeder 2009] and [Kaletha 2012]. In the real case, the results of Adams and Vogan are a central ingredient in our argument. To obtain our results, we exploit the internal structure of real L-packets using recent results of Shelstad [2008]. In the case of quasisplit p-adic symplectic and special orthogonal groups, we prove a result similar to that of Adams and Vogan using Arthur's characterization of the stable characters of L-packets on quasisplit p-adic classical groups as twisted transfers of characters of GL_n. After that, the argument is the same as for the real case. The constructions of [DeBacker and Reeder 2009] and [Kaletha 2012] are inspected directly.

The second basic operation that we explore is that of changing the Whittaker datum. To explain it, we need some notation. Let F be a local field and G a connected reductive group defined over F. Let W' be the Weil group of F if $F = \mathbb{R}$ or the Weil–Deligne group of F if F is an extension of $\mathbb{Q}_p$. Then, if G is quasisplit, it is expected that there is a bijective correspondence $(\varphi, \rho) \mapsto \pi$. The target of this correspondence is the set of equivalence classes of irreducible admissible tempered representations. The source of this correspondence is the set of pairs (φ, ρ) where $\varphi : W' \to {}^LG$ is a tempered Langlands parameter, and ρ is an irreducible representation of the finite group $\pi_0(\mathrm{Cent}(\varphi, \hat{G})/Z(\hat{G})^\Gamma)$. Here $\hat{G}$ is the complex (connected) Langlands dual group of G, and ${}^LG$ is the L-group of G. However, it is known that such a correspondence can in general not be unique. In order to hope for a unique correspondence, following Shahidi [1990, Section 9] one must choose a Whittaker datum for G, which is a $G(F)$-conjugacy class of pairs (B, ψ) where B is a Borel subgroup of G defined over F and ψ is a generic character of the F-points of the unipotent radical of B. Then it is expected that there exists a bijection $(\varphi, \rho) \mapsto \pi$ as above which has the property that π has a (B, ψ)-Whittaker functional precisely when $\rho = 1$. Let us denote this conjectural correspondence by $\iota_{B,\psi}$. We are interested in how it varies when one varies the Whittaker datum (B, ψ). We remark that there is a further normalization of $\iota_{B,\psi}$ that must be chosen. As described in [Kottwitz and Shelstad 2012, Section 4], it is expected that there will be two normalizations of the local Langlands correspondence for reductive groups, reflecting the two possible normalizations of the local Artin reciprocity map. Related to these normalizations are the different normalizations of the transfer factors Δ, Δ', Δ_D and Δ'_D for ordinary and twisted endoscopy described in [Kottwitz and Shelstad 2012, Section 5].

The reason we study these two questions together is that they appear to be related. Indeed, when one studies how the pair (φ, ρ) corresponding to a representation π changes when one takes the contragredient of π, one is led to consider $\iota_{B,\psi}$ for different Whittaker data.

We will now go into more detail and describe our expectation for the behavior of the local Langlands correspondence with respect to taking contragredient and changing the Whittaker datum. We emphasize that we claim no originality for these conjectures. Our formula in the description of the contragredient borrows greatly from the paper of Adams and Vogan, as well as from a conversation with Robert Kottwitz, who suggested taking the contragredient of ρ. After the paper was written, we were informed by Dipendra Prasad that an equation closely related to (1) is stated as a conjecture in [Gan, Gross, and Prasad 2012, Section 9], and that moreover (2) is part of a more general framework of conjectures currently being developed by him under the name “relative local Langlands correspondence”. We refer the reader to the draft [Prasad 2012].

We continue to assume that F is either real or p-adic, and G is a quasisplit connected reductive group defined over F. Fix a Whittaker datum (B, ψ). For any Langlands parameter $\varphi : W' \to {}^LG$, let $S_\varphi = \mathrm{Cent}(\varphi, \hat{G})$. The basic form of the expected tempered local Langlands correspondence is a bijection $\iota_{B,\psi}$ from the set of pairs (φ, ρ), where φ is a tempered Langlands parameter and ρ is an irreducible representation of $\pi_0(S_\varphi/Z(\hat{G})^\Gamma)$ to the set of equivalence classes of irreducible admissible tempered representations. A refinement of this correspondence is obtained when one allows ρ to be an irreducible representation of $\pi_0(S_\varphi)$ rather than its quotient $\pi_0(S_\varphi/Z(\hat{G})^\Gamma)$. The right-hand side is then the set of equivalence classes of tuples (G', ξ, u, π), where $\xi : G \to G'$ is an inner twist, $u \in Z^1(F, G)$ is an element with the property $\xi^{-1}\sigma(\xi) = \mathrm{Int}(u(\sigma))$ for all $\sigma \in \Gamma$, and π is an irreducible admissible tempered representation of $G'(F)$. The triples (G', ξ, u) are called pure inner twists of G, and the purpose of this refined version of the correspondence is to include connected reductive groups which are not quasisplit. The idea of using pure inner forms is due to Vogan, and one can find a formulation of this refinement of the correspondence in [Vogan 1993] or [DeBacker and Reeder 2009, Section 3]. A further refinement is obtained by allowing ρ to be an irreducible algebraic representation of the complex algebraic group $\bar{S}_\varphi = S_\varphi/[S_\varphi \cap \hat{G}_{\mathrm{der}}]^\circ$. The right-hand side then is the set of equivalence classes of tuples (G', ξ, b, π), where $\xi : G \to G'$ is an inner twist and b is a basic element of $Z^1(W, G(\bar{L}))$, where L is the completion of the maximal unramified extension of F, and where b gives rise to ξ as in [Kottwitz 1997]. This further refinement was introduced by Kottwitz in an attempt to include all connected reductive groups into the correspondence (it is known that not every connected reductive group is a pure inner form of a quasisplit group). Indeed, when the center of G is connected, all inner forms of G come from basic elements of $Z^1(W, G(\bar{L}))$. Moreover, one can reduce the general case to that of connected center. An exposition of this formulation of the correspondence can be found in [Kaletha 2011].

We now let $\iota_{B,\psi}$ denote any version of the above conjectural correspondence, normalized so that $\iota_{B,\psi}(\varphi, \rho)$ is (B, ψ)-generic precisely when $\rho = 1$. The set of Whittaker data for G is a torsor for the abelian group $G_{\mathrm{ad}}(F)/G(F)$. Dualizing Langlands' construction of a character on $G(F)$ for each element of $H^1(W, Z(\hat{G}))$, one obtains from each element of $G_{\mathrm{ad}}(F)/G(F)$ a character on the finite abelian group $\ker(H^1(W, Z(\hat{G}_{\mathrm{sc}})) \to H^1(W, Z(\hat{G})))$. This groups accepts a map from $\pi_0(S_\varphi/Z(\hat{G})^\Gamma)$ for every Langlands parameter φ. In this way, given a pair of Whittaker data $\mathfrak{w}$ and $\mathfrak{w}'$, the element of $G_{\mathrm{ad}}(F)/G(F)$ which conjugates $\mathfrak{w}$ to $\mathfrak{w}'$ provides a character on $\pi_0(S_\varphi/Z(\hat{G})^\Gamma)$, hence also on $\pi_0(S_\varphi)$ and $\bar{S}_\varphi$. We denote this character by $(\mathfrak{w}, \mathfrak{w}')$. Then we expect that

$$\iota_{\mathfrak{w}'}(\varphi, \rho) = \iota_{\mathfrak{w}}\big(\varphi, \rho \otimes (\mathfrak{w}, \mathfrak{w}')^\epsilon\big), \tag{1}$$

where $\epsilon = 1$ if $\iota_{\mathfrak{w}}$ and $\iota_{\mathfrak{w}'}$ are compatible with endoscopic transfer via the transfer factors Δ' or Δ'_D, and $\epsilon = -1$ if $\iota_{\mathfrak{w}}$ and $\iota_{\mathfrak{w}'}$ are compatible endoscopic transfer via the transfer factors Δ or Δ_D.

To describe how we expect $\iota_{B,\psi}$ to behave with respect to taking contragredients, we follow [Adams and Vogan 2012] and consider the Chevalley involution on $\hat{G}$: As is shown in [Adams and Vogan 2012], there exists a canonical element of $\mathrm{Out}(\hat{G})$ which contains all automorphisms of $\hat{G}$ that act as inversion on some maximal torus. This canonical element provides a canonical $\hat{G}$-conjugacy class of L-automorphisms of ${}^LG$ as follows. Fix a Γ-invariant splitting of $\hat{G}$ and let $\hat{C} \in \mathrm{Aut}(\hat{G})$ be the unique lift of the canonical element of $\mathrm{Out}(\hat{G})$ which sends the fixed splitting of $\hat{G}$ to its opposite. Then $\hat{C}$ commutes with the action of Γ, and we put ${}^LC$ to be the automorphism of $\hat{G} \rtimes W$ given by $\hat{C} \rtimes \mathrm{id}$. If we change the splitting of $\hat{G}$, there exists [Kottwitz 1984, Corollary 1.7] an element $g \in \hat{G}^\Gamma$ which conjugates it to the old splitting. This element also conjugates the two versions of $\hat{C}$, and hence also the two versions of ${}^LC$. We conclude that $\hat{G}$-conjugacy class of ${}^LC$ is indeed canonical. Thus, for any Langlands parameter $\varphi : W' \to {}^LG$, we have a well-defined (up to equivalence) Langlands parameter ${}^LC \circ \varphi$. The automorphism $\hat{C}$ restricts to an isomorphism $S_\varphi \to S_{{}^LC\circ\varphi}$ and for each representation ρ of $\bar{S}_\varphi$ we can consider the representation $\rho\circ\hat{C}^{-1}$ of $\bar{S}_{{}^LC\circ\varphi}$. When φ is tempered, we expect

$$\iota_{B,\psi}(\varphi,\rho)^\vee = \iota_{B,\psi^{-1}}\big({}^LC\circ\varphi,\rho^\vee\circ\hat{C}^{-1}\big). \tag{2}$$

For this formula it is not important whether $\iota_{B,\psi}$ is normalized with respect to the classical or Deligne's normalization of the local Artin map, as long as $\iota_{B,\psi^{-1}}$ is normalized in the same way.

We will now briefly describe the contents of this paper. In Section 3, we recall the fundamental results of Arthur and Shelstad on the endoscopic classification of tempered representations of real and classical p-adic groups. In Section 4 we will describe more precisely the construction of the character $(\mathfrak{w},\mathfrak{w}')$ alluded to in this introduction, and will then prove (1). Section 5 is devoted to the proof of (2) for tempered representations of quasisplit real K-groups and quasisplit symplectic and special orthogonal p-adic groups. Finally, in Section 6 we consider depth-zero and epipelagic L-packets on general p-adic groups and prove (2) for those cases as well.

The arguments in Sections 4 and 5 are quite general and we expect them to provide a proof of (1) and (2) for other p-adic groups besides symplectic and orthogonal, as soon as Arthur's work has been extended to them. For example, we expect that the case of unitary groups will follow directly from our arguments.

2. Notation

Throughout this paper, F will denote either the field $\mathbb{R}$ or a finite extension of the field $\mathbb{Q}_p$. We write W for the absolute Weil group of F, and Γ for the absolute Galois group. We let W' stand for the Weil group of F when $F=\mathbb{R}$ and for the Weil–Deligne group of F if F is an extension of $\mathbb{Q}_p$.

Given a connected reductive group G defined over F, we will write $\hat{G}$ for the complex connected Langlands dual group of G, and ${}^LG$ for the L-group. Given a maximal torus $S\subset G$, we write $R(S,G)$ for the set of roots of S in G, $N(S,G)$ for the normalizer of S in G, and $\Omega(S,G)$ for the Weyl group $N(S,G)/S$. We will write $Z(G)$ for the center of G, and G_{sc} and G_{ad} for the simply connected cover and the adjoint quotient of the derived subgroup G_{der} of G.

Given a finite group $\mathcal{S}$, we will write $\mathrm{Irr}(\mathcal{S})$ for the set of isomorphism classes of irreducible representations of $\mathcal{S}$. The subset consisting of the one-dimensional representations will be called $\mathcal{S}^D$. Given a complex algebraic group $\mathcal{S}$, we will write $\mathrm{Irr}(\mathcal{S})$ for the set of isomorphism classes of irreducible algebraic representations of $\mathcal{S}$.

We will use freely the language and basic constructions in the theory of endoscopy. We refer the reader to [Langlands and Shelstad 1987] and [Kottwitz and Shelstad 1999] for the foundations of the theory.

3. Results of Arthur and Shelstad

In this section we will recall the results of Arthur and Shelstad on endoscopic transfer and its inversion, which will be an essential ingredient in our proofs. The formulation in the real case is slightly more complicated due to the fact that semisimple simply connected real groups can have nontrivial Galois cohomology, so we will describe the p-adic case first.

Let F be a p-adic field. Arthur's results apply to groups G which are either the symplectic group, or the split special odd orthogonal group, or the split or quasisplit special even orthogonal groups, as well as to products of such groups with copies of GL_n. If G is such a group, Arthur fixes a maximal compact subgroup K of $G(F)$ and denotes by $\mathcal{H}(G)$ the Hecke algebra of smooth, compactly supported, right and left K-finite functions on $G(F)$. To describe the results of his that we'll need, let us first assume that G has no even orthogonal factors, as the case of even orthogonal groups is slightly more subtle. Fix a Whittaker datum (B, ψ). Let $\varphi : W' \to {}^LG$ be a tempered Langlands parameter and put $\mathcal{S}_\varphi = \pi_0(\mathrm{Cent}(\varphi, \hat{G})/Z(\hat{G})^\Gamma)$. Arthur's recent results [2013, Section 2] imply that there exists an L-packet Π_φ of representations of $G(F)$ and a canonical bijection

$$\iota_{B,\psi} : \mathrm{Irr}(\mathcal{S}_\varphi) \to \Pi_\varphi, \quad \rho \mapsto \pi_\rho,$$

which sends the trivial representation to a (B, ψ)-generic representation. This bijection can also be written as a pairing $\langle \cdot, \cdot \rangle : \mathcal{S}_\varphi \times \Pi_\varphi \to \mathbb{C}$, and this is the language adopted by Arthur. A semisimple element $s \in \mathrm{Cent}(\varphi, \hat{G})$ gives rise to an endoscopic datum $\mathfrak{e} = (H, \mathcal{H}, s, \xi)$ for G. We briefly recall the construction, following [Kottwitz and Shelstad 1999, Section 2]: $\hat{H} = \hat{G}_s^\circ$, $\mathcal{H} = \hat{H} \cdot \varphi(W)$, and ξ is the inclusion map $\mathcal{H} \to {}^LG$. The group $\mathcal{H}$ can be shown to be a split extension of W by $\hat{H}$, and hence provides a homomorphism $\Gamma \to \mathrm{Out}(\hat{H})$. The group H is the unique quasisplit group with complex dual $\hat{H}$ for which the homomorphism $\Gamma \to \mathrm{Out}(H)$ given by the rational structure coincides under the canonical isomorphism $\mathrm{Out}(H) \cong \mathrm{Out}(\hat{H})$ with the homomorphism $\Gamma \to \mathrm{Out}(\hat{H})$ given by $\mathcal{H}$. In addition to the datum $(H, \mathcal{H}, s, \xi)$, Arthur chooses [2013, Section 1.2] an L-isomorphism $\xi_{H_1} : \mathcal{H} \to {}^LH$. By construction, φ factors through ξ and we obtain $\varphi_s = \xi_{H_1} \circ \varphi$ which is a Langlands parameter for H. The group H is again of the same type as G — a product of symplectic, orthogonal, and general linear groups (it can also have even orthogonal factors, which we will discuss momentarily). Associated to the Langlands parameter φ_s is an L-packet on H, whose stable character we denote by $S\Theta_{\varphi_s}$ (this is the stable form (2.2.2) in [Arthur 2013]). Let $\mathfrak{z}_\mathfrak{e}$ denote the pair (H, ξ_{H_1}). This is strictly speaking not a z-pair in the sense of [Kottwitz and Shelstad 1999, Section 2.2], because H will in general not have a simply connected derived group, but this will not cause any trouble. Let $\Delta[\psi, \mathfrak{e}, \mathfrak{z}_\mathfrak{e}]$ denote the Whittaker normalization of the Langlands–Shelstad transfer factor. Arthur shows that if $f \in \mathcal{H}(G)$ and $f^s \in \mathcal{H}(H)$ have $\Delta[\psi, \mathfrak{e}, \mathfrak{z}_\mathfrak{e}]$-matching orbital integrals, then

$$S\Theta_{\varphi_s}(f^s) = \sum_{\rho \in \mathrm{Irr}(\mathcal{S}_\varphi)} \langle s, \rho \rangle \Theta_{\pi_\rho}(f).$$

The group $\mathcal{S}_\varphi$ is finite and abelian, and $\mathrm{Irr}(\mathcal{S}_\varphi)$ is the set of characters of $\mathcal{S}_\varphi$, which is also a finite abelian group. Performing Fourier-inversion on these finite abelian groups one obtains

$$\Theta_{\pi_\rho}(f) = |\mathcal{S}_\varphi|^{-1} \sum_{s \in \mathcal{S}_\varphi} \overline{\langle s, \rho \rangle} S\Theta_{\varphi^s}(f^s).$$

This formula is the inversion of endoscopic transfer in the p-adic case.

If G is an even orthogonal group, the following subtle complication occurs: the group $\mathbb{Z}/2\mathbb{Z}$ acts on both G and $\hat{G}$ by outer automorphisms, and Theorem 8.4.1 of [Arthur 2013] associates to a given tempered Langlands parameter φ not one, but two L-packets $\Pi_{\varphi,1}$ and $\Pi_{\varphi,2}$. Each of them comes with a canonical bijection $\iota_{B,\psi,i} : \mathrm{Irr}(\mathcal{S}_\varphi) \to \Pi_{\varphi,i}$, and for each $\rho \in \mathrm{Irr}(\mathcal{S}_\varphi)$ the two representations $\iota_{B,\psi,1}(\rho)$ and $\iota_{B,\psi,2}(\rho)$ are an orbit under the action of $\mathbb{Z}/2\mathbb{Z}$. For each φ, there is a dichotomy: either $\Pi_{\varphi,1} = \Pi_{\varphi,2}$, and $\mathbb{Z}/2\mathbb{Z}$ acts trivially on this L-packet; or $\Pi_{\varphi,1} \cap \Pi_{\varphi,2} = \varnothing$, and the action of $1 \in \mathbb{Z}/2\mathbb{Z}$ interchanges $\Pi_{\varphi,1}$ and $\Pi_{\varphi,2}$. In this situation, we will take $\iota_{B,\psi}(\rho)$ to mean the pair of representations $\{\iota_{B,\psi,1}(\rho), \iota_{B,\psi,2}(\rho)\}$. Following Arthur, we will use the notation $\tilde{\mathcal{H}}(G)$ to denote the subalgebra of $\mathbb{Z}/2\mathbb{Z}$-fixed functions in $\mathcal{H}(G)$ if G is a p-adic even orthogonal group. For all other simple groups G, we set $\tilde{\mathcal{H}}(G)$ equal to $\mathcal{H}(G)$. If G is a product of simple factors G_i, then $\tilde{\mathcal{H}}(G)$ is determined by $\tilde{\mathcal{H}}(G_i)$. All constructions, as well as the two character identities displayed above, continue to hold, but only for functions $f \in \tilde{\mathcal{H}}(G)$. Notice that on $f \in \tilde{\mathcal{H}}(G)$, the characters of the two representations $\iota_{B,\psi,1}(\rho)$ and $\iota_{B,\psi,2}(\rho)$ evaluate equally, and moreover $f^s \in \tilde{\mathcal{H}}(H)$, so the above character relations do indeed make sense.

We will now describe the analogous formulas in the real case, which are results of Shelstad [2008]. Let G be a quasisplit connected reductive group defined over $F = \mathbb{R}$ and fix a Whittaker datum (B, ψ). Let $\varphi : W \to {}^LG$ be a tempered Langlands parameter, and $\mathcal{S}_\varphi$ as above. One complicating factor in the real case is that, while there is a canonical map

$$\Pi_\varphi \to \mathrm{Irr}(\mathcal{S}_\varphi),$$

it is not bijective, but only injective. It was observed by Adams, Barbasch, and Vogan that, in order to obtain a bijective map, one must replace Π_φ by the disjoint union of multiple L-packets. All these L-packets correspond to φ, but belong to different inner forms of G. The correct inner forms to take are the ones parametrized by $H^1(F, G_{\mathrm{sc}})$. The disjoint union of these inner forms is sometimes called the K-group associated to G, and denoted ${}^K G$. For an exposition on K-groups we refer the reader to [Arthur 1999, Section 2] and [Shelstad 2008]. Writing $\mathbf{\Pi}_\varphi$ for the disjoint union of L-packets over all inner forms in the K-group, one now has again a bijection

$$\mathbf{\Pi}_\varphi \to \mathrm{Irr}(\mathcal{S}_\varphi)$$

(see [Shelstad 2008, Section 11]) whose inverse we will denote by $\iota_{B,\psi}$, and we denote by $\langle \cdot, \cdot \rangle$ again the pairing between $\mathcal{S}_\varphi$ and $\mathbf{\Pi}_\varphi$ given by this bijection. Note that Shelstad uses a variant of $\mathcal{S}_\varphi$ involving the simply connected cover of $\hat{G}$. Since we are only considering quasisplit K-groups (that is, those which contain a quasisplit form), this variant will not be necessary and the group $\mathcal{S}_\varphi$ will be enough.

From a semisimple element $s \in \mathrm{Cent}(\varphi, \hat{G})$ we obtain an endoscopic datum $\mathfrak{e}$ by the same procedure as in the p-adic case just described. A second complicating factor is that, contrary to p-adic case discussed above, there will in general be no L-isomorphism $\mathcal{H} \to {}^L H$. Instead, one chooses a z-extension H_1 of H. Then there exists an L-embedding $\xi_{H_1} : \mathcal{H} \to {}^L H_1$. We let $\mathfrak{z}_\mathfrak{e}$ denote the datum (H_1, ξ_{H_1}). Then $\varphi_s = \xi_{H_1} \circ \varphi$ is a tempered Langlands parameter for H_1 and Shelstad [2008, Section 11] shows that for any two functions $\boldsymbol{f} \in \mathcal{H}({}^K G)$ and $f^s \in \mathcal{H}(H_1)$ whose orbital integrals are $\Delta[\psi, \mathfrak{e}, \mathfrak{z}_\mathfrak{e}]$-matching, one has

$$S\Theta_{\varphi_s}(f^s) = \sum_{\rho \in \mathrm{Irr}(\mathcal{S}_\varphi)} \langle s, \rho \rangle \Theta_{\pi_\rho}(\boldsymbol{f})$$

and

$$\Theta_{\pi_\rho}(\boldsymbol{f}) = |\mathcal{S}_\varphi|^{-1} \sum_{s \in \mathcal{S}_\varphi} \overline{\langle s, \rho \rangle} S\Theta_{\varphi^s}(f^s).$$

In the following sections, we will not use the notation ${}^K G$ for a K-group and the boldface symbols for objects associated with it. Rather, we will treat it like a regular group and denote it by G, in order to simplify the statements of the results. We also note that the finite abelian groups $\mathcal{S}_\varphi$ occurring here are in fact 2-groups, so we may remove the complex conjugation from $\langle s, \rho \rangle$ in the inversion formulas.

4. Change of Whittaker data

Let G be a quasisplit connected reductive group defined over a real or p-adic field F. Given a finite abelian group A, we will write A^D for its group of characters. To save notation, we will write $\hat{Z}$ for the center of $\hat{G}$, and $\hat{Z}_{\mathrm{sc}}$ for the center of $\hat{G}_{\mathrm{sc}}$.

Lemma 4.1. *There exists a canonical injection* (*bijection*, *if F is p-adic*)

$$G_{\mathrm{ad}}(F)/G(F) \to \ker\big(H^1(W, \hat{Z}_{\mathrm{sc}}) \to H^1(W, \hat{Z})\big)^D.$$

Proof. We will write $G(F)^{\tilde{D}}$ for the set of continuous characters on $G(F)$ which are trivial on the image of $G_{\mathrm{sc}}(F)$. Recall that Langlands [1989] has constructed surjective homomorphisms $H^1(W, \hat{Z}) \to G(F)^{\tilde{D}}$ and $H^1(W, \hat{Z}_{\mathrm{sc}}) \to G_{\mathrm{ad}}(F)^{\tilde{D}}$ (see [Borel 1979, Section 10] for an exposition of the construction). If F is p-adic, they are also bijective and the statement follows right away, because the finite abelian group $G_{\mathrm{ad}}(F)/G(F)$ is Pontryagin dual to

$$\ker\big(G_{\mathrm{ad}}(F)^{\tilde{D}} \to G(F)^{\tilde{D}}\big). \tag{3}$$

If F is real, the kernel of $H^1(W, \hat{Z}_{\mathrm{sc}}) \to G_{\mathrm{ad}}(F)^{\tilde{D}}$ maps onto the kernel of $H^1(W, \hat{Z}) \to G(F)^{\tilde{D}}$ (this is obvious from the reinterpretation of these homomorphisms given in [Kaletha 2012, Section 3.5]). This implies that the kernel of

$$H^1(W, \hat{Z}_{\mathrm{sc}}) \to H^1(W, \hat{Z})$$

surjects onto (3). □

Let $\mathfrak{w}, \mathfrak{w}'$ be two Whittaker data for G. We denote by $(\mathfrak{w}, \mathfrak{w}')$ the unique element of $G_{\mathrm{ad}}(F)/G(F)$ which conjugates $\mathfrak{w}$ to $\mathfrak{w}'$. We view this element as a character on the finite abelian group

$$\ker\big(H^1(W, \hat{Z}_{\mathrm{sc}}) \to H^1(W, \hat{Z})\big) \tag{4}$$

via Lemma 4.1. Given a Langlands parameter $\varphi : W' \to {}^LG$, we consider the composition

$$H^0(W, \varphi, \hat{G}) \to H^0(W, \varphi, \hat{G}_{\mathrm{ad}}) \to H^1(W, \hat{Z}_{\mathrm{sc}}),$$

where $H^0(W, \varphi, -)$ denotes the set of invariants of W with respect to the action given by φ. This map is continuous, hence it kills the connected component of the algebraic group $H^0(W, \varphi, \hat{G})$. Furthermore, it kills $H^0(W, \hat{Z})$. Thus we obtain a map

$$\pi_0\big(\mathrm{Cent}(\varphi(W), \hat{G})/Z(\hat{G})^{\Gamma}\big) \to \ker\big(H^1(W, \hat{Z}_{\mathrm{sc}}) \to H^1(W, \hat{Z})\big).$$

Composing this map with the map

$$\pi_0\big(S_\varphi/Z(\hat{G})^{\Gamma}\big) \to \pi_0\big(\mathrm{Cent}(\varphi(W), \hat{G})/Z(\hat{G})^{\Gamma}\big)$$

induced by the inclusion $S_\varphi \to \mathrm{Cent}(\varphi(W), \hat{G})$, we see that $(\mathfrak{w}, \mathfrak{w}')$ gives rise to a character on $\pi_0(S_\varphi/Z(\hat{G})^{\Gamma})$, which we again denote by $(\mathfrak{w}, \mathfrak{w}')$.

Now let $s \in S_\varphi$. Consider the endoscopic datum $\mathfrak{e} = (H, \mathcal{H}, s, \xi)$ determined by s, as described in Section 3. Let $\mathfrak{z}_{\mathfrak{e}}$ be any z-pair for $\mathfrak{e}$. We denote by $\Delta[\mathfrak{w}, \mathfrak{e}, \mathfrak{z}_{\mathfrak{e}}]$ the Langlands–Shelstad transfer factor [Langlands and Shelstad 1987], normalized with respect to $\mathfrak{w}$ (whose definition we will briefly recall in the following proof).

Lemma 4.2.

$$\Delta[\mathfrak{w}', \mathfrak{e}, \mathfrak{z}_\mathfrak{e}] = \big\langle (\mathfrak{w}, \mathfrak{w}'), s\big\rangle \cdot \Delta[\mathfrak{w}, \mathfrak{e}, \mathfrak{z}_\mathfrak{e}].$$

Proof. Write $\mathfrak{w} = (B, \psi)$. Let $\mathrm{spl} = (T, B, \{X_\alpha\})$ be a splitting of G containing the Borel subgroup B given by $\mathfrak{w}$ and $\psi_F : F \to \mathbb{C}^\times$ be a character with the property that spl and ψ_F give rise to ψ as in [Kottwitz and Shelstad 1999, Section 5.3]. Then $\Delta[\mathfrak{w}, \mathfrak{e}, \mathfrak{z}_\mathfrak{e}]$ is defined as the product

$$\epsilon(V_{G,H}, \psi_F) \cdot \Delta[\mathrm{spl}, \mathfrak{e}, \mathfrak{z}_\mathfrak{e}],$$

where $\Delta[\mathrm{spl}, \mathfrak{e}, \mathfrak{z}_\mathfrak{e}]$ is the normalization of the transfer factor relative to the splitting spl as constructed in [Langlands and Shelstad 1987, Section 3.7] (where it is denoted by Δ_0), and $\epsilon(V_{G,H}, \psi_F)$ is the epsilon factor (with Langlands' normalization; see for example [Tate 1979, (3.6)]) of the degree-zero virtual Γ representation

$$V_{G,H} = X^*(T) \otimes \mathbb{C} - X^*(T^H) \otimes \mathbb{C},$$

where T^H is any maximally split maximal torus of H.

Let $g \in G_{\mathrm{ad}}(F)$ be an element with $\mathrm{Ad}(g)\mathfrak{w} = \mathfrak{w}'$. Put $\mathrm{spl}' = \mathrm{Ad}(g)\mathrm{spl}$. Then spl' and ψ_F give rise to the Whittaker datum $\mathfrak{w}'$, and consequently we have

$$\Delta[\mathfrak{w}', \mathfrak{e}, \mathfrak{z}_\mathfrak{e}] = \epsilon(V_{G,H}, \psi_F) \cdot \Delta[\mathrm{spl}', \mathfrak{e}, \mathfrak{z}_\mathfrak{e}].$$

Let $z = g^{-1}\sigma(g) \in H^1(F, Z(G_{\mathrm{sc}}))$. Choose any maximal torus S of G coming from H (that is, S is the image of an admissible embedding into G of a maximal torus of H). According to [Langlands and Shelstad 1987, Section 2.3], we have

$$\Delta[\mathrm{spl}', \mathfrak{e}, \mathfrak{z}_\mathfrak{e}] = \langle z, s\rangle \Delta[\mathrm{spl}, \mathfrak{e}, \mathfrak{z}_\mathfrak{e}],$$

where z is mapped under $H^1(F, Z(G_{\mathrm{sc}})) \to H^1(F, S_{\mathrm{sc}})$ and s is mapped under $Z(\hat{H})^\Gamma \to \hat{S}^\Gamma \to [\hat{S}_{\mathrm{ad}}]^\Gamma$, and the pairing uses Tate–Nakayama duality. The number $\langle z, s\rangle$ can also be obtained by mapping s under

$$H^0(W, Z(\hat{H})) \to H^0(W, \varphi, \hat{G}) \to H^0(W, \varphi, \hat{G}_{\mathrm{ad}}) \to H^1(W, Z(\hat{G}_{\mathrm{sc}}))$$

and pairing it with z, using the duality between $H^1(F, Z(G_{\mathrm{sc}})) = H^1(F, S_{\mathrm{sc}} \to S_{\mathrm{ad}})$ and $H^1(W, Z(\hat{G}_{\mathrm{sc}})) = H^1(W, \hat{S}_{\mathrm{sc}} \to \hat{S}_{\mathrm{ad}})$. Using [Kaletha 2012, Section 3.5], one sees that this is the same as the number $\langle (\mathfrak{w}, \mathfrak{w}'), s\rangle$. □

Theorem 4.3. *Let G be a quasisplit real K-group, or a quasisplit symplectic or special orthogonal p-adic group. For any tempered Langlands parameter $\varphi : W' \to {}^LG$ and every $\rho \in \mathrm{Irr}(\mathcal{S}_\varphi)$, we have*

$$\iota_{\mathfrak{w}'}(\varphi, \rho) = \iota_{\mathfrak{w}}(\varphi, \rho \otimes (\mathfrak{w}, \mathfrak{w}')^{-1}),$$

provided that $\iota_\mathfrak{w}$ and $\iota_{\mathfrak{w}'}$ are normalized to satisfy the endoscopic character identities with respect to the transfer factors $\Delta[\mathfrak{w}, -, -]$ and $\Delta[\mathfrak{w}', -, -]$.

Proof. Fix a semisimple $s \in S_\varphi$. As described in Section 3, the pair (φ, s) gives rise to an endoscopic datum $\mathfrak{e}$, and after a choice of a z-pair $\mathfrak{z}_\mathfrak{e} = (H_1, \xi_{H_1})$ for $\mathfrak{e}$, it further gives rise to a Langlands parameter φ_s for H_1. If the functions $f \in \tilde{\mathscr{H}}(G)$ and $f^s \in \tilde{\mathscr{H}}(H_1)$ have $\Delta[\mathfrak{w}, \mathfrak{e}, \mathfrak{z}_\mathfrak{e}]$-matching orbital integrals, then by Lemma 4.2 the functions f and $\langle(\mathfrak{w}, \mathfrak{w}'), s\rangle \cdot f^s$ have $\Delta[\mathfrak{w}', \mathfrak{e}, \mathfrak{z}_\mathfrak{e}]$-matching orbital integrals. Thus

$$\begin{aligned}\sum_\rho \langle s, \rho\rangle \Theta_{\iota_{\mathfrak{w}'}(\varphi,\rho)}(f) &= \langle(\mathfrak{w}, \mathfrak{w}'), s\rangle\, S\Theta_{\varphi^s}(f^s)\\ &= \langle(\mathfrak{w}, \mathfrak{w}'), s\rangle \sum_\rho \langle s, \rho\rangle \Theta_{\iota_{\mathfrak{w}}(\varphi,\rho)}(f)\\ &= \sum_\rho \langle s, \rho \otimes (\mathfrak{w}, \mathfrak{w}')\rangle \Theta_{\iota_{\mathfrak{w}}(\varphi,\rho)}(f)\\ &= \sum_\rho \langle s, \rho\rangle \Theta_{\iota_{\mathfrak{w}}(\varphi,\rho\otimes(\mathfrak{w},\mathfrak{w}')^{-1})}(f),\end{aligned}$$

where the sums run over $\rho \in \mathrm{Irr}(\mathscr{S}_\varphi)$. Since this is true for all s, Fourier inversion gives the result. □

5. Tempered representations and their contragredient

In this section, we will prove formula (2) for quasisplit real K-groups and quasisplit p-adic symplectic and special orthogonal groups. The bulk of the work lies in an analysis of some properties of transfer factors. We refer the reader to [Langlands and Shelstad 1987, Sections 2–3] and [Kottwitz and Shelstad 1999, Sections 3–4] for the construction of transfer factors and the associated cohomological data.

Let F be $\mathbb{R}$ or a finite extension of $\mathbb{Q}_p$, and G a quasisplit connected reductive group over F. We fix an F-splitting $\mathrm{spl} = (T, B, \{X_\alpha\})$ of G. We write $\hat{G}$ for the complex dual of G and fix a splitting $\widehat{\mathrm{spl}} = (\hat{T}, \hat{B}, \{X_{\hat{\alpha}}\})$. We assume that the action of Γ on $\hat{G}$ preserves $\widehat{\mathrm{spl}}$, and that there is an isomorphism $X_*(T) \cong X^*(\hat{T})$ which identifies the B-positive cone with the $\hat{B}$-positive cone. Let $\hat{C}$ be the Chevalley involution on $\hat{G}$ which sends $\widehat{\mathrm{spl}}$ to the opposite splitting [Adams and Vogan 2012, Section 2]. The automorphism $\hat{C}$ commutes with the action of Γ and thus ${}^LC = \hat{C} \times \mathrm{id}_W$ is an L-automorphism of ${}^LG$.

Consider a function $c : R(T, G) \to \bar{F}^\times$ which is invariant under the action of $\Omega(T, G) \times \{\pm 1\}$ on $R(T, G)$ and equivariant under the action of Γ. Then $(T, B, \{c_\alpha X_\alpha\})$ is another F-splitting of G, which we will denote by $c \cdot \mathrm{spl}$. Given any maximal torus $S \subset G$ and any Borel subgroup B_S containing S and defined over $\bar{F}$, the admissible isomorphism $T \to S$ which sends B to B_S transports c to a function $c : R(S, G) \to \bar{F}^\times$ which is again $(\Omega(S, G) \times \{\pm 1\})$-invariant and Γ-equivariant. Moreover, the latter function is independent of the choice of B_S (and also of B). If $A = (a_\alpha)_{\alpha \in R(S,G)}$ is a set of a-data for $R(S, G)$, then $c \cdot A = (c_\alpha a_\alpha)_{\alpha \in R(S,G)}$ is also a set of a-data.

Let λ denote the splitting invariant constructed in [Langlands and Shelstad 1987, Section 2.3].

Lemma 5.1. $$\lambda(S, A, c \cdot \mathrm{spl}) = \lambda(S, c \cdot A, \mathrm{spl}).$$

Proof. We begin by recalling the construction of the splitting invariant. For a simple root $\alpha \in R(T, G)$, let $\eta_\alpha^{\mathrm{spl}} : \mathrm{SL}_2 \to G$ be the homomorphism determined by the splitting spl. We put

$$n_{\mathrm{spl}}(s_\alpha) = \eta_\alpha^{\mathrm{spl}} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

For any $w \in \Omega(T, G)$ choose a reduced expression $w = s_{\alpha_1} \cdots s_{\alpha_n}$ and set

$$n_{\mathrm{spl}}(w) = n_{\mathrm{spl}}(s_{\alpha_1}) \cdots n_{\mathrm{spl}}(s_{\alpha_n}).$$

By [Springer 1981, Section 11.2.9], this product is independent of the choice of reduced expression.

We choose a Borel subgroup $B_S \subset G$ defined over $\bar{F}$ and containing S, and an element $h \in G(\bar{F})$ such that $\mathrm{Ad}(h)(T, B) = (S, B_S)$. Then, for $\sigma \in \Gamma$ and $s \in S$, we have

$$\mathrm{Ad}(h^{-1})[{}^\sigma s] = {}^{w_S(\sigma)\sigma} \mathrm{Ad}(h^{-1})[s]$$

for some $w_S(\sigma) \in \Omega(T, G)$. Then $\lambda(S, A, \mathrm{spl}) \in H^1(F, S)$ is the element whose image under $\mathrm{Ad}(h^{-1})$ is represented by the cocycle

$$\sigma \mapsto \prod_{\substack{\alpha > 0 \\ (w_S(\sigma)\sigma)^{-1}\alpha < 0}} \alpha^\vee(a_{\mathrm{Ad}(h)\alpha}) \cdot n_{\mathrm{spl}}(w_S(\sigma)) \cdot \sigma(h^{-1})h, \tag{5}$$

where $\alpha > 0$ means that $\alpha \in R(T, G)$ is B-positive.

We now examine the relationship between n_{spl} and $n_{c \cdot \mathrm{spl}}$. Recall the standard triple (E, H, F) in $\mathrm{Lie}(\mathrm{SL}_2)$, where

$$E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \text{and} \quad F = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

The differential $d\eta_\alpha^{\mathrm{spl}}$ sends (E, H, F) to $(X_\alpha, H_\alpha, X_{-\alpha})$, where $H_\alpha = d\alpha^\vee(1)$ and $X_{-\alpha} \in \mathfrak{g}_{-\alpha}$ is determined by $[X_\alpha, X_{-\alpha}] = H_\alpha$. On the other hand, the differential $d\eta_\alpha^{c \cdot \mathrm{spl}}$ sends (E, H, F) to $(c_\alpha \cdot X_\alpha, H_\alpha, c_\alpha^{-1} X_{-\alpha})$. Thus

$$\eta_\alpha^{c \cdot \mathrm{spl}} = \eta_\alpha^{\mathrm{spl}} \circ \mathrm{Ad} \begin{bmatrix} \sqrt{c_\alpha} & 0 \\ 0 & \sqrt{c_\alpha}^{-1} \end{bmatrix}$$

for an arbitrary choice of a square root of c_α. It follows that

$$n_{c \cdot \mathrm{spl}}(s_\alpha) = \alpha^\vee(c_\alpha) \cdot n_{\mathrm{spl}}(s_\alpha).$$

Using induction and [Bourbaki 2002, Chapter VI, Section 1.6, Corollary 2], we conclude that for any $w \in \Omega(T, G)$, we have

$$n_{c\cdot\mathrm{spl}}(w) = \prod_{\substack{\alpha>0 \\ w^{-1}\alpha<0}} \alpha^\vee(c_\alpha) \cdot n_{\mathrm{spl}}(w).$$

From this equation and the fact that any $\sigma \in \Gamma$ preserves the sets of B-positive and B-negative roots, we see that

$$n_{c\cdot\mathrm{spl}}(w_S(\sigma)) = \prod_{\substack{\alpha>0 \\ (w_S(\sigma)\sigma)^{-1}\alpha<0}} \alpha^\vee(c_\alpha) \cdot n_{\mathrm{spl}}(w_S(\sigma)).$$

The statement follows by comparing (5) for $\lambda(S, A, c\cdot\mathrm{spl})$ and $\lambda(S, c\cdot A, \mathrm{spl})$. $\square$

Given a torus S defined over F, we will denote by -1 the homomorphism $S \to S$ which sends $s \in S$ to s^{-1}. It is of course defined over F. Its dual ${}^LS \to {}^LS$ is given by $(s, w) \mapsto (s^{-1}, w)$ and will also be denoted by -1. Given a maximal torus $S \subset G$ and a set of χ-data $X = \{\chi_\alpha \mid \alpha \in R(S, G)\}$ for $R(S, G)$, we denote by $-X$ the set $\{\chi_\alpha^{-1} \mid \alpha \in R(S, G)\}$. This is also a set of χ-data. It is shown in [Langlands and Shelstad 1987, Section 2.6] that X provides a canonical $\hat{G}$-conjugacy class of L-embeddings ${}^L\xi_X : {}^LS \to {}^LG$.

Lemma 5.2. *Let $S \subset G$ be a maximal torus defined over F, and let X be χ-data for $R(S, G)$. Let ${}^L\xi_X : {}^LS \to {}^LG$ be the canonical $\hat{G}$-conjugacy class of embeddings associated to X. Then the $\hat{G}$-conjugacy classes of maps ${}^L\xi_X \circ (-1)$ and ${}^LC \circ {}^L\xi_{-X}$ are equal.*

Remark. Intuitively, we can express the statement of Lemma 5.2 by the following diagram, whose commutativity is to be understood up to $\hat{G}$-conjugacy:

$$\begin{tikzcd} {}^LG \arrow[r, "{}^LC"] & {}^LG \\ {}^LS \arrow[u, "{}^L\xi_X"] \arrow[r, "-1"] & {}^LS \arrow[u, "{}^L\xi_{-X}"'] \end{tikzcd}$$

Proof. We choose a representative ${}^L\xi_X$ within its $\hat{G}$-conjugacy class by following the constructions in [Langlands and Shelstad 1987, Section 2.6]. For this, we choose a Borel subgroup defined over $\overline{F}$ and containing S. This provides an admissible isomorphism $\hat{\xi} : \hat{S} \to \hat{T}$. For $w \in W$, let $\sigma_S(w) \in \Omega(\hat{T}, \hat{G})$ be defined by

$$\hat{\xi}({}^ws) = {}^{\sigma_S(w)w}\hat{\xi}(s).$$

Then a representative of ${}^L\xi_X$ is given by

$${}^L\xi_X(s, w) = \big[\hat{\xi}(s)\, r_{\hat{B},X}(w)\, n_{\widehat{\mathrm{spl}}}(\sigma_S(w)), w\big],$$

where $r_{\hat{B},X}$ denotes the cochain r_p constructed in [Langlands and Shelstad 1987, Section 2.6] from the χ-data X and the gauge p determined by $\hat{B}$, and $n_{\widehat{\text{spl}}}$ is the section $\Omega(\hat{T},\hat{G}) \to N(\hat{T},\hat{G})$ determined by the splitting $\widehat{\text{spl}}$ as described in the proof of the previous lemma. Using the fact that $\hat{C}$ acts by inversion on $\hat{T}$ and [Adams and Vogan 2012, Lemma 5.8], we see that

$$^{L}C \circ {}^{L}\xi_X(s,w) = \big[\hat{\xi}(s)^{-1}\, r_{\hat{B},X}(w)^{-1}\, n_{\widehat{\text{spl}}}(\sigma_S(w)^{-1})^{-1}, w\big].$$

One sees that $r_{\hat{B},X}(w)^{-1} = r_{\hat{B},-X}(w)$. Moreover, by [Adams and Vogan 2012, Lemma 5.4] we have

$$n_{\text{spl}}(\sigma_S(w)^{-1})^{-1} = \big[t \cdot \sigma_S(w) t^{-1}\big] n_{\text{spl}}(\sigma_S(w)),$$

where $t \in \hat{T}$ is any lift of $\rho^\vee(-1) \in \hat{T}_{\text{ad}}$, $\rho^\vee$ being half the sum of the positive coroots. We can choose $t \in \hat{T}^\Gamma$ by choosing a root i of -1 and putting $t = \prod_{\alpha \in R(\hat{T},\hat{B})} \alpha^\vee(i)$. Then we see that

$$^{L}C \circ {}^{L}\xi_X(s,w) = \text{Ad}(t) \circ {}^{L}\xi_{-X} \circ (-1)(s,w). \qquad \square$$

Let θ be an automorphism which preserves spl, and let $\boldsymbol{a} \in H^1(W, Z(\hat{G}))$. The class $\boldsymbol{a}$ corresponds to a character $\omega : G(F) \to \mathbb{C}^\times$. Let $\hat{\theta}$ be the automorphism dual to θ, which preserves $\widehat{\text{spl}}$. Note that $\hat{\theta}$ commutes with the action of Γ. We will write ${}^{L}\theta$ for $\theta \times \text{id}_W$.

Let us recall some basic facts from [Kottwitz and Shelstad 1999, Section 1]. Let $\hat{G}^1$ be the connected component of the group $\hat{G}^{\hat{\theta}}$, let $\hat{T}^1 = \hat{T} \cap \hat{G}^1$, and let $\hat{B}^1 = \hat{B} \cap \hat{G}^1$. Then $\hat{G}^1$ is a reductive group and $(\hat{T}^1, \hat{B}^1)$ is a Borel pair for it. The set $\Delta(\hat{T}^1, \hat{B}^1)$ of $\hat{B}^1$-simple roots for $\hat{T}^1$ is the set of restrictions to $\hat{T}^1$ of the set $\Delta(\hat{T}, \hat{B})$ of $\hat{B}$-simple roots for $\hat{T}$. Moreover, the fibers of the restriction map

$$\text{res} : \Delta(\hat{T}, \hat{B}) \to \Delta(\hat{T}^1, \hat{B}^1)$$

are precisely the $\langle\hat{\theta}\rangle$-orbits in $\Delta(\hat{T}, \hat{B})$. We denote the image of α under res by α_{res}.

We can extend the pair $(\hat{T}^1, \hat{B}^1)$ to a Γ-splitting $\widehat{\text{spl}}^1 = (\hat{T}^1, \hat{B}^1, \{X_{\alpha_{\text{res}}}\})$ of $\hat{G}^1$ by setting for each $\alpha_{\text{res}} \in \Delta(\hat{T}^1, \hat{B}^1)$

$$X_{\alpha_{\text{res}}} = \sum_{\substack{\beta \in \Delta(\hat{T},\hat{B}) \\ \beta_{\text{res}} = \alpha_{\text{res}}}} X_\beta.$$

Since $\hat{\theta}$ commutes with Γ, the group $\hat{G}^1$ and the splitting just constructed is preserved by Γ. Thus, $\hat{G}^1 \rtimes W$ is the L-group of a connected reductive group G^1. Moreover, since $\hat{\theta}$ also commutes with $\hat{C}$, the automorphism $\hat{C}$ preserves the group $\hat{G}^1$ and acts by inversion on its maximal torus $\hat{T}^1$. Thus, $\hat{C}$ is a Chevalley involution for $\hat{G}^1$.

Although it will not concern us, we want to remark that it is in general not true that $\hat{C}$ sends the splitting $\widehat{\mathrm{spl}}^1$ to its opposite. Rather, it sends $\widehat{\mathrm{spl}}^1$ to the splitting of $\hat{G}^1$ constructed from the opposite of $\widehat{\mathrm{spl}}$ by the same procedure as above. That this splitting differs from the opposite of $\widehat{\mathrm{spl}}^1$ is due to the fact that for $\alpha_{\mathrm{res}} \in R(\hat{T}^1, \hat{G}^1)$, the coroot $H_{\alpha_{\mathrm{res}}}$ is not always the sum of H_β for all β in the Γ-orbit corresponding to α_{res}. In fact, we have

$$H_{\alpha_{\mathrm{res}}} = c_{\alpha_{\mathrm{res}}} \cdot \sum_{\substack{\beta \in \Delta(\hat{T}, \hat{B}) \\ \beta_{\mathrm{res}} = \alpha_{\mathrm{res}}}} H_\beta,$$

where $c_\alpha = 1$ if α_{res} is of type R_1 and $c_\alpha = 2$ if α_{res} is of type R_2.

Let $\mathfrak{e} = (H, s, \mathcal{H}, \xi)$ be an endoscopic datum for $(G, \theta, \boldsymbol{a})$. Let $\big(\hat{T}^H, \hat{B}^H, \{\hat{X}^H_\beta\}\big)$ be a Γ-fixed splitting of $\hat{H}$, and denote by $\hat{C}^H$ the corresponding Chevalley involution of $\hat{H}$ and by ${}^LC^H$ the corresponding L-automorphism of ${}^LH$. Let $\mathfrak{z}_{\mathfrak{e}} = (H_1, \xi_{H_1})$ be a z-pair for $\mathfrak{e}$. The splitting of $\hat{H}$ provides a unique one for $\hat{H}_1$ and the involutions $\hat{C}^{H_1}$ and ${}^LC^{H_1}$ restricted to $\hat{H}$ and ${}^LH$ equal $\hat{C}^H$ and ${}^LC^H$.

We write ${}^LC(\mathfrak{e})$ for the quadruple $(H, s', \mathcal{H}', \xi')$, where $s' = \hat{C}(s^{-1})$, $\mathcal{H}'$ is the same group as $\mathcal{H}$ but with the embedding $\hat{H} \to \mathcal{H}$ composed with $\hat{C}^H$, and $\xi' = {}^LC \circ {}^L\theta \circ \xi$. We write ${}^LC^H(\mathfrak{z}_{\mathfrak{e}})$ for the pair $(H_1, {}^LC^{H_1} \circ \xi_{H_1})$.

Fact 5.3. The quadruple ${}^LC(\mathfrak{e})$ is an endoscopic datum for $(G, \theta^{-1}, \boldsymbol{a})$, and ${}^LC^H(\mathfrak{z}_{\mathfrak{e}})$ is a z-pair for it. If $\mathfrak{e}'$ is an endoscopic datum for $(G, \theta, \boldsymbol{a})$ equivalent to $\mathfrak{e}$, then ${}^LC(\mathfrak{e}')$ is equivalent to ${}^LC(\mathfrak{e})$. An isomorphism $S^H \to S_\theta$ from a maximal torus of H to a maximal torus of G^1 is $\mathfrak{e}$-admissible if and only if it is ${}^LC(\mathfrak{e})$-admissible.

Proof. The proof is straightforward, but we include it for the convenience of the reader. We need to check [Kottwitz and Shelstad 1999, (2.1.1–2.1.4b)]. It is clear that $\mathcal{H}'$ remains a split extension of $\hat{H}$ by W. To check that s' is $\hat{\theta}^{-1}$-quasi-semisimple, one observes that the automorphism $\mathrm{Int}(\hat{C}(s^{-1})) \circ \hat{\theta}^{-1}$ is conjugate to $(\mathrm{Int}(s)\hat{\theta})^{-1}$ by $\hat{C} \circ \mathrm{Int}(s^{-1})$. The fact that ξ' is an isomorphism onto its image is inherited from ξ, and the image is

$$\begin{aligned}\hat{C}\big(\hat{\theta}\big(\xi(\hat{H})\big)\big) &= \hat{C}\big(\hat{\theta}\big(\mathrm{Cent}\big(\mathrm{Int}(s) \circ \hat{\theta}, \hat{G}\big)^\circ\big)\big) \\ &= \hat{C}\big(\hat{\theta}\big(\mathrm{Cent}\big(\hat{\theta}^{-1} \circ \mathrm{Int}(s^{-1}), \hat{G}\big)^\circ\big)\big) \\ &= \mathrm{Cent}\big(\mathrm{Int}(s') \circ \hat{\theta}^{-1}, \hat{G}\big)^\circ.\end{aligned}$$

Finally, we have

$$\begin{aligned}\mathrm{Int}(s') \circ {}^L\theta^{-1} \circ \xi' &= {}^LC \circ \mathrm{Int}(s^{-1}) \circ \xi \\ &= {}^LC \circ ((a')^{-1} \cdot {}^L\theta \circ \xi) \\ &= a' \cdot \xi'.\end{aligned}$$

This shows that ${}^LC(\mathfrak{e})$ is indeed an endoscopic datum for $(G, \theta^{-1}, \boldsymbol{a})$. A direct computation shows that if $g \in \hat{G}$ is an isomorphism $\mathfrak{e} \to \mathfrak{e}'$, then $\hat{C}\hat{\theta}(g)$ is an isomorphism ${}^LC(\mathfrak{e}) \to {}^LC(\mathfrak{e}')$. To check that ${}^LC^H(\mathfrak{z}_\mathfrak{e})$ is a z-pair for ${}^LC(\mathfrak{e})$, we only need to observe that, since the restriction of ${}^LC^{H_1}$ to $\hat{H}$ equals $\hat{C}^H$, the composition of ${}^LC^{H_1} \circ \xi_{H_1} : \mathcal{H}_1 \to {}^LH_1$ with the inclusion $\hat{H} \to \mathcal{H}_1$ is indeed the natural inclusion $\hat{H} \to \hat{H}_1 \to {}^LH_1$. To compare the notions of admissible isomorphisms, replace $\mathfrak{e}$ by an equivalent datum so that $s \in \hat{T}$ and $\xi(\hat{T}^H) \subset \hat{T}$. Then ξ restricts to an isomorphism $\hat{T}^H \to \hat{T}^1$. Recalling the definitions of $\mathcal{H}'$ and ξ', we see that ξ' restricts to the same isomorphism. Thus the notion of admissibility of isomorphisms of tori remains unchanged when we pass from $\mathfrak{e}$ to ${}^LC(\mathfrak{e})$. □

Assume now that spl is θ-stable and augment B to a θ-stable Whittaker datum (B, ψ). Then, associated to $(G, \theta, \boldsymbol{a})$, (B, ψ), $\mathfrak{e}$, and $\mathfrak{z}_\mathfrak{e}$, we have the Whittaker normalization of the transfer factor for G and H_1. In fact, as explained in [Kottwitz and Shelstad 2012], there are two different such normalizations — one adapted to the classical local Langlands correspondence for tori [ibid., (5.5.2)], and one adapted to the renormalized correspondence [ibid., (5.5.1)]. To be consistent with their notation, we will call these transfer factors $\Delta'[\psi, \mathfrak{e}, \mathfrak{z}_\mathfrak{e}]$ (for the classical local Langlands correspondence), and $\Delta_D[\psi, \mathfrak{e}, \mathfrak{z}_\mathfrak{e}]$ (for the renormalized correspondence). On the other hand, associated to $(G, \theta^{-1}, \boldsymbol{a})$, (B, ψ^{-1}), ${}^LC(\mathfrak{e})$, and ${}^LC^H(\mathfrak{z}_\mathfrak{e})$, we also have the Whittaker normalization of the transfer factor, again in the two versions. We will call these $\Delta'[\psi^{-1}, {}^LC(\mathfrak{e}), {}^LC^H(\mathfrak{z}_\mathfrak{e})]$ and $\Delta_D[\psi^{-1}, {}^LC(\mathfrak{e}), {}^LC^H(\mathfrak{z}_\mathfrak{e})]$. In the case $\theta = 1$ and $\boldsymbol{a} = 1$ (that is, ordinary endoscopy), one also has the normalizations Δ and Δ'_D [Section 5.1]. The normalization Δ is the one compatible with [Langlands and Shelstad 1987].

Proposition 5.4. *Let $\gamma_1 \in H_1(F)$ be a strongly G-regular semisimple element, and let $\delta \in G(F)$ be a strongly θ-regular θ-semisimple element. We have*

$$\Delta'[\psi, \mathfrak{e}, \mathfrak{z}_\mathfrak{e}](\gamma_1, \delta) = \Delta'\big[\psi^{-1}, {}^LC(\mathfrak{e}), {}^LC^H(\mathfrak{z}_\mathfrak{e})\big]\big(\gamma_1^{-1}, \theta^{-1}(\delta^{-1})\big).$$

The same equality holds with Δ_D in place of Δ'. Moreover, in the setting of ordinary endoscopy, the equality also holds for Δ and Δ'_D.

Proof. Let us first discuss the different versions of the transfer factor. In ordinary endoscopy, one obtains Δ from Δ' by replacing s with s^{-1}. Thus it is clear that the above equality will hold for the one if and only if it holds for the other. The same is true for Δ_D and Δ'_D. Returning to twisted endoscopy, the difference between Δ' and Δ_D is more subtle, and the statement for the one does not formally follow from the statement for the other. However, the proof for both cases is the same, and we will give it for the case of Δ_D.

One sees easily that γ is a θ-norm of δ precisely when γ^{-1} is a θ^{-1}-norm of $\theta^{-1}(\delta^{-1})$. We assume that this is the case. Let $S_{H_1} \subset H_1$ be the centralizer of

γ_1, let $S_H \subset H$ be the image of S_{H_1}. The torus S_H is the centralizer of the image $\gamma \in H(F)$ of γ_1. We choose a θ-admissible maximal torus $S \subset G$, an admissible isomorphism $\varphi : S_H \to S_\theta$, an element $\delta^* \in S(\bar{F})$ whose image in S_θ equals $\varphi(\gamma)$, and an element $g \in G_{\mathrm{sc}}(\bar{F})$ with $\delta^* = g\delta\theta(g^{-1})$. The objects φ, g and δ^* will enter into the construction of the transfer factor $\Delta_D[\psi, \mathfrak{e}, \mathfrak{z}_\mathfrak{e}](\gamma_1, \delta)$.

As already remarked, γ^{-1} is a θ^{-1}-norm for $\theta^{-1}(\delta^{-1})$. By Fact 5.3, φ is an ${}^L C(\mathfrak{e})$-admissible isomorphism of tori, and it is clear that $\varphi(\gamma^{-1})$ equals the image of $\theta^{-1}(\delta^{-1})$ in S_θ. Moreover $\theta^{-1}(\delta^{*-1}) = g \cdot \theta^{-1}(\delta^{-1}) \cdot \theta^{-1}(g^{-1})$. Thus we may use the objects φ, g and $\theta^{-1}(\delta^{*-1})$ when constructing the transfer factor $\Delta_D[\psi^{-1}, {}^L C(\mathfrak{e}), {}^L C^H(\mathfrak{z}_\mathfrak{e})](\gamma_1^{-1}, \theta^{-1}(\delta^{-1}))$.

We will also need θ-invariant a-data A for $R(S, G)$ and χ-data X for $R_{\mathrm{res}}(S, G)$. Moreover, we fix an additive character $\psi_F : F \to \mathbb{C}^\times$ and assume that the splitting $\mathrm{spl} = (T, B, X_\alpha)$ and the character ψ_F give rise to the fixed Whittaker datum (B, ψ). Up to equivalence of endoscopic data we may assume $s \in \hat{T}$. This implies $\hat{C}(s^{-1}) = s$. Then, by [Kottwitz and Shelstad 2012, Equation 5.5.1], we have

$$\Delta_D[\psi, \mathfrak{e}, \mathfrak{z}_\mathfrak{e}] = \epsilon(V_{G,H}, \psi_F) \cdot \Delta_{\mathrm{I}}^{\mathrm{new}}[\mathrm{spl}, A] \cdot \Delta_{\mathrm{II}}^{-1}[A, X] \cdot \Delta_{\mathrm{III}}[\mathfrak{e}, \mathfrak{z}_\mathfrak{e}, X] \cdot \Delta_{\mathrm{IV}}. \quad (6)$$

The factor $\epsilon(V_{G,H}, \psi_F)$ is the epsilon factor (with Langlands' normalization; see for example [Tate 1979, (3.6)]) for the virtual Γ-representation

$$V_{G,H} = X^*(T)^\theta \otimes \mathbb{C} - X^*(T^H) \otimes \mathbb{C},$$

where T^H is any maximally split maximal torus of H. It does not depend on any further data and is thus the same for both sides of the equality we are proving. One also sees immediately from the definition that

$$\Delta_{\mathrm{IV}}\big(\gamma_1^{-1}, \theta^{-1}(\delta^{-1})\big) = \Delta_{\mathrm{IV}}(\gamma_1, \delta). \quad (7)$$

We now examine the factors Δ_{I}, Δ_{II} and Δ_{III}, the latter requiring the bulk of the work. These factors depend on most of the objects chosen so far. We have indicated in brackets the more important objects on which they depend, as it will be necessary to keep track of them. These are not all the dependencies. For example, all factors Δ_i depend on the datum $\mathfrak{e}$, but except for Δ_{III}, this dependence is only through the datum s, which we have arranged to be equal for $\mathfrak{e}$ and ${}^L C(\mathfrak{e})$, and so we have not included $\mathfrak{e}$ in the notation for these factors.

The factor $\Delta_{\mathrm{I}}^{\mathrm{new}}[\mathfrak{e}, \mathrm{spl}, A]$ does not depend directly on γ_1 and δ, but rather only on the choices of S and φ. As we have remarked in the preceding paragraphs, these choices also serve γ_1^{-1} and $\theta^{-1}(\delta^{-1})$, and we see that

$$\Delta_{\mathrm{I}}^{\mathrm{new}}[\mathrm{spl}, A]\big(\gamma_1^{-1}, \theta^{-1}(\delta^{-1})\big) = \Delta_{\mathrm{I}}^{\mathrm{new}}[\mathrm{spl}, A](\gamma_1, \delta). \quad (8)$$

We turn to $\Delta_{\mathrm{II}}[A, X]$. Let $-A$ denote the a-data obtained from A by replacing each a_α by $-a_\alpha$. Let $-X$ denote the χ-data obtained from X by replacing each χ_α by

χ_α^{-1}. Then one checks that

$$\Delta_{\mathrm{II}}[A, X]\big(\gamma_1^{-1}, \theta^{-1}(\delta^{-1})\big) = \Delta_{\mathrm{II}}[-A, -X](\gamma_1, \delta). \tag{9}$$

Before we can examine $\Delta_{\mathrm{III}}[\mathfrak{e}, \mathfrak{z}_\mathfrak{e}, X]$, we need to recall its construction, following [Kottwitz and Shelstad 1999, Section 4.4 and Section 5.3]. We define an F-torus S_1 as the fiber product

$$\begin{tikzcd} S_1 \arrow[rr] \arrow[d] & & S \arrow[d] \\ S_{H_1} \arrow[r] & S_H \arrow[r, "\varphi"] & S_\theta. \end{tikzcd}$$

The element $\delta_1^* = (\gamma_1, \delta^*)$ belongs to S_1. The automorphism $\mathrm{id} \times \theta$ of $S_{H_1} \times S$ induces an automorphism θ_1 of S_1. This automorphism restricts trivially to the kernel of $S_1 \to S$, and hence $1 - \theta_1$ induces a homomorphism $S \to S_1$, which we can compose with $S_{\mathrm{sc}} \to S$ to obtain a homomorphism $S_{\mathrm{sc}} \to S_1$, which we still denote by $1 - \theta_1$.

The element $(\sigma(g)g^{-1}, \delta_1^*)$ belongs to $H^1(F, S_{\mathrm{sc}} \xrightarrow{1-\theta_1} S_1)$ and is called $\mathrm{inv}(\gamma_1, \delta)$. Kottwitz and Shelstad [1999, A.3] construct a pairing $\langle\cdot,\cdot\rangle_{\mathrm{KS}}$ between the abelian groups

$$H^1(F, S_{\mathrm{sc}} \xrightarrow{1-\theta_1} S_1) \quad \text{and} \quad H^1(W, \hat{S}_1 \xrightarrow{1-\hat{\theta}_1} \hat{S}_{\mathrm{ad}}).$$

Using this pairing, they define

$$\Delta_{\mathrm{III}}[\mathfrak{e}, \mathfrak{z}_\mathfrak{e}, X](\gamma_1, \delta) = \big\langle \mathrm{inv}(\gamma_1, \delta), A_0[\mathfrak{e}, \mathfrak{z}_\mathfrak{e}, X]\big\rangle_{\mathrm{KS}},$$

where $A_0[\mathfrak{e}, \mathfrak{z}_\mathfrak{e}, X]$ is an element of $H^1(F, \hat{S}_1 \xrightarrow{1-\hat{\theta}_1} \hat{S}_{\mathrm{ad}})$ constructed as follows:

The χ-data X provides an $\hat{H}$-conjugacy class of embeddings ${}^L S_H \to {}^L H$ and a $\hat{G}^1$-conjugacy class of embeddings ${}^L S_\theta \to {}^L G^1$, where $\hat{G}^1$ is the connected stabilizer of $\hat{\theta}$. Conjugating within $\hat{H}$ and $\hat{G}^1$ we arrange that these embeddings map $\hat{S}_H$ to $\hat{T}^H$ and $\hat{S}_\theta$ to $\hat{T}^1$. Composing with the canonical embeddings ${}^L H \to {}^L H_1$ and ${}^L G^1 \to {}^L G$, we obtain embeddings $\xi_1[X] : {}^L S_\theta \to {}^L G$ and $\xi_{S_H}[X] : {}^L S_H \to {}^L H_1$. There is a unique embedding $\xi_1^1[X] : {}^L S \to {}^L G$ extending $\xi_1[X]$, and there is a unique embedding $\xi_{S_H}^1[X] : {}^L S_{H_1} \to {}^L H_1$ extending $\xi_{S_H}[X]$.

Letting $\bar{x}$ denote the image of $x \in \mathcal{H}$ under the projection $\mathcal{H} \to W$, define

$$\mathcal{U} = \big\{x \in \mathcal{H} \;\big|\; \mathrm{Ad}(\xi(x))|_{\hat{T}^1} = \mathrm{Ad}(\xi_1[X](1 \times \bar{x}))|_{\hat{T}^1}\big\}.$$

Then $\mathcal{U}$ is an extension of W by $\hat{T}^H$. One can show that $\xi(\mathcal{U}) \subset \xi_1^1[X]({}^L S)$ and $\xi_{H_1}(\mathcal{U}) \subset \xi_{S_H}^1[X]({}^L S_{H_1})$. Then we can define, for any $w \in W$, an element $a_S[X](w) \in \hat{S}_1$, by choosing a lift $u(w) \in \mathcal{U}$ and letting $a_S[X](w) = (t_1^{-1}, t) \in \hat{S}_{H_1} \times \hat{S} \twoheadrightarrow \hat{S}_1$, where $t_1 \in \hat{S}_{H_1}$ and $t \in \hat{S}$ are the unique elements satisfying

$$\xi_1^1[X](t \times w) = \xi(u(w)) \quad \text{and} \quad \xi_{S_H}^1[X](t_1 \times w) = \xi_{H_1}(u(w)). \tag{10}$$

We can further define $s_S = [\xi_1^1]^{-1}(s) \in \hat{S}$ and also view it as an element of $\hat{S}_{\mathrm{ad}}$. Then

$$A_0[\mathfrak{e}, \mathfrak{z}_{\mathfrak{e}}, X] = \big(a_S[X]^{-1}, s_S\big) \in H^1\big(W, \hat{S}_1 \to \hat{S}_{\mathrm{ad}}\big).$$

We are now ready to examine $\Delta_{\mathrm{III}}[\mathfrak{e}, \mathfrak{z}_{\mathfrak{e}}, X]$. We have

$$\mathrm{inv}\big(\gamma_1^{-1}, \theta^{-1}(\delta^{-1})\big) = \big(\sigma(g)g^{-1}, \theta_1^{-1}(\delta_1^{*-1})\big). \tag{11}$$

This is an element of $H^1(F, S_{\mathrm{sc}} \xrightarrow{1-\theta_1^{-1}} S_1)$. We have

$$\begin{aligned}\Delta_{\mathrm{III}}\big[{}^LC(\mathfrak{e}), {}^LC^H(\mathfrak{e}_{\mathfrak{z}}), X\big]&\big(\gamma_1^{-1}, \theta^{-1}(\delta^{-1})\big)\\ &= \Big\langle \mathrm{inv}\big(\gamma_1^{-1}, \theta^{-1}(\delta^{-1})\big), A_0\big[{}^LC(\mathfrak{e}), {}^LC^H(\mathfrak{e}_{\mathfrak{z}}), X\big]\Big\rangle_{\mathrm{KS}}.\end{aligned}$$

Here $A_0\big[{}^LC(\mathfrak{e}), {}^LC^H(\mathfrak{e}_{\mathfrak{z}}), X\big]$ is the element of $H^1(W, \hat{S}_1 \xrightarrow{1-\hat{\theta}_1^{-1}} \hat{S}_{\mathrm{ad}})$, constructed as above, but with respect to the endoscopic datum ${}^LC(\mathfrak{e})$ and the z-pair ${}^LC^H(\mathfrak{z}_{\mathfrak{e}})$, rather than $\mathfrak{e}$ and $\mathfrak{z}_{\mathfrak{e}}$. Thus $A_0\big[{}^LC(\mathfrak{e}), {}^LC^H(\mathfrak{e}_{\mathfrak{z}}), X\big] = (\tilde{a}_S[X]^{-1}, s_S)$, with $\tilde{a}_S[X](w) = (\tilde{t}_1^{-1}, \tilde{t})$, and

$$\begin{aligned}\xi_1^1[X](\tilde{t} \times w) &= {}^L\theta \circ {}^LC \circ \xi(u(w)),\\ \xi_{S_H}^1[X](\tilde{t}_1 \times w) &= {}^LC^H \circ \xi_{H_1}(u(w)).\end{aligned}$$

Using (10) we see that

$$\begin{aligned}\xi_1^1[X](\tilde{t} \times w) &= {}^L\theta \circ {}^LC \circ \xi_1^1[X](t \times w),\\ \xi_{S_H}^1[X](\tilde{t}_1 \times w) &= {}^LC^H \circ \xi_{S_H}^1[X](t_1 \times w).\end{aligned}$$

According to Lemma 5.2 this is equivalent to

$$\begin{aligned}\xi_1^1[X](\tilde{t} \times w) &= {}^L\theta \circ \xi_1^1[-X](t^{-1} \times w),\\ \xi_{S_H}^1[X](\tilde{t}_1 \times w) &= \xi_{S_H}^1[-X](t_1^{-1} \times w).\end{aligned}$$

We conclude that

$$\tilde{a}_S[X](w) = \hat{\theta}_1\big(a_S[-X](w)^{-1}\big). \tag{12}$$

The isomorphism of complexes

$$\begin{tikzcd} S_{\mathrm{sc}} \arrow[r, "\mathrm{id}"] \arrow[d, "1-\theta_1^{-1}"'] & S_{\mathrm{sc}} \arrow[d, "1-\theta_1"] \\ S_1 \arrow[r, "\theta_1 \circ (\ )^{-1}"] & S_1 \end{tikzcd}$$

induces an isomorphism $H^1(F, S_{\mathrm{sc}} \xrightarrow{1-\theta_1^{-1}} S_1) \to H^1(F, S_{\mathrm{sc}} \xrightarrow{1-\theta_1} S_1)$ which, by (11), sends $\mathrm{inv}(\gamma_1^{-1}, \theta^{-1}(\delta^{-1}))$ to $\mathrm{inv}(\gamma_1, \delta)$. The dual isomorphism of complexes

$$\begin{tikzcd} \hat{S}_1 \arrow[d, "1-\hat{\theta}_1^{-1}"'] & \hat{S}_1 \arrow[l, "\hat{\theta}_1\circ(\ )^{-1}"'] \arrow[d, "1-\hat{\theta}_1"] \\ \hat{S}_{\mathrm{ad}} & \hat{S}_{\mathrm{ad}} \arrow[l, "\mathrm{id}"'] \end{tikzcd}$$

induces an isomorphism $H^1(W, \hat{S}_1 \xrightarrow{1-\hat{\theta}_1} \hat{S}_{\mathrm{ad}}) \to H^1(W, \hat{S}_1 \xrightarrow{1-\hat{\theta}_1^{-1}} \hat{S}_{\mathrm{ad}})$ which, by (12), sends $A_0[\mathfrak{e}, \mathfrak{z}_\mathfrak{e}, -X]$ to $A_0[{}^LC(\mathfrak{e}), {}^LC^H(\mathfrak{z}_\mathfrak{e}), X]$. We conclude that

$$\Delta_{\mathrm{III}}\big[{}^LC(\mathfrak{e}), {}^LC^H(\mathfrak{z}_\mathfrak{e}), X\big]\big(\gamma_1^{-1}, \theta^{-1}(\delta^{-1})\big) = \Delta_{\mathrm{III}}[\mathfrak{e}, \mathfrak{z}_\mathfrak{e}, -X](\gamma_1, \delta). \tag{13}$$

Combining (6), (7), (8), (9), and (13), we obtain

$$\begin{aligned} &\Delta_D\big[\psi, {}^LC(\mathfrak{e}), {}^LC^H(\mathfrak{z}_\mathfrak{e})\big]\big(\gamma_1^{-1}, \theta^{-1}(\delta^{-1})\big) \\ &\quad= \epsilon(V_{G,H}, \psi_F) \cdot \Delta_{\mathrm{I}}^{\mathrm{new}}[\mathrm{spl}, A]\big(\gamma_1^{-1}, \theta^{-1}(\delta^{-1})\big) \cdot \Delta_{\mathrm{II}}^{-1}[A, X]\big(\gamma_1^{-1}, \theta^{-1}(\delta^{-1})\big) \\ &\qquad\qquad \cdot \Delta_{\mathrm{III}}\big[{}^LC(\mathfrak{e}), {}^LC^H(\mathfrak{z}_\mathfrak{e}), X\big]\big(\gamma_1^{-1}, \theta^{-1}(\delta^{-1})\big) \cdot \Delta_{\mathrm{IV}}\big(\gamma_1^{-1}, \theta^{-1}(\delta^{-1})\big) \\ &\quad= \epsilon(V_{G,H}, \psi_F) \cdot \Delta_{\mathrm{I}}^{\mathrm{new}}[\mathrm{spl}, A](\gamma_1, \delta) \cdot \Delta_{\mathrm{II}}^{-1}[-A, -X](\gamma_1, \delta) \\ &\qquad\qquad \cdot \Delta_{\mathrm{III}}[\mathfrak{e}, \mathfrak{z}_\mathfrak{e}, -X](\gamma_1, \delta) \cdot \Delta_{\mathrm{IV}}(\gamma_1, \delta). \end{aligned}$$

Since $-X$ and $-A$ are valid choices of χ-data and a-data, according to (6) the second product is almost equal to $\Delta_\psi[\psi, \mathfrak{e}, \mathfrak{z}_\mathfrak{e}](\gamma_1, \delta)$. The only difference is that the a-data occurring in Δ_{I} is A, while the one occurring in Δ_{II} is $-A$. Let $-\mathrm{spl}$ be the splitting $(T, B, \{-X_\alpha\})$; then $-\mathrm{spl}$ and the character ψ_F^{-1} give rise to the fixed Whittaker datum (B, ψ), just like the splitting spl and the character ψ_F did. Thus

$$\begin{aligned} \epsilon(V_{G,H}, \psi_F) \cdot \Delta_{\mathrm{I}}[\mathrm{spl}, A] &= \epsilon(V_{G,H}, \psi_F^{-1}) \cdot \Delta_{\mathrm{I}}[-\mathrm{spl}, A] \\ &= \epsilon(V_{G,H}, \psi_F^{-1}) \cdot \Delta_{\mathrm{I}}[\mathrm{spl}, -A], \end{aligned}$$

with the first equality following from the argument of [Kottwitz and Shelstad 1999, Section 5.3], and the second from Lemma 5.1. Noting that spl and ψ_F^{-1} give rise to the Whittaker datum (B, ψ^{-1}), we obtain

$$\Delta_D\big[\psi, {}^LC(\mathfrak{e}), {}^LC^H(\mathfrak{z}_\mathfrak{e})\big]\big(\gamma_1^{-1}, \theta^{-1}(\delta^{-1})\big) = \Delta_D[\psi^{-1}, \mathfrak{e}, \mathfrak{z}_\mathfrak{e}](\gamma_1, \delta). \qquad \square$$

Corollary 5.5. *Let $f \in \mathcal{H}(G)$ and $f^{H_1} \in \mathcal{H}(H_1)$ be functions such that the (θ^{-1}, ω)-twisted orbital integrals of f match the stable orbital integrals of f^{H_1} with respect to $\tilde{\Delta}[\psi^{-1}, {}^LC(e), {}^LC^H(z_e)]$. Then the (θ, ω)-twisted orbital integrals of $f \circ \theta^{-1} \circ i$ match the stable orbital integrals of $f^{H_1} \circ i$ with respect to $\tilde{\Delta}[\psi, e, z_e]$. Here $\tilde{\Delta}$ stands for any of the two (respectively, four) Whittaker normalizations of the transfer* *factor for twisted (respectively, standard) endoscopy, and i is the map on $G(F)$ or $H_1(F)$ sending every element to its inverse.*

Proof. $\mathrm{SO}(\gamma_1, f^{H_1} \circ i)$

$$\begin{aligned}
&= \mathrm{SO}(\gamma_1^{-1}, f^{H_1}) \\
&= \sum_{\delta \in G(F)/\theta^{-1}\text{-}\sim} \tilde{\Delta}\big[\psi^{-1}, {}^LC(e), {}^LC^H(z_e)\big](\gamma_1^{-1}, \delta)\, O^{\theta^{-1},\omega}(\delta, f) \\
&= \sum_{\delta \in G(F)/\theta^{-1}\text{-}\sim} \tilde{\Delta}\big[\psi^{-1}, {}^LC(e), {}^LC^H(z_e)\big](\gamma_1^{-1}, \delta)\, O^{\theta,\omega}\big(\theta(\delta^{-1}), f \circ \theta^{-1} \circ i\big) \\
&= \sum_{\delta' \in G(F)/\theta\text{-}\sim} \tilde{\Delta}[\psi, e, z_e](\gamma_1, \delta')\, O^{\theta,\omega}(\delta', f \circ \theta^{-1} \circ i).
\end{aligned}$$

The last line follows from Proposition 5.4, with the substitution $\delta' = \theta(\delta^{-1})$. □

Fact 5.6. Assume that θ has finite order. Let π be an irreducible admissible tempered (B, ψ)-generic θ-stable representation of $G(F)$, and let $A : \pi \to \pi \circ \theta$ be the unique isomorphism which preserves a (B, ψ)-Whittaker functional. Then the dual map $A^\vee : (\pi \circ \theta)^\vee \to \pi^\vee$ preserves a (B, ψ^{-1})-Whittaker functional.

Proof. Let V be the vector space on which π acts. Since π is tempered, it is unitary. Let $\langle \cdot , \cdot \rangle$ be a π-invariant nondegenerate Hermitian form on V. Then

$$\overline{V} \to V^\vee, \quad w \mapsto \langle \cdot , w \rangle$$

is a π–$\pi^\vee$-equivariant isomorphism, and it identifies $A^\vee$ with the $\langle \cdot , \cdot \rangle$-adjoint of A, which we will call A^*. We claim that $A^* = A^{-1}$. Indeed, $(v, w) \mapsto \langle Av, Aw \rangle$ is another π-invariant scalar product, hence there exists a scalar $c \in \mathbb{C}^\times$ with $\langle Av, Aw \rangle = c\langle v, w \rangle$. On the one hand, since both sides are Hermitian, this scalar must belong to $\mathbb{R}_{>0}$. On the other hand, since θ has finite order, so does A, and thus c must be a root of unity. This shows that $c = 1$, hence $A^* = A^{-1}$. Let σ denote complex conjugation. If $\lambda : V \to \mathbb{C}$ is a (B, ψ)-Whittaker functional preserved by A, then $\sigma \circ \lambda : \overline{V} \to \mathbb{C}$ is a (B, ψ^{-1})-Whittaker functional preserved by $A^\vee = A^* = A^{-1}$. □

Corollary 5.7. *If $\tilde{\pi}$ is the unique extension of π to a representation of $G(F) \rtimes \langle \theta \rangle$ so that $\tilde{\pi}(\theta)$ is the isomorphism $\pi \to \pi \circ \theta$ which fixes a (B, ψ)-Whittaker functional, then $\tilde{\pi}^\vee$ is the unique extension of $\pi^\vee$ to a representation of $G(F) \rtimes \langle \theta \rangle$ so that $\tilde{\pi}^\vee(\theta)$ is the isomorphism $\pi^\vee \to \pi^\vee \circ \theta$ which fixes a (B, ψ^{-1})-Whittaker functional.*

Let us recall Theorem 7.1(a) of [Adams and Vogan 2012]. For any Langlands parameter $\varphi : W \to {}^LG$ for a real connected reductive group G with corresponding L-packet Π_φ, the theorem shows that the set $\{\pi^\vee \mid \pi \in \Pi_\varphi\}$ is also an L-packet, and its parameter is ${}^LC \circ \varphi$. Assume that φ is tempered, and denote by $S\Theta_\varphi$

the stable character of the L-packet Π_φ. Then an immediate corollary is that $S\Theta_\varphi \circ i = S\Theta_{^LC \circ \varphi}$. We will now prove this equality for quasisplit symplectic and special orthogonal p-adic groups. After that, we will use it to derive formula (2). With this formula at hand, we will derive the precise p-adic analog of Adams and Vogan's Theorem 7.1(a) as a corollary.

Theorem 5.8. *Let H be a quasisplit symplectic or special orthogonal group and $\varphi : W' \to {}^LH$ a tempered Langlands parameter. Write $S\Theta_\varphi$ for the stable character of the L-packet attached to φ. Then we have an equality of linear forms on $\tilde{\mathcal{H}}(H)$:*

$$S\Theta_\varphi \circ i = S\Theta_{^LC^H \circ \varphi}.$$

Proof. We recall very briefly the characterizing property of $S\Theta_\varphi$, following Arthur [2013, Section 1 and Section 2]. Let $G = \mathrm{GL}_n/F$ and let $\mathrm{spl} = (T, B, \{X_\alpha\})$ be the standard splitting consisting of the subgroup T of diagonal matrices, the subgroup B of upper triangular matrices, and the set $\{X_\alpha\}$ of elementary matrices whose entries are zero except for one entry in the first superdiagonal, which is equal to 1. Let θ be the outer automorphism of G preserving spl. Equip $\hat G = \mathrm{GL}_n(\mathbb{C})$ with its standard splitting $(\hat T, \hat B, \{\hat X_\alpha\})$ and let $\hat\theta$ be the outer automorphism of $\hat G$ preserving that splitting. The standard representation $\hat H \to \hat G$ can be extended to an L-embedding $\xi : {}^LH \to {}^LG$ and augmented by an element $s \in \hat T$ to provide an endoscopic datum $\mathfrak{e} = (H, {}^LH, s, \xi)$ for the triple $(G, \theta, 1)$. Then $\xi \circ \varphi$ is a Langlands parameter for G invariant under $\hat\theta$. Let π be the representation of $G(F)$ assigned to $\xi \circ \varphi$ by the local Langlands correspondence [Harris and Taylor 2001; Henniart 2000]. We have $\pi \cong \pi \circ \theta$. Choosing an additive character $\psi_F : F \to \mathbb{C}^\times$ we obtain from the standard splitting of G a θ-stable Whittaker datum (B, ψ). There is a unique isomorphism $A : \pi \to \pi \circ \theta$ which preserves one (hence all) (B, ψ)-Whittaker functionals. Then we have the distribution

$$f \mapsto T\Theta^{\psi}_{\xi\circ\varphi}(f) = \mathrm{tr}\Big(v \mapsto \int_{G(F)} f(g)\pi(g)Av\,dg\Big).$$

By construction, $S\Theta_\varphi$ is the unique stable distribution on $\tilde{\mathcal{H}}(H)$ with the property that

$$S\Theta_\varphi(f^H) = T\Theta^{\psi}_{\xi\circ\varphi}(f)$$

for all $f \in \tilde{\mathcal{H}}(G)$ and $f^H \in \tilde{\mathcal{H}}(H)$ such that the $(\theta, 1)$-twisted orbital integrals of f match the stable orbital integrals of f^H with respect to $\Delta'[\psi, \mathfrak{e}, \mathfrak{z}_\mathfrak{e}]$. Here $\mathfrak{z}_\mathfrak{e}$ stands for the tautological pair (H, id).

Now consider the transfer factor $\Delta'[\psi^{-1}, {}^LC(\mathfrak{e}), {}^LC^H(\mathfrak{z}_\mathfrak{e})]$. We have chosen both $\hat C$ and $\hat C^H$ to preserve the standard tori in $\hat G$ and $\hat H$ and act as inversion on those. Moreover the endoscopic element s belongs to $\hat T$. Using the datum ${}^LC(\mathfrak{e})$ and the pair ${}^LC^H(\mathfrak{z}_\mathfrak{e})$ has the same effect as using the datum $\big(H, \mathcal{H}, s, {}^LC \circ {}^L\theta \circ \xi \circ {}^LC^{H^{-1}}\big)$

and the pair $\mathfrak{z}_\mathfrak{e}$. We have ${}^L\theta \circ \xi = \mathrm{Int}(s^{-1})\xi$, so replacing ${}^L\theta \circ \xi$ by ξ changes the above datum to an equivalent one. An easy computation shows furthermore that ${}^LC \circ \xi \circ {}^LC^{H^{-1}} = \mathrm{Int}(t) \circ \xi$ for a suitable $t \in \hat{T}$. All in all, up to equivalence, we see that replacing $\mathfrak{e}$ and $\mathfrak{z}_\mathfrak{e}$ by ${}^LC(\mathfrak{e})$ and ${}^LC^H(\mathfrak{z}_\mathfrak{e})$ has no effect, and this implies that

$$\Delta'\big[\psi^{-1}, {}^LC(\mathfrak{e}), {}^LC^H(\mathfrak{z}_\mathfrak{e})\big] = \Delta'\big[\psi^{-1}, \mathfrak{e}, \mathfrak{z}_\mathfrak{e}\big].$$

Let us abbreviate this factor to $\Delta'[\psi^{-1}]$.

We have $S\Theta_{{}^LC^H \circ\varphi}(f^H) = T\Theta^{\psi}_{\xi\circ {}^LC^H\circ\varphi}(f)$. As we just argued, $\xi \circ {}^LC^H$ is $\hat{G}$-conjugate to ${}^LC\circ\xi$. Thus, the Galois representation $\xi \circ {}^LC^H \circ \varphi$ is the contragredient to the Galois representation $\xi \circ \varphi$. As the local Langlands correspondence for GL_n respects the operation of taking the contragredient, Corollary 5.7 implies that

$$T\Theta^{\psi}_{\xi\circ {}^LC^H\circ\varphi}(f) = T\Theta^{\psi^{-1}}_{\xi\circ\varphi}(f \circ \theta^{-1} \circ i).$$

By construction of $S\Theta_{\xi\circ\varphi}$, we have

$$T\Theta^{\psi^{-1}}_{\xi\circ\varphi}(f \circ \theta^{-1} \circ i) = S\Theta_{\xi\circ\varphi}({}'f^H)$$

whenever ${}'f^H$ is an element of $\tilde{\mathscr{H}}(H)$ whose stable orbital integrals match the $(\theta, 1)$-orbital integrals of $f \circ \theta^{-1} \circ i$ with respect to $\Delta'[\psi^{-1}]$. By Corollary 5.5, $f^H \circ i$ is such a function, and we see that the distribution $f \mapsto S\Theta_{\xi\circ\varphi}(f^H \circ i)$ satisfies the property that characterizes $S\Theta_{{}^LC_H\circ\varphi}$, hence must be equal to the latter. □

Theorem 5.9. *Let G be a quasisplit real K-group or a quasisplit symplectic or special orthogonal p-adic group, and let (B, ψ) be a Whittaker datum. Let $\varphi : W' \to {}^LG$ be a tempered Langlands parameter, and $\rho \in \mathrm{Irr}(\mathcal{S}_\varphi)$. Then*

$$\iota_{B,\psi}(\varphi, \rho)^\vee = \iota_{B,\psi^{-1}}\big({}^LC \circ \varphi, [\rho \circ \hat{C}^{-1}]^\vee\big).$$

Proof. Put $\pi = \iota_{B,\psi}(\varphi, \rho)$. For each semisimple $s \in S_\varphi$, let $\mathfrak{e}_s = (H, \mathscr{H}, s, \xi)$ be the corresponding endoscopic datum (see Section 3), and choose a z-pair $\mathfrak{z}_s = (H_1, \xi_{H_1})$. We have the Whittaker normalization $\Delta[\psi, \mathfrak{e}_s, \mathfrak{z}_s]$ of the transfer factor compatible with [Langlands and Shelstad 1987] (see the discussion before Proposition 5.4).

By construction, φ factors through ξ. Put $\varphi_s = \varphi \circ \xi_{H_1}$. For any function $f \in \tilde{\mathscr{H}}(G)$ let $f^{s,\psi} \in \tilde{\mathscr{H}}(H_1)$ be such that f and $f^{s,\psi}$ have $\Delta[\psi, \mathfrak{e}_s, \mathfrak{z}_s]$-matching orbital integrals. Then the distribution

$$f \mapsto S\Theta_{\varphi_s}(f^{s,\psi})$$

is independent of the choices of $f^{s,\psi}$ and $\mathfrak{z}_s$. As discussed in Section 3, we have the inversion of endoscopic transfer

$$\Theta_\pi(f) = |\mathcal{S}_\varphi|^{-1} \sum_{s\in\mathcal{S}_\varphi} \overline{\langle s, \rho\rangle} S\Theta_{\varphi_s}(f^{s,\psi}).$$

Thus, we need to show that

$$\Theta_{\pi^\vee}(f) = |\mathcal{S}_{{}^LC\circ\varphi}|^{-1} \sum_{s'\in\mathcal{S}_{{}^LC\circ\varphi}} \overline{\langle \hat{C}^{-1}(s'), \rho^\vee\rangle} S\Theta_{[{}^LC\circ\varphi]_{s'}}\big(f^{s',\psi^{-1}}\big).$$

Reindexing the sum using $s' = \hat{C}(s^{-1})$, we can write the right-hand side as

$$|\mathcal{S}_\varphi|^{-1} \sum_{s\in\mathcal{S}_\varphi} \overline{\langle s, \rho\rangle} S\Theta_{[{}^LC\circ\varphi]_{s'}}\big(f^{s',\psi^{-1}}\big).$$

The theorem will be proved once we show

$$S\Theta_{\varphi_s}\big([f\circ i]^{s,\psi}\big) = S\Theta_{[{}^LC\circ\varphi]_{s'}}\big(f^{s',\psi^{-1}}\big).$$

The endoscopic datum corresponding to ${}^LC\circ\varphi$ and s' is precisely ${}^LC(\mathfrak{e}_s)$ (in the sense that $1\in\mathrm{GL}_n(\mathbb{C})$ is an isomorphism between the two). We are free to choose any z-pair for it, and we choose ${}^LC^H(\mathfrak{z}_s)$. Then $[{}^LC\circ\varphi]_{s'} = {}^LC^{H_1}\circ\varphi_s$ and Theorem 5.8 in the p-adic case and [Adams and Vogan 2012, Theorem 7.1(a)] in the real case imply

$$S\Theta_{[{}^LC\circ\varphi]_{s'}}\big(f^{s',\psi^{-1}}\big) = S\Theta_{\varphi_s}\big(f^{s',\psi^{-1}}\circ i\big).$$

The functions f and $f^{s',\psi^{-1}}$ have $\Delta[\psi^{-1}, {}^LC(\mathfrak{e}_s), {}^LC^H(\mathfrak{z}_s)]$-matching orbital integrals. By Corollary 5.5, the functions $f\circ i$ and $f^{s',\psi^{-1}}\circ i$ have $\Delta[\psi, \mathfrak{e}, \mathfrak{z}_\mathfrak{e}]$-matching orbital integrals. It follows that the functions $[f\circ i]^{s,\psi}$ and $f^{s',\psi^{-1}}\circ i$ have the same stable orbital integrals, and the theorem follows. □

We alert the reader that, as was explained in Section 3, the symbol $\iota_{B,\psi}(\varphi,\rho)$ refers to an individual representation of $G(F)$ in all cases of Theorem 5.9, except possibly when G is an even orthogonal p-adic group, in which case Arthur's classification may assign to the pair (φ,ρ) a pair of representations, rather than an individual representation. In that case, the theorem asserts that if $\{\pi_1,\pi_2\}$ is the pair of representations associated with (φ,ρ), then $\{\pi_1^\vee,\pi_2^\vee\}$ is the pair of representations associated with $\iota_{B,\psi^{-1}}({}^LC\circ\varphi, [\rho\circ\hat{C}^{-1}]^\vee)$.

The following is the p-adic analog of [Adams and Vogan 2012, Theorem 7.1(a)].

Corollary 5.10. *Let G be a quasisplit symplectic or special orthogonal p-adic group, and let $\varphi : W' \to {}^LG$ be a tempered Langlands parameter. If Π is an L-packet assigned to φ, then*

$$\Pi^\vee = \big\{\pi^\vee \mid \pi\in\Pi\big\}$$

is an L-packet assigned to ${}^LC\circ\varphi$.

Proof. When G is either a symplectic or an odd orthogonal group, the statement follows immediately from Theorem 5.9. However, if G is an even orthogonal group, Π is one of two L-packets Π_1, Π_2 assigned to φ, and a priori we only know that the set $\Pi^\vee$ belongs to the union of the two L-packets Π'_1, Π'_2 assigned to ${}^LC \circ \varphi$. We claim that in fact it equals one of these two L-packets. Indeed, let $S\Theta$ be the stable character of Π. This is now a stable linear form on $\mathcal{H}(G)$, not just on $\tilde{\mathcal{H}}(G)$. The linear form $S\Theta \circ i$ is still stable. If $S\Theta'_1$ and $S\Theta'_2$ are the stable characters of Π'_1 and Π'_2 respectively, then the restrictions of $S\Theta'_1$ and $S\Theta'_2$ to $\tilde{\mathcal{H}}(G)$ are equal, and moreover according to Theorem 5.8 these restrictions are equal to the restriction of $S\Theta \circ i$ to $\tilde{\mathcal{H}}(G)$. From [Arthur 2013, Corollary 8.4.5] we conclude that

$$S\Theta \circ i = \lambda S\Theta'_1 + \mu S\Theta'_2$$

for some $\lambda, \mu \in \mathbb{C}$ with $\lambda + \mu = 1$. However, each of the three distributions $S\Theta \circ i$, $S\Theta'_1$, $S\Theta'_2$ is itself a sum of characters of tempered representations. Since Π'_1 and Π'_2 are disjoint, the linear independence of these characters then forces one of the numbers λ, μ to be equal to 1, and the other to 0. □

6. Depth-zero and epipelagic L-packets of p-adic groups

In this section, we are going to examine two constructions of L-packets on general reductive p-adic groups and show that (2) is satisfied by these L-packets.

The first construction is that of [DeBacker and Reeder 2009], in which L-packets consisting of depth-zero supercuspidal representations are constructed for each pure inner form of an unramified p-adic group. This construction was then extended to inner forms of p-adic groups arising from isocrystals with additional structure [Kaletha 2011]. The second construction is that of [Kaletha 2012], in which L-packets consisting of epipelagic representations are constructed for each tamely ramified p-adic group. The notion of epipelagic representation was introduced and studied by Reeder and Yu in [2012].

Fix a Langlands parameter $\varphi : W \to {}^LG$ of the type considered in [DeBacker and Reeder 2009] or [Kaletha 2012]. Fix a Γ-invariant splitting $(\hat{T}, \hat{B}, \{X_{\hat{\alpha}}\})$ of $\hat{G}$ and arrange that $\hat{T}$ is the unique torus normalized by φ. Choose a Chevalley involution $\hat{C}$ which sends the fixed splitting to its opposite. Then $\hat{C}$ commutes with all automorphisms preserving the fixed splitting, in particular with the action of Γ on $\hat{G}$, and hence ${}^LC = \hat{C} \times \mathrm{id}_W$ is an L-automorphism. Moreover, the action of $\hat{C}$ on $N(\hat{T}, \hat{G})$ preserves $\hat{T}$ and thus induces an action on $\Omega(\hat{T}, \hat{G})$. Since $\hat{C}(X_{\hat{\alpha}}) = X_{-\hat{\alpha}}$, this action fixes each simple reflection and is therefore trivial.

In both constructions of L-packets the first step is to form the Γ-module $\hat{S}$ with underlying abelian group $\hat{T}$ and Γ-action given by the composition $\Gamma \to \Omega(\hat{T}, \hat{G}) \rtimes \Gamma$ of φ and the natural projection $N(\hat{T}, \hat{G}) \to \Omega(\hat{T}, \hat{G})$. By the argument of the preceding paragraph, the Γ-module $\hat{S}$ for ${}^LC \circ \varphi$ is the same as the one for φ.

The next step is to obtain from φ a character $\chi : S(F) \to \mathbb{C}^\times$. This is done by factoring $\varphi = {}^Lj_X \circ \varphi_S$, where ${}^Lj_X : {}^LS \to {}^LG$ is an L-embedding constructed from φ, and $\varphi_S : W \to {}^LS$ is a Langlands parameter for S, and then letting χ be the character corresponding to φ_S. For the depth-zero case, this is the reinterpretation given in [Kaletha 2011], and the L-embedding ${}^LS \to {}^LG$ is obtained by choosing arbitrary unramified χ-data X for $R(\hat{S}, \hat{G})$. Applying Lemma 5.2 to the equation $\varphi = {}^Lj_X \circ \varphi_S$ we see that

$$ {}^LC \circ \varphi = {}^Lj_{-X} \circ (-1) \circ \varphi_S. $$

Since $-X$ is another set of unramified χ-data, and it is shown in [Kaletha 2011, Section 3.4] that φ_S is independent of the choice of X, we see that $[{}^LC \circ \varphi]_S = (-1) \circ \varphi_S$. In other words, the character of $S(F)$ constructed from ${}^LC \circ \varphi$ is χ_S^{-1}.

We claim that the same is true in the epipelagic case. That case is a bit more subtle because ${}^Lj_X$ depends on φ more strongly — the χ-data X is chosen based on the restriction of φ to wild inertia. What we need to show is that if X is chosen for φ, then the choice for ${}^LC \circ \varphi$ is $-X$. This however follows right away from the fact that the restriction of ${}^LC \circ \varphi$ to wild inertia equals the composition of (-1) with the restriction of φ to wild inertia.

The third step in the construction of both kinds of L-packets relies on a procedure (different in the two cases) which associates to an admissible embedding j of S into an inner form G' of G a representation $\pi(\chi_S, j)$ of $G'(F)$. We won't recall this procedure — for our current purposes it will be enough to treat it as a black box. The only feature of this black box that is essential for us is that the contragredient of $\pi(\chi_S, j)$ is given by $\pi(\chi_S^{-1}, j)$. Now let (B, ψ) be a Whittaker datum for G. It is shown in both cases that there exists an admissible embedding $j_0 : S \to G$, unique up to $G(F)$-conjugacy, so that the representation $\pi(\chi_S, j_0)$ of $G(F)$ is (B, ψ)-generic. Moreover, one has $S_\varphi = [\hat{S}]^\Gamma$, so that $\mathrm{Irr}(S_\varphi) = X^*(\hat{S}^\Gamma) = X_*(S)_\Gamma = \boldsymbol{B}(S)$, where $\boldsymbol{B}(S)$ is the set of isomorphism classes of isocrystals with S-structure [Kottwitz 1997]. Using j_0 one obtains a map $\mathrm{Irr}(S_\varphi) = \boldsymbol{B}(S) \to \boldsymbol{B}(G)_{\mathrm{bas}}$. Each $\rho \in \mathrm{Irr}(S_\varphi)$ provides in this way an extended pure inner twist $(G^{b_\rho}, b_\rho, \xi_\rho)$. The composition $j_\rho = \xi_\rho \circ j_0$ is an admissible embedding $S \to G^{b_\rho}$ defined over F and provides by the black box construction alluded to above a representation $\pi(\chi_S, j_\rho)$ of $G^{b_\rho}(F)$. The construction of L-packets and their internal parametrization is then realized by

$$ \iota_{B,\psi} : \mathrm{Irr}(S_\varphi) \to \Pi_\varphi, \quad \rho \mapsto \pi(\chi_S, j_\rho). $$

The contragredient of $\pi(\chi_S, j_0)$ is given by $\pi(\chi_S^{-1}, j_0)$, and the latter representation is (B, ψ^{-1})-generic. Hence, the version of j_0 associated to ${}^LC \circ \varphi$ and the Whittaker datum (B, ψ^{-1}) is equal to j_0. Using $S_{{}^LC\circ\varphi} = [\hat{S}^\Gamma]$ and $\rho^\vee \circ \hat{C}^{-1} = \rho$, and reviewing the procedure above, we see that

$$ \iota_{B,\psi^{-1}}\big({}^LC \circ \varphi, \rho^\vee \circ \hat{C}^{-1}\big) = \pi(\chi_S^{-1}, j_\rho) = \pi(\chi_S, j_\rho)^\vee = \iota_{B,\psi}(\varphi, \rho)^\vee. $$

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Homogeneous projective bundles over abelian varieties

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We consider projective bundles (or Brauer–Severi varieties) over an abelian variety which are homogeneous, that is, invariant under translation. We describe the structure of these bundles in terms of projective representations of commutative group schemes; the irreducible bundles correspond to Heisenberg groups and their standard representations. Our results extend those of Mukai on semihomogeneous vector bundles, and yield a geometric view of the Brauer group of abelian varieties.

1. Introduction

The main objects of this article are projective bundles (or Brauer–Severi varieties) over an abelian variety X which are homogeneous, that is, isomorphic to their pull-backs under all translations. Among these bundles, projectivizations of vector bundles are well understood thanks to [Mukai 1978]. Indeed, vector bundles with homogeneous projectivization are exactly the semihomogeneous vector bundles of Mukai. Those that are simple (that is, their global endomorphisms are just scalars) admit several remarkable characterizations; for example, they are all obtained as direct images of line bundles under isogenies. Moreover, every indecomposable semihomogeneous vector bundle is the tensor product of a unipotent bundle and a simple semihomogeneous bundle.

In this article, we obtain somewhat similar statements for the structure of homogeneous projective bundles. We build on the results of [Brion 2012a] about homogeneous principal bundles under an arbitrary algebraic group; here we consider of course the projective linear group PGL_n. In loose terms, the approach of our earlier paper reduces the classification of homogeneous bundles to that of commutative subgroup schemes of PGL_n. The latter, carried out in Section 2, is based on the classical construction of Heisenberg groups and their irreducible representations.

In Section 3, we introduce a notion of irreducibility for homogeneous projective bundles, which is equivalent to the group scheme of bundle automorphisms being

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finite. (The projectivization of a semihomogeneous vector bundle E is irreducible if and only if E is simple.) We characterize those projective bundles that are homogeneous and irreducible by the vanishing of all the cohomology groups of their adjoint vector bundle (Proposition 3.7). Also, we show that the homogeneous irreducible bundles are classified by the pairs (H, e), where H is a finite subgroup of the dual abelian variety, and $e : H \times H \to \mathbb{G}_m$ a nondegenerate alternating bilinear pairing (Proposition 3.1). Finally, we obtain a characterization of those homogeneous projective bundles that are projectivizations of vector bundles, first in the irreducible case (Proposition 3.10; it states in loose terms that the pairing e originates from a line bundle on X) and then in the general case (Theorem 3.11).

Irreducible homogeneous projective bundles over an elliptic curve are exactly the projectivizations of indecomposable vector bundles with coprime rank and degree, as follows from the classic work of Atiyah [1957]. But any abelian variety X of dimension at least two admits many homogeneous projective bundles that are not projectivizations of vector bundles. In fact, any class in the Brauer group $\mathrm{Br}\, X$ is represented by a homogeneous bundle (as shown by [Elencwajg and Narasimhan 1983, Theorem 1] in the setting of complex tori). Also, our approach yields a geometric view of a description of $\mathrm{Br}\, X$ due to Berkovich [1972]; this is developed in Remark 3.13.

Spaces of algebraically equivalent effective divisors on an arbitrary projective variety afford geometric examples of projective bundles. These spaces are investigated in Section 4 for abelian varieties and curves of genus $g \geq 2$; they turn out to be homogeneous in the former case, but not in the latter.

Finally, in Section 5 we investigate those homogeneous projective bundles that are self-dual, that is, equipped with an isomorphism to their dual bundle; these correspond to principal bundles under the projective orthogonal or symplectic groups. Here the main ingredients are the Heisenberg groups associated to symplectic vector spaces over a field with two elements. Also, we introduce a geometric notion of indecomposability (which differs from the group-theoretic notion of L-indecomposability defined in [Balaji et al. 2005]), and obtain a structure result for indecomposable homogeneous self-dual bundles (Proposition 5.9).

Throughout this article, the base field k is algebraically closed, of arbitrary characteristic $p \geq 0$. Most of our results on $\mathbb{P}^{n-1}$-bundles hold under the assumption that n is not a multiple of p; indeed, the structure of commutative subgroup schemes of PGL_n is much more complicated when p divides n (see [Levy et al. 2009]). For the same reason, we only consider self-dual projective bundles in characteristic other than 2. It would be interesting to extend our results to "bad" characteristics.

Notation and conventions. We use the book [Demazure and Gabriel 1970] as a general reference for group schemes. Our reference for abelian varieties is [Mumford 1970]; we generally follow its notation. In particular, the group law of an abelian variety X is denoted additively and multiplication by an integer n is denoted by n_X, with kernel X_n. For any point $x \in X$, we denote by $T_x : X \to X$ the translation $y \mapsto x + y$. The dual abelian variety is denoted by $\widehat{X}$.

2. Structure of homogeneous projective bundles

Generalities on projective bundles. Recall that a *projective bundle* over a variety X is a variety P equipped with a proper flat morphism

$$f : P \to X \tag{1}$$

with fibers at all closed points isomorphic to projective space $\mathbb{P}^{n-1}$ for some integer $n \geq 1$. Then f is a $\mathbb{P}^{n-1}$-bundle for the étale topology (see [Grothendieck 1968a, §8]).

Also, recall from [loc. cit.] that the $\mathbb{P}^{n-1}$-bundles are in a one-to-one correspondence with the torsors (or principal bundles)

$$\pi : Y \to X \tag{2}$$

under the projective linear group, $\mathrm{PGL}_n = \mathrm{Aut}(\mathbb{P}^{n-1})$. Specifically, P is the associated bundle $Y \times^{\mathrm{PGL}_n} \mathbb{P}^{n-1}$, and Y is the bundle of isomorphisms $X \times \mathbb{P}^{n-1} \to P$ over X. Thus, any representation $\rho : \mathrm{PGL}_n \to \mathrm{GL}(V)$ defines the associated vector bundle $Y \times^{\mathrm{PGL}_n} V$ over X. The representation of PGL_n in the space M_n of $n \times n$ matrices by conjugation yields a matrix bundle on X; its sheaf of local sections is an Azumaya algebra of rank n^2 over X,

$$\mathcal{A} := (\pi_*(\mathcal{O}_Y) \otimes \mathrm{M}_n)^{\mathrm{PGL}_n},$$

viewed as a sheaf of noncommutative $\mathcal{O}_X$-algebras over $\pi_*(\mathcal{O}_Y)^{\mathrm{PGL}_n} = \mathcal{O}_X$. In particular, $\mathcal{A}$ defines a central simple algebra of degree n over the function field $k(X)$. By [Grothendieck 1968a, corollaire 5.11], the assignment $P \mapsto \mathcal{A}$ yields a one-to-one correspondence between $\mathbb{P}^{n-1}$-bundles and Azumaya algebras of rank n^2. The quotient of $\mathcal{A}$ by $\mathcal{O}_X$ is the sheaf of local sections of the *adjoint bundle* $\mathrm{ad}\, P$, the vector bundle associated with the adjoint representation of PGL_n in its Lie algebra $\mathfrak{pgl}_n$ (the quotient of the Lie algebra M_n by the scalar matrices). The correspondences between $\mathbb{P}^{n-1}$-bundles, PGL_n-torsors, and Azumaya algebras of rank n^2 preserve morphisms. As a consequence, every morphism of $\mathbb{P}^{n-1}$-bundles is an isomorphism.

There is a natural operation of *product* on projective bundles: to any $\mathbb{P}^{n_i-1}$-bundles $f_i : P_i \to X$ $(i = 1, 2)$ with associated PGL_{n_i}-bundles $\pi_i : Y_i \to X$, one associates the $\mathbb{P}^{n_1 n_2-1}$-bundle

$$f : P_1 P_2 \to X$$

that corresponds to the $\mathrm{PGL}_{n_1 n_2}$-torsor obtained from the $\mathrm{PGL}_{n_1} \times \mathrm{PGL}_{n_2}$-torsor

$$\pi_1 \times \pi_2 : Y_1 \times_X Y_2 \to X$$

by the extension of structure groups

$$\mathrm{PGL}_{n_1} \times \mathrm{PGL}_{n_2} = \mathrm{PGL}(k^{n_1}) \times \mathrm{PGL}(k^{n_2}) \xrightarrow{\rho} \mathrm{PGL}(k^{n_1} \otimes k^{n_2}) = \mathrm{PGL}_{n_1 n_2},$$

where ρ stems from the natural representation $\mathrm{GL}(k^{n_1}) \times \mathrm{GL}(k^{n_2}) \to \mathrm{GL}(k^{n_1} \otimes k^{n_2})$. So $P_1 P_2$ contains the fibered product $P_1 \times_X P_2$; it may be viewed as a global analogue of the Segre product of projective spaces. The corresponding operation on Azumaya algebras is the tensor product (see [Grothendieck 1968a, §8]).

Likewise, any projective bundle $f : P \to X$ has a *dual* bundle

$$f^* : P^* \to X,$$

where P^* is the same variety as P, but the action of PGL_n is twisted by the automorphism arising from the inverse transpose; then $P^* = Y \times^{\mathrm{PGL}_n} (\mathbb{P}^{n-1})^*$, where $(\mathbb{P}^{n-1})^*$ denotes the dual projective space. The Azumaya algebra associated with P^* is the opposite algebra $\mathscr{A}^{\mathrm{op}}$. The assignment $P \mapsto P^*$ is of course contravariant, and the bidual P^{**} comes with a canonical isomorphism of bundles $P \xrightarrow{\cong} P^{**}$.

Given a positive integer $n_1 \le n$, a $\mathbb{P}^{n_1-1}$-*subbundle* $f_1 : P_1 \to X$ of the $\mathbb{P}^{n-1}$-bundle (1) corresponds to a reduction of structure group of the associated PGL_n-torsor (2) to a PGL_{n,n_1}-torsor $\pi_1 : Y_1 \to X$, where $\mathrm{PGL}_{n,n_1} \subset \mathrm{PGL}_n$ denotes the maximal parabolic subgroup that stabilizes a linear subspace $\mathbb{P}^{n_1-1}$ of $\mathbb{P}^{n-1}$. Equivalently, the subbundle P_1 corresponds to a PGL_n-equivariant morphism

$$\gamma : Y \to \mathrm{PGL}_n / \mathrm{PGL}_{n,n_1} = \mathrm{Gr}_{n,n_1}$$

(the Grassmannian parametrizing these subspaces). We have $P \cong Y_1 \times^{\mathrm{PGL}_{n,n_1}} \mathbb{P}^{n-1}$ and $P_1 \cong Y_1 \times^{\mathrm{PGL}_{n,n_1}} \mathbb{P}^{n_1-1}$ as bundles over X, where PGL_{n,n_1} acts on $\mathbb{P}^{n_1-1}$ via its quotient PGL_{n_1}.

Given two positive integers n_1 and n_2 such that $n_1 + n_2 = n$, a *decomposition* of type (n_1, n_2) of the $\mathbb{P}^{n-1}$-bundle (1) consists of two $\mathbb{P}^{n_i-1}$-subbundles $f_i : P_i \to X$ $(i = 1, 2)$ which are disjoint (as subvarieties of P). This corresponds to a reduction of structure group of the PGL_n-torsor (2) to a torsor $\pi_{12} : Y_{12} \to X$ under the maximal Levi subgroup

$$\mathrm{P}(\mathrm{GL}_{n_1} \times \mathrm{GL}_{n_2}) = \mathrm{PGL}_{n,n_1} \cap \mathrm{PGL}_{n,n_2} \subset \mathrm{PGL}_n$$

that stabilizes two disjoint linear subspaces $\mathbb{P}^{n_i-1}$ of $\mathbb{P}^{n-1}$ $(i = 1, 2)$. Then

$$P_i = Y_{12} \times^{\mathrm{P}(\mathrm{GL}_{n_1} \times \mathrm{GL}_{n_2})} \mathbb{P}^{n_i-1}$$

for $i = 1, 2$, where $\mathrm{P}(\mathrm{GL}_{n_1} \times \mathrm{GL}_{n_2})$ acts on each $\mathbb{P}^{n_i - 1}$ via its quotient PGL_{n_i}. The decompositions of type (n_1, n_2) correspond to the PGL_n-equivariant morphisms

$$\delta : Y \to \mathrm{PGL}_n / \mathrm{P}(\mathrm{GL}_{n_1} \times \mathrm{GL}_{n_2}) \tag{3}$$

to the variety of decompositions.

If the bundle (1) admits no decomposition, then we say, of course, that it is *indecomposable*. Equivalently, the associated torsor (2) admits no reduction of structure group to a proper Levi subgroup.

When P is the projectivization $\mathbb{P}(E)$ of a vector bundle E over X, the subbundles of P correspond bijectively to those of E, and the decompositions of P to the splittings $E = E_1 \oplus E_2$ of vector bundles. Also, note that $\mathbb{P}(E)\mathbb{P}(F) = \mathbb{P}(E \otimes F)$ and $\mathbb{P}(E)^* = \mathbb{P}(E^*)$, with obvious notation.

Homogeneous projective bundles. From now on, X denotes a fixed abelian variety, $f : P \to X$ a $\mathbb{P}^{n-1}$-bundle, and $\pi : Y \to X$ the corresponding PGL_n-torsor. Then P is a nonsingular projective variety and f is its Albanese morphism. In particular, f is uniquely determined by the variety P.

Since P is complete, its automorphism functor is represented by a group scheme $\mathrm{Aut}\, P$, locally of finite type. Moreover, we have a homomorphism of group schemes

$$f_* : \mathrm{Aut}(P) \to \mathrm{Aut}(X)$$

with kernel the subgroup scheme $\mathrm{Aut}_X(P) \cong \mathrm{Aut}_X^{\mathrm{PGL}_n}(Y)$ of bundle automorphisms. Also, $\mathrm{Aut}_X(P)$ is affine of finite type, and its Lie algebra is $H^0(X, \mathrm{ad}(P))$ (see, for example, [Brion 2011, §4] for these results).

We say that a $\mathbb{P}^{n-1}$-bundle (1) is *homogeneous* if the image of f_* contains the subgroup $X \subset \mathrm{Aut}(X)$ of translations; equivalently, the bundle P is isomorphic to its pull-backs under all translations. This amounts to the vector bundle $\mathrm{ad}\, P$ being homogeneous (see [Brion 2012a, Corollary 2.15]; if P is the projectivization of a vector bundle, this follows alternatively from [Mukai 1978, Theorem 5.8]).

The structure of homogeneous projective bundles is described by the following:

Theorem 2.1. (i) *A $\mathbb{P}^{n-1}$-bundle $f : P \to X$ is homogeneous if and only if there exist an exact sequence of group schemes*

$$1 \longrightarrow H \longrightarrow G \xrightarrow{\ \gamma\ } X \longrightarrow 1, \tag{4}$$

where G is antiaffine (*i.e.*, $\mathcal{O}(G) = k$), *and a faithful homomorphism $\rho : H \to \mathrm{PGL}_n$ such that P is the associated bundle $G \times^H \mathbb{P}^{n-1} \to G/H = X$, where H acts on $\mathbb{P}^{n-1}$ via ρ.*

Then the exact sequence (4) *is unique*; *the group scheme G is smooth*, *connected*, *and commutative* (*in particular*, *H is commutative*), *and the projective representation ρ is unique up to conjugacy in* PGL_n. *Moreover*, *the corresponding*

PGL_n*-torsor is the associated bundle* $G \times^H \mathrm{PGL}_n \to X$, *and the corresponding Azumaya algebra satisfies*

$$\mathcal{A} \cong (\gamma_*(\mathbb{O}_G) \otimes \mathrm{M}_n)^H \tag{5}$$

as a sheaf of algebras over $\gamma_*(\mathbb{O}_G)^H \cong \mathbb{O}_X$.

(ii) *For* P *as in* (i), *we have an isomorphism*

$$\mathrm{Aut}_X(P) \cong \mathrm{PGL}_n^H, \tag{6}$$

the right-hand side being the centralizer of H *in* PGL_n. *As a consequence*,

$$H^0(X, \mathrm{ad}(P)) = \mathfrak{pgl}_n^H. \tag{7}$$

(iii) *The homogeneous projective subbundles of* P *are exactly the bundles* $G \times^H S \to X$, *where* $S \subset \mathbb{P}^{n-1}$ *is an* H*-stable linear subspace.*

(iv) *Any decomposition of* P *consists of homogeneous subbundles.*

Proof. Part (i) follows readily from Theorem 3.1 of [Brion 2012a], and (ii) from Proposition 3.6 of the same reference.

(iii) Let $f_1 : P_1 \to X$ be a projective subbundle, and consider the corresponding reduction of structure group of the PGL_n-torsor Y to a PGL_{n,n_1}-torsor $\pi_1 : Y_1 \to X$. If f_1 is homogeneous, then again by [Brion 2012a, Theorem 3.1], we have a PGL_{n,n_1}-equivariant isomorphism

$$Y_1 \cong G_1 \times^{H_1} \mathrm{PGL}_{n,n_1}$$

for some exact sequence $0 \to H_1 \to G_1 \to X \to 0$ with G_1 antiaffine, and some faithful homomorphism $\rho_1 : H_1 \to \mathrm{PGL}_{n,n_1}$. Thus,

$$Y \cong Y_1 \times^{\mathrm{PGL}_{n,n_1}} \mathrm{PGL}_n \cong G_1 \times^{H_1} \mathrm{PGL}_n$$

equivariantly for the action of PGL_n. By the uniqueness in (i), it follows that $G_1 = G$ and $H_1 = H$; hence $P_1 = G \times^H S$ for some H-stable linear subspace $S \subset \mathbb{P}^{n-1}$.

Conversely, any H-stable linear subspace obviously yields a homogeneous projective subbundle.

(iv) A decomposition of P of type (n_1, n_2) corresponds to a PGL_n-equivariant morphism $\delta : Y \to \mathrm{PGL}_n / \mathrm{P}(\mathrm{GL}_{n_1} \times \mathrm{GL}_{n_2})$. Since the variety $\mathrm{PGL}_n / \mathrm{P}(\mathrm{GL}_{n_1} \times \mathrm{GL}_{n_2})$ is affine, the corresponding reduction of structure group $\pi_{12} : Y_{12} \to X$ is homogeneous by [loc. cit., Proposition 2.8]. Thus, the associated bundles P_1 and P_2 are homogeneous as well. $\square$

Remark 2.2. Let P_i ($i = 1, 2$) be homogeneous bundles corresponding to extensions $1 \to H_i \to G_i \to X \to 1$ and projective representations $\rho_i : H_i \to \mathrm{PGL}_{n_i}$. Then the $\mathrm{PGL}_{n_1 n_2}$-torsor that corresponds to $P_1 P_2$ is the associated bundle

$$(G_1 \times_X G_2) \times^{H_1 \times H_2} \mathrm{PGL}_{n_1 n_2} \longrightarrow (G_1 \times_X G_2)/(H_1 \times H_2) = X,$$

where the homomorphism $H_1 \times H_2 \to \mathrm{PGL}_{n_1 n_2}$ is given by the tensor product $\rho_1 \otimes \rho_2$. Thus, $P_1 P_2$ is the homogeneous bundle classified by the extension $1 \to H \to G \to X \to 1$, where $G \subset G_1 \times_X G_2$ denotes the largest antiaffine subgroup and $H = (H_1 \times H_2) \cap G$, and by the projective representation $(\rho_1 \otimes \rho_2)|_H$.

As a consequence, the m-th power P^m corresponds to the same extension as P and to the m-th tensor power of its projective representation. Likewise, the dual of a homogeneous bundle is the homogeneous bundle associated with the same extension and with the dual projective representation.

The antiaffine algebraic groups are classified in [Brion 2009] and independently [Sancho de Salas and Sancho de Salas 2009], and the antiaffine extensions (4) in [Brion 2012a, §3.3]. We now describe the other ingredients of Theorem 2.1, that is, the commutative subgroup schemes $H \subset \mathrm{PGL}_n$ up to conjugacy. Every such subgroup scheme has a unique decomposition

$$H = H_u \times H_s,$$

where H_u is unipotent and H_s is diagonalizable. Thus, H_s sits in an exact sequence

$$1 \to H_s^0 \to H_s \to F \to 1,$$

where H_s^0 is a connected diagonalizable group scheme (the neutral component of H_s), and the group of components F is finite, diagonalizable, and of order prime to p (in particular, F is smooth); this exact sequence is unique and splits noncanonically. In turn, H_s^0 is an extension of a finite diagonalizable group scheme of order a power of p, by a torus (the reduced neutral component); this extension is also unique and splits noncanonically.

Denote by $\widetilde{H} \subset \mathrm{GL}_n$ the preimage of $H \subset \mathrm{PGL}_n$. This yields a central extension

$$1 \to \mathbb{G}_m \to \widetilde{H} \to H \to 1, \tag{8}$$

where the multiplicative group $\mathbb{G}_m$ is viewed as the group of invertible scalar matrices. We say that $\widetilde{H}$ is the *theta group* of H, and define similarly $\widetilde{H}_u$, $\widetilde{H}_s$, and $\widetilde{H}_s^0$ (the latter is the neutral component of $\widetilde{H}_s$).

Given two S-valued points $\tilde{x}$ and $\tilde{y}$ of $\widetilde{H}$, where S denotes an arbitrary scheme, the commutator $\tilde{x}\tilde{y}\tilde{x}^{-1}\tilde{y}^{-1}$ is a S-valued point of $\mathbb{G}_m$ and depends only on the images of $\tilde{x}$ and $\tilde{y}$ in H. This defines a morphism

$$e : H \times H \to \mathbb{G}_m, \tag{9}$$

which is readily seen to be bilinear (that is, we have $e(xy,z)=e(x,z)e(y,z)$ and $e(x,yz)=e(x,z)e(y,z)$ for all S-valued points x, y, z of H) and alternating (that is, $e(x,x)=1$ for all x). We say that e is the *commutator pairing* of the extension (8).

Note that the dual bundle P^* has pairing e^{-1}; moreover, the power P^m, where m is a positive integer, has pairing e^m.

The center $Z(\widetilde{H})$ sits in an exact sequence of group schemes

$$1 \to \mathbb{G}_m \to Z(\widetilde{H}) \to H^{\perp} \to 1, \tag{10}$$

where the S-valued points of $H^{\perp}$ are those points of H such that $e(x,y)=1$ for all S'-valued points y of H and all schemes S' over S. In particular, $\widetilde{H}$ *is commutative if and only if* $e=1$.

We now show that the obstruction for being the projectivization of a homogeneous vector bundle is just the commutator pairing. The obstruction for being the projectivization of an arbitrary vector bundle will be determined in Theorem 3.11.

Proposition 2.3. *With the above notation, the following conditions are equivalent*:

(i) *P is the projectivization of a homogeneous vector bundle.*

(ii) *The extension* (8) *splits.*

(iii) $e=1$.

Proof. (i) $\Longrightarrow$ (ii) By [Brion 2012a, Theorem 3.1], any homogeneous vector bundle E of rank n over X is of the form $G\times^H k^n \to G/H = X$ for some antiaffine extension $1\to H\to G\to X\to 1$ and some faithful representation $\sigma : H\to \mathrm{GL}_n$. Since H is commutative, k^n contains eigenvectors of H; thus, twisting σ by a character of H (which does not change the projectivization $\mathbb{P}(E)$), we may assume that k^n contains nonzero fixed points of H. Then σ defines a faithful projective representation $\rho : H\to \mathrm{PGL}_n$. Hence G and ρ are the data associated with the homogeneous projective bundle $\mathbb{P}(E)\to X$, and σ splits the extension (8).

(ii) $\Longrightarrow$ (i) Any splitting of that extension yields a homomorphism $\sigma : H\to \mathrm{GL}_n$ that lifts ρ. Then the associated bundle $G\times^H k^n\to X$ is a homogeneous vector bundle with projectivization P.

(ii) $\Longleftrightarrow$ (iii) The forward implication is obvious. Conversely, if $e=1$, then $\widetilde{H}$ is commutative. It follows that $\widetilde{H}\cong U\times \widetilde{H}_s$, where the unipotent part U is isomorphic to H_u via the homomorphism $\widetilde{H}\to H$, and $\widetilde{H}_s$ sits in an exact sequence of diagonalizable group schemes $1\to\mathbb{G}_m\to\widetilde{H}_s\to H_s\to 1$. But every such sequence splits, since so does the dual exact sequence of character groups. □

Next, we obtain a very useful structure result for H under the assumption that n is not divisible by the characteristic:

Proposition 2.4. *Keep the above notation, and assume that* $(n,p)=1$.

(i) *The extension* $1 \to \mathbb{G}_m \to \widetilde{H}_u \to H_u \to 1$ *has a unique splitting, and the corresponding lift of* H_u *(that we still denote by* H_u*) is central in* $\widetilde{H}$*. Also, the extension* $1 \to \mathbb{G}_m \to \widetilde{H}_s^0 \to H_s^0 \to 1$ *splits noncanonically and* $\widetilde{H}_s^0$ *is central in* $\widetilde{H}$*.*

(ii) *We have canonical decompositions of group schemes*

$$\widetilde{H} = H_u \times \widetilde{H}_s, \quad Z(\widetilde{H}) = H_u \times Z(\widetilde{H}_s).$$

Moreover, $Z(\widetilde{H}_s)$ *is diagonalizable and sits in an exact sequence*

$$1 \to \widetilde{H}_s^0 \to Z(\widetilde{H}_s) \to F^{\perp} \to 1$$

which splits noncanonically.

(iii) *The commutator pairing e factors through a bilinear alternating morphism*

$$e_F : F \times F \to \mathbb{G}_m. \tag{11}$$

Proof. Since any commutator has determinant 1, we see that e takes values in the subgroup scheme $\mu_n = \mathbb{G}_m \cap \mathrm{SL}_n$ of n-th roots of unity. In other terms, e factors through the pairing

$$se : H \times H \to \mu_n$$

defined by the central extension

$$1 \to \mu_n \to S\widetilde{H} \to H \to 1,$$

where $S\widetilde{H} := H \cap \mathrm{SL}_n$. Note that μ_n is smooth by our assumption on n. Moreover, se restricts trivially to $nH \times H$, where nH denotes the image of the multiplication by n in the commutative group scheme H.

We claim that $H_u \subset nH$. This is clear if $p = 0$, since H_u is then isomorphic to the additive group of a vector space. If $p \geq 1$, then the commutative unipotent group scheme H_u is killed by some power of p. Using again the assumption that $(n, p) = 1$, it follows that $H_u = nH_u \subset nH$.

By that claim, se restricts trivially to $H_u \times H$, and hence $\widetilde{H}_u \subset Z(\widetilde{H})$; in particular, $\widetilde{H}_u$ is commutative. Thus, $\widetilde{H}_u \cong H_u \times \mathbb{G}_m$; this proves the assertion about H_u.

We already saw that the extension $1 \to \mathbb{G}_m \to \widetilde{H}_s^0 \to H_s^0 \to 1$ splits. Also, $H_s^0 \cong T \times E$, where T is a torus (the reduced neutral component), and E is a finite group scheme killed by some power of p. As above, it follows that $H_s^0 \subset nH$, and that $\widetilde{H}_s^0$ is central in $\widetilde{H}$. This completes the proof of (i).

The decompositions in (ii) are direct consequences of (i). The assertion on $Z(\widetilde{H}_s)$ follows from the exact sequence $1 \to \widetilde{H}_s^0 \to \widetilde{H}_s \to F \to 1$, since $\widetilde{H}_s^0 \subset Z(\widetilde{H}_s)$. Finally, (iii) also follows readily from (i). ☐

Remark 2.5. With the notation and assumptions of Proposition 2.4, the group scheme $\mathrm{Aut}_X(P)$ is smooth, as follows from the isomorphism (6) together with [Herpel 2013, Theorem 1.1]. Moreover, $\mathrm{Aut}\, P$ is smooth as well: indeed, we have an exact sequence of group schemes

$$1 \longrightarrow \mathrm{Aut}_X(P) \longrightarrow \mathrm{Aut}(P) \xrightarrow{f_*} \mathrm{Aut}_P(X) \longrightarrow 1,$$

where $\mathrm{Aut}_P(X)$ is a subgroup scheme of $\mathrm{Aut}\, X$ containing the group X of translations. Since $\mathrm{Aut}(X) = X \ltimes \mathrm{Aut}_{\mathrm{gp}}(X)$, where the group scheme of automorphisms of algebraic groups $\mathrm{Aut}_{\mathrm{gp}}(X)$ is étale (possibly infinite), it follows that $\mathrm{Aut}_P(X)$ is smooth, and hence so is $\mathrm{Aut}\, X$.

Nondegenerate theta groups. As in the above subsection, we consider a commutative subgroup scheme $H \subset \mathrm{PGL}_n$ and the associated theta group $\widetilde{H} \subset \mathrm{GL}_n$; we assume that $(n, p) = 1$.

We say that $\widetilde{H}$ is *nondegenerate* if $Z(\widetilde{H}) = \mathbb{G}_m$. By Proposition 2.4, this is equivalent to the assertions that H is a finite commutative group of order prime to p, and the homomorphism

$$\epsilon : H \to \mathscr{X}(H), \quad x \mapsto (y \mapsto e(x, y)) \tag{12}$$

is faithful, where $\mathscr{X}(H) := \mathrm{Hom}_{\mathrm{gp}}(H, \mathbb{G}_m)$ denotes the character group of H. It follows that ϵ is an isomorphism.

We now recall from [Mumford 1966, §1] the structure of nondegenerate theta groups. Choose a subgroup $K \subset H$ that is totally isotropic for the commutator pairing e, and maximal with this property. Then

$$\widetilde{H} \cong \mathbb{G}_m \times K \times \mathscr{X}(K),$$

where the group law on the right-hand side is given by

$$(t, x, \chi) \cdot (t', x', \chi') = (tt'\chi'(x), x + x', \chi + \chi'), \tag{13}$$

the group laws on K and $\mathscr{X}(K)$ being denoted additively. Such a group is called the *Heisenberg group* associated with the finite group K; we denote it by $\mathscr{H}(K)$ and identify the group K (resp. $\mathscr{X}(K)$) with its lift $\{1\} \times K \times \{0\}$ (resp. $\{1\} \times \{0\} \times \mathscr{X}(K)$) in $\mathscr{H}(K)$.

Also, recall that $\mathscr{H}(K)$ has a unique irreducible representation on which $\mathbb{G}_m$ acts via $t \mapsto t\,\mathrm{id}$: the *standard representation* (also called the *Schrödinger representation*) in the space $\mathbb{O}(K)$ of functions on K with values in k, on which $\widetilde{H}$ acts via

$$((t, x, \chi) \cdot f)(y) := t\chi(y) f(x + y).$$

The corresponding commutator pairing e is given by

$$e((x, \chi), (x', \chi')) := \chi'(x)\chi(x')^{-1}.$$

In particular, the standard representation $W(K)$ contains a unique line of K-fixed points and has dimension $n = \#(K)$; moreover, the group H is killed by n and has order n^2. Any finite-dimensional representation V of $\mathcal{H}(K)$ on which $\mathbb{G}_m$ acts by scalar multiplication is a direct sum of m copies of $W(K)$, where $m := \dim(V^K)$. Such a representation is called *of weight* 1.

For later use, we record the following result, which is well-known in the setting of theta structures on ample line bundles over complex abelian varieties (see [Birkenhake and Lange 2004, Lemma 6.6.6 and Exercise 6.10.14]):

Lemma 2.6. *Assume that* $(n, p) = 1$ *and let* $\widetilde{H} \subset \mathrm{GL}_n$ *be a nondegenerate theta group.*

(i) *The algebra* M_n *has a basis* $(u_h)_{h\in H}$ *such that every* u_h *is an eigenvector of* H (*acting by conjugation*) *with weight* $\epsilon(h)$, *and*

$$u_{x,\chi}u_{x',\chi'} = \chi'(x)u_{x+x',\chi+\chi'}$$

for all $h = (x, \chi)$ *and* $h' = (x', \chi')$ *in* $H = K \times \mathscr{X}(K)$. *In particular*, *the representation of* H *in* M_n *by conjugation is isomorphic to the regular representation.*

(ii) *The centralizers of* $\widetilde{H}$ *in* GL_n *and of* H *in* PGL_n *satisfy*

$$\mathrm{GL}_n^{\widetilde{H}} = \mathbb{G}_m, \quad \mathrm{PGL}_n^H = H.$$

Moreover, *the normalizers sit in exact sequences*

$$1 \to \mathbb{G}_m \to N_{\mathrm{GL}_n}(\widetilde{H}) \to N_{\mathrm{PGL}_n}(H) \to 1,$$
$$1 \to H \to N_{\mathrm{PGL}_n}(H) \to \mathrm{Aut}(H, e) \to 1.$$

Also, *we have an isomorphism*

$$\mathrm{Aut}^{\mathbb{G}_m}(\widetilde{H}) \cong N_{\mathrm{PGL}_n}(H). \tag{14}$$

Proof. (i) We may view H as a subset of M_n via $(x, \chi) \mapsto u_{x,\chi} := (1, x, \chi) \in \widetilde{H} \subset \mathrm{GL}_n$. Then the assertions follow readily from (13) for the group law of $\widetilde{H}$.

(ii) By Schur's lemma, we have $\mathrm{GL}_n^{\widetilde{H}} = \mathbb{G}_m$; this yields the first exact sequence.

In view of (i), the fixed points of H acting on $\mathbb{P}(M_n)$ by conjugation are exactly the points of $H \subset \mathrm{PGL}_n$; thus, $\mathrm{PGL}_n^H = H$. To obtain the second exact sequence, it suffices to show that the image in $\mathrm{Aut}\, H$ of $N_{\mathrm{PGL}_n}(H)$ equals $\mathrm{Aut}(H, e)$. But if $g \in \mathrm{PGL}_n$ normalizes H, then one readily checks that the conjugation $\mathrm{Int}(g)|_H$ preserves the pairing e. Conversely, let $g \in \mathrm{Aut}(H, e)$; then composing the inclusion $\rho : H \to \mathrm{PGL}_n$ with g, we obtain a projective representation ρ_g with the same commutator pairing. Thus, ρ_g lifts to a representation $\tilde{\rho}_g : \widetilde{H} \to \mathrm{GL}_n$ which is isomorphic to the standard representation. It follows that g extends to the conjugation by some $\tilde{g} \in \mathrm{GL}_n$ that normalizes H.

The isomorphism (14) follows similarly from the fact that the standard representation is the unique irreducible representation of weight 1. □

Returning to an arbitrary theta group $\widetilde{H} \subset \mathrm{GL}_n$, we now describe the representation of $\widetilde{H}$ in $k^n =: V$. Consider the decomposition

$$V = \bigoplus_{\lambda} V_\lambda \tag{15}$$

into weight spaces of the diagonalizable group $Z(\widetilde{H}_s)$, where λ runs over the characters of weight 1 of that group (those that restrict to the identity character of $\mathbb{G}_m$). By Proposition 2.4, each V_λ is stable under $\widetilde{H}$.

Proposition 2.7. *With the above notation, each quotient $\widetilde{H}_s/\ker(\lambda)$ is isomorphic to the Heisenberg group $\mathcal{H}(K/F^\perp)$, where K denotes a maximal totally isotropic subgroup scheme of F relative to e_F.*

Moreover, we have an isomorphism of representations of $\widetilde{H} \cong H_u \times \widetilde{H}_s$:

$$V_\lambda \cong U_\lambda \otimes W(K/F^\perp),$$

where U_λ is a representation of H_u and $W(K/F^\perp)$ is the standard representation of $\widetilde{H}_s/\ker(\lambda)$.

Proof. Note that λ yields a splitting of (10), and an isomorphism $Z(\widetilde{H}_s)/\ker(\lambda) \cong \mathbb{G}_m$. Also, $\widetilde{H}_s/Z(\widetilde{H}_s) \cong \widetilde{H}/Z(\widetilde{H}) \cong F/F^\perp$ by Proposition 2.4. Thus, the exact sequence

$$1 \to Z(\widetilde{H}_s)/\ker(\lambda) \to \widetilde{H}_s/\ker(\lambda) \to \widetilde{H}_s/Z(\widetilde{H}_s) \to 1$$

may be identified with the central extension

$$1 \to \mathbb{G}_m \to \widetilde{H}_s/\ker(\lambda) \to F/F^\perp \to 1,$$

and the corresponding commutator pairing is induced by e_F. This shows that $\widetilde{H}_s/\ker(\lambda)$ is a nondegenerate theta group. Now the first assertion follows from the structure of these groups.

Also, V_λ is a representation of $\widetilde{H}_s/\ker(\lambda)$ on which the center $\mathbb{G}_m$ acts with weight 1, and hence a direct sum of copies of the standard representation. This implies the second assertion in view of Proposition 2.4 again. □

Corollary 2.8. *With the above notation, the representation of $\widetilde{H}$ in V is an iterated extension of irreducible representations of the same dimension,*

$$d := [K : F^\perp] = \sqrt{[F : F^\perp]} = \sqrt{[H : H^\perp]}. \tag{16}$$

In particular, n is a multiple of d, with equality if and only if $\widetilde{H}$ is a Heisenberg group acting via its standard representation.

We say that d is the *homogeneous index* of the bundle (1); this is the minimal rank of a homogeneous subbundle of P in view of Theorem 2.1. (One can show that the homogeneous index of P is a multiple of the index of the associated central simple algebra over $k(X)$.) Note that $F/F^{\perp}$ is killed by d, and hence $e_F^d = 1$. In view of Proposition 2.3, it follows that the d-th power P^d *is the projectivization of a homogeneous vector bundle*.

Proposition 2.9. *With the notation and assumptions of Proposition 2.4, the following assertions are equivalent for a homogeneous* $\mathbb{P}^{n-1}$*-bundle* $f : P \to X$:

(i) P *is indecomposable.*

(ii) *The associated representation* $\tilde{\rho} : \widetilde{H} \to \mathrm{GL}_n$ *is indecomposable.*

(iii) $\widetilde{H}_s$ *is a Heisenberg group and* $V \cong U \otimes W$ *as representations of* $H \cong H_u \times \widetilde{H}_s$, *where* U *is an indecomposable representation of* H_u *and* W *is the standard representation of* $\widetilde{H}_s$.

(iv) *The neutral component* $\mathrm{Aut}_X^0(P)$ *is unipotent.*

Proof. (i) $\Longleftrightarrow$ (ii) The forward implication is obvious, and the converse follows from Theorem 2.1(iv).

(ii) $\Longleftrightarrow$ (iii) This is a direct consequence of Proposition 2.7.

(iii) $\Longrightarrow$ (iv) Since $(n, p) = 1$, we have $\mathrm{M}_n = k\,\mathrm{id} \oplus \mathfrak{pgl}_n$ as representations of PGL_n acting by conjugation. In view of (7), this yields

$$\mathrm{Lie}\,\mathrm{Aut}_X(P) = \mathrm{M}_n^H / k\,\mathrm{id} = \mathrm{End}^H(U \otimes W)/k\,\mathrm{id}\,.$$

Moreover, $\mathrm{End}^H(U \otimes W) \cong \mathrm{End}^{H_u}(U)$ by Schur's lemma, and hence

$$\mathrm{Lie}\,\mathrm{Aut}_X(P) \cong \mathrm{End}^{H_u}(U)/k\,\mathrm{id}\,.$$

This isomorphism of Lie algebras arises from the natural homomorphism

$$\mathrm{GL}(U)^{H_u}/\mathbb{G}_m\,\mathrm{id} \to \mathrm{Aut}_X(P).$$

Since $\mathrm{Aut}_X(P)$ is smooth (Remark 2.5), we see that its neutral component is a quotient of $\mathrm{GL}(U)^{H_u}/\mathbb{G}_m\,\mathrm{id}$. But the latter group is unipotent, since U is indecomposable.

(iv) $\Longrightarrow$ (iii) Observe that the weight space decomposition (15) is trivial: otherwise, $\mathrm{Aut}_X(P)$ contains a copy of $\mathbb{G}_m$ that fixes some weight space pointwise and acts by scalar multiplication on all the other weight spaces. Thus, $V \cong U \otimes W$, where W is irreducible. Moreover, U is indecomposable; otherwise, $\mathrm{Aut}_X(P)$ contains a copy of $\mathbb{G}_m$ by the above argument. □

Remarks 2.10. (1) The results of this subsection do not extend readily to the case where p divides n: for instance, there exists a nondegenerate theta group $\widetilde{H} \subset \mathrm{GL}_p$

with H unipotent and local. Consider indeed the group scheme α_p (the kernel of the p-th power map of $\mathbb{G}_a$) and the duality pairing

$$u : \alpha_p \times \alpha_p \to \mathbb{G}_m, \quad (x, y) \longmapsto \sum_{i=0}^{p-1} \frac{x^i}{i!}.$$

This yields a bilinear alternating pairing e on $H := \alpha_p \times \alpha_p$ via

$$e((x, y), (x', y')) := u(x, y')u(x', y)^{-1}.$$

Then we may take for $\widetilde{H}$ the associated Heisenberg group scheme (with $K = \alpha_p \times \{0\}$ and $\mathscr{X}(K) = \{0\} \times \alpha_p$), equipped with its standard representation in $\mathbb{O}(\alpha_p) \cong k^p$.

Note that the above group scheme H is contained in an abelian surface, the product of two supersingular elliptic curves. More generally, any finite commutative group scheme is contained in some abelian variety (see [Oort 1966, §15.4]).

(2) For an arbitrary homogeneous projective bundle P, each representation U_λ (with the notation of Proposition 2.7) is a direct sum of indecomposable representations with multiplicities; moreover, these indecomposable summands and their multiplicities are uniquely determined up to reordering, in view of the Krull–Schmidt theorem. Thus, the representation of $\widetilde{H}$ in V decomposes into a direct sum (with multiplicities) of tensor products $U \otimes W$, where U is an indecomposable representation of H_u and W is an irreducible representation of $\widetilde{H}_s$.

Let $L \subset \mathrm{PGL}_n$ denote the stabilizer of such a decomposition. Then L is a Levi subgroup, uniquely determined up to conjugation; moreover, the PGL_n-torsor $\pi : Y \to X$ admits a reduction of structure group to an L-torsor $\pi_L : Y_L \to X$. Arguing as in the proof that (iii) implies (iv) above, one may check that the natural homomorphism $Z(L) \to \mathrm{Aut}_X^L(Y_L)$ (where $Z(L)$ denotes the center of L, and $\mathrm{Aut}_X^L(Y_L)$ the group of bundle automorphisms of Y_L) yields an isomorphism of the reduced neutral component $Z(L)^0_{\mathrm{red}}$ to a maximal torus of $\mathrm{Aut}_X^L(Y_L)$. Thus, the torsor $\pi_L : Y_L \to X$ is *L-indecomposable* in the sense of Definition 2.1 of [Balaji et al. 2005]. Moreover, this torsor is the unique reduction of $\pi : P \to X$ to an L-indecomposable torsor for a Levi subgroup, by Theorem 3.4 of the same reference (the latter result is obtained there in characteristic zero, and generalized to arbitrary characteristics in [Balaji et al. 2006b]; see also [Balaji et al. 2006a]).

Conversely, the equivalence of statements (i) and (iv) above follows from the results of [Balaji et al. 2005; 2006b] in view of the smoothness of $\mathrm{Aut}_X(P)$.

3. Irreducible bundles

Throughout this section, we consider $\mathbb{P}^{n-1}$-bundles $f : P \to X$, and call them *bundles* for simplicity; we still assume that $(n, p) = 1$.

Structure and characterizations. We say that a homogeneous bundle P is *irreducible* if so is the projective representation $\rho : H \to \mathrm{PGL}_n$ associated with P via Theorem 2.1. By Proposition 2.7, this means that the theta group $\widetilde{H}$ is a Heisenberg group acting on k^n via its standard representation.

We now parametrize the irreducible homogeneous bundles, and describe the corresponding Azumaya algebras as well as the adjoint bundles and automorphism groups:

Proposition 3.1. (i) *The irreducible homogeneous $\mathbb{P}^{n-1}$-bundles are classified by the pairs (H, e), where $H \subset X_n$ is a subgroup of order n^2 and $e : H \times H \to \mathbb{G}_m$ is a nondegenerate alternating pairing. In particular, such bundles exist for any given n, and they form only finitely many isomorphism classes.*

(ii) *For the bundle P corresponding to (H, e), the associated Azumaya algebra $\mathcal{A}$ admits a grading by the group H, namely,*

$$\mathcal{A} \cong \bigoplus_{\mathcal{L} \in H} \mathcal{L},$$

where each element of $H \subset \widehat{X}$ is viewed as an invertible sheaf on X. In particular, we have a decomposition

$$\mathrm{ad}(P) \cong \bigoplus_{\mathcal{L} \in H, \mathcal{L} \neq 0} \mathcal{L}.$$

(iii) *For P as in* (ii), *we have $\mathrm{Aut}_X(P) \cong H$. Moreover, the neutral component $\mathrm{Aut}^0(P)$ is the extension of X by H, dual to the inclusion $\mathcal{X}(H) \cong H \subset \widehat{X}$, and $\mathrm{Aut}(P)/\mathrm{Aut}^0(P)$ is isomorphic to the subgroup of $\mathrm{Aut}_{\mathrm{gp}}(X) \cong \mathrm{Aut}_{\mathrm{gp}}(\widehat{X})$ that preserves H and e.*

Proof. (i) By the results of Section 2, the irreducible homogeneous bundles are classified by the pairs consisting of an isogeny $1 \to H \to G \to X \to 1$ and a nondegenerate alternating pairing e on H; then e provides an isomorphism of H with its character group. The assertion now follows from duality of isogenies.

(ii) This follows from the isomorphism of $\mathcal{O}_X$-algebras (5) together with the isomorphism of $\mathcal{O}_X$-H-algebras $\gamma_*(\mathcal{O}_G) \cong \bigoplus_{\mathcal{L} \in \mathcal{X}(H)} \mathcal{L}$ and with the decomposition $\mathrm{M}_n \cong \bigoplus_{h \in H} k u_h$ obtained in Lemma 2.6(i).

(iii) Combining the isomorphism (6) and Lemma 2.6(ii), we see that the natural map $H \to \mathrm{Aut}_X(P)$ is an isomorphism. In view of the commutative diagram with exact rows

$$\begin{array}{ccccccccc}
1 & \longrightarrow & H & \longrightarrow & G & \longrightarrow & X & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathrm{Aut}_X(P) & \longrightarrow & \mathrm{Aut}(P) & \xrightarrow{f_*} & \mathrm{Aut}(X) & &
\end{array}$$

and of the isomorphism $\mathrm{Aut}(X) \cong X \ltimes \mathrm{Aut}_{\mathrm{gp}}(X)$, where $\mathrm{Aut}_{\mathrm{gp}}(X)$ is étale, it follows that the natural map $G \to \mathrm{Aut}^0(P)$ is an isomorphism as well. The structure of $\mathrm{Aut}(P)/\mathrm{Aut}^0(P)$ follows from Theorem 2.1 together with Lemma 2.6(ii). $\square$

Remark 3.2. Recall from [Mumford 1966, §1] that every finite commutative group H of order prime to p, equipped with a nondegenerate alternating pairing e, admits a decomposition

$$H = H_{n_1} \times \cdots \times H_{n_r}, \quad e = (e_{d_1}, \ldots, e_{d_r}),$$

such that

$$H_{n_i} = \mathbb{Z}/n_i\mathbb{Z} \times \mathscr{X}(\mathbb{Z}/n_i\mathbb{Z}) \cong (\mathbb{Z}/n_i\mathbb{Z})^2, \quad e_{d_i}((x,\chi),(x',\chi')) = \chi'(x)^{d_i}\chi(x')^{-d_i},$$

where the n_i and d_j are integers satisfying $n_{i+1}|n_i$, $0 \le d_i < n_i$, and $(d_i, n_i) = 1$ for all i. Moreover, $n_1, \ldots, n_r$ are uniquely determined by H. Since H is a subgroup of $\widehat{X}_n \cong (\mathbb{Z}/n\mathbb{Z})^{2g}$, where $g := \dim(X)$, we see that $r \le g$; conversely, any product of r cyclic groups of order prime to p can be embedded into $\widehat{X}_n$ provided that $r \le g$.

It follows that every homogeneous irreducible bundle admits a decomposition into a product

$$P = P_1 \cdots P_r,$$

where each P_i corresponds to (H_{n_i}, e_{d_i}). Moreover, the P_i are exactly the irreducible homogeneous bundles associated with a product of two copies of a cyclic group; we may call these bundles *cyclic*.

Equivalently, the associated Azumaya algebra satisfies

$$\mathscr{A} = \mathscr{A}_1 \otimes \cdots \otimes \mathscr{A}_r,$$

where $\mathscr{A}_i$ corresponds to (H_{n_i}, e_{d_i}). Moreover, the $\mathbb{O}_X$-algebra $\mathscr{A}_i$ is generated by two invertible sheaves $\mathscr{L}$ and $\mathscr{M}$ (associated with the natural generators of $(\mathbb{Z}/n_i\mathbb{Z})^2$), with relations $x^{n_i} = \xi$, $y^{n_i} = \eta$, and $xy = \zeta^{d_i} yx$ for any local generators $x \in \mathscr{L}$ and $y \in \mathscr{M}$, where ξ (resp. η) denotes a local trivialization of $\mathscr{L}^{\otimes n}$ (resp. $\mathscr{M}^{\otimes n}$), and ζ is a fixed primitive d_i-th root of unity (this follows by combining the isomorphism of algebras (5) with the description of the H_{n_i}-algebra M_{n_i} obtained in Lemma 2.6(i)). In particular, $\mathscr{A}_i$ yields a cyclic division algebra over $k(X)$.

Example 3.3. Let X be an elliptic curve. Then X is canonically isomorphic to $\widehat{X}$ and the finite subgroups of X admitting a nondegenerate alternating pairing are exactly the n-torsion subgroups X_n. In view of the above remark, it follows that the irreducible homogeneous bundles over X are exactly the cyclic bundles. By Theorem 10 of [Atiyah 1957], they are exactly the projectivizations of the indecomposable vector bundles of coprime rank and degree, that is, of the simple vector bundles.

Example 3.4. Returning to an arbitrary abelian variety X, we recall from [Mumford 1966, §1] a geometric construction of Heisenberg groups. Let L be a line bundle on X, and $K(L)$ the kernel of the polarization homomorphism

$$\varphi_L : X \to \widehat{X}, \quad x \mapsto T_x^*(L) \otimes L^{-1}. \tag{17}$$

Denoting by $\mathscr{G}(L)$ the group scheme of automorphisms of the variety L which commute with the action of $\mathbb{G}_m$ by multiplication on fibers, we have a central extension

$$1 \to \mathbb{G}_m \to \mathscr{G}(L) \to K(L) \to 1.$$

The associated commutator pairing on $K(L)$ is denoted by e^L.

Also, recall that an effective line bundle L is ample if and only if φ_L is an isogeny; equivalently, $K(L)$ is finite. Then the theta group $\mathscr{G}(L)$ is nondegenerate, and acts on the space of global sections $H^0(X, L)$ via its standard representation. Thus, $K(L)$ acts on the associated projective space

$$|L| := \mathbb{P}(H^0(X, L))$$

and the natural map

$$f : X \times^{K(L)} |L| \to X/K(L) \cong \widehat{X}$$

is an irreducible homogeneous bundle.

As will be shown in detail in Section 4, this bundle is the projectivization of a natural vector bundle E over $\widehat{X}$. Moreover, if X is an elliptic curve (so that $X \cong \widehat{X}$) and L has degree n, then E has rank n and degree -1.

We now obtain several criteria for a homogeneous projective bundle to be irreducible:

Proposition 3.5. *The following conditions are equivalent for a homogeneous bundle P:*

(i) *P is irreducible.*

(ii) *P admits no proper homogeneous subbundle.*

(iii) ad *P splits into a direct sum of nonzero algebraically trivial line bundles.*

(iv) $H^0(X, \mathrm{ad}(P)) = 0$.

(v) $\mathrm{Aut}_X(P)$ *is finite.*

If P is the projectivization of a (*semihomogeneous*) *vector bundle E, then P is irreducible if and only if E is simple.*

Proof. (i) $\Longrightarrow$ (ii) follows from Theorem 2.1(iii).

(i) $\Longrightarrow$ (ii) follows from Proposition 3.1(ii).

(iii) $\Longrightarrow$ (iv) holds since $H^0(X, L) = 0$ for any nonzero $L \in \widehat{X}$.

(iv) $\Longrightarrow$ (v) follows from the fact that $\operatorname{Lie}\operatorname{Aut}_X(P) = H^0(X, \operatorname{ad}(P))$.

(v)$\Longrightarrow$(i) By Proposition 2.9, P is indecomposable and the quotient $\mathrm{GL}(U)^{H_u}/\mathbb{G}_m\,\mathrm{id}$ is finite, where U is the indecomposable representation of H_u given by that proposition. But $\mathrm{GL}(U)^{H_u}/\mathbb{G}_m\,\mathrm{id}$ has positive dimension for any unipotent subgroup scheme $H_u \subset \mathrm{GL}(U)$, unless $\dim(U) = 1$; in the latter case, P is clearly irreducible.

The final assertion follows from the equivalence of (i) and (iv) in view of the isomorphism

$$H^0(X, \operatorname{ad}(\mathbb{P}(E))) \cong H^0(X, \operatorname{End}(E))/k\,\mathrm{id}. \qquad \square$$

Remark 3.6. The indecomposable homogeneous bundles are exactly the products $\mathbb{P}(U)I$, where U is an indecomposable unipotent vector bundle, and I an irreducible homogeneous bundle (as follows from Proposition 2.9).

In particular, the indecomposable homogeneous bundles over an elliptic curve X are exactly the projectivizations $\mathbb{P}(U \otimes E)$, where U is as above, and E is a simple vector bundle (as in Example 3.3).

By a result of [Atiyah 1957], any indecomposable vector bundle over X is isomorphic to $U \otimes E \otimes L$ for U and E as above and L a line bundle. Also, U is uniquely determined by its rank; moreover, E is uniquely determined by its (coprime) rank and degree, up to tensoring with a line bundle of degree 0.

Next, we obtain a cohomological criterion for a bundle to be homogeneous and irreducible, thereby extending a result of Mukai [1978, Theorem 5.8] about simple semihomogeneous vector bundles:

Proposition 3.7. *A bundle P is homogeneous and irreducible if and only if we have $H^0(X, \operatorname{ad}(P)) = H^1(X, \operatorname{ad}(P)) = 0$; then $H^i(X, \operatorname{ad}(P)) = 0$ for all $i \geq 0$.*

Proof. Recall that $H^i(X, L) = 0$ for all $i \geq 0$ and all nonzero $L \in \widehat{X}$. By Proposition 3.1(ii), the same holds with L replaced with $\operatorname{ad} P$, if P is homogeneous and irreducible.

For the converse, observe that $\operatorname{ad}(P) = \pi_*(T_{Y/X})^{\mathrm{PGL}_n}$, where $\pi : Y \to X$ denotes the PGL_n-torsor associated to P, and $T_{Y/X}$ the relative tangent bundle. Thus, $\operatorname{ad} P$ sits in an exact sequence

$$0 \to \operatorname{ad}(P) \to \pi_*(T_Y)^{\mathrm{PGL}_n} \to T_X \to 0$$

obtained from the standard exact sequence $0 \to T_{Y/X} \to T_Y \to \pi^*(T_X) \to 0$ by taking the invariant direct image under π. If $H^1(X, \operatorname{ad}(P)) = 0$, then the natural map

$$H^0(Y, T_Y)^{\mathrm{PGL}_n} = H^0(X, \pi_*(T_Y)^{\mathrm{PGL}_n}) \to H^0(X, T_X)$$

is surjective. But $H^0(Y, T_Y)^{\mathrm{PGL}_n} \cong \mathrm{Lie}(\mathrm{Aut}^{\mathrm{PGL}_n}(Y))$ and $H^0(X, T_X) \cong \mathrm{Lie}(\mathrm{Aut}(X))$; moreover, $\mathrm{Aut}^{\mathrm{PGL}_n}(Y) = \mathrm{Aut}(P)$ is smooth by Remark 2.5, and $\mathrm{Aut}(X)$ is smooth as well. Hence the homomorphism $\mathrm{Aut}^{\mathrm{PGL}_n}(Y) \to \mathrm{Aut}(X)$ is surjective on neutral components, that is, Y is homogeneous. Thus, P is homogeneous, too. If in addition $H^0(X, \mathrm{ad}(P)) = 0$, then P is irreducible by Proposition 3.5. □

Remark 3.8. The above argument shows that a bundle P is homogeneous if it satisfies $H^1(X, \mathrm{ad}(P)) = 0$. This may also be seen as follows: observe that $\mathrm{ad}(P) = f_*(T_{P/X})$ (as follows, for example, by considering an étale trivialization of P). Moreover, $R^i f_*(T_{P/X}) = 0$ for all $i \geq 1$, since $H^i(\mathbb{P}^{n-1}, T_{\mathbb{P}^{n-1}}) = 0$ for all such i. As a consequence, $H^1(P, T_{P/X}) = 0$. Then f is rigid as a morphism with target X in view of [Sernesi 2006, Corollary 3.4.9]. It follows readily that P is homogeneous.

The converse statement does not hold, for example, when X is an elliptic curve in characteristic zero, U_n is the indecomposable unipotent vector bundle of rank $n \geq 2$, and $P = \mathbb{P}(U_n)$. Then

$$\mathrm{ad}(P) \cong (U_n \otimes U_n^*)/k\,\mathrm{id} \cong U_{2n-1} \oplus U_{2n-3} \oplus \cdots \oplus U_3,$$

and hence $H^0(X, \mathrm{ad}(P))$ has dimension $n-1$. By the Riemann–Roch theorem, the same holds for $H^1(X, \mathrm{ad}(P))$.

Projectivizations of vector bundles. In this subsection, we characterize those homogeneous projective bundles that are projectivizations of (not necessarily homogeneous) vector bundles. We first consider a special class of bundles, defined as follows.

Given a positive integer m, not divisible by p, we say that a bundle P is *trivialized by* m_X (the multiplication by m in X) if the pull-back bundle $m_X^*(P) \to X$ is trivial.

In fact, every such bundle is homogeneous, as a consequence of the following:

Proposition 3.9. (i) *A bundle P is trivialized by m_X if and only if $P \cong X \times^{X_m} \mathbb{P}^{n-1}$ as bundles over $X \cong X/X_m$, for some action of X_m on $\mathbb{P}^{n-1}$.*

(ii) *Any irreducible homogeneous $\mathbb{P}^{n-1}$-bundle is trivialized by n_X.*

Proof. (i) If P is trivialized by m_X, then we have a cartesian square

$$\begin{array}{ccc} X \times \mathbb{P}^{n-1} & \xrightarrow{p_1} & X \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle m_X} \\ P & \xrightarrow{f} & X, \end{array}$$

where p_1 denotes the first projection. Thus, the action of X_m by translations on X lifts to an action on $X \times \mathbb{P}^{n-1}$ such that q is invariant. This action is of the form

$$x \cdot (y, z) = (x + y, \varphi(x, y) \cdot z)$$

for some morphism $\varphi : X_m \times X \to \mathrm{Aut}(\mathbb{P}^{n-1}) = \mathrm{PGL}_n$. But every morphism from the abelian variety X to the affine variety PGL_n is constant. Thus, φ is independent of y, that is, φ yields an action of X_m on $\mathbb{P}^{n-1}$. Moreover, the X_m-invariant morphism q factors through a morphism of $\mathbb{P}^{n-1}$-bundles $X \times^{X_m} \mathbb{P}^{n-1} \to P$ which is the desired isomorphism.

The converse implication is obvious.

(ii) Write $P = G \times^H \mathbb{P}^{n-1}$ as in Theorem 2.1; then H is killed by n in view of the structure of nondegenerate theta groups. In other words, the homomorphism $\gamma : G \to X$ is an isogeny with kernel killed by n. Thus, there exists a unique isogeny $\tau : X \to G$ such that $\gamma\tau = n_X$. Then $X_n = \tau^{-1}(H)$ and hence $X = X \times^{X_n} \mathbb{P}^{n-1}$, where X_n acts on $\mathbb{P}^{n-1}$ via the surjective homomorphism $\tau_{|X_n} : X_n \to H$. □

By the above proposition, a bundle P trivialized by m_X defines an alternating bilinear map

$$e_{P,m} : X_m \times X_m \to \mu_m.$$

Moreover, the irreducible homogeneous bundles are classified by those maps such that $[X_m : X_m^{\perp}] = m^2$ (as follows from Proposition 3.5). Also, one easily checks that the assignment $P \mapsto e_{P,m}$ is multiplicative, that is, $e_{P_1 P_2,m} = e_{P_1,m} e_{P_2,m}$ and $e_{P^*,m} = e_{P,m}^{-1}$.

We may now obtain the desired characterization:

Proposition 3.10. *Let P be a bundle trivialized by m_X. Then P is the projectivization of a vector bundle if and only if there exists a line bundle L on X such that $e_{P,m} = e^{L^{\otimes m}}|_{X_m}$ (this makes sense as $K(L^{\otimes m})$ contains X_m).*

Proof. Assume that $P = \mathbb{P}(E)$ for some vector bundle E of rank n on X. Since the projective bundle $m_X^*(\mathbb{P}(E))$ is trivial, we have

$$m_X^*(E) \cong M^{\oplus n}$$

for some line bundle M on X. Replacing E with $E \otimes N$, where N is a symmetric line bundle on X, leaves $\mathbb{P}(E)$ unchanged and replaces $m_X^*(E)$ with $m_X^*(E) \otimes N^{\otimes m^2}$, and hence M with $M \otimes N^{\otimes m^2}$. Taking for N a large power of an ample symmetric line bundle, we may assume that M is very ample.

The pull-back $m_X^*(E)$ is equipped with an X_m-linearization. Equivalently, the action of X_m by translations on X lifts to an action on $M^{\oplus n}$ which is linear on fibers. In particular, $T_x^*(M^{\oplus n}) \cong M^{\oplus n}$ for any $x \in X_m$. This isomorphism is given by an $n \times n$ matrix of maps $T_x^* M \to M$; thus, $H^0(X, T_x^*(M^{-1}) \otimes M) \neq 0$. Since

$T_x^*(M^{-1}) \otimes M \in \widehat{X}$, it follows that this line bundle is trivial. In other words, $X_m \subset K(M)$; this is equivalent to the existence of a line bundle L in X such that $M = L^{\otimes m}$. Moreover, we have a representation of X_m in $H^0(X, M^{\oplus n}) \cong H^0(X, M) \otimes k^n$ that lifts the homomorphism

$$\phi : X_m \to \mathrm{PGL}(H^0(X, M)) \times \mathrm{PGL}_n \tag{18}$$

given by the X_m-action on $\mathbb{P}(H^0(X, M))$ as a subgroup of $K(M)$, and the X_m-action on $\mathbb{P}^{n-1}$ that defines P. It follows that $e^M e_{P,m} = 1$ on X_m; equivalently, $e_{P,m}$ is the restriction to X_m of $e^{M^{\otimes(-1)}} = e^{L^{\otimes(-m)}} = e^{L^{\otimes m(m-1)}}$ (since $e^{L^{\otimes m^2}} = 1$).

To show the converse, we reduce by inverting the above arguments to the case that $e^M e_{P,m} = 1$ on X_m for some line bundle M on X such that $X_m \subset K(M)$; we may also assume that M is very ample. Then X_m acts on $H^0(X, M^{\oplus n})$ by lifting the homomorphism (18). Moreover, the evaluation morphism

$$\mathbb{O}_X \otimes H^0(X, M^{\oplus n}) = \mathbb{O}_X \otimes H^0(X, M) \otimes k^n \to M \otimes k^n = M^{\oplus n}$$

is surjective and its kernel is stable under the induced action of X_m (since the analogous morphism $\mathbb{O}_X \otimes H^0(X, M) \to M$ is equivariant with respect to the theta group of $X_m \subset K(M)$). Thus, X_m acts on $M^{\oplus n}$ by lifting its action on X via translation. Now $M^{\oplus n}$ descends to a vector bundle on $X/X_m \cong X$ with projectivization P. □

Next, we extend the statement of Proposition 3.10 to all homogeneous bundles P. We use the notation of Section 2; in particular, the associated pairing e_F introduced in Proposition 2.4. Then e_F factors through a nondegenerate pairing on $F/F^\perp \cong H/H^\perp$ and this group is killed by the homogeneous index $d = d(H)$ defined by (16). Thus, the isogeny $G/H^\perp \to G/H = X$ has its kernel killed by d; as in the proof of Proposition 3.9(ii), this yields a canonical surjective homomorphism $X_d \to H/H^\perp$ and, in turn, a bilinear alternating pairing e_P on X_d.

Theorem 3.11. *With the above notation, P is the projectivization of a vector bundle if and only if $e_P = e^{L^d}|_{X_d}$ for some line bundle L on X.*

Proof. Choose a linear subspace $S \subset \mathbb{P}^{n-1}$ which is H-stable, and minimal for this property. Then S yields a homogeneous irreducible $\mathbb{P}^{d-1}$-subbundle of P and the associated pairing on X_d is just e_P. Now the statement is a consequence of Proposition 3.10 together with the following observation.

Lemma 3.12. *Let $f : P \to Z$ be a projective bundle over a nonsingular variety, and $f_1 : P_1 \to Z$ a projective subbundle. Then P is the projectivization of a vector bundle if and only if so is P_1.*

Proof. Clearly, if $P = \mathbb{P}(E)$ for some vector bundle E over Z, then $P_1 = \mathbb{P}(E_1)$ for some subbundle $E_1 \subset E$. To show the converse, consider the PGL_n-torsor

$\pi : Y \to Z$ associated with P; recall that the subbundle P_1 yields a reduction of structure group to a PGL_{n,n_1}-torsor $\pi_1 : Y_1 \to Z$, where $\mathrm{PGL}_{n,n_1} \subset \mathrm{PGL}_n$ denotes the stabilizer of $\mathbb{P}^{n_1-1} \subset \mathbb{P}^{n-1}$. We have an exact sequence of algebraic groups

$$1 \longrightarrow G_{n,n_1} \longrightarrow \mathrm{PGL}_{n,n_1} \xrightarrow{r} \mathrm{PGL}_{n_1} \longrightarrow 1,$$

where r denotes the restriction to $\mathbb{P}^{n_1-1}$ and $G_{n,n_1} \cong \mathrm{M}_{n_1,n-n_1} \ltimes \mathrm{GL}_{n-n_1}$, the semidirect product being defined by the natural action of GL_{n-n_1} on the space of matrices $M_{n_1,n-n_1}$. Also, π_1 factors as

$$Y_1 \xrightarrow{\varphi} Y_1/G_{n,n_1} \xrightarrow{\psi} Z,$$

where φ is a G_{n,n_1}-torsor and ψ is the PGL_{n_1}-torsor associated with P_1. By assumption, $P_1 = \mathbb{P}(E_1)$ for some vector bundle E_1; this is equivalent to ψ being locally trivial in view of Proposition 18 of [Serre 2001]. But φ is locally trivial as well, since the algebraic group G_{n,n_1} is special by Sections 4.3 and 4.4 of that same reference. Thus, π_1 is locally trivial, and hence so is π. We conclude that $P = \mathbb{P}(E)$ for some vector bundle E.

Alternatively, one may use the fact that P is the projectivization of a vector bundle if and only if f has a rational section [loc. cit.], and conclude by applying [Gille and Szamuely 2006, Proposition 5.3.1]. □ □

Remark 3.13. We now relate Proposition 3.10 to a description of the Brauer group $\mathrm{Br}\, X$, due to Berkovich. Recall from [Grothendieck 1968a, §8.4] that $\mathrm{Br}\, X$ may be viewed as the set of equivalence classes of projective bundles over X, where two such bundles P_1 and P_2 are equivalent if there exist vector bundles E_1 and E_2 such that $\mathbb{P}(E_1)P_1 \cong \mathbb{P}(E_2)P_2$; the group structure stems from the operations of product and duality. By [Berkovich 1972, §3], we have an exact sequence for any positive integer n:

$$0 \longrightarrow \mathrm{Pic}(X)/n\,\mathrm{Pic}(X) \xrightarrow{\varphi} \mathrm{Hom}(\Lambda^2 X_n, \mu_n) \xrightarrow{\psi} \mathrm{Br}(X)_n \longrightarrow 0,$$

where $\mathrm{Hom}(\Lambda^2 X_n, \mu_n)$ consists of the bilinear alternating pairings $X_n \times X_n \to \mu_n$ and $\mathrm{Br}(X)_n \subset \mathrm{Br}(X)$ denotes the n-torsion subgroup; the map φ sends the class of $L \in \mathrm{Pic}(X)$ to the pairing $e^{L^{\otimes n}}|_{X_n}$ and the map ψ sends e to the class of the Azumaya algebra

$$\mathscr{A} := \bigoplus_{\alpha \in \widehat{X}_n,\, \sigma \in X_n} \mathscr{L}_\alpha e_\sigma,$$

where $\mathscr{L}_\alpha$ denotes the invertible sheaf associated with α and the multiplication is defined by

$$f_\alpha e_\sigma \cdot f_\beta e_\tau = \bar{e}_n(\beta, \sigma) a_{\sigma,\tau} f_\alpha f_\beta e_{\sigma+\tau}.$$

Here f_α (resp. f_β) is a local section of $\mathscr{L}_\alpha$ (resp. $\mathscr{L}_\beta$); $\bar{e}_n$ is the canonical pairing between $\widehat{X}_n$ and X_n and $\{a_{\sigma,\tau}\} \in Z^2(X_n, \mathbb{G}_m)$ is a 2-cocycle such that $e(\sigma, \tau) = a_{\sigma,\tau} a_{\tau,\sigma}^{-1}$.

(The class of $\mathscr{A}$ in the Brauer group does not depend on the choice of the representative $\{a_{\sigma,\tau}\}$ of e viewed as an element of $H^2(X_n, \mathbb{G}_m)$.) Thus,

$$\mathscr{L} := \bigoplus_{\alpha \in \widehat{X}_n} \mathscr{L}_\alpha e_0$$

is a maximal étale subalgebra of $\mathscr{A}$ in the sense of [Grothendieck 1968a, définition 5.6]; note that $\mathscr{L} \cong (n_X)_* \mathbb{O}_X$ as $\mathbb{O}_X$-algebras. Moreover, the left $\mathscr{L}$-module $\mathscr{A}$ is free with basis $(a_\sigma)_{\sigma \in X_n}$. By [loc. cit., corollaire 5.5], it follows that $n_X^*(\mathscr{A}) \cong \mathrm{M}_m(\mathbb{O}_X)$, where $m := \#(X_n) = n^{2g}$. In other words, the projective bundle associated with $\mathscr{A}$ is trivialized by n_X. In view of Proposition 3.9, it follows that the associated projective bundle is homogeneous.

In fact, *any class in* $\mathrm{Br}(X)_n$ *is represented by an irreducible homogeneous bundle*. Indeed, given any homogeneous bundle P, we may choose an irreducible subbundle P_1; then the product $P_1 P_1^*$ is a subbundle of $P P_1^*$ and is the projectivization of a vector bundle. By Lemma 3.12, it follows that the class of $P P_1^*$ in $\mathrm{Br}\, X$ is trivial; equivalently, P and P_1 have the same class there.

Recall that the natural map $\mathrm{Br}(X) \to \mathrm{Br}(k(X))$ is injective (see [Grothendieck 1968b, §1]). Also, as a very special case of a theorem of Merkurjev and Suslin (see [Gille and Szamuely 2006, Theorem 2.5.7]), each class in $\mathrm{Br}(k(X))_n$ can be represented by a tensor product of cyclic algebras. So the decomposition of classes in $\mathrm{Br}(X)_n$ obtained in Remark 3.2 may be viewed as a global analogue of that result for abelian varieties.

Finally, note that Proposition 3.10 is equivalent to the assertion that *the image of* φ *consists of those pairings associated with projectivizations of semihomogeneous vector bundles.* In loose terms, the Brauer group is generated by homogeneous bundles and the relations arise from semihomogeneous vector bundles.

4. Examples

Let X be an abelian variety, and λ an effective class in the Néron–Severi group $NS(X)$ viewed as the group of divisors on X modulo algebraic equivalence. The effective divisors on X with class λ are parametrized by a projective scheme $\mathrm{Div}^\lambda(X)$. Indeed, the Hilbert polynomial of any such divisor D, relative to a fixed ample line bundle on X, depends only on λ; thus, $\mathrm{Div}^\lambda(X)$ is a union of connected components of the Hilbert scheme $\mathrm{Hilb}(X)$.

Also, recall that the line bundles on X with class λ are parametrized by the Picard variety $\mathrm{Pic}^\lambda(X)$. Choosing L in that variety, we have

$$\mathrm{Pic}^\lambda(X) = L \otimes \mathrm{Pic}^0(X) = L \otimes \widehat{X}.$$

On $X \times \operatorname{Pic}^\lambda(X)$ we have a universal bundle: the Poincaré bundle $\mathcal{P}$, uniquely determined up to the pull-back of a line bundle under the second projection

$$\pi : X \times \operatorname{Pic}^\lambda(X) \to \operatorname{Pic}^\lambda(X).$$

The universal family on $\operatorname{Div}^\lambda(X)$ yields a morphism

$$f : \operatorname{Div}^\lambda(X) \to \operatorname{Pic}^\lambda(X), \quad D \longmapsto \mathcal{O}_X(D). \tag{19}$$

Note that X acts on $\operatorname{Div}^\lambda(X)$ and on $\operatorname{Pic}^\lambda(X)$ via its action on itself by translations; moreover, f is equivariant. Also, the isotropy subgroup scheme in X of any point of $\operatorname{Pic}^\lambda(X)$ is the group scheme $K(L)$ that occurred in Example 3.4.

If λ is ample, then $\operatorname{Pic}^\lambda(X)$ is the X-orbit $X \cdot L \cong X/K(L)$. Thus, f is a homogeneous fiber bundle over $X/K(L)$; the latter abelian variety is isomorphic to $\widehat{X}$ via the polarization homomorphism (17).

Proposition 4.1. *Let $\lambda \in \operatorname{NS}(X)$ be an ample class, and $L \in \operatorname{Pic}^\lambda(X)$.*

(i) *We have an isomorphism*

$$\operatorname{Div}^\lambda(X) \cong X \times^{K(L)} |L|$$

of homogeneous bundles over $X/K(L)$. In particular, $\operatorname{Div}^\lambda(X)$ is a homogeneous projective bundle over $\widehat{X}$.

(ii) *The sheaf $\mathcal{E} := \pi_*(\mathcal{P})$ is locally free, and the morphism* (19) *is the projectivization of the corresponding vector bundle.*

(iii) *The group scheme $\operatorname{Aut}(\operatorname{Div}^\lambda(X))$ is the semidirect product of X (acting by translations) with the subgroup of $\operatorname{Aut}_{\mathrm{gp}}(X)$ that preserves $K(L)$ and e^L.*

Proof. (i) Clearly, the set-theoretic fiber of f at L is the projective space $|L|$, and its dimension $h^0(L) - 1 = \chi(L) - 1$ is independent of $L \in \operatorname{Pic}^\lambda(X)$. As a consequence, the scheme $\operatorname{Div}^\lambda(X)$ is irreducible of dimension $\dim(X) + h^0(L) - 1$.

To complete the proof, it suffices to show that the differential of f at any $D \in |L|$ is surjective with kernel of dimension $h^0(L) - 1$. Identifying $\operatorname{Div}^\lambda(X)$ with a union of components of $\operatorname{Hilb}(X)$, and $\operatorname{Pic}^\lambda(X)$ with $\widehat{X}$, the differential

$$T_D f : T_D \operatorname{Div}^\lambda(X) \to T_L \operatorname{Pic}^\lambda(X)$$

is identified with the boundary map $\partial : H^0(D, L_{|D}) \to H^1(X, \mathcal{O}_X)$ of the long exact sequence of cohomology associated with the short exact sequence

$$0 \to \mathcal{O}_X \to L \to L_{|D} \to 0$$

(see [Sernesi 2006, Proposition 3.3.6]). Since $H^1(X, L) = 0$, this long exact sequence begins with

$$0 \longrightarrow k \longrightarrow H^0(X, L) \longrightarrow H^0(D, L_{|D}) \xrightarrow{\partial} H^1(X, \mathcal{O}_X) \longrightarrow 0$$

which yields the desired assertion.

(ii) The vanishing of $H^1(X, L)$ also implies that $\mathscr{E}$ is locally free and satisfies $\mathscr{E}(L) \cong H^0(X, L)$. Thus, it suffices to check that the associated projective bundle $\mathbb{P}(\mathscr{E})$ is homogeneous. But for any $x \in X$, there exists an invertible sheaf L_x on $\mathrm{Pic}^\lambda(X)$ such that

$$(T_x, T_x)^*(\mathscr{P}) \cong \mathscr{P} \otimes \pi^* L_x$$

in view of the universal property of the Poincaré bundle $\mathscr{P}$. Since $\pi_*(T_x, T_x)^*(\mathscr{P}) \cong T_x^*(\pi_*(\mathscr{P})) = T_x^*(\mathscr{E})$, this yields an isomorphism

$$T_x^*(\mathscr{E}) \cong \mathscr{E} \otimes L_x.$$

In other words, $\mathscr{E}$ is semihomogeneous.

(iii) This is checked by arguing as in the proof of Proposition 3.1(iii). □

The case of an arbitrary effective class λ reduces to the ample case in view of the following:

Proposition 4.2. *Let $\lambda \in \mathrm{NS}(X)$ be an effective class, $L \in \mathrm{Pic}^\lambda(X)$, and $q : X \to \bar{X}$ the quotient map by the reduced neutral component $K(L)^0_{\mathrm{red}} \subset K(L)$. Then $\lambda = q^*(\bar{\lambda})$ for a unique ample class $\bar{\lambda} \in \mathrm{NS}(\bar{X})$, and $f : \mathrm{Div}^\lambda(X) \to \mathrm{Pic}^\lambda(X)$ may be identified with $\bar{f} : \mathrm{Div}^{\bar{\lambda}}(\bar{X}) \to \mathrm{Pic}^{\bar{\lambda}}(\bar{X})$.*

Proof. We claim that any $D \in \mathrm{Div}^\lambda(X)$ equals $q^*(\bar{D})$ for some ample effective divisor $\bar{D}$ on $\bar{X}$.

To see this, recall that nD is base-point-free for any $n \geq 2$; this yields morphisms

$$\gamma_n : X \to \mathbb{P}(H^0(X, L^{\otimes n})^*) \quad (n \geq 2),$$

which are equivariant for the action of $K(L)$. The abelian variety $K(L)^0_{\mathrm{red}}$ acts trivially on each projective space $\mathbb{P}(H^0(X, L^{\otimes n})^*)$; thus, each γ_n is invariant under $K(L)^0_{\mathrm{red}}$. In the Stein factorization of γ_n as

$$X \xrightarrow{\varphi_n} Y_n \xrightarrow{\psi_n} \mathbb{P}(H^0(X, L^{\otimes n})^*),$$

where $(\varphi_n)_*(\mathbb{O}_X) = \mathbb{O}_{Y_n}$ and ψ_n is finite, the morphism φ_n is the natural map

$$\varphi : X \to \mathrm{Proj} \bigoplus_{m=0}^{\infty} H^0(X, L^{\otimes m}) =: Y.$$

In particular, φ_n is independent of n and invariant under $K(L)^0_{\mathrm{red}}$. Moreover, since nD is the pull-back of a hyperplane under γ_n for any $n \geq 2$, we see that $D = 3D - 2D = \varphi^*(E)$ for some Cartier divisor E on Y. Then E is effective and $H^0(X, L^{\otimes n}) \cong H^0(Y, M^{\otimes n})$ for all n, where $M := \mathcal{O}_Y(E)$; it follows that E is ample. Consider the factorization

$$\bar{\varphi} : \bar{X} := X/K(L)^0_{\mathrm{red}} \to Y,$$

the effective divisor $\bar{D} := \bar{\varphi}^*(E)$, and the associated invertible sheaf $\bar{L} = \bar{\varphi}^*(M)$. Then $L = q^*(\bar{L})$. Thus, the group scheme $K(\bar{L}) = K(L)/K(L)^0_{\mathrm{red}}$ is finite and $\bar{L}$ has nonzero global sections; hence $\bar{L}$ is ample. Thus, $\bar{\varphi}$ is finite. But $\bar{\varphi}_*(\mathcal{O}_{\bar{X}}) = \mathcal{O}_Y$; it follows that $\bar{\varphi}$ is an isomorphism, and this identifies φ with q. This proves the claim.

As a consequence, $\lambda = q^*(\bar{\lambda})$ for a unique ample class $\bar{\lambda}$. We now show that the morphism

$$q^* : \mathrm{Div}^{\bar{\lambda}}(\bar{X}) \to \mathrm{Div}^{\lambda}(X)$$

is an isomorphism. By the first step, q^* is bijective. In view of Proposition 4.1, it follows that the scheme $\mathrm{Div}^{\lambda}(X)$ is irreducible of dimension $\dim(\bar{X}) + h^0(\bar{X}, \bar{L}) - 1$. On the other hand, the Zariski tangent space of $\mathrm{Div}^{\lambda}(X)$ at D equals

$$H^0(D, L_{|D}) \cong H^0(\bar{D}, \bar{L}_{|\bar{D}}) = T_{\bar{D}} \mathrm{Div}^{\bar{\lambda}}(\bar{X}).$$

Thus, q^* is étale and hence is an isomorphism. $\square$

In the above construction, one may replace the abelian variety X with any smooth projective variety; for example, a curve C. Then an effective class in $\mathrm{NS}(C) \cong \mathbb{Z}$ is just a nonnegative integer d. Moreover, $\mathrm{Div}^d(C)$ is the symmetric product $C^{(d)}$, a smooth projective variety of dimension d equipped with a morphism

$$f = f_d : C^{(d)} \to \mathrm{Pic}^d(C). \tag{20}$$

Choosing a point of C, we may identify $\mathrm{Pic}^d(C)$ with the Jacobian variety $J = J(C)$.

If $d > 2g-2$, where g denotes of course the genus of C, then f is the projectivization of a vector bundle $E = E_d$ on $\mathrm{Pic}^d(C)$, the direct image of the Poincaré bundle on $C \times \mathrm{Pic}^d(C)$ under the second projection. Moreover, E has rank $n := d - g + 1$.

Proposition 4.3. *With the above notation, the projective bundle* (20) *is homogeneous if and only if $g \le 1$.*

Proof. Assume that (20) is homogeneous. Then E is semihomogeneous; in view of [Mukai 1978, Lemma 6.11], we then have an isomorphism of vector bundles on J,

$$n_J^*(E) \cong \det(E)^{\otimes n} \otimes F,$$

for some homogeneous vector bundle F. Moreover, the Chern classes of F are algebraically trivial by [Mukai 1978, Theorem 4.17]. Thus, the total Chern class of E satisfies

$$n_J^*(c(E)) = (1 + nc_1(E))^n$$

in the cycle ring of J modulo algebraic equivalence. Since $n_J^*(c_1(E)) = n^2 c_1(E)$ in that ring, this yields

$$c(E) = \left(1 + \frac{c_1(E)}{n}\right)^n. \tag{21}$$

We now recall a formula for $c(E)$ due to [Mattuck 1961, Theorem 3]. Denoting by W_i the image of p_i for $0 \le i \le g$, we have

$$c(E) = \sum_{i=0}^{g} (-1)^i [W_{g-i}^-],$$

where W_j^- denotes the image of W_j under the involution $(-1)_J$ and the equality holds again modulo algebraic equivalence. In particular,

$$c_1(E) = -[W_{g-1}^-] = -\theta,$$

where θ denotes the Chern class of the theta divisor, and

$$c_g(E) = (-1)^g e,$$

where e denotes the class of a point. In view of (21), this yields

$$e = \binom{n}{g} \frac{\theta^g}{n^g}.$$

Since $\theta^g = g!\,e$, we obtain $n^g = n(n-1)\cdots(n-g+1)$ and hence $g \le 1$.

Conversely, if $g = 0$ then $C^{(d)} = \mathbb{P}^d$ and there is nothing to prove; if $g = 1$ then the assertion follows from Proposition 4.1. ☐

Remark 4.4. By [Ein and Lazarsfeld 1992], the vector bundle E is stable with respect to the principal polarization of J. In particular, E is simple, that is, $\mathrm{Aut}_J(P)$ is finite. This yields examples of simple vector bundles on abelian varieties which are not semihomogeneous (see [Oda 1971] for the first construction of bundles satisfying these properties).

5. Homogeneous self-dual projective bundles

Generalities on self-dual bundles. Throughout this subsection, we assume that $p \neq 2$; we consider projective bundles over a fixed variety X. Let $f : P \to X$ be a $\mathbb{P}^{n-1}$-bundle, and $f^* : P^* \to X$ the dual bundle. By contravariance, any isomorphism of bundles

$$\varphi : P \to P^* \tag{22}$$

defines a dual isomorphism $\varphi^* : P = P^{**} \to P^*$. We say that (22) is *self-dual* if $\varphi^* = \varphi$.

For later use, we now present some general results on self-dual bundles; we omit their (easy) proofs, which can be found in the arXiv version of this article [Brion 2012b].

Proposition 5.1. *Given a $\mathbb{P}^{n-1}$-bundle P, there is a bijective correspondence between the self-dual morphisms* (22) *and the reductions of structure group of the associated* PGL_n*-torsor* $\pi : Y \to X$ *to a* $\mathrm{PO}_{n,\varepsilon}$*-torsor* $\psi : Z \to X$, *where* $\varepsilon = \pm 1$ *and* $\mathrm{PO}_{n,\varepsilon} \subset \mathrm{PGL}_n$ *denotes the projective orthogonal* (*resp. symplectic*) *group if* $\varepsilon = +1$ (*resp.* -1).

We say that the self-dual morphism (22) is *symmetric* (resp. *alternating*) if $\varepsilon = 1$ (resp. $= -1$). Denote by $\mathrm{GO}_{n,\varepsilon}$ the preimage of $\mathrm{PO}_{n,\varepsilon}$ in GL_n. Then $\mathrm{GO}_{n,\varepsilon}$ is the stabilizer of a unique line in the space of bilinear forms on k^n. Moreover, any such semiinvariant form B is nondegenerate, and it is symmetric (resp. alternating) if φ has the same property.

The group $\mathrm{GO}_{n,\varepsilon}$ is connected and reductive for any n; hence so is $\mathrm{PO}_{n,\varepsilon}$. If n is odd, then we must have $\varepsilon = +1$, and $\mathrm{PO}_{n,\varepsilon} = \mathrm{SO}_n$; if n is even, then $\mathrm{PO}_{n,+1} = \mathrm{PSO}_n$ and $\mathrm{PO}_{n,-1} = \mathrm{PSp}_n$. As a consequence, $\mathrm{PO}_{n,\varepsilon}$ is semisimple of adjoint type unless $n = 2$ and $\varepsilon = 1$; then $\mathrm{PO}_{2,+1} = \mathbb{G}_m$.

Together with the results of Grothendieck recalled on pages 2477–2478 and 2496–2497, Proposition 5.1 yields one-to-one correspondences between *self-dual* $\mathbb{P}^{n-1}$*-bundles* (that is, bundles equipped with a self-dual morphism), $\mathrm{PO}_{n,\varepsilon}$-torsors, and Azumaya algebras $\mathscr{A}$ of rank n^2 equipped with an involution (as in [Parimala and Srinivas 1992]); these correspondences preserve morphisms. The $\mathrm{PO}_{n,\varepsilon}$-torsor $Z \to X$ corresponds to the associated bundle $P = Z \times^{\mathrm{PO}_{n,\varepsilon}} \mathbb{P}^{n-1} \to X$ equipped with the isomorphism to P^* arising from the $\mathrm{PO}_{n,\varepsilon}$-equivariant isomorphism $\mathbb{P}^{n-1} \xrightarrow{\cong} (\mathbb{P}^{n-1})^*$ given by B. The associated Azumaya algebra is the sheaf of local sections of the matrix bundle $Z \times^{\mathrm{PO}_{n,\varepsilon}} \mathrm{M}_n$ equipped with the involution arising from the isomorphism $\mathrm{M}_n \to (\mathrm{M}_n)^{\mathrm{op}}$ defined by the adjoint with respect to the pairing B.

Like for $\mathbb{P}^{n-1}$-bundles, we may define the *product* of the self-dual bundles (P_i, φ_i) $(i = 1, 2)$ in terms of the associated $\mathrm{PO}_{n_i,\varepsilon_i}$-torsors $Z_i \to X$. Specifically, the product $(P_1P_2, \varphi_1\varphi_2)$ corresponds to the $\mathrm{PO}_{n_1n_2,\varepsilon_1\varepsilon_2}$-torsor obtained from the $\mathrm{PO}_{n_1,\varepsilon_1} \times \mathrm{PO}_{n_2,\varepsilon_2}$-torsor $Z_1 \times_X Z_2 \to X$ by the extension of structure groups

$$\mathrm{PO}_{n_1,\varepsilon_1} \times \mathrm{PO}_{n_2,\varepsilon_2} = \mathrm{PO}_{\varepsilon_1}(k^{n_1}) \times \mathrm{PO}_{\varepsilon_2}(k^{n_2}) \xrightarrow{\rho} \mathrm{PO}_{\varepsilon_1\varepsilon_2}(k^{n_1} \otimes k^{n_2}) = \mathrm{PO}_{n_1n_2,\varepsilon_1\varepsilon_2},$$

where ρ stems from the natural map $\mathrm{GO}_{\varepsilon_1}(k^{n_1}) \times \mathrm{GO}_{\varepsilon_2}(k^{n_2}) \to \mathrm{GO}_{\varepsilon_1\varepsilon_2}(k^{n_1} \otimes k^{n_2})$. This product also corresponds to the tensor product of algebras with involutions, as considered in [Parimala and Srinivas 1992].

Next, we introduce a notion of *decomposition* of self-dual bundles; for this, we need some observations on duality for subbundles. Any $\mathbb{P}^{n_1-1}$-subbundle P_1 of a bundle P defines a $\mathbb{P}^{n-n_1-1}$-subbundle of P^*, as follows: P_1 corresponds to a PGL_n-equivariant morphism γ from Y to the Grassmannian $\mathrm{PGL}_n / \mathrm{PGL}_{n,n_1}$ and hence to an equivariant morphism γ^* from Y^* to the dual Grassmannian, $\mathrm{PGL}_n / \mathrm{PGL}_{n,n-n_1}$. The latter morphism yields the desired subbundle $P_1^\perp$. One checks that $P_1^{\perp\perp} = P_1$ under the identification of P with P^{**}. Moreover, every decomposition (P_1, P_2) of P yields a decomposition $(P_2^\perp, P_1^\perp)$ of P^*, of the same type. We may now define a *decomposition* of a self-dual bundle (P, φ) as a decomposition (P_1, P_2) of the bundle P, such that $\varphi(P_1) = P_2^\perp$; then also $\varphi(P_2) = P_1^\perp$ by self-duality.

Proposition 5.2. *Under the correspondence of Proposition 5.1, the decompositions of type* (n_1, n_2) *of* (P, φ) *correspond bijectively to the reductions of structure group of the* $\mathrm{PO}_{n,\varepsilon}$*-torsor* Z *to a* $\mathrm{P}(\mathrm{O}_{n_1,\varepsilon} \times \mathrm{O}_{n_2,\varepsilon})$*-torsor.*

Moreover, each subbundle P_i *in a decomposition of* (P, φ) *uniquely determines the other one and comes with a self-dual isomorphism* $\varphi_i : P_i \to P_i^*$ *of the same sign as* φ.

The subbundles P_i occurring in a decomposition of (P, φ) are characterized by the property that $\varphi(P_i)$ and $P_i^\perp$ are disjoint; we then say that P_i is *nondegenerate*. A self-dual bundle will be called *indecomposable* if it admits no proper decomposition; equivalently, any proper subbundle is degenerate.

Remarks 5.3. (1) We also have the notion of *L-indecomposability* from [Balaji et al. 2005], namely, a self-dual bundle is L-indecomposable if the associated $\mathrm{PO}_{n,\varepsilon}$-torsor admits no reduction of structure group to a proper Levi subgroup. The maximal Levi subgroups of $\mathrm{PO}_{n,\varepsilon}$ are exactly the subgroups $\mathrm{P}(\mathrm{O}_{n_1,\varepsilon} \times \mathrm{GL}_{n_2})$, where $n_1 \geq 0$, $n_2 \geq 1$, $n_1 + 2n_2 = n$, and $\mathrm{GL}_{n_2} \subset \mathrm{O}_{2n_2,\varepsilon}$ is the subgroup that stabilizes a decomposition $k^{2n_2} = V_1 \oplus V_2$ with V_1 and V_2 totally isotropic subspaces of dimension n_2. Thus, a self-dual bundle is L-indecomposable if and only if it admits no proper *hyperbolic* nondegenerate subbundle, where (P, φ) is called hyperbolic if the bundle P has a decomposition (P_1, P_2) such that $\varphi(P_i) = P_i^\perp$ for $i = 1, 2$.

(2) If $P = \mathbb{P}(E)$ for some vector bundle E over X, then the symmetric (resp. antisymmetric) morphisms $\varphi : P \to P^*$ correspond bijectively to the symmetric (resp. antisymmetric) nondegenerate bilinear forms $B : E \times E \to L$, where L is a line bundle and B is viewed up to multiplication by a regular invertible function on X.

Also, note that $\mathbb{P}(E)$ is hyperbolic if and only if E admits a splitting

$$E \cong V \oplus (V^* \otimes L)$$

for some vector bundle V and some line bundle L; then the bilinear form B on E takes values in L and is given by

$$b(v \oplus (\xi \otimes s), w \oplus (\eta \otimes t)) = \langle v, \eta \rangle t + \varepsilon \langle w, \xi \rangle s,$$

where $\langle -, - \rangle$ denotes the canonical pairing on $V \times V^*$.

Structure of homogeneous self-dual bundles. In this subsection, we still assume that $p \neq 2$; we denote by X a fixed abelian variety and by $f : P \to X$ a $\mathbb{P}^{n-1}$-bundle. We say that a self-dual bundle (P, φ) is *homogeneous* if the corresponding $\mathrm{PO}_{n,\varepsilon}$-torsor Z of Proposition 5.1 is homogeneous. Then the bundle P is easily seen to be homogeneous.

In view of [Brion 2012a, Theorem 3.1], the structure of homogeneous self-dual bundles is described by a completely analogous statement to Theorem 2.1, where PGL_n is replaced with $\mathrm{PO}_{n,\varepsilon}$. This reduces the classification of these bundles to that of the commutative subgroup schemes of $\mathrm{PO}_{n,\varepsilon}$ up to conjugacy. Let H be such a subgroup scheme, $\widetilde{H}$ its preimage in $\mathrm{GO}_{n,\varepsilon}$ and $e : H \times H \to \mathbb{G}_m$ the associated commutator pairing. Choose a nondegenerate bilinear form B on $k^n =: V$ which is an eigenvector of $\mathrm{GO}_{n,\varepsilon}$; such a form is unique up to scalar. We say that the pair $(\widetilde{H}, B)$ is a *self-dual theta group*, and (V, B) a *self-dual representation*. Note that $\widetilde{H}$ is equipped with a character

$$\beta : \widetilde{H} \to \mathbb{G}_m \tag{23}$$

such that

$$(\tilde{x} \cdot B)(v_1, v_2) = B(\tilde{x}^{-1} v_1, \tilde{x}^{-1} v_2) = \beta(\tilde{x}) B(v_1, v_2),$$

for all $\tilde{x} \in \widetilde{H}$ and $v_1, v_2 \in V$. In particular, $\beta(t) = t^{-2}$ for all $t \in \mathbb{G}_m$; we say that β has $\mathbb{G}_m$-*weight* -2. The existence of such a character imposes a strong restriction on the quotient $H/H^\perp = F/F^\perp$ (where F denotes the group of components of H_s and the orthogonals are relative to the pairing e):

Lemma 5.4. *With the above notation, $H/H^\perp$ is a 2-elementary finite group; in particular, the homogeneous index of P is a power of 2. Moreover, e factors through a nondegenerate alternating morphism*

$$se : H/H^\perp \times H/H^\perp \to \mu_2. \tag{24}$$

Proof. Since $\beta(e(x, y)) = \chi(\tilde{x}\tilde{y}\tilde{x}^{-1}\tilde{y}^{-1}) = 1$ for all $x, y \in H$ with lifts $\tilde{x}, \tilde{y} \in \widetilde{H}$, we see that $e(2x, y) = e(x, y)^2 = 1$. Thus, $H^\perp$ contains $2H$ (the image of the multiplication by 2 in the commutative group scheme H), that is, F is killed by 2. Since $p \neq 2$, this implies the first assertion. For the second, note that e factors through a morphism $H \times H \to \mu_2$ and hence through a bilinear alternating morphism (24), which must be nondegenerate by the definition of $H^\perp$. $\square$

In view of this result, Proposition 2.4, Lemma 2.6, and Proposition 2.7 also hold in this setting (without the assumption that $(n, p) = 1$), by the same arguments.

We now assume that e is nondegenerate; equivalently, $H^\perp$ is trivial. Then we may view H as a finite-dimensional vector space over the field $\mathbb{F}_2$ with two elements, and

se as a symplectic form (with values in $\mathbb{F}_2$), by identifying $\mathbb{F}_2$ to μ_2 via $x \mapsto (-1)^x$. We denote by $\mathrm{Sp}(H) = \mathrm{Aut}(H, se)$ the corresponding symplectic group.

Choose a maximal totally isotropic subspace $K \subset H$. Then $H \cong K \oplus K^*$ and this identifies se with the standard symplectic form ω defined by

$$\omega((x, \xi), (x', \xi')) = \langle x, \xi' \rangle + \langle x', \xi \rangle,$$

where $\langle -, - \rangle : K \times K^* \to \mathbb{F}_2$ denotes the canonical pairing. In particular, $\#(H) = \#(K)^2 = 2^{2r}$, where $r := \dim_{\mathbb{F}_2}(K)$, and $\mathrm{Sp}(H) = \mathrm{Sp}_{2r}(\mathbb{F}_2)$; we say that r is the *rank* of (H, e). Moreover, the dual K^* is identified to the character group of K, via the map $\xi \longmapsto (x \mapsto (-1)^{\langle x, \xi \rangle})$. Recall that $\widetilde{H}$ is isomorphic to the Heisenberg group $\mathcal{H}(K)$, and has a unique irreducible representation of weight 1: the standard representation in $\mathcal{O}(K)$, of dimension 2^r.

We now analyze the representation of $\widetilde{H}$ in the space of bilinear forms on W. Since $p \neq 2$, we have a decomposition of representations $W^* \otimes W^* = S^2 W^* \oplus \Lambda^2 W^*$ into the symmetric and the alternating components. For any $x \in K$, denote by $\epsilon_x \in W^*$ the evaluation at x, that is, $\epsilon_x(f) = f(x)$ for any $f \in W$. Then the ϵ_x $(x \in K)$ form a basis of W^* and satisfy

$$(t, x, \xi) \cdot \epsilon_y = t^{-1} (-1)^{\langle x+y, \xi \rangle} \epsilon_{x+y}.$$

Define bilinear forms on W by

$$B_{x,\xi} := \sum_{y \in K} (-1)^{\langle y, \xi \rangle} \epsilon_y \otimes \epsilon_{x+y} \quad (x \in K, \xi \in K^*).$$

Lemma 5.5. *With the above notation, each $B_{x,\chi}$ is an eigenvector of $\widetilde{H}$ with weight*

$$\chi_{x,\xi} : (t, y, \eta) \longmapsto t^{-2} (-1)^{\langle x, \eta \rangle + \langle y, \xi \rangle}.$$

Also, $B_{x,\xi}$ is symmetric (resp. alternating) if and only if $\langle x, \xi \rangle = 0$ (resp. $= 1$).

Moreover, the $B_{x,\xi}$ form a basis of $W^* \otimes W^*$.

Proof. The first assertion is easily checked. It implies the second assertion, since the $B_{x,\chi}$ have pairwise distinct weights and their number is $\#(K)^2 = \dim(W^* \otimes W^*)$. □

The normalizer $N_{\mathrm{GL}(W)}(\widetilde{H})$ acts on $W^* \otimes W^*$; it stabilizes $S^2 W^*$ and $\Lambda^2 W^*$, and permutes the eigenspaces of $\widetilde{H}$. Thus, $N_{\mathrm{GL}(W)}(\widetilde{H})$ acts on the set of their weights,

$$\mathcal{X} := \{\chi_{x,\xi} \mid x \in K, \xi \in K^*\}.$$

Note that $\mathcal{X}$ is exactly the set of characters of $\widetilde{H}$ with $\mathbb{G}_m$-weight -2. This is an affine space with underlying vector space the character group of H, that we identify with H via the pairing se. Also, $N_{\mathrm{GL}(W)}(\widetilde{H})$ acts on $\mathcal{X}$ by affine automorphisms, and the subgroup $\widetilde{H}$ of $N_{\mathrm{GL}(W)}(\widetilde{H})$ acts trivially, since $\widetilde{H}$ acts on itself by conjugation.

In view of Lemma 2.6, it follows that $N_{\mathrm{GL}(W)}(\widetilde{H})$ acts on $\mathscr{X}$ via its quotient $\mathrm{Sp}(H)$; the linear part of this affine action is the standard action of $\mathrm{Sp}(H)$ on H.

Proposition 5.6. *The above action of* $\mathrm{Sp}(H)$ *on* $\mathscr{X}$ *has two orbits*: *the symmetric characters* $\chi_{x,\xi}$, *where* $\langle x, \xi\rangle = 0$, *and the alternating characters. In particular*, S^2W^* *and* Λ^2W^* *are irreducible representations of* $N_{\mathrm{GL}(W)}(\widetilde{H})$.

Proof. Consider the general linear group $\mathrm{GL}(K) \cong \mathrm{GL}_r(\mathbb{F}_2)$ acting naturally on $\mathbb{O}(K) = W$. Then one readily checks that this action is faithful and normalizes $\widetilde{H}$; also, the resulting homomorphism $\mathrm{GL}(K) \to N_{\mathrm{GL}(W)}(\widetilde{H})$ lifts the (injective) homomorphism $\mathrm{GL}(K) \to \mathrm{Sp}(H)$ associated with the natural representation of $\mathrm{GL}(K)$ in $K \oplus K^*$. Moreover, the induced action of $\mathrm{GL}(K)$ on $\mathscr{X}$ is given by $\gamma \cdot \chi_{x,\xi} = \chi_{\gamma(x),\gamma(\xi)}$. Since the pairs (x, ξ) such that $\langle x, \xi\rangle = 1$ form a unique orbit of $\mathrm{GL}(K)$, we see that $\mathrm{Sp}(H)$ acts transitively on the alternating characters.

On the other hand, the pairs (x, ξ) such that $\langle x, \xi\rangle = 0$ decompose into orbits of $\mathrm{GL}(K)$ according to the (non)vanishing of x and ξ; this yields four orbits if $m \geq 2$, and three orbits if $m = 1$ (then the orbit with $x \neq 0 \neq \xi$ is missing). Note that the unique $\mathrm{GL}(K)$-fixed point $\chi_{0,0}$ (a symmetric weight) is not fixed by $\mathrm{Sp}(H)$: otherwise, the latter group would act on $\mathscr{X}$ via its representation on H, and hence would act transitively on $\mathscr{X} \setminus \{\chi_{0,0}\} \cong H \setminus \{0\}$. But this is impossible, since $\mathrm{Sp}(H)$ preserves the symmetric weights. Also, note that $\mathrm{GL}(K)$ has index 2 in its normalizer $N_{\mathrm{Sp}(H)}(\mathrm{GL}(K))$; moreover, any element of $N_{\mathrm{Sp}(H)}(\mathrm{GL}(K)) \setminus \mathrm{GL}(K)$ fixes $\chi_{0,0}$ and exchanges the $\mathrm{GL}(K)$-orbits $\{\chi_{x,0} \mid x \in K, x \neq 0\}$ and $\{\chi_{0,\xi} \mid \xi \in K^*, x \neq 0\}$.

As a consequence, $\mathrm{Sp}(H)$ acts transitively on the symmetric characters if $m = 1$. We now show that this property also holds when $m \geq 2$. In view of Lemma 2.6, it suffices to construct automorphisms $u, v \in \mathrm{Aut}^{\mathbb{G}_m}(\widetilde{H})$ such that $u(\chi_{x,0}) = \chi_{x,\xi} = v(\chi_{0,\xi})$ for some nonzero $x \in K$ and $\xi \in K^*$. For this, let $q : K \to \mathbb{F}_2$ be a quadratic form, and $\varphi : K \to K^*$ the associated alternating map, defined by $\langle \varphi(x), y\rangle = q(x + y) + q(x) + q(y)$. Let $u = u_q : \widetilde{H} \to \widetilde{H}$ be the map such that $u(t, x, \xi) = (t(-1)^{q(x)}, x, \xi + \varphi(x))$. Then one may check that $u \in \mathrm{Aut}^{\mathbb{G}_m}(\widetilde{H})$ and $u(\chi_{x,0}) = \chi_{x,\varphi(x)}$. Since we may choose q so that $\varphi(x) \neq 0$, this yields the desired automorphism u (and v by symmetry). □

By Lemma 5.5 and Proposition 5.6, *there are exactly two isomorphism classes of self-dual nondegenerate theta groups of a prescribed rank*, the isomorphism type being just the "sign". We now construct representatives of each class; we first consider the case of rank 1. Then $H = \mathbb{F}_2^2 \cong (\mathbb{Z}/2\mathbb{Z})^2$ has a faithful homomorphism to PGL_2, unique up to conjugation. Thus, H lifts to two natural subgroups of GL_2: the dihedral group $D \subset \mathrm{O}_2$, and the quaternionic group $Q \subset \mathrm{Sp}_2 = \mathrm{SL}_2$. Both groups are finite of order eight; moreover, $\widetilde{H}_1 := \mathbb{G}_m D$ (resp. $\widetilde{H}_0 := \mathbb{G}_m Q$) is a nondegenerate theta group of rank 1 equipped with a symmetric (resp. alternating) semiinvariant bilinear form.

For an arbitrary rank r, the central product $\widetilde{H}_1 \cdots \widetilde{H}_1$ of r copies of $\widetilde{H}_1$ (the quotient of the product $\widetilde{H}_1 \times \cdots \times \widetilde{H}_1$ by the subtorus $\{(t_1, \ldots, t_r) \mid t_1 \cdots t_r = 1\}$) is a self-dual nondegenerate theta group of rank r and sign $+1$. Similarly, the central product of $\widetilde{H}_0$ with $r-1$ copies of $\widetilde{H}_1$ is a self-dual nondegenerate theta group of rank r and sign -1.

Remark 5.7. The above description of the self-dual nondegenerate theta groups may also be deduced from the structure of *extraspecial* 2*-groups*, that is, of those finite groups G such that the center Z has order two, and G/Z is 2-elementary (see [Huppert 1967, Kapitel III, Satz 13.8] or [Gorenstein 1980, Chapter 5, Theorem 5.2]). Namely, by Lemma 5.4, every self-dual nondegenerate theta group yields an extension $1 \to \mu_2 \to G \to H \to 1$, where G is extraspecial. Yet the approach followed here is more self-contained.

Returning to an arbitrary self-dual theta group $(\widetilde{H} \subset \mathrm{GL}(V), B)$, we now investigate the decomposition of V into eigenspaces V_λ of $Z(\widetilde{H}_s)$. Recall from Proposition 2.7 that $V_\lambda \cong U_\lambda \otimes W_\lambda$ as a representation of $\widetilde{H} \cong H_u \times \widetilde{H}_s$, where W_λ is the standard representation of the Heisenberg group $\widetilde{H}_s/\ker(\lambda)$. Also, since B has weight β, we have $B(V_\lambda, V_\mu) = \{0\}$ unless $\lambda + \mu = -\beta$. This readily implies the following observations:

Lemma 5.8. (i) *As a self-dual representation, V is the direct sum of the pairwise orthogonal subspaces V_λ, where $2\lambda = -\beta$, and $V_\lambda \oplus V_{-\lambda-\beta}$, where $2\lambda \neq -\beta$.*

(ii) *If $2\lambda = -\beta$, then U_λ (resp. W_λ) is a self-dual representation of H_u (resp. $\widetilde{H}_s/\ker(\lambda)$). Moreover, the restriction of B to V_λ is the tensor product of the corresponding bilinear forms on U_λ, resp. W_λ.*

(iii) *If $2\lambda \neq -\beta$, then $V_{-\lambda-\beta} \cong V_\lambda^*(-\beta)$ as representations of $\widetilde{H}$. Moreover, the restriction of B to $V_\lambda \oplus V_{-\lambda-\beta}$ is given by the symmetrization or alternation of the canonical pairing $V_\lambda \otimes V_\lambda^*(-\beta) \to k(-\beta)$.*

As a direct consequence, we obtain the following analogue of the structure of indecomposable homogeneous bundles (Proposition 2.9):

Proposition 5.9. *The following assertions are equivalent for a homogeneous self-dual bundle (P, φ):*

(i) *(P, φ) is indecomposable.*

(ii) *V is indecomposable as a self-dual representation.*

(iii) *$\widetilde{H}_s$ is a Heisenberg group and one of the following cases occurs*:

(I) $V \cong U \otimes W$, *where U is an indecomposable self-dual representation of H_u and W is the standard irreducible representation of $\widetilde{H}_s$. Moreover, H_s is 2-elementary.*

(II) $V \cong (U \otimes W) \oplus (U^* \otimes W^*)(-\beta)$, *where U is an indecomposable representation of H_u, W is the standard irreducible representation of $\widetilde{H}_s$, and β is a character of $\widetilde{H}_s$ of weight -2.*

Remarks 5.10. (1) In contrast to Proposition 2.9, there exist indecomposable self-dual bundles (P, φ) such that $\mathrm{Aut}^0_X(P, \varphi)$ is not unipotent. Specifically, if (P, φ) is hyperbolic (type (II) above), then the action of $\mathbb{G}_m$ on V with weight spaces $U \otimes W$ of weight 1, and $(U^* \otimes W^*)(-\beta)$ of weight -1, yields a one-parameter subgroup of bundle automorphisms of (P, φ).

In fact, the condition that $\mathrm{Aut}^0_X(P, \varphi)$ is unipotent characterizes *L-indecomposable* self-dual bundles. Also, one easily shows that the homogeneous self-dual bundle (P, φ) is L-indecomposable if and only if the self-dual representation V contains no nontrivial direct summand of type (II).

(2) If (P, φ) is irreducible in the sense that it arises from a nondegenerate theta group, then $\mathrm{Aut}_X(P)$ is finite by Proposition 3.5; as a consequence, $\mathrm{Aut}_X(P, \varphi)$ is finite. But the converse does not hold in general, for example, for homogeneous self-dual $\mathbb{P}^2$-bundles associated with the subgroup H of PO_3 generated by the images of the diagonal matrices with coefficients ± 1 (then $H \cong (\mathbb{Z}/2\mathbb{Z})^2$ and $e = 0$). Thus, the criteria for irreducibility in Propositions 3.5 and 3.7 do not extend to self-dual bundles. In [Brion et al. 2012, §7.3], an alternative, group-theoretical notion of irreducibility is introduced for homogeneous principal bundles under a semisimple group in characteristic zero, and Propositions 3.5 and 3.7 are generalized to that setting.

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On the second Tate–Shafarevich group of a 1-motive

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We prove finiteness results for Tate–Shafarevich groups in degree 2 associated with 1-motives. We give a number-theoretic interpretation of these groups, relate them to Leopoldt's conjecture, and present an example of a semiabelian variety with an infinite Tate–Shafarevich group in degree 2. We also establish an arithmetic duality theorem for 1-motives over number fields, which complements earlier results of Harari and Szamuely.

Introduction and overview

Let k be a number field, and let X be a commutative group scheme over k. The Tate–Shafarevich group $\text{Ш}^i(k, X)$ of X is the subgroup of the étale cohomology group $H^i(k, X)$ consisting of those elements that restrict to zero over each completion of k. These groups are among the most fundamental invariants associated with commutative group schemes over number fields, and their vanishing is by definition the obstruction to various local-to-global principles.

If the group scheme X is given by a finitely generated discrete group with continuous Galois action, or if X is a group of multiplicative type, then $\text{Ш}^i(k, X)$ is finite for all i [Milne 1986, Theorem I.4.20; Neukirch et al. 2000, Theorem 8.6.8]. It

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is widely conjectured that if A is an abelian variety over k, then the group $Ш^1(k, A)$ is finite, and it is known that for $i \neq 1$ the group $Ш^i(k, A)$ is trivial. This is a nontrivial statement for $i = 2$; indeed, the proof of Corollary I.6.24 in [Milne 1986] shows that the vanishing of $Ш^2(k, A)$ is essentially equivalent to the positive answer to the congruence subgroup problem for the abelian variety dual to A, given by Serre [1964; 1971].

An evident generalisation of these finiteness results would be to show that $Ш^i(k, G)$ is finite for semiabelian varieties G over k, i.e., when G is an extension of an abelian variety A by a torus. A simple dévissage shows that $Ш^1(k, G)$ is finite [Harari and Szamuely 2005, Lemma 4.11], provided $Ш^1(k, A)$ is finite. The situation is more complicated for $i = 2$, and surprisingly, it turns out that the groups $Ш^2(k, G)$ are not always finite.

Theorem 1. *There exists a semiabelian variety G over $\mathbb{Q}$ such that the group $Ш^2(\mathbb{Q}, G)$ contains a subgroup isomorphic to $\mathbb{Q}/\mathbb{Z}$ and in particular is infinite.*

A 1-motive M over a number field k is a two-term complex of group schemes $M = [Y \to G]$ over k placed in degrees -1 and 0, where Y is given by a finitely generated free discrete group with continuous Galois action and where G is a semiabelian variety. It was asked in [Harari and Szamuely 2005, Remark 4.13] whether for all 1-motives M the group $Ш^2(k, M)$ is finite. By Theorem 1, we already know that this is not always the case even for 1-motives of the form $[0 \to G]$ over $\mathbb{Q}$. Our second result shows that even for very simple 1-motives it might be difficult to decide whether $Ш^2(k, M)$ is finite (assuming the conservation law of difficulty).

Theorem 2. *If the group $Ш^2(k, M)$ is finite for all 1-motives of the form $M = [\mathbb{Z}^r \to \mathbb{G}_m^s]$ over k, then Leopoldt's conjecture holds for k (and all prime numbers).*

The converse to this statement is not true: there exist 1-motives of this particular form over number fields for which Leopoldt's conjecture is known to hold such that $Ш^2(k, M)$ is infinite. Our third result provides conditions on a 1-motive that ensure that $Ш^2(k, M)$ is finite. It is most conveniently expressed as a duality theorem. Classical global arithmetic duality theorems are statements about the existence and nondegeneracy of canonical pairings between certain Tate–Shafarevich groups. Let X be a group of multiplicative type over k, and denote by $X^\vee$ its group of characters. The Poitou–Tate duality theorem states that there is a natural, perfect pairing of finite groups

$$Ш^i(k, X) \times Ш^{3-i}(k, X^\vee) \to \mathbb{Q}/\mathbb{Z}$$

[Milne 1986, Theorem I.4.20; Neukirch et al. 2000, Theorem 8.6.8]. The analogue of this duality theorem for abelian varieties is the Cassels–Tate duality theorem. It states that for an abelian variety A over k with dual $A^\vee$, there is a canonical pairing

$$Ш^i(k, A) \times Ш^{2-i}(k, A^\vee) \to \mathbb{Q}/\mathbb{Z}$$

whose left and right kernels are the maximal divisible subgroups [Milne 1986, Theorem I.6.26]. Conjecturally, it is a perfect pairing of finite groups.

The idea to unify and generalise these arithmetic duality theorems to duality theorems for 1-motives appeared first in [Harari and Szamuely 2005]. Deligne constructed for each 1-motive M a dual 1-motive $M^\vee$. Harari and Szamuely show that for a 1-motive M over a number field k there is a canonical pairing

$$Ш^1(k, M) \times Ш^1(k, M^\vee) \to \mathbb{Q}/\mathbb{Z}$$

that is nondegenerate modulo divisible subgroups and generalises the Cassels–Tate pairing. They also construct a pairing between a certain modification of $Ш^0(k, M)$ and $Ш^2(k, M^\vee)$ and show that it is nondegenerate modulo divisible subgroups. However, this modified $Ш^0(k, M)$ remains somehow uncontrollable, and the resulting generalised pairing does not seem to be very useful (the statement of [Harari and Szamuely 2005, Theorem 0.2] was rectified in [Harari and Szamuely 2009]).

Theorem 3. *Let k be a number field, and let $M = [u : Y \to G]$ be a 1-motive over k with dual $M^\vee$. There exists a natural pairing*

$$Ш^0(k, M) \times Ш^2(k, M^\vee) \to \mathbb{Q}/\mathbb{Z} \tag{$*$}$$

generalising the Poitou–Tate pairing for finitely generated Galois modules and tori. The group $Ш^0(k, M)$ is finite, and the pairing $(*)$ is nondegenerate on the left. If the semiabelian variety G is an abelian variety or a torus, such that the $\mathbb{Q}$-algebra $\mathrm{End}_{\bar{k}}(G) \otimes \mathbb{Q}$ is a product of division algebras, then the pairing $(*)$ is a perfect pairing of finite groups.

It was already shown in [Harari and Szamuely 2005] that $Ш^0(k, M)$ is finite. The finiteness results stated in the second part of the theorem are new and are also the essential part of the theorem. Equivalently, the condition on G is that over an algebraic closure of k either G is the multiplicative group or an abelian variety isogenous to a product of pairwise nonisogenous simple abelian varieties. Our proof uses techniques developed by Serre [1964; 1971] in his work on the congruence subgroup problem for abelian varieties.

Overview. In Section 1, we rehearse 1-motives and ℓ-adic realisations, which will play a prominent role throughout this paper. In Section 2, we construct a duality pairing that relates the ℓ-adic realisation of a 1-motive with the second Tate–Shafarevich group of its dual and obtain the pairing $(*)$ of Theorem 3. In Section 3, we compute the cohomology of some ℓ-adic Lie groups associated with 1-motives, and in Section 4, we use these computations to prove the finiteness statements in Theorem 3. We conclude the proof of Theorem 3 in Section 5. In

Sections 6 and 7, we prove Theorems 2 and 1, respectively. There remain several open questions and unsolved problems, which I state in the last section.

1. About 1-motives and their realisations

In this section, we rehearse the relevant facts about classical 1-motives and their realisations defined by Deligne [1974, §10].

1.1. Throughout this section, S is a noetherian regular scheme, $\mathcal{F}_S$ stands for the category of sheaves of commutative groups on the small fppf site over S and $\mathcal{D}\mathcal{F}_S$ for the derived category of $\mathcal{F}_S$. We identify commutative group schemes over S with objects of $\mathcal{F}_S$ via the functor of points. In particular, we say that an fppf sheaf on S is an abelian scheme, a torus, or a finite flat group scheme if it can be represented by such. By a *lattice over* S, we mean an object of $\mathcal{F}_S$ that is locally isomorphic to a finitely generated free $\mathbb{Z}$-module. Notice that if S is the spectrum of a field, then Y may be regarded as a finitely generated group on which the absolute Galois group acts continuously.

Definition 1.2. A 1*-motive* over S is a diagram

$$M = \begin{bmatrix} & & Y & & \\ & & {\scriptstyle u}\downarrow & & \\ 0 \to T & \to & G & \to A & \to 0 \end{bmatrix}$$

in the category $\mathcal{F}_S$, where Y is a lattice, T a torus, and A an abelian scheme. A morphism of 1-motives is a morphism between diagrams. The *complex associated with* M is the complex of fppf-sheaves $[M] := [Y \to G]$, placed in degrees -1 and 0. We denote by $\mathcal{M}_{1,S}$ the category of 1-motives over S.

1.3. Observe that the sheaf G is representable. Indeed, we may look at it as a T-torsor over A, and since T is affine, representability follows from [Milne 1980, Theorem III.4.3a]. Later on, 1-motives will often be given by their associated complexes and morphisms accordingly by commutative squares. This is also customary in the literature and justified by the fact that there are no nonzero morphisms from a torus to an abelian scheme.

1.4. We say that a sequence of morphisms of 1-motives $0 \to M_1 \to M_2 \to M_3 \to 0$ is a *short exact sequence* if the induced sequences of lattices, tori, and abelian schemes are exact in $\mathcal{F}_S$. Such a short exact sequence of 1-motives yields then an exact triangle

$$[M_1] \to [M_2] \to [M_3] \to [M_1][1]$$

in the derived category $\mathcal{D}\mathcal{F}_S$. With a 1-motive M over S are naturally associated several short exact sequences coming from the *weight filtration* on M. This is the natural three-term filtration given by $W_iM = 0$ if $i \leq -3$ and $W_iM = M$ if $i \geq 0$ and

$$W_{-2}M := \left[\begin{array}{c} 0 \\ \downarrow \\ 0 \to T = T \to 0 \to 0 \end{array}\right] \quad \text{and} \quad W_{-1}M := \left[\begin{array}{c} 0 \\ \downarrow \\ 0 \to T \to G \to A \to 0 \end{array}\right].$$

Although 1-motives do not form an abelian category, the quotients M/W_iM make sense in the obvious way.

Definition 1.5. Let M be a 1-motive over S, and let ℓ be a prime number that is invertible on S. The *ℓ-adic Tate module* and the *ℓ-divisible Barsotti–Tate group* associated with M are the smooth ℓ-adic sheaf

$$\mathrm{T}_\ell(M) := \lim_{i \geq 0} H^1([M] \otimes^{\mathbb{L}} \mathbb{Z}/\ell^i\mathbb{Z})$$

and the smooth ℓ-divisible group

$$\mathrm{B}_\ell(M) := \operatorname{colim}_{i \geq 0} H^1([M] \otimes^{\mathbb{L}} \mathbb{Z}/\ell^i\mathbb{Z}),$$

both over S, where the derived tensor product is taken in the derived category $\mathfrak{DF}_S$.

1.6. By construction, $\mathrm{T}_\ell M$ only depends on the complex $[M] = [Y \xrightarrow{u} G]$ up to quasi-isomorphism, and the assignment $M \mapsto \mathrm{T}_\ell M$ is functorial. Using the flat resolution $\mathbb{Z} \xrightarrow{\ell^i} \mathbb{Z}$ of $\mathbb{Z}/\ell^i\mathbb{Z}$, we see that the object $[M] \otimes^{\mathbb{L}} \mathbb{Z}/\ell^i\mathbb{Z}$ of $\mathfrak{DF}_S$ is given by the bounded complex

$$\cdots \to 0 \to Y \xrightarrow{y \mapsto (\ell^i y, u(y))} Y \oplus G \xrightarrow{(y,g) \mapsto u(y) - \ell^i g} G \to 0 \to \cdots$$

supported in degrees 0, 1, and 2. For $n \neq 1$, we have $H^n([M] \otimes^{\mathbb{L}} \mathbb{Z}/\ell^i\mathbb{Z}) = 0$ because Y is torsion-free and G is divisible as a sheaf. Hence, the object $[M] \otimes^{\mathbb{L}} \mathbb{Z}/\ell^i\mathbb{Z}$ of $\mathfrak{DF}_S$ is homologically concentrated in degree 1. Given a category $\mathscr{C}$ and a functor $F : \mathfrak{DF}_S \to \mathscr{C}$, we write

$$F(\mathrm{T}_\ell M) := \lim_{i \geq 0} F([M] \otimes^{\mathbb{L}} \mathbb{Z}/\ell^i\mathbb{Z}[-1]),$$
$$F(\mathrm{B}_\ell M) := \operatorname{colim}_{i \geq 0} F([M] \otimes^{\mathbb{L}} \mathbb{Z}/\ell^i\mathbb{Z}[-1]),$$

and interpret these expressions as either limit systems in $\mathscr{C}$ or actual objects of $\mathscr{C}$, provided limits and colimits exist in $\mathscr{C}$. This is only a notation, and we do not need or claim that F commutes with limits or colimits.

1.7. Suppose S is connected, and let $\operatorname{spec}(\bar{k}) = \bar{s} \to S$ be a geometric point where $\bar{k}$ is an algebraic closure of the residue field k at the scheme point underlying $\bar{s}$. We can describe the finite, locally constant group schemes $H^1([M] \otimes^{\mathbb{L}} \mathbb{Z}/\ell^i\mathbb{Z})$ in terms of finite $\pi_1 := \pi_1^{\text{ét}}(S, \bar{s})$-modules as follows. The underlying group is given by

$$\frac{\{(y, P) \in Y(\bar{k}) \times G(\bar{k}) \mid \ell^i P = u(y)\}}{\{(\ell^i y, u(y)) \mid y \in Y(\bar{k})\}},$$

and the action of π_1 is induced by the action of π_1 on $\bar{k}$. Taking the limit over $i \geq 0$, we find the description of the ℓ-adic sheaf $\mathrm{T}_\ell M$ as a π_1-module. The short exact sequence of 1-motives coming from the weight filtration

$$0 \to [0 \to G] \to M \to [Y \to 0] \to 0$$

induces a sequence of ℓ-adic sheaves and one of continuous π_1-representations

$$0 \to \mathrm{T}_\ell G \to \mathrm{T}_\ell M \to Y \otimes \mathbb{Z}_\ell \to 0,$$

which is exact because $G(\bar{k})$ is an ℓ-divisible group. Observe that, given $y \in Y$, a preimage of $y \otimes 1$ in $\mathrm{T}_\ell M$ is given by a sequence $(y, P_i)_{i \geq 0}$ with $P_0 = u(y)$ and $\ell P_i = P_{i-1}$ for $i \geq 1$.

1.8. Let A be a commutative group. We consider the following four operations on A relative to the prime ℓ:

$$A \mathbin{\widehat{\otimes}} \mathbb{Z}_\ell := \lim_{i \geq 0} A/\ell^i A, \qquad \mathrm{T}_\ell A := \lim_{i \geq 0} A[\ell^i],$$
$$A[\ell^\infty] := \operatorname{colim}_{i \geq 0} A[\ell^i], \qquad A \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell := \operatorname{colim}_{i \geq 0} A/\ell^i A.$$

These are the ℓ-adic completion, the ℓ-adic Tate module, extraction of ℓ-torsion, and tensorisation with $\mathbb{Q}_\ell/\mathbb{Z}_\ell$. These four operations are related, as follows. Given a short exact sequence of commutative groups $0 \to A \to B \to C \to 0$, there is a long exact sequence of $\mathbb{Z}_\ell$-modules

$$0 \to \mathrm{T}_\ell A \to \mathrm{T}_\ell B \to \mathrm{T}_\ell C \to A \mathbin{\widehat{\otimes}} \mathbb{Z}_\ell \to B \mathbin{\widehat{\otimes}} \mathbb{Z}_\ell \to C \mathbin{\widehat{\otimes}} \mathbb{Z}_\ell \to 0$$

coming from the snake lemma, identifying $- \mathbin{\widehat{\otimes}} \mathbb{Z}_\ell$ with the first right derived functor of the Tate module functor $\mathrm{T}_\ell(-)$ and vice versa. Similarly, there is a six-term exact sequence of ℓ-torsion groups

$$0 \to A[\ell^\infty] \to B[\ell^\infty] \to C[\ell^\infty] \to A \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell \to B \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell \to C \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell \to 0$$

identifying $(-)[\ell^\infty]$ with the first left derived functor of $- \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell$ and vice versa. Given a bilinear pairing of commutative groups $A \times B \to \mathbb{Q}/\mathbb{Z}$, these operations induce pairings

$$A \mathbin{\widehat{\otimes}} \mathbb{Z}_\ell \times B[\ell^\infty] \to \mathbb{Q}/\mathbb{Z} \quad \text{and} \quad \mathrm{T}_\ell A \times (B \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell) \to \mathbb{Q}/\mathbb{Z}.$$

If the original pairing was nondegenerate, these are nondegenerate pairings as well. Most of the time, we shall deal with commutative groups on which the multiplication-by-ℓ has finite kernel and cokernel. For such a group A, the $\mathbb{Z}_\ell$-modules $A \mathbin{\widehat{\otimes}} \mathbb{Z}_\ell$

and $\mathrm{T}_\ell A$ are finitely generated, and the torsion groups $A \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell$ and $A[\ell^\infty]$ are of cofinite type (meaning that their Pontryagin duals are finitely generated as $\mathbb{Z}_\ell$-modules), and there is an isomorphism of finite groups $(A \mathbin{\widehat{\otimes}} \mathbb{Z}_\ell)[\ell^\infty] \cong A[\ell^\infty] \mathbin{\widehat{\otimes}} \mathbb{Z}_\ell$. Nondegenerate pairings of such groups induce perfect pairings of topological groups.

Proposition 1.9. *Let $F : \mathcal{D}\mathcal{F}_S \to \mathcal{D}\,\mathcal{A}b$ be a triangulated functor, and let M be a 1-motive over S. There are canonical short exact sequences of $\mathbb{Z}_\ell$-modules*

$$0 \to H^{i-1}F(M) \mathbin{\widehat{\otimes}} \mathbb{Z}_\ell \to H^iF(\mathrm{T}_\ell M) \to \mathrm{T}_\ell H^iF(M) \to 0$$

and short exact sequences of ℓ-torsion groups

$$0 \to H^{i-1}F(M) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell \to H^iF(\mathrm{B}_\ell M) \to H^iF(M)[\ell^\infty] \to 0$$

both natural in M and F.

Proof. The short exact sequence of constant sheaves $0 \to \mathbb{Z} \xrightarrow{\ell^i} \mathbb{Z} \to \mathbb{Z}/\ell^i\mathbb{Z} \to 0$ induces a long exact sequence of groups, from where we can cut out the short exact sequences

$$0 \to H^iF(M) \otimes \mathbb{Z}/\ell^i\mathbb{Z} \to H^iF(M \otimes^{\mathbb{L}} \mathbb{Z}/\ell^i\mathbb{Z}) \to H^{i+1}F(M)[\ell^i] \to 0.$$

The limit system of commutative groups $(H^iF(M) \otimes \mathbb{Z}/\ell^i\mathbb{Z})_{i=0}^\infty$ has the Mittag–Leffler property, and the short exact sequences in the proposition are then obtained by taking limits or colimits, respectively, over $i \geq 0$. □

Corollary 1.10. *Let k be a number field, let ℓ be a prime number, and let $M = [u : Y \to G]$ be a 1-motive over k. Set $Z := H^{-1}(M) = \ker u$. The morphism of $\mathbb{Z}_\ell$-modules*

$$H^i(k, Z \otimes \mathbb{Z}_\ell) \to H^i(k, \mathrm{T}_\ell M)$$

induced by the morphism of 1-motives $[Z \to 0] \to [Y \to G]$ is an isomorphism for $i = 0$ and injective for $i = 1$.

Proof. Proposition 1.9 applied to the functor $\mathrm{R}\Gamma(k, -)$ yields a short exact sequence of $\mathbb{Z}_\ell$-modules

$$0 \to H^{-1}(k, M) \mathbin{\widehat{\otimes}} \mathbb{Z}_\ell \to H^0(k, \mathrm{T}_\ell M) \to \mathrm{T}_\ell H^0(k, M) \to 0.$$

Since $Z(k) = H^{-1}(k, M)$ is a finitely generated group, we can identify $Z(k) \mathbin{\widehat{\otimes}} \mathbb{Z}_\ell$ with $Z(k) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell$, so to get the statement for $i = 0$, it remains to show that the last group in this sequence vanishes. Write $\mathbb{O}_k$ for the ring of integers of k, and choose a sufficiently small open subscheme $U \subseteq \operatorname{spec} \mathbb{O}_k$ such that M extends to a 1-motive over U and such that ℓ is invertible on U. We have then $Z(U) = Z(k)$ and $H^0(U, \mathrm{T}_\ell M) = H^0(k, \mathrm{T}_\ell M)$, so we may as well show that $\mathrm{T}_\ell H^0(U, M)$ vanishes. Indeed, it follows by dévissage from the Mordell–Weil theorem, Dirichlet's unit theorem, and finiteness of $H^1(U, Y)$ that $H^0(U, M)$ is a finitely generated group

[Harari and Szamuely 2005, Lemma 3.2], so its Tate module is trivial. For the case $i = 1$, we consider the triangle

$$[Z \to 0] \to [Y \to G] \to [Y/Z \to G]$$

and observe that if we quotient both terms of the complex $[Y/Z \to G]$ by the finite torsion part of Y/Z, we get a quasi-isomorphic complex, which is the complex of a 1-motive $M' = [u' : Y' \to G']$ where now u' is injective. By the first part, we have $H^0(k, \mathrm{T}_\ell M') = 0$, and the statement can be read in the long exact cohomology sequence associated with $0 \to Z \otimes \mathbb{Z}_\ell \to \mathrm{T}_\ell M \to \mathrm{T}_\ell M' \to 0$. □

The statement of Corollary 1.10 remains true over any field k that is finitely generated over its prime field and prime number ℓ different from the characteristic of k. It is wrong in general for local fields.

We now come to the dual 1-motive: with each 1-motive M over a noetherian regular scheme S is functorially associated a dual 1-motive $M^\vee$ over S so that we get an involution of the category $\mathcal{M}_{1,S}$ of 1-motives over S. The duals of tori, lattices, and abelian schemes, if seen as 1-motives, are the usual duals, and the duality functor is compatible with the weight filtration. This is the content of the following theorem:

Theorem 1.11. *There exists an antiequivalence of categories $(-)^\vee : \mathcal{M}_{1,S} \to \mathcal{M}_{1,S}$ such that for every 1-motive M over S the following hold*:

(1) *There are canonical and natural isomorphisms of 1-motives*

$$(M/W_{-i}(M))^\vee \cong W_{i-3}(M^\vee)$$

for each i.

(2) *There is a natural isomorphism*

$$[M^\vee] \cong \mathrm{R}\,\mathcal{H}om(M, \mathbb{G}_m[1])_{\leq 0}$$

in the derived category $\mathcal{D}\mathcal{F}_S$, where $(-)_{\leq 0}$ means truncation in degree 0.

(3) *There is a natural isomorphism of 1-motives $\epsilon_M : M \to M^{\vee\vee}$ such that the induced morphism of complexes coincides with the canonical evaluation morphism in the derived category of $\mathcal{F}_S$ (explained below).*

Properties (1), (2), and (3) characterise $(-)^\vee$ up to an isomorphism of functors.

For every object X of $\mathcal{D}\mathcal{F}_S$, we have a natural morphism $X \to \mathrm{R}\,\mathcal{H}om(\mathrm{R}\,\mathcal{H}om(X, \mathbb{G}_m[1]), \mathbb{G}_m[1])$ (see [SGA 5, Exposé 1] after Proposition 1.6) as well as $X \to X_{\leq 0}$. Together, these yield the natural morphism

$$X_{\leq 0} \to \mathrm{R}\,\mathcal{H}om(\mathrm{R}\,\mathcal{H}om(X, \mathbb{G}_m[1])_{\leq 0}, \mathbb{G}_m[1])_{\leq 0},$$

which is the one we consider for $X = X_{\leq 0} = [M]$ in part (3) of the theorem.

1.12. The uniqueness of the functor $(-)^\vee$ can be shown by a simple dévissage argument. Its existence is in essence the construction of the dual 1-motive of [Deligne 1974, §10.2.11] combined with the following observations (1) and (2):

(1) If X is either a finite flat group scheme, a torus, or a lattice over S, then the sheaf $\mathcal{H}om(X, \mathbb{G}_m)$ is represented by the Cartier dual of X, and $\mathcal{E}xt^1(X, \mathbb{G}_m) = 0$.

(2) If A is an abelian scheme over S, the sheaves $\mathcal{H}om(A, \mathbb{G}_m)$ and $\mathcal{E}xt^2(A, \mathbb{G}_m)$ are trivial, and $\mathcal{E}xt^1(A, \mathbb{G}_m)$ is represented by the dual abelian scheme $A^\vee$.

(3) For all $i \geq 2$, the sheaves $\mathcal{E}xt^i(X, \mathbb{G}_m)$ and $\mathcal{E}xt^i(A, \mathbb{G}_m)$ are torsion. If ℓ is invertible on S, these sheaves contain no ℓ-torsion.

In the case X is a finite flat group scheme, the statements of (1) can be found in [Oort 1966, Theorem III.16.1]. For locally constant group schemes and tori, these follow from [SGA 3 II, Exposé XIII, Corollaire 1.4] and [SGA 7 I, Exposé VIII, Proposition 3.3.1], respectively. In (2), we have $\mathcal{H}om(A, \mathbb{G}_m) = 0$ because A is proper and geometrically connected, and $\mathbb{G}_m$ is affine. The isomorphism $\mathcal{E}xt^1(A, \mathbb{G}_m) \cong A^\vee$ is given by the classical Barsotti–Weil formula [Oort 1966, Theorem III.18.1].[1] It is shown in [Breen 1969a] that (over a noetherian regular base scheme as we suppose S to be) the sheaves $\mathcal{E}xt^i(A, \mathbb{G}_m)$ are torsion for all $i > 1$. Using the second statement of (1), we see that for $n \neq 0$ the multiplication-by-n map on $\mathcal{E}xt^2(A, \mathbb{G}_m)$ is injective; hence, $\mathcal{E}xt^2(A, \mathbb{G}_m) = 0$. Finally, the torsion sheaves $\mathcal{E}xt^i(X, \mathbb{G}_m)$ and $\mathcal{E}xt^i(A, \mathbb{G}_m)$ contain no ℓ-torsion because if F is a finite flat group scheme over S annihilated by ℓ, then $\mathcal{E}xt^i(F, \mathbb{G}_m) = 0$ for all $i \geq 1$. Indeed, such a group scheme is locally constant and locally presented as $0 \to \mathbb{Z}^r \to \mathbb{Z}^r \to F \to 0$, and the functor $\mathcal{H}om(\mathbb{Z}^r, -)$ is exact.

The reason why we need the truncation operations in Theorem 1.11(1) is that in general we do not know whether the sheaves $\mathcal{E}xt^i(F, \mathbb{G}_m)$ vanish for $i > 1$ if F is a finite flat group scheme over S that is not locally constant. This is presumably not the case, as an explicit example of Breen [1969b] suggests (he works with sheaves for the étale topology, but it seems that his example also works in the fppf setting). Over a field of characteristic 0, or after inverting all residual characteristics of S, the truncation is not needed.

[1] The additional hypothesis that either A is projective over S or that S is artinian is superfluous. The trouble is caused only by Oort's Proposition I.5.3, where representability of the Picard functor $T \mapsto \operatorname{Pic} A/T$ is known just in these cases. This problem has been overcome by M. Raynaud [Faltings and Chai 1990, Theorem 1.9].

Oort proves that the Barsotti–Weil formula over a general scheme holds if it holds over all of its residue fields. He then says that the formula is known to hold over any field and quotes Serre's *Groupes algébriques et corps de classes*, VII.16, Théorème 6. But Serre makes right at the beginning of Chapter VII the hypothesis that the ground field is algebraically closed. Hence, as long as all residue fields of S are perfect, Oort's proof is fine. The general case follows by checking that Serre's arguments also work verbatim over separably closed fields.

Proposition 1.13. *Let M be a 1-motive over S with dual $M^\vee$, and let $n \geq 1$ be an integer that is invertible on S. The Cartier dual of the finite flat group scheme $H^1([M] \otimes^{\mathbb{L}} \mathbb{Z}/n\mathbb{Z})$ is naturally isomorphic to $H^1([M^\vee] \otimes^{\mathbb{L}} \mathbb{Z}/n\mathbb{Z})$. In particular, there is a canonical, natural isomorphism $\mathrm{T}_\ell(M^\vee) \cong \mathcal{H}om(\mathrm{T}_\ell M, \mathbb{Z}_\ell(1))$ of ℓ-adic sheaves on S for every prime number ℓ invertible on S.*

Proof. This follows from Theorem 1.11 and the statement (3) of Section 1.12. □

2. The pairing between $\text{Ш}^0(k, M)$ and $\text{Ш}^2(k, M^\vee)$

We fix number field k with algebraic closure $\bar{k}$ and write Ω for the set of all places of k. For $v \in \Omega$, we denote by k_v the completion of k at v. After recalling the definition of Tate–Shafarevich groups, we use the Poitou–Tate duality theorem for finite Galois modules to identify the ℓ-torsion part of $\text{Ш}^2(k, M^\vee)$ with the dual of $\text{Ш}^1(k, \mathrm{T}_\ell M)$ for any 1-motive $M = [Y \to G]$ over k. Then we show that the group $\text{Ш}^0(k, M)$ is finite and that its ℓ-part canonically injects into $\text{Ш}^1(k, \mathrm{T}_\ell M)$.

2.1. Let C be a bounded complex of continuous $\mathrm{Gal}(\bar{k}\,|\,k)$-modules. The Tate–Shafarevich groups $\text{Ш}^i(k, C)$ of C are defined by

$$\text{Ш}^i(k, C) := \ker\left(H^i(k, C) \to \prod_{v \in \Omega} H^i(k_v, C)\right),$$

where H^i is continuous cochain cohomology with the convention that for archimedean v the group $H^i(k_v, C) = H^i(\mathrm{Gal}(\mathbb{C}\,|\,k_v), C)$ stands for Tate modified cohomology [Neukirch et al. 2000, I§2]. The Tate–Shafarevich groups $\text{Ш}^i(k, M)$ of a 1-motive $M = [Y \to G]$ over k are those of the complex of discrete Galois modules $Y(\bar{k}) \to G(\bar{k})$ placed in degrees -1 and 0.

Proposition 2.2. *Let M be a 1-motive over k, and let ℓ be a prime number. There is a canonical, perfect pairing of topological groups*

$$\text{Ш}^1(k, \mathrm{T}_\ell M) \times \text{Ш}^2(k, M^\vee)[\ell^\infty] \to \mathbb{Q}/\mathbb{Z}$$

where $\text{Ш}^1(k, \mathrm{T}_\ell M)$ is profinite and $\text{Ш}^2(k, M^\vee)[\ell^\infty]$ is discrete. In particular, $\text{Ш}^2(k, M^\vee)[\ell^\infty]$ is finite or zero if and only if $\text{Ш}^1(k, \mathrm{T}_\ell M)$ is so.

Proof. By Poitou–Tate duality for finite Galois modules [Neukirch et al. 2000, Theorem 8.6.8], we have a natural, perfect duality between finite groups

$$\text{Ш}^1(k, M \otimes^{\mathbb{L}} \mathbb{Z}/\ell^i\mathbb{Z}) \times \text{Ш}^2(k, M^\vee \otimes^{\mathbb{L}} \mathbb{Z}/\ell^i\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z},$$

noting Proposition 1.13. The functor $\text{Ш}^2(k, -)$ commutes with arbitrary colimits, and $\text{Ш}^1(k, -)$ commutes with limits of finite Galois modules by [Serre 1964, Proposition 7]. We obtain thus, without violating the notational conventions from Section 1.6, a perfect pairing of topological groups

$$Ш^1(k, \mathrm{T}_\ell M) \times Ш^2(k, \mathrm{B}_\ell M^\vee) \to \mathbb{Q}/\mathbb{Z},$$

and it remains to show that $Ш^2(k, \mathrm{B}_\ell M^\vee)$ is canonically isomorphic to the ℓ-part of the torsion group $Ш^2(k, M^\vee)$. Indeed, from Proposition 1.9 we get the following commutative diagram of torsion groups with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^1(k, M^\vee) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell & \longrightarrow & H^2(k, \mathrm{B}_\ell M^\vee) & \longrightarrow & H^2(k, M^\vee)[\ell^\infty] & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \prod_{v\in\Omega} H^1(k_v, M^\vee) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell & \to & \prod_{v\in\Omega} H^2(k_v, \mathrm{B}_\ell M^\vee) & \to & \prod_{v\in\Omega} H^2(k_v, M^\vee)[\ell^\infty] & \to & 0 \end{array}$$

Because $H^1(k, M^\vee)$ and $H^1(k_v, M^\vee)$ are torsion, the first terms of both rows are zero; hence, the canonical isomorphism $Ш^2(k, \mathrm{B}_\ell M^\vee) \cong Ш^2(k, M^\vee)[\ell^\infty]$, as required. □

2.3. Let M be a 1-motive over k, and let ℓ be a prime number. From Proposition 1.9, we get a commutative diagram of $\mathbb{Z}_\ell$-modules with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^0(k, M) \widehat{\otimes} \mathbb{Z}_\ell & \longrightarrow & H^1(k, \mathrm{T}_\ell M) & \longrightarrow & \mathrm{T}_\ell H^1(k, M) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha_\ell} & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \prod_{v\in\Omega} H^0(k_v, M) \widehat{\otimes} \mathbb{Z}_\ell & \longrightarrow & \prod_{v\in\Omega} H^1(k_v, \mathrm{T}_\ell M) & \longrightarrow & \prod_{v\in\Omega} \mathrm{T}_\ell H^1(k_v, M) & \longrightarrow & 0 \end{array}$$

The kernel of the rightmost vertical map is the Tate module of $Ш^1(k, M)$, which is torsion free, and even trivial if $Ш^1(k, M)$ is finite (which conjecturally is always the case; compare [Harari and Szamuely 2005, Corollary 4.9]). In any case, the map $\ker \alpha_\ell \to Ш^1(k, \mathrm{T}_\ell M)$ is an isomorphism on torsion elements. The kernel of α_ℓ contains $Ш^0(k, M) \otimes \mathbb{Z}_\ell$ and hence an injection

$$Ш^0(k, M) \otimes \mathbb{Z}_\ell \to Ш^1(k, \mathrm{T}_\ell M). \tag{†}$$

In [Harari and Szamuely 2005, Section 5], a profinite group $Ш^0_\wedge(k, M)$ was introduced. Its pro-ℓ part is $\ker \alpha_\ell$ and hence equal to $Ш^1(k, \mathrm{T}_\ell M)$ in the case $Ш^1(k, A)$ is finite. This relates Proposition 5.1 of [loc. cit.] to our Proposition 2.2. There is a canonical isomorphism $Ш^0(k, M) \otimes \mathbb{Z}_\ell \cong Ш^0(k, M)[\ell^\infty]$ because $Ш^0(k, M)$ is finite as we shall see in Proposition 2.5. These observations yield the following corollary to Proposition 2.2:

Corollary 2.4. *The pairing of Proposition 2.2 induces a pairing*

$$Ш^0(k, M)[\ell^\infty] \times Ш^2(k, M^\vee)[\ell^\infty] \to \mathbb{Q}/\mathbb{Z},$$

which is nondegenerate on the left. Its right kernel is divisible if and only if the map (†) *induces an isomorphism* $Ш^0(k, M) \otimes \mathbb{Z}_\ell \to Ш^1(k, \mathrm{T}_\ell M)_{\mathrm{tor}}$, *and this pairing is a perfect pairing of finite groups if and only if the map* (†) *is an isomorphism.*

We end the section with the following proposition, which explains the group $Ш^0(k, M)$ and shows that it is finite [Harari and Szamuely 2005, Lemma 4.11]:

Proposition 2.5. *Let* $M = [u : Y \to G]$ *be a* 1*-motive over* k, *and set* $Z := \ker u$. *The morphism of* 1*-motives* $[Z \to 0] \to [Y \to G]$ *induces an isomorphism of finite groups*

$$Ш^1(k, Z) \xrightarrow{\cong} Ш^0(k, M).$$

For any prime number ℓ, *there are canonical isomorphisms of finite groups*

$$Ш^1(k, Z)[\ell^\infty] \cong Ш^1(k, Z) \otimes \mathbb{Z}_\ell \cong Ш^1(k, Z \otimes \mathbb{Z}_\ell) \cong Ш^1(k, \mathrm{T}_\ell[Z \to 0]).$$

All these groups are annihilated by the order of any finite Galois extension $k' \,|\, k$ *over which* Z *is constant.*

Proof. By diagram chase, using one finite place $v \in \Omega$, we see that the map $Ш^0(k, M) \to Ш^1(k, Y)$ is injective. It follows in particular that $Ш^0(k, M)$ is zero if the Galois action on Y is trivial. In general, let $k' \,|\, k$ be a finite Galois extension such that $\mathrm{Gal}(\bar{k} \,|\, k')$ acts trivially on Y, and let Ω' be the set of places of k'. For $w \in \Omega'$ lying over $v \in \Omega$, we write k'_w for the completion of k' at w and k_w for the completion of k at v. As a Galois module, $Z := \ker(u)$ can be interpreted as $Z = H^{-1}(k', M)$. From the Hochschild–Serre spectral sequence, we get then a commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & H^1(\mathrm{Gal}(k' \,|\, k), Z) \arrow[r] \arrow[d] & H^0(k, M) \arrow[r] \arrow[d] & H^0(k', M) \arrow[d] \\
0 \arrow[r] & \prod_{w \in \Omega'} H^1(\mathrm{Gal}(k'_w \,|\, k_w), Z) \arrow[r] & \prod_{w \in \Omega'} H^0(k_w, M) \arrow[r] & \prod_{w \in \Omega'} H^0(k'_w, M)
\end{tikzcd}$$

Because $\mathrm{Gal}(\bar{k} \,|\, k')$ acts trivially on Y, we have $Ш^0(k', M) = 0$ by our previous observation; hence,

$$Ш^0(k, M) \cong \ker\Bigl(H^1(\mathrm{Gal}(k' \,|\, k), Z) \to \prod_{w \in \Omega'} H^1(\mathrm{Gal}(k'_w \,|\, k_w), Z) \Bigr),$$

the product running over all $w \in \Omega'$ or alternatively over all decomposition subgroups of $\mathrm{Gal}(k' \,|\, k)$. The finiteness statement follows as $H^1(\mathrm{Gal}(k' \,|\, k), Z)$ is finite and annihilated by the order of $\mathrm{Gal}(k'|k)$ [Weibel 1994, Theorem 6.5.8 and Corollary 6.5.10]. Repeating the arguments for the 1-motive $[Z \to 0]$ yields the first isomorphism of the proposition.

Now let ℓ be a prime number. The first isomorphism from the left exists for any finite commutative group in place of $Ш^1(k, Z)$. For the next one, choose a finite Galois extension $k'|k$ such that Z is constant over k'. We can proceed as before in order to express $Ш^1(k, Z)$ and $Ш^1(k, Z \otimes \mathbb{Z}_\ell)$ in terms of cohomology groups of the finite group $\mathrm{Gal}(k'|k)$ and its subgroups. It remains to show that, given a finite group Γ acting on Z, the canonical map $H^1(\Gamma, Z) \otimes \mathbb{Z}_\ell \to H^1(\Gamma, Z \otimes \mathbb{Z}_\ell)$ is an isomorphism. This is indeed so for any flat $\mathbb{Z}$-algebra in place of $\mathbb{Z}_\ell$ by the universal coefficient theorem. The last isomorphism holds because $Z \otimes \mathbb{Z}_\ell \cong \lim Z/\ell^i Z$ and because limits are left exact and commute with continuous H^1. □

3. Lie algebra cohomology of the Tate module

We fix a number field k with algebraic closure $\bar{k}$ and a prime number ℓ. With every 1-motive M over k is associated a continuous $\mathbb{Q}_\ell$-linear representation $V_\ell M = \mathrm{T}_\ell M \otimes \mathbb{Q}_\ell$ of $\mathrm{Gal}(\bar{k}|k)$. The image of $\mathrm{Gal}(\bar{k}|k)$ in $\mathrm{GL}(\mathrm{V}_\ell M)$ is an ℓ-adic Lie group L^M, whose Lie algebra we denote by $\mathfrak{l}^M$. An idea going back to Serre and Tate, used by Serre [1964] to solve the congruence subgroup problem for abelian varieties over number fields, is to consider the vector space $H^1_*(\mathfrak{l}^M, \mathrm{V}_\ell M)$ consisting of those elements of $H^1(\mathfrak{l}^M, \mathrm{V}_\ell M)$ that restrict to zero on each one-dimensional subalgebra of $\mathfrak{l}^M$. Our goal is to describe $H^1_*(\mathfrak{l}^M, \mathrm{V}_\ell M)$.

We will work only with 1-motives $M = [Y \to G]$ where G is either an abelian variety or a torus rather than a general semiabelian variety. This brings considerable simplifications in both statements and proofs. I will comment at the end of Section 3.11 on this hypothesis and on the modifications that are necessary in order to compute $H^1_*(\mathfrak{l}^M, \mathrm{V}_\ell M)$ for general 1-motives. The following theorem is the crucial ingredient for our finiteness results:

Theorem 3.1. *Let $M = [u := Y \to G]$ be a 1-motive over k where G is an abelian variety or a torus. Set $E_\ell := \mathrm{End}_{\bar{k}}(G) \otimes \mathbb{Q}_\ell$ and $X_\ell := \mathrm{im}(u) \otimes \mathbb{Q}_\ell$, denote by D_ℓ the E_ℓ-submodule of $G(\bar{k}) \otimes \mathbb{Q}_\ell$ generated by X_ℓ, and define*

$$\overline{X}_\ell := \{x \in D_\ell \mid f(x) \in f(X_\ell) \text{ for all } f \in \mathrm{Hom}_{E_\ell}(D_\ell, \mathrm{V}_\ell G)\}.$$

There is a canonical isomorphism of $\mathbb{Q}_\ell$-vector spaces $\overline{X}_\ell / X_\ell \cong H^1_*(\mathfrak{l}^M, \mathrm{V}_\ell M)$.

The proof of this theorem relies on a structure result for the Lie algebra $\mathfrak{l}^M$, which in turn relies on Faltings's theorems on endomorphisms of abelian varieties over number fields. Observe that the object $\overline{X}_\ell / X_\ell$ can be calculated by means of ordinary linear algebra. The statement of the theorem is wrong for general semiabelian varieties G.

3.2. We recall some definitions and results from [Serre 1964]. Let L be a profinite group, and let T be a continuous L-module. We write $H^1(L, T)$ for the group of continuous cocycles $L \to T$ modulo coboundaries and define

$$H^1_*(L, T) := \ker\Bigl(H^1(L, T) \to \prod_{x \in L} H^1(\langle x \rangle, T)\Bigr),$$

where $\langle x \rangle$ denotes the closed subgroup of L generated by x. If N is a closed normal subgroup of L acting trivially on T, then the inflation map induces an isomorphism $H^1_*(L/N, T) \to H^1_*(L, T)$ [loc. cit., Proposition 6]. If T is a profinite L-module, say $T = \lim T_i$ where the T_i are finite discrete L-modules, then the canonical map $H^1(L, T) \to \lim H^1(L, T_i)$ is an isomorphism [loc. cit., Proposition 7]. Because the limit functor is left exact, also the canonical map $H^1_*(L, T) \to \lim H^1_*(L, T_i)$ is an isomorphism in that case.

For a Lie algebra $\mathfrak{l}$ acting on a vector space V, we denote by $H^1_*(\mathfrak{l}, V)$ the subspace of $H^1(\mathfrak{l}, V)$ consisting of those elements that restrict to zero in $H^1(\langle x \rangle, V)$ for every one-dimensional subalgebra $\langle x \rangle$ of $\mathfrak{l}$.

Lemma 3.3. *Let L be a compact ℓ-adic Lie group with Lie algebra $\mathfrak{l}$ acting on a finite-dimensional $\mathbb{Q}_\ell$-vector space V. For any open subgroup N of L, we have*

$$H^1_*(L, V) = \ker\Bigl(H^1(L, V) \to \prod_{x \in N} H^1(\langle x \rangle, V)\Bigr).$$

If N is normal, there is a canonical isomorphism $H^1_*(L, V) \cong H^1_*(N, V)^{L/N}$. If N is sufficiently small, there is a canonical isomorphism $H^1_*(N, V) \cong H^1_*(\mathfrak{l}, V)$.

Proof. Let N be an open subgroup of L, and let c be an element of $H^1(L, V)$ restricting to zero in $H^1(\langle x \rangle, V)$ for each $x \in N$. Fix an element $x \in L$, and let us show that c restricts to zero in $H^1(\langle x \rangle, V)$. Because $\langle x \rangle$ is compact, the quotient $\langle x \rangle/(N \cap \langle x \rangle)$ is finite. By a restriction-corestriction argument and using that V is uniquely divisible, we see that the restriction map $H^1(\langle x \rangle, V) \to H^1(\langle x \rangle \cap N, V)$ is injective, hence the first claim. Now suppose that N is open and normal. Since L is compact, the quotient L/N is finite and we have $H^i(L/N, V) = 0$ for all $i > 0$, and the Hochschild–Serre spectral sequence yields an isomorphism $H^1(L, V) \cong H^1(N, V)^{L/N}$. We must show that in the diagram

$$\begin{tikzcd}
0 \arrow[r] & H^1_*(L, V) \arrow[r] \arrow[d] & H^1(L, V) \arrow[r] \arrow[d, "\cong"'] \arrow[dr, dashed, "\delta"] & \prod_{x \in L} H^1(\langle x \rangle, V) \arrow[d] \\
0 \arrow[r] & H^1_*(N, V)^{L/N} \arrow[r] & H^1(N, V)^{L/N} \arrow[r] & \prod_{x \in N} H^1(\langle x \rangle, V)
\end{tikzcd}$$

the leftmost vertical map is an isomorphism, i.e., that the kernel of the diagonal map δ is exactly $H^1_*(L, V)$. But this is again the first statement of the lemma. Finally, if N is sufficiently small, we have an isomorphism $H^1(N, V) \cong H^1(\mathfrak{l}, V)$ by a well-known theorem of Lazard [1965, V.2.4.10], from which the last statement follows. $\square$

3.4. Let $M = [u : Y \to G]$ be a 1-motive over k where G is an abelian variety or a torus. The Tate module $\mathrm{T}_\ell M$ of M is an extension of $Y \otimes \mathbb{Z}_\ell$ by the Tate module $\mathrm{T}_\ell G$ of G as we have seen in Section 1.7, so we get an extension continuous of Galois representations

$$0 \to \mathrm{V}_\ell G \to \mathrm{V}_\ell M \to Y \otimes \mathbb{Q}_\ell \to 0.$$

We denote by $\mathfrak{l}^M$ and $\mathfrak{l}^G$ the Lie algebras of the image of $\Gamma := \mathrm{Gal}(\bar{k} \mid k)$ in the groups $\mathrm{GL}(\mathrm{V}_\ell M)$ and $\mathrm{GL}(\mathrm{V}_\ell G)$, respectively. The Galois group Γ acts continuously on these Lie algebras by conjugation, and we have a canonical surjection $\mathfrak{l}^M \to \mathfrak{l}^G$. Let $\mathfrak{n}^M$ denote its kernel, so $\mathfrak{n}^M$ consists of those elements of $\mathfrak{l}^M$ that act trivially on $\mathrm{V}_\ell G$. The Lie algebra $\mathfrak{n}^M$ is commutative, and we can identify it with a linear subspace of $\mathrm{Hom}(Y \otimes \mathbb{Q}_\ell, \mathrm{V}_\ell G)$ via the map

$$\vartheta : \mathfrak{n}^M \to \mathrm{Hom}_{\mathbb{Q}_\ell}(Y \otimes \mathbb{Q}_\ell, \mathrm{V}_\ell G)$$

given by $\vartheta(x)(y) = x.v$ where $v \in \mathrm{V}_\ell M$ is any element mapping to $y \in Y \otimes \mathbb{Q}_\ell$. Routine checking shows that this map is well-defined, injective, and Γ-equivariant. We can describe the image of ϑ as follows. Look at u as being a k-rational point on the abelian variety or torus $\mathcal{H}om(Y, G)$, and denote by B the connected component of the smallest algebraic subgroup of $\mathcal{H}om(Y, G)$ containing u. Then B is also an abelian variety or a torus, and we have an inclusion

$$\mathrm{V}_\ell B \subseteq \mathrm{V}_\ell\, \mathcal{H}om(Y, G) \cong \mathrm{Hom}_{\mathbb{Q}_\ell}(Y \otimes \mathbb{Q}_\ell, \mathrm{V}_\ell G).$$

The following theorem is a special case of [Jossen 2013b, Theorem 6.2]. In the case G is an abelian variety, it goes back to a result of Ribet [1976] (see [Hindry 1988, Appendix 2]).

Theorem 3.5. *The map ϑ induces an isomorphism of Galois representations $\vartheta : \mathfrak{n}^M \xrightarrow{\cong} \mathrm{V}_\ell B$.*

In particular, it follows that the dimension of $\mathfrak{n}^M$ is independent of ℓ. If G is an abelian variety, it is still unknown whether the dimension of $\mathfrak{l}^G$ is independent of ℓ.

Lemma 3.6. *Let $M = [u : Y \to G]$ be a 1-motive over k where G is an abelian variety or a torus. Denote by D the $\mathrm{End}_{\bar{k}}(G)$ submodule of $G(\bar{k})$ generated by $\mathrm{im}(u)$, and define $B \subseteq \mathcal{H}om(Y, G)$ as in Section 3.4. The linear map*

$$h : \mathrm{Hom}_{\bar{k}}(B, G) \otimes \mathbb{Q} \to G(\bar{k}) \otimes \mathbb{Q}$$

given by $h(\psi \otimes 1) = \psi(nu) \otimes n^{-1}$, *where* $n \geq 1$ *is any integer such that* $nu \in B(k)$, *induces an isomorphism* $\mathrm{Hom}_{\bar{k}}(B, G) \otimes \mathbb{Q} \cong D \otimes \mathbb{Q}$.

Proof. The homomorphism h is injective. Indeed, if $\psi : B \to G$ is such that $h(\psi) = 0$, then $\ker \psi$ is a subgroup of B containing a nonzero multiple of u; hence, $\ker \psi = B$ by minimality of B. By Poincaré's complete reducibility theorem [Mumford 1970, IV.19, Theorem 1], the inclusion $B \subseteq \mathcal{H}om(Y, G)$ induces a surjection

$$Y \otimes \mathrm{End}_{\bar{k}}(G) \cong \mathrm{Hom}_{\bar{k}}(\mathcal{H}om(Y, G), G) \xrightarrow{\text{res}} \mathrm{Hom}_{\bar{k}}(B, G)$$

sending $y \otimes \varphi$ to the unique homomorphism $\psi : B \to G$ with $\psi(nu) = n\varphi(u(y))$ where $n \geq 1$ is sufficiently big that $nu \in B(k)$, so the remaining statements follow. $\square$

Lemma 3.7. *Let the* 1*-motive* $M = [Y \to G]$, *the subgroup* $D \subseteq G(\bar{k})$, *and the algebraic subgroup* $B \subseteq G$ *be as in Lemma 3.6. There is a commutative diagram*

$$\begin{tikzcd}
Y \otimes \mathbb{Q}_\ell \arrow[r, "u\otimes\mathrm{id}"] \arrow[d, "\cong"] & D \otimes \mathbb{Q}_\ell \arrow[r, "(1)", "\cong"'] \arrow[d, "(2)"', "\cong"] & \mathrm{Hom}_{\mathfrak{l}^G}(\mathrm{V}_\ell B, \mathrm{V}_\ell G) \arrow[d, "\cong"', "\vartheta^*"] \\
H^0(\mathfrak{l}^M, Y \otimes \mathbb{Q}_\ell) \arrow[r, "\partial"] & H^1(\mathfrak{l}^M, \mathrm{V}_\ell G) \arrow[r, "\text{res}", "\cong"'] & \mathrm{Hom}_{\mathfrak{l}^G}(\mathfrak{n}^M, \mathrm{V}_\ell G)
\end{tikzcd}$$

with canonical isomorphisms where indicated.

Proof. We start with the left-hand square. The leftmost vertical isomorphism is tautological because $\mathfrak{l}^M$ acts trivially on $Y \otimes \mathbb{Q}_\ell$. The map ∂ is the connecting morphism in the long exact cohomology sequence coming from the weight filtration of M. The vector spaces $D \otimes \mathbb{Q}_\ell$ and $H^1(\mathfrak{l}^M, \mathrm{V}_\ell G)$ are naturally E_ℓ-modules, where $E_\ell := \mathrm{End}_{\bar{k}}(G) \otimes \mathbb{Q}_\ell$ — the first one by definition and the second one via the canonical action of E_ℓ on $\mathrm{V}_\ell G$. The map (2) is then given by E_ℓ-linearity and sending $u(y) \otimes 1$ to $\partial(y \otimes 1)$ for all $y \in Y$. By definition of D, this indeed describes a unique map such that the left-hand square commutes. We will see in a moment that it is well-defined and an isomorphism.

We now come to the right-hand square, starting with the description of the map (1). Every element of $D \otimes \mathbb{Q}_\ell$ is a linear combination of elements of the form $\psi(u) \otimes 1$ for some $\psi \in \mathrm{Hom}_{\bar{k}}(B, G)$ by Lemma 3.6. The map (1) is given by linearity and sends $\psi(u) \otimes 1$ to the $\mathfrak{l}^G$-equivariant map $\mathrm{V}_\ell \psi : \mathrm{V}_\ell B \to \mathrm{V}_\ell G$. This map is an isomorphism by Lemma 3.6 and by Faltings's theorem on homomorphisms of abelian varieties. The rightmost vertical map is given by precomposition with the isomorphism ϑ, hence an isomorphism. The lower horizontal map is given by restriction of cocycles and an isomorphism because $H^i(\mathfrak{l}^G, \mathrm{V}_\ell G)$ vanishes for $i = 1, 2$ [Serre 1971, Théorème 2].

By definition of ϑ, the big square commutes. Moreover, the isomorphisms (1) and ϑ^* and the inverse of res are all isomorphisms of E_ℓ-modules. Hence, so is their composition, which is then an isomorphism of E_ℓ-modules $D\otimes\mathbb{Q}_\ell \to H^1(\mathfrak{l}^M, \mathrm{V}_\ell G)$, which must coincide with (2). $\square$

Proposition 3.8 [Jossen 2013a, Corollary 2.19]. *The Lie algebra extension*

$$0 \to \mathfrak{n}^M \to \mathfrak{l}^M \to \mathfrak{l}^G \to 0$$

is split. There exist a Lie algebra section $\sigma : \mathfrak{l}^G \to \mathfrak{l}^M$ and a $\mathbb{Q}_\ell$-linear section $s : Y \otimes \mathbb{Q} \to \mathrm{V}_\ell M$ such that the action of $\mathfrak{l}^M$ on $\mathrm{V}_\ell M$ is given by

$$(f + \sigma(g)).(v + s(y)) = g.v + f.y$$

for all $f \in \mathfrak{n}^M$, all $g \in \mathfrak{l}^G$, all $v \in \mathrm{V}_\ell A$, and $y \in Y \otimes \mathbb{Q}_\ell$.

Proof. This is essentially a consequence of Theorem 3.5, semisimplicity of $\mathrm{V}_\ell G$ as $\mathfrak{l}^G$-module (by the assumption on G and [Faltings 1983]), and the vanishing of $H^i(\mathfrak{l}^G, \mathrm{V}_\ell G)$ for $i = 1, 2$ [Serre 1971, Théorème 2]. $\square$

Lemma 3.9. *Let $M = [u := Y \to G]$ be a 1-motive over k where G is an abelian variety or a torus. In order that an element $h \in H^1(\mathfrak{l}^M, \mathrm{V}_\ell G)$ belongs to $H^1_*(\mathfrak{l}^M, \mathrm{V}_\ell M)$, it suffices that it maps to zero in $H^1(\mathfrak{c}, \mathrm{V}_\ell M)$ for each one-dimensional subalgebra $\mathfrak{c}$ of $\mathfrak{n}^M$.*

Proof. Represent $h \in H^1(\mathfrak{l}^M, \mathrm{V}_\ell G)$ by a cocycle $c : \mathfrak{l}^M \to \mathrm{V}_\ell G \subseteq \mathrm{V}_\ell M$, and choose a linear section $s : Y \otimes \mathbb{Q}_\ell \to \mathrm{V}_\ell M$ and a Lie algebra section $\sigma : \mathfrak{l}^G \to \mathfrak{l}^M$ as in Proposition 3.8. Since $H^1(\mathfrak{l}^G, \mathrm{V}_\ell G)$ vanishes [Serre 1971, Théorème 2], the cocycle $c \circ \sigma$ is a coboundary. Thus, changing c by a coboundary, we may suppose that $c \circ \sigma = 0$. Let $\mathfrak{c}$ be a one-dimensional subalgebra of $\mathfrak{l}^M$ generated by an element $x \in \mathfrak{l}^M$. We have to show that there exists an element $v \in \mathrm{V}_\ell M$ such that $c(x) = x.v$. We can write x as $x = f + \sigma(g)$ for some $f \in \mathfrak{n}^M$ and $g \in \mathfrak{l}^G$. By hypothesis, there exists an element $v \in \mathrm{V}_\ell M$ with $c(f) = f.v$. We can write v as $v = v' + s(y)$ for some $v' \in \mathrm{V}_\ell G$ and $y \in Y \otimes \mathbb{Q}_\ell$. We then have

$$c(x) = c(f + \sigma(g)) = c(f) = f.v = f.s(y) = (\sigma(g) + f).s(y) = x.s(y),$$

and this proves the lemma. $\square$

Lemma 3.10. *Let $M = [u : Y \to G]$ be a 1-motive over k where G is an abelian variety or a torus. Denote by D the $E := \mathrm{End}_{\bar{k}}(G)$ submodule of $G(\bar{k})$ generated by $X := \mathrm{im}(u)$. The isomorphism $D \otimes \mathbb{Q}_\ell \to H^1(\mathfrak{l}^M, \mathrm{V}_\ell G)$ from Lemma 3.7 induces an isomorphism between the kernels of the maps*

$$D \otimes \mathbb{Q}_\ell \to \prod_{h \in H} \mathrm{V}_\ell G / h(X \otimes \mathbb{Q}_\ell) \quad \textit{and} \quad H^1(\mathfrak{l}^M, \mathrm{V}_\ell G) \to \prod_{\mathfrak{c} \subseteq \mathfrak{l}^M} H^1(\mathfrak{c}, \mathrm{V}_\ell M)$$

where the leftmost product runs over all $h \in H := \mathrm{Hom}_{E\otimes\mathbb{Q}_\ell}(D \otimes \mathbb{Q}_\ell, \mathrm{V}_\ell G)$.

Proof. Lemma 3.9 shows that if on the right side we let the product only run over $\mathfrak{c} \in \mathfrak{n}^M$ we still get the same kernel. For every $\mathfrak{c} = \langle x \rangle \subseteq \mathfrak{n}^M$, we have

$$H^1(\mathfrak{c}, \mathrm{V}_\ell M) \cong \frac{\mathrm{V}_\ell M}{\{x.v \mid v \in \mathrm{V}_\ell M\}} = \frac{\mathrm{V}_\ell M}{\mathrm{im}(\vartheta(x))},$$

where $\vartheta : \mathfrak{n} \to \mathrm{Hom}(Y \otimes \mathbb{Q}, \mathrm{V}_\ell G)$ is as defined in Section 3.4. The map

$$H^1(\mathfrak{l}^M, \mathrm{V}_\ell G) \to \mathrm{Hom}_{\mathfrak{l}^G}(\mathfrak{n}^M, \mathrm{V}_\ell G)$$

given by restriction of cocycles is an isomorphism; thus, we have to show that the kernels of the maps

$$D \otimes \mathbb{Q}_\ell \to \prod_{h \in H} \mathrm{V}_\ell G / h(X \otimes \mathbb{Q}_\ell) \quad \text{and} \quad \mathrm{Hom}_{\mathfrak{l}^G}(\mathfrak{n}^M, \mathrm{V}_\ell G) \to \prod_{x \in \mathfrak{n}^M} \mathrm{V}_\ell G / \mathrm{im}(\vartheta(x))$$

correspond under the isomorphism $D \otimes \mathbb{Q}_\ell \cong \mathrm{Hom}_{\mathfrak{l}^G}(\mathfrak{n}^M, \mathrm{V}_\ell G)$ sending $\psi(u) \otimes 1$ to $\mathrm{V}_\ell \psi \circ \vartheta$ for all $\psi \in \mathrm{Hom}_{\bar{k}}(B, G)$. Here, $B \subseteq \mathcal{H}om(Y, G)$ is defined as in Theorem 3.5. The map on the right sends an $\mathfrak{l}^G$-module homomorphism $c : \mathfrak{n}^M \to \mathrm{V}_\ell G$ to the class of $c(x)$ in the factor corresponding to x. As for the map on the left, by Lemma 3.7, we may as well take $\mathrm{Hom}_{\mathfrak{l}^G}(\mathrm{V}_\ell B, \mathrm{V}_\ell G)$ in place of $D \otimes \mathbb{Q}_\ell$ as the domain. Then we must show that the kernels of the maps

$$\mathrm{Hom}_{\mathfrak{l}^G}(\mathrm{V}_\ell B, \mathrm{V}_\ell G) \to \prod_h \mathrm{V}_\ell G / f(X \otimes \mathbb{Q}_\ell)$$

and

$$\mathrm{Hom}_{\mathfrak{l}^G}(\mathfrak{n}^M, \mathrm{V}_\ell G) \to \prod_{x \in \mathfrak{n}^M} \mathrm{V}_\ell G / \mathrm{im}(\vartheta(x))$$

correspond to each other via composition with the isomorphism $\vartheta : \mathfrak{n}^M \to \mathrm{V}_\ell B$, the first of these products now running over all $E \otimes \mathbb{Q}_\ell$-module morphisms $h : \mathrm{Hom}_{\mathfrak{l}^G}(\mathrm{V}_\ell B, \mathrm{V}_\ell G) \to \mathrm{V}_\ell G$. The canonical map

$$\mathrm{V}_\ell B \xrightarrow{\cong} \mathrm{Hom}_{E \otimes \mathbb{Q}_\ell}(\mathrm{Hom}_{\mathfrak{l}^G}(\mathrm{V}_\ell B, \mathrm{V}_\ell G), \mathrm{V}_\ell G), \qquad v \mapsto [f \mapsto f(v)]$$

is an isomorphism by Schur's lemma, so all these $E \otimes \mathbb{Q}_\ell$-module homomorphisms h are given by evaluation in an element $v \in \mathrm{V}_\ell B$. If h is the evaluation in $v = \vartheta(x)$ for some $x \in \mathfrak{n}^M$, then $h(X \otimes \mathbb{Q}_\ell) = \mathrm{im}\, \vartheta(x)$, hence the claim of the lemma. □

Proof of Theorem 3.1. We consider the following diagram, where the exact row is induced by the weight filtration on the $\mathfrak{l}^M$-module $\mathrm{V}_\ell M$ and where the column is exact by definition:

$$
\begin{tikzcd}
 & & H^1_*(\mathfrak{l}^M, \mathrm{V}_\ell M) \arrow[d, "\subseteq"] \arrow[dr, "0"] & \\
H^0(\mathfrak{l}^M, Y\otimes\mathbb{Q}_\ell) \arrow[r, "\partial"] & H^1(\mathfrak{l}^M, \mathrm{V}_\ell G) \arrow[r] \arrow[dr, "\delta"'] & H^1(\mathfrak{l}^M, \mathrm{V}_\ell M) \arrow[r] \arrow[d] & H^1(\mathfrak{l}^M, Y\otimes\mathbb{Q}_\ell) \\
 & & \prod_{\mathfrak{c}\subseteq\mathfrak{l}^M} H^1(\langle x\rangle, \mathrm{V}_\ell M) &
\end{tikzcd}
$$

The upper diagonal map is zero because $\mathfrak{l}^M$ acts trivially on $Y\otimes\mathbb{Q}_\ell$; hence, $H^1_*(\mathfrak{l}^M, Y\otimes\mathbb{Q}_\ell)$ is trivial. This shows that every element of $H^1_*(\mathfrak{l}^M, \mathrm{V}_\ell M)$ comes from an element in $H^1(\mathfrak{l}^M, \mathrm{V}_\ell G)$, so we find an isomorphism

$$\ker\delta/\operatorname{im}\partial \cong H^1_*(\mathfrak{l}^M, \mathrm{V}_\ell M)$$

induced by the inclusion $\mathrm{V}_\ell G\subseteq \mathrm{V}_\ell M$. Lemmas 3.7 and 3.10 respectively show that the isomorphism $D\otimes\mathbb{Q}_\ell\to H^1(\mathfrak{l}^M, \mathrm{V}_\ell G)$ induces isomorphisms $X\otimes\mathbb{Q}_\ell\cong\operatorname{im}\partial$ and

$$
\begin{aligned}
\overline{X}_\ell &:= \bigl\{x\in D\otimes\mathbb{Q}_\ell \bigm| f(x)\in f(X\otimes\mathbb{Q}_\ell) \text{ for all } f\in\operatorname{Hom}_{E\otimes\mathbb{Q}_\ell}(D\otimes\mathbb{Q}_\ell, \mathrm{V}_\ell G)\bigr\} \\
&\cong \ker\delta,
\end{aligned}
$$

as needed. □

3.11. Throughout this section, we have always supposed that the semiabelian variety G is either an abelian variety or a torus. Most statements and constructions, notably Theorems 3.1 and 3.5, remain true if G is isogenous to a product of an abelian variety and a torus, and the proofs require only small additional arguments, but the statements are wrong for general semiabelian varieties. The main problem here is that a general semiabelian variety G is not a semisimple object, so the analogue of Poincaré's complete reducibility theorem fails, and the Galois representation $\mathrm{V}_\ell G$ is not semisimple either.

In a general setting, the Lie algebra $\mathfrak{n}^M$ should be replaced by the subalgebra of $\mathfrak{l}^M$ consisting of those elements of which act trivially on $\mathrm{V}_\ell A$ and $\mathrm{V}_\ell T$, where A and T are respectively the abelian and torus parts of M. This is then in general not a commutative but just a nilpotent Lie algebra. The generalisation of Theorem 3.5 is [Jossen 2013b, Theorem 6.2]. The subgroup D of $G(\bar{k})$ has to be replaced by the group of so-called *deficient points* [loc. cit., Definition 6.2], and the generalisation of Lemma 3.6 is [loc. cit., Theorem 8.10]. Finally, E-linearity should be reformulated in terms of derivations. With these settings, it should be possible to generalise Theorem 3.1 to arbitrary 1-motives.

4. Finiteness results

In this section, we prove the finiteness statements of Theorem 3 stated in the introduction. We fix a number field k with algebraic closure $\bar{k}$ and a prime number ℓ and write $\Gamma := \mathrm{Gal}(\bar{k}\,|\,k)$ for the absolute Galois group of k and Ω for the set of all places of k. For a 1-motive M over k, we write $\mathrm{V}_\ell M := \mathrm{T}_\ell M \otimes \mathbb{Q}_\ell$ and denote by L^M the image of Γ in $\mathrm{GL}(\mathrm{T}_\ell M)$ and by $\mathfrak{l}^M \subseteq \mathrm{End}(\mathrm{V}_\ell M)$ the Lie algebra of L^M.

Theorem 4.1. *Let $M = [u : Y \to G]$ be a 1-motive over k. The $\mathbb{Z}_\ell$-module $Ш^1(k, \mathrm{T}_\ell M)$ is finitely generated. If G is an abelian variety or a torus, such that $\mathrm{End}_{\bar{k}}(G) \otimes \mathbb{Q}$ is a product of division algebras, then $Ш^1(k, \mathrm{T}_\ell M)$ is finite.*

Observe that if G is an abelian variety, then $\mathrm{End}_{\bar{k}}(G) \otimes \mathbb{Q}$ is a product of division algebras precisely if $G_{\bar{k}}$ is isogenous to a product of pairwise nonisogenous simple abelian varieties over $\bar{k}$. If G is a torus, then $\mathrm{End}_{\bar{k}}(G) \otimes \mathbb{Q}$ is a product of division algebras precisely if G is of dimension ≤ 1.

The plan of this section is as follows. First we show that for every 1-motive M over k there is a canonical injection of $Ш^1(k, \mathrm{T}_\ell M)$ into $H^1_*(L^M, \mathrm{T}_\ell M)$. We continue with some elementary linear algebra and prove, using Theorem 3.1, that the group $H^1_*(L^M, \mathrm{T}_\ell M)$, and hence $Ш^1(k, \mathrm{T}_\ell M)$, is finite for all ℓ if M is a 1-motive satisfying the condition in the theorem.

Proposition 4.2. *Let M be a 1-motive over k. There is a canonical injective $\mathbb{Z}_\ell$-linear map $Ш^1(k, \mathrm{T}_\ell M) \to H^1_*(L^M, \mathrm{T}_\ell M)$. The $\mathbb{Z}_\ell$-module $H^1_*(L^M, \mathrm{T}_\ell M)$ is finitely generated, and its rank is bounded by the dimension of $H^1_*(\mathfrak{l}^M, \mathrm{V}_\ell M)$.*

Proof. For every finite Galois module F, the subgroup $Ш^1(k, F)$ of $H^1(k, F) = H^1(\Gamma, F)$ is contained in $H^1_*(\Gamma, F)$ by [Serre 1964, Proposition 8], which is essentially a consequence of Chebotarev's density theorem. Because $H^1(k, -)$ commutes with limits of finite Galois modules and by left exactness of the limit functor, we can deduce that $Ш^1(k, \mathrm{T}_\ell M)$ is contained in $H^1_*(\Gamma, \mathrm{T}_\ell M)$, and $H^1_*(\Gamma, \mathrm{T}_\ell M)$ is isomorphic to $H^1_*(L^M, \mathrm{T}_\ell M)$ by [Serre 1964, Proposition 6], hence the canonical injection. By [loc. cit., Proposition 9], the $\mathbb{Z}_\ell$-module $H^1(L^M, \mathrm{T}_\ell M)$ is finitely generated, and we have an isomorphism of finite-dimensional vector spaces

$$H^1(L^M, \mathrm{T}_\ell M) \otimes \mathbb{Q}_\ell \cong H^1(L^M, \mathrm{V}_\ell M).$$

This identifies $H^1_*(L^M, \mathrm{T}_\ell M) \otimes \mathbb{Q}_\ell$ with a subspace of $H^1_*(L^M, \mathrm{V}_\ell M)$, which in turn is a subspace of $H^1_*(\mathfrak{l}^M, \mathrm{V}_\ell M)$ by Lemma 3.3. □

Lemma 4.3. *Let $K_1|K_0$ be an extension of fields (think of $\mathbb{Q}_\ell|\mathbb{Q}$). Let E_0 be a K_0-algebra, let D_0 and V_0 be E_0-modules, and let X_0 be a K_0-linear subspace of D_0. Denote by E_1, D_1, V_1, and X_1 the corresponding objects obtained by tensoring* *with K_1. Define*

$$\overline{X}_0 \colon= \{x \in D_0 \mid f(x) \in f(X_0) \textit{ for all } f \in \operatorname{Hom}_{E_0}(D_0, V_0)\},$$
$$\overline{X}_1 \colon= \{x \in D_1 \mid f(x) \in f(X_1) \textit{ for all } f \in \operatorname{Hom}_{E_1}(D_1, V_1)\}.$$

Then, the inclusion $\overline{X}_1 \subseteq \overline{X}_0 \otimes K_1$ holds. In particular, if the equality $X_0 = \overline{X}_0$ holds, then the equality $X_1 = \overline{X}_1$ holds as well.

Proof. Let x be an element of $\overline{X}_1 \subseteq D_1$, and let us show that x belongs to $\overline{X}_0 \otimes K_1$. Every E_0-linear map $D_0 \to V_0$ gives rise by K_1-linear extension to an E_1-linear map $D_1 \to V_1$, so by definition of $\overline{X}_1$, there exists in particular for every $f \in \operatorname{Hom}_{E_0}(D_0, V_0)$ an element $x^f \in X_1$ such that $f(x) = f(x^f)$. Let $(t_i)_{i \in I}$ be a K_0-basis of K_1, so we can write x and x^f as sums

$$x = \sum_{i \in I} x_i \otimes t_i \quad \text{and} \quad x^f = \sum_{i \in I} x_i^f \otimes t_i$$

for unique elements $x_i \in D_0$ and $x_i^f \in X_0$, almost all zero. We have to show that the x_i belong to $\overline{X}_0$ for all $i \in I$. The equality $f(x) = f(x^f)$ reads

$$\sum_{i \in I} f(x_i) \otimes t_i = \sum_{i \in I} f(x_i^f) \otimes t_i.$$

Linear independence of the t_i over K_0 implies that we have in fact $f(x_i) = f(x_i^f)$ for all i. Hence, for every $i \in I$ and every $f \in \operatorname{Hom}_{E_0}(D_0, V_0)$, we have $f(x_i) \in f(X_0)$, that is, $x_i \in \overline{X}_0$ as we wanted to show. As for the additional statement, if we have $X_0 = \overline{X}_0$, then the inclusions

$$X_0 \otimes K_1 \overset{\text{def}}{=} X_1 \subseteq \overline{X}_1 \subseteq \overline{X}_0 \otimes K_1$$

must all be equalities. □

Lemma 4.4. *Let K be a field of characteristic 0, let E be a finite product of finite-dimensional division algebras over K, let D and V be finite-dimensional E-modules, and suppose that V is faithful. Let X be a K-linear subspace of D. An element $v \in D$ belongs to X if and only if $f(v)$ belongs to $f(X)$ for all E-linear maps $f : D \to V$.*

Proof. We only show the case where E is a division algebra over K; the proof of the general case is similar. That V is faithful means then just that V is nonzero, and without loss of generality, we may suppose that V is E, so we are considering E-linear forms $f : D \to E$. Let $\operatorname{tr}_{E|K} : E \to K$ be a trace map, which for our purpose can be just any K-linear map with the property that

$$\operatorname{tr}_{E|K}(yx) = 0 \text{ for all } y \in E \implies x = 0.$$

Such a trace map always exists (see, e.g., [Gille and Szamuely 2006, Section 2.6]). Consider then the K-linear map

$$\mathrm{Hom}_E(D, E) \to \mathrm{Hom}_K(D, K), \quad f \mapsto \mathrm{tr}_{E|K} \circ f.$$

We claim that this is an isomorphism of K-vector spaces. We only have to show injectivity; surjectivity follows then by dimension-counting. To show injectivity, we can suppose that $D = E$. The above map sends then an E-linear endomorphism of E, which is just multiplication on the right by some $x \in E$, to the K-linear map $y \mapsto \mathrm{tr}(yx)$. If this map is zero, then x must be zero by the above property of the trace, hence injectivity. The hypothesis on v implies that

$$\mathrm{tr}_{E|K}\, f(v) \in \mathrm{tr}_{E|K}\, f(X)$$

for all $f \in \mathrm{Hom}_E(D, E)$; hence, $f(v) \in f(X)$ for all $f \in \mathrm{Hom}_K(D, K)$; hence, $v \in X$ by standard linear algebra. □

Proof of Theorem 4.1. By Proposition 4.2, it is enough to show that the vector space $H^1_*(\mathfrak{l}^M, \mathrm{V}_\ell M)$ is trivial. Set $E_\ell := \mathrm{End}_{\bar{k}}(G) \otimes \mathbb{Q}_\ell$ and $X_\ell := \mathrm{im}(u) \otimes \mathbb{Q}_\ell$, denote by D_ℓ the E_ℓ-submodule of $G(\bar{k}) \otimes \mathbb{Q}_\ell$ generated by X_ℓ, and define

$$\overline{X}_\ell := \{x \in D_\ell \mid f(x) \in f(X_\ell) \text{ for all } f \in \mathrm{Hom}_{E_\ell}(D_\ell, \mathrm{V}_\ell G)\}.$$

By Theorem 3.1, we have to check that the equality $X_\ell = \overline{X}_\ell$ holds. Fix an embedding of k into the field of complex numbers $\mathbb{C}$. Set $\mathrm{V}_0 G := H_1(G(\mathbb{C}), \mathbb{Q})$ and $E_0 := \mathrm{End}_{\bar{k}}(G) \otimes \mathbb{Q}$ and $X_0 := \mathrm{im}(u) \otimes \mathbb{Q}$, and denote by D_0 the E_0-submodule of $G(\bar{k}) \otimes \mathbb{Q}$ generated by X_0. Note that $\mathrm{V}_0 G$ is a faithful E_0-module and that there is a natural isomorphism $\mathrm{V}_\ell G \cong \mathrm{V}_0 G \otimes \mathbb{Q}_\ell$. By Lemma 4.3, it is now enough to check the equality $X_0 = \overline{X}_0$ for

$$\overline{X}_0 := \{x \in D_0 \mid f(x) \in f(X_0) \text{ for all } f \in \mathrm{Hom}_{E_0}(D_0, \mathrm{V}_0 G)\}.$$

By hypothesis, the $\mathbb{Q}$-algebra E_0 is a product of division algebras; hence, the equality $X_0 = \overline{X}_0$ indeed holds by Lemma 4.4. □

4.5. One can think of other linear algebra conditions on the objects E, D, V, and X than those in Lemma 4.4 that ensure the equality $X = \overline{X}$. For instance, the conclusion of the lemma holds true for any finite-dimensional semisimple algebra E over K and faithful V if X is of dimension ≤ 1 or if X is an E-submodule or D. One can conclude along the same lines that if $M = [u : Y \to G]$ is a 1-motive where G is an abelian variety or a torus, such that the image of u generates an $\mathrm{End}_{\bar{k}} \otimes \mathbb{Q}$-submodule of $G(\bar{k}) \otimes \mathbb{Q}$ or such that $u(Y)$ is of rank ≤ 1, then $Ш^1(k, \mathrm{T}_\ell M)$ is finite.

4.6. Our strategy of proving finiteness of $Ш^1(k, \mathrm{T}_\ell M)$ consisted of showing that the a priori larger group $H^1_*(L^M, \mathrm{T}_\ell M)$ is finite. This strategy does not succeed always; indeed, there exist 1-motives M such that the group $H^1_*(L^M, \mathrm{T}_\ell M)$ is infinite, yet $Ш^1(k, \mathrm{T}_\ell M)$ is finite. The point here is that $H^1_*(\Gamma, \mathrm{T}_\ell M)$ only sees the primes at which $\mathrm{T}_\ell M$ is unramified, whereas $Ш^1(k, \mathrm{T}_\ell M)$ sees all primes.

5. The torsion of $Ш^1(k, \mathrm{T}_\ell M)$

In this section, we complete the proof of Theorem 3 by examining the finite torsion part of the group $Ш^1(k, \mathrm{T}_\ell M)$. The key ingredient for this is the following lemma:

Lemma 5.1. *Let T be a finitely generated free $\mathbb{Z}_\ell$-module, and set $V := T \otimes \mathbb{Q}_\ell$. Let $D \subseteq L \subseteq \mathrm{GL}(T)$ be Lie subgroups with Lie algebras $\mathfrak{d}$ and $\mathfrak{l}$, respectively. If*

(1) *the set $\{\pi \circ x \mid x \in \mathfrak{l},\ \pi \in V^*\}$ is a linear subspace of V^* and*

(2) *for all open subgroups $H \subseteq L$ containing D the equality $T^H = T^L$ holds,*

then the map $r : H^1_*(L, T) \to H^1_*(D, T)$ given by restriction of cocycles is injective on torsion elements.

This generalises Lemma 4.1 in [Jossen 2013a], which we get back by taking for D the trivial group. In our application, T will be $\mathrm{T}_\ell M$ for a 1-motive M, L will be L^M, i.e., the image of $\Gamma := \mathrm{Gal}(\bar{k}\,|\,k)$ in $\mathrm{GL}(\mathrm{T}_\ell M)$, and D will be the image in $\mathrm{GL}(\mathrm{T}_\ell M)$ of a decomposition group $D_v \subseteq \Gamma$.

Proof of Lemma 5.1. Let $c : L \to T$ be a cocycle representing an element of order ℓ in $\ker(r)$, and let us show that c is a coboundary. Because c represents a torsion element in $H^1(L, T)$, its image in $H^1(L, V)$ is trivial. Thus, identifying T with a subset of V, there exists $v \in V$ such that $c(g) = gv - v$ for all $g \in L$. The cocycle c is a coboundary if v belongs to $v \in V^L + T$. In fact, we will show that

$$v \in (T + V^D) \cap (T + V^{\mathfrak{l}}). \tag{‡}$$

Since the restriction of c to D is a coboundary, there exists $t \in T$ such that $c(g) = gt - t$ for all $g \in D$; hence, $v - t \in V^D$, and so $v \in T + V^D$ as needed. To say that the cohomology class of c belongs to $H^1_*(L, T)$ is to say that for each $g \in L$ there exists an element $t_g \in T$ such that $c(g) = gt_g - t_g$. Let N be an open normal subgroup of $\mathfrak{l}$ on which the exponential map $\exp : N \to \mathfrak{l}$ is defined so that $V^{\langle g\rangle} = \ker(\exp(g))$ for all $g \in N$. We then have

$$v \in \bigcap_{g\in L}(T + V^{\langle g\rangle}) \subseteq \bigcap_{g\in N}(T + V^{\langle g\rangle}) = \bigcap_{x\in\mathfrak{l}}(T + \ker(x)).$$

Because of the hypothesis (1), Lemma 4.4 of [Jossen 2013a] applies, which yields $v \in T + \bigcap_{x\in\mathfrak{l}} \ker(x) = T + V^{\mathfrak{l}}$ and completes the proof of (‡).

By modifying v by an element of T, we may suppose without loss of generality that v belongs to V^D and in particular to $V^{\mathfrak{d}}$. The finite group $G := D/(N \cap D)$ acts on $V^{\mathfrak{d}}$ as well as on $V^{\mathfrak{l}}$. By Maschke's theorem, there exists a $\mathbb{Q}_\ell$-linear, G-equivariant retraction map $r : V^{\mathfrak{d}} \to V^{\mathfrak{l}}$ of the inclusion $V^{\mathfrak{l}} \to V^{\mathfrak{d}}$. Restricting r to $V^{\mathfrak{l}} + (T \cap V^{\mathfrak{d}})$, we find a decomposition of G-modules

$$V^{\mathfrak{l}} + (T \cap V^{\mathfrak{d}}) = V^{\mathfrak{l}} \oplus (\ker r \cap (T \cap V^{\mathfrak{d}})).$$

Writing $v = v_1 + t_1$ with $v_1 \in V^{\mathfrak{l}}$ and $t_1 \in \ker r \cap T \cap V^{\mathfrak{d}}$ according to this decomposition, we see that v_1 (and also t_1) is fixed under G because v is so; hence, we have

$$v \in (V^{\mathfrak{l}} \cap (V^{\mathfrak{d}})^G) + T = (V^N \cap V^D) + T = V^{ND} + T.$$

The subgroup ND of L is open and contains D; hence, $v \in V^L + T$ by hypothesis (2). □

Lemma 5.2. *Let $M = [u : Y \to G]$ be a 1-motive where G is an abelian variety or a torus such that $\mathrm{End}_{\bar{k}}(G) \otimes \mathbb{Q}$ is a product of division algebras. If the Galois action on Y is trivial, then $\text{Ш}^1(k, \mathrm{T}_\ell M)$ is trivial.*

Proof. For every finite Galois extension $k' \mid k$, we have

$$H^0(k, \mathrm{T}_\ell M) = H^0(k', \mathrm{T}_\ell M) \cong H^{-1}(M) \otimes \mathbb{Z}_\ell$$

by Corollary 1.10. Hence, we have $(\mathrm{T}_\ell M)^{L^M} = (\mathrm{T}_\ell M)^U$ for all open subgroups U of L^M. It follows from [Jossen 2013a, Propositions 3.1 and 3.2], which use the hypothesis on $\mathrm{End}_{\bar{k}}(G) \otimes \mathbb{Q}$ that the image of the bilinear map

$$\mathfrak{l}^M \times (\mathrm{V}_\ell M)^* \to (\mathrm{V}_\ell M)^*, \qquad (x, \pi) \mapsto \pi \circ x$$

is a linear subspace of $(\mathrm{V}_\ell M)^*$. The hypotheses of Lemma 5.1 are thus satisfied, and taking for D the trivial group, it shows that $H^1_*(L^M, \mathrm{T}_\ell M)$ is torsion-free. By Theorem 4.1, this group is also finite, hence trivial, and we conclude by Proposition 4.2. □

Proof of Theorem 3. Let $M = [u : Y \to G]$ be a 1-motive over k. We have constructed the pairing of the theorem and shown in Corollary 2.4 that it is nondegenerate on the left and in Proposition 2.5 that $\text{Ш}^0(k, M)$ is finite. Suppose then that G is an abelian variety or a torus such that $\mathrm{End}_{\bar{k}}(G) \otimes \mathbb{Q}$ is a product of division algebras. By Corollary 2.4, it remains to prove that the canonical map

$$\text{Ш}^0(k, M) \otimes \mathbb{Z}_\ell \to \text{Ш}^1(k, \mathrm{T}_\ell M) \tag{$\star$}$$

constructed in Section 2.3 is an isomorphism. We define $Z := H^{-1}(M) = \ker u$ and use Proposition 2.5 to identify $\text{Ш}^1(k, M) \otimes \mathbb{Z}_\ell$ with $\text{Ш}^1(k, Z \otimes \mathbb{Z}_\ell)$. Fix a finite Galois extension $k' \mid k$ over which Z is constant. For every place w of k', we write

k_w for the completion of k at the restriction of w to k, and $D_w := \mathrm{Gal}(k'_w \,|\, k_w)$. From the Hochschild–Serre spectral sequence, we get a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & H^1(\mathrm{Gal}(k'\,|\,k), Z \otimes \mathbb{Z}_\ell) \arrow[r] \arrow[d] & H^1(k, \mathrm{T}_\ell M) \arrow[r] \arrow[d] & H^1(k', \mathrm{T}_\ell M) \arrow[d] \\
0 \arrow[r] & \prod_w H^1(G_w, H^0(k'_w, \mathrm{T}_\ell M)) \arrow[r] & \prod_w H^1(k_w, \mathrm{T}_\ell M) \arrow[r] & \prod_w H^1(k'_w, \mathrm{T}_\ell M)
\end{tikzcd}$$

The rightmost vertical map is injective by Lemma 5.2; hence, every element of $Ш^1(k, \mathrm{T}_\ell M)$ comes from a unique element of $H^1(\mathrm{Gal}(k'\,|\,k), Z \otimes \mathbb{Z}_\ell)$, hence from $H^1(k, Z \otimes \mathbb{Z}_\ell)$. It remains to show that this element is in $Ш^1(k, Z \otimes \mathbb{Z}_\ell)$. To this end, we consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & H^1(k, Z \otimes \mathbb{Z}_\ell) \arrow[r] \arrow[d] \arrow[dr, dashed, "\delta"] & H^1(k, \mathrm{T}_\ell M) \arrow[d] \\
& \prod H^1(k_v, Z \otimes \mathbb{Z}_\ell) \arrow[r] & \prod H^1(k_v, \mathrm{T}_\ell M)
\end{tikzcd}$$

where the horizontal maps are induced by the morphism of 1-motives $[Z \to 0] \to [Y \to G]$. Injectivity of the top horizontal map follows from Corollary 1.10. We have thus $\ker \delta \cong Ш^1(k, \mathrm{T}_\ell M)$ and must show that every element of $\ker \delta$ maps already to zero in $H^1(k_v, Z \otimes \mathbb{Z}_\ell)$ for all $v \in \Omega$; that is, $\ker \delta = Ш^1(k, Z \otimes \mathbb{Z}_\ell)$.

Fix an element x of $Ш^1(k, \mathrm{T}_\ell M)$ and a place v, and let D_v be a decomposition group for v. We know that x comes via inflation from an element z of the finite group $H^1_*(L^M, \mathrm{T}_\ell M)$. Write D for the image of D_v in $\mathrm{GL}(\mathrm{T}_\ell M)$. This D is a Lie subgroup of L^M, and by hypothesis, z restricts to zero in $H^1(D, \mathrm{T}_\ell M)$. By Lemma 5.1 (using again [Jossen 2013a, Propositions 3.1 and 3.2]), we conclude that there is an open subgroup U of L^M containing D such that z is already zero in $H^1(U, \mathrm{T}_\ell M)$. This shows as well that there is an open subgroup Γ' of Γ containing D_v such that x maps to zero in $H^1(\Gamma', \mathrm{T}_\ell M)$. Consider then the diagram

$$\begin{tikzcd}
0 \arrow[r] & H^1(\Gamma, Z \otimes \mathbb{Z}_\ell) \arrow[r] \arrow[d] \arrow[dr, dashed, "\delta'"] & H^1(\Gamma, \mathrm{T}_\ell M) \arrow[d, "x \mapsto 0"] \\
0 \arrow[r] & H^1(\Gamma', Z \otimes \mathbb{Z}_\ell) \arrow[r] \arrow[d] & H^1(\Gamma', \mathrm{T}_\ell M) \arrow[d] \\
& H^1(D_v, Z \otimes \mathbb{Z}_\ell) \arrow[r] & H^1(D_v, \mathrm{T}_\ell M)
\end{tikzcd}$$

We know that the element $x \in Ш^1(k, \mathrm{T}_\ell M)$ comes from an element of $\ker \delta'$. The middle row is exact by Corollary 1.10 and because Γ' is the Galois group of a number field so that this element maps to zero in $H^1(\Gamma', Z \otimes \mathbb{Z}_\ell)$, hence in $H^1(D_v, Z \otimes \mathbb{Z}_\ell)$. $\square$

6. Tate 1-motives and Leopoldt's conjecture

In this section, we study the pairing of Theorem 3 in the case where M is a *Tate* 1-*motive* over k, that is, a 1-motive of the form $M = [\mathbb{Z}^r \to \mathbb{G}_m^s]$. I will show the following sharper version of Theorem 2 stated in the introduction:

Theorem 6.1. *Let k be a number field with ring of integers $\mathcal{O}_k$, and let ℓ be a prime number. If for every* 1*-motive of the form $M = [\mathbb{Z}^r \to \mathbb{G}_m^2]$ over* $\operatorname{spec}(\mathcal{O}_k)$ *the group* $Ш^2(k, M^\vee)[\ell^\infty]$ *is trivial, then the statement of Leopoldt's conjecture is true for k and ℓ.*

6.2. We work with the following formulation of Leopoldt's conjecture [Neukirch et al. 2000, Theorem 10.3.6(iii)]. For a finite prime $\mathfrak{p}$ of k, let $\mathcal{O}_{k,\mathfrak{p}}$ denote the ring of integers of the completion of k at $\mathfrak{p}$. There is a canonical map

$$i_\ell : \mathcal{O}_k^* \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \to \prod_{\mathfrak{p}|\ell} \mathcal{O}_{k,\mathfrak{p}}^* \mathbin{\widehat{\otimes}} \mathbb{Z}_\ell,$$

which on each component $i_\mathfrak{p} : \mathcal{O}_k^* \otimes \mathbb{Z}_\ell \to \mathcal{O}_{k,\mathfrak{p}} \mathbin{\widehat{\otimes}} \mathbb{Z}_\ell$ is obtained by applying $- \mathbin{\widehat{\otimes}} \mathbb{Z}_\ell$ to the inclusion $\mathcal{O}_k^* \subseteq \mathcal{O}_{k,\mathfrak{p}}^*$. Leopoldt's conjecture asserts that the map i_ℓ is injective. Note that i_ℓ is injective on torsion elements and injective if $\mathcal{O}_k^*$ is of rank ≤ 1.

Proof of Theorem 6.1. We suppose Leopoldt's conjecture is false for k and ℓ, so there exists a nontorsion element $z \in \ker(i_\ell) \subseteq \mathcal{O}_k^* \otimes_{\mathbb{Z}} \mathbb{Z}_\ell$, which we may write as

$$z = \sum_{i=1}^{n} \varepsilon_i \otimes \lambda_i,$$

where $n \geq 2$ is the rank of $\mathcal{O}_k^*$ and $\varepsilon_1, \ldots, \varepsilon_n$ are $\mathbb{Z}$-linearly independent elements of $\mathcal{O}_k^*$. By reordering the ε_i and replacing ε_1 by ε_1^{-1} if necessary, we may as well assume $\lambda_1 + \lambda_2 \neq 0$. We will now construct a 1-motive M of the form $M = [u : \mathbb{Z}^{2n-1} \to \mathbb{G}_m^2]$ over $\operatorname{spec}(\mathcal{O}_k)$ such that the group $Ш^1(k, \mathrm{T}_\ell M)$ is infinite. The 1-motive dual to M is then of the form $M^\vee = [\mathbb{Z}^2 \to \mathbb{G}_m^{2n-1}]$, and $Ш^2(k, M^\vee)$ will be infinite by Proposition 2.2. Let $Y \simeq \mathbb{Z}^{2n-1}$ be the group matrices

$$y = \begin{pmatrix} y_{11} & y_{12} & y_{13} & \cdots & y_{1n} \\ y_{21} & y_{22} & y_{23} & \cdots & y_{2n} \end{pmatrix}$$

with integer coefficient satisfying $y_{11} + y_{22} = 0$, and define the morphism u by

$$u(y) = \begin{pmatrix} y_{11}\varepsilon_1 + y_{12}\varepsilon_2 + y_{13}\varepsilon_3 + \cdots + y_{1n}\varepsilon_n \\ y_{21}\varepsilon_1 + y_{22}\varepsilon_2 + y_{23}\varepsilon_3 + \cdots + y_{2n}\varepsilon_n \end{pmatrix} \in \mathbb{G}_m^2(\mathcal{O}_k),$$

where we decided to write the group $\mathbb{G}_m(\mathbb{O}_k) = \mathbb{O}_k^*$ additively. So if ε denotes the column vector of the ε_i, we have just $u(y) = y\varepsilon$. We will prove the following lemma later:

Lemma 6.3. *For each $1 \le i \le n$, there exists $y \in Y$ such that $\binom{\varepsilon_i}{\varepsilon_i} \equiv u(y) \bmod \mathfrak{p}$ holds in $\mathbb{G}_m^2(\kappa_\mathfrak{p})$, where $\kappa_\mathfrak{p} =$ is the residue field at $\mathfrak{p}$.*

Set $U := \operatorname{spec}(\mathbb{O}_k[\ell^{-1}])$, and denote by c_i and c, respectively, the images of $\binom{\varepsilon_i}{\varepsilon_i} \otimes 1$ and $\binom{z}{z}$ under the composite map

$$\mathbb{G}_m^2(U) \otimes \mathbb{Z}_\ell \to H^0(U, M) \otimes \mathbb{Z}_\ell \to H^1(U, \mathrm{T}_\ell M)$$

where the first map is induced by the projection $\mathbb{G}_m^2(U) \to \mathbb{G}_m^2(U)/u(Y) \cong H^0(U, M)$ and the second map is the injection defined in Proposition 1.9. The $\mathbb{Z}_\ell$-submodules of $\mathbb{G}_m^2(\mathbb{O}_k) \otimes \mathbb{Z}_\ell$ generated by $\binom{z}{z}$ and by $u(Y)$ intersect trivially because $\lambda_1 + \lambda_2 \neq 0$; hence, c is of infinite order in $H^1(U, \mathrm{T}_\ell M) \subseteq H^1(k, \mathrm{T}_\ell M)$. I claim that c belongs to $Ш^1(k, \mathrm{T}_\ell M)$. Fix a place $\mathfrak{p}$ of k of residual characteristic p, and let us show that the restriction of c to $H^1(k_\mathfrak{p}, \mathrm{T}_\ell M)$ is zero. In the case $p = \ell$, this is true by construction, considering the commutative diagram

$$\begin{tikzcd}
(\mathbb{O}_k^*)^2 \otimes \mathbb{Z}_\ell \arrow[r] \arrow[d] & H^1(U, \mathrm{T}_\ell M) \arrow[d] \\
(\mathbb{O}_{k,\mathfrak{p}}^*)^2 \widehat{\otimes} \mathbb{Z}_\ell \arrow[r] & H^1(k_\mathfrak{p}, \mathrm{T}_\ell M)
\end{tikzcd}$$

and that the image of $\binom{z}{z}$ is already zero in $(\mathbb{O}_{k,\mathfrak{p}}^*)^2 \widehat{\otimes} \mathbb{Z}_\ell$. Suppose now that $p \neq \ell$, so $\mathrm{T}_\ell M$ is unramified at $\mathfrak{p}$. Because $c = \lambda_1 c_1 + \cdots + \lambda_n c_n$, it suffices to show that the restriction of each c_i to $H^1(\kappa_\mathfrak{p}, \mathrm{T}_\ell M)$ is zero. In view of the commutative diagram

$$\begin{tikzcd}
 & (\mathbb{O}_k^*)^2 \otimes \mathbb{Z}_\ell \arrow[r] \arrow[d] & H^1(U, \mathrm{T}_\ell M) \arrow[d] \\
Y \otimes \mathbb{Z}_\ell \arrow[r, "u \bmod \mathfrak{p}"] & (\kappa_\mathfrak{p}^*)^2 \otimes \mathbb{Z}_\ell \arrow[r] & H^1(\kappa_\mathfrak{p}, \mathrm{T}_\ell M)
\end{tikzcd}$$

this amounts to show that there exists $y \in Y$ such that $\binom{\varepsilon_i}{\varepsilon_i} \equiv u(y) \bmod \mathfrak{p}$ holds in $\mathbb{G}_m^2(\kappa_\mathfrak{p})$, which is what we claimed in Lemma 6.3. Hence, c belongs indeed to $Ш^1(k, \mathrm{T}_\ell M)$ and is of infinite order, and thus, $Ш^1(k, \mathrm{T}_\ell M)$ is infinite. □

Proof of Lemma 6.3. Fix $1 \le i \le n$ and a maximal ideal $\mathfrak{p}$ of $\mathbb{O}_k$ with residue field $\kappa_\mathfrak{p}$. We have to find a matrix $y \in Y$ such that $\binom{\varepsilon_i}{\varepsilon_i}$ is congruent to $u(y)$ modulo $\mathfrak{p}$. For $i \neq 1, 2$, such a y exists trivially. Let $J_1 \subseteq \mathbb{Z}$ be the ideal consisting of those $m \in \mathbb{Z}$ such that $m\varepsilon_1 \bmod \mathfrak{p}$ is in the subgroup of $\kappa_\mathfrak{p}^*$ generated by ε_2, and let $a_1 \ge 1$ be the positive generator of J_1. Similarly, define J_2 and a_2. There exist $b_1, b_2 \in \mathbb{Z}$ such that the linear dependence relations

$$a_1\varepsilon_1 + b_2\varepsilon_2 = 1 \quad \text{and} \quad b_1\varepsilon_1 + a_2\varepsilon_2 = 1$$

hold in the finite group $\kappa_{\mathfrak{p}}^*$, written additively. Note that b_i is a multiple of a_i. We claim that the integers a_1 and a_2 are coprime. Indeed, suppose there exists a prime ℓ dividing a_1 and a_2 so that we can write $a_i = \ell a_i'$ and $b_i = \ell b_i'$. Let Z be the subgroup of κ_ℓ^* generated by ε_1 and ε_2. Since $\kappa_{\mathfrak{p}}^*[\ell]$ is cyclic of order ℓ, we may suppose that $\kappa_{\mathfrak{p}}^*[\ell] \cap Z$ is contained in the subgroup of κ_ℓ^* generated by, say, ε_2. Thus, the point

$$T := a_1'\varepsilon_1 + b_2'\varepsilon_2 \in \kappa_{\mathfrak{p}}^*[\ell] \cap Z$$

can be written as $T = c\varepsilon_2$, and we get the relation $a_1'\varepsilon_1 + (b_2' - c)\varepsilon_2 = 1$, which contradicts the minimality of a_1. Therefore, a_1 and a_2 are coprime as claimed, and we can choose integers c_1 and c_2 such that $a_1c_1 + a_2c_2 = 1$. The matrices

$$y_1 = \begin{pmatrix} 1 - a_1c_1 & -c_1b_2 & 0 & \cdots & 0 \\ 1 - c_2b_1 & -a_2c_2 & 0 & \cdots & 0 \end{pmatrix} \quad \text{and} \quad y_2 = \begin{pmatrix} -a_1c_1 & 1 - c_1b_2 & 0 & \cdots & 0 \\ -c_2b_1 & 1 - a_2c_2 & 0 & \cdots & 0 \end{pmatrix}$$

belong to Y, and we have $\binom{\varepsilon_i}{\varepsilon_i} \equiv u(y_i) \bmod \mathfrak{p}$ as desired. □

Remark 6.4. If k is a number field whose group of global units $\mathbb{O}_k^*$ has rank ≤ 1, one can show that $Ш^1(k, \mathrm{T}_\ell M) = 0$ holds for every Tate 1-motive M over k. On the other hand, a construction analogous to the one used in the next section produces Tate 1-motives over particular number fields k with infinite $Ш^1(k, \mathrm{T}_\ell M)$.

7. A semiabelian variety with infinite $Ш^2$

In this section, we prove Theorem 1 by producing a semiabelian variety G over $\mathbb{Q}$ such that $Ш^2(\mathbb{Q}, G)$ contains $\mathbb{Q}/\mathbb{Z}$ as a subgroup and hence in particular is infinite. The technique is similar to that in the previous paragraph, and here we exploit now that for elliptic curves of sufficiently big rank the statement analogue to Leopoldt's conjecture trivially fails.

7.1. Let E be an elliptic curve over $\mathbb{Q}$ of rank at least 3, and let $P_1, P_2, P_3 \in E(\mathbb{Q})$ be $\mathbb{Z}$-linearly independent rational points. Let us write A for the abelian threefold E^3 over $\mathbb{Q}$ and Y for the group of 3×3 matrices of trace 0 with integer coefficients. Looking at $Y \simeq \mathbb{Z}^8$ as a Galois module with trivial Galois action, we consider the 1-motive

$$M = [u : Y \to A], \qquad u(y) = yP = \begin{pmatrix} y_{11}P_1 + y_{12}P_2 + y_{13}P_3 \\ y_{21}P_1 + y_{22}P_2 + y_{23}P_3 \\ y_{31}P_1 + y_{32}P_2 + y_{33}P_3 \end{pmatrix} \in E(\mathbb{Q})^3 = A(\mathbb{Q}).$$

The map u is injective, and I will use X as a shorthand for the group $u(Y) \subseteq A(\mathbb{Q})$. This 1-motive M is of special interest because it produces a counterexample to the so-called *problem of detecting linear dependence*: although $P \notin X$ and even $nP \notin X$ for all $n \neq 0$, there exists for every prime p where E has good reduction an element $x \in X$ such that P is congruent to x modulo p. The verification of this is similar to the proof of Lemma 6.3; see [Jossen and Perucca 2010]. Using Theorem 3.1, one shows that $H^1_*(\mathfrak{l}^M, \mathrm{V}_\ell M)$ is nontrivial — this is what makes the counterexample work and also how it was found in the first place.

7.2. I claim that the Tate–Shafarevich group in degree 2 of the semiabelian variety dual to the 1-motive M constructed in the previous paragraph contains a subgroup isomorphic to $\mathbb{Q}/\mathbb{Z}$. By Proposition 2.2, this amounts to say that for each prime number ℓ the Tate–Shafarevich group

$$Ш^1(\mathbb{Q}, \mathrm{T}_\ell M)$$

is of rank ≥ 1 as a $\mathbb{Z}_\ell$-module. Fix a prime ℓ, and let us denote by $[c_P]$ the cohomology class of $P \otimes 1$ via the injection $H^0(\mathbb{Q}, M) \otimes \mathbb{Z}_\ell \to H^1(\mathbb{Q}, \mathrm{T}_\ell M)$ from Proposition 1.9. A cocycle c_P representing $[c_P]$ is explicitly given by

$$c_P(\sigma) = (\sigma P_i - P_i)_{i=0}^{\infty},$$

where $(P_i)_{i=0}^{\infty}$ are elements of $A(\overline{\mathbb{Q}})$ such that $P_0 = P$ and $\ell P_i = P_{i-1}$. Up to a coboundary, c_P does not depend on the choice of the division points P_i. As the class $[P]$ of P in $H^0(\mathbb{Q}, M) \cong A(\mathbb{Q})/X$ is of infinite order, the element $[c_P] \in H^1(\mathbb{Q}, \mathrm{T}_\ell M)$ is of infinite order too. We claim that $n[c_P]$ belongs to $Ш^1(\mathbb{Q}, \mathrm{T}_\ell M)$ for some integer $n \geq 1$ (depending on ℓ). To check this, we must show that for every finite place p of $\mathbb{Q}$ the restriction of nc_P to a decomposition group D_p is a coboundary. In the case where $\ell = 2$ and $p = \infty$, we should also demand that the restriction of $n[c_P]$ to $H^1(\mathrm{Gal}(\mathbb{C}|\mathrm{R}), \mathrm{T}_\ell M)$ is zero, but we can ignore this by choosing n to be even. So from now on, we will stick to finite primes p only.

Lemma 7.3. *Let p be a prime, and let $D_p \subseteq \mathrm{Gal}(\overline{\mathbb{Q}} \,|\, \mathbb{Q})$ be a decomposition group at p. The restriction of c_P to D_p is a coboundary if and only if the class of P in $A(\mathbb{Q}_p)/X = H^0(\mathbb{Q}_p, M)$ is ℓ-divisible.*

Proof. Choose an algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$ and an embedding of $\overline{\mathbb{Q}}$ into $\overline{\mathbb{Q}}_p$ in such a way that the given decomposition group D_p equals $\mathrm{Gal}(\overline{\mathbb{Q}}|(\overline{\mathbb{Q}} \cap \mathbb{Q}_p))$. Consider the commutative diagram with exact rows

$$\begin{array}{ccccc}
0 \longrightarrow & H^0(\mathbb{Q}, M) \otimes \mathbb{Z}_\ell & \longrightarrow & H^1(\mathbb{Q}, \mathrm{T}_\ell M) \\
& \downarrow & & \downarrow \\
0 \longrightarrow & H^0(\mathbb{Q}_p, M) \mathbin{\widehat{\otimes}} \mathbb{Z}_\ell & \longrightarrow & H^1(\mathbb{Q}_p, \mathrm{T}_\ell M)
\end{array}$$

The restriction of c_P to D_p is a coboundary if and only if $[P]\otimes 1 \in H^0(\mathbb{Q}, M)\otimes\mathbb{Z}_\ell$ maps to zero in $H^0(\mathbb{Q}_p, M)\widehat{\otimes}\mathbb{Z}_\ell$, that is, if and only if the class of P in $H^0(\mathbb{Q}_p, M)$ is ℓ-divisible. □

Lemma 7.4. *For every prime p, the closure of X in $A(\mathbb{Q}_p)$ for the p-adic topology is an open subgroup of $A(\mathbb{Q}_p)$ of finite index.*

Proof. Because $E(\mathbb{Q}_p)$ has the structure of a compact p-adic Lie group of dimension 1, there exists an open subgroup of $E(\mathbb{Q}_p)$ isomorphic to $\mathbb{Z}_p$, and because $E(\mathbb{Q}_p)$ is compact, any such subgroup has finite index [Silverman 1986, Proposition 6.3]. We find thus a short exact sequence of profinite groups

$$0 \to \mathbb{Z}_p^3 \to A(\mathbb{Q}_p) \to F \to 0$$

for some finite group F. Let $m \geq 1$ be an integer annihilating F so that mX is contained in $\mathbb{Z}_p^3$. The elements

$$\begin{pmatrix} 0 \\ mP_1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ mP_2 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} mP_3 \\ 0 \\ 0 \end{pmatrix}$$

of $mX \subseteq \mathbb{Z}_p^3 \subseteq A(\mathbb{Q}_p)$ are linearly independent over $\mathbb{Z}_p$ because each $mP_i \in \mathbb{Z}_p$ is nonzero. The closure of mX in $\mathbb{Z}_p^3$ contains the $\mathbb{Z}_p$-submodule generated by these three points, hence is of finite index in $\mathbb{Z}_p^3$. We conclude that the closure of X in $A(\mathbb{Q})$ has finite index. Every closed subgroup of finite index is also open. □

7.5. We now come to the proof of the claims made in Section 7.2. First of all, let us choose an integer $n \geq 1$ such that the following conditions are met:

(0) If $\ell = 2$, then n is even.

(1) For every prime $p \neq \ell$ where E has bad reduction, the point nP is ℓ-divisible in $A(\mathbb{Q}_p)$.

(2) For $p = \ell$, the point nP belongs to the closure of X in $A(\mathbb{Q}_p)$ for the p-adic topology.

Such an integer n exists. Indeed, start with, say, $n = 2$, so condition (0) is satisfied. We have already observed that $A(\mathbb{Q}_p)$ is an extension of a finite discrete group F by $\mathbb{Z}_p^3$, so by replacing n by some sufficiently high multiple of n, we can assure that nP belongs to the subgroup $\mathbb{Z}_p^3$ of $A(\mathbb{Q}_p)$, which is ℓ-divisible. We do this for all the finitely many primes of bad reduction, so condition (1) is met. As for the last condition, we know that the closure of X in $A(\mathbb{Q}_p)$ has finite index by Lemma 7.4, so we again replace n by some sufficiently high multiple if necessary. In order to show that $n[c_P]$ belongs to $Ш^1(\mathbb{Q}, \mathrm{T}_\ell M)$, it remains to show by Lemma 7.3 that for each prime p the class of nP in $H^0(\mathbb{Q}, M) = A(\mathbb{Q}_p)/X$ is ℓ-divisible. In other words, we must show:

Claim. *For every $i \geq 0$, there exist elements $Q_i \in A(\mathbb{Q}_p)$ and $x_i \in X$ such that $\ell^i Q_i + x_i = nP$.*

We have already ruled out the case $p = \infty$, and for finite p, we will distinguish three cases: first, the case where p is a place of good reduction for E and $p \neq \ell$, second, the case where p is a place of bad reduction and $p \neq \ell$, and finally, the case $p = \ell$. All but finitely many primes p fall in the first case. For the finitely many primes that remain, the claim will hold by our particular choice of n.

Case 1 (good reduction at p and $p \neq \ell$). In this case, we can consider the surjective reduction map $\mathrm{red}_p : E(\mathbb{Q}_p) \to E(\mathbb{F}_p)$. Its kernel is isomorphic to $\mathbb{Z}_p$, so we get a short exact sequence

$$0 \to \mathbb{Z}_p^3 \to A(\mathbb{Q}_p) \xrightarrow{\mathrm{red}_p} A(\mathbb{F}_p) \to 0.$$

By [Jossen and Perucca 2010], there exists an element $x \in X$ such that $\mathrm{red}_p(P) = \mathrm{red}_p(x)$ in $A(\mathbb{F}_p)$. Because $\mathbb{Z}_p$ is uniquely ℓ-divisible, we can define $Q_i := \ell^{-i} n(P - x)$ and get $\ell^i Q_i + nx = nP$.

Case 2 (bad reduction at p and $p \neq \ell$). Condition (1) in Section 7.5 ensures that nP is ℓ-divisible in $A(\mathbb{Q}_p)$ for bad $p \neq \ell$, so the class of P in $A(\mathbb{Q}_p)/X$ is ℓ-divisible as well.

Case 3 ($p = \ell$). For all $i \geq 0$, the subgroup $p^i A(\mathbb{Q}_p)$ is open in $A(\mathbb{Q}_p)$; hence, by condition (2) in Section 7.5, the intersection $X \cap (nP + p^i A(\mathbb{Q}_p))$ is nonempty. But that means that there exists an element $Q_i \in A(\mathbb{Q}_p)$ and an element $x_i \in X$ such that $p^i Q_i + x_i = nP$, just as needed.

8. Open questions and problems

I present three open arithmetic questions and an elementary problem in linear algebra, which so far have defied all attempts of being solved. The first question is about how far finitely generated subgroups of a Mordell–Weil group are detectable by reduction maps. It is a sharpened version of the problem that in the literature is named the problem of *detecting linear dependence*.

Question 8.1. Let G be a semiabelian variety defined over a number field k, and let X be a finitely generated subgroup of $G(k)$. Denote by $\overline{X} \subseteq G(k)$ the subgroup of those points P such that for almost all finite primes $\mathfrak{p}$ of k the reduction $P \bmod \mathfrak{p}$ belongs to $X \bmod \mathfrak{p}$ in $G(\kappa_\mathfrak{p})$. Let $M = [u : Y \to G]$ be a 1-motive where Y is constant and $X = u(Y)$. Is it true that the map

$$\overline{X}/X \otimes \mathbb{Z}_\ell \to H^1_*(k, \mathrm{T}_\ell M)$$

induced by the injection $H^0(k, M) \otimes \mathbb{Z}_\ell \to H^1(k, \mathrm{T}_\ell M)$ is an isomorphism?

A positive answer to this question was given in [Jossen 2013a] in the case where G is a geometrically simple abelian variety. In this case, we know that $H^1_*(k, \mathrm{T}_\ell M)$ is trivial and get a nice local-global principle for subgroups of Mordell–Weil groups. Apart from a few other isolated examples, the question remains open, even in the cases where G is an abelian variety or a torus. The second question is similar in nature, but we impose a stronger local condition.

Question 8.2. Let G be a semiabelian variety defined over a number field k, let X be a finitely generated subgroup of $G(k)$, and let $P \in G(k)$ be a rational point. Suppose that for *all* finite primes $\mathfrak{p}$ of k the point P belongs to the closure of X in $G(k_\mathfrak{p})$ for the p-adic topology on $G(k_\mathfrak{p})$. Does then P belong to X?

We know the answer to be positive if G is a simple abelian variety and in some other scattered examples. If we could choose the integer n in Section 7.5 *independently of* ℓ, the answer to the question would be negative. Thirdly, I would like to ask for a converse to Theorem 6.1.

Question 8.3. Let k be a number field for which the statement of Leopoldt's conjecture holds. For which 1-motives $M = [Y \to G]$, where G is a torus, is the pairing

$$Ш^0(k, M) \times Ш^2(k, M^\vee) \to \mathbb{Q}/\mathbb{Z}$$

of Theorem 3 a perfect pairing of finite groups? In general, can we compute the dimension of $Ш^1(k, \mathrm{V}_\ell M)$? How does this dimension vary with ℓ?

The second part of this question can as well be formulated for general mixed Artin–Tate motives. At last, motivated by the proof of Theorem 4.1, let me state a problem in linear algebra that any first-year student can understand.

Problem. Let K be a field, and write E for the K-algebra of $n \times n$ matrices with coefficients in K. Denote by V and V_0 the E-modules of $n \times m$ and $n \times m_0$ matrices, respectively. Finally, let W be a K-linear subspace of V, and define

$$\overline{W} := \{v \in V \mid f(v) \in f(W) \text{ for all } f \in \mathrm{Hom}_E(V, V_0)\},$$

so $\overline{W}$ is a linear subspace of V containing W. Observe that elements of $\mathrm{Hom}_E(V, V_0)$ are just $m \times m_0$ matrices by Schur's lemma. The problem is to compute $\overline{W}$. This means find an algorithm that takes as an input a K-basis of W (this will be some finitely many $n \times m$ matrices) and provides a basis of $\overline{W}$ or equivalently provides some finitely many $f_1, \dots, f_r \in \mathrm{Hom}_E(V, V_0)$ such that

$$\overline{W} = \{v \in V \mid f_i(v) \in f_i(W) \text{ for } i = 1, 2, \dots, r\}.$$

Changing scalars from K to a bigger field may result in a smaller dimensional $\overline{W}$ (i.e., the inclusion of Lemma 4.3 may be strict). Yet, I don't know of a solution to the problem even in the case where K is algebraically closed.

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Triangulable $\mathcal{O}_F$-analytic (φ_q,Γ)-modules of rank 2

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The theory of (φ_q,Γ)-modules is a generalization of Fontaine's theory of (φ,Γ)-modules, which classifies G_F-representations on $\mathcal{O}_F$-modules and F-vector spaces for any finite extension F of $\mathbb{Q}_p$. In this paper following Colmez's method we classify triangulable $\mathcal{O}_F$-analytic (φ_q,Γ)-modules of rank 2. In the process we establish two kinds of cohomology theories for $\mathcal{O}_F$-analytic (φ_q,Γ)-modules. Using them, we show that if D is an étale $\mathcal{O}_F$-analytic (φ_q,Γ)-module such that $D^{\varphi_q=1,\Gamma=1}=0$ (i.e., $V^{G_F}=0$, where V is the Galois representation attached to D), then any overconvergent extension of the trivial representation of G_F by V is $\mathcal{O}_F$-analytic. In particular, contrary to the case of $F=\mathbb{Q}_p$, there are representations of G_F that are not overconvergent.

Introduction

This paper depends heavily on the theory of (φ,Γ)-modules for Lubin–Tate extensions, a generalization of Fontaine's theory of (φ,Γ)-modules. The existence of this generalization was more or less implicit in [Fontaine 1990; Colmez 2002]. See also [Fourquaux 2005; Scholl 2006, Remark 2.3.1]. Kisin and Ren [2009] provided details, where (φ,Γ)-modules for Lubin–Tate extensions are called (φ_q,Γ)-modules.

To recall this theory, let F be a finite extension of $\mathbb{Q}_p$, $\mathcal{O}_F$ the ring of integers in F, and π a uniformizer of $\mathcal{O}_F$. Fix an algebraic closure of F denoted by $\bar{F}$, and put $G_F=\mathrm{Gal}(\bar{F}/F)$. Let k_F be the residue field of F and set $q=\#k_F$. Let $\mathrm{W}=\mathrm{W}(k_F)$ be the ring of Witt vectors over k_F. Then $F_0:=\mathrm{W}[1/p]$ is the maximal absolutely unramified subfield of F. Let $\mathcal{F}$ be a Lubin–Tate group over F corresponding to the uniformizer π. Then $\mathcal{F}$ is a formal $\mathcal{O}_F$-module. Let X be a local coordinate on $\mathcal{F}$. Then the formal Hopf algebra $\mathcal{O}_{\mathcal{F}}$ may be identified with $\mathcal{O}_F[\![X]\!]$. For any $a\in\mathcal{O}_F$, let $[a]_{\mathcal{F}}\in\mathcal{O}_F[\![X]\!]$ be the power series giving the endomorphism a of $\mathcal{F}$. If $n\geq 1$, let $F_n\subset\bar{F}$ be the subfield generated by the π^n-torsion points of $\mathcal{F}$. Write $F_\infty=\bigcup_n F_n$, $\Gamma=\mathrm{Gal}(F_\infty/F)$ and $G_{F_\infty}=\mathrm{Gal}(\bar{F}/F_\infty)$. For any integer $n\geq 0$, let $\Gamma_n\subset\Gamma$ be the subgroup $\mathrm{Gal}(F_\infty/F_n)$. Let $T\mathcal{F}$ be the Tate module of $\mathcal{F}$. It

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is a free $\mathcal{O}_F$-module of rank 1. The action of G_F on $T\mathcal{F}$ factors through Γ and induces an isomorphism $\chi_{\mathcal{F}} : \Gamma \to \mathcal{O}_F^\times$. For any $a \in \mathcal{O}_F^\times$ we write $\sigma_a := \chi_{\mathcal{F}}^{-1}(a)$. Using the periods of $T\mathcal{F}$, one can construct a ring $\mathcal{O}_{\mathscr{E}}$ with actions of $\varphi_q = \varphi^{\log_p q}$ and Γ. We will recall the construction in Section 1. Kisin and Ren [2009] defined étale (φ_q, Γ)-modules over $\mathcal{O}_{\mathscr{E}}$ and classified G_F-representations on $\mathcal{O}_F$-modules in terms of these modules.

Here we are interested in triangulable $\mathcal{O}_F$-analytic (φ_q, Γ)-modules over a Robba ring $\mathscr{R}_L$, where L is a finite extension of F. A *triangulable* (φ_q, Γ)*-module* over $\mathscr{R}_L$ means a (φ_q, Γ)-module D that has a filtration consisting of (φ_q, Γ)-submodules $0 = D_0 \subset D_1 \subset \cdots \subset D_d = D$ such that D_i/D_{i-1} is free of rank 1 over $\mathscr{R}_L$.

In the spirit of [Colmez 2008] on the classification of triangulable (φ, Γ)-modules of rank 2, in the present paper we will classify triangulable $\mathcal{O}_F$-analytic (φ_q, Γ)-modules over $\mathscr{R}_L$ of rank 2. One motivation for doing this is our belief that under the hypothetical p-adic local Langlands correspondence these (φ_q, Γ)-modules should correspond to certain unitary principal series of $\mathrm{GL}_2(F)$. Colmez [2010a] and Liu, Xie, and Zhang [Liu et al. 2012] determined the spaces of locally analytic vectors of the unitary principal series of $\mathrm{GL}_2(\mathbb{Q}_p)$ based on this kind of (φ, Γ)-module. Our computations of dimensions of $\mathrm{Ext}^1_{\mathrm{an}}$ match those of [Kohlhaase 2011] on extensions of locally analytic representations. Nakamura [2009] gave a generalization of Colmez's work in another direction. But we think that Nakamura's point of view is probably not the best one for applications to the p-adic local Langlands correspondence.

For our purpose we consider two kinds of cohomology theories for $\mathcal{O}_F$-analytic (φ_q, Γ)-modules.

For a (φ_q, Γ)-module D over $\mathscr{R}_L$, we define $H^\bullet(D)$ by the cohomology of the semigroup $\varphi_q^{\mathbb{N}} \times \Gamma$ as in [Colmez 2010a]. Then the first cohomology group $H^1(D)$ is isomorphic to $\mathrm{Ext}(\mathscr{R}_L, D)$, the L-vector space of extensions of $\mathscr{R}_L$ by D in the category of (φ_q, Γ)-modules.

If D is $\mathcal{O}_F$-analytic, we consider the complex

$$C^\bullet_{\varphi_q,\nabla}(D): \quad 0 \to D \xrightarrow{f_1} D \oplus D \xrightarrow{f_2} D \to 0,$$

where $f_1 : D \to D \oplus D$ is the map defined as $m \mapsto ((\varphi_q - 1)m, \nabla m)$, and $f_2 : D \oplus D \to D$ is $(m, n) \mapsto \nabla m - (\varphi_q - 1)n$. The operator ∇ is defined in Section 1C. Put $H^i_{\varphi_q,\nabla}(D) := H^i(C^\bullet_{\varphi_q,\nabla}(D))$, for $i = 0, 1, 2$. Each of these modules admits a Γ-action. We set $H^i_{\mathrm{an}}(D) = H^i_{\varphi_q,\nabla}(D)^\Gamma$.

Theorem 0.1. *Let D be an $\mathcal{O}_F$-analytic (φ_q, Γ)-module over $\mathscr{R}_L$. Then there is a natural isomorphism* $\mathrm{Ext}_{\mathrm{an}}(\mathscr{R}_L, D) \to H^1_{\mathrm{an}}(D)$, *where* $\mathrm{Ext}_{\mathrm{an}}(\mathscr{R}_L, D)$ *is the L-vector space that consists of extensions of $\mathscr{R}_L$ by D in the category of $\mathcal{O}_F$-analytic (φ_q, Γ)-modules.*

The proof of Theorem 0.1 is given in Section 4; it is due to the referee, and is much simpler than that in our original version.

Theorem 0.2. *Let D be an $\mathcal{O}_F$-analytic (φ_q,Γ)-module over $\mathcal{R}_L$. The codimension of* $\mathrm{Ext}_{\mathrm{an}}(\mathcal{R}_L, D)$ *in* $\mathrm{Ext}(\mathcal{R}_L, D)$ *is* $([F:\mathbb{Q}_p]-1)\dim_L D^{\varphi_q=1,\Gamma=1}$. *In particular, if* $D^{\varphi_q=1,\Gamma=1}=0$, *then* $\mathrm{Ext}_{\mathrm{an}}(\mathcal{R}_L, D)=\mathrm{Ext}(\mathcal{R}_L, D)$.

To prove this, we will construct a (noncanonical) projection from $\mathrm{Ext}(\mathcal{R}_L, D)$ onto $\mathrm{Ext}_{\mathrm{an}}(\mathcal{R}_L, D)$ whose kernel is of dimension $([F:\mathbb{Q}_p]-1)\dim_L D^{\varphi_q=1,\Gamma=1}$.

If V is an overconvergent L-representation of G_F (in the sense of Definition 1.4), Δ is the (φ_q,Γ)-module over $\mathscr{E}_L^\dagger$ attached to V, and $D=\mathcal{R}_L\otimes_{\mathscr{E}_L^\dagger}\Delta$, then $\mathrm{Ext}(\mathcal{R}_L, D)$ measures the set of extensions of the trivial representation by V that are overconvergent (see Proposition 1.5 and Proposition 1.6). Theorem 0.2 tells us that if $V^{G_F}=D^{\varphi_q=1,\Gamma=1}=0$, then any such extension is $\mathcal{O}_F$-analytic.

Let $\mathscr{I}(L)$ (resp. $\mathscr{I}_{\mathrm{an}}(L)$) be the set of continuous (resp. locally F-analytic) characters $\delta: F^\times\to L^\times$. Let δ_{unr} denote the character of $F^\times$ such that $\delta_{\mathrm{unr}}(\pi)=q^{-1}$ and $\delta_{\mathrm{unr}}|_{\mathcal{O}_F^\times}=1$. Then δ_{unr} is a locally F-analytic character. If $\delta\in\mathscr{I}(L)$, let $\mathcal{R}_L(\delta)$ be the (φ_q,Γ)-module over $\mathcal{R}_L$ of rank 1 that has a basis e_δ such that $\varphi_q(e_\delta)=\delta(\pi)e_\delta$ and $\sigma_a(e_\delta)=\delta(a)e_\delta$. If $\delta\in\mathscr{I}_{\mathrm{an}}(L)$, then $\mathcal{R}_L(\delta)$ is $\mathcal{O}_F$-analytic.

For locally F-analytic characters we have the following:

Theorem 0.3. *For any $\delta\in\mathscr{I}_{\mathrm{an}}(L)$, we have*

$$\dim_L H^1_{\mathrm{an}}(\mathcal{R}_L(\delta))=\begin{cases}2 & \text{if } \delta=x^{-i},\ i\in\mathbb{N} \text{ or } x^i\delta_{\mathrm{unr}},\ i\in\mathbb{Z}_+,\\ 1 & \text{otherwise},\end{cases}$$

$$\dim_L H^1(\mathcal{R}_L(\delta))=\begin{cases}[F:\mathbb{Q}_p]+1 & \text{if } \delta=x^{-i},\ i\in\mathbb{N},\\ 2 & \text{if } \delta=x^i\delta_{\mathrm{unr}},\ i\in\mathbb{Z}_+,\\ 1 & \text{otherwise.}\end{cases}$$

For the proof of Theorem 0.3 we follow Colmez's method. Colmez [2008] used the theory of p-adic Fourier transform for $\mathbb{Z}_p$. For our case we use the p-adic Fourier transform for $\mathcal{O}_F$ developed by Schneider and Teitelbaum [2001] instead. But this transform can not be applied to our situation directly because, except for the case of $F=\mathbb{Q}_p$, it is defined over $\mathbb{C}_p$ and can not be defined over any finite extension L of F. We overcome this difficulty by applying it to $\mathcal{R}_{\mathbb{C}_p}$ and then descending certain results to $\mathcal{R}_L$. As a result, we obtain that if δ_1 and δ_2 are in $\mathscr{I}_{\mathrm{an}}(L)$, then $\mathcal{R}_L(\delta_1)^{\psi=0}$ and $\mathcal{R}_L(\delta_2)^{\psi=0}$ are isomorphic to each other as $L[\Gamma]$-modules. This is exactly what we need. In fact, we will show that $S_\delta:=(\mathcal{R}_Le_\delta/\mathcal{R}_L^+e_\delta)^{\psi=0,\Gamma=1}$ is 1-dimensional over L for any $\delta\in\mathscr{I}_{\mathrm{an}}(L)$, and that $H^1_{\mathrm{an}}(\mathcal{R}_L(\delta))$ is isomorphic to S_δ when $v_\pi(\delta(\pi))<1-v_\pi(q)$ and δ is not of the form x^i.

For characters that are not locally F-analytic we have the following:

Theorem 0.4. *For any $\delta\in\mathscr{I}(L)\backslash\mathscr{I}_{\mathrm{an}}(I)$ we have $H^1(\mathcal{R}_L(\delta))=0$. Consequently, every extension of $\mathcal{R}_L$ by $\mathcal{R}_L(\delta)$ splits.*

To state our result on the classification, we need some parameter spaces. These parameter spaces are analogues of Colmez's parameter spaces [Colmez 2008]. Let $\mathcal{S}$ be the analytic variety over $\mathcal{I}_{\mathrm{an}}(L) \times \mathcal{I}_{\mathrm{an}}(L)$ whose fiber over (δ_1, δ_2) is isomorphic to $\mathrm{Proj}(H^1(\delta_1\delta_2^{-1}))$, $\mathcal{S}_{\mathrm{an}}$ the analytic variety over $\mathcal{I}_{\mathrm{an}}(L) \times \mathcal{I}_{\mathrm{an}}(L)$ whose fiber over (δ_1, δ_2) is isomorphic to $\mathrm{Proj}(H^1_{\mathrm{an}}(\delta_1\delta_2^{-1}))$. There is a natural inclusion $\mathcal{S}_{\mathrm{an}} \hookrightarrow \mathcal{S}$. Let $\mathcal{S}_+$, $\mathcal{S}_+^{\mathrm{an}}$, $\mathcal{S}_+^{\mathrm{ng}}$, $\mathcal{S}_+^{\mathrm{cris}}$, $\mathcal{S}_+^{\mathrm{st}}$, $\mathcal{S}_+^{\mathrm{ord}}$ and $\mathcal{S}_+^{\mathrm{ncl}}$ be the subsets of $\mathcal{S}$ defined in Section 6. We can assign to any $s \in \mathcal{S}$ (resp. $s \in \mathcal{S}_{\mathrm{an}}$) a triangulable (resp. triangulable and $\mathcal{O}_F$-analytic) (φ_q, Γ)-module $D(s)$.

Theorem 0.5. (a) *For $s \in \mathcal{S}$, $D(s)$ is of slope zero if and only if s is in $\mathcal{S}_+ - \mathcal{S}_+^{\mathrm{ncl}}$; $D(s)$ is of slope zero and the Galois representation attached to $D(s)$ is irreducible if and only if s is in $\mathcal{S}_* - (\mathcal{S}_*^{\mathrm{ord}} \cup \mathcal{S}_*^{\mathrm{ncl}})$; $D(s)$ is of slope zero and $\mathcal{O}_F$-analytic if and only if s is in $\mathcal{S}_+^{\mathrm{an}} - \mathcal{S}_+^{\mathrm{ncl}}$.*

(b) *Let $s = (\delta_1, \delta_2, \mathcal{L})$ and $s' = (\delta_1', \delta_2', \mathcal{L}')$ be in $\mathcal{S}_+ - \mathcal{S}_+^{\mathrm{ncl}}$. If $\delta_1 = \delta_1'$, then $D(s) \cong D(s')$ if and only if $s = s'$. If $\delta_1 \neq \delta_1'$, then $D(s) \cong D(s')$ if and only if $s, s' \in \mathcal{S}_+^{\mathrm{cris}} \cup \mathcal{S}_+^{\mathrm{ord}}$, with $\delta_1' = x^{w(s)}\delta_2$, $\delta_2' = x^{-w(s)}\delta_1$.*

In the case when $F = \mathbb{Q}_p$, this becomes Colmez's result [Colmez 2008]. The proof of Theorem 0.5 will be given at the end of Section 6.

We give another application of Theorem 0.3. In the case of $F = \mathbb{Q}_p$ — the cyclotomic extension case — Cherbonnier and Colmez [1998] showed that all representations of $G_{\mathbb{Q}_p}$ are overconvergent. But our following result shows that this is not the case when $[F : \mathbb{Q}_p] \geq 2$.

Theorem 0.6. *Suppose that $[F : \mathbb{Q}_p] \geq 2$. Then there exist 2-dimensional L-representations of G_F that are not overconvergent (in the sense of Definition 1.4).*

By Kedlaya's theorem [2004], any (φ_q, Γ)-module of slope zero $D(s)$ in Theorem 0.5(a) comes from a 2-dimensional L-representation of G_F that is overconvergent.

We outline the structure of this paper. We recall Fontaine's rings, the theory of (φ_q, Γ)-modules and the relation between (φ_q, Γ)-modules and Galois representations in Section 1A and Section 1B, and then define $\mathcal{O}_F$-analytic (φ_q, Γ)-modules over the Robba ring $\mathcal{R}_L$ in Section 1C. We define ψ in Section 2A, and study the properties of ∂ and Res in Section 2B. In Section 3A we extend ψ to $\mathcal{R}_{\mathbb{C}_p}$, in Section 3B we define operators m_α on $\mathcal{R}_{\mathbb{C}_p}$, and then in Section 3C we study the Γ-action on $\mathcal{R}_L(\delta)^{\psi=0}$ for all $\delta \in \mathcal{I}_{\mathrm{an}}(L)$. The cohomology theories for $\mathcal{O}_F$-analytic (φ_q, Γ)-modules are given in Section 4. In Section 5 we compute $H^1_{\mathrm{an}}(\mathcal{R}_L(\delta))$ and $H^1(\mathcal{R}_L(\delta))$ for all $\delta \in \mathcal{I}_{\mathrm{an}}(L)$. After providing preliminary lemmas in Section 5A, we compute $H^0(\delta)$ for all $\delta \in \mathcal{I}(L)$ in Section 5B and $H^1_{\mathrm{an}}(\delta)$ for all $\delta \in \mathcal{I}_{\mathrm{an}}(L)$ satisfying $v_\pi(\delta(\pi)) < 1 - v_\pi(q)$ in Section 5C. For the purpose of computing $H^1_{\mathrm{an}}(\delta)$ for all $\delta \in \mathcal{I}_{\mathrm{an}}(L)$, we construct a transition map $\partial : H^1_{\mathrm{an}}(x^{-1}\delta) \to H^1_{\mathrm{an}}(\delta)$, which is done in Section 5D. The computation of $H^1_{\mathrm{an}}(\delta)$ is given in Section 5E.

In Section 5F we define two maps ι_k and $\iota_{k,\mathrm{an}}$. Applying results in Section 5, we classify triangulable $\mathcal{O}_F$-analytic (φ_q,Γ)-modules in Section 6.

1. (φ_q,Γ)-modules and $\mathcal{O}_F$-analytic (φ_q,Γ)-modules

In this section we recall the theory of (φ_q,Γ)-modules built in [Colmez 2002; Fourquaux 2005; Kisin and Ren 2009]. We keep using notation from the introduction.

1A. ***The rings of formal series.*** Put $\widetilde{\mathrm{E}}^+=\varprojlim \mathcal{O}_{\overline{F}}/p$ with the transition maps given by Frobenius, and let $\widetilde{\mathrm{E}}$ be the fractional field of $\widetilde{\mathrm{E}}^+$. We may also identify $\widetilde{\mathrm{E}}^+$ with $\varprojlim \mathcal{O}_{\overline{F}}/\pi$ with the transition maps given by the q-Frobenius $\varphi_q=\varphi^{\log_p q}$. Evaluation of X at π^∞-torsion points induces a map $\iota: T\mathcal{F}\to\widetilde{\mathrm{E}}^+$. Precisely, if $v=(v_n)_{n\geq 0}\in T\mathcal{F}$, with $v_n\in\mathcal{F}[\pi^n](\mathcal{O}_{\overline{F}})$, and $\pi\cdot v_{n+1}=v_n$, then we have $\iota(v)=(v_n^*(X)+\pi\mathcal{O}_{\overline{F}})_{n\geq 0}$.

Let $\{\,\cdot\,\}$ be the unique lifting map $\widetilde{\mathrm{E}}^+\to \mathrm{W}(\widetilde{\mathrm{E}}^+)_F:=\mathrm{W}(\widetilde{\mathrm{E}}^+)\otimes_{\mathcal{O}_{F_0}}\mathcal{O}_F$ such that $\varphi_q\{x\}=[\pi]_{\mathcal{F}}(\{x\})$ (see [Colmez 2002, Lemma 9.3]). When $\mathcal{F}$ is the cyclotomic Lubin–Tate group $\mathbb{G}_{\mathrm{m}}$, we have $\{x\}=[1+x]-1$, where $[1+x]$ is the Teichmüller lifting of $1+x$. This map respects the action of G_F. If $v\in T\mathcal{F}$ is an $\mathcal{O}_F$-generator, there is an embedding $\mathcal{O}_F[\![u_{\mathcal{F}}]\!]\hookrightarrow \mathrm{W}(\widetilde{\mathrm{E}}^+)_F$ sending $u_{\mathcal{F}}$ to $\{\iota(v)\}$ which identifies $\mathcal{O}_F[\![u_{\mathcal{F}}]\!]$ with a G_F-stable and φ_q-stable subring of $\mathrm{W}(\widetilde{\mathrm{E}}^+)_F$. The G_F-action on $\mathcal{O}_F[\![u_{\mathcal{F}}]\!]$ factors through Γ. By [Colmez 2002, Lemma 9.3] we have

$$\varphi_q(u_{\mathcal{F}})=[\pi]_{\mathcal{F}}(u_{\mathcal{F}}),\quad \sigma_a(u_{\mathcal{F}})=[a]_{\mathcal{F}}(u_{\mathcal{F}}).$$

In the case of $\mathcal{F}=\mathbb{G}_{\mathrm{m}}$, $u_{\mathcal{F}}$ is denoted by T in [Colmez 2008]. Here T is used to denote the Tate module of a Lubin–Tate group.

Let $\mathcal{O}_{\mathscr{E}}$ be the π-adic completion of $\mathcal{O}_F[\![u_{\mathcal{F}}]\!][1/u_{\mathcal{F}}]$. Then $\mathcal{O}_{\mathscr{E}}$ is a complete discrete valuation ring with uniformizer π and residue field $k_F((u_{\mathcal{F}}))$. The topology induced by this valuation is called the *strong topology*. Usually we consider the *weak topology* on $\mathcal{O}_{\mathscr{E}}$, i.e., the topology with $\{\pi^i\mathcal{O}_{\mathscr{E}}+u_{\mathcal{F}}^j\mathcal{O}_F[\![u_{\mathcal{F}}]\!]: i,j\in\mathbb{N}\}$, as a fundamental system of open neighborhoods of 0. Let $\mathscr{E}$ be the field of fractions of $\mathcal{O}_{\mathscr{E}}$. Let $\mathscr{E}^+$ be the subring $F\otimes_{\mathcal{O}_F}\mathcal{O}_F[\![u_{\mathcal{F}}]\!]$ of $\mathscr{E}$.

For any $r\in\mathbb{R}_+\cup\{+\infty\}$, let $\mathscr{E}^{]0,r]}$ be the ring of Laurent series $f=\sum_{i\in\mathbb{Z}}a_iu_{\mathcal{F}}^i$ with coefficients in F that are convergent on the annulus $0<v_p(u_{\mathcal{F}})\leq r$. For any $0<s\leq r$ we define the valuation $v^{\{s\}}$ on $\mathscr{E}^{]0,r]}$ by

$$v^{\{s\}}(f)=\inf_{i\in\mathbb{Z}}(v_p(a_i)+is)\in\mathbb{R}\cup\{\pm\infty\}.$$

We equip $\mathscr{E}^{]0,r]}$ with the Fréchet topology defined by the family of valuations $\{v^{\{s\}}:0<s\leq r\}$. Then $\mathscr{E}^{]0,r]}$ is complete. We equip the Robba ring $\mathscr{R}:=\bigcup_{r>0}\mathscr{E}^{]0,r]}$ with the inductive limit topology. The subring of $\mathscr{R}$ consisting of Laurent series of the form $\sum_{i\geq 0}a_iu_{\mathcal{F}}^i$ is denoted by $\mathscr{R}^+$.

Put $\mathscr{E}^\dagger := \big\{\sum_{i\in\mathbb{Z}} a_i u_{\mathcal{F}}^i \in \mathscr{R} \,\big|\, a_i$ is bounded as $i\to+\infty\big\}$. This is a field contained in $\mathscr{E}$ and in $\mathscr{R}$.

Put $\mathscr{E}^{(0,r]} = \mathscr{E}^\dagger \cap \mathscr{E}^{]0,r]}$. Let $v^{[0,r]}$ be the valuation defined by $v^{[0,r]}(f) = \min_{0\leq s\leq r} v^{\{s\}}(f)$. Let $\mathcal{O}_{\mathscr{E}^{(0,r]}}$ be the ring of integers in $\mathscr{E}^{(0,r]}$ for the valuation $v^{[0,r]}$. We equip $\mathcal{O}_{\mathscr{E}^{(0,r]}}[1/u_{\mathcal{F}}]$ with the topology induced by the valuation $v^{\{r\}}$ and then equip

$$\mathscr{E}^{(0,r]} = \bigcup_{m\in\mathbb{N}} \pi^{-m}\mathscr{E}^{(0,r]}[1/u_{\mathcal{F}}]$$

with the inductive limit topology. The resulting topology on $\mathscr{E}^{(0,r]}$ is called the *weak topology* [Colmez 2010b]. Note that the restriction of the weak topology to the subset

$$\Big\{ f(u_{\mathcal{F}}) = \sum_{i\in\mathbb{Z}} a_i u_{\mathcal{F}}^i \in \mathscr{E}^{(0,r]} : a_i = 0 \text{ if } i\geq 0\Big\}$$

coincides with the topology defined by the valuation $v^{\{r\}}$, and its restriction to $\mathscr{E}^+$ coincides with the weak topology on $\mathscr{E}^+$. Then we equip $\mathscr{E}^\dagger = \bigcup_{r>0} \mathscr{E}^{(0,r]}$ with the inductive limit topology.

We extend the actions of φ_q and Γ on $\mathcal{O}_F[\![u_{\mathcal{F}}]\!]$ to $\mathscr{E}^+$, $\mathcal{O}_{\mathscr{E}}$, $\mathscr{E}$, $\mathscr{E}^\dagger$ and $\mathscr{R}$ continuously.

Put $t_{\mathcal{F}} = \log_{\mathcal{F}}(u_{\mathcal{F}})$, where $\log_{\mathcal{F}}$ is the logarithmic of $\mathcal{F}$. Then $t_{\mathcal{F}}$ is in $\mathscr{R}$ but not in $\mathscr{E}^\dagger$. When $\mathcal{F} = \mathbb{G}_{\mathrm{m}}$, $t_{\mathcal{F}}$ coincides with the usual t in [Colmez 2008]. Note that $\varphi_q(t_{\mathcal{F}}) = \pi t_{\mathcal{F}}$ and $\sigma_a(t_{\mathcal{F}}) = a t_{\mathcal{F}}$ for any $a\in\mathcal{O}_F^\times$. Put $Q = Q(u_{\mathcal{F}}) = [\pi]_{\mathcal{F}}(u_{\mathcal{F}})/u_{\mathcal{F}}$.

We have the following analogue of [Berger 2004, Lemma I.3.2].

Lemma 1.1. *If I is a Γ-stable principal ideal of $\mathscr{R}^+$, then I is generated by an element of the form*

$$u_{\mathcal{F}}^{j_0} \prod_{n=0}^{+\infty} \big(\varphi_q^n(Q(u_{\mathcal{F}})/Q(0))\big)^{j_{n+1}}. \tag{1-1}$$

Furthermore, if $\mathscr{R}^+\cdot\varphi_q(I)\subseteq I$, then the sequence $\{j_n\}_{n\geq 0}$ is decreasing, and if $\mathscr{R}^+\cdot\varphi_q(I)\supseteq I$, then the sequence $\{j_n\}_{n\geq 0}$ is increasing.

Proof. The argument is similar to the proof of [Berger 2004, Lemma I.3.2]. Let $f(u_{\mathcal{F}})$ be a generator of I. Put $V_\rho(I) = \{z\in\mathbb{C}_p : f(z) = 0, 0\leq |z|\leq\rho\}$ for any $\rho\in(0,1)$. If I is stable by Γ, then $V_\rho(I)$ is stable by $[a]_{\mathcal{F}}$ for any $a\in\mathcal{O}_F^\times$. As $V_\rho(I)$ is finite, for any $z\in V_\rho(I)$ there must be some element $a\in\mathcal{O}_F^\times$, $a\neq 1$ such that $[a]_{\mathcal{F}}(z) = z$. Note that $[\pi]_{\mathcal{F}}(z)$ satisfies $[a]_{\mathcal{F}}([\pi]_{\mathcal{F}}(z)) = [\pi]_{\mathcal{F}}(z)$ if $[a]_{\mathcal{F}}(z) = z$. But the cardinal number of the set $\{z\in\mathbb{C}_p : [a]_{\mathcal{F}}(z) = z, |z|\leq\rho\}$ is finite. Thus for any $z\in V_I(\rho)$ there exists a positive integer $m = m(\rho)$ such that $[\pi^m]_{\mathcal{F}}(z) = 0$. Therefore I is generated by an element of the form (1-1).

The last assertion is easy to prove. □

Corollary 1.2. *We have $(t_{\mathcal{F}}) = \Big(u_{\mathcal{F}}\prod_{n\geq 0}\varphi_q^n\big(Q(u_{\mathcal{F}})/Q(0)\big)\Big)$ in the ring $\mathscr{R}^+$.*

Proof. Because the ideal $(t_{\mathcal{F}})$ is Γ-invariant and $\mathscr{R}^+\cdot\varphi_q(t_{\mathcal{F}})=(t_{\mathcal{F}})$, by Lemma 1.1 there exists $j\in\mathbb{N}$ such that $(t_{\mathcal{F}})=\big(u_{\mathcal{F}}^j\prod_{n\geq 0}\varphi_q^n(Q(u_{\mathcal{F}})/Q(0))^j\big)$. Since $(t_{\mathcal{F}}/u_{\mathcal{F}})\equiv 1 \bmod u_{\mathcal{F}}\mathscr{R}^+$ we obtain $j=1$. $\square$

If $\mathcal{F}'$ is another Lubin–Tate group over F corresponding to π, by the theory of Lubin–Tate groups there exists a unique continuous ring isomorphism $\eta_{\mathcal{F},\mathcal{F}'}:\mathcal{O}^+_{\mathscr{E}_{\mathcal{F}}}\to\mathcal{O}^+_{\mathscr{E}_{\mathcal{F}'}}$ with

$$\eta_{\mathcal{F},\mathcal{F}'}(u_{\mathcal{F}})=u_{\mathcal{F}'}+\text{ higher degree terms in }\mathcal{O}_F[\![u_{\mathcal{F}'}]\!]$$

such that $\eta_{\mathcal{F},\mathcal{F}'}\circ[a]_{\mathcal{F}}=[a]_{\mathcal{F}'}\circ\eta_{\mathcal{F},\mathcal{F}'}$ for all $a\in\mathcal{O}_F$. We extend $\eta_{\mathcal{F},\mathcal{F}'}$ to isomorphisms

$$\mathcal{O}_{\mathscr{E}_{\mathcal{F}}}\xrightarrow{\sim}\mathcal{O}_{\mathscr{E}_{\mathcal{F}'}},\quad \mathscr{E}^+_{\mathcal{F}}\xrightarrow{\sim}\mathscr{E}^+_{\mathcal{F}'},\quad \mathscr{E}_{\mathcal{F}}\xrightarrow{\sim}\mathscr{E}_{\mathcal{F}'},\quad \mathscr{E}^\dagger_{\mathcal{F}}\to\mathscr{E}^\dagger_{\mathcal{F}'},\quad \mathscr{R}_{\mathcal{F}}\to\mathscr{R}_{\mathcal{F}'}.$$

By abuse of notation these isomorphisms are again denoted by $\eta_{\mathcal{F},\mathcal{F}'}$.

Let $\ell_u=\log u_{\mathcal{F}}$ be a variable over $\mathscr{R}[1/t_{\mathcal{F}}]$. We extend the φ_q, Γ-actions to $\mathscr{R}[1/t_{\mathcal{F}},\ell_u]$ by

$$\varphi_q(\ell_u)=q\ell_u+\log\frac{[\pi]_{\mathcal{F}}(u_{\mathcal{F}})}{u_{\mathcal{F}}^q},\quad \sigma_a(\ell_u)=\ell_u+\log\frac{[a]_{\mathcal{F}}(u_{\mathcal{F}})}{u_{\mathcal{F}}}.$$

1B. ***Galois representations and (φ_q,Γ)-modules.*** Let L be a finite extension of F. Let $\mathrm{Rep}_L G_F$ be the category of finite-dimensional L-vector spaces V equipped with a linear action of G_F.

If A is any of $\mathscr{E}^+$, $\mathscr{E}$, $\mathscr{E}^\dagger$, $\mathscr{R}$, we put $A_L=A\otimes_F L$. Then we extend the φ_q, Γ-actions on A to A_L by L-linearity. Let R denote any of $\mathscr{E}_L$, $\mathscr{E}^\dagger_L$ and $\mathscr{R}_L$. For a (φ_q,Γ)-module over R, we mean a free R-module D of finite rank together with continuous semilinear actions of φ_q and Γ commuting with each other such that φ_q sends a basis of D to a basis of D. When $R=\mathscr{E}_L$, we say that D is *étale* if D has a φ_q-stable $\mathcal{O}_{\mathscr{E}_L}$-lattice M such that the linear map $\varphi_q^*M\to M$ is an isomorphism. When $R=\mathscr{E}^\dagger_L$, we say that D is *étale* if $\mathscr{E}_L\otimes_{\mathscr{E}^\dagger_L}D$ is étale. When $R=\mathscr{R}_L$, we say that D is *étale* or *of slope* 0 if there exists an étale (φ_q,Γ)-module Δ over $\mathscr{E}^\dagger_L$ such that $D=\mathscr{R}_L\otimes_{\mathscr{E}^\dagger_L}\Delta$. Let $\mathrm{Mod}^{\varphi_q,\Gamma,\acute{e}t}_{/R}$ be the category of étale (φ_q,Γ)-modules over R.

Put $\widetilde{\mathrm{B}}=\mathrm{W}(\widetilde{\mathrm{E}})_F[1/\pi]$. Let B be the completion of the maximal unramified extension of $\mathscr{E}$ in $\widetilde{\mathrm{B}}$ for the π-adic topology. Both $\widetilde{\mathrm{B}}$ and B admit actions of φ_q and G_F. We have $\mathrm{B}^{G_{F_\infty}}=\mathscr{E}$.

For any $V\in\mathrm{Rep}_L G_F$, put $\mathrm{D}_{\mathscr{E}}(V)=(\mathrm{B}\otimes_F V)^{G_{F_\infty}}$. For any $D\in\mathrm{Mod}^{\varphi_q,\Gamma,\acute{e}t}_{/\mathscr{E}_L}$, put $\mathrm{V}(D)=(\mathrm{B}\otimes_{\mathscr{E}}D)^{\varphi_q=1}$.

Theorem 1.3 [Kisin and Ren 2009, Theorem 1.6]. *The functors* V *and* $\mathrm{D}_{\mathscr{E}}$ *are quasi-inverse equivalences of categories between* $\mathrm{Mod}^{\varphi_q,\Gamma,\acute{e}t}_{/\mathscr{E}_L}$ *and* $\mathrm{Rep}_L G_F$.

As usual, let $\widetilde{\mathrm{B}}^\dagger$ be the subring of $\widetilde{\mathrm{B}}$ consisting of overconvergent elements, and put $\mathrm{B}^\dagger = \mathrm{B} \cap \widetilde{\mathrm{B}}^\dagger$. Then $(\mathrm{B}^\dagger)^{G_{F_\infty}} = \mathscr{E}^\dagger$.

Definition 1.4. If V is an L-representation of G_F, we say that V is *overconvergent* if $\mathrm{D}_{\mathscr{E}^\dagger}(V) := (\mathrm{B}^\dagger \otimes_F V)^{G_{F_\infty}}$ contains a basis of $\mathrm{D}_{\mathscr{E}}(V)$.

When $F = \mathbb{Q}_p$, according to the Cherbonnier–Colmez theorem [1998], all L-representations are overconvergent. But in general this is not true. For details, see Remark 5.21.

Proposition 1.5. (a) *If Δ is an étale (φ_q, Γ)-module over $\mathscr{E}_L^\dagger$, then*

$$\mathrm{V}(\mathscr{E}_L \otimes_{\mathscr{E}_L^\dagger} \Delta) = (\mathrm{B}^\dagger \otimes_{\mathscr{E}^\dagger} \Delta)^{\varphi_q = 1}.$$

(b) *The functor $\Delta \mapsto \mathscr{E}_L \otimes_{\mathscr{E}_L^\dagger} \Delta$ is a fully faithful functor from the category $\mathrm{Mod}_{/\mathscr{E}_L^\dagger}^{\varphi_q,\Gamma,\text{ét}}$ to the category $\mathrm{Mod}_{/\mathscr{E}_L}^{\varphi_q,\Gamma,\text{ét}}$.*

(c) *The functor $\mathrm{D}_{\mathscr{E}^\dagger}$ is an equivalence of categories between the category of overconvergent L-representations of G_F and $\mathrm{Mod}_{/\mathscr{E}_L^\dagger}^{\varphi_q,\Gamma,\text{ét}}$.*

Proof. Without loss of generality we may assume that $L = F$. Put $\widetilde{\mathrm{B}}_{\mathbb{Q}_p} = \mathrm{W}(\widetilde{\mathrm{E}})[1/p]$ and $\widetilde{\mathrm{B}}_{\mathbb{Q}_p}^\dagger = \widetilde{\mathrm{B}}_{\mathbb{Q}_p} \cap \widetilde{\mathrm{B}}^\dagger$. The technique of almost étale descent as in [Berger and Colmez 2008] allows us to show that the functor $\Delta \mapsto \widetilde{\mathrm{B}}_{\mathbb{Q}_p} \otimes_{\widetilde{\mathrm{B}}_{\mathbb{Q}_p}^\dagger} \Delta$ from the category of étale (φ, G_F)-modules over $\widetilde{\mathrm{B}}_{\mathbb{Q}_p}^\dagger$ to the category of étale (φ, G_F)-modules over $\widetilde{\mathrm{B}}_{\mathbb{Q}_p}$ is an equivalence. For any (φ_q, G_F)-module D over $\widetilde{\mathrm{B}}^\dagger$ (resp. $\widetilde{\mathrm{B}}$), we can attach a (φ, G_F)-module $\bar{D}$ over $\widetilde{\mathrm{B}}_{\mathbb{Q}_p}^\dagger$ (resp. $\widetilde{\mathrm{B}}_{\mathbb{Q}_p}$) to D by letting $\bar{D} = \bigoplus_{i=0}^{f-1} \varphi^{i*}(D)$ with the map

$$\varphi^*(\bar{D}) = \bigoplus_{i=1}^{f} \varphi^{i*}(D) \to \bigoplus_{i=0}^{f-1} \varphi^{i*}(D) = \bar{D}$$

that sends $\varphi^{i*}(D)$ identically to $\varphi^{i*}(D)$ for $i = 1, \ldots, f-1$ and sends $\varphi^{f*}(D) = \varphi_q^*(D)$ to D using φ_q. Here $f = \log_p q$. Thus the functor $\alpha : \Delta \mapsto \widetilde{\mathrm{B}} \otimes_{\widetilde{\mathrm{B}}^\dagger} \Delta$ from the category of étale (φ_q, G_F)-modules over $\widetilde{\mathrm{B}}^\dagger$ to the category of étale (φ_q, G_F)-modules over $\widetilde{\mathrm{B}}$ is an equivalence. Now let Δ be an étale (φ_q, Γ)-module over $\mathscr{E}^\dagger$, and put $V = \mathrm{V}(\mathscr{E} \otimes_{\mathscr{E}^\dagger} \Delta)$. As $\alpha(\widetilde{\mathrm{B}}^\dagger \otimes_F V) = \widetilde{\mathrm{B}} \otimes_F V = \widetilde{\mathrm{B}} \otimes_{\mathscr{E}^\dagger} \Delta = \alpha(\widetilde{\mathrm{B}}^\dagger \otimes_{\mathscr{E}^\dagger} \Delta)$, we have $\widetilde{\mathrm{B}}^\dagger \otimes_F V = \widetilde{\mathrm{B}}^\dagger \otimes_{\mathscr{E}^\dagger} \Delta$. Thus V is contained in $\widetilde{\mathrm{B}}^\dagger \otimes_{\mathscr{E}^\dagger} \Delta \cap \mathrm{B} \otimes_{\mathscr{E}^\dagger} \Delta = \mathrm{B}^\dagger \otimes_{\mathscr{E}^\dagger} \Delta$, and $V = (\mathrm{B}^\dagger \otimes_{\mathscr{E}^\dagger} \Delta)^{\varphi_q = 1}$. This proves (a).

Next we prove (b). Let Δ_1 and Δ_2 be two objects in $\mathrm{Mod}_{/\mathscr{E}^\dagger}^{\varphi_q,\Gamma,\text{ét}}$. What we have to show is that the natural map

$$\mathrm{Hom}_{\mathrm{Mod}_{/\mathscr{E}^\dagger}^{\varphi_q,\Gamma,\text{ét}}}(\Delta_1, \Delta_2) \to \mathrm{Hom}_{\mathrm{Mod}_{/\mathscr{E}}^{\varphi_q,\Gamma,\text{ét}}}(\mathscr{E} \otimes_{\mathscr{E}^\dagger} \Delta_1, \mathscr{E} \otimes_{\mathscr{E}^\dagger} \Delta_2)$$

is an isomorphism. For this we reduce the problem to showing that

$$(\check{\Delta}_1 \otimes_{\mathscr{E}^\dagger} \Delta_2)^{\varphi_q = 1, \Gamma = 1} \to \big(\mathscr{E} \otimes_{\mathscr{E}^\dagger} (\check{\Delta}_1 \otimes_{\mathscr{E}^\dagger} \Delta_2)\big)^{\varphi_q = 1, \Gamma = 1}$$

is an isomorphism. Here $\check{\Delta}_1$ is the $\mathscr{E}^\dagger$-module of $\mathscr{E}^\dagger$-linear maps from Δ_1 to $\mathscr{E}^\dagger$, which is equipped with a natural étale (φ_q,Γ)-module structure. We have

$$\begin{aligned}\big(\mathscr{E}\otimes_{\mathscr{E}^\dagger}(\check{\Delta}_1\otimes_{\mathscr{E}^\dagger}\Delta_2)\big)^{\varphi_q=1,\Gamma=1} &= \big(\mathrm{B}\otimes_{\mathscr{E}^\dagger}(\check{\Delta}_1\otimes_{\mathscr{E}^\dagger}\Delta_2)\big)^{\varphi_q=1,G_F=1}\\ &= \mathrm{V}\big(\mathscr{E}\otimes_{\mathscr{E}^\dagger}(\check{\Delta}_1\otimes_{\mathscr{E}^\dagger}\Delta_2)\big)^{G_F=1}\\ &= \big(\mathrm{B}^\dagger\otimes_{\mathscr{E}^\dagger}(\check{\Delta}_1\otimes_{\mathscr{E}^\dagger}\Delta_2)\big)^{\varphi_q=1,G_F=1}\\ &= (\check{\Delta}_1\otimes_{\mathscr{E}^\dagger}\Delta_2)^{\varphi_q=1,\Gamma=1}. \end{aligned}\tag{1-2}$$

Finally, (c) follows from (a), (b) and Theorem 1.3. □

Proposition 1.6. *The functor* $\Delta\mapsto\mathscr{R}_L\otimes_{\mathscr{E}_L^\dagger}\Delta$ *is an equivalence of categories between* $\mathrm{Mod}^{\varphi_q,\Gamma,\text{ét}}_{/\mathscr{E}_L^\dagger}$ *and* $\mathrm{Mod}^{\varphi_q,\Gamma,\text{ét}}_{/\mathscr{R}_L}$.

Proof. Let D be an étale (φ_q,Γ)-module over $\mathscr{R}_L$. By Kedlaya's slope filtration theorem [2004], there exists a unique φ_q-stable $\mathscr{E}_L^\dagger$-submodule Δ of D that is étale as a φ_q-module such that $D=\mathscr{R}_L\otimes_{\mathscr{E}_L^\dagger}\Delta$. For any $\gamma\in\Gamma$, $\gamma(\Delta)$ also has this property. Thus, by uniqueness of Δ, we have $\gamma(\Delta)=\Delta$. This means that Δ is Γ-invariant. □

1C. *$\mathcal{O}_F$-analytic (φ_q,Γ)-modules.* For any $r\geq s>0$, let $v^{[s,r]}$ be the valuation defined by $v^{[s,r]}(f)=\inf_{r'\in[s,r]}v^{\{r'\}}(f)$. Note that

$$v^{[s,r]}(f)=\inf_{\substack{z\in\mathbb{C}_p\\ s\leq v_p(z)\leq r}} v_p(f(z)).$$

Lemma 1.7. *For any* $r>s>0$, *there exists a sufficiently large integer* $n=n(s,r)$ *such that, if* $\gamma\in\Gamma_n$, *then we have* $v^{[s,r]}\big((1-\gamma)z\big)\geq v^{[s,r]}(z)+1$ *for all* $z\in\mathscr{E}_L^{]0,r]}$.

Proof. It suffices to consider $z=u_{\mathcal{F}}^k$, $k\in\mathbb{Z}$. If $k\geq 0$, then

$$\gamma(u_{\mathcal{F}}^k)-u_{\mathcal{F}}^k=u_{\mathcal{F}}^k\bigg(\frac{\gamma(u_{\mathcal{F}})}{u_{\mathcal{F}}}-1\bigg)\bigg(\frac{\gamma(u_{\mathcal{F}}^{k-1})}{u_{\mathcal{F}}^{k-1}}+\cdots+1\bigg)$$

and

$$\gamma(u_{\mathcal{F}}^{-k})-u_{\mathcal{F}}^{-k}=u_{\mathcal{F}}^{-k}\bigg(\frac{u_{\mathcal{F}}}{\gamma(u_{\mathcal{F}})}-1\bigg)\bigg(\frac{u_{\mathcal{F}}^{k-1}}{\gamma(u_{\mathcal{F}}^{k-1})}+\cdots+1\bigg).$$

As $v^{[s,r]}(yz)\geq v^{[s,r]}(y)+v^{[s,r]}(z)$, the lemma follows from the fact that $\gamma(u_{\mathcal{F}})/u_{\mathcal{F}}$ approaches 1 as $\gamma\to 1$. □

Let D be an object in $\mathrm{Mod}^{\varphi_q,\Gamma,\text{ét}}_{/\mathscr{R}_L}$. We choose a basis $\{e_1,\ldots,e_d\}$ of D and write $D^{]0,r]}=\bigoplus_{i=1}^d\mathscr{E}_L^{]0,r]}\cdot e_i$. Our definition of $D^{]0,r]}$ depends on the choice of $\{e_1,\ldots,e_d\}$; however, if $\{e_1',\ldots,e_d'\}$ is another basis, then

$$\bigoplus_{i=1}^d\mathscr{E}_L^{]0,r]}\cdot e_i=\bigoplus_{i=1}^d\mathscr{E}_L^{]0,r]}\cdot e_i'$$

for sufficiently small $r > 0$. When $r > 0$ is sufficiently small, $D^{]0,r]}$ is stable under Γ. By Lemma 1.7 and the continuity of the Γ-action on $D^{]0,r]}$, the series

$$\log \gamma = \sum_{i=1}^{\infty} (\gamma - 1)^i (-1)^{i-1}/i$$

converges on $D^{]0,r]}$ when $\gamma \to 1$. It follows that the map

$$\mathrm{d}\Gamma : \mathrm{Lie}\,\Gamma \to \mathrm{End}_L D^{]0,r]}, \quad \beta \mapsto \log(\exp \beta)$$

is well defined for sufficiently small β, and we extend it to all of $\mathrm{Lie}\,\Gamma$ by $\mathbb{Z}_p$-linearity. As a result, we obtain a $\mathbb{Z}_p$-linear map $\mathrm{d}\Gamma_D : \mathrm{Lie}\,\Gamma \to \mathrm{End}_L D$. For any $\beta \in \mathrm{Lie}\,\Gamma$, $\mathrm{d}\Gamma_{\mathcal{R}_L}(\beta)$ is a derivation of $\mathcal{R}_L$ and $\mathrm{d}\Gamma_D(\beta)$ is a differential operator over $\mathrm{d}\Gamma_{\mathcal{R}_L}(\beta)$, which means that for any $a \in \mathcal{R}_L$, $m \in D$ and $\beta \in \mathrm{Lie}\,\Gamma$ we have

$$\mathrm{d}\Gamma_D(\beta)(am) = \mathrm{d}\Gamma_{\mathcal{R}_L}(\beta)(a)m + a \cdot \mathrm{d}\Gamma_D(\beta)(m). \tag{1-3}$$

The isomorphism $\chi_{\mathcal{F}} : \Gamma \to \mathcal{O}_F^\times$ induces an $\mathcal{O}_F$-linear isomorphism $\mathrm{Lie}\,\Gamma \to \mathcal{O}_F$. We will identify $\mathrm{Lie}\,\Gamma$ with $\mathcal{O}_F$ via this isomorphism.

We say that D is *$\mathcal{O}_F$-analytic* if the map $\mathrm{d}\Gamma_D$ is not only $\mathbb{Z}_p$-linear, but also $\mathcal{O}_F$-linear. If D is $\mathcal{O}_F$-analytic, the operator $\mathrm{d}\Gamma_D(\beta)/\beta$, $\beta \in \mathcal{O}_F$, $\beta \neq 0$, does not depend on the choice of β. The resulting operator is denoted by ∇_D or just ∇ if there is no confusion. Note that the Γ-action on $\mathcal{R}_L$ is $\mathcal{O}_F$-analytic and by [Kisin and Ren 2009, Lemma 2.1.4]

$$\nabla = t_{\mathcal{F}} \cdot \frac{\partial F_{\mathcal{F}}}{\partial Y}(u_{\mathcal{F}}, 0) \cdot \mathrm{d}/\mathrm{d}u_{\mathcal{F}}, \tag{1-4}$$

where $F_{\mathcal{F}}(X, Y)$ is the formal group law of $\mathcal{F}$. Put $\partial = (\partial F_{\mathcal{F}}/\partial Y)(u_{\mathcal{F}}, 0) \cdot \mathrm{d}/\mathrm{d}u_{\mathcal{F}}$. From the relation $\sigma_a(t_{\mathcal{F}}) = at_{\mathcal{F}}$ we obtain $\nabla t_{\mathcal{F}} = t_{\mathcal{F}}$ and $\partial\, t_{\mathcal{F}} = 1$. When $\mathcal{F} = \mathbb{G}_{\mathrm{m}}$, ∇ and ∂ are already defined in [Berger 2002]. In this case $F_{\mathcal{F}}(X, Y) = X + Y + XY$ and so $\partial = (1 + u_{\mathcal{F}})\,\mathrm{d}/\mathrm{d}u_{\mathcal{F}}$.

We end this section by classification of (φ_q, Γ)-modules over $\mathcal{R}_L$ of rank 1.

Let $\mathscr{I}(L)$ be the set of continuous characters $\delta : F^\times \to L^\times$ and $\mathscr{I}_{\mathrm{an}}(L)$ the subset of locally F-analytic characters. If δ is in $\mathscr{I}_{\mathrm{an}}(L)$, the quotient $\log \delta(a)/\log a$, for $a \in \mathcal{O}_F^\times$ (which makes sense when $\log a \neq 0$) does not depend on a. This number, denoted by w_δ, is called the *weight* of δ. Clearly $w_\delta = 0$ if and only if δ is locally constant; w_δ is in $\mathbb{Z}$ if and only if δ is locally algebraic.

If $\delta \in \mathscr{I}(L)$, let $\mathcal{R}_L(\delta)$ be the (φ_q, Γ)-module over $\mathcal{R}_L$ (of rank 1) that has a basis e_δ such that $\varphi_q(e_\delta) = \delta(\pi)e_\delta$ and $\sigma_a(e_\delta) = \delta(a)e_\delta$. It is easy to check that, if $\delta \in \mathscr{I}_{\mathrm{an}}(L)$, then $\mathcal{R}_L(\delta)$ is $\mathcal{O}_F$-analytic and $\nabla_\delta = \nabla_{\mathcal{R}_L(\delta)} = t_{\mathcal{F}}\partial + w_\delta$ (more precisely $\nabla_\delta(ze_\delta) = (t_{\mathcal{F}}\partial z + w_\delta z)e_\delta$). If $\mathcal{R}_L(\delta)$ is étale, that is, $v_p(\delta(\pi)) = 0$, we will use $L(\delta)$ to denote the Galois representation attached to $\mathcal{R}_L(\delta)$.

Remark 1.8. All 1-dimensional L-representations of G_F are overconvergent. In fact, such a representation comes from a character of $F^\times$ and thus is of the form $L(\delta)$.

Proposition 1.9. *Let D be a (φ_q, Γ)-module over $\mathcal{R}_L$ of rank* 1. *Then there exists a character $\delta \in \mathcal{I}(L)$ such that D is isomorphic to $\mathcal{R}_L(\delta)$. Furthermore, D is $\mathcal{O}_F$-analytic if and only if $\delta \in \mathcal{I}_{\mathrm{an}}(L)$.*

Proof. The argument is similar to the proof of [Colmez 2008, Proposition 3.1]. We first reduce to the case that D is étale. Then by Proposition 1.6 there exists an étale (φ_q, Γ)-module Δ over $\mathscr{E}_L^\dagger$ such that $D = \mathcal{R}_L \otimes_{\mathscr{E}_L^\dagger} \Delta$. Now the first assertion follows from Proposition 1.5 and Remark 1.8. The second assertion is obvious. □

2. The operators ψ and ∂

2A. *The operator ψ.* We define an operator ψ and study its properties.

Note that $\{u_{\mathcal{F}}^i\}_{0\le i\le q-1}$ is a basis of $\mathscr{E}_L$ over $\varphi_q(\mathscr{E}_L)$. So $\mathscr{E}_L$ is a field extension of $\varphi_q(\mathscr{E}_L)$ of degree q. Put $\mathrm{tr} = \mathrm{tr}_{\mathscr{E}_L/\varphi_q(\mathscr{E}_L)}$.

Lemma 2.1. (a) *There is a unique operator $\psi : \mathscr{E}_L \to \mathscr{E}_L$ such that $\varphi_q \circ \psi = q^{-1}\mathrm{tr}$.*

(b) *For any $a, b \in \mathscr{E}_L$ we have $\psi(\varphi_q(a)b) = a\psi(b)$. In particular, $\psi \circ \varphi_q = \mathrm{id}$.*

(c) *ψ commutes with Γ.*

Proof. Assertion (a) follows from the fact that φ_q is injective. Assertion (b) follows from the relation

$$\varphi_q\big(\psi(\varphi_q(a)b)\big) = \mathrm{tr}(\varphi_q(a)b)/q = \varphi_q(a)\mathrm{tr}(b)/q = \varphi_q(a)\varphi_q(\psi(b)) = \varphi_q(a\psi(b))$$

and the injectivity of φ_q. As φ_q commutes with Γ, $\varphi_q(\mathscr{E}_L)$ is stable under Γ. Thus $\gamma \circ \mathrm{tr} \circ \gamma^{-1} = \mathrm{tr}$ for all $\gamma \in \Gamma$. This ensures that ψ commutes with Γ. Assertion (c) follows. □

We first compute ψ in the case of the special Lubin–Tate group.

Proposition 2.2. *Suppose that $\mathcal{F}$ is the special Lubin–Tate group.*

(a) *If $\ell \ge 0$, then $\psi(u_{\mathcal{F}}^\ell) = \sum_{i=0}^{[\ell/q]} a_{\ell,i} u_{\mathcal{F}}^i$ with $v_\pi(a_{\ell,i}) \ge [\ell/q] + 1 - i - v_\pi(q)$.*

(b) *If $\ell < 0$, then $\psi(u_{\mathcal{F}}^\ell) = \sum_{i=\ell}^{[\ell/q]} b_{\ell,i} u_{\mathcal{F}}^i$ with $v_\pi(b_{\ell,i}) \ge [\ell/q] + 1 - i - v_\pi(q)$.*

Proof. First we prove (a) by induction on ℓ. As the minimal polynomial of $u_{\mathcal{F}}$ is $X^q + \pi X - (u_{\mathcal{F}}^q + \pi u_{\mathcal{F}})$, by Newton's formula we have

$$\mathrm{tr}(u_{\mathcal{F}}^i) = \begin{cases} 0 & \text{if } 1 \le i \le q-2, \\ (1-q)\pi & \text{if } i = q-1. \end{cases}$$

It follows that

$$\psi(u_{\mathcal{F}}^i) = \begin{cases} 0 & \text{if } 1 \le i \le q-2, \\ (1-q)\pi/q & \text{if } i = q-1. \end{cases}$$

Thus the assertion holds when $0 \le \ell \le q-1$. Now we assume that $\ell = j \ge q$ and the assertion holds when $0 \le \ell \le j-1$. We have

$$\begin{aligned}\psi(u_{\mathcal{F}}^{\ell}) &= \psi\big((u_{\mathcal{F}}^{q}+\pi u_{\mathcal{F}})u_{\mathcal{F}}^{\ell-q}\big) - \psi(\pi u_{\mathcal{F}}^{\ell-q+1}) = u_{\mathcal{F}}\psi(u_{\mathcal{F}}^{\ell-q}) - \pi\psi(u_{\mathcal{F}}^{\ell-q+1}) \\ &= \sum_{i=1}^{[\ell/q]} a_{\ell-q,i-1}u_{\mathcal{F}}^{i} - \sum_{i=0}^{[(\ell+1)/q]-1} \pi a_{\ell-q+1,i}u_{\mathcal{F}}^{i}.\end{aligned}$$

Thus $a_{\ell,i} = a_{\ell-q,i-1} - \pi a_{\ell-q+1,i}$. By the inductive assumption we have

$$v_{\pi}(a_{\ell-q,i-1}) \ge [(\ell-q)/q]+1-(i-1)-v_{\pi}(q) = [\ell/q]+1-i-v_{\pi}(q)$$

and

$$v_{\pi}(a_{\ell-q+1,i}) \ge [(\ell-q+1)/q]+1-i-v_{\pi}(q) \ge [\ell/q]-i-v_{\pi}(q).$$

It follows that $v_{\pi}(a_{\ell,i}) \ge [\ell/q]+1-i-v_{\pi}(q)$.

Next we prove (b). We have

$$\begin{aligned}\psi(u_{\mathcal{F}}^{\ell}) = \psi\left(\frac{(u_{\mathcal{F}}^{q-1}+\pi)^{-\ell}}{\varphi_q(u_{\mathcal{F}})^{-\ell}}\right) &= \frac{\psi\Big(\sum_{j=0}^{-\ell}\begin{bmatrix}-\ell\\ j\end{bmatrix}u_{\mathcal{F}}^{j(q-1)}\pi^{-\ell-j}\Big)}{u_{\mathcal{F}}^{-\ell}} \\ &= \sum_{i=0}^{[-\ell(q-1)/q]}\sum_{j=0}^{-\ell}\begin{bmatrix}-\ell\\ j\end{bmatrix}\pi^{-\ell-j}a_{j(q-1),i}\cdot u_{\mathcal{F}}^{i+\ell} \\ &= \sum_{i=\ell}^{[\ell/q]}\sum_{j=0}^{-\ell}\begin{bmatrix}-\ell\\ j\end{bmatrix}\pi^{-\ell-j}a_{j(q-1),i-\ell}\cdot u_{\mathcal{F}}^{i}.\end{aligned}$$

Here, $\begin{bmatrix}-\ell\\ j\end{bmatrix} = \dfrac{(-\ell)!}{j!(-\ell-j)!}$. Thus $b_{\ell,i} = \sum_{j=0}^{-\ell}\begin{bmatrix}-\ell\\ j\end{bmatrix}\pi^{-\ell-j}a_{j(q-1),i-\ell}$. Since

$$\begin{aligned}v_{\pi}(\pi^{-\ell-j}a_{j(q-1),i-\ell}) &\ge -\ell-j+\big([j(q-1)/q]+1-(i-\ell)-v_{\pi}(q)\big) \\ &= [-j/q]+1-i-v_{\pi}(q) \\ &\ge [\ell/q]+1-i-v_{\pi}(q),\end{aligned}$$

we obtain $v_{\pi}(b_{\ell,i}) \ge [\ell/q]+1-i-v_{\pi}(q)$. □

Let $\mathscr{E}_L^-$ be the subset of $\mathscr{E}_L$ consisting of elements of the form $\sum_{i\le -1} a_i u_{\mathcal{F}}^i$.

Corollary 2.3. *Suppose that $\mathcal{F}$ is the special Lubin–Tate group. Then $\psi(\mathscr{E}_L^-) \subset \mathscr{E}_L^-$.*

Proof. This follows directly from Proposition 2.2. □

Proposition 2.4. (a) *$\psi(\mathscr{E}_L^+) = \mathscr{E}_L^+$, $\psi(\mathcal{O}_{\mathscr{E}_L^+}) \subset \frac{\pi}{q}\mathcal{O}_{\mathscr{E}_L^+}$ and $\psi(\mathcal{O}_{\mathscr{E}_L}) \subset \frac{\pi}{q}\mathcal{O}_{\mathscr{E}_L}$.*

(b) *ψ is continuous for the weak topology on $\mathscr{E}_L$.*

(c) *$\mathscr{E}_L^{\dagger}$ is stable under ψ, and the restriction of ψ on $\mathscr{E}_L^{\dagger}$ is continuous for the weak topology of $\mathscr{E}_L^{\dagger}$.*

(d) *If* $f\in\mathscr{E}_L^{(0,r]}$, *then the sequence* $\big(\frac{q}{\pi}\psi\big)^n(f)$, $n\in\mathbb{N}$, *is bounded in* $\mathscr{E}_L^{(0,r]}$ *for the weak topology.*

Proof. Let $\mathcal{F}_0$ be the special Lubin–Tate group over F corresponding to π. Observe that $\psi_{\mathcal{F}}=\eta_{\mathcal{F}_0,\mathcal{F}}^{-1}\psi_{\mathcal{F}_0}\eta_{\mathcal{F}_0,\mathcal{F}}$. Since $\eta_{\mathcal{F}_0,\mathcal{F}}(u_{\mathcal{F}_0})$ equals $u_{\mathcal{F}}$ times a unit in $\mathcal{O}_F[\![u_{\mathcal{F}}]\!]$, we have

$$\eta_{\mathcal{F}_0,\mathcal{F}}\big(\mathcal{O}^{(0,r]}_{\mathscr{E}_{\mathcal{F}_0,L}}[1/u_{\mathcal{F}_0}]\big)=\mathcal{O}^{(0,r]}_{\mathscr{E}_{\mathcal{F},L}}[1/u_{\mathcal{F}}]\quad\text{for any } r>0,$$

and $\eta_{\mathcal{F}_0,\mathcal{F}}$ respects the valuation $v^{[0,r]}$. Thus $\eta_{\mathcal{F}_0,\mathcal{F}}:\mathscr{E}^{(0,r]}_{\mathcal{F}_0,L}\to\mathscr{E}^{(0,r]}_{\mathcal{F},L}$ is a topological isomorphism. It follows that $\mathscr{E}^{\dagger}_{\mathcal{F}_0,L}\to\mathscr{E}^{\dagger}_{\mathcal{F},L}$ and its inverse are continuous for the weak topology. Similarly $\eta_{\mathcal{F}_0,\mathcal{F}}:\mathscr{E}_{\mathcal{F}_0,L}\to\mathscr{E}_{\mathcal{F},L}$ and its inverse are continuous for the weak topology. Hence we only need to consider the case of the special Lubin–Tate group. Assertions (a) and (b) follow from Proposition 2.2. For (c) we only need to show that, for any $r>0$, we have $\psi(\mathscr{E}_L^{(0,r]})\subset\mathscr{E}_L^{(0,r]}$ and the restriction $\psi:\mathscr{E}_L^{(0,r]}\to\mathscr{E}_L^{(0,r]}$ is continuous. By (b) the restriction of ψ to $\mathscr{E}_L^{+}$ is continuous. By Proposition 2.2(b) and Corollary 2.3, if f is in $\mathscr{E}_L^{-}\cap\mathscr{E}_L^{(0,r]}$, then $\psi(f)$ is in $\mathscr{E}_L^{-}$ and $v^{\{r\}}(\psi(f))\geq v^{\{r\}}(f)+v_p(\pi/q)$. Thus $\psi:\mathscr{E}_L^{-}\cap\mathscr{E}_L^{(0,r]}\to\mathscr{E}_L^{-}\cap\mathscr{E}_L^{(0,r]}$ is continuous, which proves (c). As $(q/\pi)\psi(\mathcal{O}_{\mathscr{E}_L^+})\subset\mathcal{O}_{\mathscr{E}_L^+}$ and $v^{\{r\}}\big((q/\pi)\psi(f)\big)\geq v^{\{r\}}(f)$ for any $f\in\mathscr{E}_L^{-}\cap\mathscr{E}_L^{(0,r]}$, (d) follows. □

Next we extend ψ to $\mathscr{R}_L$.

Proposition 2.5. *We can extend* tr *continuously to* $\mathscr{R}_L$. *The resulting operator* tr *satisfies* $\mathrm{tr}|_{\varphi_q(\mathscr{R}_L)}=q\cdot\mathrm{id}$ *and* $\mathrm{tr}(\mathscr{R}_L)=\varphi_q(\mathscr{R}_L)$.

Proof. Let $\mathscr{E}_L^{\gg-\infty}$ denote the subset of $\mathscr{E}_L$ consisting of $f\in\mathscr{E}_L$ of the form $\sum_{n\gg-\infty}a_nu_{\mathcal{F}}^n$. If $f\in\mathscr{E}_L^{\gg-\infty}$, then

$$\mathrm{tr}(f)=\sum_{\eta\in\ker[\pi]_{\mathcal{F}}}f(u_{\mathcal{F}}+_{\mathcal{F}}\eta).$$

If η is in $\ker[\pi]_{\mathcal{F}}$, then $v_p(\eta)\geq\frac{1}{(q-1)e_F}$, where $e_F=[F:F_0]$. Thus, if r and $s\in\mathbb{R}_+$ satisfy $1/((q-1)e_F)>r\geq s$, the morphisms $u_{\mathcal{F}}\mapsto u_{\mathcal{F}}+_{\mathcal{F}}\eta$ ($\eta\in\ker[\pi]_{\mathcal{F}}$) keep the annulus $\{z\in\mathbb{C}_p:p^{-r}\leq|z|\leq p^{-s}\}$ stable. So for any $f\in\mathscr{E}_L^{\gg-\infty}$ we have $v^{[s,r]}(f(u_{\mathcal{F}}+_{\mathcal{F}}\eta))=v^{[s,r]}(f)$ and $v^{[s,r]}(\mathrm{tr}(f))\geq v^{[s,r]}(f)$. Hence there exists a unique continuous operator $\mathrm{Tr}:\mathscr{R}_L\to\mathscr{R}_L$ such that $\mathrm{Tr}(f)=\mathrm{tr}(f)$ for any $f\in\mathscr{E}_L^{\gg-\infty}$. (For any $f\in\mathscr{R}_L$, choosing a positive real number r such that $f\in\mathscr{E}_L^{]0,r]}$, we can find a sequence $\{f_i\}_{i\geq1}$ in $\mathscr{E}_L^{\gg-\infty}$ such that $f_i\to f$ in $\mathscr{E}^{]0,r]}$; then $\{\mathrm{tr}(f_i)\}_{i\geq1}$ is a Cauchy sequence in $\mathscr{E}_L^{[s,r]}$ for any s satisfying $0<s\leq r$, and we let $\mathrm{Tr}(f)$ be their limit in $\mathscr{E}^{]0,r]}$; it is easy to show that $\mathrm{Tr}(f)$ does not depend on any choice.) From the continuity of Tr we obtain that $\mathrm{Tr}|_{\mathscr{E}_L^\dagger}=\mathrm{tr}$ and $\mathrm{Tr}|_{\varphi_q(\mathscr{R}_L)}=q\cdot\mathrm{id}$. By Lemma 2.6 below, $\varphi_q:\mathscr{R}_L\to\mathscr{R}_L$ is strict and thus has a closed image. Since $\mathscr{E}_L^\dagger$ is dense in $\mathscr{R}_L$ and $\mathrm{Tr}(\mathscr{E}_L^\dagger)=\varphi_q(\mathscr{E}_L^\dagger)\subset\varphi_q(\mathscr{R}_L)$, we have $\mathrm{Tr}(\mathscr{R}_L)\subseteq\varphi_q(\mathscr{R}_L)$. □

Lemma 2.6. *If* $\dfrac{q}{(q-1)e_F} > r \geq s > 0$ *and* $f \in \mathscr{E}_L^{]0,r]}$, *then we have*

- $v^{[s,r]}(\gamma(f)) = v^{[s,r]}(f)$ *for all* $\gamma \in \Gamma$;
- $v^{[s,r]}(\varphi_q(f)) = v^{[qs,qr]}(f)$ *if* $r < 1/((q-1)e_F)$.

Proof. Since $[\chi_{\mathcal{F}}(\gamma)]_{\mathcal{F}}(u_{\mathcal{F}}) \in u_{\mathcal{F}}\mathcal{O}_F[\![u_{\mathcal{F}}]\!]$, we have $v_p([\chi_{\mathcal{F}}(\gamma)]_{\mathcal{F}}(z)) \geqslant v_p(z)$ for all $z \in \mathbb{C}_p$ such that $v_p(z) > 0$. By the same reason we have $v_p([\chi_{\mathcal{F}}(\gamma^{-1})]_{\mathcal{F}}(z)) \geqslant v_p(z)$ and thus $v_p([\chi_{\mathcal{F}}(\gamma)]_{\mathcal{F}}(z)) \leqslant v_p(z)$. So $v_p([\chi_{\mathcal{F}}(\gamma)]_{\mathcal{F}}(z)) = v_p(z)$.

If $z \in \mathbb{C}_p$ satisfies

$$p^{-\frac{1}{(q-1)e_F}} < p^{-r} \leqslant |z| \leqslant p^{-s} < 1,$$

then $v_p([\pi]_{\mathcal{F}}(z)) = q v_p(z)$. Thus, the image by $z \mapsto [\pi]_{\mathcal{F}}(z)$ of the annulus $\{z \in \mathbb{C}_p : p^{-r} \leqslant |z| \leqslant p^{-s}\}$ is inside the annulus $\{z \in \mathbb{C}_p : p^{-qr} \leqslant |z| \leqslant p^{-qs}\}$. Conversely, if $w \in \mathbb{C}_p$ is such that $p^{-qr} \leqslant |w| \leqslant p^{-qs}$, then $v_p(w) < q/((q-1)e_F)$. The Newton polygon of the polynomial $-w + [\pi]_{\mathcal{F}}(u_{\mathcal{F}})$ shows that this polynomial has q roots of valuation $\frac{1}{q} v_p(w)$. If $z \in \mathbb{C}_p$ is such a root, we have $p^{-r} \leqslant |z| \leqslant p^{-s}$. Thus, the image of the annulus $p^{-r} \leqslant |z| \leqslant p^{-s}$ is the annulus $p^{-qr} \leqslant |z| \leqslant p^{-qs}$. □

We define $\psi : \mathscr{R}_L \to \mathscr{R}_L$ by $\psi = \frac{1}{q}\, \varphi_q^{-1} \circ \mathrm{tr}$.

Lemma 2.7. *If* $q/((q-1)e_F) > r \geq s > 0$ *and* $f \in \mathscr{E}_L^{]0,r]}$, *then*

$$v^{[s,r]}(\psi(f)) \geqslant v^{[s/q,r/q]}(f) - v_p(q).$$

Proof. By Lemma 2.6 it suffices to show that

$$v^{[s/q,r/q]}\big(\varphi_q(\psi(f))\big) = v^{[s/q,r/q]}(q^{-1}\mathrm{tr}(f)) \geq v^{[s/q,r/q]}(f) - v_p(q).$$

But this follows from Proposition 2.5 and its proof. □

As a consequence, $\psi : \mathscr{R}_L \to \mathscr{R}_L$ is continuous.

Corollary 2.8. (a) $\{u_{\mathcal{F}}^i\}_{0 \leq i \leq q-1}$ *is a basis of* $\mathscr{E}_L^\dagger$ *over* $\varphi_q(\mathscr{E}_L^\dagger)$, *and*

$$\mathrm{tr}|_{\mathscr{E}_L^\dagger} = \mathrm{tr}_{\mathscr{E}_L^\dagger/\varphi_q(\mathscr{E}_L^\dagger)}.$$

(b) $\{u_{\mathcal{F}}^i\}_{0 \leq i \leq q-1}$ *is a basis of* $\mathscr{R}_L$ *over* $\varphi_q(\mathscr{R}_L)$.

Proof. Let $\{b_i\}_{0 \leq i \leq q-1}$ be the dual basis of $\{u_{\mathcal{F}}^i\}_{0 \leq i \leq q-1}$ relative to $\mathrm{tr}_{\mathscr{E}_L/\varphi_q(\mathscr{E}_L)}$. Let B be the inverse of the matrix $(\mathrm{tr}(u_{\mathcal{F}}^{i+j}))_{i,j}$. Then $B \in \mathrm{GL}_q(\mathscr{E}_L^\dagger)$ and

$$(b_0, b_1, \ldots, b_{q-1})^t = B(1, u_{\mathcal{F}}, \ldots, u_{\mathcal{F}}^{q-1})^t.$$

So $b_0, b_1, \ldots, b_{q-1}$ are in $\mathscr{E}_L^\dagger$. Then $f = \sum_{i=0}^{q-1} u_{\mathcal{F}}^i \psi(b_i f)$ for any $f \in \mathscr{E}_L$, $\mathscr{E}_L^\dagger$ or $\mathscr{R}_L$. (For the former two cases, this follows from the definition of $\{b_i\}_{0 \leq i \leq q-1}$; for the last case, we apply the continuity of ψ.) Thus $\{u_{\mathcal{F}}^i\}_{0 \leq i \leq q-1}$ generate $\mathscr{E}_L^\dagger$ (resp. $\mathscr{R}_L$) over $\varphi_q(\mathscr{E}_L^\dagger)$ (resp. $\varphi_q(\mathscr{R}_L)$). In either case, to prove the independence of

$\{u_{\mathcal{F}}^i\}_{0\leq i\leq q-1}$, we only need to use the fact $\psi(b_i u_{\mathcal{F}}^j)=\delta_{ij}$ $(i,j\in\{0,1,\dots,q-1\})$, where δ_{ij} is the Kronecker sign. Finally we note that the second assertion of (a) follows from the first one. □

We apply the above to (φ_q,Γ)-modules.

Proposition 2.9. *If D is a (φ_q,Γ)-module over R where $R=\mathscr{E}_L$, $\mathscr{E}_L^\dagger$ or $\mathscr{R}_L$, then there is a unique operator $\psi: D\to D$ such that*

$$\psi(a\varphi_q(x))=\psi(a)x \quad \textit{and} \quad \psi(\varphi_q(a)x)=a\psi(x) \tag{2-1}$$

for any $a\in R$ and $x\in D$. Moreover ψ commutes with Γ.

Proof. Let $\{e_1,e_2,\dots,e_d\}$ be a basis of D over R. By the definition of (φ_q,Γ)-modules, $\{\varphi_q(e_1),\varphi_q(e_2),\dots,\varphi_q(e_d)\}$ is also a basis of D. For any $m\in D$, writing $m=a_1\varphi_q(e_1)+a_2\varphi_q(e_2)+\cdots+a_d\varphi_q(e_d)$, we put

$$\psi(m)=\psi(a_1)e_1+\psi(a_2)e_2+\cdots+\psi(a_d)e_d.$$

Then ψ satisfies (2-1). It is easy to prove the uniqueness of ψ. Observe that for any $\gamma\in\Gamma$, $\gamma\psi\gamma^{-1}$ also satisfies (2-1). Thus $\gamma\psi\gamma^{-1}=\psi$ by uniqueness of ψ. This means that ψ commutes with Γ. □

2B. ***The operator ∂ and the map* Res.** Recall that $\partial=(\partial F_{\mathcal{F}}/\partial Y)(u_{\mathcal{F}},0)\cdot \mathrm{d}/\mathrm{d}u_{\mathcal{F}}$. So $\mathrm{d}t_{\mathcal{F}}=(\partial F_{\mathcal{F}}/\partial Y)(u_{\mathcal{F}},0)\,\mathrm{d}u_{\mathcal{F}}$ and $(\mathrm{d}t_{\mathcal{F}}/\mathrm{d}u_{\mathcal{F}})=\big((\partial F_{\mathcal{F}}/\partial Y)(u_{\mathcal{F}},0)\big)^{-1}$.

Lemma 2.10. *If $r\geq s>0$ and $f\in\mathscr{R}_L^{]0,r]}$, then $v^{[s,r]}(\partial f)\geqslant v^{[s,r]}(f)-r$.*

Proof. Observe that $v_p\big((\partial F_{\mathcal{F}}/\partial Y)(z,0)\big)=0$ for all z in the disk $|z|<1$. Thus $v^{[s,r]}(\partial f)=v^{[s,r]}(\mathrm{d}f/\mathrm{d}u_{\mathcal{F}})$. Write $f=\sum_{n\in\mathbb{Z}}a_n u_{\mathcal{F}}^n$. Then we have

$$\begin{aligned} v^{[s,r]}\Big(\frac{\mathrm{d}f}{\mathrm{d}u_{\mathcal{F}}}\Big) &= \inf_{\substack{r\geqslant v_p(z)\geqslant s\\ n\in\mathbb{Z}}} v_p(na_nz^{n-1}) \\ &\geqslant \inf_{\substack{r\geqslant v_p(z)\geqslant s\\ n\in\mathbb{Z}}} \big(v_p(a_n)+nv_p(z)-v_p(z)\big) \\ &\geqslant \inf_{\substack{r\geqslant v_p(z)\geqslant s\\ n\in\mathbb{Z}}} \big(v_p(a_n)+nv_p(z)\big)-r\geqslant v^{[s,r]}(f)-r, \end{aligned}$$

as desired. □

Lemma 2.11. *We have*

$$\partial\circ\sigma_a=a\sigma_a\circ\partial,\quad \partial\circ\varphi_q=\pi\varphi_q\circ\partial,\quad \partial\circ\psi=\pi^{-1}\psi\circ\partial.$$

Proof. From the definition of ∇ we see that $\nabla=t_{\mathcal{F}}\partial$ commutes with Γ, φ_q and ψ. Hence the lemma follows from the equalities

$$\sigma_a(t_{\mathcal{F}})=at_{\mathcal{F}},\quad \varphi_q(t_{\mathcal{F}})=\pi t_{\mathcal{F}},\quad \psi(t_{\mathcal{F}})=\psi(\pi^{-1}\varphi_q(t_{\mathcal{F}}))=\pi^{-1}t_{\mathcal{F}}.$$ □

Let $\mathrm{res} : \mathcal{R}_L \mathrm{d}u_{\mathcal{F}} \to L$ be the residue map $\mathrm{res}\big(\sum_{i\in\mathbb{Z}} a_i u_{\mathcal{F}}^i \,\mathrm{d}u_{\mathcal{F}}\big) = a_{-1}$, and let $\mathrm{Res} : \mathcal{R}_L \to L$ be the map defined by $\mathrm{Res}(f) = \mathrm{res}(f\,\mathrm{d}t_{\mathcal{F}})$.

Proposition 2.12. *We have the exact sequence*

$$0 \to L \to \mathcal{R}_L \xrightarrow{\partial} \mathcal{R}_L \xrightarrow{\mathrm{Res}} L \to 0,$$

where $L \to \mathcal{R}_L$ is the inclusion map.

Proof. The kernel of ∂ is just the kernel of $\mathrm{d}/\mathrm{d}u_{\mathcal{F}}$ and thus is L. For any $a \in L$ we have $\mathrm{Res}\big((a/u_{\mathcal{F}}) \cdot (\mathrm{d}t_{\mathcal{F}}/\mathrm{d}u_{\mathcal{F}})^{-1}\big) = a$, which implies that Res is surjective. If $f = \partial g$, then $f\,\mathrm{d}t_{\mathcal{F}} = \mathrm{d}g$ and so $\mathrm{Res}(f) = \mathrm{res}(\mathrm{d}g) = 0$. It follows that $\mathrm{Res} \circ \partial = 0$. Conversely, if $f \in \mathcal{R}_L$ satisfies $\mathrm{Res}(f) = 0$, then f can be written as

$$f = \Big(\frac{\mathrm{d}t_{\mathcal{F}}}{\mathrm{d}u_{\mathcal{F}}}\Big)^{-1} \cdot \sum_{i\neq -1} a_i u_{\mathcal{F}}^i.$$

Put $g = \sum_{i\neq -1} \dfrac{a_i}{i+1} u_{\mathcal{F}}^{i+1}$. Then $f = \partial g$. □

Proposition 2.13. (a) $\mathrm{Res} \circ \sigma_a = a^{-1}\mathrm{Res}$.

(b) $\mathrm{Res} \circ \varphi_q = (q/\pi)\mathrm{Res}$ *and* $\mathrm{Res} \circ \psi = (\pi/q)\mathrm{Res}$.

Proof. First we prove (a). Let g be in $\mathcal{R}_L$ and put $f = \partial g$. By Lemma 2.11 we have

$$\sigma_a(f) = \sigma_a \circ \partial(g) = a^{-1}\partial(\sigma_a(g)), \qquad \psi(f) = \psi \circ \partial(g) = \pi\,\partial(\psi(g)).$$

Thus by Proposition 2.12 we have $\mathrm{Res} \circ \sigma_a = a^{-1}\mathrm{Res} = 0$ and $\mathrm{Res} \circ \psi = \frac{\pi}{q}\mathrm{Res} = 0$ on $\partial\mathcal{R}_L$. From

$$\sigma_a(1/u_{\mathcal{F}}) = \frac{1}{[a]_{\mathcal{F}}(u_{\mathcal{F}})} \equiv \frac{1}{a u_{\mathcal{F}}} \mod \mathcal{R}_L^+,$$

we see that $\mathrm{Res} \circ \sigma_a(1/u_{\mathcal{F}}) = a^{-1}\mathrm{Res}(1/u_{\mathcal{F}})$. Assertion (a) follows.

To prove $\mathrm{Res} \circ \psi = (\pi/q)\mathrm{Res}$, without loss of generality we suppose that $\mathcal{F}$ is the special Lubin–Tate group. In this case $\psi(1/u_{\mathcal{F}}) = \pi/(q u_{\mathcal{F}})$, and so $\mathrm{Res}(\psi(1/u_{\mathcal{F}})) = (\pi/q)\mathrm{Res}(1/u_{\mathcal{F}})$. It follows that $\mathrm{Res} \circ \psi = (\pi/q)\mathrm{Res}$. Finally we have $\mathrm{Res}(\varphi_q(z)) = (q/\pi)\mathrm{Res}\big(\psi(\varphi_q(z))\big) = (q/\pi)\mathrm{Res}(z)$ for any $z \in \mathcal{R}_L$. In other words, $\mathrm{Res} \circ \varphi_q = (q/\pi)\mathrm{Res}$. □

Using Res we can define a pairing $\{\cdot,\cdot\} : \mathcal{R}_L \times \mathcal{R}_L \to L$ by $\{f, g\} = \mathrm{Res}(fg)$.

Proposition 2.14. *The pairing $\{\cdot,\cdot\}$ is perfect and induces a continuous isomorphism from $\mathcal{R}_L$ to its dual. Moreover we have*

$$\{\sigma_a(f),\sigma_a(g)\} = a^{-1}\{f,g\}, \quad \{\varphi_q(f),\varphi_q(g)\} = \frac{q}{\pi}\{f,g\}, \quad \{f,\psi(g)\} = \frac{\pi}{q}\{\varphi_q(f),g\}.$$

Proof. The first assertion follows from [Colmez 2010d, Remark I.1.5]; the formulas from Proposition 2.13. □

3. Operators on $\mathcal{R}_{\mathbb{C}_p}$

3A. ***The operator ψ on $\mathcal{R}_{\mathbb{C}_p}$.*** First we define $\mathcal{R}_{\mathbb{C}_p}$. For any $r \geq 0$, let

$$\mathscr{E}^{]0,r]}_{\mathbb{C}_p} := \mathscr{E}^{]0,r]} \widehat{\otimes}_F \mathbb{C}_p$$

be the topological tensor product, i.e., the Hausdorff completion of the projective tensor product $\mathscr{E}^{]0,r]} \otimes_F \mathbb{C}_p$ (see [Schneider 2002]). Then $\mathscr{E}^{]0,r]}_{\mathbb{C}_p}$ is the ring of Laurent series $f = \sum_{i\in\mathbb{Z}} a_i u_{\mathcal{F}}^i$ with coefficients in $\mathbb{C}_p$ that are convergent on the annulus $0 < v_p(u_{\mathcal{F}}) \leq r$. We also write $\mathcal{R}^+_{\mathbb{C}_p}$ for $\mathscr{E}^{]0,+\infty]}_{\mathbb{C}_p}$. Then we define $\mathcal{R}_{\mathbb{C}_p}$ to be the inductive limit $\lim_{r\to 0} \mathscr{E}^{]0,r]}_{\mathbb{C}_p}$.

We recall how the p-adic Fourier theory of [Schneider and Teitelbaum 2001] shows that $\mathcal{R}^+_{\mathbb{C}_p}$ is isomorphic to the ring $\mathfrak{D}(\mathcal{O}_F, \mathbb{C}_p)$ of $\mathbb{C}_p$-valued locally F-analytic distributions on $\mathcal{O}_F$. From that reference we know that there exists a rigid analytic group variety $\mathfrak{X}$ such that $\mathfrak{X}(L)$, for any extension $L \subseteq \mathbb{C}_p$ of F, is the set of L-valued locally F-analytic characters. For $\lambda \in \mathfrak{D}(\mathcal{O}_F, L)$, put $F_\lambda(\chi) = \lambda(\chi)$, $\chi \in \mathfrak{X}(L)$. Then F_λ is in $\mathcal{O}(\mathfrak{X}/L)$, and the map $\mathfrak{D}(\mathcal{O}_F, L) \to \mathcal{O}(\mathfrak{X}/L)$, $\lambda \mapsto F_\lambda$, is an isomorphism of L-Fréchet algebras.

Let $\mathcal{F}'$ be the p-divisible group dual to $\mathcal{F}$ and $T\mathcal{F}'$ the Tate module of $\mathcal{F}'$. Then $T\mathcal{F}'$ is a free $\mathcal{O}_F$-module of rank 1; the Galois action on $T\mathcal{F}'$ is given by the continuous character $\tau := \chi_{\mathrm{cyc}} \cdot \chi_{\mathcal{F}}^{-1}$, where χ_{cyc} is the cyclotomic character. By Cartier duality, we obtain a Galois equivariant pairing $\langle\ ,\ \rangle : \mathcal{F}(\mathbb{C}_p) \otimes_{\mathcal{O}_F} T\mathcal{F}' \to \mathrm{B}_1(\mathbb{C}_p)$, where $\mathrm{B}_1(\mathbb{C}_p)$ is the multiplicative group $\{z \in \mathbb{C}_p : |z-1| < 1\}$. Fixing a generator t' of $T\mathcal{F}'$, we obtain a map $\mathcal{F}(\mathbb{C}_p) \to \mathrm{B}_1(\mathbb{C}_p)$. As a formal series, this morphism can be written as $\beta_{\mathcal{F}}(X) := \exp(\Omega \log_{\mathcal{F}}(X))$ for some $\Omega \in \mathbb{C}_p$, and it lies in $1 + X\mathbb{O}_{\mathbb{C}_p}[\![X]\!]$. Moreover, we have

$$v_p(\Omega) = \frac{1}{p-1} - \frac{1}{(q-1)e_F}$$

(see the appendix of [Schneider and Teitelbaum 2001] or [Colmez 1993]) and $\sigma(\Omega) = \tau(\sigma)\Omega$ for all $\sigma \in G_F$. Using $\langle \cdot, \cdot \rangle$ we obtain an isomorphism of rigid analytic group varieties

$$\kappa : \mathcal{F}(\mathbb{C}_p) \xrightarrow{\sim} \mathfrak{X}(\mathbb{C}_p), \quad z \mapsto \kappa_z(i) := \langle t', [i]_{\mathcal{F}}(z)\rangle = \beta_{\mathcal{F}}([i]_{\mathcal{F}}(z)).$$

Passing to global sections, we obtain the desired isomorphism

$$\mathfrak{D}(\mathcal{O}_F, \mathbb{C}_p) \cong \mathcal{O}(\mathfrak{X}/\mathbb{C}_p) \cong \mathcal{R}^+_{\mathbb{C}_p}.$$

We extend φ_q, ψ and the Γ-action $\mathbb{C}_p$-linearly and continuously to $\mathcal{R}_{\mathbb{C}_p}$. By continuity we have $\psi(\varphi_q(f)g) = f\psi(g)$ for any $f, g \in \mathcal{R}_{\mathbb{C}_p}$. All these actions keep $\mathcal{R}^+_{\mathbb{C}_p}$ invariant.

Lemma 3.1. *We have*

$$\begin{aligned}
\sigma_a(\beta_{\mathcal{F}}([i]_{\mathcal{F}})) &= \beta_{\mathcal{F}}([ai]_{\mathcal{F}}),\\
\varphi_q(\beta_{\mathcal{F}}([i]_{\mathcal{F}})) &= \beta_{\mathcal{F}}([\pi i]_{\mathcal{F}}),\\
\psi(\beta_{\mathcal{F}}([i]_{\mathcal{F}})) &= \begin{cases} 0 & \text{if } i \notin \pi\mathcal{O}_F,\\ \beta_{\mathcal{F}}([i/\pi]_{\mathcal{F}}) & \text{if } i \in \pi\mathcal{O}_F,\end{cases}\\
\partial(\beta_{\mathcal{F}}([i]_{\mathcal{F}})) &= i\Omega\beta_{\mathcal{F}}([i]_{\mathcal{F}}).
\end{aligned}$$

Proof. The formulae for σ_a and φ_q are obvious. The formula for ∂ follows from

$$\partial \exp(i\Omega\log_{\mathcal{F}}(u_{\mathcal{F}})) = \exp(i\Omega\log_{\mathcal{F}}(u_{\mathcal{F}}))\cdot\partial(i\Omega t_{\mathcal{F}}) = i\Omega\exp(i\Omega\log_{\mathcal{F}}(u_{\mathcal{F}})).$$

If $i \in \pi\mathcal{O}_F$, then $\psi(\beta_{\mathcal{F}}([i]_{\mathcal{F}})) = \psi\circ\varphi_q(\beta_{\mathcal{F}}([i/\pi]_{\mathcal{F}})) = \beta_{\mathcal{F}}([i/\pi]_{\mathcal{F}})$. For any $i \notin \pi\mathcal{O}_F$, we have

$$\begin{aligned}
\psi(\beta_{\mathcal{F}}([i]_{\mathcal{F}})) &= \frac{1}{q}\varphi_q^{-1}\Bigg(\sum_{\eta\in\ker[\pi]_{\mathcal{F}}}\beta_{\mathcal{F}}([i]_{\mathcal{F}}(u_{\mathcal{F}} +_{\mathcal{F}} \eta))\Bigg)\\
&= \frac{1}{q}\varphi_q^{-1}\Bigg(\beta_{\mathcal{F}}([i]_{\mathcal{F}})\sum_{\eta\in\ker[\pi]_{\mathcal{F}}}\beta_{\mathcal{F}}([i]_{\mathcal{F}}(\eta))\Bigg) = 0
\end{aligned}\tag{3-1}$$

because $\{\beta_{\mathcal{F}}([i]_{\mathcal{F}}(\eta)) : \eta\in\ker[\pi]_{\mathcal{F}}\} = \{\beta_{\mathcal{F}}(\eta) : \eta\in\ker[\pi]_{\mathcal{F}}\}$ take values in the set of p-th roots of unity and each of these p-th roots of unity appears q/p times. □

The isomorphism $\mathscr{R}^+_{\mathbb{C}_p} \cong \mathfrak{D}(\mathcal{O}_F, \mathbb{C}_p)$ transfers the actions of φ_q, ψ and Γ to $\mathfrak{D}(\mathcal{O}_F, \mathbb{C}_p)$.

Lemma 3.2. *For any $\mu \in \mathfrak{D}(\mathcal{O}_F, \mathbb{C}_p)$, we have*

$$\sigma_a(\mu)(f) = \mu(f(a\,\cdot)),\quad \varphi_q(\mu)(f) = \mu(f(\pi\,\cdot)).$$

Proof. Note that the action of φ_q and Γ on $\mathscr{R}^+_{\mathbb{C}_p}$ comes, by passing to global sections, from the (φ_q, Γ)-action on $\mathcal{F}(\mathbb{C}_p)$ with $\varphi_q = [\pi]_{\mathcal{F}}$ and $\sigma_a = [a]_{\mathcal{F}}$. The isomorphism κ transfers the action to $\mathfrak{X}(\mathbb{C}_p)$: $\varphi_q(\chi)(x) = \chi(\pi x)$ and $\sigma_a(\chi)(x) = \chi(ax)$. Passing to global sections yields what we want. □

Lemma 3.3. *The family $(\beta_{\mathcal{F}}([i]_{\mathcal{F}}))_{\bar{i}\in\mathcal{O}_F/\pi}$ is a basis of $\mathscr{R}_{\mathbb{C}_p}$ over $\varphi_q(\mathscr{R}_{\mathbb{C}_p})$. Moreover, if*

$$f = \sum_{\bar{i}\in\mathcal{O}_F/\pi}\beta_{\mathcal{F}}([i]_{\mathcal{F}})\varphi_q(f_i),$$

then the terms of the sum do not depend on the choice of the liftings i, and

$$f_i = \psi(\beta_{\mathcal{F}}([-i]_{\mathcal{F}})f).$$

Proof. What we need to show is that

$$f = \sum_{\bar{i}\in\mathcal{O}_F/\pi} \beta_{\mathcal{F}}([i]_{\mathcal{F}}) \cdot \varphi_q \circ \psi(\beta_{\mathcal{F}}([-i]_{\mathcal{F}}) f) \tag{3-2}$$

for all $f \in \mathcal{R}_{\mathbb{C}_p}$. Indeed, (3-2) implies that $\{\beta_{\mathcal{F}}([i]_{\mathcal{F}})\}_{\bar{i}\in\mathcal{O}_F/\pi}$ generate $\mathcal{R}_{\mathbb{C}_p}$ over $\varphi_q(\mathcal{R}_{\mathbb{C}_p})$. On the other hand, if

$$f = \sum_{\bar{i}\in\mathcal{O}_F/\pi} \beta_{\mathcal{F}}([i]_{\mathcal{F}})\varphi_q(f_i),$$

using (3-1) we obtain $f_i = \psi(\beta_{\mathcal{F}}([-i]_{\mathcal{F}}) f)$, which implies the linear independence of $\{\beta_{\mathcal{F}}([i]_{\mathcal{F}})\}_{\bar{i}\in\mathcal{O}_F/\pi}$ over $\varphi_q(\mathcal{R}_{\mathbb{C}_p})$. As the map

$$f \mapsto \sum_{\bar{i}\in\mathcal{O}_F/\pi} \beta_{\mathcal{F}}([i]_{\mathcal{F}}) \cdot \varphi_q \circ \psi(\beta_{\mathcal{F}}([-i]_{\mathcal{F}}) f)$$

is $\varphi_q(\mathcal{R}_{\mathbb{C}_p})$-linear and continuous, we only need to prove (3-2) for a subset that topologically generates $\mathcal{R}_{\mathbb{C}_p}$ over $\varphi_q(\mathcal{R}_{\mathbb{C}_p})$. For example, $\{u_{\mathcal{F}}^i\}_{0\leq i\leq q-1}$ is such a subset. So it is sufficient to prove (3-2) for $f \in \mathcal{R}^+_{\mathbb{C}_p}$. For any $i \in \mathcal{O}_F$, let δ_i be the Dirac distribution such that $\delta_i(f) = f(i)$. Then $\kappa^*(\delta_i) = \beta_{\mathcal{F}}([i]_{\mathcal{F}})$. Indeed, we have

$$\kappa^*(\delta_i)(z) = \delta_i(z) = \kappa_z(i) = \beta_{\mathcal{F}}([i]_{\mathcal{F}}(z)).$$

It is easy to see that $(\delta_i)_{\bar{i}\in\mathcal{O}_F/\pi}$ is a basis of $\mathcal{D}(\mathcal{O}_F, \mathbb{C}_p)$ over $\varphi_q(\mathcal{D}(\mathcal{O}_F, \mathbb{C}_p))$. Thus every $f \in \mathcal{R}^+_{\mathbb{C}_p}$ can be written uniquely in the form $f = \sum_{\bar{i}\in\mathcal{O}_F/\pi} \beta_{\mathcal{F}}([i]_{\mathcal{F}})\varphi_q(f_i)$ with $f_i \in \mathcal{R}^+_{\mathbb{C}_p}$. As observed above, from (3-1) we deduce that $f_i = \psi(\beta_{\mathcal{F}}([-i]_{\mathcal{F}}) f)$. □

Next we define operators Res_U, analogous to the operators defined in [Colmez 2010d].

For any $f \in \mathcal{R}_{\mathbb{C}_p}$, $i \in \mathcal{O}_F$ and integer $m \geqslant 0$, put

$$\mathrm{Res}_{i+\pi^m\mathcal{O}_F}(f) = \beta_{\mathcal{F}}([i]_{\mathcal{F}})(\varphi_q^m \circ \psi^m)(\beta_{\mathcal{F}}([-i]_{\mathcal{F}}) f).$$

Lemma 3.3 says that

$$f = \sum_{\bar{i}\in\mathcal{O}_F/\pi} \mathrm{Res}_{i+\pi\mathcal{O}_F}(f).$$

This implies that the operators $\mathrm{Res}_{i+\pi^m\mathcal{O}_F}$ are well defined (i.e., independent of the choice of i in the ball $i + \pi^m\mathcal{O}_F$). Applying Lemma 3.3 recursively we get

$$f = \sum_{\bar{i}\in\mathcal{O}_F/\pi^m} \mathrm{Res}_{i+\pi^m\mathcal{O}_F}(f).$$

Finally, if U is a compact open subset of $\mathcal{O}_F$, it is a finite disjoint union of balls $i_k + \pi^{m_k}\mathcal{O}_F$. Define $\mathrm{Res}_U = \sum_k \mathrm{Res}_{i_k+\pi^{m_k}\mathcal{O}_F}$. The map $\mathrm{Res}_U : \mathcal{R}_{\mathbb{C}_p} \to \mathcal{R}_{\mathbb{C}_p}$ does not depend on the choice of these balls, and we have $\mathrm{Res}_{\mathcal{O}_F} = 1$, $\mathrm{Res}_{\varnothing} = 0$ and $\mathrm{Res}_{U\cup U'} + \mathrm{Res}_{U\cap U'} = \mathrm{Res}_U + \mathrm{Res}_{U'}$.

3B. ***The operator m_α.*** Let $\alpha : \mathcal{O}_F \to \mathbb{C}_p$ be a locally (F-)analytic function. In this subsection, we define an operator $m_\alpha : \mathcal{R}_{\mathbb{C}_p} \to \mathcal{R}_{\mathbb{C}_p}$ similar to the one defined in [Colmez 2010c, V.2].

Since α is a locally analytic function on $\mathcal{O}_F$, there is an integer $m \geqslant 0$ such that

$$\alpha(x) = \sum_{n=0}^{+\infty} a_{i,n}(x-i)^n \quad \text{for all } x \in i + \pi^m \mathcal{O}_F,$$

with $a_{i,n} = (1/n!)(\mathrm{d}^n/\mathrm{d}x^n)\alpha(x)|_{x=i}$. Let $\ell \geqslant m$ be an integer. Define

$$m_\alpha(f) = \sum_{\bar{i}\in\mathcal{O}_F/\pi^\ell} \beta_{\mathcal{F}}([i]_{\mathcal{F}})\left(\varphi_q^\ell \circ \left(\sum_{n=0}^{+\infty} a_{i,n}\pi^{\ell n}\Omega^{-n}\partial^n\right)\circ \psi^\ell\right)(\beta_{\mathcal{F}}([-i]_{\mathcal{F}})\cdot f).$$

(Formally, this definition can be seen as saying that $m_\alpha = \alpha(\Omega^{-1}\partial)$). According to Lemmas 2.6, 2.7 and 2.10, if $r < 1/(q^{\ell-1}(q-1)e_F)$ then we have

$$v^{[s,r]}\big((\varphi_q^\ell \circ \Omega^{-n}\partial^n \circ \psi^\ell)(g)\big) \geqslant -nq^\ell r - nv_p(\Omega) + v^{[s,r]}(g) - \ell v_p(q),$$

and thus $\sum_{n=0}^{+\infty} a_{n,i}\pi^{\ell n}(\varphi_q^\ell \circ \Omega^{-n}\partial^n \circ \psi^\ell)(g)$ converges when ℓ and r satisfy

$$\frac{\ell}{e_F} - q^\ell r - \frac{1}{p-1} + \frac{1}{(q-1)e_F} \geqslant \frac{m}{e_F}.$$

If we choose $\ell > m + e_F/(p-1) - 1/(q-1)$ and r close enough to 0, then this condition is satisfied. Hence, we have indeed defined a continuous operator $m_\alpha : \mathcal{R}_{\mathbb{C}_p} \to \mathcal{R}_{\mathbb{C}_p}$.

Now, let us prove that $m_\alpha(f)$ neither depend on the choice of ℓ, nor on that of the liftings i for $\bar{i} \in \mathcal{O}_F/\pi^\ell$. By linearity and continuity, we may assume that $f = 1_{i+\pi^m\mathcal{O}_F}(x-i)^k$. Note that we have

$$a_{i+\pi^m v,n} = \begin{bmatrix} k \\ n \end{bmatrix} \pi^{(k-n)m} v^{k-n}.$$

It suffices to show that

$$\sum_{\bar{v}\in\mathcal{O}_F/\pi^{\ell-m}} \beta_{\mathcal{F}}([\pi^m v]_{\mathcal{F}})\left(\varphi_q^\ell \circ \left(\sum_{n=0}^{k} a_{i+\pi^m v,n}\pi^{\ell n}\Omega^{-n}\partial^n\right)\circ\psi^\ell\right)\big(\beta_{\mathcal{F}}([-\pi^m v]_{\mathcal{F}})\cdot f\big) = \big(\varphi_q^m \circ (\pi^{mk}\Omega^{-k}\partial^k)\circ\psi^m\big)f,$$

and for this it is sufficient to prove that

$$\sum_{\overline{v}\in\mathcal{O}_F/\pi^{\ell-m}} \beta_{\mathcal{F}}([v]_{\mathcal{F}})\left(\varphi_q^{\ell-m}\circ\left(\sum_{n=0}^{k} a_{i+\pi^m v,n}\pi^{\ell n}\Omega^{-n}\,\partial^n\right)\circ\psi^{\ell-m}\right)\bigl(\beta_{\mathcal{F}}([-v]_{\mathcal{F}})\cdot f\bigr)$$
$$=\pi^{mk}\Omega^{-k}\,\partial^k f.$$

As

$$\sum_{n=0}^{k} a_{i+\pi^m v,n}\pi^{\ell n}\Omega^{-n}\,\partial^n=\sum_{n=0}^{k}\begin{bmatrix}k\\ n\end{bmatrix}\pi^{(k-n)m}v^{k-n}\cdot\pi^{\ell n}\Omega^{-n}\,\partial^n$$
$$=\pi^{mk}\left(\pi^{\ell-m}\Omega^{-1}\,\partial+v\right)^k,$$

it suffices to prove that

$$\Omega^{-k}\,\partial^k f=\sum_{\overline{v}\in\mathcal{O}_F\pi^{\ell-m}}\beta_{\mathcal{F}}([v]_{\mathcal{F}})\bigl(\varphi_q^{\ell-m}\circ(\pi^{\ell-m}\Omega^{-1}\,\partial+v)^k\circ\psi^{\ell-m}\bigr)\bigl(\beta_{\mathcal{F}}([-v]_{\mathcal{F}})f\bigr).$$

Since $(\pi^{\ell-m}\Omega^{-1}\,\partial+v)^k\circ\psi^{\ell-m}=\psi^{\ell-m}\circ(\Omega^{-1}\,\partial+v)^k$ and

$$(\Omega^{-1}\,\partial+v)(\beta_{\mathcal{F}}([-v]_{\mathcal{F}})f)=\beta_{\mathcal{F}}([-v]_{\mathcal{F}})\Omega^{-1}\,\partial f$$

(which follows from Lemma 3.1), the problem reduces to proving

$$f=\sum_{\overline{v}\in\mathcal{O}_F/\pi^{\ell-m}}\beta_{\mathcal{F}}([v]_{\mathcal{F}})(\varphi^{\ell-m}\circ\psi^{\ell-m})(\beta_{\mathcal{F}}([-v]_{\mathcal{F}})f).$$

But this can be deduced from Lemma 3.1 and Lemma 3.3.

Lemma 3.4. *If $\alpha,\beta:\mathcal{O}_F\to\mathbb{C}_p$ are locally analytic functions, then $m_\alpha\circ m_\beta=m_{\alpha\beta}$.*

Proof. We can choose ℓ sufficiently large, so that the same value can be used to define $m_\alpha(f)$ and $m_\beta(f)$. Since $\psi^\ell\circ\varphi_q^\ell=1$, the equality in the lemma reduces to the expression of the product of two power series. □

Lemma 3.5.

- $m_1=\mathrm{id}$.
- *If U is a compact open subset of $\mathcal{O}_F$, then* $\mathrm{Res}_U=m_{\mathbf{1}_U}$.
- *If $\lambda\in\mathbb{C}_p$, then $m_{\lambda\alpha}=\lambda m_\alpha$.*
- $\varphi_q\circ m_\alpha=m_{x\mapsto\mathbf{1}_{\pi\mathcal{O}_F}(x)\alpha(\pi^{-1}x)}\circ\varphi_q$.
- $\psi\circ m_\alpha=m_{x\mapsto\alpha(\pi x)}\circ\psi$.
- *For any $a\in\mathcal{O}_F^\times$, we have $\sigma_a\circ m_\alpha=m_{x\mapsto\alpha(a^{-1}x)}\circ\sigma_a$.*
- *$\mathcal{R}^+_{\mathbb{C}_p}$ is stable under m_α.*

Proof. These are easy consequences of the definition of m_α. □

Remark 3.6. The notation m_α stands for "multiply by α": for any $\mu \in \mathfrak{D}(\mathcal{O}_F, \mathbb{C}_p)$ we have $m_\alpha \kappa^*(F_\mu) = \kappa^*(F_{\alpha\mu})$, where $\alpha\mu$ is the distribution such that $(\alpha\mu)(f) = \mu(\alpha f)$ for all locally F-analytic function f.

The operator m_α has been defined over $\mathcal{R}_{\mathbb{C}_p}$, using a period $\Omega \in \mathbb{C}_p$ that is transcendental over F. However, in some cases, it is possible to construct related operators over $\mathcal{R}_L$, for L smaller than $\mathbb{C}_p$. This is done using the following lemma.

Lemma 3.7. *Let σ be in G_L. Consider the action of σ over $\mathcal{R}_{\mathbb{C}_p}$ given by*

$$f^\sigma(u_{\mathcal{F}}) = \sum_{n\in\mathbb{Z}} \sigma(a_n) u_{\mathcal{F}}^n \quad \text{if } f(u_{\mathcal{F}}) = \sum_{n\in\mathbb{Z}} a_n u_{\mathcal{F}}^n \in \mathcal{R}_{\mathbb{C}_p}.$$

Then $m_\alpha(f)^\sigma = m_\beta(f^\sigma)$ for $\beta(x) = \sigma\left(\alpha\left(\dfrac{\chi_{\mathcal{F}}(\sigma)}{\chi_{\mathbb{G}_m}(\sigma)}x\right)\right)$.

Proof. This can be deduced easily from the definition of m_α and the action of σ on Ω. □

3C. *The $L[\Gamma]$-module $\mathcal{R}_L(\delta)^{\psi=0}$.*

Let $\delta : F^\times \to L^\times$ be a locally F-analytic character. Then the map $x \mapsto \mathbf{1}_{\mathcal{O}_F^\times}(x)\delta(x)$ is locally analytic on $\mathcal{O}_F$. Thus, we have an operator $m_{\mathbf{1}_{\mathcal{O}_F^\times}\delta}$ on $\mathcal{R}_{\mathbb{C}_p}$.

Lemma 3.8. *Let f be in $\mathcal{R}_L$. If $m_{\mathbf{1}_{\mathcal{O}_F^\times}\delta}(f) = \sum_{n\in\mathbb{Z}} a_n u_{\mathcal{F}}^n \in \mathcal{R}_{\mathbb{C}_p}$, the coefficients a_n are all on the same line of the L-vector space $\mathbb{C}_p$. Moreover, this line does not depend on f.*

Proof. Let σ be in G_L. From Lemma 3.7 and Lemma 3.5 we see that

$$m_{\mathbf{1}_{\mathcal{O}_F^\times}\delta}(f)^\sigma = \delta\left(\frac{\chi_{\mathcal{F}}(\sigma)}{\chi_{\mathbb{G}_\mathrm{m}}(\sigma)}\right) m_{\mathbf{1}_{\mathcal{O}_F^\times}\delta}(f),$$

and thus $\sigma(a_n) = \delta\left(\dfrac{\chi_{\mathcal{F}}(\sigma)}{\chi_{\mathbb{G}_\mathrm{m}}(\sigma)}\right) a_n$ for all n.

The Ax–Sen–Tate theorem (see [Ax 1970] or [Le Borgne 2010], for example) says that $\mathbb{C}_p^{G_L} = L$. Hence,

$$\left\{ z \in \mathbb{C}_p : \sigma(z) = \delta\left(\frac{\chi_{\mathcal{F}}(\sigma)}{\chi_{\mathbb{G}_\mathrm{m}}(\sigma)}\right) z \text{ for all } \sigma \in G_L \right\}$$

is an L-vector subspace of $\mathbb{C}_p$ with dimension 0 or 1, which proves the lemma. □

Since

$$m_{\mathbf{1}_{\mathcal{O}_F^\times}\delta} \circ m_{\mathbf{1}_{\mathcal{O}_F^\times}\delta^{-1}} = \mathrm{Res}_{\mathcal{O}_F^\times} = 1 - \varphi_q \circ \psi$$

is not null, there is a unique L-line in $\mathbb{C}_p$ (which depends only on δ) in which all the coefficients of the series $m_{\mathbf{1}_{\mathcal{O}_F^\times}\delta}(f)$, for $f \in \mathcal{R}_L$, lie. Choose some nonzero a_δ on this line.

Since $\varphi_q \circ \psi \circ m_{\mathbf{1}_{\mathcal{O}_F^\times}\delta} = m_{\mathbf{1}_{\pi\mathcal{O}_F}\mathbf{1}_{\mathcal{O}_F^\times}\delta} = 0$ and φ_q is injective, $m_{\mathbf{1}_{\mathcal{O}_F^\times}\delta}(f)$ is in $\mathcal{R}_{\mathbb{C}_p}^{\psi=0}$.

Lemma 3.9. *Define*

$$M_\delta : \mathcal{R}_L^{\psi=0} \to \mathcal{R}_L^{\psi=0}, \quad f \mapsto a_\delta^{-1} m_{\mathbf{1}_{\mathcal{O}_F^\times}\delta}(f).$$

(*These maps are defined up to homothety, with ratio in L, because of the choice of constants a_δ*).

- *M_1 is a homothety (with ratio in $L^\times$) of $\mathcal{R}_L^{\psi=0}$.*
- *$M_{\delta_1} \circ M_{\delta_2} = M_{\delta_1\delta_2}$, up to homothety.*
- *M_δ is a bijection, and its inverse is $M_{\delta^{-1}}$ up to homothety.*
- *For all $\gamma \in \Gamma$, we have $\delta(\gamma)\gamma \circ M_\delta = M_\delta \circ \gamma$.*
- *$(\mathcal{R}_L^+)^{\psi=0}$ is stable under M_δ.*

Proof. This follows from Lemma 3.5 and the equalities

$$\operatorname{Im}\operatorname{Res}_{\mathcal{O}_F^\times} = \operatorname{Ker}\operatorname{Res}_{\pi\mathcal{O}_F} = \mathcal{R}_{\mathbb{C}_p}^{\psi=0}. \qquad \square$$

If δ is in $\mathscr{I}_{\text{an}}(L)$, we put $\mathcal{R}_L^-(\delta) = \mathcal{R}_L(\delta)/\mathcal{R}_L^+(\delta)$. Since $\mathcal{R}_L^+(\delta)$ is φ_q, ψ, Γ-stable, $\mathcal{R}_L^-(\delta)$ also has φ_q, ψ, Γ-actions.

Lemma 3.10. *We have an exact sequence*

$$0 \to \mathcal{R}_L^+(\delta)^{\psi=0} \to \mathcal{R}_L(\delta)^{\psi=0} \to \mathcal{R}_L^-(\delta)^{\psi=0} \to 0.$$

Proof. This follows from the snake lemma and the surjectivity of the map

$$\psi : \mathcal{R}_L^+(\delta) \to \mathcal{R}_L^+(\delta). \qquad \square$$

Observe that $\mathcal{R}_L(\delta)^{\psi=0} = \mathcal{R}_L^{\psi=0} \cdot e_\delta$ and $\mathcal{R}_L^+(\delta)^{\psi=0} = (\mathcal{R}_L^+)^{\psi=0} \cdot e_\delta$. As ψ commutes with Γ, $\mathcal{R}_L(\delta)^{\psi=0}$, $\mathcal{R}_L^+(\delta)^{\psi=0}$ and $\mathcal{R}_L^-(\delta)^{\psi=0}$ are all Γ-invariant.

Proposition 3.11. *Let δ_1 and δ_2 be two locally F-analytic characters $F^\times \to L^\times$. Then as $L[\Gamma]$-modules, $\mathcal{R}_L(\delta_1)^{\psi=0}$ is isomorphic to $\mathcal{R}_L(\delta_2)^{\psi=0}$, $\mathcal{R}_L^+(\delta_1)^{\psi=0}$ is isomorphic to $\mathcal{R}_L^+(\delta_2)^{\psi=0}$, and $\mathcal{R}_L^-(\delta_1)^{\psi=0}$ is isomorphic to $\mathcal{R}_L^-(\delta_2)^{\psi=0}$.*

Proof. All of the isomorphisms in question are induced by $M_{\delta_1^{-1}\delta_2}$. $\square$

Proposition 3.12. *The map ∂ induces Γ-equivariant isomorphisms*

$$(\mathcal{R}_L(\delta))^{\psi=0} \to (\mathcal{R}_L(x\delta))^{\psi=0},$$
$$(\mathcal{R}_L^+(\delta))^{\psi=0} \to (\mathcal{R}_L^+(x\delta))^{\psi=0},$$
$$(\mathcal{R}_L^-(\delta))^{\psi=0} \to (\mathcal{R}_L^-(x\delta))^{\psi=0}.$$

Proof. We first show that the maps in question are bijective. For this we only need to consider the case of $\delta = 1$. Since $\operatorname{Ker}\partial = L$, ∂ is injective on $\mathcal{R}_L^{\psi=0}$. For any

$z \in \mathcal{R}_L^{\psi=0}$, $\mathrm{Res}(z) = (q/\pi)\mathrm{Res}(\psi(z)) = 0$. Thus by Proposition 2.12 there exists $z' \in \mathcal{R}_L$ such that $\partial z' = z$. As $\partial(\psi(z')) = \frac{1}{\pi}\psi(\partial z') = 0$, we have $\psi(z) = c$ for some $c \in L$. Then $z' - c \in \mathcal{R}_L^{\psi=0}$ and $\partial(z' - c) = z$. This shows that the map $\mathcal{R}_L^{\psi=0} \to \mathcal{R}_L^{\psi=0}$ is bijective. It is clear that, for any $z \in \mathcal{R}_L^{\psi=0}$, $\partial z \in \mathcal{R}_L^+$ if and only if $z \in \mathcal{R}_L^+$. Thus the restriction $\partial : (\mathcal{R}_L^+)^{\psi=0} \to (\mathcal{R}_L^+)^{\psi=0}$ and the induced map $\partial : (\mathcal{R}_L^-)^{\psi=0} \to (\mathcal{R}_L^-)^{\psi=0}$ are also bijective.

That these isomorphisms are Γ-equivariant follows from Lemma 2.11. □

Put

$$S_\delta := \mathcal{R}_L^-(\delta)^{\Gamma=1,\psi=0}. \tag{3-3}$$

As before, let ∇_δ be the operator on $\mathcal{R}_L^+$ or $\mathcal{R}_L$ such that $(\nabla_\delta a)e_\delta = \nabla(ae_\delta)$, i.e., $\nabla_\delta = t_{\mathcal{F}}\partial + w_\delta$. The set $\mathcal{R}_L^+(\delta)/\nabla_\delta\mathcal{R}_L^+(\delta)$ also admits actions of Γ, φ_q and ψ. Put

$$T_\delta := (\mathcal{R}_L^+(\delta)/\nabla_\delta\mathcal{R}_L^+(\delta))^{\Gamma=1,\psi=0}.$$

Both S_δ and T_δ are L-vector spaces and only depend on $\delta|_{\mathcal{O}_F^\times}$.

Lemma 3.13. $S_\delta = \mathcal{R}_L^-(\delta)^{\psi=0,\nabla_\delta=0,\ \Gamma=1}$; *that is, S_δ coincides with the set of Γ-invariant solutions of $\nabla_\delta z = 0$ in $\mathcal{R}_L^-(\delta)^{\psi=0}$.*

Proof. In fact, if $z \in \mathcal{R}_L^-(\delta)^{\Gamma=1}$, then $\nabla_\delta z = 0$. □

Corollary 3.14. $\dim_L S_\delta = \dim_L S_1$ *and* $\dim_L T_\delta = \dim_L T_1$ *for all* $\delta \in \mathcal{I}_{\mathrm{an}}(L)$.

Proof. This follows directly from Proposition 3.11. □

Corollary 3.15. *The map $z \mapsto \partial^n z$ induces isomorphisms $S_\delta \to S_{x^n\delta}$ and $T_\delta \to T_{x^n\delta}$.*

Proof. This follows directly from Proposition 3.12. □

We determine $\dim_L S_\delta$ and $\dim_L T_\delta$ below.

Lemma 3.16. *The map ∇_δ induces an injection $\overline{\nabla}_\delta : S_\delta \to T_\delta$.*

Proof. By Proposition 3.11 we only need to consider the case of $\delta = 1$.

Let z be an element of S_1. Let $\tilde{z} \in \mathcal{R}_L^{\psi=0}$ be a lifting of z. By Lemma 3.13, $\nabla\tilde{z}$ is in $\mathcal{R}_L^+$. We show that the image of $\nabla\tilde{z}$ in $\mathcal{R}_L^+/\nabla\mathcal{R}_L^+$ belongs to T_1. Since $\psi(\tilde{z}) = 0$, $\psi(\nabla\tilde{z}) = \nabla(\psi(\tilde{z})) = 0$. For any $\gamma \in \Gamma$ there exists $a_\gamma \in \mathcal{R}_L^+$ such that $\gamma\tilde{z} = \tilde{z} + a_\gamma$. Thus $\gamma(\nabla\tilde{z}) = \nabla\tilde{z} + \nabla a_\gamma$. Hence the image of $\tilde{z}$ in $\mathcal{R}_L^+/\nabla\mathcal{R}_L^+(\delta)$ is fixed by Γ. Furthermore, the image only depends on z: if $\tilde{z}' \in \mathcal{R}_L^{\psi=0}$ is another lifting of z, then $\nabla(\tilde{z}' - \tilde{z})$ is in $\nabla\mathcal{R}_L^+$. Therefore we obtain a map $\overline{\nabla} : S_1 \to T_1$.

We prove that $\overline{\nabla}$ is injective. Suppose that $z \in S_1$ satisfies $\overline{\nabla}z = 0$. Let $\tilde{z} \in \mathcal{R}_L^{\psi=0}$ be a lifting of z. Since $\nabla\tilde{z}$ is in $\nabla\mathcal{R}_L^+$, there exists $y \in \mathcal{R}_L^+$ such that $\nabla y = \nabla\tilde{z}$. Thus $\nabla(\tilde{z} - y) = 0$. Then $\tilde{z} - y$ is in L, which implies that $\tilde{z} \in \mathcal{R}_L^+$, or equivalently $z = 0$. □

Lemma 3.17. $$\dim_L T_1 = 1.$$

Proof. Note that $T_1=(\mathcal{R}_L^+/\mathcal{R}_L^+t_{\mathcal{F}})^{\Gamma=1,\psi=0}$. As $\mathcal{R}_L^+$ is a Fréchet–Stein algebra, from the decomposition of the ideal $(t_{\mathcal{F}})$ given by Corollary 1.2 we obtain an isomorphism

$$\jmath:\mathcal{R}_L^+/\mathcal{R}_L^+t_{\mathcal{F}}\xrightarrow{\sim}\mathcal{R}_L^+/([\pi]_{\mathcal{F}}(u_{\mathcal{F}}))\times\prod_{n\geq 1}\mathcal{R}_L^+/(\varphi_q^n(Q)). \tag{3-4}$$

The operator ψ induces maps

$$\psi_0:\mathcal{R}_L^+/([\pi]_{\mathcal{F}}(u_{\mathcal{F}}))\to\mathcal{R}_L^+/\mathcal{R}_L^+u_{\mathcal{F}}\quad\text{and}\quad\psi_n:\mathcal{R}_L^+/(\varphi_q^n(Q))\to\mathcal{R}_L^+/(\varphi_q^{n-1}(Q)),$$

for $n\geq 1$. Thus $\jmath((\mathcal{R}_L^+/\mathcal{R}_L^+t_{\mathcal{F}})^{\Gamma=1,\psi=0})$ is exactly the subset of the codomain of (3-4) consisting of $(y_n)_{n\geq 0}$ such that $y_0\in(\mathcal{R}_L^+/([\pi]_{\mathcal{F}}(u_{\mathcal{F}})))^{\Gamma}$, $\psi_0(y_0)=0$, and

$$y_n\in(\mathcal{R}_L^+/(\varphi_q^n(Q)))^{\Gamma},\quad\psi_n(y_n)=0,\quad\text{for all } n\geq 1.$$

If $n\geq 1$, then $\mathcal{R}_F^+/\varphi_q^n(Q)$ is a finite extension of F and the action of Γ factors through the whole Galois group of this extension. Thus $(\mathcal{R}_F^+/(\varphi_q^n(Q)))^{\Gamma}=F$ and $(\mathcal{R}_L^+/(\varphi_q^n(Q)))^{\Gamma}=L$. Since $\psi_n(a)=a$ for any $a\in L$, $(\mathcal{R}_F^+/(\varphi_q^n(Q)))^{\Gamma}\cap\ker(\psi_n)=0$ for any $n\geq 1$. Similarly $(\mathcal{R}_L^+/([\pi]_{\mathcal{F}}(u_{\mathcal{F}})))^{\Gamma}=(\mathcal{R}_L^+/(u_{\mathcal{F}}))^{\Gamma}\times(\mathcal{R}_L^+/(Q))^{\Gamma}$ has dimension 2 over L. As $\psi_0(1)=1$ and the image $\mathcal{R}_L^+/\mathcal{R}_L^+u_{\mathcal{F}}$ of ψ_0 has dimension 1 over L, the kernel of $\psi_0|_{(\mathcal{R}_L^+/([\pi]_{\mathcal{F}}(u_{\mathcal{F}})))^{\Gamma}}$ is of dimension 1. It follows that $T_1=(\mathcal{R}_L^+/\mathcal{R}_L^+t_{\mathcal{F}})^{\Gamma=1,\psi=0}$ is of dimension 1. □

Corollary 3.18. $\dim_L S_1=1.$

Proof. The map ∇ injects S_1 into T_1 with image of dimension 1. □

Remark 3.19. If $z\in T_1$ is nonzero, then any lifting $\tilde{z}\in\mathcal{R}_L^+$ of z is not in $u_{\mathcal{F}}\mathcal{R}_L^+$, or equivalently $\tilde{z}|_{u_{\mathcal{F}}=0}\neq 0$. We only need to verify this for the special Lubin–Tate group. In this case, $\mathcal{R}_L^+/([\pi]_{\mathcal{F}}(u_{\mathcal{F}}))=\bigoplus_{i=0}^{q-1}Lu_{\mathcal{F}}^i$. We have

$$\big(\mathcal{R}_L^+/([\pi]_{\mathcal{F}}(u_{\mathcal{F}}))\big)^{\Gamma}=L\oplus Lu_{\mathcal{F}}^{q-1}.$$

Indeed, an element of $\mathcal{R}_L^+/([\pi]_{\mathcal{F}}(u_{\mathcal{F}}))$ is fixed by Γ if and only if it is fixed by the operators $z\mapsto\sigma_\xi(z)$ with $\xi\in\mu_{q-1}$; but $\sigma_\xi(u_{\mathcal{F}})=[\xi]_{\mathcal{F}}(u_{\mathcal{F}})=\xi u_{\mathcal{F}}$, so $\sigma_\xi(u_{\mathcal{F}}^i)=\xi^i u_{\mathcal{F}}^i$ for any $i\in\mathbb{N}$. Then $(\mathcal{R}_L^+/([\pi]_{\mathcal{F}}(u_{\mathcal{F}})))^{\Gamma=1,\psi=0}=L\cdot(u_{\mathcal{F}}^{q-1}-(1-q)\pi/q)$.

Proposition 3.20. *For any $\delta\in\mathcal{I}_{\mathrm{an}}(L)$, $\dim_L S_\delta=\dim_L T_\delta=1$ and the map $\overline{\nabla}_\delta$ is an isomorphism.*

Proof. Use Corollary 3.14, Lemma 3.16, Lemma 3.17 and Corollary 3.18. □

4. Cohomology theories for (φ_q,Γ)-modules

For a (φ_q,Γ)-module D over $\mathcal{R}_L$, the (φ_q,Γ)-module structure induces an action of the semigroup $G^+:=\varphi_q^{\mathbb{N}}\times\Gamma$ on D. Following [Colmez 2010a] we define $H^\bullet(D)$

as the cohomology of the semigroup G^+. Let $C^\bullet(G^+, D)$ be the complex

$$0 \to C^0(G^+, D) \xrightarrow{d_1} C^1(G^+, D) \xrightarrow{d_2} \cdots,$$

where $C^0(G^+, D) = D$, $C^n(G^+, D)$ for $n \geq 1$ is the set of continuous functions from $(G^+)^n$ to D, and d_{n+1} is the differential

$$d_{n+1}c(g_0, \dots, g_n) =$$
$$g_0 \cdot c(g_1, \dots, g_n) + \sum_{i=1}^{n-1} (-1)^{i+1} c(g_0, \dots, g_i g_{i+1}, \dots g_n) + (-1)^{n+1} c(g_0, \dots g_{n-1}).$$

Then $H^i(D) = H^i(C^\bullet(G^+, D))$.

If D_1 and D_2 are (φ_q, Γ)-modules over $\mathscr{R}_L$, we use $\mathrm{Ext}(D_1, D_2)$ to denote the set, in fact an L-vector space, of extensions of D_1 by D_2 in the category of (φ_q, Γ)-modules over $\mathscr{R}_L$.

We construct a natural map $\Theta^D : \mathrm{Ext}(\mathscr{R}_L, D) \to H^1(D)$ for any (φ_q, Γ)-module D. Let $\widetilde{D}$ be an extension of $\mathscr{R}_L$ by D. Let $e \in \widetilde{D}$ be a lifting of $1 \in \mathscr{R}_L$. Then $g \mapsto g(e) - e$, $g \in G^+$, is a 1-cocycle, and induces an element of $H^1(D)$ independent of the choice of e. Thus we obtain the desired map

$$\Theta^D : \mathrm{Ext}(\mathscr{R}_L, D) \to H^1(D).$$

Proposition 4.1. *For any (φ_q, Γ)-module D over $\mathscr{R}_L$, Θ^D is an isomorphism.*

Proof. Let $\widetilde{D}$ be an extension of $\mathscr{R}_L$ by D in the category of (φ_q, Γ)-modules whose image under Θ^D is zero. Let $e \in \widetilde{D}$ be a lifting of $1 \in \mathscr{R}_L$. As the image of $g \mapsto g(e) - e$, $g \in G^+$, in $H^1(D)$ is zero, there exists some $d \in D$ such that $(g-1)e = (g-1)d$ for all $g \in G^+$. Then $g(e-d) = e-d$ for all $g \in G^+$. Thus $\widetilde{D} = D \oplus \mathscr{R}_L(e-d)$ as a (φ_q, Γ)-module. This proves the injectivity of Θ^D. Next we prove the surjectivity of Θ^D. Given a 1-cocycle $g \mapsto c(g) \in D$, correspondingly we can extend the (φ_q, Γ)-module structure on D to the $\mathscr{R}_L$-module $\widetilde{D} = D \oplus \mathscr{R}_L e$ such that $\varphi_q(e) = e + c(\varphi_q)$ and $\gamma(e) = e + c(\gamma)$ for $\gamma \in \Gamma$. □

If D_1 and D_2 are $\mathcal{O}_F$-analytic (φ_q,Γ)-modules over $\mathscr{R}_L$, we use $\mathrm{Ext}_{\mathrm{an}}(D_1, D_2)$ to denote the L-vector space of extensions of D_1 by D_2 in the category of $\mathcal{O}_F$-analytic (φ_q, Γ)-modules over $\mathscr{R}_L$. We will introduce another cohomology theory $H^*_{\mathrm{an}}(-)$, wherein for any $\mathcal{O}_F$-analytic (φ_q, Γ)-module D the first cohomology group $H^1_{\mathrm{an}}(D)$ coincides with $\mathrm{Ext}_{\mathrm{an}}(\mathscr{R}_L, D)$.

If D is $\mathcal{O}_F$-analytic, we consider the complex

$$C^\bullet_{\varphi_q,\nabla}(D): \quad 0 \to D \xrightarrow{f_1} D \oplus D \xrightarrow{f_2} D \to 0,$$

where $f_1 : D \to D \oplus D$ is the map $m \mapsto ((\varphi_q - 1)m, \nabla m)$ and $f_2 : D \oplus D \to D$ is $(m, n) \mapsto \nabla m - (\varphi_q - 1)n$. As f_1 and f_2 are Γ-equivariant, Γ acts on the cohomology groups $H^i_{\varphi_q,\nabla}(D) := H^i(C^\bullet_{\varphi_q,\nabla}(D))$, $i = 0, 1, 2$. Put $H^i_{\mathrm{an}}(D) := H^i_{\varphi_q,\nabla}(D)^\Gamma$.

By a simple calculation we obtain

$$H^0(D) = H^0_{\mathrm{an}}(D) = D^{\varphi_q=1,\Gamma=1}.$$

Note that $D^{\varphi_q=1}$ is finite-dimensional over L, and so is $H^0(D)$. If D is étale and if V is the L-linear Galois representation of G_F attached to D, then

$$H^0(D) = H^0_{\mathrm{an}}(D) = H^0(G_F, V) = V^{G_F}.$$

We introduce some convenient notation. Put $Z^1_{\varphi_q,\nabla}(D) := \ker(f_2)$ and $B^1(D) := \mathrm{im}(f_1)$. For any (m_1, n_1) and (m_2, n_2) in $Z^1_{\varphi_q,\nabla}(D)$, we write $(m_1, n_1) \sim (m_2, n_2)$ if $(m_1 - m_2, n_1 - n_2) \in B^1(D)$. Put

$$Z^1(D) := \{(m, n) \in Z^1_{\varphi_q,\nabla}(D) : (m, n) \sim \gamma(m, n) \text{ for any } \gamma \in \Gamma\}.$$

Then $H^1_{\mathrm{an}}(D) = Z^1(D)/B^1(D)$.

Let $\widetilde{D}$ be an $\mathcal{O}_F$-analytic extension of $\mathcal{R}_L$ by D. Let $e \in \widetilde{D}$ be a lifting of $1 \in \mathcal{R}_L$. Then $((\varphi_q - 1)e, \nabla_{\widetilde{D}} e)$ belongs to $Z^1(D)$ and induces an element of $H^1_{\mathrm{an}}(D)$ independent of the choice of e. In this way we obtain a map

$$\Theta^D_{\mathrm{an}} : \mathrm{Ext}_{\mathrm{an}}(\mathcal{R}_L, D) \to H^1_{\mathrm{an}}(D).$$

Theorem 4.2 (= Theorem 0.1). *For any $\mathcal{O}_F$-analytic (φ_q, Γ)-module D over $\mathcal{R}_L$, Θ^D_{an} is an isomorphism.*

The proof below is due to the referee and is much simpler than that in our original version.

Proof. First we show that Θ^D_{an} is injective. Let $\widetilde{D}$ be an $\mathcal{O}_F$-analytic extension of $\mathcal{R}_L$ by D whose image under Θ^D_{an} is zero. Let $e \in \widetilde{D}$ be a lifting of $1 \in \mathcal{R}_L$. As the image of $((\varphi_q - 1)e, \nabla_{\widetilde{D}} e)$ in $H^1_{\varphi_q,\nabla}(D)$ is zero, there exists some $d \in D$ such that $(\varphi_q - 1)e = (\varphi_q - 1)d$ and $\nabla_{\widetilde{D}} e = \nabla_{\widetilde{D}} d$. Then $e - d$ is in $\widetilde{D}^{\varphi_q=1,\nabla=0}$. The Γ-action on $\widetilde{D}^{\varphi_q=1,\nabla=0}$ is locally constant and thus is semisimple. So $1 \in \mathcal{R}_L$ has a lifting $e' \in \widetilde{D}^{\varphi_q=1,\nabla=0}$ fixed by Γ. This proves the injectivity of Θ^D_{an}.

Next we prove the surjectivity of Θ^D_{an}.

Let z be in $H^1_{\mathrm{an}}(D)$ and let (x, y) represent z, so that $\nabla x = (\varphi_q - 1)y$. The invariance of z by Γ ensures the existence of $y_\sigma \in D$ for each $\sigma \in \Gamma$ such that $(\sigma - 1)(x, y) = ((\varphi_q - 1)y_\sigma, \nabla y_\sigma)$. As y_σ is unique up to an element of $D^{\varphi_q=1,\nabla=0}$, the 2-cocycle $y_{\sigma,\tau} = y_{\sigma\tau} - \sigma y_\tau - y_\sigma$ takes values in $D^{\varphi_q=1,\nabla=0}$. If $z = 0$, then there exists $a \in D$ such that $x = (\varphi_q - 1)a$ and $y = \nabla a$. We have $\nabla(y_\sigma - (\sigma - 1)a) = 0$. In other words, we can write $y_\sigma = (\sigma - 1)a + a_\sigma$ with $a_\sigma \in D^{\varphi_q=1,\nabla=0}$. Then

$y_{\sigma,\tau} = a_{\sigma\tau} - \sigma a_\tau - a_\sigma$ and thus $y_{\bullet,\bullet}$ is a coboundary. So we obtain a map $H^1_{\mathrm{an}}(D) \to H^2(\Gamma, D^{\varphi_q=1,\nabla=0})$.

We will show that the image of z by this map is zero. Fix a basis $\{e_1, \dots, e_d\}$ of D over $\mathcal{R}_L$. Let $r > 0$ be sufficiently small such that the matrices of φ_q and $\sigma \in \Gamma$ relative to $\{e_i\}_{i=1}^d$ are all in $\mathrm{GL}_d(\mathscr{E}_L^{]0,r]})$. Put $D^{]0,r]} = \bigoplus_{i=1}^d \mathscr{E}_L^{]0,r]} e_i$; if $s \in (0, r]$ put $D^{[s,r]} = \bigoplus_{i=1}^d \mathscr{E}_L^{[s,r]} e_i$. Then $D^{]0,r]}$ and $D^{[s,r]}$ are stable by Γ. As the matrix of φ_q is invertible in $\mathrm{M}_d(\mathscr{E}_L^{]0,r]})$, $\{\varphi_q(e_i)\}_{i=1}^d$ is also a basis of $D^{]0,r]}$. Shrinking r if necessary we may assume that φ_q maps $D^{[s,r]}$ to $D^{[s/q,r/q]}$; we may also suppose that x and y are in $D^{]0,r]}$, and that $t_{\mathcal{F}} \in \mathscr{E}_L^{]0,r]}$. By the relation $\nabla = t_{\mathcal{F}} \partial$ on $\mathscr{E}_L^{[s,r]}$, Lemma 2.10 and the fact that ∇ is a differential operator, that is, satisfies a relation similar to (1-3), we can show that the action of Γ induces a bounded infinitesimal action ∇ on the Banach space $D^{[s,r]}$. We leave this to the reader. Let us denote $\ell(\sigma) = \log(\chi_{\mathcal{F}}(\sigma))$. For σ close enough to 1 (depending on D and s, r) the series of operators

$$E(\sigma) = \ell(\sigma) + \frac{\ell(\sigma)^2}{2}\nabla + \frac{\ell(\sigma)^3}{3!}\nabla^2 + \cdots$$

converges on $D^{[s,r]}$ and also on $D^{[s/q,r/q]}$. Note that for σ close enough to 1 we have $\sigma = \exp(\ell(\sigma)\nabla)$ on $D^{[s/q,r/q]}$. Let Γ' be an open subgroup of Γ such that for $\sigma \in \Gamma'$ the above two facts hold. Then for $\sigma \in \Gamma'$ we have

$$(\varphi_q - 1)(E(\sigma)y) = E(\sigma)(\varphi_q - 1)y = E(\sigma)\nabla x = \nabla E(\sigma)x = (\sigma - 1)x. \tag{4-1}$$

Note that $\varphi_q(E(\sigma)y)$ is in $D^{[s/q,r/q]}$. So by (4-1) we have

$$E(\sigma)y \in D^{[s/q,r/q]} \cap D^{[s,r]} = D^{[s/q,r]}$$

if s is chosen such that $s < r/q$. Doing this repeatedly we will obtain $E(\sigma)y \in D^{]0,r]}$. Taking $y_\sigma = E(\sigma)y$ for $\sigma \in \Gamma'$ we will have $y_{\sigma,\tau} = 0$ for $\sigma, \tau \in \Gamma'$. In other words, the restriction to Γ' of the image of z in $H^2(\Gamma, D^{\varphi_q=1,\nabla=0})$ is 0. Since Γ/Γ' is finite and $D^{\varphi_q=1,\nabla=0}$ is a $\mathbb{Q}$-vector space, the image of z is itself 0. So we can modify y_σ by an element of $D^{\varphi_q=1,\nabla=0}$ so that $y_{\sigma,\tau}$ is identically 0. But this means that $(\sigma - 1)y_\tau = (\tau - 1)y_\sigma$, so the 1-cocycle $\varphi_q \mapsto x$, $\sigma \mapsto y_\sigma$ defines an element of $H^1(D)$, hence also an extension of $\mathcal{R}_L$ by D.

We will show that the resulting extension in fact belongs to $\mathrm{Ext}^1_{\mathrm{an}}(\mathcal{R}_L, D)$. As Γ is locally constant on $D^{\varphi_q=1,\nabla=0}$, shrinking Γ' if necessary we may assume that Γ' acts trivially on $D^{\varphi_q=1,\nabla=0}$. Then $\sigma \mapsto y_\sigma - E(\sigma)y$ is a continuous homomorphism from Γ' to $D^{\varphi_q=1,\nabla=0}$. Note that any continuous homomorphism from Γ' to $D^{\varphi_q=1,\nabla=0}$ can be extended to Γ. Thus $y_\sigma - E(\sigma)y = \lambda(\sigma)$ for some $\lambda \in \mathrm{Hom}(\Gamma, D^{\varphi_q=1,\nabla=0})$ and all $\sigma \in \Gamma'$. If S is a set of representatives of Γ/Γ' in Γ, the map

$$T_S = \frac{1}{|\Gamma : \Gamma'|} \sum_{\sigma \in S} \sigma$$

is the identity on $H^1_{\mathrm{an}}(D)$ and a projection from $D^{\varphi_q=1,\nabla=0}$ to $H^0(D)$; moreover it commutes with φ_q, ∇ and Γ. This means that we can apply T_S to (x,y) and y_σ; then we have $y_\sigma - E(\sigma)y = \lambda(\sigma)$ for some $\lambda\in \mathrm{Hom}(\Gamma, H^0(D))$ and all $\sigma\in\Gamma'$. As $\sigma\mapsto E(\sigma)y$ is analytic, the extension in question is $\mathcal{O}_F$-analytic. □

As above, let $\mathrm{Hom}(\Gamma, H^0(D))$ be the set of continuous homomorphisms of groups from Γ to $H^0(D)$. An element $h:\Gamma\to H^0(D)$ of this set is said to be *locally analytic* if $h(\exp(a\beta)) = ah(\exp\beta)$ for all $a\in\mathcal{O}_F$ and $\beta\in\mathrm{Lie}\,\Gamma$. Let $\mathrm{Hom}_{\mathrm{an}}(\Gamma, H^0(D))$ be the subset of $\mathrm{Hom}(\Gamma, H^0(D))$ consisting of locally analytic homomorphisms. We have natural injections

$$\mathrm{Hom}_{\mathrm{an}}(\Gamma, H^0(D))\to \mathrm{Ext}^1_{\mathrm{an}}(\mathcal{R}_L, D)\quad\text{and}\quad \mathrm{Hom}(\Gamma, H^0(D))\to \mathrm{Ext}^1(\mathcal{R}_L, D).$$

Theorem 4.3. *Assume that D is an $\mathcal{O}_F$-analytic (φ_q,Γ)-module over $\mathcal{R}_L$. Then we have an exact sequence*

$$0\to \mathrm{Hom}_{\mathrm{an}}(\Gamma, H^0(D))\to \mathrm{Hom}(\Gamma, H^0(D))\oplus \mathrm{Ext}^1_{\mathrm{an}}(\mathcal{R}_L, D)\to \mathrm{Ext}^1(\mathcal{R}_L, D)\to 0.$$

For the proof we introduce an auxiliary cohomology theory. Let γ be an element of Γ of infinite order, i.e., $\log(\chi_{\mathcal{F}}(\gamma))\neq 0$. We consider the complex

$$C^\bullet_{\varphi_q,\gamma}(D):\quad 0\to D\xrightarrow{g_1} D\oplus D\xrightarrow{g_2} D\to 0,$$

where $g_1: D\to D\oplus D$ is the map $m\mapsto((\varphi_q-1)m, (\gamma-1)m)$ and $g_2: D\oplus D\to D$ is $(m,n)\mapsto(\gamma-1)m-(\varphi_q-1)n$. As g_1 and g_2 are Γ-equivariant, Γ acts on $H^i_{\varphi_q,\gamma}(D) := H^i(C^\bullet_{\varphi_q,\gamma}(D))$, $i=0,1,2$. Put $H^i_{\mathrm{an},\gamma}(D) := H^i_{\varphi_q,\gamma}(D)^\Gamma$. A simple calculation shows that $H^0_{\mathrm{an},\gamma}(D) = H^0_{\mathrm{an}}(D)$.

For any $\gamma\in\Gamma$ we use $\overline{\langle\gamma\rangle}$ to denote the closed subgroup of Γ topologically generated by γ. If γ is of infinite order and if D is an $\mathcal{R}_L$-module together with a semilinear $\overline{\langle\gamma\rangle}$-action, let ∇_γ be the operator on D that can be written as $\lim_{\gamma'\to 1}(\log(\gamma')/\log(\chi_{\mathcal{F}}(\gamma')))$ formally, where γ' runs through all elements of $\overline{\langle\gamma\rangle}$ with $\log\chi_{\mathcal{F}}(\gamma')\neq 0$. (For a precise definition we only need to imitate the definition of ∇.)

Let $\widetilde{D}$ be an $\mathcal{O}_F$-analytic extension of $\mathcal{R}_L$ by D. Let $e\in\widetilde{D}$ be a lifting of $1\in\mathcal{R}_L$. Then $((\varphi_q-1)e, (\gamma-1)e)$ induces an element of $H^1_{\mathrm{an},\gamma}(D)$ independent of the choice of e. This yields a map $\Theta^D_{\mathrm{an},\gamma}:\mathrm{Ext}_{\mathrm{an}}(\mathcal{R}_L, D)\to H^1_{\mathrm{an},\gamma}(D)$. Given an element of $H^1_{\mathrm{an},\gamma}(D)$, we can attach to it an extension $\widetilde{D}$ of $\mathcal{R}_L$ by D in the category of free $\mathcal{R}_L$-modules of finite rank together with semilinear actions of φ_q and $\overline{\langle\gamma\rangle}$. Let $e\in\widetilde{D}$ be a lifting of $1\in\mathcal{R}_L$. Then $\big((\varphi_q-1)e, \nabla_\gamma e\big)$ belongs to $Z^1(D)$ and induces an element of $H^1_{\mathrm{an}}(D)$ independent of the choice of e. This gives a map $\Upsilon^D_{\mathrm{an},\gamma}: H^1_{\mathrm{an},\gamma}(D)\to H^1_{\mathrm{an}}(D)$. Observe that $\Upsilon^D_{\mathrm{an},\gamma}\circ\Theta^D_{\mathrm{an},\gamma} = \Theta^D_{\mathrm{an}}$. By an argument similar to the proof of the injectivity of Θ^D_{an}, we can show that both $\Theta^D_{\mathrm{an},\gamma}$ and

$\Upsilon^D_{\mathrm{an},\gamma}$ are injective. Hence it follows from Theorem 4.2 that $\Theta^D_{\mathrm{an},\gamma}$ and $\Upsilon^D_{\mathrm{an},\gamma}$ are isomorphisms.

If c is a 1-cocycle representing an element z of $H^1(D)$, then $(c(\varphi_q), c(\gamma))$ induces an element in $H^1_{\mathrm{an},\gamma}(D)$ which only depends on z. This yields a map $\Upsilon^D_\gamma : H^1(D) \to H^1_{\mathrm{an},\gamma}(D)$. Hence, $\Theta^D_{\mathrm{an},\gamma} : \mathrm{Ext}_{\mathrm{an}}(\mathcal{R}_L, D) \to H^1_{\mathrm{an},\gamma}(D)$ extends to a map $\mathrm{Ext}(\mathcal{R}_L, D) \to H^1_{\mathrm{an},\gamma}(D)$, which will also be denoted by $\Theta^D_{\mathrm{an},\gamma}$. We have the following commutative diagram:

$$\begin{tikzcd} \mathrm{Ext}(\mathcal{R}_L, D) \arrow[r, "\Theta^D", "\sim"'] \arrow[dr, "\Theta^D_{\mathrm{an},\gamma}"] & H^1(D) \arrow[d, "\Upsilon^D_\gamma"] \\ \mathrm{Ext}_{\mathrm{an}}(\mathcal{R}_L, D) \arrow[u, hook] \arrow[r, "\sim", "\Theta^D_{\mathrm{an},\gamma}"'] & H^1_{\mathrm{an},\gamma}(D) \end{tikzcd} \tag{4-2}$$

The composition $(\Theta^D_{\mathrm{an},\gamma^{-1}})^{-1} \circ \Upsilon^D_\gamma \circ \Theta^D$ is a projection from $\mathrm{Ext}(\mathcal{R}_L, D)$ to $\mathrm{Ext}_{\mathrm{an}}(\mathcal{R}_L, D)$, which depends on γ.

Proof of Theorem 4.3. We only need to prove the surjectivity of

$$\mathrm{Hom}(\Gamma, H^0(D)) \oplus \mathrm{Ext}^1_{\mathrm{an}}(\mathcal{R}_L, D) \to \mathrm{Ext}^1(\mathcal{R}_L, D).$$

Let $\widetilde{D}$ be in $\mathrm{Ext}^1(\mathcal{R}_L, D)$. Without loss of generality we may assume that the image of $\widetilde{D}$ by the projection $(\Theta^D_{\mathrm{an},\gamma^{-1}})^{-1} \circ \Upsilon^D_\gamma \circ \Theta^D$ is zero. Let $e \in \widetilde{D}$ be a lifting of $1 \in \mathcal{R}_L$. Then let c be the 1-cocycle defined by $\varphi_q \mapsto (\varphi_q - 1)e$, $\sigma \mapsto (\sigma - 1)e$ for $\sigma \in \Gamma$, so that $\bar{c}$, the class of c in $H^1(D)$, corresponds to $\widetilde{D}$. So the image of $\bar{c}$ by the map Υ^D_γ is zero. This means that there exists $d \in D$ such that $(\varphi_q - 1)d = c(\varphi_q)$ and $(\gamma - 1)d = c(\gamma)$. Replacing e by $e - d$, we may assume that $c(\varphi_q) = c(\gamma) = 0$. Then for any $\sigma \in \Gamma$, we have $(\varphi_q - 1)c(\sigma) = (\sigma - 1)c(\varphi_q) = 0$ and $(\gamma - 1)c(\sigma) = (\sigma - 1)c(\gamma) = 0$. This means that $c(\sigma) \in D^{\varphi_q=1,\gamma=1}$. Note that $M := D^{\varphi_q=1,\gamma=1}$ is of finite rank over L. We write $M = H^0(D) \oplus \bigoplus_j M_j$ as a Γ-module, where each M_j is an irreducible Γ-module. Write $c = c' + \sum_j c_j$ by this decomposition. Observe that c' and c_j are all 1-cocycles. As M_j is irreducible and the Γ-action on M_j is nontrivial, there exists some $\gamma_j \in \Gamma$ such that $\gamma_j - 1$ is invertible on M_j. Then there exists $m_j \in M_j$ such that $c_j(\gamma_j) = (\gamma_j - 1)m_j$. A simple calculation shows that $c_j(\sigma) = (\sigma - 1)m_j$ for all $\sigma \in \Gamma$. Replacing e by $e - \sum_j m_j$, we may assume that $c = c'$. Then $c(\varphi_q) = 0$ and $c|_\Gamma$ is a homomorphism from Γ to $H^0(D)$. □

Corollary 4.4 (= Theorem 0.2). $\mathrm{Ext}_{\mathrm{an}}(\mathcal{R}_L, D)$ *is of codimension*

$$([F : \mathbb{Q}_p] - 1) \dim_L H^0(D)$$

in $\mathrm{Ext}(\mathcal{R}_L, D)$. *In particular*, *if* $H^0(D) = 0$, *then* $\mathrm{Ext}_{\mathrm{an}}(\mathcal{R}_L, D) = \mathrm{Ext}(\mathcal{R}_L, D)$; *in other words*, *all extensions of* $\mathcal{R}_L$ *by* D *are* $\mathcal{O}_F$*-analytic.*

Proof. This follows from Theorem 4.3 and the equalities $\dim_L \operatorname{Hom}(\Gamma, H^0(D)) = [F:\mathbb{Q}_p]\dim_L H^0(D)$ and $\dim_L \operatorname{Hom}_{\mathrm{an}}(\Gamma, H^0(D)) = \dim_L H^0(D)$. □

5. Computation of $H^1_{\mathrm{an}}(\delta)$ and $H^1(\delta)$

In the case of $F=\mathbb{Q}_p$, Colmez [2008] computed H^1 for not necessarily étale (φ,Γ)-modules of rank 1 over the Robba ring. In this case, Liu [2008] computed H^2 for this kind of (φ,Γ)-modules, and used it and Colmez's result to build analogues, for not necessarily étale (φ,Γ)-modules over the Robba ring, of the Euler–Poincaré characteristic formula and Tate local duality. Later, Chenevier [2013] obtained the Euler–Poincaré characteristic formula for families of triangulable (φ,Γ)-modules and some related results.

In this section we compute $H^1_{\mathrm{an}}(\delta)=H^1_{\mathrm{an}}(\mathcal{R}_L(\delta))$ (for $\delta\in\mathcal{I}_{\mathrm{an}}(L)$) and $H^1(\delta)=H^1(\mathcal{R}_L(\delta))$ (for $\delta\in\mathcal{I}(L)$) following Colmez's approach. In Sections 5B and 5E we assume that δ is in $\mathcal{I}(L)$; in Sections 5C, 5D and 5F we assume that δ is in $\mathcal{I}_{\mathrm{an}}(L)$.

5A. *Preliminary lemmas.*

Lemma 5.1. (a) *If $\alpha\in L^\times$ is not of the form π^{-i}, $i\in\mathbb{N}$, then $\alpha\varphi_q-1:\mathcal{R}_L^+\to\mathcal{R}_L^+$ is an isomorphism.*

(b) *If $\alpha=\pi^{-i}$ with $i\in\mathbb{N}$, then the kernel of $\alpha\varphi_q-1:\mathcal{R}_L^+\to\mathcal{R}_L^+$ is $L\cdot t_{\mathcal{F}}^i$, and $a\in\mathcal{R}_L^+$ is in the image of $\alpha\varphi_q-1$ if and only if $\partial^i a|_{u_{\mathcal{F}}=0}=0$. Further, $\alpha\varphi_q-1$ is bijective on the subset $\{a\in\mathcal{R}_L^+:\partial^i a|_{u_{\mathcal{F}}=0}=0\}$.*

Proof. The argument is similar to the proof of [Colmez 2008, Lemma A.1]. If $k>-v_\pi(\alpha)$, then $-\sum_{n=0}^{+\infty}(\alpha\varphi_q)^n$ is the continuous inverse of $\alpha\varphi_q-1$ on $u_{\mathcal{F}}^k\mathcal{R}_L^+$. The assertions follow from the fact that $\mathcal{R}_L^+=\bigoplus_{i=0}^{k-1}L\cdot t_{\mathcal{F}}^i\oplus u_{\mathcal{F}}^k\mathcal{R}_L^+$ and the formula $\varphi_q(t_{\mathcal{F}}^i)=\pi^i t_{\mathcal{F}}^i$. We just need to remark that $\partial^i a|_{u_{\mathcal{F}}=0}=0$ if and only if a is in $\bigoplus_{j=0}^{i-1}Lt_{\mathcal{F}}^j\oplus u_{\mathcal{F}}^{i+1}\mathcal{R}_L^+$. □

Lemma 5.2. *If $\alpha\in L$ satisfies $v_\pi(\alpha)<1-v_\pi(q)$, then for any $b\in\mathcal{E}_L^\dagger$ there exists $c\in\mathcal{E}_L^\dagger$ such that $b'=b-(\alpha\varphi_q-1)c$ is in $(\mathcal{E}_L^\dagger)^{\psi=0}$.*

Proof. By Proposition 2.4(d), $c=\sum_{k=1}^{+\infty}\alpha^{-k}\psi^k(b)$ is convergent in $\mathcal{E}_L^\dagger$. It is easy to check that $\alpha c-\psi(c)=\psi(b)$, which proves the lemma. □

Corollary 5.3. *If $\alpha\in L$ satisfies $v_\pi(\alpha)<1-v_\pi(q)$, then for any $b\in\mathcal{R}_L$ there exists $c\in\mathcal{R}_L$ such that $b'=b-(\alpha\varphi_q-1)c$ is in $(\mathcal{E}_L^\dagger)^{\psi=0}$.*

Proof. Let k be an integer $>-v_\pi(\alpha)$. By Lemma 5.1, there exists $c_1\in\mathcal{R}_L$ such that $b-(\alpha\varphi_q-1)c_1$ is of the form $\sum_{i<k}a_iu_{\mathcal{F}}^i$ and thus is in $\mathcal{E}_L^\dagger$. Then we apply Lemma 5.2. □

Lemma 5.4. *If $\alpha\in L$ satisfies $v_\pi(\alpha)<1-v_\pi(q)$, and if $z\in\mathcal{R}_L$ satisfies $\psi(z)-\alpha z\in\mathcal{R}_L^+$, then $z\in\mathcal{R}_L^+$.*

Proof. Write z in the form $\sum_{k\in\mathbb{Z}} a_k u_{\mathcal{F}}^k$ and put $y = \sum_{k\le -1} a_k u_{\mathcal{F}}^k \in \mathscr{E}_L^\dagger$. If $y \neq 0$, multiplying z by a scalar in L we may suppose that $\inf_{k\le -1} v_p(a_k) = 0$. Then

$$y - \alpha^{-1}\psi(y) = \alpha^{-1}(\alpha z - \psi(z)) + \sum_{k\ge 0} a_k(\alpha^{-1}\psi(u_{\mathcal{F}}^k) - u_{\mathcal{F}}^k)$$

belongs to $\mathcal{O}_{\mathscr{E}_L^\dagger} \cap \mathscr{R}_L^+ = \mathcal{O}_L[\![u_{\mathcal{F}}]\!]$. But this is a contradiction since $y - \alpha^{-1}\psi(y) \equiv y \bmod \pi$. Hence $y = 0$. □

Corollary 5.5. *If $\alpha \in L$ satisfies $v_\pi(\alpha) < 1 - v_\pi(q)$ and if $z \in \mathscr{R}_L$ is such that $(\alpha\varphi_q - 1)z \in \mathscr{R}_L^{\psi=0}$, then z is in $\mathscr{R}_L^+$.*

Proof. We have $\psi(z) - \alpha z = \psi(z - \alpha\varphi_q(z)) = 0$. Then we apply Lemma 5.4. □

5B. ***Computation of $H^0(\delta)$.*** Recall that if $\delta \in \mathscr{I}_{\mathrm{an}}(L)$, then $H^0_{\mathrm{an}}(\delta) = H^0(\delta)$.

Proposition 5.6. *Let δ be in $\mathscr{I}(L)$.*

(a) *If δ is not of the form x^{-i} with $i \in \mathbb{N}$, then $H^0(\delta) = 0$.*

(b) *If $i \in \mathbb{N}$, then $H^0(x^{-i}) = Lt_{\mathcal{F}}^i$.*

Proof. Observe that

$$\mathscr{R}_L^-(\delta)^{\varphi_q=1} = (\mathscr{R}_L^-)^{\delta(\pi)\varphi_q=1} \cdot e_\delta = 0,$$

where $\mathscr{R}_L^-(\delta) = \mathscr{R}_L(\delta)/\mathscr{R}_L^+(\delta)$. Thus $\mathscr{R}_L(\delta)^{\varphi_q=1,\Gamma=1} = \mathscr{R}_L^+(\delta)^{\varphi_q=1,\Gamma=1}$. If $\delta(\pi)$ is not of the form π^{-i}, with $i \in \mathbb{N}$, by Lemma 5.1(a) we have $\mathscr{R}_L^+(\delta)^{\varphi_q=1} = 0$ and so $\mathscr{R}_L^+(\delta)^{\varphi_q=1,\Gamma=1} = 0$. If $\delta(\pi) = \pi^{-i}$, then

$$\mathscr{R}_L^+(\delta)^{\varphi_q=1,\Gamma=1} = (Lt_{\mathcal{F}}^i \cdot e_\delta)^{\Gamma=1} = \begin{cases} Lt_{\mathcal{F}}^i \cdot e_\delta & \text{if } \delta = x^{-i}, \\ 0 & \text{otherwise,} \end{cases}$$

as desired. □

Corollary 5.7. *If δ_1 and δ_2 are two different characters in $\mathscr{I}(L)$, then $\mathscr{R}_L(\delta_1)$ is not isomorphic to $\mathscr{R}_L(\delta_2)$.*

Proof. We only need to show that $\mathscr{R}_L(\delta_1\delta_2^{-1})$ is not isomorphic to $\mathscr{R}_L$. By Proposition 5.6, $\mathscr{R}_L(\delta_1\delta_2^{-1})$ is not generated by $H^0(\delta_1\delta_2^{-1})$, but $\mathscr{R}_L$ is generated by $H^0(1)$. Thus $\mathscr{R}_L(\delta_1\delta_2^{-1})$ is not isomorphic to $\mathscr{R}_L$. □

5C. ***Computation of $H^1_{\mathrm{an}}(\delta)$ for $\delta \in \mathscr{I}_{\mathrm{an}}(L)$ with $v_\pi(\delta(\pi)) < 1 - v_\pi(q)$.*** Until the end of Section 5 we will write $\mathscr{R}_L(\delta)$ as $\mathscr{R}_L$ with the twisted (φ_q, Γ)-action given by

$$\varphi_{q;\delta}(x) = \delta(\pi)\varphi_q(x), \quad \sigma_{a;\delta}(x) = \delta(a)\sigma_a(x).$$

Recall that $\nabla_\delta = t_{\mathcal{F}}\partial + w_\delta$. Write $\delta(\sigma_a) = \delta(a)$.

Lemma 5.8. *Suppose that $\delta \in \mathscr{I}_{\mathrm{an}}(L)$ satisfies $v_\pi(\delta(\pi)) < 1 - v_\pi(q)$. For any $(a,b) \in Z^1_{\varphi_q,\nabla}(\delta)$, there exists $(m,n) \in Z^1_{\varphi_q,\nabla}(\delta)$ with $m \in (\mathscr{E}_L^\dagger)^{\psi=0}$ and $n \in \mathscr{R}_L^+$ such that $(a,b) \sim (m,n)$.*

Proof. As $v_\pi(\delta(\pi)) < 1 - v_\pi(q)$, by Corollary 5.3 there exists $c \in \mathscr{R}_L$ such that

$$m = a - (\delta(\pi)\varphi_q - 1)c$$

is in $(\mathscr{E}_L^\dagger)^{\psi=0}$. Put $n = b - \nabla_\delta c$. Then (m,n) is in $Z^1_{\varphi_q,\nabla}(\delta)$ and $(m,n) \sim (a,b)$. As $(\delta(\pi)\varphi_q - 1)n = \nabla_\delta m = t_{\mathcal{F}}\partial m + w_\delta m$ is in $\mathscr{R}_L^{\psi=0}$, by Corollary 5.5, n is in $\mathscr{R}_L^+$. □

Lemma 5.9. *Suppose that $v_\pi(\delta(\pi)) < 1 - v_\pi(q)$ and δ is not of the form x^{-i}. Let (m,n) be in $Z^1_{\varphi_q,\nabla}(\delta)$ with $m \in (\mathscr{E}_L^\dagger)^{\psi=0}$ and $n \in \mathscr{R}_L^+$. Then (m,n) is in $B^1(\delta)$ if and only if*

- *$m \in (\mathscr{E}_L^+)^{\psi=0}$ when $\delta(\pi)$ is not of the form π^{-i}, $i \in \mathbb{N}$;*
- *$m \in (\mathscr{E}_L^+)^{\psi=0}$ and $\partial^i m|_{u_{\mathcal{F}}=0} = 0$ when $\delta(\pi) = \pi^{-i}$ and $w_\delta \neq -i$ for some $i \in \mathbb{N}$;*
- *$m \in (\mathscr{E}_L^+)^{\psi=0}$ and $\partial^i m|_{u_{\mathcal{F}}=0} = \partial^i n|_{u_{\mathcal{F}}=0} = 0$ when $\delta(\pi) = \pi^{-i}$ and $w_\delta = -i$ for some $i \in \mathbb{N}$.*

Proof. We only prove the assertion for the case that $\delta(\pi) = \pi^{-i}$ and $w_\delta \neq -i$ for some $i \in \mathbb{N}$. The arguments for the other two cases are similar.

If (m,n) is in $B^1(\delta)$, then there exists $z \in \mathscr{R}_L$ such that $(\delta(\pi)\varphi_q - 1)z = m$ and $\nabla_\delta z = n$. Since m is in $\mathscr{R}_L^{\psi=0}$, by Corollary 5.5 we have $z \in \mathscr{R}_L^+$. It follows that m is in $\mathscr{R}_L^+ \cap \mathscr{E}_L^\dagger = \mathscr{E}_L^+$. By Lemma 5.1(b), we have $\partial^i m|_{u_{\mathcal{F}}=0} = 0$.

Now we assume that m is in $\mathscr{E}_L^+$ and $\partial^i m|_{u_{\mathcal{F}}=0} = 0$. By Lemma 5.1(b), there exists $z \in \mathscr{R}_L^+$ with $\partial^i z|_{u_{\mathcal{F}}=0} = 0$ such that $(\delta(\pi)\varphi_q - 1)z = m$. Then

$$\begin{aligned}(\delta(\pi)\varphi_q - 1)(\nabla_\delta z - n) &= \nabla_\delta(\delta(\pi)\varphi_q - 1)z - (\delta(\pi)\varphi_q - 1)n \\ &= \nabla_\delta m - (\delta(\pi)\varphi_q - 1)n = 0.\end{aligned}$$

Again by Lemma 5.1(b), we have $\nabla_\delta z - n = c\, t_{\mathcal{F}}^i$ for some $c \in L$. Put $z' = z - c\, t_{\mathcal{F}}^i/(w_\delta + i)$. Then $(\delta(\pi)\varphi_q - 1)z' = m$ and $\nabla_\delta z' = n$. Hence (m,n) is in $B^1(\delta)$. □

Recall that $S_\delta = \mathscr{R}_L^-(\delta)^{\Gamma=1,\psi=0}$.

Proposition 5.10. *Suppose that $v_\pi(\delta(\pi)) < 1 - v_\pi(q)$.*

(a) *If δ is not of the form x^{-i}, then $H^1_{\mathrm{an}}(\delta)$ is isomorphic to the L-vector space S_δ and is 1-dimensional.*

(b) *If $\delta = x^{-i}$, then $H^1_{\mathrm{an}}(\delta)$ is 2-dimensional over L and is generated by the images of $(t_{\mathcal{F}}^i, 0)$ and $(0, t_{\mathcal{F}}^i)$.*

Proof. For (a) we only consider the case that $\delta(\pi)=\pi^{-i}$ and $w_\delta=-i$ for some $i\in\mathbb{N}$. The arguments for the other cases are similar. As $\delta\neq x^{-i}$, there exists an element $\gamma_1\in\Gamma$ of infinite order such that $\delta(\gamma_1)\neq\chi_{\mathcal{F}}(\gamma_1)^{-i}$.

We give two useful facts: for any $z\in\mathcal{R}_L^+$, $\partial^i z|_{u_{\mathcal{F}}=0}=0$ if and only if

$$\partial^i(\delta(\gamma_1)\gamma_1-1)z|_{u_{\mathcal{F}}=0}=0;$$

if $\partial^i z|_{u_{\mathcal{F}}=0}=0$, then $\partial^i(\delta(\gamma)\gamma-1)z|_{u_{\mathcal{F}}=0}=0$ for any $\gamma\in\Gamma$. Both of these two facts follow from Lemma 5.1(b). We will use them freely below.

Let (m,n) be in $Z^1(\delta)$ with $m\in(\mathcal{E}_L^\dagger)^{\psi=0}$ and $n\in\mathcal{R}_L^+$. For any $\gamma\in\Gamma$, since $\gamma(m,n)-(m,n)\in B^1(\delta)$, by Lemma 5.9, $(\delta(\gamma)\gamma-1)m$ is in $\mathcal{R}_L^+$; i.e., the image of m in $\mathcal{R}_L^-(\delta)$ belongs to S_δ.

We will show that, for any $\bar{m}\in S_\delta$, there exists a lifting $m\in(\mathcal{E}_L^\dagger)^{\psi=0}$ of $\bar{m}$ such that $\partial^i(\delta(\gamma)\gamma-1)m|_{u_{\mathcal{F}}=0}=0$ for all $\gamma\in\Gamma$. Let $m'\in(\mathcal{E}_L^\dagger)^{\psi=0}$ be an arbitrary lifting of $\bar{m}$. Assume that $\partial^i(\delta(\gamma_1)\gamma_1-1)m'|_{u_{\mathcal{F}}=0}=c$. Put

$$m=m'-\frac{1}{i!}\frac{ct_{\mathcal{F}}^i}{\delta(\gamma_1)\chi_{\mathcal{F}}(\gamma_1)^i-1}.$$

Then $\partial^i(\delta(\gamma_1)\gamma_1-1)m|_{u_{\mathcal{F}}=0}=0$ and thus $\partial^i\nabla_\delta m|_{u_{\mathcal{F}}=0}=0$. Hence, by Lemma 5.1(b) there exists $n\in\mathcal{R}_L^+$ with $\partial^i n|_{u_{\mathcal{F}}=0}=0$ such that $(\delta(\pi)\varphi_q-1)n=\nabla_\delta m$. This means that $(m,n)\in Z^1_{\varphi_q,\nabla}(\delta)$. For any $\gamma\in\Gamma$, since

$$\partial^i(\delta(\gamma_1)\gamma_1-1)(\delta(\gamma)\gamma-1)m|_{u_{\mathcal{F}}=0}=\partial^i(\delta(\gamma)\gamma-1)(\delta(\gamma_1)\gamma_1-1)m|_{u_{\mathcal{F}}=0}=0,$$

we have $\partial^i(\delta(\gamma)\gamma-1)m|_{u_{\mathcal{F}}=0}=0$. In a word, for any $\gamma\in\Gamma$, $(\delta(\gamma)\gamma-1)m$ is in $\mathcal{R}_L^+$ and $\partial^i(\delta(\gamma)\gamma-1)m|_{u_{\mathcal{F}}=0}=\partial^i(\delta(\gamma)\gamma-1)n|_{u_{\mathcal{F}}=0}=0$. This means that $\gamma(m,n)-(m,n)$ is in $B^1(\delta)$ for any $\gamma\in\Gamma$. In other words, (m,n) is in $Z^1(\delta)$.

Now let (m_1,n_1) and (m_2,n_2) be two elements of $Z^1(\delta)$ with $m_1,m_2\in(\mathcal{E}_L^\dagger)^{\psi=0}$ and $n_1,n_2\in\mathcal{R}_L^+$. By Lemma 5.9,

$$\begin{aligned}\partial^i(\delta(\gamma_1)\gamma_1-1)m_1|_{u_{\mathcal{F}}=0}&=\partial^i(\delta(\gamma_1)\gamma_1-1)m_2|_{u_{\mathcal{F}}=0}=\partial^i(\delta(\gamma_1)\gamma_1-1)n_1|_{u_{\mathcal{F}}=0}\\&=\partial^i(\delta(\gamma_1)\gamma_1-1)n_2|_{u_{\mathcal{F}}=0}=0.\end{aligned}$$

Suppose that the image of m_1 in S_δ coincides with that of m_2, which implies that $m_1-m_2\in\mathcal{E}_L^+$. From

$$\partial^i(\delta(\gamma_1)\gamma_1-1)(m_1-m_2)|_{u_{\mathcal{F}}=0}=\partial^i(\delta(\gamma_1)\gamma_1-1)(n_1-n_2)|_{u_{\mathcal{F}}=0}=0$$

we obtain $\partial^i(m_1-m_2)|_{u_{\mathcal{F}}=0}=\partial^i(n_1-n_2)|_{u_{\mathcal{F}}=0}=0$. Thus $(m_1,n_1)\sim(m_2,n_2)$.

Combining all of the above discussions, we obtain an isomorphism $S_\delta\xrightarrow{\sim}H^1_{\mathrm{an}}(\delta)$. Then by Proposition 3.20, $\dim_L H^1_{\mathrm{an}}(\delta)=\dim_L S_\delta=1$.

Next we prove (b). Again let (m,n) be in $Z^1(\delta)$ with $m\in(\mathscr{E}_L^\dagger)^{\psi=0}$ and $n\in\mathscr{R}_L^+$. Then the image of m in $\mathscr{R}_L^-(\delta)$, denoted by $\bar{m}$, is in S_δ. We show that m in fact belongs to $(\mathscr{R}_L^+)^{\psi=0}$, i.e., $\bar{m}=0$. By Corollary 3.15, $\partial^i:S_\delta\to S_1$ is an isomorphism. So we only need to prove that the image of $\partial^i m$ in S_1 is zero. By Remark 3.19, it suffices to show that $\nabla\partial^i m|_{u_{\mathcal{F}}=0}=0$. But $\nabla\partial^i m=\partial^i\nabla_\delta m$. Since $\nabla_\delta m=(\delta(\pi)\varphi_q-1)n$, by Lemma 5.1(b) we have $\partial^i\nabla_\delta m|_{u_{\mathcal{F}}=0}=0$.

Write $m=at_{\mathcal{F}}^i+m'$ with $a\in L$ and $m'\in\mathscr{R}_L^+$ satisfying $\partial^i m'|_{u_{\mathcal{F}}=0}=0$. By Lemma 5.1(b) there exists $z\in\mathscr{R}_L^+$ such that $(\delta(\pi)\varphi_q-1)z=m'$. Then $(m,n)\sim(at_{\mathcal{F}}^i,n-\nabla_\delta z)$. Thus we may suppose that $m=at_{\mathcal{F}}^i$. Then

$$(\delta(\pi)\varphi_q-1)n=\nabla_\delta(at_{\mathcal{F}}^i)=0.$$

So, by Lemma 5.1(b), we have $n=bt_{\mathcal{F}}^i$ for some $b\in L$. Suppose that $(at_{\mathcal{F}}^i,bt_{\mathcal{F}}^i)$ is in $B^1(\delta)$. Then there exists $z\in\mathscr{R}_L$ such that $(\delta(\pi)\varphi_q-1)z=at_{\mathcal{F}}^i$ and $\nabla_\delta z=bt_{\mathcal{F}}^i$. So $\psi(z)-\delta(\pi)z=\psi\big((1-\delta(\pi)\varphi_q)z\big)=\psi(-at_{\mathcal{F}}^i)\in\mathscr{R}_L^+$. By Lemma 5.4 we get $z\in\mathscr{R}_L^+$. By Lemma 5.1(b) again we have $a=0$ and $z\in Lt_{\mathcal{F}}^i$. Then $bt_{\mathcal{F}}^i=\nabla_\delta z=0$. □

5D. $\partial:H^1_{\varphi_q,\nabla}(x^{-1}\delta)\to H^1_{\varphi_q,\nabla}(\delta)$ *and* $\partial:H^1_{\mathrm{an}}(x^{-1}\delta)\to H^1_{\mathrm{an}}(\delta)$. Observe that, if (m,n) is in $Z^1_{\varphi_q,\nabla}(x^{-1}\delta)$ (resp. $B^1(x^{-1}\delta)$), then $(\partial m,\partial n)$ is in $Z^1_{\varphi_q,\nabla}(\delta)$ (resp. $B^1(\delta)$). Thus we have a map $\partial:H^1_{\varphi_q,\nabla}(x^{-1}\delta)\to H^1_{\varphi_q,\nabla}(\delta)$. Further, the map is Γ-equivariant and thus induces a map $\partial:H^1_{\mathrm{an}}(x^{-1}\delta)\to H^1_{\mathrm{an}}(\delta)$.

Put

$$\begin{aligned}\bar{Z}^1_{\varphi_q,\nabla}(\delta)&:=\{(m,n)\in Z^1_{\varphi_q,\nabla}(\delta):\mathrm{Res}(m)=\mathrm{Res}(n)=0\},\\ \bar{B}^1(\delta)&:=\{(m,n)\in B^1(\delta):\mathrm{Res}(m)=\mathrm{Res}(n)=0\}.\end{aligned}$$

Then $\bar{H}^1_{\varphi_q,\nabla}(\delta):=\bar{Z}^1_{\varphi_q,\nabla}(\delta)/\bar{B}^1_{\varphi_q,\nabla}(\delta)$ is a subspace of $H^1_{\varphi_q,\nabla}(\delta)$.

Lemma 5.11. *If $\delta(\pi)\neq\pi/q$ or $w_\delta\neq1$, then for any $(m,n)\in Z^1_{\varphi_q,\nabla}(\delta)$, there exists $(m_1,n_1)\in\bar{Z}^1_{\varphi_q,\nabla}(\delta)$ such that $(m,n)\sim(m_1,n_1)$, and so $H^1_{\varphi_q,\nabla}(\delta)=\bar{H}^1_{\varphi_q,\nabla}(\delta)$.*

Proof. Let (m,n) be in $Z^1_{\varphi_q,\nabla}(\delta)$. Then $\nabla_\delta m=(\delta(\pi)\varphi_q-1)n$. If $\delta(\pi)\neq\pi/q$, by Proposition 2.13 and the definition of Res we have

$$\mathrm{Res}\bigg(m-(\delta(\pi)\varphi_q-1)\bigg(\mathrm{Res}(m)\frac{(\mathrm{d}t_{\mathcal{F}}/\mathrm{d}u_{\mathcal{F}})^{-1}}{(\delta(\pi)\frac{q}{\pi}-1)u_{\mathcal{F}}}\bigg)\bigg)=0.$$

Replacing (m,n) by

$$\bigg(m-(\delta(\pi)\varphi_q-1)\bigg(\mathrm{Res}(m)\frac{(\mathrm{d}t_{\mathcal{F}}/\mathrm{d}u_{\mathcal{F}})^{-1}}{(\delta(\pi)\frac{q}{\pi}-1)u_{\mathcal{F}}}\bigg),n-\nabla_\delta\bigg(\mathrm{Res}(m)\frac{(\mathrm{d}t_{\mathcal{F}}/\mathrm{d}u_{\mathcal{F}})^{-1}}{(\delta(\pi)\frac{q}{\pi}-1)u_{\mathcal{F}}}\bigg)\bigg),$$

we may assume that $\mathrm{Res}(m)=0$. Then

$$\begin{aligned}\big(\tfrac{q}{\pi}\delta(\pi)-1\big)\mathrm{Res}(n)&=\mathrm{Res}\big((\delta(\pi)\varphi_q-1)n\big)=\mathrm{Res}(\nabla_\delta m)\\&=\mathrm{Res}\big(\partial(t_{\mathcal{F}}m)+(w_\delta-1)m\big)=(w_\delta-1)\mathrm{Res}(m)=0,\end{aligned}$$

and so $\mathrm{Res}(n) = 0$.

The argument for the case of $w_\delta \neq 1$ is similar. $\square$

The map $\partial : H^1_{\varphi_q,\nabla}(x^{-1}\delta) \to H^1_{\varphi_q,\nabla}(\delta)$ factors through $\partial : H^1_{\varphi_q,\nabla}(x^{-1}\delta) \to \overline{H}^1_{\varphi_q,\nabla}(\delta)$, since $\mathrm{Res} \circ \partial = 0$.

Lemma 5.12. (a) *If $\delta(\pi) \neq \pi$ or $w_\delta \neq 1$, then $\partial : H^1_{\varphi_q,\nabla}(x^{-1}\delta) \to \overline{H}^1_{\varphi_q,\nabla}(\delta)$ is surjective.*

(b) *If $\delta(\pi) = \pi$ and $w_\delta = 1$, then we have an exact sequence of Γ-modules*

$$H^1_{\varphi_q,\nabla}(x^{-1}\delta) \xrightarrow{\partial} \overline{H}^1_{\varphi_q,\nabla}(\delta) \to L(x^{-1}\delta) \to 0.$$

Proof. Let (m, n) be in $\bar{Z}^1_{\varphi_q,\nabla}(\delta)$. Then there exist m' and n' such that $\partial m' = m$ and $\partial n' = n$. Then $\nabla_{x^{-1}\delta} m' - (\pi^{-1}\delta(\pi)\varphi_q - 1)n' = c$ is in L. If $\delta(\pi) \neq \pi$, we replace n' by $n' + c/(\pi^{-1}\delta(\pi) - 1)$. If $w_\delta \neq 1$, we replace m' by $m' - c/(w_\delta - 1)$. Then (m', n') is in $Z^1_{\varphi_q,\nabla}(x^{-1}\delta)$. This proves (a). When $\delta(\pi) = \pi$ and $w_\delta = 1$, $\nabla m' - (\varphi_q - 1)n'$ does not depend on the choice of m' and n'. This induces a map $\overline{H}^1_{\varphi_q,\nabla}(\delta) \to L$ whose kernel is exactly $\partial H^1_{\varphi_q,\nabla}(x^{-1}\delta)$. We show that $\overline{H}^1_{\varphi_q,\nabla}(\delta) \to L$ is surjective. Put $m' = \log(\varphi_q(u_{\mathcal{F}})/u_{\mathcal{F}}^q)$. A simple calculation shows that

$$\nabla m' = \left(\frac{t_{\mathcal{F}} \cdot [\pi]'_{\mathcal{F}}(u_{\mathcal{F}})}{[\pi]_{\mathcal{F}}(u_{\mathcal{F}})} - q\frac{t_{\mathcal{F}}}{u_{\mathcal{F}}}\right)\partial u_{\mathcal{F}} \equiv (1-q) \mod u_{\mathcal{F}}\mathcal{R}_L^+.$$

Thus by Lemma 5.1(b) there exists $n' \in u_{\mathcal{F}}\mathcal{R}_L^+$ such that $(\varphi_q - 1)n' = \nabla m' - (1-q)$. Put $m = \partial m'$ and $n = \partial n'$. Then (m, n) is in $\bar{Z}^1_{\varphi_q,\nabla}(\delta)$, whose image in L is nonzero. The Γ-action on $\overline{H}^1_{\varphi_q,\nabla}(\delta)$ induces an action on L. From

$$\big(\delta(a)\sigma_a(m), \delta(a)\sigma_a(n)\big) = \big(\partial(a^{-1}\delta(a)\sigma_a(m')), \partial(a^{-1}\delta(a)\sigma_a(n'))\big)$$

and

$$\begin{aligned}\nabla(a^{-1}\delta(a)\sigma_a(m')) - (\varphi_q - 1)(a^{-1}\delta(a)\sigma_a(n')) &= a^{-1}\delta(a)\sigma_a(\nabla m' - (\varphi_q - 1)n') \\ &\equiv a^{-1}\delta(a)(1-q) \mod u_{\mathcal{F}}\mathcal{R}_L^+,\end{aligned}$$

we see that the induced action comes from the character $x^{-1}\delta$. $\square$

Sublemma 5.13. *Let a, b be in L. If (a, b) is in $Z^1_{\varphi_q,\nabla}(x^{-1}\delta)$ but not in $B^1(x^{-1}\delta)$, then $\delta(\pi) = \pi$ and $w_\delta = 1$.*

Proof. If $\delta(\pi) \neq \pi$, then $(a, b) \sim \left(0, b - \dfrac{\nabla_{x^{-1}\delta}}{\pi^{-1}\delta(\pi) - 1}a\right)$. So

$$\begin{aligned}(\pi^{-1}\delta(\pi) - 1)\left(b - \frac{\nabla_{x^{-1}\delta}}{\pi^{-1}\delta(\pi) - 1}a\right) &= (\pi^{-1}\delta(\pi)\varphi_q - 1)\left(b - \frac{\nabla_{x^{-1}\delta}}{\pi^{-1}\delta(\pi) - 1}a\right) \\ &= 0.\end{aligned}$$

As $\delta(\pi) \neq \pi$, we have $b - \dfrac{\nabla_{x^{-1}\delta}}{\pi^{-1}\delta(\pi) - 1}a = 0$. Similarly, if $w_\delta \neq 1$, then (a, b) is in $Z^1_{\varphi_q,\nabla}(x^{-1}\delta)$ if and only if $(a,b)\sim(0,0)$. $\square$

Recall that δ_{unr} is the character of $F^\times$ such that $\delta_{\mathrm{unr}}(\pi)=q^{-1}$ and $\delta_{\mathrm{unr}}|_{\mathcal{O}_F^\times}=1$.

Sublemma 5.14. *The pair*

$$(m,n):=\left(\frac{1}{q}\log\frac{\varphi_q(u_{\mathcal{F}})}{u_{\mathcal{F}}^q},\frac{t_{\mathcal{F}}\partial u_{\mathcal{F}}}{u_{\mathcal{F}}}\right)$$

induces a nonzero element of $H^1_{\mathrm{an}}(\delta_{\mathrm{unr}})$.

Proof. Note that $m=(\delta_{\mathrm{unr}}(\pi)\varphi_q-1)\log u_{\mathcal{F}}$ and $n=\nabla\log u_{\mathcal{F}}$. Thus (m,n) is in $Z^1_{\varphi_q,\nabla}(\delta_{\mathrm{unr}})$. For any $\gamma\in\Gamma$ we have $\gamma(m,n)\sim(m,n)$. Indeed,

$$\gamma(m,n)-(m,n)=\left((\delta_{\mathrm{unr}}(\pi)\varphi_q-1)\log\frac{\gamma(u_{\mathcal{F}})}{u_{\mathcal{F}}},\nabla\log\frac{\gamma(u_{\mathcal{F}})}{u_{\mathcal{F}}}\right).$$

So (m,n) is in $Z^1(\delta_{\mathrm{unr}})$. We show that (m,n) is not in $B^1(\delta_{\mathrm{unr}})$. Otherwise there exists $z\in\mathcal{R}_L$ such that $m=(\delta_{\mathrm{unr}}(\pi)\varphi_q-1)z$ and $n=\nabla z$. This implies that $\nabla(\log u_{\mathcal{F}}-z)=0$, or equivalently $\log u_{\mathcal{F}}-z$ is in L, a contradiction. $\square$

Corollary 5.15. *If $\delta(\pi)=\pi/q$ and $w_\delta=1$, then $\left(\frac{1}{q}\log(\varphi_q(u_{\mathcal{F}})/u_{\mathcal{F}}^q),t_{\mathcal{F}}\partial u_{\mathcal{F}}/u_{\mathcal{F}}\right)$ is in $Z^1_{\varphi_q,\nabla}(x^{-1}\delta)$ but not in $B^1(x^{-1}\delta)$.*

Lemma 5.16. (a) *If $\delta(\pi)\neq\pi,\pi/q$ or if $w_\delta\neq1$, then $\partial:H^1_{\varphi_q,\nabla}(x^{-1}\delta)\to\overline{H}^1_{\varphi_q,\nabla}(\delta)$ is injective.*

(b) *If $\delta(\pi)=\pi$ and $w_\delta=1$, then we have an exact sequence of Γ-modules*

$$0\to L(x^{-1}\delta)\oplus L(x^{-1}\delta)\to H^1_{\varphi_q,\nabla}(x^{-1}\delta)\xrightarrow{\partial}\overline{H}^1_{\varphi_q,\nabla}(\delta).$$

(c) *If $\delta(\pi)=\pi/q$ and $w_\delta=1$, then we have an exact sequence of Γ-modules*

$$0\to L(x^{-1}\delta)\to H^1_{\varphi_q,\nabla}(x^{-1}\delta)\xrightarrow{\partial}\overline{H}^1_{\varphi_q,\nabla}(\delta).$$

Proof. Let (m,n) be in $Z^1_{\varphi_q,\nabla}(x^{-1}\delta)$, and suppose that $(\partial m,\partial n)\in\bar{B}^1(\delta)$. Let z be an element of $\mathcal{R}_L$ such that $(\delta(\pi)\varphi_q-1)z=\partial m$ and $\nabla_\delta z=\partial n$. If $\mathrm{Res}(z)=0$, then there exists $z'\in\mathcal{R}_L$ such that $\partial z'=z$. Then $m-(\delta(\pi)\pi^{-1}\varphi_q-1)z'$ and $n-\nabla_{x^{-1}\delta}z'$ are in $\{(a,b):a,b\in L\}$, i.e., (m,n) is in $B^1(x^{-1}\delta)\oplus L(0,1)\oplus L(1,0)$.

If either $\delta(\pi)\neq\frac{\pi}{q}$ or $w_\delta\neq1$, we always have $\mathrm{Res}(z)=0$. Indeed, this follows from

$$(\delta(\pi)\tfrac{q}{\pi}-1)\mathrm{Res}(z)=\mathrm{Res}\big((\delta(\pi)\varphi_q-1)z\big)=\mathrm{Res}(\partial m)=0$$

and

$$(w_\delta-1)\mathrm{Res}(z)=\mathrm{Res}\big(\partial(t_{\mathcal{F}}z)+(w_\delta-1)z\big)=\mathrm{Res}(\nabla_\delta z)=\mathrm{Res}(\partial n)=0.$$

In the case of $\delta(\pi)=\frac{\pi}{q}$ and $w_\delta=1$, if $z\in L(\partial u_{\mathcal{F}}/u_{\mathcal{F}})$, then (m,n) is in

$$L(0,1)\oplus L(1,0)\oplus L\left(\frac{1}{q}\log\frac{\varphi_q(u_{\mathcal{F}})}{u_{\mathcal{F}}^q},\frac{t_{\mathcal{F}}\partial u_{\mathcal{F}}}{u_{\mathcal{F}}}\right).$$

Now our lemma follows from Sublemma 5.13 and Corollary 5.15. □

Proposition 5.17. (a) *If $\delta(\pi)\neq\pi,\pi/q$ or if $w_\delta\neq 1$, then*

$$\partial: H^1_{\varphi_q,\nabla}(x^{-1}\delta)\to H^1_{\varphi_q,\nabla}(\delta)$$

is an isomorphism of Γ-modules.

(b) *If $\delta(\pi)=\pi$ and $w_\delta=1$, then we have an exact sequence of Γ-modules*

$$0\to L(x^{-1}\delta)\oplus L(x^{-1}\delta)\to H^1_{\varphi_q,\nabla}(x^{-1}\delta)\xrightarrow{\partial} H^1_{\varphi_q,\nabla}(\delta)\to L(x^{-1}\delta)\to 0.$$

(c) *If $\delta(\pi)=\pi/q$ and $w_\delta=1$, then we have an exact sequence of Γ-modules*

$$0\to L(x^{-1}\delta)\to H^1_{\varphi_q,\nabla}(x^{-1}\delta)\xrightarrow{\partial} H^1_{\varphi_q,\nabla}(\delta)\to L(x^{-1}\delta)\oplus L(x^{-1}\delta)\to 0.$$

Proof. Assertions (a) and (b) follow from Lemmas 5.11, 5.12 and 5.16. Based on these lemmas, for (c) we only need to show that we have an exact sequence of Γ-modules

$$0\to \overline{H}^1_{\varphi_q,\nabla}(\delta)\to H^1_{\varphi_q,\nabla}(\delta)\xrightarrow{\mathrm{Res}} L(x^{-1}\delta)\oplus L(x^{-1}\delta)\to 0,$$

where Res is induced by $(m,n)\mapsto(\mathrm{Res}(m),\mathrm{Res}(n))$, which is Γ-equivariant by Proposition 2.13. Here we prove this under the assumption that q is not a power of π. We will see in Section 5F that it also holds without this assumption. Put $m_1=1/u_{\mathcal{F}}$. Then $\nabla_\delta m_1=t_{\mathcal{F}}\,\partial(1/u_{\mathcal{F}})+1/u_{\mathcal{F}}=\partial(t_{\mathcal{F}}/u_{\mathcal{F}})$ is in $\mathcal{R}_L^+$. As q is not a power of π, the map $\frac{\pi}{q}\varphi_q-1:\mathcal{R}_L^+\to\mathcal{R}_L^+$ is an isomorphism. Let n_1 be the unique solution of $(\frac{\pi}{q}\varphi_q-1)n_1=t_{\mathcal{F}}\partial m_1+m_1$ in $\mathcal{R}_L^+$. Then $c_1=(m_1,n_1)$ is in $Z^1_{\varphi_q,\nabla}(\delta)$ and $\mathrm{Res}(m_1,n_1)=(1,0)\neq 0$. For any $\ell\in\mathbb{N}$ we choose a root ξ_ℓ of $Q_\ell=\varphi_q^{\ell-1}(Q)$. For any $f(u_{\mathcal{F}})\in\mathcal{R}_L^+$, the value of f at ξ_ℓ is an element $f(\xi_\ell)$ in $L\otimes_F F_\ell$. By (3-4) there exists an element $z\in\mathcal{R}_L^+$ whose value at ξ_ℓ is $1\otimes\log\xi_\ell$. Put $m_2=t_{\mathcal{F}}^{-1}(q^{-1}\varphi_q-1)(\log u_{\mathcal{F}}-z)$ and $n_2=\partial(\log u_{\mathcal{F}}-z)$. Then (m_2,n_2) is in $Z^1_{\varphi_q,\nabla}(\delta)$ and $\mathrm{Res}(n_2)=1$. □

Proposition 5.18. (a) *If $\delta\neq x, x\delta_{\mathrm{unr}}$, then $\partial: H^1_{\mathrm{an}}(x^{-1}\delta)\to H^1_{\mathrm{an}}(\delta)$ is an isomorphism.*

(b) *If $\delta=x$, then $\partial: H^1_{\mathrm{an}}(x^{-1}\delta)\to H^1_{\mathrm{an}}(\delta)$ is zero, and $\dim_L H^1_{\mathrm{an}}(\delta)=1$.*

(c) *If $\delta=x\delta_{\mathrm{unr}}$, then $\partial: H^1_{\mathrm{an}}(x^{-1}\delta)\to H^1_{\mathrm{an}}(\delta)$ is zero, and $\dim_L H^1_{\mathrm{an}}(\delta)=2$.*

Proof. We apply Proposition 5.17. There is nothing to prove for the case that $\delta(\pi)\neq\pi,\pi/q$ or $w_\delta\neq 1$. Combining the assertions in this case and Proposition 5.10 we obtain that $\dim_L H^1_{\mathrm{an}}(\delta_{\mathrm{unr}})=1$. This fact is useful below.

Next we consider the case of $\delta(\pi)=\pi/q$ and $w_\delta=1$. The argument for the case of $\delta(\pi)=\pi$ and $w_\delta=1$ is similar.

Let M be the image of $\partial: H^1_{\varphi_q,\nabla}(x^{-1}\delta)\to H^1_{\varphi_q,\nabla}(x)$. Then we have two short exact sequences of Γ-modules

$$0\to L(x^{-1}\delta)\to H^1_{\varphi_q,\nabla}(x^{-1}\delta)\xrightarrow{\partial} M\to 0$$

and

$$0\to M\to H^1_{\varphi_q,\nabla}(\delta)\to L(x^{-1}\delta)\oplus L(x^{-1}\delta)\to 0.$$

We will show that taking Γ-invariants yields two exact sequences

$$0\to L(x^{-1}\delta)^\Gamma\to H^1_{\mathrm{an}}(x^{-1}\delta)\xrightarrow{\partial} M^\Gamma\to 0$$

and

$$0\to M^\Gamma\to H^1_{\mathrm{an}}(\delta)\to L(x^{-1}\delta)^\Gamma\oplus L(x^{-1}\delta)^\Gamma\to 0.$$

If the Γ-actions on $H^1_{\varphi_q,\nabla}(x^{-1}\delta)$ and $H^1_{\varphi_q,\nabla}(\delta)$ are semisimple, then there is nothing to prove. However we will avoid this by an alternative argument. It suffices to prove the surjectivity of $H^1_{\varphi_q,\nabla}(x^{-1}\delta)^\Gamma\to M^\Gamma$ and $H^1_{\varphi_q,\nabla}(\delta)^\Gamma\to L(x^{-1}\delta)^\Gamma\oplus L(x^{-1}\delta)^\Gamma$. The latter follows from the proof of Proposition 5.17. In fact, if $\delta=x\delta_{\mathrm{unr}}$, then (m_1,n_1) and (m_2,n_2) constructed there are in $Z^1(\delta)$. Now let c be any element of M^Γ. Then the preimage $\partial^{-1}(Lc)$ is two-dimensional over L and Γ-invariant. From the definition of $H^1_{\varphi_q,\nabla}$, we obtain that the induced ∇-action on $\partial^{-1}(Lc)$ is zero and thus $\partial^{-1}(Lc)$ is a semisimple Γ-module, as wanted.

If $\delta=x\delta_{\mathrm{unr}}$, then $\dim_L L(x^{-1}\delta)^\Gamma=\dim_L H^1_{\mathrm{an}}(x^{-1}\delta)=1$, and so $M^\Gamma=0$. Thus $\partial: H^1_{\mathrm{an}}(x^{-1}\delta)\to H^1_{\mathrm{an}}(\delta)$ is zero and $\dim_L H^1_{\mathrm{an}}(\delta)=2$. If $\delta\neq x\delta_{\mathrm{unr}}$, then $\partial: H^1_{\mathrm{an}}(x^{-1}\delta)\to H^1_{\mathrm{an}}(\delta)$ is an isomorphism since both $H^1_{\mathrm{an}}(x^{-1}\delta)\to M^\Gamma$ and $M^\Gamma\to H^1_{\mathrm{an}}(\delta)$ are isomorphisms. □

5E. *Dimension of $H^1(\delta)$ for $\delta\in\mathscr{I}(L)$.*

Theorem 5.19. *(= Theorem 0.3) Let δ be in $\mathscr{I}_{\mathrm{an}}(L)$.*

(a) *If δ is not of the form x^{-i} with $i\in\mathbb{N}$, or the form $x^i\delta_{\mathrm{unr}}$ with $i\in\mathbb{Z}_+$, then $H^1_{\mathrm{an}}(\delta)$ and $H^1(\delta)$ are 1-dimensional over L.*

(b) *If $\delta=x^i\delta_{\mathrm{unr}}$ with $i\in\mathbb{Z}_+$, then $H^1_{\mathrm{an}}(\delta)$ and $H^1(\delta)$ are 2-dimensional over L.*

(c) *If $\delta=x^{-i}$ with $i\in\mathbb{N}$, then $H^1_{\mathrm{an}}(\delta)$ is 2-dimensional over L and $H^1(\delta)$ is $(d+1)$-dimensional over L, where $d=[F:\mathbb{Q}_p]$.*

Proof. The assertions for $H^1_{\mathrm{an}}(\delta)$ follow from Propositions 5.10 and 5.18. By Proposition 5.6 we have

$$\dim_L \mathscr{R}_L(\delta)^{\varphi_q=1,\Gamma=1}=\begin{cases}1 & \text{if } \delta=x^{-i} \text{ with } i\in\mathbb{N},\\ 0 & \text{otherwise.}\end{cases}$$

So the assertions for $H^1(\delta)$ come from those for $H^1_{\mathrm{an}}(\delta)$ and Corollary 4.4. □

When $\delta = x^{-i}$ with $i \in \mathbb{N}$, $H^1_{\mathrm{an}}(\delta)$ is generated by the classes of $(t^i_{\mathcal{F}}, 0)$ and $(0, t^i_{\mathcal{F}})$. Let ρ_i $(i = 1, \dots, d)$ be a basis of $\mathrm{Hom}(\Gamma, Lt^i_{\mathcal{F}})$. Then the class of the 1-cocycle c_0 with $c_0(\varphi_q) = t^i_{\mathcal{F}}$ and $c_0|\Gamma = 0$, and the classes of 1-cocycles c_i with $c_i(\varphi_q) = 0$ and $c_i|\Gamma = \rho_i$ $(i = 1, \dots, d)$, form a basis of $H^1(\delta)$.

Theorem 5.20. (= *Theorem 0.4*) *If $\delta \in \mathscr{I}(L)$ is not locally F-analytic, then $H^1(\delta) = 0$.*

Proof. As the maps $\gamma - 1$, $\gamma \in \Gamma$, are null on $H^1(\delta)$, by the definition of H^1, so are the maps $\mathrm{d}\Gamma_{\mathcal{R}_L(\delta)}(\beta)$, $\beta \in \mathrm{Lie}\,\Gamma$ and the differences $\beta^{-1}\mathrm{d}\Gamma_{\mathcal{R}_L(\delta)}(\beta) - \beta'^{-1}\mathrm{d}\Gamma_{\mathcal{R}_L(\delta)}(\beta')$. Note that $\beta^{-1}\mathrm{d}\Gamma_{\mathcal{R}_L(\delta)}(\beta) - \beta'^{-1}\mathrm{d}\Gamma_{\mathcal{R}_L(\delta)}(\beta')$ are $\mathcal{R}_L$-linear on $\mathcal{R}_L(\delta)$. So

$$\beta^{-1}\mathrm{d}\Gamma_{\mathcal{R}_L(\delta)}(\beta) - \beta'^{-1}\mathrm{d}\Gamma_{\mathcal{R}_L(\delta)}(\beta')$$

are multiplications by scalars in L, since $\beta^{-1}\mathrm{d}\Gamma_{\mathcal{R}_L(\delta)}(\beta)e_\delta - \beta'^{-1}\mathrm{d}\Gamma_{\mathcal{R}_L(\delta)}(\beta')e_\delta$ is in Le_δ. If the intersection of their kernels is null, then the cohomology $H^1(\delta)$ vanishes. Thus, either the intersection of their kernels is 0 and so the cohomology vanishes, or they are all null and δ is of the form $x \mapsto x^w$ for x close to 1 with $w = \frac{\log \delta(\beta)}{\log \beta}$ for β close to 1 (i.e., δ is locally F-analytic). □

Remark 5.21. Suppose that $[F : \mathbb{Q}_p] \geq 2$. Let $\delta \neq 1$ be a character of $F^\times$ with $\delta(\pi) \in \mathcal{O}_L^\times$, and let $L(\delta)$ be the L-representation of G_F induced by δ. Suppose that $\delta \neq x^2\delta_{\mathrm{unr}}$ when $[F : \mathbb{Q}_p] = 2$. Combining Theorem 5.19 and the Euler–Poincaré characteristic formula [Tate 1963] we obtain that there exist Galois representations in $\mathrm{Ext}(L, L(\delta))$ that are not overconvergent. Theorem 5.20 tells us that if further δ is not locally analytic, then there is no nontrivial overconvergent extension of L by $L(\delta)$.

5F. *The maps $\iota_k : H^1(\delta) \to H^1(x^{-k}\delta)$ and $\iota_{k,\mathrm{an}} : H^1_{\mathrm{an}}(\delta) \to H^1_{\mathrm{an}}(x^{-k}\delta)$.* Let k be a positive integer.

Proposition 5.22. *Let δ be in $\mathscr{I}_{\mathrm{an}}(L)$.*

(a) *If $w_\delta \notin \{1-k, \dots, 0\}$, then $H^0_{\mathrm{an}}(\mathcal{R}_L(\delta)/t^k_{\mathcal{F}}\mathcal{R}_L(\delta)) = 0$.*

(b) *If $w_\delta \in \{1-k, \dots, 0\}$, then $H^0_{\mathrm{an}}(\mathcal{R}_L(\delta)/t^k_{\mathcal{F}}\mathcal{R}_L(\delta))$ is a 1-dimensional L-vector space.*

Proof. We have $\mathcal{R}^+_L/t^k_{\mathcal{F}}\mathcal{R}^+_L = \mathcal{R}^+_L/(u^k_{\mathcal{F}}) \times \prod_{n=1}^\infty \mathcal{R}^+_L/(\varphi_q^{n-1}(Q))^k$. As Γ-modules, $\mathcal{R}^+_L/(u^k_{\mathcal{F}}) = \bigoplus_{i=0}^{k-1} Lt^i_{\mathcal{F}}$ and $\mathcal{R}^+_L/(\varphi_q^n(Q))^k = \bigoplus_{i=0}^{k-1}(L \otimes_F F_n)t^i_{\mathcal{F}}$. Thus as a Γ-module, $\mathcal{R}^+_L/t^k_{\mathcal{F}}\mathcal{R}^+_L$ is isomorphic to $\bigoplus_{i=0}^{k-1}(\mathcal{R}^+_L/\mathcal{R}^+_L t_{\mathcal{F}}) \otimes_L Lt^i_{\mathcal{F}}$. Note that the natural map $\mathcal{R}^+_L/\mathcal{R}^+_L t^k_{\mathcal{F}} \to \mathcal{R}_L/\mathcal{R}_L t^k_{\mathcal{F}}$ is surjective. Furthermore, two sequences $(y_n)_{n\geq 0}$ and $(z_n)_{n\geq 0}$ in $\mathcal{R}^+_L/\mathcal{R}^+_L u^k_{\mathcal{F}} \times \prod_{n=1}^\infty \mathcal{R}^+_L/(\varphi_q^{n-1}(Q))^k$ have the same image in $\mathcal{R}_L/\mathcal{R}_L t^k_{\mathcal{F}}$, if and only if there exists $N > 0$ such that $y_n = z_n$ when $n \geq N$.

Since the action of Γ on $(\mathcal{R}^+_L/t_{\mathcal{F}}\mathcal{R}^+_L)t^i_{\mathcal{F}}$ twisted by the character x^{-i} is smooth, (a) follows.

For (b) we only need to consider the case of $w_\delta = 0$ and $k = 1$. The operator φ_q induces injections $\mathcal{R}_L^+/(\varphi_q^n(Q)) \to \mathcal{R}_L^+/(\varphi_q^{n+1}(Q))$ denoted by $\varphi_{q,n}$. The action of φ_q on $\mathcal{R}_L/\mathcal{R}_L t_{\mathcal{F}}$ is given by $\varphi_q(y_n)_n = (\varphi_{q,n}(y_n))_{n+1}$. For any $n \geq 0$, the Γ-action on $L \otimes_F F_n$ factors through Γ/Γ_n, and the resulting Γ/Γ_n-module $L \otimes_F F_n$ is isomorphic to the regular one. Thus for any discrete character δ of Γ, $\dim_L(L \otimes_F F_n)^{\Gamma=\delta^{-1}} = 1$ when n is sufficiently large. Then from the fact that the $\varphi_{q,n}$ $(n \geq 1)$ are injective, we obtain $\dim_L(\mathcal{R}_L/t_{\mathcal{F}}\mathcal{R}_L)^{\Gamma=\delta^{-1},\varphi_q=\delta(\pi)^{-1}} = 1$. □

Corollary 5.23. *Let δ be in $\mathcal{I}_{\mathrm{an}}(L)$.*

(a) *If $w_\delta \notin \{1,\dots,k\}$, then $H^0_{\mathrm{an}}(t_{\mathcal{F}}^{-k}\mathcal{R}_L(\delta)/\mathcal{R}_L(\delta)) = 0$.*

(b) *If $w_\delta \in \{1,\dots,k\}$, then $H^0_{\mathrm{an}}(t_{\mathcal{F}}^{-k}\mathcal{R}_L(\delta)/\mathcal{R}_L(\delta))$ is a 1-dimensional L-vector space.*

Note that $\mathcal{R}_L(x^{-k}\delta)$ is canonically isomorphic to $t_{\mathcal{F}}^{-k}\mathcal{R}_L(\delta)$. When $k \geq 1$, the inclusion $\mathcal{R}_L(\delta) \hookrightarrow t_{\mathcal{F}}^{-k}\mathcal{R}_L(\delta)$ induces maps $\iota_{k,\mathrm{an}} : H^1_{\mathrm{an}}(\delta) \to H^1_{\mathrm{an}}(x^{-k}\delta)$ and $\iota_k : H^1(\delta) \to H^1(x^{-k}\delta)$. If $\gamma \in \Gamma$ is of infinite order, then we have this commutative diagram:

$$\begin{array}{ccc} H^1(\delta) & \xrightarrow{\iota_k} & H^1(x^{-k}\delta) \\ \Big\downarrow{\scriptstyle \Upsilon^\delta_{\mathrm{an},\gamma}\circ\Upsilon^\delta_\gamma} & & \Big\downarrow{\scriptstyle \Upsilon^{x^{-k}\delta}_{\mathrm{an},\gamma}\circ\Upsilon^{x^{-k}\delta}_\gamma} \\ H^1_{\mathrm{an}}(\delta) & \xrightarrow{\iota_{k,\mathrm{an}}} & H^1_{\mathrm{an}}(x^{-k}\delta). \end{array} \tag{5-1}$$

Lemma 5.24. *We have the exact sequence*

$$0 \to H^0_{\mathrm{an}}(\delta) \to H^0_{\mathrm{an}}(x^{-k}\delta) \to H^0_{\mathrm{an}}(t_{\mathcal{F}}^{-k}\mathcal{R}_L(\delta)/\mathcal{R}_L(\delta)) \to H^1_{\mathrm{an}}(\delta) \xrightarrow{\iota_{k,\mathrm{an}}} H^1_{\mathrm{an}}(x^{-k}\delta).$$

Proof. From the short exact sequence

$$0 \to \mathcal{R}_L(\delta) \to \mathcal{R}_L(x^{-k}\delta) \to \mathcal{R}_L(x^{-k}\delta)/\mathcal{R}_L(\delta) \to 0, \tag{5-2}$$

we deduce an exact sequence

$$\begin{aligned} 0 \to H^0_{\varphi_q,\nabla}(\delta) \to H^0_{\varphi_q,\nabla}(x^{-k}\delta) \to H^0_{\varphi_q,\nabla}(t_{\mathcal{F}}^{-k}\mathcal{R}_L(\delta)/\mathcal{R}_L(\delta)) \\ \to H^1_{\varphi_q,\nabla}(\delta) \to H^1_{\varphi_q,\nabla}(x^{-k}\delta). \end{aligned} \tag{5-3}$$

Being finite-dimensional, $H^0_{\varphi_q,\nabla}(\delta)$ and $H^0_{\varphi_q,\nabla}(x^{-k}\delta)$ are semisimple Γ-modules; since $t_{\mathcal{F}}^{-k}\mathcal{R}_L(\delta)/\mathcal{R}_L(\delta)$ is a semisimple Γ-module, so is $H^0_{\varphi_q,\nabla}(t_{\mathcal{F}}^{-k}\mathcal{R}_L(\delta)/\mathcal{R}_L(\delta))$. Hence, taking Γ-invariants of each term in (5-3), we obtain the desired exact sequence. □

Proposition 5.25. *Let δ be in $\mathcal{I}_{\mathrm{an}}(L)$, $k \in \mathbb{Z}_+$. If $w_\delta \notin \{1,\dots,k\}$, then $\iota_{k,\mathrm{an}}$ and ι_k are isomorphisms.*

Proof. We only prove the assertion for $\iota_{k,\mathrm{an}}$. The proof of the assertion for ι_k is similar. By Theorem 5.19, $\dim_L H^1_{\mathrm{an}}(\delta) = \dim_L H^1_{\mathrm{an}}(x^{-k}\delta)$ when $w_\delta \notin \{1, \dots, k\}$. Combining Lemma 5.24 with the fact that $H^0_{\mathrm{an}}(t_{\mathcal{F}}^{-k}\mathcal{R}_L(\delta)/\mathcal{R}_L(\delta)) = 0$ and that $\dim_L H^1_{\mathrm{an}}(\delta) = \dim_L H^1_{\mathrm{an}}(x^{-k}\delta)$, we obtain the assertion. □

We assign to any nonzero $c \in H^1_{\mathrm{an}}(\delta)$ an $\mathscr{L}$-invariant in $\mathrm{P}^1(L) = L \cup \{\infty\}$. In the case of $\delta = x^{-k}$ with $k \in \mathbb{N}$, put $\mathscr{L}((at_{\mathcal{F}}^k, bt_{\mathcal{F}}^k)) = a/b$. If $\delta = x\delta_{\mathrm{unr}}$, then any $c \in H^1_{\mathrm{an}}(\delta)$ can be written as

$$c = t_{\mathcal{F}}^{-1}\big((q^{-1}\varphi_q - 1)(\lambda G(1,1) + \mu(\log u_{\mathcal{F}} - z)), t_{\mathcal{F}}\partial(\lambda G(1,1) + \mu(\log u_{\mathcal{F}} - z))\big)$$

with $\lambda, \mu \in L$. Here $G(1,1)$ is an element of $\mathcal{R}_L$ which induces a basis of $(\mathcal{R}_L/\mathcal{R}_L t_{\mathcal{F}})^\Gamma$ and whose value at ξ_n is $1 \otimes 1 \in L \otimes_F F_n$ when n is large enough; z is an element of $\mathcal{R}_L$ whose value at ξ_n is $1 \otimes \log(\xi_n) \in L \otimes_F F_n$ for any n. We put $\mathscr{L}(c) = -(e_F(q-1)/q) \cdot (\lambda/\mu)$. In the case of $\delta = x^k\delta_{\mathrm{unr}}$ with $k \geq 2$, for any $c \in H^1_{\mathrm{an}}(x^k\delta_{\mathrm{unr}})$, put $\mathscr{L}(c) = \mathscr{L}(\iota_{k-1}(c))$. In the case that δ is not of the form x^{-k} with $k \in \mathbb{N}$ or the form $x^k\delta_{\mathrm{unr}}$ with $k \in \mathbb{Z}_+$, we put $\mathscr{L}(c) = \infty$.

Proposition 5.26. *Let δ be in $\mathscr{I}_{\mathrm{an}}(L)$, $k \in \mathbb{Z}_+$.*

(a) *If $w_\delta \in \{1, \dots, k\}$ and if $\delta \neq x^{w_\delta}, x^{w_\delta}\delta_{\mathrm{unr}}$, then $\iota_{k,\mathrm{an}}$ and ι_k are zero.*

(b) *If $\delta = x^{w_\delta}\delta_{\mathrm{unr}}$ with $1 \leq w_\delta \leq k$, then $\iota_{k,\mathrm{an}}$ and ι_k are surjective, and the kernel of $\iota_{k,\mathrm{an}}$ is the 1-dimensional subspace $\{c \in H^1_{\mathrm{an}}(\delta) : c = 0$ or $\mathscr{L}(c) = \infty\}$.*

(c) *If $\delta = x^{w_\delta}$ with $1 \leq w_\delta \leq k$, then $\iota_{k,\mathrm{an}}$ and ι_k are injective, and the image of $\iota_{k,\mathrm{an}}$ is $\{c \in H^1_{\mathrm{an}}(x^{-k}\delta) : c = 0$ or $\mathscr{L}(c) = \infty\}$.*

Proof. We will use Lemma 5.24 frequently without mentioning it.

First we prove (a). From $\dim_L H^0_{\mathrm{an}}(t_{\mathcal{F}}^{-k}\mathcal{R}_L(\delta)/\mathcal{R}_L(\delta)) = \dim_L H^1_{\mathrm{an}}(\delta) = 1$ and $H^0_{\mathrm{an}}(x^{-k}\delta) = 0$ we obtain the assertion for $\iota_{k,\mathrm{an}}$. The assertion for ι_k follows from this and the commutative diagram (5-1), where the two vertical maps are isomorphisms.

Next we prove (b). From $H^0_{\mathrm{an}}(x^{-k}\delta) = 0$, $\dim_L H^0_{\mathrm{an}}(t_{\mathcal{F}}^{-k}\mathcal{R}_L(\delta)/\mathcal{R}_L(\delta)) = 1$, $\dim_L H^1_{\mathrm{an}}(\delta) = 2$, and $\dim_L H^1_{\mathrm{an}}(x^{-k}\delta) = 1$, we obtain the surjectivity of $\iota_{k,\mathrm{an}}$. The surjectivity of ι_k follows from this and the commutative diagram (5-1), where the two vertical maps are isomorphisms. We show that if $c \in H^1_{\mathrm{an}}(\delta)$ satisfies $\mathscr{L}(c) = \infty$, then $\iota_{k,\mathrm{an}}(c) = 0$. As $\mathscr{L}(\iota_{w_\delta-1,\mathrm{an}}(c)) = \infty$ and $\iota_{k,\mathrm{an}} = \iota_{k+1-w_\delta,\mathrm{an}}\iota_{w_\delta-1,\mathrm{an}}$, we reduce to the case of $\delta = x\delta_{\mathrm{unr}}$. In this case, $c = t_{\mathcal{F}}^{-1}\lambda\big((q^{-1}\varphi_q - 1)G(1,1), \nabla G(1,1)\big)$ with $\lambda \in L$. Thus $\iota_{1,\mathrm{an}}(c) = \lambda\big((q^{-1}\varphi_q - 1)G(1,1), \nabla G(1,1)\big) \sim (0,0)$. Hence $\iota_{k,\mathrm{an}}(c) = 0$ for any integer $k \geq 1$.

Finally we prove (c). From the equalities $H^0_{\mathrm{an}}(\delta) = 0$ and $\dim_L H^0_{\mathrm{an}}(x^{-k}\delta) = \dim_L H^0_{\mathrm{an}}(t_{\mathcal{F}}^{-k}\mathcal{R}_L(\delta)/\mathcal{R}_L(\delta)) = 1$, we obtain the injectivity of $\iota_{k,\mathrm{an}}$. The injectivity of ι_k follows from this and the commutative diagram (5-1), where the vertical map $\Upsilon^\delta_{\mathrm{an},\gamma} \circ \Upsilon^\delta_\gamma$ is an isomorphism. For the second assertion, let (m, n) be in $Z^1(x^{w_\delta})$.

Then

$$\iota_{w_\delta-1}(m,n)=(t_{\mathcal{F}}^{w_\delta-1}m,\, t_{\mathcal{F}}^{w_\delta-1}n)\in Z^1(x).$$

In other words, $\partial(t_{\mathcal{F}}^{w_\delta}m)=\nabla_x(t_{\mathcal{F}}^{w_\delta-1}m)=(\pi\varphi_q-1)(t_{\mathcal{F}}^{w_\delta-1}n)$. Thus $\mathrm{Res}(t_{\mathcal{F}}^{w_\delta-1}n)=0$ and there exists $z\in\mathcal{R}_L$ such that $\partial z=t_{\mathcal{F}}^{w_\delta-1}n$ or equivalently $\nabla z=t_{\mathcal{F}}^{w_\delta}n$. It follows that $\nabla_{x^{w_\delta-k}}(t_{\mathcal{F}}^{k-w_\delta}z)=\big(\nabla+(w_\delta-k)\big)(t_{\mathcal{F}}^{k-w_\delta}z)=t_{\mathcal{F}}^{k-w_\delta}\nabla z=t_{\mathcal{F}}^{k}n$. Thus

$$\iota_{k,\mathrm{an}}(m,n)=(t_{\mathcal{F}}^{k}m,\, t_{\mathcal{F}}^{k}n)\sim\big(t_{\mathcal{F}}^{k}m-(\pi^{w_\delta-k}\varphi_q-1)(t_{\mathcal{F}}^{k-w_\delta}z),0\big).$$

So we have $\iota_{k,\mathrm{an}}(m,n)=(at_{\mathcal{F}}^{k-w_\delta},0)$. If $\iota_{k,\mathrm{an}}(m,n)\neq 0$, or equivalently $a\neq 0$, then $\mathcal{L}(\iota_{k,\mathrm{an}}(m,n))=\infty$. □

6. Triangulable (φ_q,Γ)-modules of rank 2

Colmez [2008] classified 2-dimensional trianguline representations of the Galois group $G_{\mathbb{Q}_p}$. Generalizing his work, Nakamura [2009] classified 2-dimensional trianguline representations of the Galois group of a p-adic local field that is finite over $\mathbb{Q}_p$.

In this section we classify triangulable $\mathcal{O}_F$-analytic (φ_q,Γ)-modules of rank 2 following Colmez's method. First we recall the definition.

Definition 6.1. A (φ_q,Γ)-module over $\mathcal{R}_L$ is called *triangulable* if there exists a filtration of D consisting of (φ_q,Γ)-submodules $0=D_0\subset D_1\subset\cdots\subset D_d=D$ such that D_i/D_{i-1} is free of rank 1 over $\mathcal{R}_L$.

Note that if D is $\mathcal{O}_F$-analytic, then so is D_i/D_{i-1} for any i.

If $\delta_1,\delta_2\in\mathcal{I}_{\mathrm{an}}(L)$, then $\mathrm{Ext}(\mathcal{R}_L(\delta_2),\mathcal{R}_L(\delta_1))$ is isomorphic to $\mathrm{Ext}(\mathcal{R}_L,\mathcal{R}_L(\delta_1\delta_2^{-1}))$ or $H^1(\delta_1\delta_2^{-1})$. The isomorphism only depends on the choices of e_{δ_1}, e_{δ_2} and $e_{\delta_1\delta_2^{-1}}$. Thus it is unique up to a nonzero multiple and induces an isomorphism from $\mathrm{Proj}\big(\mathrm{Ext}(\mathcal{R}_L(\delta_2),\mathcal{R}_L(\delta_1))\big)$ to $\mathrm{Proj}(H^1(\delta_1\delta_2^{-1}))$ independent of the choices of e_{δ_1}, e_{δ_2} and $e_{\delta_1\delta_2^{-1}}$. Similarly, there is a natural isomorphism from

$$\mathrm{Proj}\big(\mathrm{Ext}_{\mathrm{an}}(\mathcal{R}_L(\delta_2),\mathcal{R}_L(\delta_1))\big)$$

to $\mathrm{Proj}(H^1_{\mathrm{an}}(\delta_1\delta_2^{-1}))$. Hence the set of triangulable (resp. triangulable and $\mathcal{O}_F$-analytic) (φ_q,Γ)-modules D of rank 2 satisfying the following two conditions is classified by $\mathrm{Proj}(H^1(\delta_1\delta_2^{-1}))$ (resp. $\mathrm{Proj}(H^1_{\mathrm{an}}(\delta_1\delta_2^{-1}))$):

- $\mathcal{R}_L(\delta_1)$ is a saturated (φ_q,Γ)-submodule of D and $\mathcal{R}_L(\delta_2)$ is the quotient module.
- D is not isomorphic to $\mathcal{R}_L(\delta_1)\oplus\mathcal{R}_L(\delta_2)$.

Let $\mathscr{S}^{\mathrm{an}} = \mathscr{S}^{\mathrm{an}}(L)$ be the analytic variety obtained by blowing up $(\delta_1, \delta_2) \in \mathscr{I}_{\mathrm{an}}(L) \times \mathscr{I}_{\mathrm{an}}(L)$ along the subvarieties $\delta_1\delta_2^{-1} = x^i\delta_{\mathrm{unr}}$ for $i \in \mathbb{Z}_+$ and the subvarieties $\delta_1\delta_2^{-1} = x^{-i}$ for $i \in \mathbb{N}$. The fiber over the point (δ_1, δ_2) is isomorphic to $\mathrm{Proj}(H^1_{\mathrm{an}}(\delta_1\delta_2^{-1}))$. Similarly, let $\mathscr{S} = \mathscr{S}(L)$ be the analytic variety over $\mathscr{I}_{\mathrm{an}}(L) \times \mathscr{I}_{\mathrm{an}}(L)$ whose fiber over (δ_1, δ_2) is isomorphic to $\mathrm{Proj}(H^1(\delta_1\delta_2^{-1}))$. The inclusions $\mathrm{Ext}_{\mathrm{an}}(\mathscr{R}_L(\delta_1), \mathscr{R}_L(\delta_2)) \hookrightarrow \mathrm{Ext}(\mathscr{R}_L(\delta_1), \mathscr{R}_L(\delta_2))$ for $\delta_1, \delta_2 \in \mathscr{I}_{\mathrm{an}}(L)$ induce a natural injective map $\mathscr{S}^{\mathrm{an}} \hookrightarrow \mathscr{S}$. We write points of $\mathscr{S}$ (resp. $\mathscr{S}^{\mathrm{an}}$) in the form (δ_1, δ_2, c) with $c \in \mathrm{Proj}(H^1(\delta_1\delta_2^{-1}))$ (resp. $c \in \mathrm{Proj}(H^1_{\mathrm{an}}(\delta_1\delta_2^{-1}))$). If $(\delta_1, \delta_2, c) \in \mathscr{S}$ is in the image of $\mathscr{S}_{\mathrm{an}}$, for our convenience we use c_{an} to denote the element in $\mathrm{Proj}(H^1_{\mathrm{an}}(\delta_1\delta_2^{-1}))$ corresponding to c. For $(\delta_1, \delta_2, c) \in \mathscr{S}^{\mathrm{an}}$, since the $\mathscr{L}$-invariant induces an inclusion $\mathrm{Proj}(H^1_{\mathrm{an}}(\delta_1\delta_2^{-1})) \hookrightarrow \mathrm{P}^1(L)$, we also use $(\delta_1, \delta_2, \mathscr{L}(c))$ to denote (δ_1, δ_2, c).

If $s \in \mathscr{S}$, we assign to s the invariant $w(s) \in L$ by $w(s) = w_{\delta_1} - w_{\delta_2}$. Let $\mathscr{S}_+$ be the subset of $\mathscr{S}$ consisting of elements $s \in \mathscr{S}$ with

$$v_\pi(\delta_1(\pi)) + v_\pi(\delta_2(\pi)) = 0, \quad v_\pi(\delta_1(\pi)) \geq 0.$$

If $s \in \mathscr{S}_+$, we assign to s the invariant $u(s) \in \mathbb{Q}_+$ by

$$u(s) = v_\pi(\delta_1(\pi)) = -v_\pi(\delta_2(\pi)).$$

Put $\mathscr{S}_0 = \{s \in \mathscr{S}_+ \mid u(s) = 0\}$ and $\mathscr{S}_* = \{s \in \mathscr{S}_+ \mid u(s) > 0\}$. Then $\mathscr{S}_+$ is the disjoint union of $\mathscr{S}_0$ and $\mathscr{S}_*$. For $? \in \{+, 0, *\}$ we put $\mathscr{S}^{\mathrm{an}}_? = \mathscr{S}^{\mathrm{an}} \cap \mathscr{S}_?$. We decompose the set $\mathscr{S}^{\mathrm{an}}_?$ as $\mathscr{S}^{\mathrm{an}}_? = \mathscr{S}^{\mathrm{ng}}_? \amalg \mathscr{S}^{\mathrm{cris}}_? \amalg \mathscr{S}^{\mathrm{st}}_? \amalg \mathscr{S}^{\mathrm{ord}}_? \amalg \mathscr{S}^{\mathrm{ncl}}_?$, where

$$\begin{aligned}
\mathscr{S}^{\mathrm{ng}}_? &= \{s \in \mathscr{S}_? \mid w(s) \text{ is not an integer } \geq 1\},\\
\mathscr{S}^{\mathrm{cris}}_? &= \{s \in \mathscr{S}_? \mid w(s) \text{ is an integer } \geq 1, u(s) < w(s), \mathscr{L} = \infty\},\\
\mathscr{S}^{\mathrm{st}}_? &= \{s \in \mathscr{S}_? \mid w(s) \text{ is an integer } \geq 1, u(s) < w(s), \mathscr{L} \neq \infty\}\\
\mathscr{S}^{\mathrm{ord}}_? &= \{s \in \mathscr{S}_? \mid w(s) \text{ is an integer } \geq 1, u(s) = w(s)\},\\
\mathscr{S}^{\mathrm{ncl}}_? &= \{s \in \mathscr{S}_? \mid w(s) \text{ is an integer } \geq 1, u(s) > w(s)\}.
\end{aligned}$$

Note that $\mathscr{S}^{\mathrm{ord}}_0$ and $\mathscr{S}^{\mathrm{ncl}}_0$ are empty.

Let D be an extension of $\mathscr{R}_L(\delta_2)$ by $\mathscr{R}_L(\delta_1)$. For any $k \in \mathbb{N}$, the preimage of $t^k_{\mathcal{F}}\mathscr{R}_L(\delta_2)$ is a (φ_q, Γ)-submodule of D, which is denoted by D'. Then D' is an extension of $\mathscr{R}_L(x^k\delta_2)$ by $\mathscr{R}_L(\delta_1)$. If D is $\mathcal{O}_F$-analytic, then so is D'.

Lemma 6.2. (a) *The class of D' in $H^1(\delta_1\delta_2^{-1}x^{-k})$ coincides with $\iota_k(c)$ up to a nonzero multiple, where c is the class of D in $H^1(\delta_1\delta_2^{-1})$.*

(b) *If D is $\mathcal{O}_F$-analytic, the class of D' in $H^1_{\mathrm{an}}(\delta_1\delta_2^{-1}x^{-k})$ coincides with $\iota_{k,\mathrm{an}}(c)$ up to a nonzero multiple, where c is the class of D in $H^1_{\mathrm{an}}(\delta_1\delta_2^{-1})$.*

Proof. We only prove (b). The proof of (a) is similar. Let e be a basis of $\mathscr{R}_L(\delta_2)$ such that $\varphi_q(e) = \delta_2(\pi)e$ and $\sigma_a e = \delta_2(a)e$. Let $\tilde{e}$ be a lifting of e in D. The class of D, or the same, c, coincides with the class of $\big((\delta_2(\pi)^{-1}\varphi_q-1)\tilde{e}, (\nabla-w_{\delta_2})\tilde{e}\big)$ up to a nonzero multiple. Similarly, up to a nonzero multiple, the class of D' coincides with the class of

$$\begin{aligned}\big((\pi^{-k}\delta_2(\pi)^{-1}\varphi_q-1)(t_{\mathcal{F}}^k\tilde{e}), (\nabla-w_{\delta_2}-k)(t_{\mathcal{F}}^k\tilde{e})\big)&\\ =\big(t_{\mathcal{F}}^k(\delta_2(\pi)^{-1}\varphi_q-1)\tilde{e}, t_{\mathcal{F}}^k(\nabla-w_{\delta_2})\tilde{e}\big)&,\end{aligned}$$

which is exactly $\iota_{k,\mathrm{an}}(c)$. □

Proposition 6.3. *Put $D=D(s)$ with $s=(\delta_1,\delta_2,c)\in\mathscr{S}$. The following two conditions are equivalent*:

(a) *$D(s)$ has a (φ_q,Γ)-submodule M of rank 1 such that $M\cap\mathscr{R}_L(\delta_1)=0$.*

(b) *s is in $\mathscr{S}^{\mathrm{an}}$ and satisfies $w(s)\in\mathbb{Z}_+$, $\delta_1\delta_2^{-1}\neq x^{w(s)}$ and $\mathscr{L}(c_{\mathrm{an}})=\infty$.*

Among all such M there exists a unique one, M_{sat}, that is saturated; *M_{sat} is isomorphic to $\mathscr{R}_L(x^{w(s)}\delta_2)$. For any M that satisfies condition* (a), *there exists some $i\in\mathbb{N}$ such that $M=t_{\mathcal{F}}^iM_{\mathrm{sat}}$.*

Proof. Assume that $D(s)$ satisfies (a). Since the intersection of M and $\mathscr{R}_L(\delta_1)$ is zero, the image of M in $\mathscr{R}_L(\delta_2)$ is a nonzero (φ_q,Γ)-submodule of $\mathscr{R}_L(\delta_2)$, and so must be of the form $t_{\mathcal{F}}^k\mathscr{R}_L(\delta_2)$ with $k\in\mathbb{N}$. Since $D(s)$ does not split, we have $k\geq 1$. The preimage of $t_{\mathcal{F}}^k\mathscr{R}_L(\delta_2)$ in D is exactly $M\oplus\mathscr{R}_L(\delta_1)$. Since $M\oplus\mathscr{R}_L(\delta_1)$ splits, by Lemma 6.2 we have $\iota_k(c)=0$. By Proposition 5.26 this happens only if $w(s)\in\{1,\dots,k\}$ and $\delta_1\delta_2^{-1}\neq x^{w(s)}$. Note that, when $w(s)\in\{1,\dots,k\}$ and $\delta_1\delta_2^{-1}\neq x^{w(s)}$, $D(s)$ is automatically $\mathcal{O}_F$-analytic. Again by Proposition 5.26 we obtain $\mathscr{L}(c_{\mathrm{an}})=\infty$. This proves (a)⇔(b).

If (a) holds, then the preimage of $t_{\mathcal{F}}^{w(s)}\mathscr{R}_L(\delta_2)$ splits as $\mathscr{R}_L(\delta_1)\oplus M_0$, where M_0 is isomorphic to $\mathscr{R}_L(x^{w(s)}\delta_2)$. We show that M_0 is saturated. Note that M_0 is not included in $t_{\mathcal{F}}D(s)$. Otherwise, the preimage of $t_{\mathcal{F}}^{w(s)-1}\mathscr{R}_L(\delta_2)$ will split, which contradicts Proposition 5.26. Let e_1 (resp. e_2, e) be a basis of $\mathscr{R}_L(\delta_1)$ (resp. $\mathscr{R}_L(\delta_2)$, M_0) such that Le_1 (resp. Le_2, Le) is stable under φ_q and Γ. Let $\tilde{e}_2$ be a lifting of e_2. Write $e=ae_1+b\tilde{e}_2$. Then $a\notin t_{\mathcal{F}}\mathscr{R}_L$ and $b\in t_{\mathcal{F}}^{w(s)}\mathscr{R}_L$. Observe that the ideal I generated by a and $t_{\mathcal{F}}^{w(s)}$ satisfies $\varphi_q(I)=I$ and $\gamma(I)=I$ for all $\gamma\in\Gamma$. Thus by Lemma 1.1, $I=\mathscr{R}_L$. It follows that M_0 is saturated. If M is another (φ_q,Γ)-submodule of $D(s)$ such that $M\cap\mathscr{R}_L(\delta_1)=0$, then the image of M in $\mathscr{R}_L(\delta_2)$ is $t_{\mathcal{F}}^k\mathscr{R}_L(\delta_2)$ for some integer $k\geq w(s)$. Then $M\subset\mathscr{R}_L(\delta_1)\oplus M_0$. Since $\delta_1\neq\delta_2x^{w(s)}$, $\mathscr{R}_L(\delta_1)$ has no nonzero (φ_q,Γ)-submodule isomorphic to $\mathscr{R}_L(x^k\delta_2)$. It follows that $M\subset M_0$ and thus $M=t_{\mathcal{F}}^{k-w(s)}M_0$. □

Corollary 6.4. *Let $s=(\delta_1,\delta_2,c)$ be in $\mathscr{S}$. If s is in $\mathscr{S}^{\mathrm{an}}$ and satisfies $w(s)\in\mathbb{Z}_+$, $\delta_1\delta_2^{-1}\neq x^{w(s)}$ and $\mathscr{L}(c_{\mathrm{an}})=\infty$, then $D(s)$ has exactly two saturated (φ_q,Γ)-submodules of $D(s)$ of rank* 1, *one being $\mathscr{R}_L(\delta_1)$ and the other isomorphic to*

$\mathcal{R}_L(x^{w(s)}\delta_2)$. *Otherwise*, $D(s)$ *has exactly one saturated* (φ_q, Γ)-*submodule of rank* 1, *which is* $\mathcal{R}_L(\delta_1)$.

Corollary 6.5. *Let* $s = (\delta_1, \delta_2, c)$ *and* $s' = (\delta_1', \delta_2', c')$ *be in* $\mathcal{S}(L)$.

(a) *If* $\delta_1 = \delta_1'$, *then* $D(s) \cong D(s')$ *if and only if* $s = s'$.

(b) *If* $\delta_1 \neq \delta_1'$, *then* $D(s) \cong D(s')$ *if and only if* s *and* s' *are in* $\mathcal{S}^{\mathrm{an}}$ *and satisfy* $w(s) \in \mathbb{Z}_+$, $\delta_1' = x^{w(s)}\delta_2$, $\delta_2' = x^{-w(s)}\delta_1$ *and* $\mathcal{L}(c_{\mathrm{an}}) = \mathcal{L}(c_{\mathrm{an}}') = \infty$.

Proof. Assertion (a) is clear. We prove (b). Since $D(s) \cong D(s')$, there exists a (φ_q, Γ)-submodule M of $D(s)$ such that $M \cong \mathcal{R}_L(\delta_1')$ and $D(s)/M \cong \mathcal{R}_L(\delta_2')$. Since both $\mathcal{R}_L(\delta_1)$ and M are saturated (φ_q, Γ)-submodules of D, $\mathcal{R}_L(\delta_1) \cap M = 0$. By Proposition 6.3 we have $w(s) \in \mathbb{Z}_+$, $\delta_1\delta_2^{-1} \neq x^{w(s)}$, $\mathcal{L}(c_{\mathrm{an}}) = \infty$ and $\delta_1' = x^{w(s)}\delta_2$. Similarly, $\delta_1 = x^{w(s')}\delta_2'$. As $\delta_1\delta_2 = \delta_1'\delta_2'$, we have $w(s) = w(s')$. □

Proposition 6.6. *Let* $s = (\delta_1, \delta_2, c)$ *be in* $\mathcal{S}$. *Then* $D(s)$ *is of slope zero if and only if* $s \in \mathcal{S}_+ - \mathcal{S}_+^{\mathrm{ncl}}$; $D(s)$ *is of slope zero and the Galois representation attached to* $D(s)$ *is irreducible if and only if* s *is in* $\mathcal{S}_* - (\mathcal{S}_*^{\mathrm{ord}} \cup \mathcal{S}_*^{\mathrm{ncl}})$; $D(s)$ *is of slope zero and* $\mathcal{O}_F$-*analytic if and only if* $s \in \mathcal{S}_+^{\mathrm{an}} - \mathcal{S}_+^{\mathrm{ncl}}$.

Proof. By Kedlaya's slope filtration theorem, $D(s)$ is of slope zero if and only if $v_\pi(\delta_1(\pi)\delta_2(\pi)) = 0$ and $D(s)$ has no (φ_q, Γ)-submodule of rank 1 that is of slope < 0. In particular, if $D(s)$ is of slope zero, then $v_\pi(\delta_1(\pi)) \geq 0$ and thus $s \in \mathcal{S}_+$. Hence we only need to consider the case of $s \in \mathcal{S}_+$. Assume that $D(s)$ has a (φ_q, Γ)-submodule of rank 1, say M, that is of slope < 0. Then the intersection of M and $\mathcal{R}_L(\delta_1)$ is zero. By Proposition 6.3, we may suppose that M is saturated. By Corollary 6.4, this happens if and only if s is in $\mathcal{S}^{\mathrm{an}}$ and satisfies $w(s) \in \mathbb{Z}_+$, $\delta_1\delta_2^{-1} \neq x^{w(s)}$, $\mathcal{L}(c_{\mathrm{an}}) = \infty$ and $w(s) < u(s)$. Note that $\delta_1\delta_2^{-1} \neq x^{w(s)}$ and $\mathcal{L}(c_{\mathrm{an}}) = \infty$ automatically hold when $0 < w(s) < u(s)$. The first assertion follows. Similarly, $D(s)$ has a saturated (φ_q, Γ)-submodule of rank 1 that is of slope zero if and only if $u(s) = 0$ or $u(s) = w(s)$. By Proposition 1.5(c) and Remark 1.8, we know that the Galois representation attached to an étale (φ_q, Γ)-module D over $\mathcal{R}_L$ of rank 2 is irreducible if and only if D has no étale (φ_q, Γ)-submodule of rank 1. This shows the second assertion. The third assertion follows from the first one. □

Proof of Theorem 0.5. Assertion (a) follows from Proposition 6.6, and (b) follows from Corollary 6.5. □

Remark 6.7. Let $s \neq s'$ be as in Theorem 0.5(b). Then $s \in \mathcal{S}_*^{\mathrm{cris}}$ if and only if $s' \in \mathcal{S}_*^{\mathrm{cris}}$; $s \in \mathcal{S}_+^{\mathrm{ord}}$ if and only if $s' \in \mathcal{S}_0^{\mathrm{cris}}$.

Remark 6.8. By an argument similar to that in [Colmez 2008] one can show that if s is in $\mathcal{S}_+^{\mathrm{cris}}$ (resp. $\mathcal{S}_+^{\mathrm{ord}}$, $\mathcal{S}_+^{\mathrm{st}}$), then $D(s)$ comes from a crystalline (resp. ordinary, semistable but noncrystalline) L-representation twisted by a character.

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